EE2 Mathematics : Complex Variables

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1 Analytic Functions and the Cauchy-Riemann equations

1.1 Derivation of the Cauchy-Riemann equations

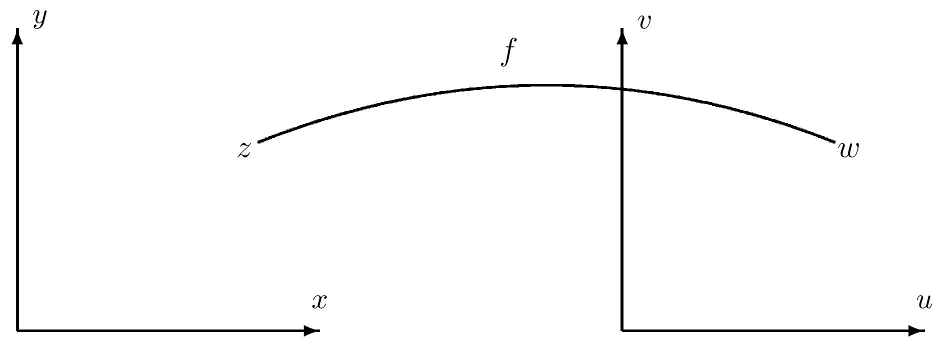
We consider a function of the complex variable $z = x + iy$

$$w = f(z)\,, \tag{1.1}$$

expressed in the usual manner except that the independent variable $z = x + iy$ is complex. So w will also be complex, and $w = f(z)$ has a real part $u(x, y)$ and an imaginary part $v(x, y)$:

$$f(z) = u(x, y) + iv(x, y)\,. \tag{1.2}$$

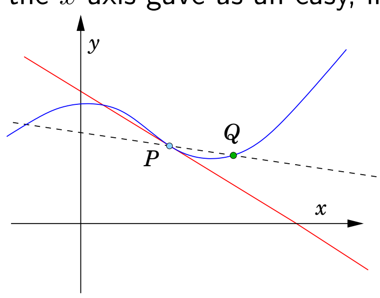
We think of this as a mapping from one plane to another:



Example 1.1.

$$\begin{aligned} w &= f(z) = z^2 \\ &= (x+iy)^2 \\ &= x^2 - y^2 + i2xy \\ &\ \Rightarrow u(x,y) = x^2 - y^2\,, \quad v(x,y) = 2xy \end{aligned} \tag{1.3}$$

Extra difficulties appear in differentiating and integrating such functions because, for one thing, z varies in a plane and not on a line. What do we mean by a derivative? For functions of a single real variable $x \in \mathbb{R}$, the idea of an incremental change δx along the x-axis gave us an easy, intuitive grasp of differentiation:

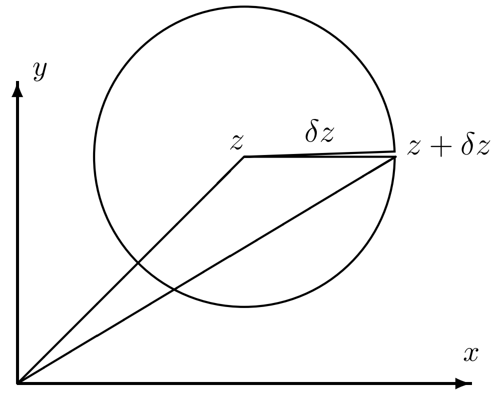


In the complex plane, the increment δx has to be replaced by an incremental change δz. Because δz is a vector the question of the direction of this limit becomes an issue.

Firstly we look at the concept of differentiation. The definition of a derivative at a point z_0 remains the same as usual ; namely

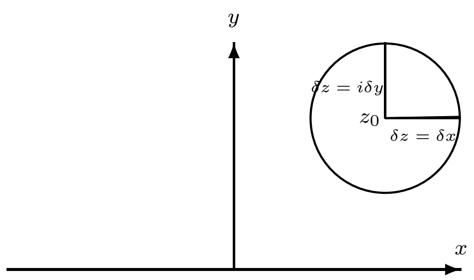
$$\left.\frac{df(z)}{dz}\right|_{z=z_0} = \lim_{\delta z \to 0}\left(\frac{f(z_0+\delta z) - f(z_0)}{\delta z}\right)\,. \tag{1.4}$$

In $\mathbb{R}$, the idea of the limit as $\delta x \to 0$ was simple: all we required is that we have the same limit approaching 0 from both sides. The subtlety here lies in the limit $\delta z \to 0$ because δz is itself a vector and therefore the limit $\delta z \to 0$ may be taken in many directions. If the limit in (1.4) is to be unique (to make any sense) *it is required that it be independent of the direction in which the limit $\delta z \to 0$ is taken.* If this is the case then it is said that **$f(z)$ is differentiable at the point z_0.**



There is a general test on functions to determine whether (1.4) is independent of the direction of the limit. The simplest way is to firstly take the limit in the horizontal direction : that is $\delta z = \delta x$, in which case

$$\frac{df(z)}{dz} = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) + iv(x+\delta x, y) - u(x,y) - iv(x,y)}{\delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \equiv u_x + iv_x\,. \tag{1.5}$$



The z-plane with a point at z_0 and a circle of radius $|\delta z|$ around it. The horizontal radius is drawn for the case when $\delta z = \delta x$ and the vertical for the case when $\delta z = i\delta y$.

Next we take the limit in the vertical direction : that is $\delta z = i\delta y$

$$\frac{df(z)}{dz} = \lim_{i\delta y \to 0} \frac{u(x, y+i\delta y) + iv(x, y+i\delta y) - u(x,y) - iv(x,y)}{i\delta y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \equiv -iu_y + v_y\,. \tag{1.6}$$

If the limits in both directions are to be equal, then df/dz in (1.5) and (1.6) must be equal:

$$-iu_y + v_y = u_x + iv_x$$

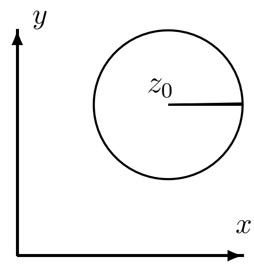
and equating real and imaginary parts we have:

$$\boxed{u_x = v_y\,, \qquad u_y = -v_x\,.} \tag{1.7}$$

The boxed pair of equations above are known as **the Cauchy-Riemann equations**. If these hold at a point z then $f(z)$ is said to be differentiable at z: they are a **necessary** condition for the limits to be the same and the function to be differentiable. As it turns out, they are also a **sufficient** condition, but deriving this requires more work. The interested reader may look in Churchill/Ward Brown, or other texts.

There is no similar requirement for the existence of a limit in single variable calculus. Thus the CR equations bring us to a further idea regarding differentiation in the complex plane:

Definition: *If $f(z)$ is differentiable at all points in a neighbourhood of a point z_0 then $f(z)$ is said to be* **analytic** *(regular) at z_0.*



A neighbourhood of z_0.

A function which is analytic for all $z \in \mathbb{C}$ is said to be **holomorphic** *(entire).*

Example 1.2. *Show that $f(z) = z^2$ is holomorphic.*

We already have

$$u(x,y) = x^2 - y^2\,, \qquad v(x,y) = 2xy\,. \tag{1.8}$$

Clearly, four trivial partial derivatives show that

$$u_x = 2x\,, \quad u_y = -2y\,, \quad v_x = 2y\,, \quad v_y = 2x\,,$$

thus demonstrating that the CR equations hold for all values of x and y. It follows that $f(z) = z^2$ is differentiable at all points in the z-plane and every point in this plane has an (infinite) neighbourhood in which $f(z) = z^2$ is differentiable. Clearly $f(z) = z^2$ is analytic everywhere: it is holomorphic.

Some functions are analytic everywhere in the complex plane except at certain points: these points are called *singularities*. The following examples illustrate this.

Example 1.3. *Consider $f(z) = \frac{1}{z}$. Show the function is analytic everywhere except the origin.*

Writing

$$\frac{1}{z} = \frac{1}{x+iy}\left(\frac{x-iy}{x-iy}\right) = \frac{x-iy}{x^2+y^2}$$

we have

$$u(x,y) = \frac{x}{x^2+y^2} \qquad v(x,y) = -\frac{y}{x^2+y^2}\,. \tag{1.9}$$

Exercise: show that the CR equations hold everywhere except at the origin $z = 0$ where the limit is indeterminate: $z = 0$ is the point where it fails to be differentiable. Hence $w = z^{-1}$ is analytic everywhere except at $z = 0$.

Differentiate:

$$u_x = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}\,, \quad \text{and} \quad v_y = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = u_x\,,$$

and similarly

$$u_y = \frac{2xy}{(x^2+y^2)^2} = -v_x\,.$$

The equality holds for all x, y, except when $x = y = 0$, the indeterminate "0/0" form, so the function is analytic everywhere except the origin.

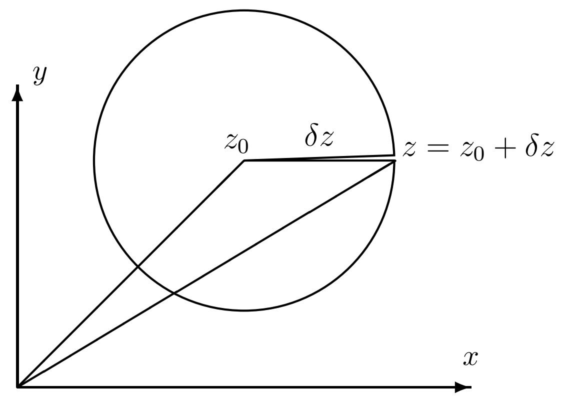
Example 1.4. *Show that $f(z) = |z|^2$ is analytic nowhere in the complex plane.*

We have
$$u(x,y) = x^2 + y^2\,, \qquad\qquad v(x,y) = 0\,, \tag{1.10}$$
and so
$$u_x = 2x\,, \qquad u_y = 2y\,, \qquad v_x = v_y = 0\,. \tag{1.11}$$
Clearly the CR equations do **not** hold anywhere except at $z = 0$. Therefore $f(z) = |z|^2$ is not differentiable anywhere except at $z = 0$ and there is no neighbourhood around $z = 0$ in which it is differentiable. Thus the function is analytic nowhere in the z-plane.

