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Fourier Integrals and Fourier Transforms (UNIT-I)

Fourier integral of $f(x)$ is given by

$$f(x) = \frac{1}{\pi}\int_{\omega=0}^{\infty}\int_{t=-\infty}^{\infty} f(t)\cos\omega(t-x)\,dt\,d\omega \quad \text{———} \quad (1)$$

Also, when $f(x)$ satisfies the Dirichlet's conditions, then Fourier integral of $f(x)$ converges to $f(x_0)$ if it is continuous at $x = x_0$ and converges to $\frac{1}{2}\left[f(x_0-0)+f(x_0+0)\right]$ if $f(x)$ is discontinuous at $x = x_0$.

Fourier cosine integral of $f(x)$ is given by (for even function)

$$f(x) = \frac{2}{\pi}\int_{\omega=0}^{\infty}\int_{t=0}^{\infty} f(t)\cos(\omega t)\cos(\omega x)\,dt\,d\omega \quad \text{———} \quad (2)$$

Same as above for continuous and discontinuous points.

Fourier sine integral of $f(x)$ is given by (for odd function)

$$f(x) = \frac{2}{\pi}\int_{\omega=0}^{\infty}\int_{t=0}^{\infty} f(t)\sin(\omega t)\sin(\omega x)\,dt\,d\omega \quad \text{———} \quad (3)$$

Same as above for continuous and discontinuous points.

Fourier transform and its inverse

We define $\mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = F(\omega)$ ——— (4)

as Fourier transform of $f(x)$.

and $f(x) = \mathcal{F}^{-1}(F(\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,e^{i\omega x}\,d\omega$

is called inverse Fourier transform of $F(\omega)$.

Fourier cosine transform

We define $\mathcal{F}_c(f(x)) = \int_0^\infty f(x)\cos(\omega x)\,dx = F_c(\omega)$ ——— (5)

as Fourier cosine transform of $f(x)$

and $f(x) = \mathcal{F}_c^{-1}(F_c(\omega)) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos(\omega x)\,d\omega$

is called inverse Fourier cosine transform of $F_c(\omega)$.

Fourier sine transform

We define $\mathcal{F}_s(f(x)) = \int_0^\infty f(x)\sin(\omega x)\,dx = F_s(\omega)$ ——— (6)

as Fourier sine transform of $f(x)$ and

$$f(x) = \mathcal{F}_s^{-1}(F_s(\omega)) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin(\omega x)\,d\omega$$

is called inverse Fourier sine transform of $F_s(\omega)$.

Remark (1) Some authors also take the following pairs of Fourier transforms

	Fourier transform	Inverse Fourier transform
(a)	$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{-i\omega x}\,dx$	$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(\omega)e^{i\omega x}\,d\omega$ ——(7)
(b)	$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{i\omega x}\,dx$	$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(\omega)e^{-i\omega x}\,d\omega$
(c)	$F(\omega) = \int_{-\infty}^\infty f(x)e^{i\omega x}\,dx$	$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty F(\omega)e^{-i\omega x}\,d\omega$

(2) Some authors take the formulae as

(a) $F_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(\omega x)\,dx$, $\quad f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(\omega)\cos(\omega x)\,d\omega$

(b) $F_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\omega x)\,dx$, $\quad f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(\omega)\sin(\omega x)\,d\omega$

Definition Function $f(x)$ is called self reciprocal if

$\mathcal{F}(f(x)) = \sqrt{2\pi}\,f(\omega)$ (according to def. given by (4))

or $\mathcal{F}(f(x)) = f(\omega)$ (according to def. given by (7))

Some Important results to remember

(1) $\int_0^\infty \frac{\sin mx}{x}\,dx = \frac{\pi}{2} \quad (m>0) \qquad = -\frac{\pi}{2} \quad (m<0)$

(2) $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

(3) **Differentiation under the sign of integration**

Leibnitz's rule

$$\frac{d}{d\alpha}\left\{\int_{\phi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx\right\} = \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f(x,\alpha)}{\partial \alpha}\,dx + \frac{d\psi}{d\alpha}\cdot f[\psi(\alpha),\alpha] - \frac{d\phi}{d\alpha}\cdot f[\phi(\alpha),\alpha]$$

If $\phi(\alpha)$ and $\psi(\alpha)$ are both constants, then

$$\frac{d}{d\alpha}\left\{\int_a^b f(x,\alpha)\,dx\right\} = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}\,dx$$

(4) **Unit step function** $\quad u_a(t) = U(t-a) = \begin{cases} 0, & t<a \\ 1, & t \geqslant a \end{cases}$

In particular $u_0(t) = \begin{cases} 0, & t<0 \\ 1, & t\geqslant 0 \end{cases} = U(t)$

(5) **Dirac delta function**

$$\delta(t-a) = \lim_{\varepsilon\to 0+} \delta_\varepsilon(t-a)$$

where $\delta_\varepsilon(t-a) = \begin{cases} \frac{1}{\varepsilon} & \text{for } a<t<a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$

Que 1 Express the function $f(x) = \begin{cases} 1; & |x| \le 1 \\ 0; & |x| > 1 \end{cases}$ as a Fourier integral and hence evaluate

(i) $\displaystyle\int_0^\infty \frac{\sin\omega \cos\omega x}{\omega}\, d\omega$ (ii) $\displaystyle\int_0^\infty \frac{\sin\omega}{\omega}\, d\omega$

Sol Fourier integral of $f(x)$ is given by.

$$f(x) = \frac{1}{\pi}\int_{\omega=0}^{\infty}\int_{t=-\infty}^{\infty} f(t)\cos\omega(t-x)\, dt\, d\omega$$

$$\therefore f(x) = \frac{1}{\pi}\int_{\omega=0}^{\infty}\left\{\int_{-1}^{1} 1.\cos\omega(t-x)\, dt\right\} d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \left\{\frac{\sin\omega(t-x)}{\omega}\right\}_{-1}^{1} d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin\omega(1-x) + \sin\omega(1+x)}{\omega}\, d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\, d\omega$$

(i) $\displaystyle\therefore \int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\, d\omega = \frac{\pi}{2} f(x)$ at the point x of continuity

$\displaystyle= \frac{\pi}{2}\left[\frac{f(x+0)+f(x-0)}{2}\right]$ at the point x of discontinuity

$\because f(x)$ is discontinuous at $x = \pm 1$

$$\therefore \frac{f(1+0)+f(1-0)}{2} = \frac{0+1}{2} = \frac{1}{2}$$

$$\frac{f(-1+0)+f(-1-0)}{2} = \frac{1+0}{2} = \frac{1}{2}$$

$$\therefore \int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\, d\omega = \begin{cases} \frac{\pi}{2} & \text{when } |x|<1 \\ 0 & \text{when } |x|>1 \\ \frac{\pi}{4} & \text{when } x=1 \text{ and } x=-1 \end{cases}$$

(ii) By (i) part, $\displaystyle\int_0^\infty \frac{\sin\omega}{\omega}\, d\omega = \frac{\pi}{2}$ (Put $x=0$ on both sides)

Que Find the Fourier cosine integral of the function e^{-ax}.

