Two viewpoints on complex numbers / functions:

$\mathbb{C}$	$\mathbb{R}^2$
$z = x + iy$ $\longleftrightarrow$	$z = \begin{bmatrix} x \\ y \end{bmatrix}$

$\boxed{\text{Fix } w = a+bi}$ complex linear map $z \mapsto wz$ $\longleftrightarrow$ real linear map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} ax - by \\ bx + ay \end{bmatrix}$

$$wz = (ax-by) + i(bx+ay)$$

$$\begin{bmatrix} ax - by \\ bx + ay \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Observe:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \underbrace{(a^2+b^2)}_{\text{"dilation"}} \underbrace{\begin{bmatrix} \frac{a}{a^2+b^2} & \frac{-b}{a^2+b^2} \\ \frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{bmatrix}}_{\text{rotation}}$$

"dilation" in $\mathbb{R}_{\geq 0} = \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} : \lambda \geq 0 \right\}$

rotation in $SO(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} : \theta \in \mathbb{R} \right\}$

$\therefore$ Complex linear maps can be thought of as elements of $\mathbb{R}_{\geq 0} \times SO(2)$

(a rotation composed with a dilation)

They are angle preserving and orientation preserving.

Prop: $f : \mathbb{C} \to \mathbb{C}$ is complex-linear

iff $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by a matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

iff $f : \mathbb{R}^2 \to \mathbb{R}^2$ is real-linear and $f(iz) = i f(z)\ \forall z$.

Pf: Check that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

iff $c = -b$ and $d = a$.

Question: What does complex linear mean in higher dimensions?

i.e. If $f : \mathbb{C}^n \to \mathbb{C}^n$ is complex linear, what can we say about $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$?

$U \subset \mathbb{C}$ open

Def: $f: U \to \mathbb{C}$ is <u>complex differentiable</u> at $a \in \mathbb{C}$ if

$$\lim_{\Delta z \to 0} \frac{f(a+\Delta z) - f(a)}{\Delta z}$$

exists. In this case we write $f'(a)$ to denote this limit.

Exercise: (easy, but we will see is clear also)

Say $f'(a) = \alpha + i\beta$.

Then $Df(a) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$

<u>A very surprising fact about complex differentiable functions:</u>

FACT: $U \subseteq \mathbb{C}$ $f: U \to \mathbb{C}$

If $f'(a)$ exists $\forall a \in U$

then (i) f is conformal and orientation-preserving (we've already seen this)

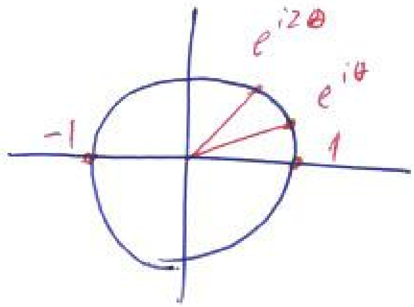
very surprising. follow from Cauchy Integral Formula

$\left\{ \begin{array}{l} \text{(ii) } f \text{ is } \underline{\text{infinitely}} \text{ differentiable } \forall a \in U \\ \text{(iii) } f \text{ is complex analytic on } U \end{array} \right.$

FACT: Complex linear maps are the only 1-1 complex differentiable maps from $\mathbb{C}$ to $\mathbb{C}$

Let's examine polynomial maps:

Example: $z \mapsto z^2$



- NOT linear:

  (it rotates $e^{i\theta}$ by θ, if it were linear it would rotate all vectors by some fixed angle)

- NOT 1-1:

  $1 \mapsto 1$ $\quad i \mapsto -1$ $\quad e^{i\theta} \mapsto e^{i2\theta}$

  $-1 \mapsto 1$ $\quad -i \mapsto -1$ $\quad -e^{i\theta} \mapsto e^{i2\theta}$

(actually 2-1 away from zero)

<u>Def</u>: A <u>complex power series</u> $P(z)$ <u>centered at 0</u> is an expression of the form

$$P(z) = c_0 + c_1 z + c_2 z^2 + \dots$$

where $c_j \in \mathbb{C}$ $\forall j$ and z is a complex variable,

<u>Def</u>: $P(z)$ <u>converges</u> to A at $z = a$ if the partial sums $P_j(a)$ converge to A as $j \to \infty$

(*) $\mathbb{C}$ has a norm given by the <u>modulus</u> that coincides with the Euclidean norm on $\mathbb{R}^2$.