As a final remark on this example let us look again at the limit $\delta z \to 0$, by considering $f(z) = |z|^2 = z\overline{z}$:

$$\begin{aligned}
\left.\frac{df}{dz}\right|_{z=z_0} &= \lim_{\delta z\to 0}\left(\frac{|z+\delta z|^2 - |z|^2}{\delta z}\right)_{z=z_0} \\
&= \lim_{\delta z\to 0}\left(\frac{(z+\delta z)(\overline{z+\delta z}) - z\overline{z}}{\delta z}\right)_{z=z_0} \\
&= \lim_{\delta z\to 0}\left(\frac{\overline{z}\delta z + z\overline{\delta z} + \delta z\overline{\delta z}}{\delta z}\right)_{z=z_0} \\
&= \overline{z_0} + z_0 \lim_{\delta z\to 0}\left(\frac{\overline{\delta z}}{\delta z}\right)_{z=z_0} + 0\,.
\end{aligned} \tag{1.12}$$

Now consider the last limit, with $z = z_0 + \delta z$ describing a circle of radius δz around z_0.



If we let $r = |\delta z|$ we can describe the circle in complex exponential form as $z = z_0 + re^{i\theta}$, with $0 \le \theta < 2\pi$ giving the whole circle.

Then

$$\delta z = re^{i\theta} \Rightarrow \overline{\delta z} = re^{-i\theta}\,,$$

so that
$$\frac{\overline{\delta z}}{\delta z} = e^{-2i\theta}$$
and (1.12) can be rewritten as
$$\left.\frac{df}{dz}\right|_{z=z_0} = \overline{z_0} + z_0 \lim_{\delta z\to 0}\left(e^{-2i\theta}\right)_{z=z_0}\,.$$

As $\delta z \to 0$, we have $r \to 0$, but there is no r in the expression for the limit, leaving

$$\left.\frac{df}{dz}\right|_{z=z_0} = \overline{z_0} + z_0 e^{-2i\theta}\,. \tag{1.13}$$

This result illustrates the problem: as θ, and with it the direction of the limit, varies, so does the limit. **This is clearly not unique except when $z_0 = 0$.**

1.2 Properties of analytic functions

Let us consider the CR equations $u_x = v_y$ and $u_y = -v_x$ as a condition for the analyticity of a function $w = u(x,y) + iv(x,y)$. If we differentiate the first equation wrt x and the second equation wrt y we have

$$u_{xx} = v_{xy}\,, \quad \text{and} \quad u_{yy} = -v_{yx}$$

Equating $v_{xy} = v_{yx}$ we get

$$u_{xx} + u_{yy} = 0 \tag{1.14}$$

thus showing that u must always be a solution of Laplace's equation. Similarly, if we differentiate the first equation wrt y and the second equation wrt x, we get $v_{xx} + v_{yy} = 0$.

Laplace's equation is used to describe the behaviour of many physical entities: steady heat diffusion, incompressible fluid flow, elasticity, and, more to the point, electrostatics and magnetostatics. Solutions to Laplace's equation are called **harmonic functions**. It is also said that $u(x,y)$ and $v(x,y)$ are **conjugate** to one another. In the following set of examples it will be shown how, given a harmonic function $u(x,y)$, its conjugate $v(x,y)$ can be constructed. The pair can then be put together as $u + iv = f(z)$ to ultimately find $f(z)$. Given a harmonic function $u(x,y$, we proceed as follows:

- Check that it is harmonic: $\nabla^2 u = 0$;
- If yes, integrate the Cauchy-Riemann equations to find the harmonic conjugate $v(x,y)$;
- Join together to form $f(z) = u + iv$.

[Note: it is equally simple to begin with a harmonic functtion $v(x,y)$ and obtain its conjugate $u(x,y)$.]

Example 1.5. *Given that $u = x^2 - y^2$ show (i) that it is harmonic; (ii) find $v(x,y)$ and then (iii) construct the corresponding complex function $f(z)$.*

With $u = x^2 - y^2$ we have $u_x = 2x$, $u_{xx} = 2$, $u_y = -2y$ and $u_{yy} = -2$. Therefore $u_{xx} + u_{yy} = 0$ so it satisfies Laplace's equation. This is a sufficient condition for v to exist and for us to write $v_y = u_x = 2x$ and $v_x = -u_y = 2y$. While there are two PDEs here there can only be one solution compatible with both, recall the solution of exact first-order ODEs. Integrating them both in turn gives

$$v = 2xy + A(x)\,, \qquad v = 2xy + B(y)\,. \tag{1.15}$$

It is clear that they are compatible if $A(x) = B(y) = \text{const} = c$ making the result

$$v = 2xy + c\,, \tag{1.16}$$

with

$$f(z) = x^2 - y^2 + 2ixy + ic = z^2 + ic\,. \tag{1.17}$$

The constant ic simply moves $f(z)$ an arbitrary distance along the imaginary axis.

Example 1.6. *Given that $u = x^3 - 3xy^2$ find its conjugate function $v(x,y)$ and the corresponding complex function $f(z)$.*

We first check that $u = x^3 - 3xy^2$ satisfies Laplace's equation: $u_x = 3x^2 - 3y^2$; $u_{xx} = 6x$; $u_y = -6xy$ and $u_{yy} = -6x$. Thus $u_{xx} + u_{yy} = 0$ and so v exists and is found from the CR equations:

$$v_y = 3x^2 - 3y^2 \qquad\qquad v_x = 6xy\,. \tag{1.18}$$

Partially integrating these gives

$$v = 3x^2y - y^3 + A(x) \qquad\qquad v = 3x^2y + B(y)\,. \tag{1.19}$$

The way to make these compatible is to choose $B(y) = -y^3 + c$ and $A(x) = c$ finally giving

$$v = 3x^2y - y^3 + c \tag{1.20}$$

with

$$\begin{aligned} f(z) &= x^3 - 3xy^2 + i(3x^2y - y^3 + c) \\ &= z^3 + ic\,. \end{aligned} \tag{1.21}$$

The last step is *not* inspired guesswork! Coefficients $1, 3, 3, 1$ and powers x^3, x^2y, xy^2, y^3 suggest looking for z^3 or something similar.

Exercise: For $v(x,y) = x^3 - 3xy^2$, find the conjugate function u, and hence $f(z)$.

Example 1.7. *Given that $u = e^x\big(x\cos y - y\sin y\big)$ show that it satisfies Laplace's equation. Also find its conjugate v and then $f(z)$.*

We find that

$$u_{xx} = e^x\big[(x+2)\cos y - y\sin y\big]\,; \qquad u_{yy} = -e^x\big[(x+2)\cos y - y\sin y\big]\,, \tag{1.22}$$

and so Laplace's equation is satisfied. Then

$$v_y = u_x = e^x\big[(x+1)\cos y - y\sin y\big]\,; \qquad v_x = -u_y = e^x\big[(x+1)\sin y + y\sin y\big]\,. \tag{1.23}$$

Using the indefinite integrals $\int y\sin y\,dy = \sin y - y\cos y$ and $\int x\,e^x dx = e^x(x-1)$ we find

$$v = e^x\big(x\sin y + y\cos y\big) + A(x)\,; \qquad v = e^x\big(y\cos y + x\sin y\big) + B(y)\,. \tag{1.24}$$

For compatibility we take $A(x) = B(y) = \text{const} = c$. Then

$$\begin{aligned} w &= e^x\big[(x+iy)\cos y - (y-ix)\sin y\big] + ic \\ &= e^x\big[z\cos y + iz\sin y\big] + ic \\ &= ze^{x+iy} + ic \\ &= ze^z + ic\,. \end{aligned} \tag{1.25}$$

Fact (not shown, see literature): all polynomial functions $P(z)$, trigonometric functions $\cos z$ and $\sin z$ and the exponential function e^z are holomorphic, i.e. analytic everywhere. The tangent function is also analytic wherever $\cos z \neq 0 \Rightarrow z \neq (2n+1)\pi/2$. In fact, all rational functions $\dfrac{P(z)}{Q(z)}$ where P and Q are analytic, are also analytic wherever $Q(z) \neq 0$. Similarly the compositin of two analytic functions is analytic, for example $\sin(z^2 + iz)$.

Differentiation thus follows familiar patterns for analytic functions: $f(z) = \sin z \Rightarrow f'(z) = \cos z$, and so on. Showing these facts is beyond the time available for this module.

1.3 Orthogonality

Let us finally consider the family of curves on which $u =$ const. Recall the total differential from year one:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{1.26}$$

Consider the curves where u is constant. Here we have $du = 0$, giving the gradient on this family as

$$\left.\frac{dy}{dx}\right|_{u=const} = -\frac{u_x}{u_y}\,. \tag{1.27}$$

Likewise, on the family of curves of constant v

$$\left.\frac{dy}{dx}\right|_{v=const} = -\frac{v_x}{v_y} \tag{1.28}$$

giving

$$\left.\frac{dy}{dx}\right|_{u=const} \times \left.\frac{dy}{dx}\right|_{v=const} = \frac{v_x u_x}{v_y u_y}\,. \tag{1.29}$$

Now if $f(z)$ is analytic in a region R then the C-R equations hold there, $u_x = v_y$ and $u_y = -v_x$, and (1.29) becomes

$$\left.\frac{dy}{dx}\right|_{u=const} \times \left.\frac{dy}{dx}\right|_{v=const} = -1\,. \tag{1.30}$$

The final result is that in regions of analyticity curves of constant u and curves of constant v are always orthogonal.

Example 1.8. *We know $f(z) = z^2$ is analytic everywhere. Sketch the curves of constant u and the curves of constant v and confirm visually that they are orthogonal.*

We have $f(z) = x^2 - y^2 + 2xyi$ so that curves where u is constant are

$$u = x^2 - y^2 = C\,,$$

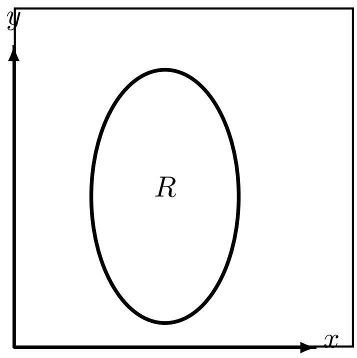
all hyperbolae. Similarly, curves of constant v are

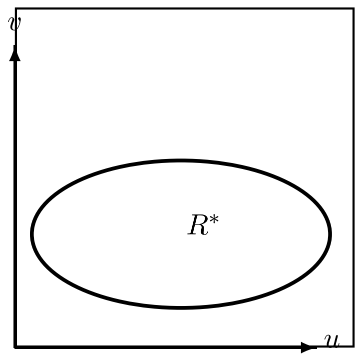
$$v = 2xy = C \Rightarrow y = \frac{c}{x}\,,$$

also hyperbolae.

