FINAL EXAM

INTRODUCTION TO MANIFOLDS

The exam consists of seven questions worth a total of 105 points. You may take up to three hours to complete the exam.

Problem 1 (15 pts)**.** Let x, y, z be the standard coordinates on $\mathbb{R}^3$. A plane in $\mathbb{R}^3$ is vertical if it is defined by $ax + by = 0$ for some $a, b \neq 0, 0$. Prove that restricted to a vertical plane, $dx \wedge dy = 0$.

Problem 2 (15 pts)**.** Determine whether each of the following statements is true or false. If false, provide a counterexample. If true, briefly justify your answer.

(a) The sphere S^2 cannot be given the structure of a Lie group.
(b) The top de Rham cohomology of an orientable smooth manifold is $\mathbb{R}$.
(c) If the first de Rham cohomology of a manifold M is zero, then M is simply connected.

Problem 3 (15 pts)**.** Let X be a smooth vector field on a smooth manifold M. Let $c(t)$ be an integral curve of X whose maximal domain of definition is *not* all of $\mathbb{R}$. Show that the image of c cannot lie in any compact subset of M.

Problem 4 (15 pts)**.** Let M be a smooth manifold. Recall that the Lie bracket of two smooth vector fields is again a smooth vector field. Is the Lie bracket C^∞-linear in both variables? Justify your answer.

Problem 5 (15 pts)**.** Let π be the covering projection $S^2 \to \mathbb{R}P^2$. Prove that $\pi^* : H^*(\mathbb{R}P^2) \to H^*(S^2)$ is injective.

Problem 6 (15 pts)**.** Let M be a smooth manifold. Let $f : M \to \mathbb{R}$ be a smooth function, and suppose c is a regular value of f. Prove that $f^{-1}([c, \infty))$ is a smooth manifold with boundary.

Problem 7 (15 pts)**.** Let $B \subset \mathbb{R}^3$ be a solid ball of radius 2 centered at the origin. Let S^1 denote the unit circle in the xy plane, with open tubular neighborhood

$$S := \{p \in \mathbb{R}^3 : m(S^1, p) < 0.1\},$$

where m is the standard Euclidean metric on $\mathbb{R}^3$. Let $M := B \setminus S$.

(a) Compute the de Rham cohomology vector spaces $H^2(M)$ and $H^3(M)$.
(b) Compute the de Rham cohomology vector space $H^1(M)$.

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