Hence show that $\int_0^\infty \frac{\cos sx}{(s^2+1)} ds = \frac{\pi}{2} e^{-x}, x \geq 0, a > 0.$

Sol Fourier cosine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi} \int_{s=0}^{\infty} \int_{t=0}^{\infty} f(t) \cos st \cos sx \, dt \, ds$$

$\therefore$ Fourier cosine integral of e^{-ax} is given by

$$f(x) = \frac{2}{\pi} \int_{s=0}^{\infty} \cos sx \left\{ \int_{t=0}^{\infty} e^{-at} \cos st \, dt \right\} ds$$

$$= \frac{2}{\pi} \int_{s=0}^{\infty} \cos sx \left\{ \frac{e^{-at}}{a^2+s^2} (-a \cos st + s \sin st) \right\}_{t=0}^{\infty} ds$$

$$= \frac{2}{\pi} \int_{s=0}^{\infty} \frac{a}{a^2+s^2} \cos sx \, ds$$

$$\therefore \int_0^\infty \frac{\cos sx}{(a^2+s^2)} ds = \frac{\pi}{2a} e^{-ax} \quad \forall x \geq 0 \ (\because f(x) = e^{-ax} \text{ is continuous for all } x)$$

Replace a by 1 on both sides, we get

$$\int_0^\infty \frac{\cos sx}{(s^2+1)} ds = \frac{\pi}{2} e^{-x}, \quad x \geq 0$$

Que Using Fourier sine integral, show that

$$\int_0^\infty \frac{1-\cos \pi\lambda}{\lambda} \sin x\lambda \, d\lambda = \begin{cases} \pi/2, & \text{when } 0 < x < \pi \\ 0, & \text{when } x > \pi \end{cases}$$

Sol Consider the function $f(x) = \begin{cases} \pi/2, & \text{when } 0 < x < \pi \\ 0, & \text{when } x > \pi \end{cases}$

Its Fourier sine integral is given by

$$f(x) = \frac{2}{\pi} \int_{\lambda=0}^{\infty} \int_{t=0}^{\infty} f(t) \sin \lambda t \sin \lambda x \, dt \, d\lambda$$

$$= \frac{2}{\pi} \int_{\lambda=0}^{\infty} \sin \lambda x \left\{ \int_0^\pi \frac{\pi}{2} \sin \lambda t \, dt \right\} d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin \lambda x \cdot \left\{ \frac{-\pi}{2} \frac{\cos \lambda t}{\lambda} \right\}_{t=0}^{\pi} d\lambda = \int_0^\infty \frac{1-\cos \pi\lambda}{\lambda} \sin x\lambda \, d\lambda$$

Hence proved.

Que Find the Fourier transform of the following function defined on $(-\infty, \infty)$

$$f(t) = \begin{cases} 1, & |t| < a \\ 0, & |t| > a \end{cases}$$ and hence evaluate

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin a\omega \cos \omega t}{\omega}\, d\omega$ (b) $\displaystyle\int_{0}^{\infty} \frac{\sin x}{x}\, dx$

Sol $$F(f(t)) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-a}^{a} 1.e^{-i\omega t}\, dt = \left(\frac{e^{-i\omega t}}{-i\omega}\right)_{t=-a}^{a} = \frac{1}{i\omega}\left(e^{i\omega a} - e^{-i\omega a}\right)$$

$$= \frac{2\sin a\omega}{\omega} = F(\omega)$$

Taking inverse Fourier transform,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} F(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} \cdot \frac{2\sin a\omega}{\omega}\, d\omega$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin a\omega\,(\cos \omega t + i \sin \omega t)}{\omega}\, d\omega$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin a\omega \cos \omega t}{\omega}\, d\omega$$

(a) $$\therefore \int_{-\infty}^{\infty} \frac{\sin a\omega \cos \omega t}{\omega}\, d\omega = \pi f(t) = \begin{cases} \pi, & |t| < a \\ 0, & |t| > a \end{cases}$$

(b) Take $t = 0$ and $a = 1$,

$$\int_{-\infty}^{\infty} \frac{\sin \omega}{\omega}\, d\omega = \pi \Rightarrow 2\int_{0}^{\infty} \frac{\sin \omega}{\omega}\, d\omega = \pi$$

or $$\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

Que Find Fourier transform of unit step function $u_a(t)$ and dirac delta function $\delta(t-a)$. Also mention the case when $a=0$.

Sol $$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$\therefore F(u_a(t)) = \int_{a}^{\infty} 1.e^{-i\omega t}\, dt = \left(\frac{e^{-i\omega t}}{-i\omega}\right)_{a}^{\infty}$$

But $\lim_{t\to\infty} e^{-i\omega t}$ does not exist as $\lim_{t\to\infty} \cos\omega t$ and $\lim_{t\to\infty} \sin\omega t$ do not exist.

Hence $\mathcal{F}(u_a(t))$ or $\mathcal{F}(u_0(t))$ does not exist in terms of definition of function.

Now, $\mathcal{F}\{\delta(t-a)\} = \int_{-\infty}^{\infty} e^{-i\omega t}\,\delta(t-a)\,dt$

$$= \lim_{\epsilon\to 0^+} \int_a^{a+\epsilon} \frac{1}{\epsilon} e^{-i\omega t}\,dt$$

$$= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left(\frac{e^{-i\omega t}}{-i\omega}\right)_a^{a+\epsilon}$$

$$= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left(\frac{e^{-i\omega a} - e^{-i\omega(a+\epsilon)}}{i\omega}\right)$$

$$= \frac{e^{-i\omega a}}{i\omega} \lim_{\epsilon\to 0^+}\left(\frac{1-e^{-i\omega\epsilon}}{\epsilon}\right) = \frac{e^{-i\omega a}}{i\omega}\lim_{\epsilon\to 0^+}\left(\frac{i\omega e^{-i\omega\epsilon}}{1}\right)$$

(by L' Hospital rule)

$\therefore\ \mathcal{F}\{\delta(t-a)\} = e^{-i\omega a}$

In particular $\mathcal{F}\{\delta(t)\} = 1$

Properties of Fourier Transforms

Linearity property For any functions $f(x)$ and $g(x)$ and any constants a and b

(a) $\mathcal{F}(af+bg)(x) = a\,\mathcal{F}(f(x)) + b\,\mathcal{F}(g(x))$

(b) $\mathcal{F}_c(af+bg)(x) = a\,\mathcal{F}_c(f(x)) + b\,\mathcal{F}_c(g(x))$; f and g are even functions

(c) $\mathcal{F}_s(af+bg)(x) = a\,\mathcal{F}_s(f(x)) + b\,\mathcal{F}_s(g(x))$; f and g are odd functions

