PHYS370 – Advanced Electromagnetism

Part 3: Electromagnetic Waves in Conducting Media

Electromagnetic Wave Equation

Recall that in a "simple" dielectric material, we derived the wave equations:

$$\nabla^2 \vec{E} - \mu\varepsilon \ddot{\vec{E}} = 0 \quad (1)$$

$$\nabla^2 \vec{B} - \mu\varepsilon \ddot{\vec{B}} = 0 \quad (2)$$

To derive these equations, we used Maxwell's equations with the assumptions that the charge density ρ and current density J were zero, and that the permeability μ and permittivity ε were constants.

We found that the above equations had plane-wave solutions, with phase velocity:

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \quad (3)$$

Maxwell's equations imposed additional constraints on the directions and relative amplitudes of the electric and magnetic fields.

How are the wave equations (and their solutions) modified for the case of electrically conducting media?

We shall restrict our analysis to the case of ohmic conductors, which are defined by:

$$\vec{J} = \sigma \vec{E} \tag{4}$$

where σ is a constant, the conductivity of the material.

All we need to do is substitute from equation (4) into Maxwell's equations, then proceed as for the case of a dielectric...

In our "simple" conductor, Maxwell's equations take the form:

$$\nabla \cdot \vec{E} = 0 \tag{5}$$
$$\nabla \cdot \vec{B} = 0 \tag{6}$$
$$\nabla \times \vec{E} = -\dot{\vec{B}} \tag{7}$$
$$\nabla \times \vec{B} = \mu\varepsilon\dot{\vec{E}} + \mu\vec{J} \tag{8}$$

where $\vec{J}$ is the current density. Assuming an ohmic conductor, we can write:

$$\vec{J} = \sigma \vec{E} \tag{9}$$

so equation (8) becomes:

$$\nabla \times \vec{B} = \mu\varepsilon\dot{\vec{E}} + \mu\sigma\vec{E} \tag{10}$$

Taking the curl of equation (7) and making appropriate substitutions as before, we arrive at the wave equation:

$$\nabla^2\vec{E} - \mu\sigma\dot{\vec{E}} - \mu\varepsilon\ddot{\vec{E}} = 0 \tag{11}$$

The wave equation for the electric field in a conducting material is (11):

$$\nabla^2 \vec{E} - \mu\sigma \dot{\vec{E}} - \mu\varepsilon \ddot{\vec{E}} = 0 \tag{12}$$

Let us try a solution of the same form as before:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{j(\omega t - \vec{k} \cdot \vec{r})} \tag{13}$$

Remember that to find the physical field, we have to take the real part. Substituting (13) into the wave equation (11) gives the dispersion relation:

$$-\vec{k}^2 - j\omega\mu\sigma + \omega^2 \mu\varepsilon = 0 \tag{14}$$

Compared to the dispersion relation for a dielectric, the new feature is the presence of an imaginary term in σ. This means the relationship between the wave vector $\vec{k}$ and the frequency ω is a little more complicated than before.

From the dispersion relation (14), we can expect the wave vector $\vec{k}$ to have real and imaginary parts. Let us write:

$$\vec{k} = \vec{\alpha} - j\vec{\beta} \tag{15}$$

for parallel real vectors $\vec{\alpha}$ and $\vec{\beta}$.

Substituting (15) into the dispersion relation (14) and taking real and imaginary parts, we find:

$$\alpha = \omega\sqrt{\mu\varepsilon} \left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\sigma^2}{\omega^2 \varepsilon^2}} \right]^{1/2} \tag{16}$$

and:

