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Fourier Series

Periodic function A function $f(x)$ is called periodic if it is defined for all real x (except perhaps for some isolated x such as $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ for $\tan x$) and there exists $T>0$ such that $f(x+T) = f(x)$ for all x.

This number T is called a period of $f(x)$.

The smallest positive T, if exists, is called the fundamental period of $f(x)$.

Examples

1. $\sin x$, $\cos x$, $\operatorname{cosec} x$ and $\sec x$ are 2π-periodic.
2. $\tan x$ and $\cot x$ are π-periodic.
3. A constant function is periodic without fundamental period.
4. x, x^2, e^x are few examples which are not periodic.

Some useful formulae

1. For $m, n \in N$,

(a) $\displaystyle\int_c^{c+2l} \cos\frac{n\pi x}{l}\,dx = 0$ (b) $\displaystyle\int_c^{c+2l} \sin\frac{n\pi x}{l}\,dx = 0$

(c) $\displaystyle\int_c^{c+2l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = 0;\ m\neq n$ (d) $\displaystyle\int_c^{c+2l} \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = 0;\ m\neq n$

(e) $\displaystyle\int_c^{c+2l} \sin\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = 0$ for all $m, n \in N$

(f) $\displaystyle\int_c^{c+2l} \cos^2\frac{n\pi x}{l}\,dx = l$ (g) $\displaystyle\int_c^{c+2l} \sin^2\frac{n\pi x}{l}\,dx = l$

2. If a and b are constants, then

(a) $\displaystyle\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{(a^2+b^2)}(a\cos bx + b\sin bx) + c$

(b) $\displaystyle\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{(a^2+b^2)}(a\sin bx - b\cos bx) + c.$

where c is the constant of integration.

(3) $\int uv\,dx = u\,(Iv) - (Du)(I^2v) + (D^2u)(I^3v) - (D^3u)(I^4v) + \cdots$

In the above formula u and v are functions of x. This formula is useful when one of the function is some +ive integral power of x, say $u = x^m$ (m positive integer).

Ex: $\int x^3 \cos 2x\,dx = (x^3)\cdot\left(\frac{\sin 2x}{2}\right) - (3x^2)\cdot\left(-\frac{\cos 2x}{4}\right) + (6x)\left(-\frac{\sin 2x}{8}\right)$

$$- (6)\cdot\left(\frac{\cos 2x}{16}\right) + c$$

$$= \frac{x}{2}\left(x^2 - \frac{3}{2}\right)\sin 2x + \frac{3}{4}\left(x^2 - \frac{1}{2}\right)\cos 2x + c$$

(4) For $n \in N$, we have $\sin n\pi = 0$, $\cos n\pi = (-1)^n$

$\therefore\ e^{in\pi} = (-1)^n$ i.e, $e^{in\pi} = 1$ if n is even $= -1$ if n is odd

Even and odd functions A function $f(x)$ in an interval I is said to be even if $f(-x) = f(x)$ $\forall\, x \in I$ and it is said to be odd if $f(-x) = -f(x)$ $\forall\, x \in I$

By the properties of definite integral,

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \quad \text{if } f(-x) = f(x) \text{ i.e, } f \text{ is even}$$

$$= 0 \quad \text{if } f(-x) = -f(x) \text{ i.e, } f \text{ is odd.}$$

Ex:- (i) $f(x) = x^2$ is even in the interval $[-\pi, \pi]$ but neither even nor odd in the interval $[0, 2\pi]$.

(ii) $f(x) = x$ is odd in the interval $[-\pi, \pi]$ but neither even nor odd in the interval $(0, 2)$.

Fourier Series Let $f(x)$ be a periodic function with period $2l$ is defined in the interval $[c, c+2l]$. Then the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) \qquad (1)$$

where the coefficients a_0, a_n and b_n are given by the Euler's formulae

$$\left.\begin{aligned} a_0 &= \frac{1}{l} \int_c^{c+2l} f(x)\, dx \\ a_n &= \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l}\, dx \\ b_n &= \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l}\, dx, \quad n = 1, 2, 3, \ldots \end{aligned}\right\} \qquad (2)$$

(Also give proof of these formulae)

is called the Fourier series of $f(x)$.