Shifting property If $\mathcal{F}(f(x)) = F(\omega)$ and a is any real number, then

$\boxed{\mathcal{F}(f(x-a)) = e^{-i\omega a}F(\omega)}$ (Shifting on x-axis)

and $\boxed{\mathcal{F}(e^{iax}f(x)) = F(\omega-a)}$ (Frequency shifting)

Remark From the above result

$$\mathcal{F}^{-1}\left(e^{-i\omega a} F(\omega)\right) = f(x-a)$$

and $\mathcal{F}^{-1}\left(F(\omega-a)\right) = e^{iax} f(x)$

Change of scale property For any $a>0$

(a) If $\mathcal{F}(f(x)) = F(\omega)$ then $\mathcal{F}(f(ax)) = \frac{1}{a} F\left(\frac{\omega}{a}\right)$

(b) If $\mathcal{F}_c(f(x)) = F_c(\omega)$ then $\mathcal{F}_c(f(ax)) = \frac{1}{a} F_c\left(\frac{\omega}{a}\right)$

(c) If $\mathcal{F}_s(f(x)) = F_s(\omega)$ then $\mathcal{F}_s(f(ax)) = \frac{1}{a} F_s\left(\frac{\omega}{a}\right)$

Modulus property (or Modulation theorem)

If $\mathcal{F}\{f(x)\} = F(\omega)$, and a is any real number, then

(a) $\mathcal{F}\{f(x)\cos ax\} = \frac{1}{2}\left[F(\omega+a) + F(\omega-a)\right]$ [First Term Sep. 2017] Proof (2.5)

(b) $\mathcal{F}\{f(x)\sin ax\} = \frac{i}{2}\left[F(\omega+a) - F(\omega-a)\right]$

(c) $\mathcal{F}^{-1}\{F(\omega)\cos a\omega\} = \frac{1}{2}\left[f(x+a) + f(x-a)\right]$

(d) $\mathcal{F}^{-1}\{F(\omega)\sin a\omega\} = \frac{-i}{2}\left[f(x+a) - f(x-a)\right]$

$\mathcal{F}_s\{f(x)\cos ax\} = \frac{1}{2}\left[F_s(\omega+a) + F_s(\omega-a)\right]$

$\mathcal{F}_s\{f(x)\sin ax\} = \frac{1}{2}\left[F_s(\omega+a) - F_s(\omega-a)\right]$

$\mathcal{F}_c\{f(x)\sin ax\} = \frac{1}{2}\left[F_s(\omega+a) - F_s(\omega-a)\right]$

Differentiations of transforms w.r.t. frequency

If $\mathcal{F}(f(x)) = F(\omega)$, $\mathcal{F}_s(f(x)) = F_s(\omega)$, $\mathcal{F}_c(f(x)) = F_c(\omega)$ exist,

then $\mathcal{F}(x^n f(x)) = i^n \frac{d^n}{d\omega^n} F(\omega)$ [End Term Dec 14-15, 4 marks]

$$\mathcal{F}_s(x^n f(x)) = \begin{cases} (-1)^{n/2} \frac{d^n}{d\omega^n} F_s(\omega) & \text{if } n \text{ is even} \\ (-1)^{(n+1)/2} \frac{d^n}{d\omega^n} F_c(\omega) & \text{if } n \text{ is odd} \end{cases}$$

and

$$\mathcal{F}_c(x^n f(x)) = \begin{cases} (-1)^{n/2} \frac{d^n}{d\omega^n} F_c(\omega) & \text{if } n \text{ is even} \\ (-1)^{(n-1)/2} \frac{d^n}{d\omega^n} F_s(\omega) & \text{if } n \text{ is odd} \end{cases}$$

Transform of derivatives

(a) Let $f(x)$ be continuous and $f^{(k)}(x)$ be piecewise continuous on every finite interval $(-l, l)$ and $\int_{-\infty}^{\infty} |f^{(k-1)}(x)\,dx|$ converges for $k = 1, 2, \ldots, n$ and $f^{(k-1)}(x) \to 0$ as $x \to \pm\infty$ for $k = 1, 2, \ldots n$ then

$$\boxed{\mathcal{F}(f^{(n)}(x)) = (i\omega)^n \mathcal{F}(f(x))}$$

(b) Let $f(x)$ be continuous on $[0, \infty)$, $f(x) \to 0$ as $x \to \infty$ and $f'(x)$ is piecewise continuous on every finite interval $[0, l]$ then

$$\boxed{\mathcal{F}_c(f'(x)) = \omega \mathcal{F}_s(f(x)) - f(0)}$$

and

$$\boxed{\mathcal{F}_s(f'(x)) = -\omega \mathcal{F}_c(f(x))}$$

(c) If $f(x)$ and $f'(x)$ are continuous in $[0, \infty)$, $f(x) \to 0$, $f'(x) \to 0$ as $x \to \infty$ and $f''(x)$ is piecewise continuous on every subinterval $[0, l]$ then

$$\boxed{\mathcal{F}_c(f''(x)) = -\omega^2 \mathcal{F}_c(f(x)) - f'(0)}$$

and

$$\boxed{\mathcal{F}_s(f''(x)) = -\omega^2 \mathcal{F}_s(f(x)) + \omega f(0)}$$

Fourier Transform of Integral Function (No need)

Let $f(x)$ be piecewise continuous on every interval $(-l, l)$ and $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges, then

$$\boxed{\mathcal{F}\left[\int_{-\infty}^{x} f(t)\,dt\right] = \frac{1}{i\omega}\mathcal{F}(f(x))}$$

provided $\mathcal{F}(f(x)) = F(\omega) = 0$ for $\omega = 0$

Convolution Let $f(x)$ and $g(x)$ be piecewise continuous on every interval $(-l, l)$ and $\int_{-\infty}^{\infty} |f(x)|dx$, $\int_{-\infty}^{\infty} |g(x)|dx$ converge then convolution of f and g denoted by $(f*g)(x)$ is defined as

$$(f*g)(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

Convolution Theorem (or Faltung theorem) for Fourier transforms

Let $f(x)$ and $g(x)$ be piecewise continuous on every interval $(-l, l)$ and $\int_{-\infty}^{\infty} |f(x)|\,dx$, $\int_{-\infty}^{\infty} |g(x)|\,dx$ converge.

