Fourier half-range series

(a) **Fourier cosine series** Suppose that a function $f(x)$ is defined on some finite interval $[0,l]$. It is possible to extend the definition of $f(x)$ to the interval $[-l,0]$ such that $f(x)$ is periodic function of period $2l$ and $f(x)$ is an even function in $[-l,l]$. This can be done by defining $f(x) = f(-x)$; $-l \leq x \leq 0$. Then, we can have Fourier series expansion of this even extension of $f(x)$ which is called half range cosine series of $f(x)$ or Fourier cosine series of $f(x)$.

Thus, half range cosine series of $f(x)$ in $[0,l]$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

and $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$

(b) **Fourier sine series** Suppose that a function $f(x)$ is defined on some finite interval $[0,l]$. It is possible to extend the definition of $f(x)$ to the interval $[-l,0]$ such that $f(x)$ is periodic function of period $2l$ and $f(x)$ is an odd function in $[-l,l]$. This can be done by defining $f(x) = -f(-x)$; $-l \leq x \leq 0$. Then, we can have Fourier series expansion of this odd extension of $f(x)$ which is called half range sine series of $f(x)$ or Fourier sine series of $f(x)$.

Thus, half range sine series of $f(x)$ in $[0,l]$ is

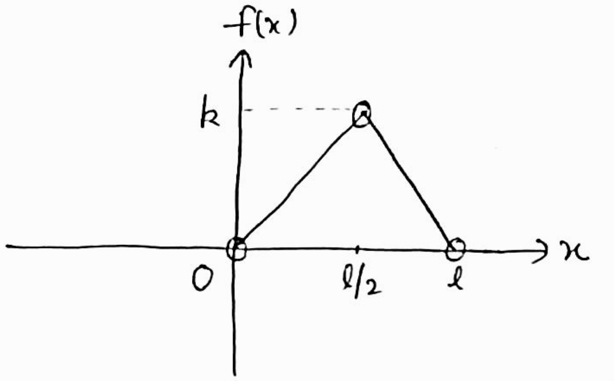
$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

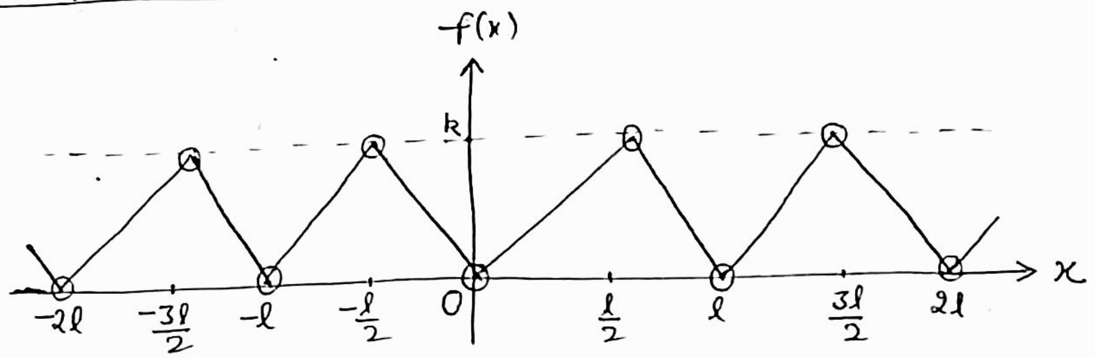
Que Sketch f(x) and its two periodic extensions and find the Fourier sine series for the function f(x) where

$$f(x) = \begin{cases} \frac{2kx}{l} \ ; & 0 < x < \frac{l}{2} \\ \frac{2k}{l}(l-x) \ ; & \frac{l}{2} < x < l \end{cases}$$