2 Mappings

2.1 Conformal mappings





A complex mapping $w = f(z)$ maps a region R in the z-plane to a different region R^* in the w-plane.

A complex function $w = f(z)$ can be thought of as a mapping from the z-plane to the w-plane. Depending on $f(z)$ the mapping may not be unique. For instance, for $w = z^2$ for the values $\pm z_0$ there is one value w_0, so the function is not one-to-one. Complex mappings do not necessarily behave in an expected way. The concept of analyticity intrudes into these ideas in the following way.

Definition A mapping is said to be *conformal* if it preserves angles in magnitude and sense. Moreover, a mapping has a *fixed point* when $w = f(z) = z$.

Theorem 1. *The mapping defined by an analytic function $w = f(z)$ is conformal except at points where $f'(z) = 0$.*

Example 2.1. *$w = z^2$ is conformal everywhere except at $z = 0$ because $f'(0) = 0$. Plotting contours of $u = x^2 - y^2$ and $v = 2xy$ shows that conformality fails at the origin.*

Contours of $u, v = const$ in the z-plane: note their orthogonality except at $z = 0$ where conformality fails.

Example 2.2. *Consider $w = \frac{1}{z-1}$ which is analytic everywhere except at $z = 1$.*

We have

$$w = \frac{1}{z-1} = \frac{1}{(x-1)+iy} = \frac{(x-1)-iy}{(x-1)^2+y^2} \tag{2.1}$$

in which case

$$u(x,y) = \frac{x-1}{(x-1)^2+y^2}\,, \qquad v(x,y) = -\frac{y}{(x-1)^2+y^2}\,. \tag{2.2}$$

It is clear from (2.2) that it is always true that

$$u^2+v^2 = \frac{1}{(x-1)^2+y^2}\,. \tag{2.3}$$

So far we have specified no shape in the z-plane on which this map operates. Some examples of what this map will do are these:

1. Consider the family of circles in the z-plane: $(x-1)^2+y^2 = a^2$. These circles are centred at $(1,\,0)$ of radius a. Clearly they map to

   $$u^2+v^2 = \frac{1}{a^2}\,, \tag{2.4}$$

   which is a family of circles in the w-plane centred at $(0,\,0)$ of radius a^{-1}. As the value of a is increased the circles in the z-plane widen and those in the w-plane decrease. It is not difficult to show that the interior (exterior) of the circles in the z-plane map to the exterior (interior) of those in the w-plane. Thus we have

   $$\begin{array}{cc} \underline{z\text{-plane}} & \underline{w\text{-plane}} \\ \text{interior} \to & \text{exterior} \\ \text{exterior} \to & \text{interior} \end{array} \tag{2.5}$$

   The circle centre $(1,\,0)$ in the z-plane maps to *the point at infinity* in the w-plane.

2. The line $x = 0$ in the z-plane maps to what? From (2.2) and (2.3) we know that

   $$u(x,y) = -\frac{1}{1+y^2}\,, \qquad v(x,y) = -\frac{y}{1+y^2}\,, \qquad u^2+v^2 = \frac{1}{1+y^2}\,. \tag{2.6}$$

   $$u^2+v^2 = \frac{1}{1+y^2} = -u\,, \qquad \Rightarrow \qquad \left(u+\tfrac{1}{2}\right)^2 + v^2 = \tfrac{1}{4}\,. \tag{2.7}$$

   In the w-plane this is a circle of radius $\frac{1}{2}$ centred at $(-\frac{1}{2},\,0)$.

Thus we conclude that some circles can map to other circles but also straight lines can also map to circles. This is investigated in the next subsection.

Exercise:

1) Show that the $x-$axis in the $z-$plane is mapped to a line in the $w-$plane.
2) Show that the line $y = x$ in the $z-$plane is mapped to a circle in the $w-$plane.

2.2 $w = \frac{1}{z}$ maps lines/circles to lines/circles

The general equation for straight lines and circles in the z-plane can be written as

$$\alpha(x^2+y^2) + \beta x + \gamma y + \Delta = 0. \tag{2.8}$$

where $\alpha,\ \beta,\ \gamma$ and Δ are constants. If $\alpha = 0$ this represents a straight line but when $\alpha \neq 0$ (2.8) represents a circle. Writing (2.8) in terms of z

$$\alpha|z|^2 + \frac{\beta}{2}(z + z^*) + \frac{\gamma}{2i}(z - z^*) + \Delta = 0\,. \tag{2.9}$$

and then transforming to an equation w and w through $w = \frac{1}{z}$ and $w^* = \frac{1}{z^*}$, (2.9) becomes

$$\frac{\alpha}{ww^*} + \frac{\beta}{2}\left(\frac{1}{w} + \frac{1}{w^*}\right) + \frac{\gamma}{2i}\left(\frac{1}{w} - \frac{1}{w^*}\right) + \Delta = 0\,, \tag{2.10}$$

or

$$\alpha + \frac{\beta}{2}(w + w^*) - \frac{i\gamma}{2}(w^* - w) + \Delta ww^* = 0\,. \tag{2.11}$$

Since $w = u + iv$ we have

$$\alpha + \beta u - \gamma v + \Delta(u^2 + v^2) = 0\,. \tag{2.12}$$

This represents a family of circles in the $u - v$ plane when $\Delta \neq 0$ and a family of lines when $\Delta = 0$. Notice, that when $\alpha \neq 0$ and $\Delta \neq 0$ then the mapping maps circles to circles but a family of lines in the z-plane ($\alpha = 0$) also maps to a family of circles in the w-plane. However, there is also the case of a family of circles in the z-plane for which $\Delta = 0$ which map to a family of lines in the w-plane. **Thus we conclude that $w = \frac{1}{z}$ maps lines/circles to lines/circles but not necessarily lines to lines and circles to circles.**

In addition to this we now study the *fractional linear* or *Möbius* transformation

$$w = \frac{az + b}{cz + d}\,, \qquad ad \neq bc\,. \tag{2.13}$$

This includes cases such as:

(i) $w = \frac{1}{z}$ when $a = d = 0,\ b/c = 1$.

(ii) $w = \frac{1}{z-1}$ as in our example above where $a = 0,\ b = 1,\ c = 1,\ d = -1$.

(2.13) can be re-written as

$$w = c^{-1}\left\{a + \frac{bc - ad}{cz + d}\right\}\,. \tag{2.14}$$

For various special cases:

1. $w = z + b\,;\ \ (a = d = 1,\ c = 0)$ – translation.
2. $w = az\,;\ \ (b = c = 0,\ d = 1)$ – contraction/expansion + rotation
3. $w = \frac{1}{z}\,;\ \ (a = d = 0,\ b = c)$ – maps lines/circles to lines/circles.

Thus a Möbius transformation maps lines/circles to lines/circles with contraction/expansion, rotation and translation on top.

2.3 Extra: Mappings of the type $w = \frac{e^z - 1}{e^z + 1}$

Consider a map $w = \frac{e^z-1}{e^z+1}$ which can be re-written as

$$e^z = \frac{1 + w}{1 - w} = \frac{(1 + u + iv)(1 - u + iv)}{(1 - u)^2 + v^2}\,. \tag{2.15}$$

Real and imaginary parts give

$$e^x \cos y = \frac{1 - u^2 - v^2}{(1 - u)^2 + v^2} \qquad e^x \sin y = \frac{2v}{(1 - u)^2 + v^2}\,. \tag{2.16}$$

From these we conclude that

1. The family of lines $y = n\pi$ in the z-plane map to the line $v = 0$ for n integer. Thus an infinite number of horizontal lines in the z-plane all map to the u-axis in the w-plane.
2. The family of lines $y = \frac{1}{2}(2n + 1)\pi$ in the z-plane map to the unit circle $u^2 + v^2 = 1$ in the w-plane.

2.4 Conformal mappings and fluid dynamics

Why is the result of Theorem 1 on conformal mappings important? In $2D$ fluid incompressible dynamics the velocity field $\boldsymbol{u}(x,y)$ must satisfy the incompressibility condition div $\boldsymbol{u}=0$, in which case $\boldsymbol{u}$ can be written as $\boldsymbol{u}=(-\psi_y,\,\psi_x)$ where ψ is known as a *stream function*. If the fluid is also irrotational (no vorticity) then curl $\boldsymbol{u}=0$. Therefore

$$\begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ -\psi_y & \psi_x & 0 \end{vmatrix} = 0\,, \tag{2.17}$$

in which case

$$\psi_{xx}+\psi_{yy}=0\,. \tag{2.18}$$

This means that $\psi(x,y)$ satisfies Laplace's equation and is therefore a harmonic function. It also means that there must be a conjugate function, designated as $\phi(x,y)$, *the potential*, that satisfies the Cauchy-Riemann equations

$$\phi_x=\psi_y\,,\qquad\qquad \phi_y=-\psi_x\,, \tag{2.19}$$

and that ϕ too must be harmonic. We've now reached the situation where $2D$ incompressible ideal fluid dynamics can be cast is the language of complex mappings by writing a complex function $f(z)$ as

$$w=f(z)=\phi(x,y)+i\psi(x,y)\,. \tag{2.20}$$

With $\phi\equiv u$ and $\psi\equiv v$ we are now free to choose mappings $w=f(z)$, as in the previous section, to see how these may be used to solve flows around complicated shapes. To proceed in this direction requires the following result:

Theorem 2. *Harmonic functions remain harmonic under a conformal mapping.*

Proof: Let H be a harmonic function in the z-plane: that is, H satisfies

$$H_{xx}+H_{yy}=0\,. \tag{2.21}$$

Now take $w=u+iv$ and with $f(z)$ analytic then the CR equations hold: $u_x=v_y$ and $v_x=-u_y$ and also $u_{xx}+u_{yy}=0$ with $v_{xx}+v_{yy}=0$. The chain rule says that:

$$H_x=u_xH_u+v_xH_v\qquad H_y=u_yH_u+v_yH_v \tag{2.22}$$

and

$$H_{xx}=u_{xx}H_u+v_{xx}H_v+u_x^2H_{uu}+v_x^2H_{vv}+2u_xv_xH_{uv} \tag{2.23}$$

$$H_{yy}=u_{yy}H_u+v_{yy}H_v+u_y^2H_{uu}+v_y^2H_{vv}+2u_yv_yH_{uv} \tag{2.24}$$

and so

$$H_{xx}+H_{yy}=(u_{xx}+u_{yy})H_u+(v_{xx}+v_{yy})H_v+(u_x^2+u_y^2)H_{uu}+(v_x^2+v_y^2)H_{vv}+2(u_xv_x+u_yv_y)H_{uv} \tag{2.25}$$