$$\beta = \frac{\omega\mu\sigma}{2\alpha} \tag{17}$$

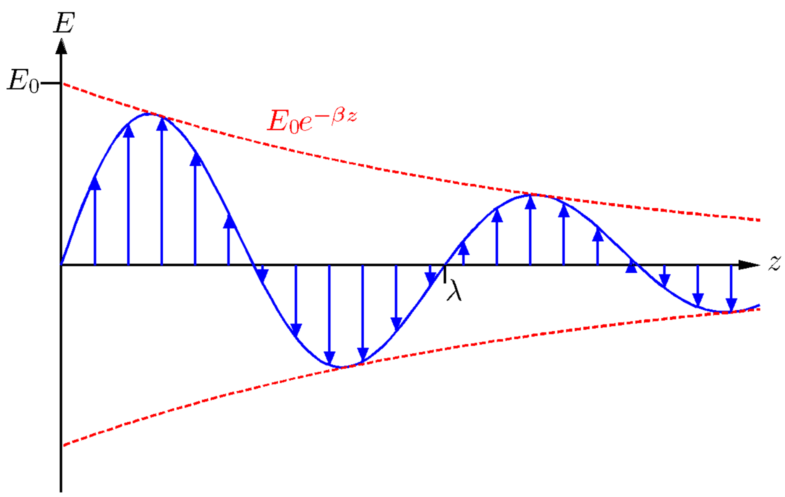
Equations (16) and (17) give the real and imaginary parts of the wave vector $\vec{k}$ in terms of the frequency ω, and the material properties μ, ε and σ.

Using equation (15) the solution (13) to the wave equation in a conducting material can be written:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{j(\omega t - \vec{\alpha}\cdot\vec{r})} e^{-\vec{\beta}\cdot\vec{r}} \tag{18}$$

The first exponential factor, $e^{j(\omega t - \vec{\alpha}\cdot\vec{r})}$ gives the usual plane-wave variation of the field with position $\vec{r}$ and time t; note that the conductivity of the material affects the wavelength for a given frequency.

The second exponential factor, $e^{-\vec{\beta}\cdot\vec{r}}$ gives an exponential decay in the amplitude of the wave...



In a "simple" non-conducting material there is no exponential decay of the amplitude: electromagnetic waves can travel for ever, without any loss of energy.

If the wave enters an electrical conductor, however, we can expect very different behaviour. The electrical field in the wave will cause currents to flow in the conductor. When a current flows in a conductor (assuming it is not a superconductor) there will be some energy changed into heat. This energy must come from the wave. Therefore, we expect the wave gradually to decay.

The varying electric field must have a magnetic field associated with it. Presumably, the magnetic field has the same wave vector and frequency as the electric field: this is the only way we can satisfy Maxwell's equations for all positions and times. Therefore, we try a solution of the form:

$$\vec{B}(\vec{r},t) = \vec{B}_0 e^{j(\omega t - \vec{k}\cdot\vec{r})} \tag{19}$$

Now we use Maxwell's equation (7):

$$\nabla \times \vec{E} = -\dot{\vec{B}} \tag{20}$$

which gives:

$$\vec{k} \times \vec{E}_0 = \omega \vec{B}_0 \tag{21}$$

or:

$$\vec{B}_0 = \frac{\vec{k}}{\omega} \times \vec{E}_0 \tag{22}$$

The magnetic field in a wave in a conducting material is related to the electric field by (22):

$$\vec{B}_0 = \frac{\vec{k}}{\omega} \times \vec{E}_0 \qquad (23)$$

As in a non-conducting material, the electric and magnetic fields are perpendicular to the direction of motion (the wave is a transverse wave) and are perpendicular to each other.

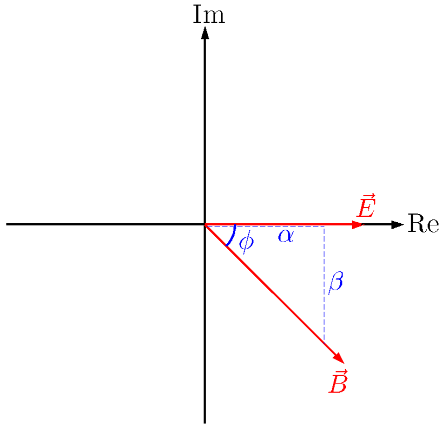
But there is a new feature, because the wave vector is complex.

