Homeworks

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1 Homework 2

1.1 Q1. CES Utility

Given

$$u = \left(\gamma\, y_1^{\frac{\sigma-1}{\sigma}} + y_2^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \qquad \text{budget: } p_1 y_1 + p_2 y_2 = I.$$

Find the ratio y_1/y_2 that maximizes u.

Solution
1. Define the intermediate function

$$V = \gamma\, y_1^{\frac{\sigma-1}{\sigma}} + y_2^{\frac{\sigma-1}{\sigma}}$$

and set up the Lagrangian

$$\mathcal{L} = V + \lambda\,(I - p_1 y_1 - p_2 y_2).$$

2. First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial y_1} = \gamma\, \frac{\sigma-1}{\sigma}\, y_1^{\frac{\sigma-1}{\sigma}-1} - \lambda\, p_1 = 0, \quad \frac{\partial \mathcal{L}}{\partial y_2} = \frac{\sigma-1}{\sigma}\, y_2^{\frac{\sigma-1}{\sigma}-1} - \lambda\, p_2 = 0.$$

3. Divide the two equations to eliminate λ:

$$\frac{\gamma\,\frac{\sigma-1}{\sigma}\,y_1^{\frac{\sigma-1}{\sigma}-1}}{\frac{\sigma-1}{\sigma}\,y_2^{\frac{\sigma-1}{\sigma}-1}} = \frac{p_1}{p_2} \implies \gamma\,y_1^{-\frac{1}{\sigma}}\,p_2 = y_2^{-\frac{1}{\sigma}}\,p_1 \implies \Big(\frac{y_1}{y_2}\Big)^{\frac{1}{\sigma}} = \frac{\gamma\,p_2}{p_1}.$$

4. Therefore, the optimal ratio is

$$\boxed{\frac{y_1}{y_2} = \Big(\gamma\,\frac{p_2}{p_1}\Big)^{\sigma}}.$$

1.2 Q2. Cobb–Douglas in an Open Economy

Given

$$U(y_1, y_2) = y_1^{\alpha}\,y_2^{\beta}, \quad \alpha + \beta = 1,$$

endowment (y_1^e, y_2^e), world prices (p_1, p_2).

1. **Income** (value of endowment):

$$I = p_1\,y_1^e + p_2\,y_2^e.$$

2. **Budget constraint**

$$p_1\,y_1 + p_2\,y_2 = I.$$

3. **Maximization** yields standard expenditure shares:

$$p_1\,y_1 = \alpha\,I, \quad p_2\,y_2 = \beta\,I.$$

4. **Demand functions**:

$$\boxed{y_1^* = \frac{\alpha\,I}{p_1}, \quad y_2^* = \frac{\beta\,I}{p_2}}.$$

5. Substituting $I = p_1\,y_1^e + p_2\,y_2^e$ gives:

$$y_1^* = \alpha\Big(y_1^e + \tfrac{p_2}{p_1}\,y_2^e\Big), \quad y_2^* = \beta\Big(\tfrac{p_1}{p_2}\,y_1^e + y_2^e\Big).$$

2 Homework 3 – Equivalence of the Two Equilibria

Claim. The home country's equilibrium under
1. "Separate free-trade" (Situation 1) and
2. "Integrated world economy" (Situation 2)

coincide. In both cases the relative price p is the same, so home produces and consumes the same bundle.

2.1 Definitions

- Let

$$\begin{aligned} RS_H(p) \;&=\; \text{home's relative-supply of good 1 (units of 1 per 2) at price } p, \\ RS_F(p) \;&=\; \text{foreign's relative-supply of good 1 at price } p, \\ RS_W(p) \;&=\; RS_H(p) + RS_F(p) \quad \text{(world relative-supply).} \end{aligned}$$

- Let $RD(p)$ be world relative-demand (units of 1 per 2) at price p.

2.2 Situation 1: Separate Free Trade

1. **World price determination.**
   Since both countries take the same world price p and supply goods competitively, the **aggregate equilibrium** price p^* solves
   $$RS_W(p^*) \;=\; RD(p^*)\,.$$
2. **Country-specific outcome.**
   At p^*, home chooses its profit-maximizing production point on its PPF and its utility-maximizing consumption on its indifference map. Call this bundle (y_1^H, y_2^H).

2.3 Situation 2: Integrated World Economy

1. **Aggregate economy.**
   Treat home + foreign as one big country with relative-supply
   $$RS_W(p) = RS_H(p) + RS_F(p).$$
2. **World equilibrium.**
   The integrated economy's equilibrium price $\tilde{p}$ solves the **same** equation
   $$RS_W(\tilde{p}) \;=\; RD(\tilde{p}).$$
3. **Specialization assumption.**
   At $\tilde{p}$ (between the two autarky prices), home fully specializes in good 1 and foreign in good 2, but the determining equation for $\tilde{p}$ is identical.

2.4 Conclusion

- Both situations pick the **same** relative price:
  $$p^* = \tilde{p}.$$
- Home faces the same price in both cases, so by price-taking behavior its production & consumption plan (y_1^H, y_2^H) is **identical**.
- **Therefore** the two equilibria coincide for the home country.

2.5 Graphical Intuition

- In both pictures you draw the world relative-supply curve $RS_W(p)$ and the relative-demand curve $RD(p)$.
- The intersection $RS_W = RD$ is **exactly** the same point in Situation 1 and Situation 2.
- Home then "picks off" its point on its own PPF at that price—so you end up at the same coordinate in both diagrams.