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$\therefore$ Convergence of a sequence in $\mathbb{C}$ is the same as in $\mathbb{R}^2$

<u>Lemma</u>: $P(a)$ converges iff $\forall \epsilon, \exists N$ s.t.

$$\underbrace{|P_n(a) - P_m(a)|}_{\text{Partial Sums}} < \epsilon \quad \text{whenever } n, m \geq N$$

Since $\mathbb{C} \overset{\shortparallel}{} \mathbb{R}^2$ is complete metric space
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<u>Def</u>: $P(z)$ is <u>absolutely convergent</u> at $z=a$ if the real series

$$\tilde{P}(a) := \sum |c_j a^j| \text{ converges.}$$

<u>Prop</u>: If $P(z)$ is absolutely convergent at $z=a$ then $P(a)$ converges.

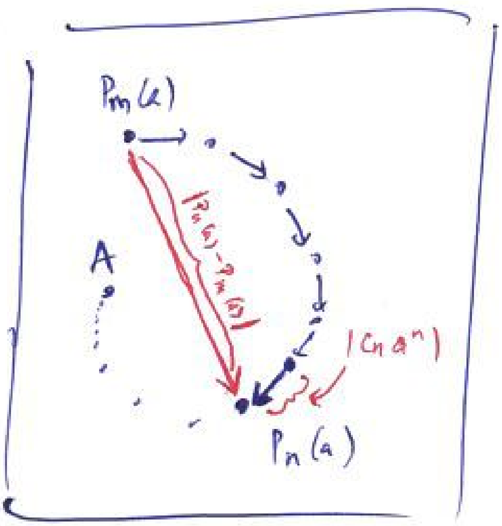
<u>Proof</u>:
$$|P_n(a) - P_m(a)| = |c_{m+1} a^{m+1} + \dots + c_n a^n|$$

triangle inequality

$$\leq |c_{m+1} a^{m+1}| + \dots + |c_n a^n|$$

$$= \underbrace{\tilde{P}_n(a) - \tilde{P}_m(a)}$$

Can be made arbitrarily small by assumption of absolute convergence



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Prop: If $P(z)$ converges at $z = a$ then it converges absolutely for all $|z| < |a|$

Pf: Say $|z| < |a|$.

Let $\rho = \frac{|z|}{|a|} < 1$

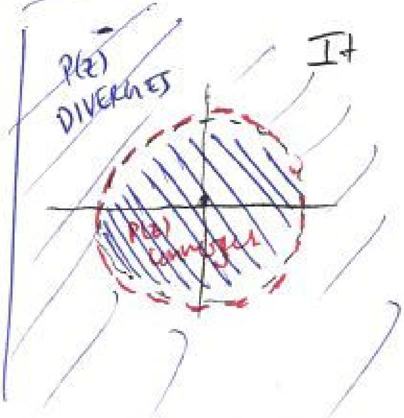
Let $M > |c_n a^n| \quad \forall n$

Then $|c_n z^n| = \left| c_n a^n \frac{z^n}{a^n} \right| < M \left| \frac{z^n}{a^n} \right| = M \rho^n \quad \forall n$

$$\Rightarrow \tilde{P}_n(z) - \tilde{P}_m(z) = |c_{m+1} z^{m+1}| + \dots + |c_n z^n|$$

$$< M(\rho^{m+1} + \dots + \rho^n)$$

$$= \underbrace{\frac{M}{1-\rho}(\rho^{m+1} - \rho^{n+1})}_{\text{can be made arbitrarily small}}$$



It follows that a power series $P(z)$ will have an associated "circle of doubt" inside of which it will converge (absolutely) and outside of which it will diverge.

On this "circle of doubt", $P(z)$ may converge everywhere, nowhere, or at only some points...

Exercise: Examine convergence of the following power series:

(i) $\displaystyle\sum_{n=0}^{\infty} z^n$

(ii) $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n}$

(iii) $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n^2}$