In a non-conducting material, the electric and magnetic fields were in phase: the expressions for the fields both had the same phase angle ϕ_0. In complex notation, the complex phase angles of the field amplitudes $\vec{E}_0$ and $\vec{B}_0$ were the same.

In a conductor, the complex phase of $\vec{k}$ gives a phase difference between the electric and magnetic fields.

In a conducting material, there is a difference between the phase angles of $\vec{E}_0$ and $\vec{B}_0$, given by the phase angle ϕ of $\vec{k}$. This is:

$$\tan\phi = \frac{\beta}{\alpha} \qquad (24)$$



Let us consider the special case of a good insulator. In this case:

$$\sigma \ll \omega\varepsilon \tag{25}$$

From equation (16), we then have:

$$\alpha \approx \omega\sqrt{\mu\varepsilon} \tag{26}$$

and from equation (17) we have:

$$\beta \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\alpha}{2}\frac{\sigma}{\omega\varepsilon} \tag{27}$$

It follows that $\beta \ll \alpha$. We recover the same situation as in the case of a non-conducting material. The decay of the wave is very slow (in terms of the number of wavelengths); the magnetic and electric components of the wave are approximately in phase ($\phi \approx 0$), and are related by:

$$B_0 \approx \frac{\alpha}{\omega}E_0 \approx \frac{E_0}{v_p} \tag{28}$$

where the phase velocity v_p is, as before, given by $v_p = 1/\sqrt{\mu\varepsilon}$.

Let us consider the special case of a very good conductor. In this case:

$$\sigma \gg \omega\varepsilon \tag{29}$$

From equation (16), we then have:

$$\alpha \approx \sqrt{\frac{\omega\mu\sigma}{2}} \tag{30}$$

and from equation (17) we have:

$$\beta \approx \sqrt{\frac{\omega\mu\sigma}{2}} \approx \alpha \tag{31}$$

In the case of a very good conductor, the real and imaginary parts of the wave vector $\vec{k}$ become equal. This means that the decay of the wave is very fast in terms of the number of wavelengths.

Note that the vectors $\vec{\alpha}$ and $\vec{\beta}$ have the same units as $\vec{k}$, i.e. meters^{-1}.

The electric field in the wave varies as (18):

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\omega t - \vec{\alpha}\cdot\vec{r})} e^{-\vec{\beta}\cdot\vec{r}} \tag{32}$$

The phase velocity is the velocity of a point that stays in phase with the wave. Consider a wave moving in the $+z$ direction:

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\omega t - \alpha z)} e^{-\beta z} \tag{33}$$

For a point staying at a fixed phase, we must have:

$$\omega t - \alpha z(t) = \text{constant} \tag{34}$$

So the phase velocity is given by:

$$v_p = \frac{dz}{dt} = \frac{\omega}{\alpha} \tag{35}$$

But note that in a good conductor, α is itself a function of ω...

For a poor conductor ($\sigma \ll \omega\varepsilon$), we have:

$$\alpha \approx \omega\sqrt{\mu\varepsilon} \tag{36}$$

so the phase velocity in a poor conductor is:

$$v_p = \frac{\omega}{\alpha} \approx \frac{1}{\sqrt{\mu\varepsilon}} \tag{37}$$

If μ and ε are constants (i.e. are independent of ω) then the phase velocity is independent of the frequency: there is no dispersion.

However, in a good conductor ($\sigma \gg \omega\varepsilon$), we have:

$$\alpha \approx \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\mu\varepsilon}\sqrt{\frac{\omega\sigma}{2\varepsilon}} \tag{38}$$

Then the phase velocity is given by:

$$v_p = \frac{\omega}{\alpha} \approx \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\frac{2\omega\varepsilon}{\sigma}} \tag{39}$$

The phase velocity depends on the frequency: there is dispersion!

The presence of dispersion means that the group velocity v_g (the velocity of a wave pulse) can differ from the phase velocity v_p (the velocity of a point staying at a fixed phase of the wave).

