PROBLEM SET 2

AFS-1, TOPOLOGY

Problem 1. Suppose (X, d) is a metric space. Show that the following metrics are equivalent to d

1. $\rho = 2d$
2. $\rho = \frac{d}{1+d}$.

Problem 2. Show that the metric $d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ on $(0, \infty)$ is equivalent to the standard metric on $(0, \infty)$.

Problem 3. Construct an example of a countable family of open dense sets in a complete metric space whose intersection is not open.

Problem 4. For any subset A of a topological space X, prove the following.

1. $\operatorname{int}(A) = (\overline{A^c})^c$,
2. $\partial A = \overline{A} \setminus \operatorname{int}(A)$,
3. $\operatorname{int}(A \setminus \operatorname{int}(A)) = \varnothing$.

In part (b), ∂A denotes the *boundary* of A, and is defined as $\overline{A} \cap \overline{A}^c$.

Problem 5. Show that the following are equivalent for a function $f : (X, \tau) \to (Y, \tau_1)$ between topological spaces.

1. f is continuous.
2. For any open set $\mathcal{O} \subset Y$, $f^{-1}(\mathcal{O})$ is open in X.
3. For any subset A of X, $f(\overline{A}) \subset \overline{f(A)}$.
4. For any closed set $C \subset Y$, $f^{-1}(C)$ is closed in X.

Problem 6. The set of accumalation points of a subset $A \subset X$ is called its *derived set*. Show that if X is Hausdorff, the derived set is closed.

Problem 7. If A is a dense subset of a topological space X, then show that $\mathcal{O} \subset \overline{\mathcal{O} \cap A}$ holds for every open set $\mathcal{O} \subset X$.

Problem 8. If $\{\mathcal{O}_i\}_{i \in I}$ is an open cover for a topological space X, show that a set $A \subset X$ is closed if and only if $A \cap \mathcal{O}_i$ is closed in $\mathcal{O}_i$ for all $i \in I$.

Problem 9. In a Hausdorff topological space X, show that

1. Every finite subset of X is closed.
2. Every sequence converges to at most one point.

Problem 10. For a function $f : (X, \tau) \to (Y, \tau_1)$ show the following.

1. If τ is the discrete topology, then F is continuous.
2. If τ is the indiscrete topology and τ_1 is a Hausdorff topology, then f is continuous if and only if it is a constant function.

Problem 11. Let $f, g : (X, \tau) \to (Y, \tau_1)$ be two continuous functions and suppose Y is Hausdorff. Suppose there is a dense subset $A \subset X$ such that $f(x) = g(x)$. The, $f = g$ on X.

Problem 12. Suppose $f : (X, \tau) \to (Y, \tau_1)$ and $g : (Y, \tau_1) \to (Z, \tau_2)$ are continuous functions. Then, show that $g \circ f$ is continuous.

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