POWER SERIES PRACTICE

INTRO TO REAL ANALYSIS

Problem 1. Recall the construction of the series representation

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

for all $x \in (-1, 1)$. Notice that the series converges at $x = 1$. Using the fact that $\arctan(x)$ is continuous, prove that the series at $x = 1$ is equal to $\arctan(1)$. What identity does this equality yield?

Problem 2. Write down Taylor series for $\frac{1}{1-x}$ and $\sin(x)$. Manipulate these to produce Taylor series for the following functions. For which values of x is each series representation valid?

(a) $x \cos(x^2)$
(b) $x/(1 + 4x^2)^2$
(c) $\ln(1 + x^2)$

Problem 3. Find an example of each of the following or explain why no such function exists.

(a) An infinitely differentiable function $g(x)$ on all of $\mathbb{R}$ with a Taylor series that converges to $g(x)$ only for $x \in (-1, 1)$.
(b) An infinitely differentiable function $h(x)$ with the same Taylor series as $\sin(x)$ but such that $h(x) \neq \sin(x)$ for all $x \neq 0$.
(c) An infinitely differentiable function $f(x)$ on all of $\mathbb{R}$ with a Taylor series that converges to $f(x)$ if and only if $x \leq 0$.

Problem 4. Let $\sum a_n x^n$ be a power series with $a_n \neq 0$ for all n. Assume that

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists.

(a) Show that if $L \neq 0$, then the series converges for all x in $(-1/L, 1/L)$.
(b) Show that if $L = 0$, then the series converges for all $x \in \mathbb{R}$.
(c) What can we say about the power series $\sum a_n x^n$ if the sequence $\left(\left| \frac{a_{n+1}}{a_n} \right| \right)$ is unbounded?

Extra, for those who have looked at $\limsup$ **already**:

(d) Show that (a) and (b) continue to hold if L is replaced by the limit

$$L' = \lim_{n\to\infty} \text{ where } s_n = \sup\left\{ \left| \frac{a_{k+1}}{a_k} \right| : k \geq n \right\}.$$

(Recall that this *limit superior* exists if the sequence $\left(\left| \frac{a_{n+1}}{a_n} \right| \right)$ is bounded.)

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