Also, the coefficients a_0, a_n and b_n are called Euler's coefficients or Fourier coefficients.

Remark We can also find a_n and b_n simultaneously by writing $a_n + i b_n = \frac{1}{l} \int_c^{c+2l} f(x)\, e^{\frac{in\pi x}{l}}\, dx$ and equating real and imaginary parts.

Dirichlet's conditions for convergence of Fourier series of $f(x)$ in $[c, c+2l]$: The conditions under which the expansion of given function $f(x)$ in Fourier series is possible are known as Dirichlet's conditions. These are :

(1) $f(x)$ is single-valued and periodic with period $2l$

(2) $f(x)$ is piecewise continuous in $[c, c+2l]$

(3) $f(x)$ has finite number of maxima or minima in $[c, c+2l]$

Remark: If a function $f(x)$ satisfies above mentioned Dirichlet's conditions then the Fourier series given by (1) is convergent and its sum is $f(x)$, except at a point x_0 at which $f(x)$ is not continuous and sum of series at discontinuity $x = x_0$ is $\frac{1}{2}\left[\lim_{x\to x_0^-} f(x) + \lim_{x\to x_0^+} f(x)\right]$ which we write as $\frac{1}{2}[f(x_0-0)+f(x_0+0)]$.

Thus, $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$ at the points where the series may or may not converge but $\sim$ can be replaced by equality sign $=$ at the points where the series converges.

Example The function $f(x) = \frac{1}{(3-x)}$, $0 < x < 2\pi$ does not satisfy Dirichlet's conditions as

$$\lim_{x\to 3^-} f(x) = \infty \quad \text{and} \quad \lim_{x\to 3^+} f(x) = -\infty$$

Both the limits are infinite, so $f(x)$ is not piecewise continuous in $(0, 2\pi)$ as $3 \in (0, 2\pi)$. Thus, Fourier series expansion of $f(x)$ in $(0, 2\pi)$ does not exist.

Fourier series of even and odd functions

Case I If $f(x)$ is an even function in $[-l, l]$, then its Fourier series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$, $\quad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$

Case II If $f(x)$ an odd function in $[-l, l]$, then its Fourier series is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{where } b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Que If $f(x)$ is a periodic function in $[-l, l]$. Prove that at both the end points the Fourier series has the same value
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n (-1)^n$$

Sol Fourier series of $f(x)$ in $[-l, l]$ is
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) \quad \text{---} (1)$$

where $a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx$$

$$\therefore f(-l) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi(-l)}{l} + b_n \sin \frac{n\pi(-l)}{l} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\pi + b_n \sin n\pi \right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n (-1)^n$$

$[\because \cos n\pi = (-1)^n, \sin n\pi]$

Similarly $f(l) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\pi + b_n \sin n\pi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n (-1)^n$

$\therefore$ At both ends Fourier series has the same value $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n (-1)^n$.

Que 1 Find the Fourier series of $f(x) = x$, $0 < x < 2\pi$ and sketch the graph from $x = -4\pi$ to $x = 4\pi$.

Sol Taking $f(x)$ to be periodic function with period 2π, Fourier series expansion of $f(x) = x$ is
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi} \int_0^{2\pi} x\, dx = \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{2\pi} = 2\pi$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} x \cos nx\, dx = \frac{1}{\pi} \left[x \left(\frac{\sin nx}{n} \right) - 1 \left(-\frac{\cos nx}{n^2} \right) \right]_0^{2\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin nx \, dx = \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1 \cdot \left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi} = -\frac{2}{n}$$

Other method to find a_n and b_n

$$a_n + i b_n = \frac{1}{\pi}\int_0^{2\pi} x e^{inx} \, dx = \frac{1}{\pi}\left[x\left(\frac{e^{inx}}{in}\right) - 1 \cdot \left(\frac{e^{inx}}{-n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left(-\frac{2\pi}{n} i\right) = -\frac{2}{n} i$$