To understand what this means, consider the superposition of two waves with equal amplitudes, both moving in the $+z$ direction, and with similar wave numbers:

$$E_x = E_0 \cos\left(\omega_+ t - [k_0 + \Delta k]\, z\right) + E_0 \cos\left(\omega_- t - [k_0 - \Delta k]\, z\right) \tag{40}$$

Using a trigonometric identity:

$$\cos A + \cos B \equiv 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \tag{41}$$

the electric field can be written:

$$E_x = 2E_0 \cos\left(\omega_0 t - k_0 z\right) \cos\left(\Delta\omega\, t - \Delta k\, z\right) \tag{42}$$

where:

$$\omega_0 = \frac{1}{2}\left(\omega_+ + \omega_-\right) \qquad \Delta\omega = \omega_+ - \omega_- \tag{43}$$

We have written the total electric field in our superposed waves as (42):

$$E_x = 2E_0 \cos\left(\omega_0 t - k_0 z\right) \cos\left(\Delta\omega\, t - \Delta k\, z\right) \tag{44}$$

Assuming that $\Delta k \ll k_0$, the first trigonometric factor represents a wave of (short) wavelength $2\pi/k_0$ and phase velocity:

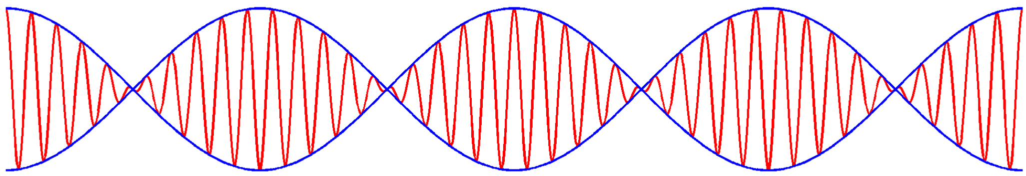
$$v_p = \frac{\omega_0}{k_0} \tag{45}$$

while the second trigonometric factor represents a modulation of (long) wavelength $2\pi/\Delta k$, which travels with velocity:

$$v_g = \frac{\Delta\omega}{\Delta k} \tag{46}$$

v_g is called the group velocity. Since $\Delta\omega$ represents the change in frequency that corresponds to a change Δk in wave number, we can write:

$$v_g = \frac{d\omega}{dk} \tag{47}$$



The red wave moves with the phase velocity v_p; the modulation (represented by the blue line) moves with group velocity v_g.

Since the energy in a wave depends on the local amplitude of the wave, the energy in the wave is carried at the group velocity v_g.

If there is no dispersion, then the phase velocity is independent of frequency:

$$v_p = \frac{\omega}{k} = \text{constant} \tag{48}$$

and the group velocity is equal to the phase velocity:

$$v_g = \frac{d\omega}{dk} = v_p \tag{49}$$

In the absence of dispersion, a modulation resulting from the superposition of two waves with similar frequencies will travel at the same speed as the waves themselves.

However, if there is dispersion, then the group velocity can differ from the phase velocity...

Group Velocity of an EM Wave in a Good Conductor

The dispersion relation for an electromagnetic wave in a good conductor is, from (38):

$$\omega = \frac{1}{\mu\varepsilon}\frac{2\varepsilon}{\sigma}\alpha^2 \tag{50}$$

where α is the real part of the wave vector. The group velocity is then:

$$\begin{aligned} v_g &= \frac{d\omega}{d\alpha} \\ &\approx \frac{1}{\mu\varepsilon}\frac{4\varepsilon}{\sigma}\alpha \\ &\approx \frac{2}{\sqrt{\mu\varepsilon}}\sqrt{\frac{2\omega\varepsilon}{\sigma}} \end{aligned} \tag{51}$$

Comparing with equation (39) for the phase velocity of an electromagnetic wave in a good conductor, we find that:

$$v_g \approx 2v_p \tag{52}$$

In other words, the group velocity is approximately twice the phase velocity.

