PROBLEM SET 1

AFS-1, TOPOLOGY

In all the following problems (X, d) is a metric space.

Problem 1. Suppose $A \subset X$. Show that

(a) the interior $\mathrm{int}(A)$ is an open set.
(b) $\mathrm{int}(A) = A$ if and only if A is open.
(c) $\mathrm{int}(A)$ is the largest (with respect to inclusion) open set contained in A.

Can you state the analogous properties for the closure of A?

Problem 2. (Another characterization of closed sets) Suppose $A \subset X$ is a subset. Show that for any point x in the closure $\overline{A}$, there is a sequence in A converging to x. Conversely show that any limit point of A is contained in $\overline{A}$.

Problem 3. (Continuity of functions on metric spaces) Suppose (X, d) and (Y, ρ) are metric spaces, and $f : X \to Y$ is a function. Show that the following are equivalent.

(a) For any $x \in X$, $\epsilon > 0$, there is a $\delta > 0$ such that for any $x_0 \in X$
$$d(x, x_0) < \delta \implies \rho(f(x), f(x_0)) < \epsilon.$$
(b) For any open set $U \subset Y$, $f^{-1}(U)$ is open.
(c) If $x_n \to x$ in X, then $f(x_n) \to f(x)$ in Y.
(d) For any subset $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.
(e) For any closed set $C \subset Y$, $f^{-1}(C)$ is closed.

If any one (and equivalently all) of the conditions holds, then $f : X \to Y$ is continuous.

Problem 4. (Continuity of the distance function) Fix $x \in X$. Show that the function
$$d_x : X \to \mathbb{R}, \quad y \mapsto d(x, y)$$
is continuous.

Problem 5. (Defining connectedness) A metric space (X, d) is said to be *connected* whenever $\varnothing$ and X are the only subsets of X that are simultaneously open and closed. A subset A of a metric space (X, d) is said to be connected whenever (A, d) is itself a connected metric space. Is the two-point set $\{0, 1\}$ equipped with discrete metric connected?

Problem 6. Establish the following properties regarding connected metric spaces and connected sets.

(a) A metric space (X, d) is connected if and only if every continuous function $f : X \to \{0, 1\}$ is constant.
(b) If in a metric space (X, d) we have $B \subset A \subset X$, then the set B is a connected subset of (A, d) if and only if B is a connected subset of (X, d).
(c) If $f : (X, d) \to (Y, \rho)$ is a continuous function and A is a connected subset of X, then $f(A)$ is a connected subset of Y.
(d) If $\{A_i\}_{i \in I}$ is a family of connected subsets of a metric space such that the intersection $\cap_{i \in I} A_i$ is non-empty; then the union $\cup_{i \in I} A_i$ is connected.
(e) If $A \subset X$ and $a \in A$, then there exists a largest (with respect to inclusion) connected subset C_a of A that contains a. (The connected set C_a is called the *component* of a with respect to A.)
(f) If a, b belong to a subset A of a metric space and C_a and C_b are the components of a and b in A, then either $C_a = C_b$ or else $C_a \cap C_b = \varnothing$. Hence, the identity $A = \cap_{a \in A} C_a$ shows that A can be written as a disjoint union of connected sets.
(g) A nonempty subset of $\mathbb{R}$ with at least two elements is a connected set if and only if it is an interval. Use this and the conclusion of (f) to infer that every open subset of $\mathbb{R}$ can be written as an at-most countable union of disjoint open intervals.

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