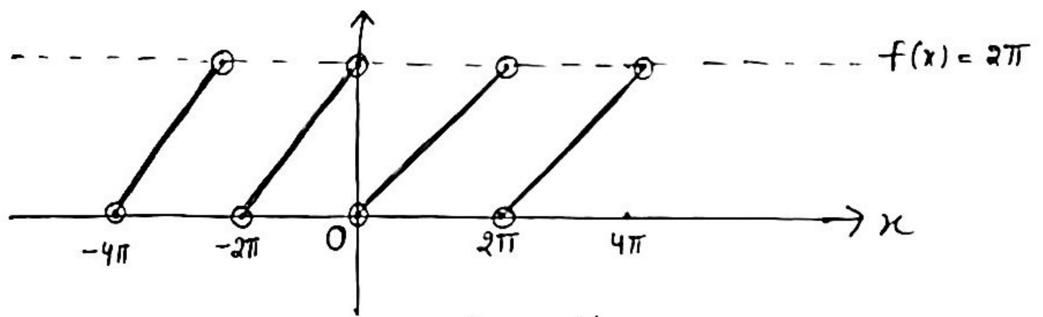
Equate real and imaginary parts

$$a_n = 0, \quad b_n = -\frac{2}{n}$$

$\therefore$ Fourier series expansion of $f(x)$ is

$$f(x) \sim \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

Graph of $f(x) = x, \; -4\pi < x < 4\pi$ is



⊙ denotes that the point is not in the graph.

Note If graph of the function is given and we have to write Fourier series, then by the graph first we write $f(x)$ and then solve it as above

Que 2 Find a Fourier series to represent $x - x^2$ from $-\pi$ to π.

Hence show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$.

Sol Let $f(x) = x - x^2$

Considering $f(x)$ to be periodic with period 2π, Fourier series expansion of $f(x)$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \, dx = -\frac{2}{\pi}\int_0^{\pi} x^2 \, dx = -\frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = -\frac{2\pi^2}{3}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx = -\frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= -\frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= -\frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^{n+1}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

$\therefore$ Fourier series expansion of $f(x)$ is

$$f(x) \sim -\frac{\pi^2}{3} + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}(2\cos nx + n\sin nx)$$

Taking $x=0$, $\quad 0 = -\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

Que 3 Expand $f(x)$ as a Fourier series in the interval $(0, 2\pi)$

if $f(x) = \begin{cases} -\pi, & 0 < x < \pi \\ x-\pi, & \pi < x < 2\pi \end{cases}$

and hence show that $\sum_{r=0}^{\infty}\frac{1}{(2r+1)^2} = \frac{\pi^2}{8}$

Sol Considering $f(x)$ to be periodic function of period 2π, let the Fourier series of $f(x)$ be

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi}(-\pi)\,dx + \int_{\pi}^{2\pi}(x-\pi)\,dx\right]$

$$= \frac{1}{\pi}\left[(-\pi x)_0^{\pi} + \left(\frac{x^2}{2} - \pi x\right)_{\pi}^{2\pi}\right] = \frac{1}{\pi}\left[-\pi^2 + (2\pi^2 - 2\pi^2) - \left(\frac{\pi^2}{2} - \pi^2\right)\right]$$

$$= -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi}(-\pi)\cos nx\,dx + \int_{\pi}^{2\pi}(x-\pi)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}(\sin nx)_0^{\pi} + \left\{(x-\pi)\frac{\sin nx}{n} - 1\left(-\frac{\cos nx}{n^2}\right)\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}\{1-(-1)^n\}\right] = \frac{1-(-1)^n}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi}(-\pi)\sin nx\,dx + \int_{\pi}^{2\pi}(x-\pi)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left(-\frac{\cos nx}{n}\right)_0^{\pi} + \left\{(x-\pi)\left(-\frac{\cos nx}{n}\right) - 1\cdot\left(-\frac{\sin nx}{n^2}\right)\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\{(-1)^n-1\} - \frac{\pi}{n}\right] = \frac{1}{n}[(-1)^n-2]$$

$\therefore$ Fourier expansion of $f(x)$ is

$$f(x) \sim -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{\{1-(-1)^n\}}{\pi n^2}\cos nx + \frac{\{(-1)^n-2\}}{n}\sin nx\right] \quad \text{——— (1)}$$

Now, $f(0) = \dfrac{f(0+0)+f(0-0)}{2} = \dfrac{f(0+0)+f(2\pi-0)}{2} = \dfrac{-\pi+\pi}{2} = 0$

Put $x=0$ in (1),

$$0 = -\frac{\pi}{4} + \left[\frac{2}{\pi}\left\{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right\}\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

Que 4 Find the Fourier series representation for

$$f(x) = |\cos x|,\ -\pi < x < \pi$$

Also draw the graph.

