MIDTERM EXAM

INTRO TO REAL ANALYSIS II

The exam consists of five questions, each worth 20 points (but there are also 5 bonus points available). You may take up to three hours to complete the exam.

Problem 1. Consider a function $f : \mathbb{R}^m \to \mathbb{R}$. Determine whether each statement about f is true or false. Give a brief explanation (one or two lines). If false, then try to correct the statement and provide a counterexample.

(a) Suppose all directional derivatives of f exist at a point $a \in \mathbb{R}^m$. Then $D(f, \mathbf{u} + \mathbf{v})(a) = D(f, \mathbf{u})(a) + D(f, \mathbf{v})(a)$ for any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^m$.
(b) Suppose f is differentiable at a point a. Then f is continuous at a.
(c) Suppose f is differentiable everywhere. Then f has an extreme value (i.e. a maximum or a minimum) at $a \in \mathbb{R}^m$ if and only if all the partial derivatives of f vanish at a.

Problem 2. Find partial derivatives for the following functions, where $g : \mathbb{R} \to \mathbb{R}$ is continuous:

(a) $f(x, y) = \int_a^{x+y} g$.
(b) $f(x, y) = \int_a^{xy} g$.

Problem 3. (a) Show that the sup norm and the euclidean norm on $\mathbb{R}^n$ are *topologically equivalent*, in the sense that a set $U \subset \mathbb{R}^n$ is open under the sup norm if and only if it is open under the euclidean norm.
(b) Show that any linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ is continuous (you can use part (a), but you are also free to supply a different proof).

Problem 4. (a) We say a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ is an *antiderivative* of a pair of functions g_1 and g_2 from $\mathbb{R}^2 \to \mathbb{R}$ if $D_1 f = g_1$ and $D_2 f = g_2$. For each pair of functions g_1 and g_2, construct an antiderivative f, or show that no such f exists.
(i) $g_1(x, y) = x$, $g_2(x, y) = y$.
(ii) $g_1(x, y) = y$, $g_2(x, y) = x$.
(iii) $g_1(x, y) = y^2$, $g_2(x, y) = x^2$.
(b) Assume g_1 and g_2 are differentiable everywhere. Conjecture a necessary and sufficient condition on g_1 and g_2 for the existence of an antiderivative f. Prove that it is necessary. For extra credit, prove that it is sufficient.

Problem 5. Let g_n and g be uniformly bounded on $[0, 1]$ (i.e. there exists a number $M > 0$ such that $|g(x)| \leq M$ and $|g_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in [0, 1]$). Assume $g_n \to g$ pointwise on $[0, 1]$ and uniformly on any set of the form $[0, \alpha]$, where $0 < \alpha < 1$. Finally, assume all g_n and g are integrable. Show that

$$\lim_{n \to \infty} \int_0^1 g_n = \int_0^1 g.$$

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