$\boxed{\text{Hence the equilibria are equal.}}$

3 Homework 4

3.1 Q1. Equilibrium in DFS (1977) with $U = \int_0^1 b(z)\ln c(z)\,dz$

1. **Preferences & demand**
   - Utility:
     $$U = \int_0^1 b(z)\ln c(z)\,dz, \quad \int_0^1 b(z)\,dz = 1.$$

- FOC w.r.t. $c(z)$:

$$\frac{b(z)}{c(z)} = \lambda\, p(z) \implies c(z) = \frac{b(z)}{\lambda\, p(z)}.$$

- Budget $\int_0^1 p(z)c(z)\,dz = I$ implies $\lambda = 1/I$, so

$$c(z) = I\,\frac{b(z)}{p(z)}.$$

- **Expenditure share** on good z:
  $p(z)\,c(z) = I\,b(z) \to$ “Cobb–Douglas” across the continuum.

2. **Production & zero profit**
   - Home has unit-labour requirements $a(z)$, foreign $a^*(z)$.
   - Let relative wage $\omega = w/w^*$.
   - Zero-profit prices:

$$p(z) = \begin{cases} w\,a(z), & z < z^*, \\ w^*\,a^*(z), & z > z^*, \end{cases}$$

   where the **cutoff** z^* solves

$$w\,a(z^*) = w^*\,a^*(z^*) \quad \implies \quad \omega = \frac{a^*(z^*)}{a(z^*)}.$$

3. **World relative-supply & demand**
   - **Supply** of home goods (z [0,z*]) vs. foreign goods (z [z*,1]):

$$RS(\omega) = \frac{L\int_0^{z^*} 1/a(z)\,dz}{L^*\int_{z^*}^1 1/a^*(z)\,dz}.$$

   - **Demand** for home goods:

$$RD(\omega) = \frac{\displaystyle\int_0^{z^*} b(z)\,dz}{\displaystyle\int_{z^*}^1 b(z)\,dz}.$$

   - **Equilibrium** requires simultaneously

$$RS(\omega) = RD(\omega), \quad \omega = \frac{a^*(z^*)}{a(z^*)}.$$

4. **Effect of $b(z)$**
   - Only the **shape** of the relative-demand curve $RD(\omega)$ changes (weights $b(z)$ instead of uniform).
   - The **qualitative** features—existence of a unique cutoff z^*, full specialization on each side—remain intact.

3.2 Q2. Intuitive transition when $L^* \to L^{*\prime}$

1. **Foreign labor rises supply shift**
   - Larger L^* raises foreign output of every good world **relative supply** of home goods falls.
2. **Relative-price adjustment**
   - To clear the market, the relative price $\omega = w/w^*$ must **decline**.

- Graphically, the supply curve $A(z)$ shifts so its intersection with demand $B(z)$ moves from E to E'.

3. **New cutoff** z^*
   - Lower ω solves $\omega = a^*(z^*)/a(z^*)$ at a **smaller** z^*.

   - Home now specializes in a narrower set of goods (only the lowest-z range).
4. **Partial-equilibrium analogy**
   - Just like $\uparrow$ supply in a single market $\to \downarrow$ price + new quantity, here $\uparrow$ foreign labor endowment $\to \downarrow$ relative price of home goods + adjusted specialization until the new intersection E' clears the world market.

4 Homework 5

Answer. Since

$$a_i(j) \sim \text{Weibull}(\text{shape} = \alpha_i,\ \text{scale} = \tfrac{1}{\lambda_i})$$

and

$$p_{ni}(j) = \frac{w_i}{d_{ni}}\, a_i(j),$$

it follows that $p_{ni}(j)$ is also Weibull with

- **Shape parameter:** α_i
- **Scale parameter:** $\dfrac{w_i}{d_{ni}\,\lambda_i}$

Its CDF is

$$\begin{aligned} F_p(x) &= \Pr(p_{ni} \le x) = \Pr\Big(a_i \le \tfrac{d_{ni}}{w_i}\, x\Big) \\ &= 1 - \exp\Big[-\big(\lambda_i \cdot \tfrac{d_{ni}}{w_i}\, x\big)^{\alpha_i}\Big]. \end{aligned}$$

5 Homework 6

Let

$$V(y_1; L, K) \;=\; \max_{\substack{L_1+L_2\le L\\ K_1+K_2\le K\\ f_1(L_1,K_1)\ge y_1}} \; f_2(L_2, K_2)$$

be the PPF viewed as a function of endowments (L, K) at fixed y_1. Then:

1. **Envelope theorem**
   At the optimum,
   $$\frac{\partial V}{\partial L} = \lambda_L, \quad \frac{\partial V}{\partial K} = \lambda_K,$$
   where λ_L, λ_K are the Lagrange multipliers on the labor and capital constraints.

2. **CRS homogeneity**
   Because each sector's technology is CRS,
   $$\lambda_L\, L \;+\; \lambda_K\, K \;=\; f_2(L_2^*, K_2^*) \;=\; V(y_1; L, K).$$

3. **Tripling endowments**
   Take $(L, K) \to (3L, 3K)$. Then by the envelope theorem and homogeneity,
   $$V(y_1; 3L, 3K) - V(y_1; L, K) = 2(\lambda_L L + \lambda_K K) = 2\, V(y_1; L, K).$$
   Hence
   $$V(y_1; 3L, 3K) = 3\, V(y_1; L, K).$$

4. **Conclusion: outward shift**
   For every y_1, the maximum feasible y_2 triples. Equivalently,

$$\boxed{\text{New PPF: } y_2^{\text{new}}(y_1) = 3\, y_2^{\text{old}}(y_1)}.$$

   The frontier simply scales out by a factor of 3, with its slope at each point unchanged.