Sol $f(x) = |\cos x|$ is an even function in $(-\pi,\pi)$.

$\therefore$ Considering $f(x)$ to be periodic function with period 2π, let its Fourier series be

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx$$

where $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi}|\cos x|\,dx = \dfrac{2}{\pi}\left[\int_0^{\pi/2}\cos x\,dx - \int_{\pi/2}^{\pi}\cos x\,dx\right]$

$$= \frac{2}{\pi}\left[(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}\right] = \frac{2}{\pi}[(1-0)-(0-1)] = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx \, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} 2\cos nx \cos x \, dx - \int_{\pi/2}^{\pi} 2\cos nx \cos x \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} \{\cos(n+1)x + \cos(n-1)x\} dx - \int_{\pi/2}^{\pi} \{\cos(n+1)x + \cos(n-1)x\} dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)x}{(n+1)} + \frac{\sin(n-1)x}{(n-1)}\right\}_0^{\pi/2} - \left\{\frac{\sin(n+1)x}{(n+1)} + \frac{\sin(n-1)x}{(n-1)}\right\}_{\pi/2}^{\pi}\right] ; n \neq 1$$

$$= \frac{2}{\pi}\left[\frac{1}{(n+1)}\sin(n+1)\frac{\pi}{2} + \frac{1}{(n-1)}\sin(n-1)\frac{\pi}{2}\right] ; n \neq 1$$

$$= \frac{2}{\pi}\left[\frac{1}{(n+1)} - \frac{1}{(n-1)}\right]\cos\frac{n\pi}{2} \quad ; n \neq 1$$

$$= -\frac{4}{\pi(n^2-1)}\cos\frac{n\pi}{2} \quad ; n \neq 1$$

$\therefore \; a_{2n+1} = 0 \; ; \; n = 1, 2, 3, \ldots$

and $a_{2n} = \dfrac{-4}{\pi(4n^2-1)}\cos n\pi = \dfrac{4}{\pi}\dfrac{(-1)^{n+1}}{(4n^2-1)} \; ; \; n = 1, 2, 3, \ldots$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos x \, dx = \frac{1}{\pi}\left[\int_0^{\pi/2} 2\cos^2 x \, dx - \int_{\pi/2}^{\pi} 2\cos^2 x \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} (1+\cos 2x) dx - \int_{\pi/2}^{\pi} (1+\cos 2x) dx\right]$$

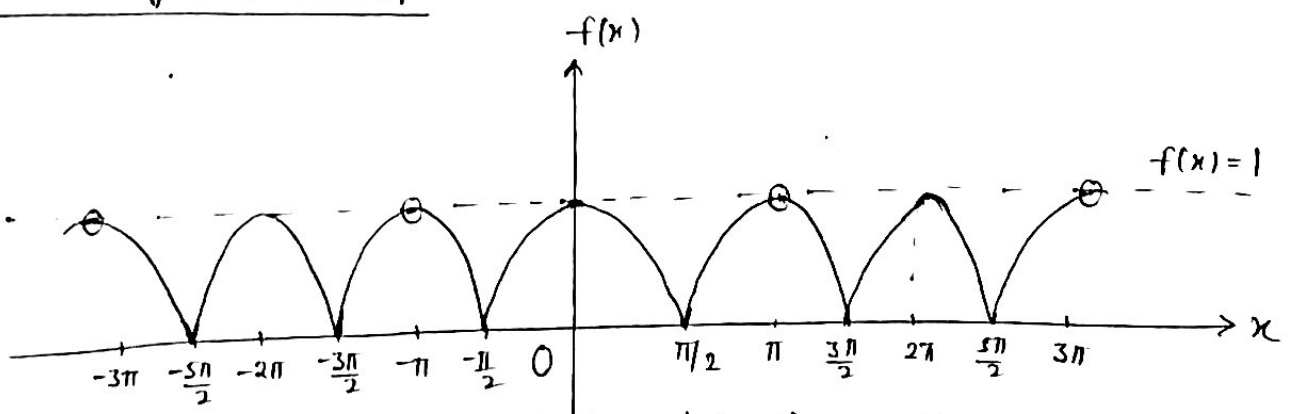
$$= \frac{1}{\pi}\left[\left(x + \frac{1}{2}\sin 2x\right)_0^{\pi/2} - \left(x + \frac{1}{2}\sin 2x\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \left(\pi - \frac{\pi}{2}\right)\right] = 0$$

