MIDTERM EXAM

INTRODUCTION TO MANIFOLDS

The exam consists of seven questions worth a total of 105 points. You may take up to three hours to complete the exam.

Problem 1 (15 pts)**.** Let I be the 3×3 identity matrix, and let $A = \begin{bmatrix} 3 & 2 & 6 \\ 2 & 2 & 5 \\ -2 & -1 & -4 \end{bmatrix}$. Let $\Gamma_I, \Gamma_A \subset \mathbb{R}^6$ denote the graphs of the corresponding linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$. Do Γ_I and Γ_A have transverse intersection? Justify your answer using the definition of transverse intersection: namely that

$$\Gamma_I \pitchfork \Gamma_A \Longleftrightarrow T_p\Gamma_I + T_p\Gamma_A = T_p\mathbb{R}^6 \text{ for all } p \in \Gamma_I \cap \Gamma_A.$$

Problem 2 (15 pts)**.** Let M be a *compact* smooth manifold. Show that there is no smooth submersion $F : M \to \mathbb{R}^k$ for any $k > 0$.

Problem 3 (15 pts)**.** Let $F : N \to M$ be a smooth map of smooth manifolds. Show that if $F_{*,p}$ has rank k for some $p \in N$, then there exists an open neighborhood U of p such that the differential $F_{*,q}$ has rank at least k, for any $q \in U$.

Problem 4 (15 pts)**.** Let $X = y^2 \frac{\partial}{\partial x}$ and $Y = x^2 \frac{\partial}{\partial y}$ be vector fields on $\mathbb{R}^2$. Compute the Lie bracket $[X, Y]$. Namely, determine smooth functions $f(x, y)$ and $g(x, y)$ for which $[X, Y] = f(x, y)\frac{\partial}{\partial x} + g(x, y)\frac{\partial}{\partial y}$.

Problem 5 (15 pts)**.** Let M be a smooth manifold. Let $F : M \to M$ be a smooth map, and suppose $c \in M$ is a regular value of F. Show that $F^{-1}(c)$ cannot have a convergent subsequence.

Problem 6 (15 pts)**.** Determine whether each of the following statements is true or false. If false, provide a counterexample. If true, briefly justify your answer.

(a) Let $F : N \to M$ be a one-to-one immersion, and let n and m denote the dimensions of N and M respectively. For any point $q \in F(N) \subset M$, there exists a coordinate neighborhood $(U, x^1, \ldots, x^m)$ about q such that $F(N) \cap U$ is defined by the vanishing of $x^1, \ldots, x^{m-n}$.

(b) Any alternating bilinear map $\omega \in A_2(\mathbb{R}^3)$ is *decomposable*, in the sense that $\omega = \alpha_1 \wedge \alpha_2$ for some covectors $\alpha_1, \alpha_2 \in V^*$.

(c) Any smooth vector field on the torus vanishes on at least one point.

Problem 7 (15 pts)**.** Let G be a Lie group, and let H be a subgroup of G. Let h_0 be a point of H. Suppose there exists an adapted chart $(U, \phi) = (U, x^1, \ldots, x^n)$ relative to H about the point h_0. In particular, suppose $H \cap U$ is defined by the vanishing of $x^1, \ldots, x^k$ for some $1 \le k \le n$. Prove that H is a regular submanifold of G.

Date: September 24, 2015.