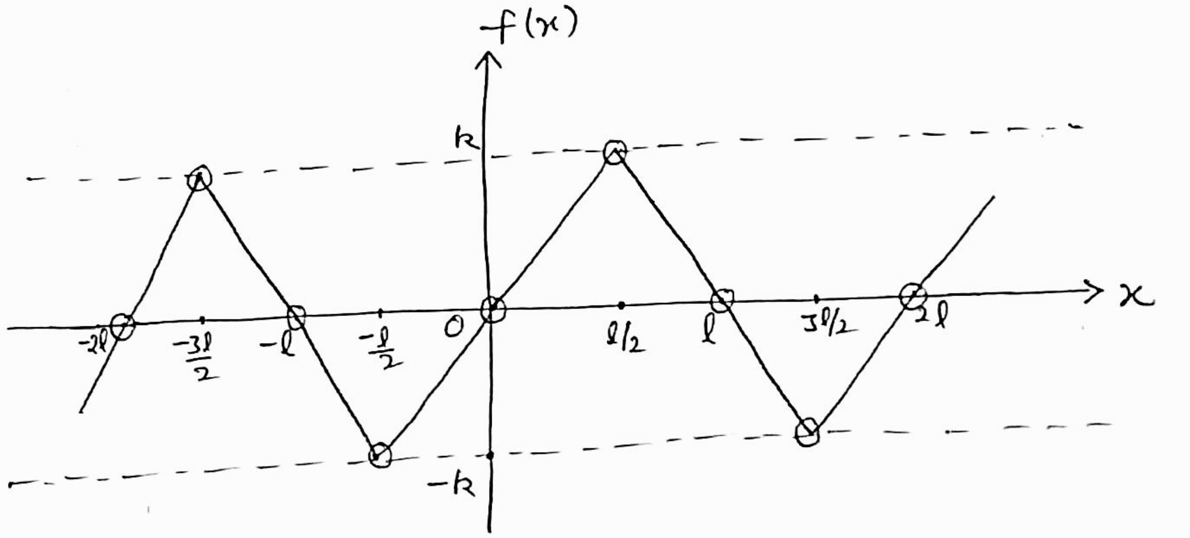
Sol (i) Graph of f(x)



(ii) Graph of even extension



(iii) Graph of odd extension



⊙ indicates the point is not in the graph.

Fourier sine series of $f(x)$ is given by

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$

$$\therefore\ b_n = \frac{2}{l}\cdot\frac{2k}{l}\left[\int_0^{l/2} x \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} (l-x) \sin\frac{n\pi x}{l}\, dx\right]$$

$$= \frac{4k}{l^2}\left[\left\{x\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right\}_0^{l/2} + \left\{(l-x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) + \left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right\}_{l/2}^{l}\right]$$

$$= \frac{4k}{l^2}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore\ b_{2n} = 0,\quad b_{2n-1} = \frac{8k}{\pi^2(2n-1)^2}(-1)^{n+1}\ ;\ n = 1, 2, 3, \ldots$$

$\therefore$ Fourier sine series of $f(x)$ is $f(x) \sim \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{l}$

<u>Que</u> Prove that for $0 \leq x \leq \pi$,

$$x(\pi - x) = \frac{\pi^2}{6} - \left[\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \cdots\right]$$

<u>Sol</u> Fourier cosine series of $x(\pi - x)$ in $0 \leq x \leq \pi$ is

$$x(\pi - x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\,dx = \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\cos nx\,dx = \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{1}{n}\sin nx\right) - (\pi - 2x)\left(-\frac{1}{n^2}\cos nx\right) + (-2)\left(-\frac{1}{n^3}\sin nx\right)\right]_0^{\pi}$$

$$= \frac{2}{n^2}\left[(-1)^{n+1} - 1\right]$$

$$\therefore\ a_{2n} = -\frac{1}{n^2},\quad a_{2n-1} = 0\ ;\ n = 1, 2, 3, \ldots$$

$\therefore$ Fourier cosine series of $x(\pi - x)$ in $0 \leq x \leq \pi$ is

$$x(\pi - x) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty}\frac{1}{n^2}\cos 2nx = \frac{\pi^2}{6} - \left[\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \cdots\right]$$

Harmonic Analysis: Sometimes the function is not given by a formula, but by a table of corresponding values. The process of finding the Fourier series for a function given by such values of the function and independent variable is known as Harmonic Analysis.

Fourier series: Let $f(x)$ be a periodic function of period $2l$ in the interval $[0, 2l]$. If the interval $[0, 2l]$ is divided into k subintervals of equal length i.e, $2l/k$ by $(k+1)$ points and corresponding values of $f(x)$ for those points are given i.e,

x	$x_0 = 0$	x_1	x_2	------	x_{k-1}	$x_k = 2l$
$f(x)$	f_0	f_1	f_2	------	f_{k-1}	$f_k = f_0$