$\therefore$ Fourier series representation of $f(x)$ is

$$f(x) \sim \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cos 2nx}{(4n^2-1)}$$

Graph of $f(x) = |\cos x|$



o indicates that the point is not in the graph.

Que Show that for $-\pi < x < \pi$,

$$\sin ax = \frac{2\sin a\pi}{\pi}\left(\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - \cdots\right); \text{ a is fraction.}$$

What will happen when a is integer.

or

Expand $f(x) = \sin ax$ as a Fourier series in $(-\pi, \pi)$ when a is fraction. What will happen to Fourier series if a is integer.

Sol When a is fraction

$f(x) = \sin ax$ is odd function of x.

Considering $f(x)$ to be periodic function of period 2π, Fourier series expansion of $f(x)$ is $f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$

where
$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin ax \sin nx \, dx$$
$$= \frac{1}{\pi}\int_0^{\pi} [\cos(n-a)x - \cos(n+a)x] \, dx$$
$$= \frac{1}{\pi}\left[\frac{\sin(n-a)x}{(n-a)} - \frac{\sin(n+a)x}{(n+a)}\right]_0^{\pi}$$
$$= \frac{1}{\pi}\left[\frac{(-1)^{n+1}\sin a\pi}{(n-a)} - \frac{(-1)^n \sin a\pi}{(n+a)}\right]$$
$$= \frac{(-1)^{n+1}}{\pi}\sin a\pi\left[\frac{1}{(n-a)} + \frac{1}{(n+a)}\right] = \frac{2n(-1)^{n+1}\sin a\pi}{\pi(n^2-a^2)}$$

$\therefore$ Fourier series expansion of $f(x)$ is

$$f(x) \sim \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1} \cdot n}{(n^2-a^2)}\sin nx$$

$\therefore$ for $-\pi < x, \pi$, $f(x) = \frac{2\sin a\pi}{\pi}\left(\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - \cdots\right)$

When a is an integer

Fourier series expansion of $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \begin{cases} 0 & \text{if } a = 0 \\ \sin ax & \text{if a is a positive integer} \\ -\sin(-ax) & \text{if a is a negative integer} \end{cases}$$

Que $f(x) = \begin{cases} -k \,; & -\pi < x < 0 \\ k \,; & 0 < x < \pi \end{cases}$ and $f(x+2\pi) = f(x)$ for all x.

Obtain the Fourier series for $f(x)$. Deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Sol $f(-x) = \begin{cases} -k \,; & -\pi < -x < 0 \text{ i.e., } 0 < x < \pi \\ k \,; & 0 < -x < \pi \text{ i.e., } -\pi < x < 0 \end{cases} = -f(x)$

$\therefore$ $f(x)$ is odd function.

Fourier series for $f(x)$ is $f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$

where $b_n = \frac{2}{\pi}\int_0^{\pi} k \sin nx \, dx = \frac{2k}{\pi}\left(-\frac{1}{n}\cos nx\right)_0^{\pi} = \frac{2k}{n\pi}\{1-(-1)^n\}$

$\therefore$ $b_{2n} = 0$, $b_{2n-1} = \frac{4k}{\pi(2n-1)}$; $n = 1, 2, 3, \ldots$

$\therefore$ Fourier series for $f(x)$ is

$$f(x) \sim \frac{4k}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)}$$

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)} \Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Que Find the Fourier series for the function

$$f(x) = 2x - x^2, \quad 0 < x < 3$$

and deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Sol length of period $= 2l = 3$ $\therefore$ $l = 3/2$

