FINAL EXAM

INTRODUCTION TO MANIFOLDS

The exam consists of five questions worth a total of 100 points. You may take up to three hours to complete the exam.

Problem 1 (20 pts)**.** Let x, y, z be the standard coordinates on $\mathbb{R}^3$. A plane in $\mathbb{R}^3$ is vertical if it is defined by $ax + by = 0$ for some $(a, b) \neq (0, 0)$. Prove that restricted to a vertical plane, $dx \wedge dy = 0$.

Problem 2 (20 pts)**.** Determine whether each of the following statements is true or false. If false, provide a counterexample. If true, briefly justify your answer.

(a) Let $F : N \to M$ be a one-to-one immersion, and let n and m denote the dimensions of N and M respectively. For any point $q \in F(N) \subset M$, there exists a coordinate neighborhood $(U, x^1, \ldots, x^m)$ about q such that $F(N) \cap U$ is defined by the vanishing of $x^1, \ldots, x^{m-n}$.

(b) Any alternating bilinear map $\omega \in A_2(\mathbb{R}^3)$ is *decomposable*, in the sense that $\omega = \alpha_1 \wedge \alpha_2$ for some covectors $\alpha_1, \alpha_2 \in V^*$.

Problem 3 (20 pts)**.** Let M be a smooth manifold. Recall that the Lie bracket of two smooth vector fields is again a smooth vector field. Is the Lie bracket $C^\infty(M)$-linear in both variables? Justify your answer.

Problem 4 (20 pts)**.** Let M be a smooth manifold. Let $f : M \to \mathbb{R}$ be a smooth function, and suppose c is a regular value of f. Prove that $f^{-1}([c, \infty))$ is a smooth manifold with boundary.

Problem 5 (20 pts)**.** Suppose p is a point in a compact oriented surface M without boundary, and that $i : C \to M \setminus \{p\}$ is the inclusion of a small circle around the puncture. Prove that the restriction map $i^* : H^1(M \setminus \{p\}) \to H^1(C)$ is the zero map. (Hint: You may need to use Stokes' Theorem).

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