Let $\mathcal{F}(f(x)) = F(\omega)$, $\mathcal{F}(g(x)) = G(\omega)$ then

$$\mathcal{F}[(f*g)(x)] = F(\omega).G(\omega)$$

i.e, If $\mathcal{F}^{-1}\{F(\omega)\} = f(x)$ and $\mathcal{F}^{-1}\{G(\omega)\} = g(x)$, then

$$\mathcal{F}^{-1}\{F(\omega).G(\omega)\} = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

Proof $\mathcal{F}\left\{\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du\right\}$

$$= \int_{-\infty}^{\infty} e^{-i\omega x}\left\{\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du\right\}dx$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-i\omega u}\left\{\int_{-\infty}^{\infty} g(x-u)\, e^{-i\omega(x-u)}\, dx\right\}du$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-i\omega u}\left\{\int_{-\infty}^{\infty} g(t)\, e^{-i\omega t}\, dt\right\}du \quad (\text{taking } x-u=t)$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-i\omega u}\, du \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t}\, dt$$

$$= F(\omega).G(\omega)$$

Convolution Let $f(x)$ and $g(x)$ be piecewise continuous on every interval $(-l, l)$ and $\int_{-\infty}^{\infty} |f(x)|\,dx$, $\int_{-\infty}^{\infty} |g(x)|\,dx$ converge then convolution of f and g denoted by $(f*g)(x)$ is defined as

$$(f*g)(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

Convolution Theorem (or Faltung-theorem) for Fourier transforms

Let $f(x)$ and $g(x)$ be piecewise continuous on every interval $(-l, l)$ and $\int_{-\infty}^{\infty} |f(x)|\,dx$, $\int_{-\infty}^{\infty} |g(x)|\,dx$ converge.

Let $\mathcal{F}(f(x)) = F(\omega)$, $\mathcal{F}(g(x)) = G(\omega)$ then

$$\mathcal{F}[(f*g)(x)] = F(\omega) \cdot G(\omega)$$

i.e, If $\mathcal{F}^{-1}\{F(\omega)\} = f(x)$ and $\mathcal{F}^{-1}\{G(\omega)\} = g(x)$, then

$$\mathcal{F}^{-1}\{F(\omega) \cdot G(\omega)\} = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

Proof $\mathcal{F}\left\{\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du\right\}$

$$= \int_{-\infty}^{\infty} e^{-i\omega x} \left\{\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du\right\} dx$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-i\omega u} \left\{\int_{-\infty}^{\infty} g(x-u)\, e^{-i\omega(x-u)}\, dx\right\} du$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-i\omega u} \left\{\int_{-\infty}^{\infty} g(t)\, e^{-i\omega t}\, dt\right\} du \quad (\text{taking } x-u=t)$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-i\omega u}\, du \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t}\, dt$$

$$= F(\omega) \cdot G(\omega)$$

Que Find the Fourier transform of $f(x) = e^{-ax^2}$, $a>0$ and hence find the Fourier transform of $e^{-x^2/2}$ and $xe^{-ax^2}, a>0$.

Sol
$$F\{e^{-ax^2}\} = \int_{-\infty}^{\infty} e^{-ax^2} \cdot e^{-i\omega x}\, dx$$
$$= \int_{-\infty}^{\infty} e^{-\left(\sqrt{a}x + \frac{i\omega}{2\sqrt{a}}\right)^2} \cdot e^{-\omega^2/4a}\, dx$$
$$= \int_{-\infty}^{\infty} e^{-y^2} \cdot e^{-\omega^2/4a} \cdot \frac{1}{\sqrt{a}}\, dy \qquad \left(y = \sqrt{a}\,x + \frac{i\omega}{2\sqrt{a}} \right)$$
$$= \frac{1}{\sqrt{a}} e^{-\omega^2/4a} \int_{-\infty}^{\infty} e^{-y^2} dy$$
$$= \frac{2}{\sqrt{a}} e^{-\omega^2/4a} \int_{0}^{\infty} e^{-y^2} dy$$
$$= \frac{2}{\sqrt{a}} e^{-\omega^2/4a} \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$$

Taking $a = \frac{1}{2}$, we get $F\{e^{-x^2/2}\} = \sqrt{2\pi}\, e^{-\omega^2/2}$ <u>A</u>

Now, if $f(x) = e^{-ax^2}$

then $f'(x) = -2axe^{-ax^2}$

$$\therefore F(f'(x)) = F(-2axe^{-ax^2})$$
$$= (i\omega)\, F(f(x))$$
$$= (i\omega) \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$$

$$\therefore F(xe^{-ax^2}) = -\frac{i\omega}{2a} \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}$$ <u>A</u>

Que Find $\mathcal{F}\{f(t)\}$ where $f(t) = \begin{cases} 0, & t<0 \\ e^{-\alpha t}, & t \geq 0 \end{cases}$, $\alpha > 0$

or

Find $\mathcal{F}\{f(t)\}$ where $f(t) = e^{-\alpha t} U(t)$, $\alpha > 0$ where $U(t)$ is unit step function.

Hence prove that $\mathcal{F}(t^n e^{-\alpha t} U(t)) = \dfrac{n!}{(\alpha + i\omega)^{n+1}}$

Sol

$$\mathcal{F}(f(t)) = \int_{-\infty}^{\infty} e^{-\alpha t} U(t) e^{-i\omega t} dt$$

$$= \int_0^{\infty} e^{-\alpha t} \cdot e^{-i\omega t} dt = \int_0^{\infty} e^{-(\alpha + i\omega)t} dt$$

$$= \left\{ \frac{e^{-(\alpha+i\omega)t}}{-(\alpha+i\omega)} \right\}_0^{\infty} = \frac{1}{(\alpha+i\omega)} \quad \text{------ (1)}$$

Differentiate n times w.r. to ω,

$$\int_0^{\infty} e^{-\alpha t} (-it)^n e^{-i\omega t} dt = \frac{(-1)^n n! \, i^n}{(\alpha+i\omega)^{n+1}}$$

$$\therefore \int_0^{\infty} e^{-\alpha t} t^n e^{-i\omega t} dt = \frac{n!}{(\alpha+i\omega)^{n+1}}$$

$$\Rightarrow \mathcal{F}(e^{-\alpha t} t^n U(t)) = \frac{n!}{(\alpha+i\omega)^{n+1}}$$

Remark $\boxed{\mathcal{F}^{-1}\left\{\frac{1}{\alpha+i\omega}\right\} = e^{-\alpha t} U(t), \ \alpha > 0}$ by (1)

Also note that $\boxed{\mathcal{F}^{-1}\left\{\frac{1}{\alpha-i\omega}\right\} = e^{\alpha t} U(-t), \ \alpha > 0}$ [because

If $\mathcal{F}\{f(t)\} = F(\omega)$ then $\mathcal{F}\{f(-t)\} = F(-\omega)$]

$$\left[\because \mathcal{F}\{f(-t)\} = \int_{-\infty}^{\infty} f(-t) e^{-i\omega t} dt \right.$$ Put $t = -u$

$$= \int_{\infty}^{-\infty} f(u) e^{i\omega u} (-du) = \int_{-\infty}^{\infty} f(u) e^{-i(-\omega)u} du$$

$$\left. = F(-\omega) \right]$$

Que Find the Fourier transform of

(i) $f(t) = e^{-|t|}$ (ii) $f(t) = e^{-a|t|}$, $a > 0$ and deduce the value of $F\{e^{-a|t-b|}\}$.