The Skin Depth of a Good Conductor

The real part, α, of the wave vector k in a conductor gives the wavelength of the wave. β measures the distance that the wave travels before its amplitude falls to $1/e$ of its original value. Let us write the solution (18) for a wave travelling in the z direction in a good conductor as:

$$\vec{E}(\vec{r},t) = \vec{E}_0'(\vec{r})e^{j(\omega t - \vec{\alpha}\cdot\vec{r})} \tag{53}$$

where:

$$\vec{E}_0'(\vec{r}) = \vec{E}_0 e^{-\vec{\beta}\cdot\vec{r}} \tag{54}$$

The amplitude of the wave falls by a factor $1/e$ in a distance $1/\beta$. We define the *skin depth* δ:

$$\delta = \frac{1}{\beta} \tag{55}$$

From equation (31), we see that for a good conductor ($\sigma \gg \omega\varepsilon$), the skin depth is given by:

$$\delta \approx \sqrt{\frac{2}{\omega\mu\sigma}} \tag{56}$$

For example, consider silver, which has conductivity $\sigma \approx 6.30 \times 10^7\,\Omega^{-1}\mathrm{m}^{-1}$, and permittivity $\varepsilon \approx \varepsilon_0 \approx 8.85 \times 10^{-12}\,\mathrm{Fm}^{-1}$.

For radiation of frequency 10^{10} Hz, the "good conductor" condition is satisfied, and the skin depth of the radiation is approximately 0.6 micron (0.6×10^{-6} m).

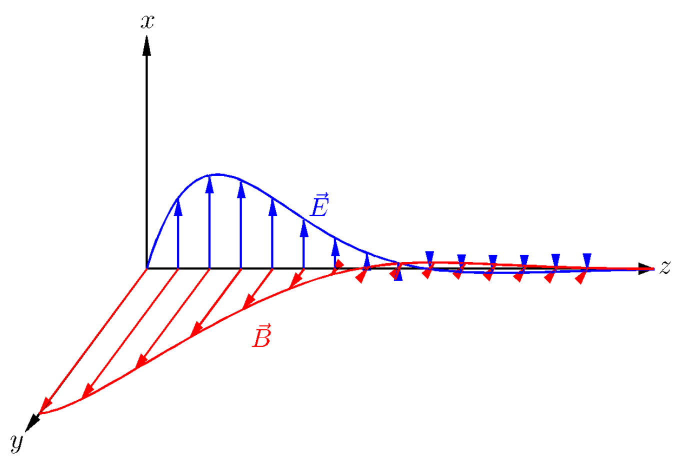
Note that in vacuum, the wavelength of radiation of frequency 10^{10} Hz is about 3 cm; but in silver, the wavelength is:

$$\lambda = \frac{2\pi}{\alpha} \approx 2\pi\delta \approx 4\,\text{micron} \tag{57}$$

The phase difference between the electric and magnetic fields in a good conductor is given by:

$$\tan\phi = \frac{\beta}{\alpha} \approx 1 \tag{58}$$

So the phase difference is approximately 45°.



Using the plane wave solutions:

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\omega t - \vec{k}\cdot\vec{r})} \quad (59)$$

$$\vec{B}(\vec{r},t) = \vec{B}_0 e^{j(\omega t - \vec{k}\cdot\vec{r})} \quad (60)$$

in Maxwell's equation:

$$\nabla \times \vec{E} = -\dot{\vec{B}} \quad (61)$$

and using also the relation $\vec{B} = \mu\vec{H}$, we find the relation between the electric field and magnetic intensity:

$$\vec{k} \times \vec{E}_0 = \omega\mu\vec{H}_0 \quad (62)$$

The vectors $\vec{k}$, $\vec{E}_0$ and $\vec{H}_0$ are mutually perpendicular. Therefore, we can write for the wave impedance:

$$Z = \frac{E_0}{H_0} = \frac{\omega\mu}{\alpha - j\beta} \quad (63)$$

In a good conductor ($\sigma \gg \omega\varepsilon$), we have (31):

$$\alpha \approx \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}} \quad (64)$$

It then follows that the wave impedance (63) in a good conductor is given by:

$$Z \approx \frac{1}{1-j}\sqrt{\frac{2\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} \quad (65)$$

Note that the impedance is now a complex number. As we shall see later, the behaviour of waves on a boundary depends on the impedances of the media on either side of the boundary.

