MIDTERM EXAM

INTRODUCTION TO MANIFOLDS

The exam consists of six questions worth a total of 100 points. You may take up to three hours to complete the exam.

Problem 1 (15 pts)**.** Let $F : N \to M$ be a smooth map of smooth manifolds. Show that if $F_{*,p}$ has rank k for some $p \in N$, then there exists an open neighborhood U of p such that the differential $F_{*,q}$ has rank at least k, for any $q \in U$.

Problem 2 (15 pts)**.** Let M be a smooth manifold. Let $F : M \to M$ be a smooth map, and suppose $c \in M$ is a regular value of F. Show that the subspace topology on $F^{-1}(c)$ is in fact the discrete topology.

Problem 3 (15 pts)**.** Let G be a Lie group, and let H be a subgroup of G. Let h_0 be a point of H. Suppose there exists an adapted chart $(U, \phi) = (U, x^1, \ldots, x^n)$ relative to H about the point h_0. In particular, suppose $H \cap U$ is defined by the vanishing of $x^1, \ldots, x^k$ for some $1 \leq k \leq n$. Prove that H is a regular submanifold of G.

Problem 4 (15 pts)**.** Let M be a smooth manifold. Prove that the projection map $\pi : TM \to M$ is a submersion.

Problem 5 (20 pts)**.** We identify the set of $n \times n$ matrices with the Euclidean space $\mathbb{R}^{n^2}$. Let $A \subset \mathbb{R}^{n^2}$ denote the set of symmetric matrices and let $B \subset \mathbb{R}^{n^2}$ denote the set of skew-symmetric matrices.

(a) Prove that A and B are both regular submanifolds of $\mathbb{R}^{n^2}$.

(b) Prove that A and B intersect transversely, in the sense that for any point $p \in A \cap B$, we have $T_pA + T_pB = T_p(\mathbb{R}^{n^2})$.

Problem 6 (20 pts)**.** Let $S^1 \subset \mathbb{R}^2$ denote the unit circle, and let $M := S^1 \times S^1 \subset \mathbb{R}^4$ denote the torus.

(a) Construct a C^∞ atlas for M.

(b) Prove that the tangent bundle TM is trivial.

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