Sol (i) $F\{f(t)\} = \int_{-\infty}^{\infty} e^{-|t|} e^{-i\omega t} dt$

$$= \left[\int_{-\infty}^{0} e^{t} . e^{-i\omega t} dt + \int_{0}^{\infty} e^{-t} . e^{-i\omega t} dt \right]$$

$$= \left\{ \frac{e^{(1-i\omega)t}}{(1-i\omega)} \right\}_{-\infty}^{0} + \left\{ \frac{e^{-(1+i\omega)t}}{-(1+i\omega)} \right\}_{0}^{\infty}$$

$$= \frac{1}{(1-i\omega)} + \frac{1}{(1+i\omega)} = \frac{2}{(1+\omega^2)} \quad \text{Ans}$$

$$= F(\omega)$$

(ii) $F\{e^{-a|t|}\} = F\{e^{-|at|}\}$ $(\because a > 0)$

$$= \frac{1}{a} F\left(\frac{\omega}{a}\right) \quad \text{by change of scale prop.}$$

$$= \frac{1}{a} \cdot \frac{2}{1+\left(\frac{\omega}{a}\right)^2} = \frac{2a}{(\omega^2+a^2)} \quad \text{Ans}$$

$$= G(\omega)$$

By shifting property on t-axis

$$F\{e^{-a|t-b|}\} = e^{-ib\omega} G(\omega) = \frac{2a e^{-ib\omega}}{(\omega^2+a^2)} \quad \text{Ans}$$

Que Find $F^{-1}\left[\frac{1}{12+7is-s^2}\right]$

Sol $F^{-1}\left[\frac{1}{12+7is-s^2}\right] = F^{-1}\left[\frac{1}{(4+is)(3+is)}\right] = F^{-1}\left[\frac{1}{(3+is)} - \frac{1}{(4+is)}\right]$

But $F^{-1}\left\{\frac{1}{4+is}\right\} = e^{-4t} U(t)$ $\quad \left\{ \because F^{-1}\left\{\frac{1}{\alpha+i\omega}\right\} = e^{-\alpha t} U(t) \right.$

and $F^{-1}\left\{\frac{1}{3+is}\right\} = e^{-3t} U(t)$

$$\therefore\ f^{-1}\left[\frac{1}{12+7is-s^2}\right] = (e^{-3t} - e^{-4t})\,U(t)$$
$$= \begin{cases} e^{-3t}(1-e^{-t}) & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$$

<u>Que</u> Using Convolution theorem, find
$$f^{-1}\left[\frac{1}{12+7is-s^2}\right]$$

<u>Sol</u> We have $f^{-1}\left[\frac{1}{12+7is-s^2}\right] = f^{-1}\left[\frac{1}{(4+is)(3+is)}\right]$

But $f^{-1}\left\{\frac{1}{4+is}\right\} = e^{-4t}\,U(t)$

and $f^{-1}\left\{\frac{1}{3+is}\right\} = e^{-3t}\,U(t)$

$\therefore$ By Convolution theorem
$$f^{-1}\left[\frac{1}{12+7is-s^2}\right] = \int_{-\infty}^{\infty} e^{-4u}\,U(u)\,e^{-3(t-u)}\,U(t-u)\,du$$
$$= e^{-3t}\int_{-\infty}^{\infty} e^{-u}\,U(u)\,U(t-u)\,du$$

But $U(u) = \begin{cases} 0 \text{ if } u<0 \\ 1 \text{ if } u \geq 0 \end{cases}$ and $U(t-u) = \begin{cases} 0 \text{ if } t<u \\ 1 \text{ if } t \geq u \end{cases}$

$\therefore\ U(u)\,U(t-u) = \begin{cases} 1 \text{ if } 0 \leq u \leq t \\ 0 \text{ if } t<u<0 \end{cases}$

$$\therefore\ f^{-1}\left[\frac{1}{12+7is-s^2}\right] = \begin{cases} e^{-3t}\int_0^t e^{-u}\,du & ; t \geq 0 \\ 0 & ; t<0 \end{cases}$$
$$= \begin{cases} e^{-3t}\left(-e^{-u}\right)_0^t & ; t \geq 0 \\ 0 & ; t<0 \end{cases}$$
$$= \begin{cases} e^{-3t}(1-e^{-t}) & ; t \geq 0 \\ 0 & ; t<0 \end{cases}$$

A

As Q.9

Que Find $F^{-1}\left\{\frac{1}{4+\omega^2}\right\}$

Sol $$f^{-1}\left\{\frac{1}{4+\omega^2}\right\} = f^{-1}\left\{\frac{1}{(2+i\omega)(2-i\omega)}\right\}$$

$$= f^{-1}\left\{\frac{1}{4}\left(\frac{1}{2+i\omega}+\frac{1}{2-i\omega}\right)\right\}$$

Now, $f^{-1}\left(\frac{1}{\alpha+i\omega}\right) = e^{-\alpha t}\,U(t) \quad , \alpha > 0$

and $f^{-1}\left(\frac{1}{\alpha-i\omega}\right) = e^{\alpha t}\,U(-t) \, , \quad \alpha > 0$

$$\therefore f^{-1}\left(\frac{1}{4+\omega^2}\right) = \frac{1}{4}\left(e^{-2t}\,U(t) + e^{2t}\,U(-t)\right)$$

$U(t) = \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t \geq 0 \end{cases}$

$$= \begin{cases} \frac{1}{4}e^{-2t} & \text{if } t>0 \\ \frac{1}{4}e^{2t} & \text{if } t<0 \\ \underbrace{\frac{1}{2}\cosh 2t}_{\frac{1}{2}} & \text{if } t=0 \end{cases}$$

$U(-t) = \begin{cases} 0 & \text{if } t>0 \\ 1 & \text{if } t \leq 0 \end{cases}$

$\cosh 0 = \frac{e^0+e^0}{2} = 1$

Que Do Fourier sine and cosine transforms of $\exp(x)$ exist? Explain. [Ist term, Sep 14, 3 or 4 marks]

Sol If Fourier sine transform of e^x exists then

$$F_s(e^x) = \int_0^\infty e^x \sin(\omega x)\,dx$$

$$= \left[\frac{e^x}{(1+\omega^2)}\left[\sin(\omega x) - \omega\cos(\omega x)\right]\right]_0^\infty$$

But $\lim\limits_{x\to\infty} e^x\left(\sin(\omega x) - \omega\cos(\omega x)\right)$ does not exist

$\therefore$ $F_s(e^x)$ does not exist.