Then the Fourier series for $f(x)$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) \quad \text{——— (1)}$$

where $a_0 = \frac{1}{l} \int_0^{2l} f(x)\, dx$

$$= \frac{1}{l} \left[\int_{x_0}^{x_1} f(x)\, dx + \int_{x_1}^{x_2} f(x)\, dx + \cdots + \int_{x_{k-1}}^{x_k} f(x)\, dx \right]$$

$$= \frac{1}{l} \left[\frac{f_0 + f_1}{2} \cdot (x_1 - x_0) + \frac{f_1 + f_2}{2} \cdot (x_2 - x_1) + \cdots + \frac{f_{k-1} + f_k}{2} \cdot (x_k - x_{k-1}) \right]$$

$$= \frac{1}{l} \cdot \frac{2l}{k} \left[\frac{f_0 + f_k}{2} + f_1 + f_2 + \cdots + f_{k-1} \right]$$

$$= \frac{2}{k} \left[f_0 + f_1 + f_2 + \cdots + f_{k-1} \right] \qquad (\because f_k = f_0)$$

$$= 2 \times \text{Mean value of } f(x) \text{ in } [0, 2l)$$

Similarly $a_n = 2 \times$ Mean value of $f(x)\cos\frac{n\pi x}{l}$ in $[0, 2l)$

$b_n = 2 \times$ Mean value of $f(x)\sin\frac{n\pi x}{l}$ in $[0, 2l)$

In equation (1), the term $\left(a_1\cos\frac{\pi x}{l} + b_1\sin\frac{\pi x}{l}\right)$ is called the fundamental or first harmonic and its amplitude $= \sqrt{a_1^2+b_1^2}$

The term $\left(a_2\cos\frac{2\pi x}{l} + b_2\sin\frac{2\pi x}{l}\right)$ is called the second harmonic and its amplitude $= \sqrt{a_2^2+b_2^2}$ and so on.

Remark If $f_k \neq f_0$, then $x_{k+1} = 2l$.

Half range Fourier sine series Suppose $f(x)$ is given at equidistant points as

x	$x_0=0$	x_1	x_2	------	x_{k-1}	x_k
$f(x)$	f_0	f_1	f_2	------	f_{k-1}	f_k

and we are to find half range sine series. Then for odd extension of $f(x)$, we must have $f_0 = 0$. Further, if $f_k = 0$ then $x_k = l$ otherwise x_{k+1} will be l.

$\therefore$ Half range sine series is given by

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

$$= \frac{2}{l}\left[\int_{x_0}^{x_1} f(x)\sin\frac{n\pi x}{l}\,dx + \int_{x_1}^{x_2} f(x)\sin\frac{n\pi x}{l}\,dx + \cdots + \int_{x_{k-1}}^{x_k} f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\left(\frac{f_0\sin\frac{n\pi x_0}{l} + f_1\sin\frac{n\pi x_1}{l}}{2}\right)(x_1-x_0) + \left(\frac{f_1\sin\frac{n\pi x_1}{l} + f_2\sin\frac{n\pi x_2}{l}}{2}\right)(x_2-x_1) + \cdots + \left(\frac{f_{k-1}\sin\frac{n\pi x_{k-1}}{l} + f_k\sin\frac{n\pi x_k}{l}}{2}\right)\cdot(x_k - x_{k-1})\right]$$

$$= \frac{2}{l}\cdot\frac{l}{k}\left[\left(\frac{f_0\sin\frac{n\pi x_0}{l} + f_k\sin\frac{n\pi x_k}{l}}{2}\right) + f_1\sin\frac{n\pi x_1}{l} + \cdots + f_{k-1}\frac{\sin n\pi x_{k-1}}{l}\right] \quad (1)$$

$\left(\because x_1 - x_0 = x_2 - x_1 = \cdots = x_k - x_{k-1} = \frac{l}{k}\right)$

$\therefore\ b_n = \frac{2}{k}\left[\frac{1}{2} f_0 \sin\frac{n\pi x_0}{l} + f_1 \sin\frac{n\pi x_1}{l} + \cdots + f_{k-1}\sin\frac{n\pi x_{k-1}}{l}\right]$

$$\boxed{b_n = 2 \times \text{Mean value of } f(x)\sin\frac{n\pi x}{l} \text{ in } [0, l]}$$ $(\because f_k = 0)$ $(\& f_0 = 0)$

Half range Fourier cosine series suppose $f(x)$ is given at equidistant points as

x	$x_0 = 0$	x_1	x_2	------	x_{k-1}	$x_k = l$
$f(x)$	f_0	f_1	f_2	------	f_{k-1}	f_k

and we are to find half range cosine series. Then for even extension of $f(x)$, f_0 and f_k may be any value and x_k is always l.