From the CR equations $u_x^2+u_y^2=v_x^2+v_y^2$ and $u_xv_x+u_yv_y=0$. Hence we are left with

$$H_{xx}+H_{yy}=(u_x^2+u_y^2)(H_{uu}+H_{vv})\,. \tag{2.26}$$

Thus if H satisfies Laplace's equation in the z-plane $H_{xx}+H_{yy}=0$ then it must satisfy Laplace's equation

$$H_{uu}+H_{vv}=0\,, \tag{2.27}$$

in the w-plane. The reason for the importance of this result lies in the fact that in incompressible inviscid fluid dynamics both the stream function and potential $(\psi, \, \phi)$ satisfy Laplace's equation. When this flows around a shape, such as an aerofoil, in the $z - plane$, under a conformal mapping we still have solutions of Laplace's equation in the w-plane. $\square$

The most famous of mappings in this area is **Joukowsi's** aerofoil transformation

$$w = z + \frac{1}{z}\,. \tag{2.28}$$

Firstly we use polar co-ordinates $x = r\cos\theta$ and $y = r\sin\theta$ so that $z = re^{i\theta}$ and

$$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta\,. \tag{2.29}$$

Therefore, with

$$a = r + \frac{1}{r} \qquad\qquad b = r - \frac{1}{r}\,, \tag{2.30}$$

u and v satisfy the equation for an ellipse

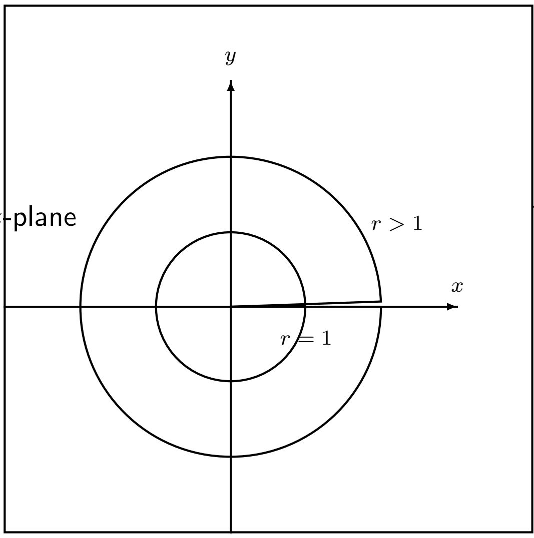
$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1\,. \tag{2.31}$$

Conclusions to be drawn are :

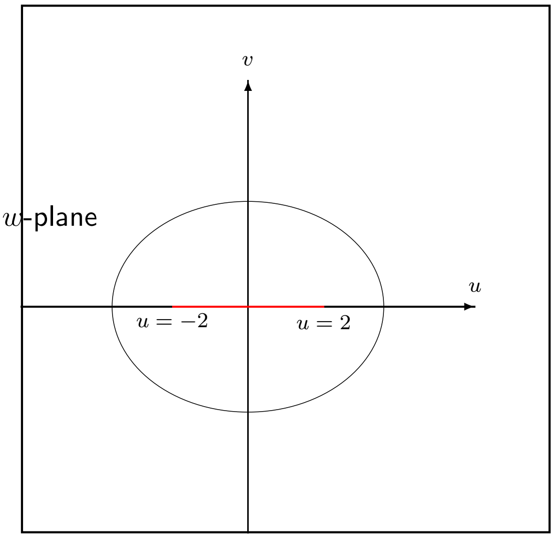
1. Circles $|z| = r = const$ centred at $z = 0$ in the z-plane map to ellipses in the w-plane.

2. The special circle $r = 1$ for which $a = 2$ and $b = 0$ maps to the red segment of the u-axis $-2 \le u \le 2$ as in the pair of figures below.

The z-plane



The w-plane

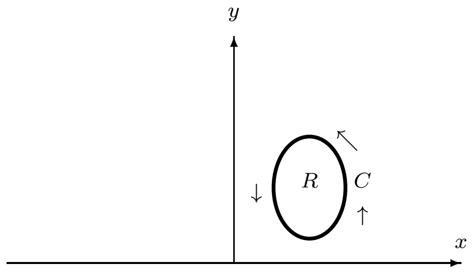


3. For circles passing through $x = \pm 1$ but not centred at $z = 0$ and for circles passing through $x = -1$ but passing outside $x = 1$ the figure below illustrated how the mapping works. Note that $dw/dz = 1 - z^{-2}$ which is zero when $z = \pm 1$. Hence conformality fails at these points.

3 Contour Integration

3.1 Cauchy's Theorem

How do we integrate a complex function? Let's begin with an easier question: where do we integrate it? We have seen a number of variations on the theme of integration in vector calculus, including contour integrals. This approach turns out to be particularly fruitful in complex analysis.



A closed contour C enclosing a region R in the z-plane around which the line integral is considered in the counter-clockwise direction

$$\oint_C F(z)\,dz\,. \tag{3.1}$$

With

$$F(z) = u + iv \qquad\qquad z = x + iy \tag{3.2}$$

we have

$$\begin{aligned}\oint_C F(z)\,dz &= \oint_C (u+iv)(dx+idy)\\ &= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)\,.\end{aligned} \tag{3.3}$$

Now Green's Theorem in a plane says that for differentiable functions $P(x,y)$ and $Q(x,y)$

$$\oint_C (P\,dx + Q\,dy) = \int\int_R (Q_x - P_y)\,dxdy\,. \tag{3.4}$$

For the real part, we take $P = u$ and $Q = -v$, so that $Q_x - P_y = -v_x - u_y$. Similarly, for the imaginary part, we take $P = v$ and $Q = u$, so that $Q_x - P_y = u_x - v_y$. Therefore we have

$$\begin{aligned}\oint (u\,dx - v\,dy) &= \int\int_R (-v_x - u_y)\,dxdy\,,\\ \oint (v\,dx + u\,dy) &= \int\int_R (u_x - v_y)\,dxdy\,,\end{aligned} \tag{3.5}$$

which turns (3.3) into

$$\oint_C F(z)\,dz = -\int\int_R (v_x + u_y)\,dxdy + i\int\int_R (u_x - v_y)\,dxdy\,. \tag{3.6}$$

If $F(z)$ is analytic everywhere within and on C then u and v must satisfy the CR equations: $u_x = v_y$ and $v_x = -u_y$, in which case both the real and imaginary parts on the RHS of (3.6) must be zero. We have established *Cauchy's Theorem:*

Theorem 3. *If $F(z)$ is analytic everywhere within and on a closed, piecewise smooth contour C then*

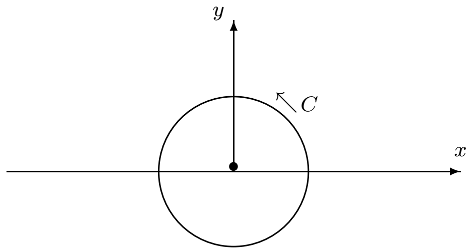
$$\oint_C F(z)\,dz = 0\,. \tag{3.7}$$

The key point is that provided $F(z)$ is analytic everywhere within and on C, any singularities of $F(z)$ *outside* of C are irrelevant.

Example 3.1. *The function*

$$f(z) = \frac{1}{z}$$

has its only singularity at the origin, so the integral of $f(z)$ around a contour which doesn't include the origin will be zero, by Cauchy's theorem. What about a contour that includes the origin?



For $f(z) = z^{-1}$, the circular contour C of radius a encloses a singularity • at the origin in the z-plane. The line integral is no longer zero because of this singularity.

Now write the circular contour C as $z = a\exp(i\theta)$ for $0 \le \theta < 2\pi$, a circle around the origin of radius a. Then

$$\frac{dz}{d\theta} = aie^{i\theta} \Rightarrow \frac{1}{z}dz = id\theta$$

so that

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{ia\exp(i\theta)d\theta}{a\exp(i\theta)} = i\int_0^{2\pi} d\theta = 2\pi i\,. \tag{3.8}$$

The singularity at $z = 0$ contributes a non-zero value of $2\pi i$ to the integral. Note that it is independent of the value of a, which is consistent with this being the only non-zero contribution to the integral: it is the same, for any circle centred at the origin.

Clearly, the only interesting behaviour happens with singularities in or on a contour. Given the powerful result of Cauchy's Theorem, our task is to see, in a more formal manner, how singularities contribute to complex integrals. Before this their nature and classification is necessary.

3.2 Poles and Residues

If a complex function fails to be analytic at a point, we call this point a *singularity*. Singularities can take many forms but the simplest class is what are called **simple poles**. A function $F(z)$ has a simple pole at $z = a$ (which could be real) if it can be written in the form

$$F(z) = \frac{g(z)}{z-a} \tag{3.9}$$

where $g(z)$ is analytic at $z = a$. Likewise, $F(z)$ has a pole of multiplicity m at $z = a$ if it can be written in the form

$$F(z) = \frac{g(z)}{(z-a)^m} \tag{3.10}$$

where, again, $g(z)$ is analytic at $z = a$, and $m = 1, 2, 3, 4, \ldots$; when $m = 2$ we have a double pole, etc.

Example 3.2. *The function*

$$f(z) = \frac{z^2}{(z-1)^2}$$

has a double pole at $z = 1$.

While all poles are singularities not all singularities are poles. For instance, $\ln z$ has a singularity at $z = 0$ but this is not a pole. Similarly, $z = a$ is not a pole when m is not an integer: $f(z) = z^2/\sqrt{z-1}$.

Definition : The **residue** of $F(z)$ at a simple pole at $z = a$ is

$$\lim_{z\to a}\{(z-a)F(z)\} = \lim_{z\to a}\left\{(z-a)\frac{g(z)}{z-a}\right\} = g(a)\,. \tag{3.11}$$

Definition : The **residue** of $F(z)$ at a pole of multiplicity[1] m at $z = a$ is

$$\lim_{z\to a}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m F(z)\right]\right\} = \lim_{z\to a}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m\frac{g(z)}{(z-a)^m}\right]\right\} = \frac{1}{(m-1)!}\frac{d^{m-1}g}{dz^{m-1}}\bigg]_{z=a}\,. \tag{3.12}$$

Note that a function may have many poles and each pole has its own residue.