Similarly if Fourier cosine transform of e^x exists then

$$F_c(e^x) = \int_0^\infty e^x\cos(\omega x)\,dx = \left[\frac{e^x}{1+\omega^2}\left\{\cos(\omega x) + \omega\sin(\omega x)\right\}\right]_0^\infty$$

But $\lim\limits_{x\to\infty} e^x\left(\cos(\omega x) + \omega\sin(\omega x)\right)$ does not exist

$\therefore$ $F_c(e^x)$ does not exist

Que Find the Fourier sine transform of $e^{-|x|}$. Hence evaluate

$$\int_0^\infty \frac{x \sin mx}{(1+x^2)} dx \, ; \quad (m>0)$$

Sol $$f_s\{e^{-|x|}\} = \int_0^\infty e^{-|x|} \sin \omega x \, dx = \int_0^\infty e^{-x} \sin \omega x \, dx$$

$$= \left[\frac{e^{-x}}{(1+\omega^2)} \{-\sin \omega x - \omega \cos \omega x\} \right]_0^\infty = \frac{\omega}{(1+\omega^2)}$$

Taking the inverse Fourier sine transform on both sides, we get

$$e^{-|x|} = f_s^{-1}\left\{ \frac{\omega}{1+\omega^2} \right\} = \frac{2}{\pi} \int_0^\infty \frac{\omega}{1+\omega^2} \sin \omega x \, d\omega$$

Now, replacing x by $m>0$ on both sides, we get

$$e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{\omega \sin m\omega}{(1+\omega^2)} d\omega = \frac{2}{\pi} \int_0^\infty \frac{x \sin mx}{(1+x^2)} dx$$

$$\Rightarrow \int_0^\infty \frac{x \sin mx}{(1+x^2)} dx = \frac{\pi}{2} e^{-m} \qquad A$$

Que Find the Fourier cosine transform of e^{-x^2}

Sol $$f_c(e^{-x^2}) = \int_0^\infty e^{-x^2} \cos(\omega x) \, dx \qquad \text{———— (1)}$$

$$\therefore \frac{d}{d\omega} f_c(e^{-x^2}) = \int_0^\infty -x e^{-x^2} \sin(\omega x) \, dx$$

$\int -2x e^{-x^2} dx$
$= \int e^t dt \qquad -x^2 = t$
$-2x dx = dt$

$$= \frac{1}{2} \int_0^\infty \sin(\omega x) (-2x e^{-x^2}) \, dx$$

$$= \frac{1}{2} \{ \sin(\omega x) . e^{-x^2} \}_0^\infty - \frac{1}{2} \int_0^\infty \omega \cos(\omega x) e^{-x^2} dx$$

(Integrating by parts)

$$= -\frac{\omega}{2} f_c(e^{-x^2})$$

or $\frac{dy}{d\omega} + \frac{\omega}{2} y = 0$ where $y = f_c(e^{-x^2})$

It is Leibnitz's linear diff. equation

$$I.F = e^{\int \frac{\omega}{2} d\omega} = e^{\omega^2/4}$$

$\therefore$ Sol. is $\quad y \cdot e^{\omega^2/4} = c$

$$\therefore \quad y = F_c(e^{-x^2}) = c e^{-\omega^2/4} \text{ ——— } (2)$$

For $\omega = 0$, $F_c(e^{-x^2}) = \int_0^\infty e^{-x^2}\,dx$ by (1)

$$= \frac{\sqrt{\pi}}{2}$$

$\therefore$ From (2) $\quad \frac{\sqrt{\pi}}{2} = c \quad$ (Put $\omega = 0$)

$$\therefore \boxed{F_c(e^{-x^2}) = \frac{\sqrt{\pi}}{2} e^{-\omega^2/4}}$$

A

Application of Fourier Transform to Differential Equations

Let $u = u(x,t)$ be a function of two variables. Then,

$$\mathcal{F}\{u(x,t)\} = \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx = U(\omega,t) \quad \text{(Fourier transform w.r.to } x\text{)}$$

and the corresponding inverse Fourier transform is

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega,t)\, e^{i\omega x}\, d\omega$$

Also, $\mathcal{F}_c\{u(x,t)\} = \int_{0}^{\infty} u(x,t)\cos\omega x\, dx = U_c(\omega,t)$ (w.r to x)

and the corresponding inverse Fourier cosine transform is

$$u(x,t) = \frac{2}{\pi}\int_{0}^{\infty} U_c(\omega,t)\cos\omega x\, d\omega$$

Similarly $\mathcal{F}_s\{u(x,t)\} = \int_{0}^{\infty} u(x,t)\sin\omega x\, dx = U_s(\omega,t)$ (w.r.to x)

and the corresponding inverse Fourier sine transform is

$$u(x,t) = \frac{2}{\pi}\int_{0}^{\infty} U_s(\omega,t)\sin\omega x\, d\omega$$

Remark (1) Formulae w.r.to x

(a) $\mathcal{F}\left(\frac{\partial^n u}{\partial x^n}\right) = (i\omega)^n \mathcal{F}\{u(x,t)\}$ and $\mathcal{F}\left(\frac{\partial^n u}{\partial t^n}\right) = \frac{d^n}{dt^n}\mathcal{F}\{u(x,t)\}$

(b) $\mathcal{F}_c\left(\frac{\partial u}{\partial x}\right) = \omega \mathcal{F}_s\{u(x,t)\} - u(0,t)$ and $\mathcal{F}_s\left(\frac{\partial u}{\partial x}\right) = -\omega\, \mathcal{F}_c\{u(x,t)\}$

(c) $\mathcal{F}_c\left(\frac{\partial^2 u}{\partial x^2}\right) = -\omega^2 \mathcal{F}_c\{u(x,t)\} - \left(\frac{\partial u}{\partial x}\right)(0,t)$

$\mathcal{F}_s\left(\frac{\partial^2 u}{\partial x^2}\right) = -\omega^2 \mathcal{F}_s\{u(x,t)\} + \omega . u(0,t)$

(d) $\mathcal{F}_c\left(\frac{\partial^n u}{\partial t^n}\right) = \frac{d^n}{dt^n}\mathcal{F}_c\{u(x,t)\}$, $\mathcal{F}_s\left(\frac{\partial^n u}{\partial t^n}\right) = \frac{d^n}{dt^n}\mathcal{F}_s\{u(x,t)\}$

(2) In most of the cases, we choose the transform according as
If $(u)_{x=0}$ i.e., $u(0,t)$ is given as constant then use Fourier sine transform w.r.to x. If $\left(\frac{\partial u}{\partial x}\right)_{x=0}$ i.e., $\left(\frac{\partial u}{\partial x}\right)(0,t)$ is given as constant then use Fourier cosine transform w.r.to x otherwise use Fourier transform w.r.to x.

(3) All the above transforms and above formulae written in remark (1) can also be use w. r. to t if needed

For example if $\left(\frac{\partial u}{\partial t}\right)_{t=0}$ i.e, $\left(\frac{\partial u}{\partial t}\right)(x,0)$ is given as constant then use Fourier cosine transform w. r. to t. In this case all the above formulae will be interchange in x and t.

(3) $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$ (It is called error-function)

and $\operatorname{erf}_c(x) = 1 - \operatorname{erf}(x)$ (It is called complementary error-function)

Also, note that

$$\int_0^\infty \frac{e^{-y^2}}{y} \sin(xy)\, dy = \frac{\pi}{2} \operatorname{erf}\left(\frac{x}{2}\right)$$

Que If the initial temperature of an infinite bar is given by

$$\theta(x) = \begin{cases} \theta_0 , & \text{for } |x| < a \\ 0 , & \text{for } |x| > a \end{cases}$$

determine the temperature at any point x and at any instant t.

