FINAL EXAM

INTRO TO REAL ANALYSIS

The exam consists of five questions, each worth 20 points. You may take up to three hours to complete the exam.

Problem 1. Consider the functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq n \\ x - n & \text{if } n < x \leq n+1 \\ 1 & \text{if } x > n+1. \end{cases}$$

Is the sequence $\{f_n\}_{n\in\mathbb{N}}$ uniformly convergent? Justify your answer.

Problem 2. Determine whether each statement is true or false. If false, provide a counterexample. If true, then very briefly justify your answer. (As usual, all functions are assumed to be real-valued with domains contained in $\mathbb{R}$, and all sets assumed to be subsets of $\mathbb{R}$.)

(a) If a sequence of functions (f_n) converges pointwise to a function f on a compact set K, then $f_n \to f$ uniformly on K.

(b) If (x_n) is a Cauchy sequence, then its image $(f(x_n))$ under a continuous function f is also a Cauchy sequence.

(c) A continuous function f on a compact set K always attains a maximum value and a minimum value on K.

(d) If a function f is represented by a power series $\sum a_n x^n$ on its interval of convergence I, then $\sum a_n x^n$ converges uniformly to f on I.

Problem 3. Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; i.e. that $f(x) = x$ for some $x \in [0, 1]$.

Problem 4. Assume f is continuous on an interval containing zero and differentiable for all $x \neq 0$. If $\lim_{x\to 0} f'(x) = L$, show that $f'(0)$ exists and equals L.

Problem 5. (a) Suppose $f : A \to \mathbb{R}$ is uniformly continuous and $(x_n) \subset A$ is a Cauchy sequence. Show that $(f(x_n))$ is a Cauchy sequence.

(b) Let g be a continuous function on an open interval (a, b). Show that g is uniformly continuous on (a, b) if and only if it is possible to define values $g(a)$ and $g(b)$ at the endpoints so that the extended function g is continuous on $[a, b]$.

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