Example 3.3. *Consider* $F(z) = \dfrac{2z}{(z-1)(z-2)}$ *which has simple poles at* $z = 1$ *and* $z = 2$.

$$\textit{Residue at } z = 1 = \lim_{z\to 1}\{(z-1)F(z)\} = -2\,. \tag{3.13}$$

$$\textit{Residue at } z = 2 = \lim_{z\to 1}\{(z-2)F(z)\} = \ 4\,. \tag{3.14}$$

Example 3.4. $F(z) = \dfrac{2z}{(z-1)^2(z+4)}$ *has a double pole at* $z = 1$ *and a simple pole at* $z = -4$.

$$\textit{Residue at } z = -4 = \lim_{z\to -4}\{(z+4)F(z)\} = -8/25\,. \tag{3.15}$$

$$\begin{aligned}\textit{Residue at double pole at } z = 1 \quad &= \quad \lim_{z\to 1}\frac{1}{1!}\left\{\frac{d}{dz}\left[(z-1)^2F(z)\right]\right\} \\ &= \quad \lim_{z\to 1}\frac{d}{dz}\left[\frac{2z}{z+4}\right] \\ &= \quad 2\lim_{z\to 1}\left[\frac{(z+4)-z}{(z+4)^2}\right] = 8/25\,. \end{aligned} \tag{3.16}$$

It is of no significance that the residues have opposite signs.

Example 3.5. *The function*

$$f(z) = \frac{3z}{3z-i}$$

has a simple pole at $z = \dfrac{1}{3}i$.

The residue is

$$\lim_{z\to i/3}\left\{(z-i/3)\frac{3z}{3z-i}\right\} = \lim_{z\to i/3}\left\{\frac{3z}{3}\right\} = \frac{1}{3}i$$

[1] This formula will be given in an exam question , if necessary; it is found from a coefficient in what is known as a Laurent expansion – see Kreyszig's book.

3.3 The residue of $F(z) = \frac{h(z)}{g(z)}$ when $g(z)$ has a simple zero at $z = a$

Taylor series exist for analytic functions, much as for functions of a single variable. Again, this is a fact we use without proof; the interested reader may consult the literature. We expand $g(z)$ about its zero at $z = a$ in a Taylor series

$$g(z) = g(a) + (z-a)g'(a) + \frac{1}{2}(z-a)^2 g''(a) + \ldots \tag{3.17}$$

Thus, noting that $g(a) = 0$ we have

$$\begin{aligned} \textit{Residue at } z = a &= \lim_{z\to a}\left\{\frac{(z-a)h(z)}{g(z)}\right\} \\ &= \lim_{z\to a}\left\{\frac{(z-a)h(z)}{(z-a)g'(a) + \frac{1}{2}(z-a)^2 g''(a) + \ldots}\right\} \\ &= \lim_{z\to a}\left\{\frac{h(z)}{g'(a) + \frac{1}{2}(z-a)g''(a) + \ldots}\right\} = \frac{h(a)}{g'(a)}\,. \end{aligned} \tag{3.18}$$

If $z = a$ were a double pole, we would also have $g'(a) = 0$ and the method would work with the obvious extension, and so on for poles of higher multiplicity.

Example 3.6. *Consider*

$$f(z) = \frac{1}{z^3 - 1}$$

where $h = 1$ and $g(z) = z^3 - 1$, with three roots $z = 1, e^{\pm 2i\pi/3}$.

So the residue at $z = 1$ can be calculated as

$$\frac{h(a)}{g'(a)} = \left.\frac{1}{3z^2}\right|_{z=1} = \frac{1}{3}\,.$$

Exercise: Obtain the residues at $z = e^{\pm 2i\pi/3}$.

3.4 The Residue Theorem

Now consider a simple pole at $z = a$ as in the Figure below, showing the full contour C comprising the two edges of the cuts $C^{\pm}$ running in opposite directions, the small circle C_a and then C_1 which is the rest of C with the small piece ϵ removed.

The pole is isolated by a device which consists of taking a "cut" into the contour and inscribing a small circle of radius r around it. The full *closed* contour C consists of the two edges of the cuts $C^{\pm}$ running in opposite directions, the small circle C_a and then C_1 which is the rest of C with the small piece ϵ removed. Thus we have

$$C\ :\quad C_1 + C_a + C^+ + C^- \,. \tag{3.19}$$

This device ensures that the pole lies outside of C as it has been constructed, in which case Cauchy's Theorem says that

$$\oint_C F(z)\,dz = 0 \tag{3.20}$$

in which case

$$0 = \oint_C F(z)\,dz = \left(\int_{C_1} + \int_{C_a} + \int_{C^+} + \int_{C^-}\right) F(z)\,dz\,. \tag{3.21}$$

Two points to note are:

1. The four integrals are not closed so they don't have the $\oint$ notation. In the limit $\epsilon \to 0$ the integrals $\int_{C^+}$ and $\int_{C^-}$ cancel as they go in opposite directions;
2. The integral over C_a is clockwise, not counter-clockwise.

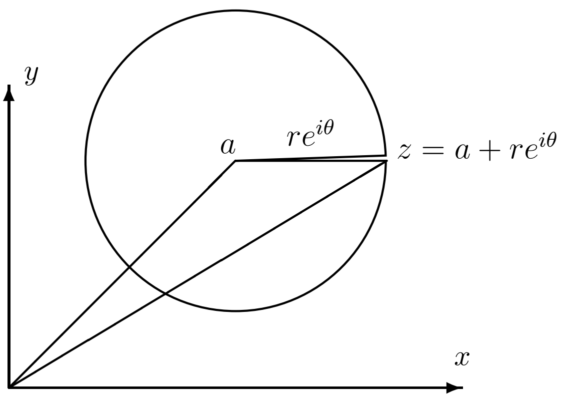
We are left with

$$\lim_{\epsilon\to 0}\int_{C_1} F(z)\,dz = -\lim_{\epsilon\to 0}\int_{C_a\hookleftarrow} F(z)\,dz = \int_{C_a\hookrightarrow} F(z)\,dz \tag{3.22}$$

We now write

$$\int_{C_a\hookrightarrow} F(z)\,dz = \int_{C_a\hookrightarrow} \frac{g(z)}{z-a}\,dz\,. \tag{3.23}$$

where $g(z)$ is analytic at $z = a$.



As in the figure above we write the equation of the circle of radius r, centre a in the complex plane as $z = a + re^{i\theta}$, where $0 \le \theta < 2\pi$ gives the circle, anticlockwise. So $dz = rie^{i\theta}d\theta$ and we can simplify:

$$\int_{C_a\hookrightarrow} F(z)\,dz = \int_{C_a\hookrightarrow} \frac{g(z)}{z-a}\,dz = \int_{C_a\hookrightarrow} \frac{g(a+re^{i\theta})}{re^{i\theta}} ire^{i\theta}\,d\theta\,. \tag{3.24}$$

Our next step at this stage is to take the limit $r \to 0$

$$\lim_{r\to 0}\int_{\substack{C_a\\ \hookrightarrow}} F(z)\,dz = \lim_{r\to 0}\int_{\substack{C_a\\ \hookrightarrow}} \frac{g(a+re^{i\theta})}{re^{i\theta}} ire^{i\theta}\,d\theta = ig(a)\int_o^{2\pi} d\theta = 2\pi i g(a) \tag{3.25}$$

which gives **Cauchy's integral formula**

$$\oint_C F(z)\,dz = \lim_{r\to 0}\int_{C_a\hookrightarrow} F(z)\,dz = 2\pi i g(a)\,. \tag{3.26}$$

However, because $z = a$ is a simple pole

$$\textit{At the pole at } z = a \textit{ the residue of } F(z) = \lim_{z \to a} \{(z-a)F(z)\} = g(a) \tag{3.27}$$

We have proved

$$\oint_C F(z)\, dz = 2\pi i \times \{\text{Residue of } F(z) \text{ at the simple pole at } z = a\} \; . \tag{3.28}$$

It is clear that this procedure of making a cut and ring-fencing a pole can be performed for many simple poles and the individual residues added. The result can also be proved (not here) when poles have higher multiplicity. Altogether we have :

Theorem 4. ***(Cauchy's) Residue Theorem :*** *If the only singularities of $F(z)$ within C are poles then*

$$\oint_C F(z)\, dz = 2\pi i \times \{\textit{Sum of the residues of } F(z) \textit{ at its poles within } C\} \; . \tag{3.29}$$

Some examples of this immensely powerful theorem follow.

Example 3.7. *Obtain*

$$\oint_C \frac{2z}{(z-1)^2(z+4)}\, dz$$

where C is the circle of radius 5, centred at the origin.

In example (3.4) we found the residues: the single pole at $z = -4$ and the double pole at $z = 1$ had residues $\pm\dfrac{8}{25}$. Hence, Cauchy's residue theorem gives the integral as

$$\oint_C f(z)\; dz = 2\pi i \left(\frac{8}{25} - \frac{8}{25}\right) = 0\, .$$

[*Note:* The zero result is a coincidence.]

Example 3.8. *Find*

$$\oint_{C_i} \frac{2z\, dz}{(z-1)(z-2)} \tag{3.30}$$

where (i) C_1 is the circle centred at $(0,0)$ of radius 3 and (ii) C_2 is the circle centred at $(0,0)$ of radius $3/2$.

$F(z)$ has two simple poles : the first at $z = 1$ and the second at $z = 2$. Their residues have been found in (3.13) and (3.14). For C_1 both poles lie inside C_1

$$\oint_{C_1} F(z)\, dz = 2\pi i \times (-2+4) = 4\pi i\, , \tag{3.31}$$

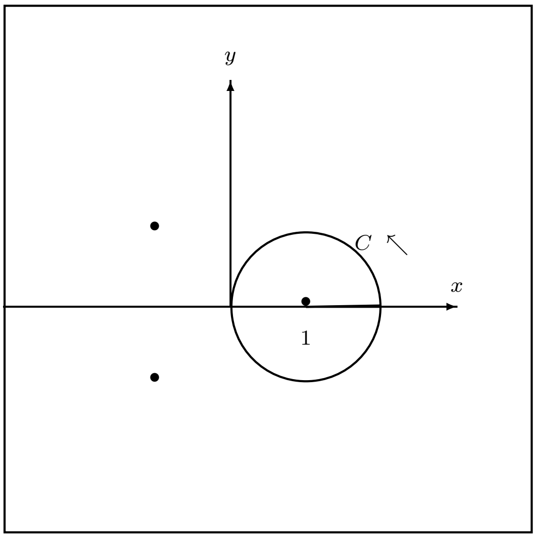
whereas for C_2 only the pole at $z = 1$ lie inside, thus

$$\oint_{C_2} F(z)\, dz = 2\pi i \times (-2) = -4\pi i\, . \tag{3.32}$$

Example 3.9. *Find*

$$\oint_C \frac{dz}{(z^3-1)^2} \tag{3.33}$$

where C is the circle $|z-1|=1$.