Sol To determine the temperature $\theta(x,t)$ we have to solve the heat equation $\frac{\partial \theta}{\partial t} = c^2 \frac{\partial^2 \theta}{\partial x^2}$, $t > 0$ ——— (1)

subject to the initial condition

$$\theta(x,0) = \begin{cases} \theta_0 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases} \quad \text{———} (2)$$

Taking Fourier transform of both sides of (1) w.r.t x,

$$\frac{d}{dt} F\{\theta(x,t)\} = c^2 \left[(i\omega)^2 F\{\theta(x,t)\} \right]$$

$$\left[\because F(f^{(n)}(x)) = (i\omega)^n F(f(x)) \right]$$

$$\Rightarrow \frac{dI}{dt} + c^2 \omega^2 I = 0 \quad \text{where } I = F\{\theta(x,t)\}$$

$\therefore$ Sol. is $I = Ae^{-c^2\omega^2 t}$ $\quad$ ($\because$ A.E. is $m + c^2\omega^2 = 0$ $\Rightarrow m = -c^2\omega^2$)

$$\Rightarrow F\{\theta(x,t)\} = Ae^{-c^2\omega^2 t} \quad \text{———} (3)$$

Now, taking the Fourier transform of (2), we get

$$F\{\theta(x,0)\} = \int_{-a}^{a} \theta_0 e^{-i\omega x} dx = \theta_0 \left(\frac{e^{-i\omega x}}{-i\omega} \right)_{-a}^{a} = \frac{\theta_0}{i\omega} \left(e^{ia\omega} - e^{-ia\omega} \right)$$

$$= \frac{2\theta_0 \sin a\omega}{\omega} \quad \text{———} (4)$$

From (3) and (4), when $t = 0$,

$$A = \frac{2\theta_0 \sin a\omega}{\omega}$$

$$\therefore F\{\theta(x,t)\} = \frac{2\theta_0 \sin a\omega}{\omega} e^{-c^2\omega^2 t}$$

Now, taking the inverse Fourier transform, we get

$$\theta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\theta_0 \sin a\omega}{\omega} e^{-c^2\omega^2 t} \cdot e^{i\omega x} d\omega$$

$$= \frac{\theta_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin a\omega}{\omega} e^{-c^2\omega^2 t}(\cos\omega x + i\sin\omega x)\, d\omega$$

$$= \frac{2\theta_0}{\pi}\int_{0}^{\infty} \frac{\sin a\omega}{\omega} e^{-c^2\omega^2 t}\cos\omega x\, d\omega$$ (by the prop. of definite integral)

$$= \frac{\theta_0}{\pi}\int_{0}^{\infty} e^{-c^2\omega^2 t} \cdot \left\{\frac{\sin(a+x)\omega + \sin(a-x)\omega}{\omega}\right\} d\omega$$

Put $\omega = \frac{1}{c\sqrt{t}}y$, $\theta(x,t) = \frac{\theta_0}{\pi}\int_0^{\infty} e^{-y^2}\left[\frac{1}{y}\sin\frac{(a+x)y}{c\sqrt{t}} + \frac{1}{y}\sin\frac{(a-x)y}{c\sqrt{t}}\right] dy$

$$= \frac{\theta_0}{\pi}\left[\frac{\pi}{2}\operatorname{erf}\frac{(a+x)}{2c\sqrt{t}} + \frac{\pi}{2}\operatorname{erf}\frac{(a-x)}{2c\sqrt{t}}\right] = \frac{\theta_0}{2}\left[\operatorname{erf}\frac{(a+x)}{2c\sqrt{t}} + \operatorname{erf}\frac{(a-x)}{2c\sqrt{t}}\right]$$

Que Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $x>0, t>0$ subject to the conditions (i) $u=0$ when $x=0, t>0$

(ii) $u = \begin{cases} 1, & 0<x<1 \\ 0, & x \geq 1. \end{cases}$ when $t=0$

and (iii) $u(x,t)$ is bounded.

Sol Take Fourier sine transform of both sides w.r.t x,

$$F_s\left(\frac{\partial u}{\partial t}\right) = F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$ ($\because u(x,t)_{x=0}$ is given)

$\Rightarrow \frac{d}{dt}U_s(\omega,t) = -\omega^2 U_s(\omega,t) + \omega u(0,t)$ where $F_s\, u(x,t) = U_s(\omega,t)$

But $u(0,t)=0$ for $t>0$

$\therefore \frac{d}{dt}U_s(\omega,t) = -\omega^2 U_s(\omega,t)$

Solution of this first order ordinary differential equation is

$$U_s(\omega,t) = A e^{-\omega^2 t} \quad \text{——— (1)}$$ ($\because$ A.E. is $m+\omega^2=0 \Rightarrow m=-\omega^2$)

Now $u(x,0) = \begin{cases} 1, & 0<x<1 \\ 0, & x \geq 1 \end{cases}$

$\therefore \; U_s(\omega,0) = \int_0^1 \sin\omega x \, dx = -\frac{1}{\omega}(\cos\omega x)_0^1 = \frac{1-\cos\omega}{\omega}$

$\therefore$ From (1), $\; U_s(\omega,0) = A = \frac{1-\cos\omega}{\omega}$

$\therefore \; U_s(\omega,t) = \frac{1-\cos\omega}{\omega} e^{-\omega^2 t}$

Take inverse Fourier sine transform

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos\omega}{\omega} e^{-\omega^2 t} \sin\omega x \, d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \frac{e^{-\omega^2 t}}{\omega}\left[2\sin\omega x - 2\sin\omega x\cos\omega\right] d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \frac{e^{-\omega^2 t}}{\omega}\left[2\sin\omega x - \sin\omega(x+1) - \sin\omega(x-1)\right] d\omega$$

Put $\omega = y/\sqrt{t}$

$$u(x,t) = \frac{1}{\pi}\int_0^\infty e^{-y^2}\left[\frac{2\sin(xy/\sqrt{t})}{y} - \frac{\sin((x+1)y/\sqrt{t})}{y} - \frac{\sin((x-1)y/\sqrt{t})}{y}\right] dy$$

$$= \frac{1}{\pi}\cdot\frac{\pi}{2}\left[2\,\mathrm{erf}\frac{x}{2\sqrt{t}} - \mathrm{erf}\frac{x+1}{2\sqrt{t}} - \mathrm{erf}\frac{x-1}{2\sqrt{t}}\right]$$

(from remark(3))

$$= \mathrm{erf}\frac{x}{2\sqrt{t}} - \frac{1}{2}\mathrm{erf}\frac{x+1}{2\sqrt{t}} - \frac{1}{2}\mathrm{erf}\frac{x-1}{2\sqrt{t}}$$