The complex phase of the impedance will tell us about the phases of the waves reflected from and transmitted across the boundary.

The time averaged energy densities in the electric and magnetic fields are:

$$\langle U_E \rangle_t = \frac{1}{2}\varepsilon \langle \vec{E}^2 \rangle_t = \frac{1}{4}\varepsilon E_0^2 e^{-2\vec{\beta}\cdot\vec{r}} \quad (66)$$

$$\langle U_H \rangle_t = \frac{1}{2}\mu \langle \vec{H}^2 \rangle_t = \frac{1}{4}\mu H_0^2 e^{-2\vec{\beta}\cdot\vec{r}} \quad (67)$$

The ratio is:

$$\frac{\langle U_E \rangle_t}{\langle U_H \rangle_t} = \frac{\varepsilon}{\mu}\frac{E_0^2}{H_0^2} = \frac{\varepsilon}{\mu}|Z|^2 \quad (68)$$

In a good conductor, the square of the magnitude of the impedance is:

$$|Z|^2 \approx \frac{\omega\mu}{\sigma} \quad (69)$$

Hence, in a good conductor, most of the energy is in the magnetic field:

$$\frac{\langle U_E \rangle_t}{\langle U_H \rangle_t} \approx \frac{\omega\varepsilon}{\sigma} \ll 1 \quad (70)$$

So far, we have assumed that the conductivity is a real number, and is independent of frequency. This is approximately true for low frequencies.

However, at high frequencies (visible frequencies and above) the behaviour of electromagnetic waves in many conductors is best described by a complex conductivity that is a function of frequency. Recall that the conductivity gives the relationship between the current density and the electric field:

$$\vec{J} = \sigma \vec{E} \quad (71)$$

So a complex conductivity indicates a phase difference between the current density and an oscillating electric field.

A model to describe this behaviour, based on the dynamics of the free electrons in the conductor, was developed in the 1900's by the German physicist Paul Drude. The detailed behaviour can get quite complicated, so we will just sketch out the main ideas.

Electrical conductors have both bound and free electrons. The bound electrons behave the same way as in a dielectric, and are subject to a binding force $-Kx$. The free electrons have no binding force. The equation of motion for free electrons in an electromagnetic wave is therefore:

$$\ddot{x} + \Gamma\dot{x} = \frac{e}{m}E_0 e^{j\omega t} \tag{72}$$

which has the solution:

$$x = \frac{e/m}{-\omega^2 + j\omega\Gamma}E_0 e^{j\omega t} \tag{73}$$

Now, the current density J depends on the conductivity σ:

$$J = \sigma E = Ne\dot{x} \tag{74}$$

where N is the number of free electrons per unit volume. From equation (73), we find:

$$\dot{x} = \frac{je\omega/m}{-\omega^2 + j\omega\Gamma}E_0 e^{j\omega t} \tag{75}$$

Therefore, we can write for the conductivity:

$$\sigma = \frac{j\omega Ne^2/m}{-\omega^2 + j\omega\Gamma} = \frac{Ne^2/m}{\Gamma + j\omega} \tag{76}$$

The conductivity is a complex number:

$$\sigma = \sigma_1 - j\sigma_2 \tag{77}$$

where:

$$\sigma_1 = \frac{Ne^2\Gamma/m}{\Gamma^2 + \omega^2}, \qquad \sigma_2 = \frac{Ne^2\omega/m}{\Gamma^2 + \omega^2} \tag{78}$$

Note that we can relate the "damping constant" Γ of the electron motion to the dc conductivity σ_0 (the conductivity at zero frequency):