The contour is the circle $|z-1|=1$ in the z-plane. The double pole lies at $z=1$ whereas the two other double poles lies outside C at $z=\exp 2\pi i/3$ and $z=\exp -2\pi i/3$.

$z^3-1=0$ factors into $(z-1)(z^2+z+1)=0$ so it has zeroes at 1, $z=\exp(\pm 2\pi i/3)$. These are double poles for $F(z)$ but only the double pole at $z=1$ lies inside C. Its residue there is

$$\begin{aligned}\lim_{z\to 1}\frac{d}{dz}\left\{\frac{(z-1)^2}{(z^3-1)^2}\right\} &= \lim_{z\to 1}\frac{d}{dz}\left\{\frac{1}{(z^2+z+1)^2}\right\} \\ &= -2\lim_{z\to 1}\left\{\frac{2z+1}{(z^2+z+1)^3}\right\} = -2/9\,. \end{aligned}\tag{3.34}$$

Therefore we deduce from the Residue Theorem that

$$\oint_C \frac{dz}{(z^3-1)^2} = -4\pi i/9\,. \tag{3.35}$$

Example 3.10. *(Exam 2006): Find*

$$\oint_C \frac{z\,dz}{(z-1)^2(z-i)} \tag{3.36}$$

where C is the circle $|z|=2$.

For the double pole at $z=1$, the residue there is

$$\begin{aligned}\lim_{z\to 1}\frac{d}{dz}\left\{\frac{z(z-1)^2}{(z-1)^2(z-i)}\right\} &= \lim_{z\to 1}\frac{d}{dz}\left\{\frac{z}{(z-i)}\right\} \\ &= -\frac{i}{(1-i)^2} = \tfrac{1}{2}\,. \end{aligned}\tag{3.37}$$

For the simple pole at $z = i$ the residue there is

$$\lim_{z\to i}\left\{\frac{z(z-i)}{(z-1)^2(z-i)}\right\} \quad = \quad \frac{i}{(1-i)^2} = -\tfrac{1}{2}\,. \tag{3.38}$$

Both poles must be included within C so we conclude from the Residue Theorem that

$$\oint_C \frac{z\,dz}{(z-1)^2(z-i)} = (\tfrac{1}{2} - \tfrac{1}{2}) = 0\,. \tag{3.39}$$

Example 3.11. *(Exam 2006): Find*

$$\oint_C \frac{z^2 dz}{(z-i)^3} \tag{3.40}$$

where C is the circle $|z| = 2$ as above.

For the triple pole at $z = i$ the residue there is

$$\lim_{z\to 1}\frac{1}{2!}\frac{d^2}{dz^2}\left\{\frac{z^2(z-i)^3}{(z-1)^3}\right\} = 1\,. \tag{3.41}$$

Hence

$$\oint_C \frac{z^2 dz}{(z-i)^3} = 2\pi i\,. \tag{3.42}$$

3.5 Improper integrals and Jordan's Lemma

We consider integrals of the type

$$\int_{-\infty}^{\infty} f(x)\,dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)\sin x\,dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)\cos x\,dx$$

All are instances of, *real integrals* of the type

$$\int_{-\infty}^{\infty} e^{imx}F(x)\,dx\,, \qquad\qquad m \geq 0\,, \tag{3.43}$$

and the Residue Theorem can be used to evaluate them, provided $F(x)$ has certain convergence properties: these are called *improper integrals* because of the infinite nature of their limits. Formally we write them as

$$\int_{-\infty}^{\infty} e^{imx}F(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} e^{imx}F(x)\,dx\,. \tag{3.44}$$

The main idea is to consider a class of *complex integrals*

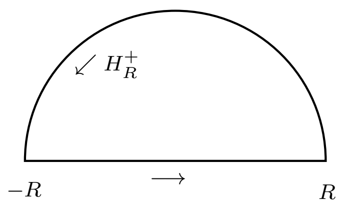
$$\oint_C e^{imz}F(z)\,dz \tag{3.45}$$

where C consists of the semi-circular arc as in the figure below. The two essential parts are the arc of the semicircle of radius R denoted by H_R and the real axis $[-R,\,R]$. Hence we can re-write (4.1) as

$$\oint_C e^{imz}F(z)\,dz = \underbrace{\int_{-R}^{R} e^{imx}F(x)\,dx}_{real\ integral} + \underbrace{\int_{H_R} e^{imz}F(z)\,dz}_{complex\ integral} \tag{3.46}$$

In principle the closed complex integral over C on the LHS can be evaluated by the Residue Theorem. Our next aim is to evaluate the real integral on the RHS in the limit $R \to \infty$. This requires a result which is called Jordan's Lemma.

Jordan's Lemma



Jordan's Lemma deals with the problem of how a contour integral behaves on the semi-circular arc H_R^+ of a closed contour C.

Lemma (**Jordan**) If the only singularities of $F(z)$ are poles, then

$$\lim_{R\to\infty}\int_{H_R} e^{imz}F(z)\,dz = 0 \tag{3.47}$$

provided that $m > 0$ and $|F(z)| \to 0$ as $R \to \infty$. If $m = 0$ then a faster convergence to zero is required for $F(z)$.

Proof: Since H_R is the semi-circle $z = Re^{i\theta} = R(\cos\theta + i\sin\theta)$ and $dz = iRe^{i\theta}d\theta$

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{H_R} e^{imz}F(z)\,dz\right| &= \lim_{R\to\infty}\left|\int_{H_R} e^{imR\cos\theta - mR\sin\theta}F(z)R\,e^{i\theta}d\theta\right| \\ &\le \lim_{R\to\infty}\int_{H_R} e^{-mR\sin\theta}|F(z)|R\,d\theta \end{aligned}\tag{3.48}$$

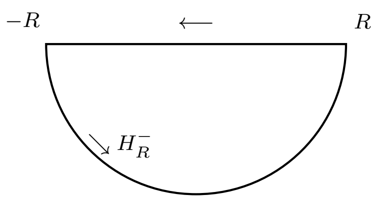
having recalled that $|e^{i\alpha}| = 1$ for any real α and $\left|\int f(z)\,dz\right| \le \int |f(z)|\,dz$. Note that in the exp-term on the RHS of (4.4), $\sin\theta > 0$ in the upper half plane. Hence, provided $m > 0$, the exponential ensures that the RHS is zero in the limit $R \to \infty$ (see remarks below). $\square$

Remarks:

a) When $m > 0$ forms of $F(z)$ such as $F(z) = \frac{1}{z}$, $F(z) = \frac{1}{z+a}$ or rational functions of z such as $F(z) = \frac{z^p \ldots}{z^q + \ldots}$ (for $0 \le p < q$ and p and q integers) will all converge fast enough as these all have simple poles and $|F(z)| \to 0$ as $R \to \infty$.

b) If, however, $m = 0$ then a modification is needed: e.g. if $F(z) = \frac{1}{z}$ then $|F(z)| \to 0$ but $\lim_{R\to\infty} z|F(z)| = 1$. We need to alter the restriction on the integers p and q to $0 \le p < q-1$ which excludes cases like $F(z) = \frac{1}{z}$, $F(z) = \frac{1}{z+a}$.

c) What about $m < 0$? To ensure that the exponential is decreasing for $R \to \infty$ we need $\sin\theta < 0$. This is true in the *lower* half plane. Hence in this case we take our contour in the *lower* half plane (call this H_R^- as opposed to H_R^+ in the upper) but still in an anti-clockwise direction.



A contour in the lower $\frac{1}{2}$-plane with semi-circle H_R^- taken in the counter-clockwise direction which is used for cases when $m < 0$. See the notes on Fourier Transforms for cases when this is useful.

The conclusion is that if $F(z)$ satisfies the conditions for Jordan's Lemma then

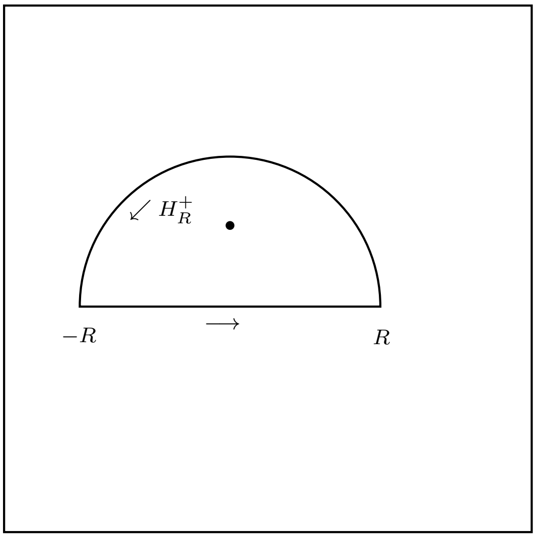
$$\int_{-\infty}^{\infty} e^{imx} F(x)\, dx = 2\pi i \times \{\text{Sum of the residues of the poles of } e^{imz}F(z) \text{ in the upper } \tfrac{1}{2}\text{-plane}\} \,. \tag{3.49}$$

Example 3.12. *Show that*

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi \,. \tag{3.50}$$

This integral can be solved with simple methods:

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^2} = \lim_{R\to\infty} \left[\tan^{-1} x\right]_{-R}^{R} = \frac{\pi}{2} + \frac{\pi}{2} \,.$$



C is comprised of a semi-circular arc H_R^+ and a section on the real axis from $-R$ to R. Only the simple pole at $z = i$ lies within C.