Que Use Fourier sine transform to solve the equation

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2}, \quad x>0, t>0 \qquad \text{—— (1)} \quad [\text{Tut Q.8}]$$

subject to the conditions

$V = V_0$ when $x = 0$, $t>0$ and $V = 0$ when $t = 0$, $x > 0$

Sol Taking Fourier sine transform on both sides of (1) w.r.t. x, we get

$$\therefore \quad F_s\left(\frac{\partial V}{\partial t}\right) = k F_s\left(\frac{\partial^2 V}{\partial x^2}\right)$$

$$\Rightarrow \frac{d}{dt} V_s(\omega,t) = k\left[\omega V(0,t) - \omega^2 V_s(\omega,t)\right] \quad \text{where } F_s V(x,t) = V_s(\omega,t)$$

$$\Rightarrow \frac{d}{dt} V_s(\omega,t) + k\omega^2 V_s(\omega,t) = k\omega V_0 \quad (\because V(0,t) = V_0)$$

It is linear in $V_s(\omega,t)$

$$\text{I.F.} = e^{\int k\omega^2 dt} = e^{k\omega^2 t}$$

$\therefore$ Solution is

$$\{V_s(\omega,t)\} \cdot e^{k\omega^2 t} = \int k\omega V_0 e^{k\omega^2 t} dt + A$$

$$\Rightarrow V_s(\omega,t) = \frac{V_0}{\omega} \cancel{e^{k\omega^2 t}} + A e^{-k\omega^2 t} \qquad (1)$$

Now, $V(x,0) = 0$

$$\therefore V_s(\omega,0) = \int_0^\infty 0 . \sin \omega x \, dx = 0$$

$\therefore$ From (1), $V_s(\omega,0) = \frac{V_0}{\omega} + A = 0 \Rightarrow A = -\frac{V_0}{\omega}$

Hence $V_s(\omega,t) = \frac{V_0}{\omega}\left(1 - e^{-k\omega^2 t}\right)$

Take inverse Fourier ~~sine transform~~

$$V(x,t) = \frac{2}{\pi}\int_0^\infty \frac{V_0}{\omega}\left(1 - e^{-k\omega^2 t}\right) \sin \omega x \, d\omega$$

$$= \frac{2V_0}{\pi}\int_0^\infty \frac{\sin \omega x}{\omega} d\omega - \frac{2V_0}{\pi}\int_0^\infty e^{-k\omega^2 t} \frac{\sin \omega x}{\omega} d\omega$$

$$= V_0 - \frac{2V_0}{\pi}\int_0^\infty e^{-k\omega^2 t} \frac{\sin \omega x}{\omega} d\omega$$

$$= V_0 - \frac{2V_0}{\pi}\int_0^\infty e^{-y^2} \frac{1}{y} \sin \frac{xy}{\sqrt{kt}} dy \qquad \left(\omega = \frac{y}{\sqrt{kt}}\right)$$

$$= V_0 - \frac{2V_0}{\pi} \cdot \frac{\pi}{2} \operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right)$$

$$= V_0\left(1 - \operatorname{erf}\frac{x}{2\sqrt{kt}}\right) = V_0 \operatorname{erfc}\frac{x}{2\sqrt{kt}}$$

Que The steady state temperature distribution $u(x,y)$ in a thin homogeneous semi-infinite plate is governed by the boundary value problem $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $0 < x < l$, $0 < y < \infty$ ——— (1)

$$u(0, y) = \underset{y>0}{e^{-5y}}, \quad u(l, y) = 0, \overset{y>0}{,} \quad \left(\frac{\partial u}{\partial y}\right)(x, 0) = 0, \quad 0 < x < l$$

Find the temperature distribution $u(x,y)$.

Sol Since $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is given, therefore we take the Fourier cosine transform on both sides of equ.(1) w.r.to y

$$\therefore F_c\left(\frac{\partial^2 u}{\partial x^2}\right) + F_c\left(\frac{\partial^2 u}{\partial y^2}\right) = 0$$

$$\Rightarrow \frac{d^2}{dx^2} U_c(x, \omega) + \left[-\omega^2 U_c(x,\omega) - \left(\frac{\partial u}{\partial y}\right)_{y=0}\right] = 0$$

where $F_c\{u(x,y)\} = U_c(x,\omega)$

$$\Rightarrow \frac{d^2}{dx^2} U_c(x,\omega) - \omega^2 U_c(x,\omega) = 0 \qquad \left(\because \left(\frac{\partial u}{\partial y}\right)(x,0) = 0\right)$$

$\therefore$ Sol. is $U_c(x, \omega) = c_1 e^{-\omega x} + c_2 e^{\omega x}$ ——— (1)

But $u(l, y) = 0$

$$\therefore U_c(l, \omega) = \int_0^\infty 0 \cdot \cos \omega y \, dy = 0$$

$\therefore$ from (1), $U_c(l, \omega) = c_1 e^{-\omega l} + c_2 e^{\omega l} = 0$ ——— (2)

Also, $u(0, y) = e^{-5y}$, $y > 0$

$$\therefore U_c(0, \omega) = \int_0^\infty e^{-5y} \cos \omega y \, dy$$

$$= \left\{\frac{e^{-5y}}{(25+\omega^2)}(-5\cos\omega y + \omega \sin \omega y)\right\}_0^\infty = \frac{5}{\omega^2+25}$$

$\therefore$ from (1), $U_c(0, \omega) = c_1 + c_2 = \frac{5}{(\omega^2+25)}$ ——— (3)

Solving (2) and (3),

$$c_1 = \frac{-5e^{\omega l}}{(\omega^2+25)(e^{-\omega l}-e^{\omega l})} = \frac{5e^{\omega l}}{2(\omega^2+25)\sinh(\omega l)}$$

$$c_2 = \frac{5e^{-\omega l}}{(\omega^2+25)(e^{-\omega l}-e^{\omega l})} = \frac{5e^{-\omega l}}{2(\omega^2+25)\sinh(\omega l)}$$

Substituting in (1),

$$U_c(x,\omega) = \frac{5[e^{\omega(l-x)}-e^{-\omega(l-x)}]}{2(\omega^2+25)\sinh(\omega l)} = \frac{5\sinh(\omega l-\omega x)}{(\omega^2+25)\sinh(\omega l)}$$

$$= \frac{5[\sinh(\omega l)\cosh(\omega x)-\cosh(\omega l)\sinh(\omega x)]}{(\omega^2+25)\sinh(\omega l)}$$

$$= \frac{5}{(\omega^2+25)}[\cosh(\omega x)-\coth(\omega l)\sinh(\omega x)]$$

Take inverse Fourier cosine transform

$$u(x,y) = \frac{10}{\pi}\int_0^\infty \frac{1}{(\omega^2+25)}[\cosh(\omega x)-\coth(\omega l)\sinh(\omega x)]\cos(\omega y)\,d\omega$$

which gives the required solution.