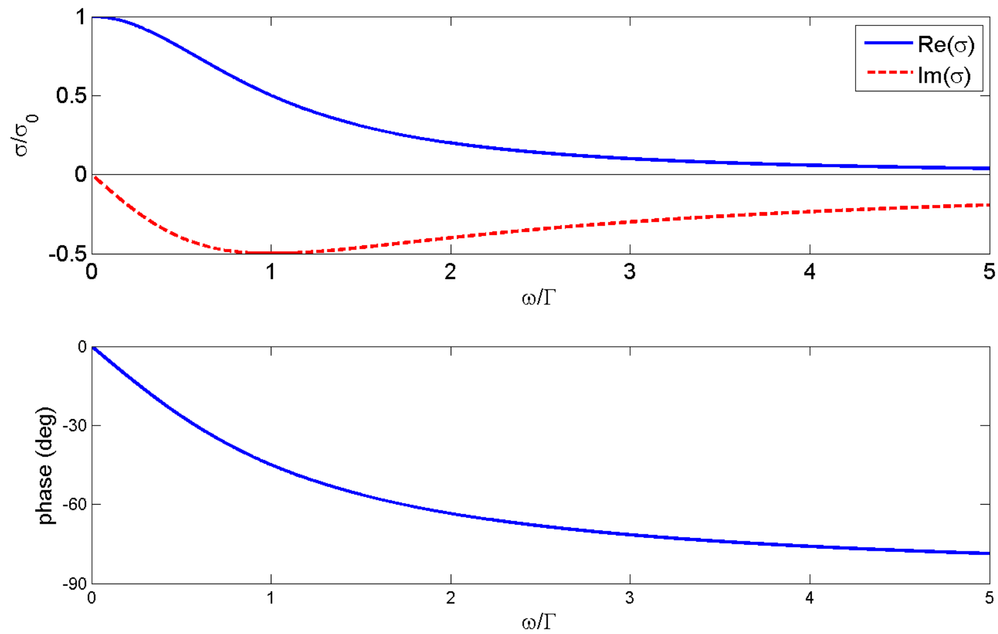
$$\sigma_0 = \lim_{\omega \to 0} \sigma = \frac{Ne^2}{\Gamma m} \tag{79}$$

In terms of σ_0, the conductivity can be written:

$$\sigma = \frac{\sigma_0}{1 + j\omega/\Gamma} \tag{80}$$

Equation (80) describes how the conductivity of a conductor varies with frequency, and is the main result of the Drude model. The constant σ_0 can be determined by experiment; if N, e and m are known, then Γ can then be calculated from (79):

$$\Gamma = \frac{Ne^2}{\sigma_0 m} \tag{81}$$



Complex Conductivity: the Drude Model

- At very low frequencies:

$$\omega \to 0, \qquad \sigma_1 \to \frac{Ne^2}{m\Gamma}, \qquad \sigma_2 \to 0 \tag{82}$$

  i.e. σ is real and constant, as for dc conductivity.

- At low frequencies ($\omega \ll \sigma/\varepsilon$, up to the infra-red range) the free electron term dominates.

- In the visible region ($\omega \approx \sigma/\varepsilon$), both terms contribute, and the formulae (78) for the conductivity agree quite well with the experimental results.

- At high frequencies ($\omega \gg \sigma/\varepsilon$, X-rays and γ-rays) the free electron term is small, and the material behaves like a dielectric.

Summary of Part 3

You should be able to:

- Derive, from Maxwell's equations, the wave equations for the electric and magnetic fields in conducting media.
- Explain the origin of the "good conductor condition" $\sigma \gg \omega\varepsilon$ for an electromagnetic plane wave.
- Derive the relationships (amplitude, phase, direction) between the electric and magnetic fields in a plane wave in conducting media.
- Derive expressions for the phase and group velocities of an electromagnetic wave in a good conductor.
- Show that in a conductor the amplitude decays exponentially and explain what happens to the energy of the wave.
- Derive an expression for the "skin depth" in the case of a plane wave travelling through a conductor.
- Explain that when an electromagnetic wave moves through a conducting medium, the conductivity of the medium can be written as a complex number, with a dependence on the frequency of the wave.