Thus we consider the complex integral over the semicircle C in the upper half-plane

$$\oint_C \frac{dz}{1+z^2} \tag{3.51}$$

with $m = 0$. The only singularities are simple poles at $z = \pm i$, one in the upper half-plane. For Jordan's lemma, we have $m = 0$, but $|1/(1+z^2)| \to 0$ faster than $1/z$, so the quadratic nature of the denominator is enough for convergence, and by Jordan's Lemma

$$\lim_{R\to\infty} \int_{H_R} \frac{dz}{1+z^2} = 0 \,. \tag{3.52}$$

The residue of $F(z)$ at the pole in the upper-half-plane at $z = i$ is

$$\lim_{z\to i} (z-i) \frac{1}{1+z^2} = \lim_{z\to i} \frac{1}{z+i} = \frac{1}{2i}$$

and so from the Residue Theorem,

$$\lim_{R\to\infty} \left(\int_{H_R} \frac{dz}{1+z^2} + \int_{-R}^{R} \frac{dx}{1+x^2} \right) = 0 + I = \oint_C \frac{dz}{1+z^2} = 2\pi i \times 1/2i = \pi \,. \tag{3.53}$$

Finally we have the result

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi\,. \tag{3.54}$$

Example 3.13. *Show that*

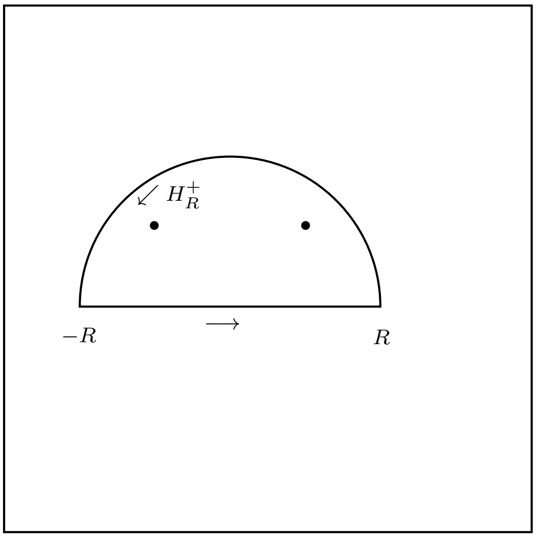
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \pi/\sqrt{2}\,. \tag{3.55}$$

As before, we consider the complex integral over the semicircle C in the upper half-plane

$$\oint_C \frac{dz}{1+z^4} \tag{3.56}$$

with $m = 0$. The existence of poles as the only singularities and the quartic nature of the denominator allows us to appeal to Jordan's Lemma:

$$\lim_{R\to\infty} \int_{H_R} \frac{dz}{1+z^4} = 0\,. \tag{3.57}$$



The only simple poles in the upper half-plane at $e^{i\pi/4}$, $e^{3i\pi/4}$ lie within C.

$z^4 = -1$ has four zeroes lying at $e^{i\pi/4}$, $e^{3i\pi/4}$ in the upper half-plane and $e^{-i\pi/4}$, $e^{-3i\pi/4}$ in the lower half-plane. Only the first two are relevant. Now we use the trick in (3.18) to find the residues of the two poles in the upper half-plane: when $f(z) = h/g$, if g has a simple zero and h is analytic at a, then the residue is $gh(a)/g'(a)$. Using $h(z) = 1$ and $g(z) = 1 + z^4 \Rightarrow g'(z) = 4z^3$, the residues at $e^{i\pi/4}$ and $e^{3i\pi/4}$ are

$$\left.\frac{1}{4z^3}\right|_{z=e^{i\pi/4}} = \frac{1}{4}e^{-3i\pi/4} \quad \text{and} \quad \left.\frac{1}{4z^3}\right|_{z=e^{3i\pi/4}} = \frac{1}{4}e^{-9i\pi/4} \tag{3.58}$$

Thus our final result is

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = (2\pi i)\frac{1}{4}\left(e^{-3i\pi/4} + e^{-9i\pi/4}\right) = \frac{\pi i}{2}\left(-e^{i\pi/4} + e^{-i\pi/4}\right) = \pi \sin\left(\frac{\pi}{4}\right) = \pi/\sqrt{2}\,. \tag{3.59}$$

Example 3.14. *Show that*

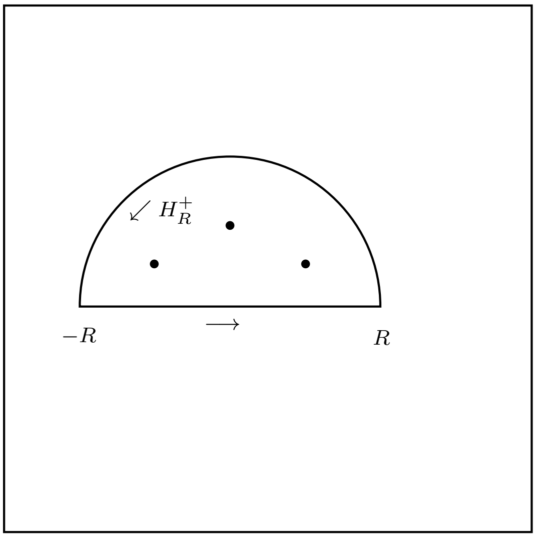
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^6} = 2\pi/3\,. \tag{3.60}$$

Thus we consider the complex integral over the semicircle C in the upper half-plane

$$\oint_C \frac{dz}{1+z^6} \tag{3.61}$$

with $m = 0$. The sextic nature of the denominator is enough for fast convergence and so from Jordan's Lemma

$$\lim_{R\to\infty} \int_{H_R} \frac{dz}{1+z^6} = 0\,. \tag{3.62}$$



The only simple poles in the upper half-plane at $e^{i\pi/6}$, $e^{i\pi/2}$ and $e^{5i\pi/6}$ lie within C.

$z^6 = -1$ has six zeroes lying at $e^{i\pi/6}$, $e^{i\pi/2}$ and $e^{5i\pi/6}$ in the upper half-plane and a further three in the lower half-plane. Now we use the trick in (3.18) to find the residues of the three poles in the upper half-plane. Using $h(z) = 1$ and $g(z) = 1 + z^6$ the residues at $e^{i\pi/6}$ and $e^{i\pi/2}$ and $e^{5i\pi/6}$ are

$$\left.\frac{1}{6z^5}\right|_{z=e^{i\pi/6}} = \frac{1}{6}e^{-5i\pi/6}\,; \qquad \left.\frac{1}{6z^5}\right|_{z=e^{i\pi/2}} = \frac{1}{6}e^{-5i\pi/2}\,; \qquad \left.\frac{1}{6z^5}\right|_{z=e^{5i\pi/6}} = \frac{1}{6}e^{-25i\pi/6} \tag{3.63}$$

Thus our final result is

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{1+x^6} &= \frac{2\pi i}{6}\left(e^{-5i\pi/6} + e^{-5i\pi/2} + e^{-i\pi/6}\right) \\ &= -\frac{2\pi i^2}{6}\left(2\sin\frac{1}{6}\pi + \sin\pi/2\right) = 2\pi/3\,. \end{aligned} \tag{3.64}$$

Example 3.15. *For $m > 0$ show that*

$$\int_{-\infty}^{\infty} \frac{\cos mx\,dx}{1+x^2} = \pi e^{-m}\,. \tag{3.65}$$

We consider the complex integral over the semicircle C in the upper half-plane

$$\oint_C \frac{e^{imz}dz}{1+z^2}\,. \tag{3.66}$$

The integrand has only one simple pole at $z = i$ in the upper half-plane whose residue is

$$\lim_{z\to i}\left\{\left(\frac{z-i}{1+z^2}\right)e^{imz}\right\} = \frac{e^{-m}}{2i}\,. \tag{3.67}$$

Therefore, from the Residue Theorem

$$\oint_C \frac{e^{imz}\,dz}{1+z^2} = 2\pi i \times \frac{e^{-m}}{2i} = \pi e^{-m}\,. \tag{3.68}$$

Moreover, from Jordan's Lemma, we have only simple poles, $m > 0$, and $|f(z)| \to 0$, so

$$\lim_{R\to\infty}\int_{H_R} \frac{e^{imz}dz}{1+z^2} = 0\,. \tag{3.69}$$

Therefore

$$\pi e^{-m} = \int_{-\infty}^{\infty} \frac{e^{imx}\,dx}{1+x^2} + 0 = \int_{-\infty}^{\infty} \frac{\cos mx\,dx}{1+x^2} + i\int_{-\infty}^{\infty} \frac{\sin mx\,dx}{1+x^2} \tag{3.70}$$

What happens to the imaginary part

$$\int_{-\infty}^{\infty} \frac{\sin mx\,dx}{1+x^2}\,? \tag{3.71}$$

Notice that the integrand is an *odd function*: therefore, over $(-\infty,\,\infty)$ the part over $(-\infty,\,0)$ will cancel with that over $(0,\,\infty)$, leaving zero as a result. Thus we have

$$\int_{-\infty}^{\infty} \frac{\cos mx\,dx}{1+x^2} = \pi e^{-m}\,. \tag{3.72}$$

Example 3.16. *For $m > 0$ show that*

$$\int_{-\infty}^{\infty} \frac{\cos mx\,dx}{(a^2+x^2)^2} = \frac{\pi e^{-ma}}{2a^3}(1+ma)\,. \tag{3.73}$$

We consider the complex integral over the semicircle C in the upper half-plane

$$\oint_C \frac{e^{imz}dz}{(a^2+z^2)^2}\,. \tag{3.74}$$

The integrand has only one double pole at $z = ia$ in the upper half-plane whose residue is

$$\lim_{z\to ia}\frac{d}{dz}\left[\frac{(z-ia)^2e^{imz}}{(a^2+z^2)^2}\right] = -\frac{ie^{-ma}}{4a^3}(1+ma)\,. \tag{3.75}$$

Therefore, from the Residue Theorem

$$\begin{aligned}\oint_C \frac{e^{imz}\,dz}{(a^2+z^2)^2} &= -2\pi i \times \frac{ie^{-ma}}{4a^3}(1+ma)\\ &= \frac{\pi e^{-ma}}{2a^3}(1+ma)\end{aligned} \tag{3.76}$$

Moreover, from Jordan's Lemma, with $m > 0$

$$\lim_{R\to\infty}\int_{H_R} \frac{e^{imz}dz}{(a^2+z^2)^2} = 0\,. \tag{3.77}$$

Therefore

$$\frac{\pi e^{-ma}}{2a^3}(1+ma) = \int_{-\infty}^{\infty} \frac{e^{imx}\,dx}{(a^2+x^2)^2} + 0\,. \tag{3.78}$$

As in the previous example the imaginary part is zero because the integrand is an odd function leaving

$$\int_{-\infty}^{\infty} \frac{\cos mx\,dx}{(a^2+x^2)^2} = \frac{\pi e^{-ma}}{2a^3}(1+ma)\,. \tag{3.79}$$

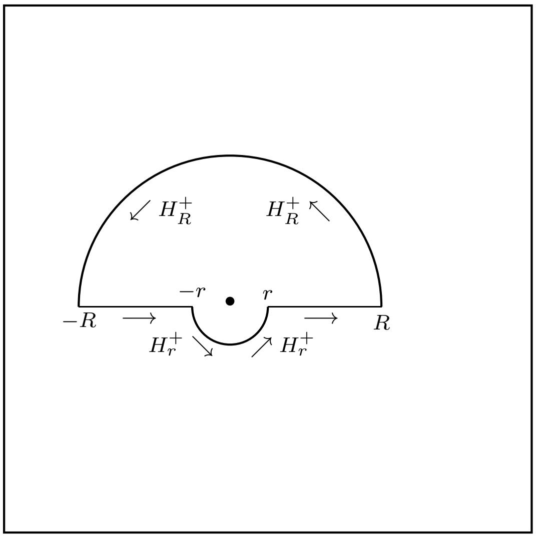
3.6 Poles on the real axis

When an integrand has a pole on the real axis this causes problems by sitting on the semicircular contour, because Cauchy's Residue theorem only applies to poles *within* the contour. The trick is to change the contour to include or exclude the pole on the axis. Let us do this by example.