$\therefore$ Half range cosine series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$= \frac{2}{k}\left[\frac{f_0 + f_k}{2} + f_1 + f_2 + \cdots + f_{k-1}\right]$$ (Solving as above equation (1))

and $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$

$$= \frac{2}{k}\left[\frac{f_0\cos\frac{n\pi x_0}{l} + f_k\cos\frac{n\pi x_k}{l}}{2} + f_1\cos\frac{n\pi x_1}{l} + f_2\cos\frac{n\pi x_2}{l} + \cdots + f_{k-1}\cos\frac{n\pi x_{k-1}}{l}\right]$$

Que Find Fourier series of $f(\theta)$ upto first three harmonics where $f(\theta)$ is given by the following table

θ°	0	60	120	180	240	300	360
$f(\theta)$	0.8	0.6	0.4	0.7	0.9	1.1	0.8

Also find the amplitude of third harmonic

Sol Fourier series of $f(\theta)$ is

$$f(\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

θ°	$f(\theta)$	$\cos\theta$	$\sin\theta$	$\cos 2\theta$	$\sin 2\theta$	$\cos 3\theta$	$\sin 3\theta$
0	0.8	1	0	1	0	1	0
60	0.6	$1/2$	$\sqrt{3}/2$	$-1/2$	$\sqrt{3}/2$	-1	0
120	0.4	$-1/2$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}/2$	1	0
180	0.7	-1	0	1	0	-1	0
240	0.9	$-1/2$	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}/2$	1	0
300	1.1	$1/2$	$-\sqrt{3}/2$	$-1/2$	$-\sqrt{3}/2$	-1	0
	4.5						

$$a_0 = \frac{2}{6}\Sigma f(\theta) = \frac{1}{3}(4.5) = 1.5$$

$$a_1 = \frac{2}{6}\Sigma f(\theta)\cos\theta = \frac{1}{3}\left[0.8-0.7+\frac{1}{2}(0.6-0.4-0.9+1.1)\right] = 0.1$$

$$b_1 = \frac{2}{6}\Sigma f(\theta)\sin\theta = \frac{1}{3}\left[\frac{\sqrt{3}}{2}(0.6+0.4-0.9-1.1)\right] = -0.3$$

$$a_2 = \frac{2}{6}\Sigma f(\theta)\cos 2\theta = \frac{1}{3}\left[0.8+0.7-\frac{1}{2}(0.6+0.4+0.9+1.1)\right] = 0$$

$$b_2 = \frac{2}{6}\Sigma f(\theta)\sin 2\theta = \frac{1}{3}\left[\frac{\sqrt{3}}{2}(0.6-0.4+0.9-1.1)\right] = 0$$

$$a_3 = \frac{2}{6}\Sigma f(\theta)\cos 3\theta = \frac{1}{3}\left[0.8-0.6+0.4-0.7+0.9-1.1\right] = -0.1$$

$$b_3 = \frac{2}{6}\Sigma f(\theta)\sin 3\theta = 0$$

Amplitude of third harmonic $= \sqrt{a_3^2+b_3^2} = 0.1$

& Fourier series of $f(\theta)$ upto first three harmonics is

$$f(\theta) \sim \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta + a_3\cos 3\theta + b_3\sin 3\theta$$

i.e, $f(\theta) \sim 0.75 + (0.1)\cos\theta - (0.3)\sin\theta - (0.1)\cos 3\theta$

Que Obtain the first three coefficients in the Fourier cosine series for y, where y is given in the following table

x	0	1	2	3	4	5
y	4	8	15	7	6	2

Sol Here $l = 5$

$\therefore$ The Fourier cosine series for y is given by

$$y = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{5} + a_2 \cos\frac{2\pi x}{5} + \cdots$$

x	y	$\pi x/5$	$\cos(\pi x/5)$	$\cos(2\pi x/5)$
0	4	0	1	1
1	8	$\pi/5$	0.8090	0.3090
2	15	$2\pi/5$	0.3090	−0.8090
3	7	$3\pi/5$	−0.3090	−0.8090
4	6	$4\pi/5$	−0.8090	0.3090
5	2	π	−1	1

$$\therefore \frac{a_0}{2} = \frac{1}{2}\cdot\frac{2}{5}\left[\frac{4+2}{2} + (8+15+7+6)\right] = \frac{1}{5}(39) = 7.8$$

$$a_1 = \frac{2}{5}\left[\frac{4+(-2)}{2} + \{(8-6)0.8090 + (15-7)\,0.3090\}\right]$$

$$= \frac{2}{5}(5.0900) = 2.036$$

$$a_2 = \frac{2}{5}\left[\frac{4+2}{2} + \{(8+6)0.3090 - (15+7)\,0.8090\}\right] = \frac{2}{5}(-10.4720)$$

$$= -4.1888$$

$\therefore$ The first three coefficients in the Fourier cosine series are

$\frac{a_0}{2} = 7.8$, $a_1 = 2.036$ and $a_2 = -4.1888$