Fourier series for $f(x)$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi x}{3} + b_n \sin\frac{2n\pi x}{3}\right)$$

where $a_0 = \frac{2}{3}\int_0^3 (2x - x^2)\,dx = \frac{2}{3}\left(x^2 - \frac{x^3}{3}\right)_0^3 = 0$

$$a_n + i b_n = \frac{2}{3}\int_0^3 (2x - x^2)\, e^{\frac{i2n\pi x}{3}}\, dx$$

$$= \frac{2}{3}\left[(2x - x^2)\left(\frac{-3i}{2n\pi} e^{\frac{i2n\pi x}{3}}\right) - (2-2x)\left(\frac{-9}{4n^2\pi^2} e^{\frac{i2n\pi x}{3}}\right) + (-2)\left(\frac{27i}{8n^3\pi^3} e^{\frac{i2n\pi x}{3}}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{9i}{2n\pi} - \frac{9}{n^2\pi^2} - \frac{9}{2n^2\pi^2}\right] = \frac{3i}{n\pi} - \frac{9}{n^2\pi^2}$$

Equate real and imaginary parts

$$a_n = \frac{-9}{n^2\pi^2} \; , \; b_n = \frac{3}{n\pi} \; ; \; n = 1, 2, 3, ---$$

$\therefore$ Fourier series for $f(x)$ is

$$f(x) \sim \frac{3}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n} \sin \frac{2n\pi x}{3} - \frac{3}{\pi n^2} \cos \frac{2n\pi x}{3} \right] ;$$

For $x = 0$,

$$\frac{f(0-0)+f(0+0)}{2} = \frac{-9}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{f(3-0)+f(0+0)}{2} = -\frac{9}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{\pi^2}{9}\left(\frac{-3+0}{2}\right) = \frac{\pi^2}{6}$$

Que Find the Fourier series to represent

$$f(x) = \begin{cases} 0 & ; -2 \leq x \leq -1 \\ 1+x & ; -1 \leq x \leq 0 \\ 1-x & ; 0 \leq x \leq 1 \\ 0 & ; 1 \leq x \leq 2 \end{cases}$$

and sketch the graph of the function. Also deduce that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + --- = \frac{\pi^2}{8}$$

Sol $\because f(-x) = f(x)$

$\therefore f(x)$ is an even function with period 4.

$\therefore l = 2$

$\therefore$ Fourier series of $f(x)$ is

$$f(x) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$$

where $a_0 = \int_0^2 f(x)\,dx = \int_0^1 (1-x)\,dx = \left(x - \frac{x^2}{2}\right)_0^1 = \frac{1}{2}$

$$a_n = \int_0^2 f(x) \cos \frac{n\pi x}{2}\,dx = \int_0^1 (1-x) \cos \frac{n\pi x}{2}\,dx$$

$$= \left[(1-x)\left(\frac{2}{n\pi} \sin \frac{n\pi x}{2}\right) + \left(\frac{-4}{n^2\pi^2} \cos \frac{n\pi x}{2}\right) \right]_0^1 = \frac{4}{n^2\pi^2}\left(1 - \cos \frac{n\pi}{2}\right)$$

$$\therefore a_{2n} = \frac{1}{n^2\pi^2}\left[1 - (-1)^n\right] \; ; \; a_{2n-1} = \frac{4}{(2n-1)^2\pi^2} \; ; \; n = 1, 2, 3, \ldots$$

$$\therefore a_{4n} = 0 \; , \; a_{2(2n-1)} = \frac{2}{(2n-1)^2\pi^2} \; ; \; a_{2n-1} = \frac{4}{(2n-1)^2\pi^2} \; ; n = 1, 2, 3.$$

$\therefore$ Fourier series for $f(x)$ is

$$f(x) = \frac{1}{4} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \left[2\cos\frac{(2n-1)\pi x}{2} + \cos(2n-1)\pi x \right]$$

For $x = 0$,

$$1 = \frac{1}{4} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{3}{(2n-1)^2} \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Graph of function $f(x)$