We would like to calculate an improper integral of the type

$$\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x} \tag{3.80}$$

using the same approach as before, with Jordan's lemma. The problem is the pole at the origin, lying on the real axis.



The contour is deformed by a small semi-circle of radius r, indented into the lower half-plane, centred at the origin, thus including the pole at $z = 0$. Following the direction of the arrows, the big semicircle of radius R is designated as H_R^+ $(\theta : 0 \to \pi)$ and the little semicircle of radius r is designated as H_r^+ $(\theta : \pi \to 2\pi)$; the superscript $^+$ indicates counterclockwise orientation.

In this version, a small semi-circular indentation into the lower half-plane has been added to the contour, thus including the pole at $z = 0$. The alternative is to have a small semi-circular indentation into the upper half-plane, thus excluding the pole from the contour.

To calculate an improper integral of the type

$$\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x} \tag{3.81}$$

we consider:

$$2\pi i \times \text{ [Sum of residues in upper half-plane and at origin]} = \oint_C \frac{f(z)\,dz}{z} = \mathcal{I}$$

and split the contour integral up into four distinct integrals

$$\mathcal{I} = \int_{-R}^{-r} \frac{f(x)\,dx}{x} + \int_{\underset{\hookrightarrow}{H_r}} \frac{f(z)\,dz}{z} + \int_{r}^{R} \frac{f(x)\,dx}{x} + \int_{\underset{\hookrightarrow}{H_R}} \frac{f(z)\,dz}{z}\,. \tag{3.82}$$

For the last of the four integrals, we have Jordan's lemma giving value zero, as $R \to \infty$. The first and the third integral together give us, in the limit:

$$\lim_{\substack{r\to 0\\ R\to\infty}} \int_{-R}^{-r} \frac{f(x)\,dx}{x} + \int_{r}^{R} \frac{f(x)\,dx}{x} = \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x}\,.$$

What's new here is what happens to the second integral, around the small indented semi-circle H_r, as $r \to 0$.

Example 3.17. *Use Cauchy's residue theorem, with Jordan's lemma and a semi-circular indentation into the lower half-plane to calculate*

$$\int_{-\infty}^{\infty} \frac{\sin x\, dx}{x} \tag{3.83}$$

As previously, we consider the complex integral

$$\oint_C \frac{e^{iz}\, dz}{z}\,. \tag{3.84}$$

The integrand has a simple pole in C as $z = 0$ is included in the above construction. The pole at the origin has residue

$$\lim_{z\to 0} z\left(\frac{e^{iz}}{z}\right) = 1\,.$$

Thus, Cauchy's Theorem can be invoked to give

$$2\pi i(\text{Residue at origin}) = \oint_C \frac{e^{iz}\, dz}{z} = \int_{-R}^{-r} \frac{e^{ix}\, dx}{x} + \int_{\underset{\hookrightarrow}{H_r}} \frac{e^{iz}\, dz}{z} + \int_{r}^{R} \frac{e^{ix}\, dx}{x} + \int_{\underset{\hookrightarrow}{H_R}} \frac{e^{iz}\, dz}{z}\,. \tag{3.85}$$

or

$$2\pi i = \int_{-\infty}^{\infty} \frac{e^{ix}\, dx}{x} + \lim_{R\to\infty} \int_{\underset{\hookrightarrow}{H_R}} \frac{e^{iz}\, dz}{z} + \lim_{r\to 0} \int_{\underset{\hookrightarrow}{H_r}} \frac{e^{iz}\, dz}{z}\,.$$

In the second integral we take the limit $R \to \infty$ and, with $m = 1$, Jordan's Lemma tell us that $\int_{H_R} = 0$ because the only singularity is a pole and the integrand decays to zero as $R \to \infty$.

For the last integral, we note that the small semicircle H_r has the equation $z = re^{i\theta}$ for $\theta : \pi \to 2\pi$, and we proceed as several times previously, with $dz/d\theta = ire^{i\theta} \Rightarrow dz = iz\; d\theta$ so we get

$$\lim_{r\to 0} \int_{\underset{\hookrightarrow}{H_r}} \frac{e^{iz}\, dz}{z} = \lim_{r\to 0} \int_{\pi}^{2\pi} \frac{e^{ire^{i\theta}}\; iz\; d\theta}{z} = i\int_{\pi}^{2\pi} d\theta \;=\; \pi i$$

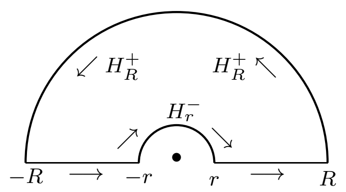
Taking the two limits $R \to \infty$ and $r \to 0$ together, we have

$$2\pi i = \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx + \pi i + 0\,. \Rightarrow \pi i = \int_{-\infty}^{\infty} \frac{\cos x}{x} dx + i\int_{-\infty}^{\infty} \frac{\sin x}{x} dx\,. \tag{3.86}$$

As before, we note that the integrand $\cos x/x$ is odd so the contributions on $(-\infty, 0)$ and $(0, \infty)$ cancel leaving us with

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi\,. \tag{3.87}$$

Exercise: Repeat the previous example, with H_r a small semi-circular indentation into the upper half-plane:



The contour is deformed by a small semi-circle of radius r, indented into the upper half-plane, centred at the origin, thus excluding the pole at $z = 0$. Following the direction of the arrows, H_R^+ is as before, but the small semicircle H_r^- has $\theta : \pi \to 0$; the superscript $^+, ^-$ indicate (counter)clockwise orientation.The same result is obtained, but care needs to be taken with the limits for θ when changing variables in the integral for H_r.

3.7 Integrals around the unit circle

We consider here integrals of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$. The idea is best illustrated with an example.

Example 3.18. *Obtain*

$$I = \int_0^{2\pi} \frac{d\theta}{a + \cos\theta}, \qquad a > 1. \tag{3.88}$$

Take C as the unit circle $z = e^{i\theta}$. Therefore $dz = ie^{i\theta}\, d\theta = iz d\theta \Rightarrow d\theta = dz/iz$. Recall that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

with a similar identity available for $\sin\theta$, so that

$$\begin{aligned} I &= \oint_C \frac{dz}{iz(a + \frac{1}{2}\left(z + \frac{1}{z}\right)} \\ &= -2i \oint_C \frac{dz}{z^2 + 2az + 1}. \end{aligned} \tag{3.89}$$

The next task is to determine the roots of $z^2 + 2az + 1 = 0$.

$$z = \frac{-2a \pm \sqrt{4a^2 - 4}}{2} = -a \pm \sqrt{a^2 - 1}. \tag{3.90}$$

We note that for $a > 1$, while $z_1 = -a + \sqrt{a^2 - 1}$ lies *within* C, the other root, $z_2 = -a - \sqrt{a^2 - 1}$ lies *without*. Therefore we exclude the pole at z_2 and compute the residue of the integrand at z_1, which is

$$\lim_{z\to z_1} (z - z_1)\frac{1}{z^2 + 2az + 1} = \lim_{z\to z_1} (z - z_1)\frac{1}{(z - z_1)(z - z_2)} = \lim_{z\to z_1} \frac{1}{z - z_2} = \frac{1}{z_1 - z_2} = \frac{1}{2\sqrt{a^2 - 1}} \tag{3.91}$$

The Residue Theorem then gives

$$I = (2\pi i) \times (-2i) \times (\text{ Residue at } z_1) = \frac{2\pi}{\sqrt{a^2 - 1}}. \tag{3.92}$$

4 An application to Fourier Transforms

Now apply this to the Fourier Transform of $f(t) = \frac{1}{1+t^2}$ as an example

$$\overline{f}(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{1 + t^2}\, dt. \tag{4.1}$$

The need for Jordan's Lemma arises (with the necessity for upper and lower half-plane contours) because ω takes both positive and negative values. It obviously plays the role of $-m$ in Jordan's Lemma so we are forced to split the calculation into two halves: take a contour in the upper half-plane H_R^+ for $\omega < 0$ and take a contour in the lower half-plane H_R^- for $\omega > 0$. Hence we need to consider the complex plane where the real axis is the t-axis with the pair of complex integrals

$$\int_{H_R^+} \frac{e^{-i\omega z}}{1 + z^2}\, dz \quad (\omega < 0) \qquad\qquad \int_{H_R^-} \frac{e^{-i\omega z}}{1 + z^2}\, dz \quad (\omega > 0) \tag{4.2}$$

Use the Residue Theorem and Jordan's Lemma on H_R^+ with a simple pole at $z = i$ to obtain, in the limit $R \to \infty$,

$$2\pi i \left(\frac{e^{\omega}}{2i}\right) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{1 + t^2}\, dt \quad (\omega < 0) \tag{4.3}$$

and, on H_R^- with a simple pole at $z=-i$, in the limit $R\to\infty$,

$$2\pi i\left(-\frac{e^{-\omega}}{2i}\right)=\int_{\infty}^{-\infty}\frac{e^{-i\omega t}}{1+t^2}\,dt\quad(\omega>0)\tag{4.4}$$

Note the limits in the integral in (4.4) are reversed because of the contour in the lower half plane. We arrive at the result

$$\overline{f}(\omega)=\int_{-\infty}^{\infty}\frac{e^{-i\omega t}}{1+t^2}\,dt=\pi\left\{\begin{array}{ll}e^{\omega}, & \omega<0\\ e^{-\omega}, & \omega>0\end{array}\right.=\pi e^{-|\omega|}.\tag{4.5}$$