SAMPLE PROOF

INTRO TO REAL ANALYSIS

Problem 1. Prove the following statement: If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

Proof. Since (x_n) is bounded, there exists a number M such that $x_n \in [-M, M]$ for all natural numbers n. Moreover, by the Bolzano-Weierstrass Theorem, there exists a convergent subsequence (x_{n_k}) converging to some real number L. Increasing M if necessary, we can assume $L \in (-M, M)$.

Since (x_n) diverges, there exists a real number $\epsilon > 0$ such that the set $\{x_n : x_n \notin (L-\epsilon, L+\epsilon)\}$ is infinite. Shrinking ϵ if necessary, we can assume $(L-\epsilon, L+\epsilon) \subset [-M, M]$. Then, either $[-M, L-\epsilon]$ or $[L+\epsilon, M]$ contains infinitely many points of (x_n).

Without loss of generality, we can assume $[L+\epsilon, M]$ contains infinitely many points of (x_n). Consider the subsequence of (x_n) consisting of all points lying in $[L+\epsilon, M]$. By Bolzano-Weierstrass, it has a convergent subsequence (x_{m_k}) converging to some limit $L' \in [L+\epsilon, M]$. Clearly $L' > L$, so we have produced subsequences converging to different limits, as desired. $\square$

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