Topic 4: Optical Diffraction II
Workshop Solutions

4.1 Resolution Limit of a Single-Grating Spectrometer

Calculate the absolute resolution limit for a simple grating spectrometer in terms of *frequency* and comment on your answer. Assume that the light is incident perpendicular to the surface of the grating.
Hint: The maximum possible diffraction angle is 90°.

Solution

The angle of diffraction in a spectrometer with normal incidence of the light is

$$\sin\theta_m = \frac{m\,\lambda}{d},$$

where m is the order, and d is the line spacing of the grating. The maximum possible angle for θ_m is 90°, so the maximum possible usable order is

$$m = \frac{d}{\lambda}.$$

The maximum resolving power of a grating is then

$$\frac{\lambda}{\Delta\lambda} = mN = \frac{d\,N}{\lambda}$$

where $d\,N = D$ is the size of the grating, and we have that

$$\Delta\lambda = \frac{\lambda^2}{D}.$$

In the solution of problem 3.2 (Analysis of a Laser Diode) we have derived a relation between differences in wavelength and differences in frequency,

$$\Delta\lambda = \frac{\lambda^2}{c}\Delta\nu\,,$$

from which we get the final result that

$$\Delta\nu = \frac{c}{D}.$$

This shows, somewhat surprisingly, that the frequency resolution $\Delta\nu$ depends only on the (illuminated) size of the grating, but not on the density of its grooves.

In the *optical* region, the size of reasonable, practical gratings is limited to $D \approx 100\,\text{mm}$, so giving $\Delta\nu \approx 3\,\text{GHz}$.

Aside: in spectroscopy, frequencies are often quoted in *wave numbers*, k, being

$$k = \frac{1}{\lambda} = c\,\nu$$

so noting that $\Delta k = c\,\Delta\nu$, we get the resolution limit in terms of *wave number* to be

$$\Delta k = \frac{1}{D}$$

which has the traditional (non-SI) units of cm^{-1}.

4.2 Bessel Functions

Given that the expansion of Bessel functions of integer order is

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!\Gamma(n+k+1)} \quad ,$$

write out the first **three** terms of the expansion for $J_0(x)$ and $J_1(x)$.
Use this expansion to determine the value of

$$\frac{J_1(x)}{x} \quad \text{when } x = 0$$

Plot the functions $J_0(x)$, $J_1(x)$ and $J_1(x)/x$ for $x = -10 \rightarrow 10$. (Using MAPLE or the like would be a good idea).
Hint: The Gamma function, $\Gamma(x)$ is a generalisation of the factorial function to real and complex values x. If n is a positive integer, then $\Gamma(n) = (n-1)!$

Solution

Part a: Using

$$\Gamma(n+1) = n!$$

for integer n, we can write the expansion as:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!(n+k)!}$$

So the first three terms are

$$\begin{aligned} J_0(x) &= 1 - \frac{x^2}{4} + \frac{x^4}{64} \dots \\ J_1(x) &= \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} \end{aligned}$$

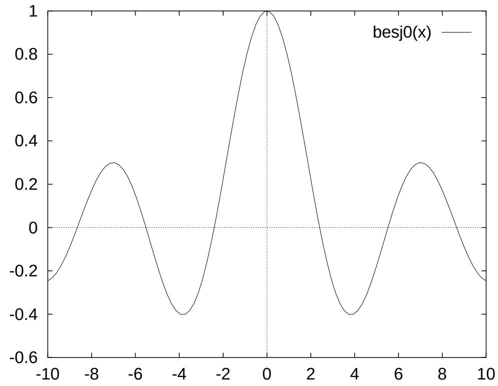
From these expansions we get that

$$\frac{J_1(x)}{x} = \frac{1}{2} - \frac{x^2}{16} + \frac{x^4}{384}$$

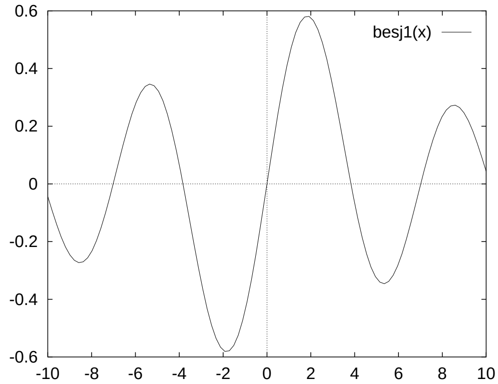
so that when $x = 0$,

$$\frac{J_1(0)}{0} = \frac{1}{2}$$

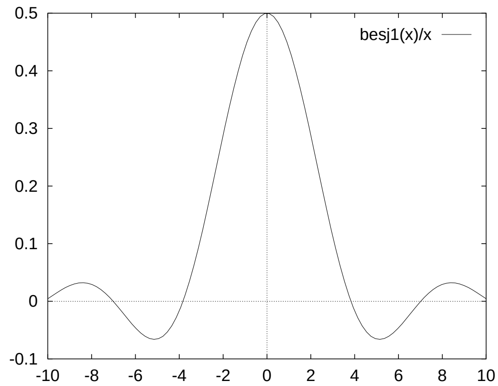
Part b: The plots are:
Plot of $J_0(x)$ from $-10 \rightarrow 10$:



Plot of $J_1(x)$ from $-10 \to 10$:



Plot of $J_1(x)/x$ from $-10 \to 10$:



There are plotted from `gnuplot` available on the CPLAB systems.

4.3 Diffraction Resolution of the eye

Estimate the diffraction resolution limit of the eye in *bright sun light*. What happens at low light levels when the pupil size increases?

Solution

We have from the *Rayleigh Limit* that the angular resolution is

$$\Delta\theta_0 = 1.22\frac{\lambda}{d}.$$

Take $\lambda = 560\,\text{nm}$, which is the peak of the eye sensitivity. The *iris* of the eye expands and contracts to control the amount of light entering the eye, ranging in diameter from about 8 mm in low light levels down to about 2 mm in the most extreme bright conditions. About 3 mm is a good estimate for *bright sunlight*. Putting in the numbers gives

$$\Delta\theta_0 = 2.3 \times 10^{-4}\,\text{Rad} = 0.78\,\text{min of arc}$$

which is very close to the 1 min of arc resolution typically quoted for the human eye. It shows that the human eye is *diffraction limited* at high light levels, with the optical system well matched to the sensor spacing of about 6 μm.

In extremely bright light, with the pupil closed to 2 mm, the corresponding values are:

$$\Delta\theta_0 = 3.4 \times 10^{-4}\,\text{Rad} = 1.2\,\text{min of arc}$$

which suggest that diffraction will limit the eye resolution. However, these conditions are somewhat unusual, being either desert sun or sunlit snow fields.

As the pupil size increases, you would *expect* the resolution to increase. However, at large diameters the aberrations in the eye, mainly spherical aberration, dominate. This abberation rises at the *fourth power* of the radius of the pupil, and has the effect of blurring the point spread function, and thus the resolution actually drops.