Problem Set 3

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1 Problem Set 3

1.1 Task 1: Solow Growth Model

Given: - Production function: $Y_t = A \cdot K_t^{0.4}$, initially $A = 1$ - Saving rate: $s = 0.4$ - Depreciation rate: $\delta = 0.1$ - Initial capital stock: $K_1 = 4$

1.1.1 Capital Accumulation Equation

The general capital accumulation equation is:

$$K_{t+1} = (1 - \delta)K_t + sY_t$$

Substituting the production function $Y_t = A \cdot K_t^{0.4}$ and $A = 1$:

$$K_{t+1} = 0.9K_t + 0.4 \cdot K_t^{0.4}$$

1.1.2 Compute Capital K_t for $t = 2$ to $t = 5$

Using the formula from (1.1.1):

- $K_1 = 4$

Step 1: $t = 2$

$$K_2 = 0.9 \cdot 4 + 0.4 \cdot 4^{0.4} \approx 3.6 + 0.4 \cdot 1.740 = 3.6 + 0.696 = 4.296$$

Step 2: $t = 3$

$$K_3 = 0.9 \cdot 4.296 + 0.4 \cdot 4.296^{0.4} \approx 3.866 + 0.716 = 4.582$$

Step 3: $t = 4$

$$K_4 = 0.9 \cdot 4.582 + 0.4 \cdot 4.582^{0.4} \approx 4.124 + 0.732 = 4.856$$

Step 4: $t = 5$

$$K_5 = 0.9 \cdot 4.856 + 0.4 \cdot 4.856^{0.4} \approx 4.370 + 0.746 = 5.116$$

1.1.3 Compute Output Y_t for $t = 2$ to $t = 5$

Use the production function: $Y_t = K_t^{0.4}$

- $Y_2 = 4.296^{0.4} \approx 1.789$
- $Y_3 = 4.582^{0.4} \approx 1.829$
- $Y_4 = 4.856^{0.4} \approx 1.864$
- $Y_5 = 5.116^{0.4} \approx 1.892$

1.1.4 Technological Progress: $A = 1.5$

Now the production function becomes:

$$Y_t = 1.5 \cdot K_t^{0.4}$$

New capital accumulation equation:

$$K_{t+1} = 0.9K_t + 0.6 \cdot K_t^{0.4}$$

Recalculate K_t and Y_t

Step 1: $t = 2$

$$K_2 = 0.9 \cdot 4 + 0.6 \cdot 4^{0.4} \approx 3.6 + 1.044 = 4.644$$

Step 2: $t = 3$

$$K_3 = 0.9 \cdot 4.644 + 0.6 \cdot 4.644^{0.4} \approx 4.180 + 1.096 = 5.276$$

Step 3: $t = 4$

$$K_4 = 0.9 \cdot 5.276 + 0.6 \cdot 5.276^{0.4} \approx 4.748 + 1.131 = 5.879$$

Step 4: $t = 5$

$$K_5 = 0.9 \cdot 5.879 + 0.6 \cdot 5.879^{0.4} \approx 5.291 + 1.160 = 6.451$$

Output with technology $A = 1.5$

- $Y_2 = 1.5 \cdot 4.644^{0.4} \approx 2.741$
- $Y_3 = 1.5 \cdot 5.276^{0.4} \approx 2.827$
- $Y_4 = 1.5 \cdot 5.879^{0.4} \approx 2.899$
- $Y_5 = 1.5 \cdot 6.451^{0.4} \approx 2.957$

Summary Tables

Without Technological Progress:

Period (t)	Capital K_t	Output Y_t
2	4.296	1.789
3	4.582	1.829
4	4.856	1.864
5	5.116	1.892

With Technological Progress (A = 1.5):

Period (t)	Capital K_t	Output Y_t
2	4.644	2.741
3	5.276	2.827
4	5.879	2.899
5	6.451	2.957

1.2 Task 2: Consumption Under Tax Policy (3-Period Model)

Given: - Utility function: $u(c) = c - \frac{1}{100}c^2$ - Time horizon: 3 periods, $t = 1, 2, 3$ - Income: $y_1 = y_2 = y_3 = 20$ - No interest rate or discounting assumed (implied by problem unless stated otherwise)

1.2.1 If the consumer chooses to consume:

- $c_1 = 15$
- $c_2 = 20$
- $c_3 = 20$

Find: Savings s_1, s_2, s_3

Savings in each period is:

$$s_t = y_t - c_t \text{ (can be negative)}$$

- $s_1 = 20 - 15 = 5$
- $s_2 = 20 - 20 = 0$
- $s_3 = 20 - 20 = 0$

Answer: - $s_1 = 5$, $s_2 = 0$, $s_3 = 0$

1.2.2 Income tax of 20% in period 1 only

New income: - $y_1 = 20 \times (1 - 0.2) = 16$ - $y_2 = 20$, $y_3 = 20$

Budget constraint:

$$c_1 + c_2 + c_3 = 16 + 20 + 20 = 56$$

Maximize utility:

$$U = u(c_1) + u(c_2) + u(c_3) = \sum_{t=1}^{3} \left(c_t - \frac{1}{100}c_t^2 \right)$$

This is a concave utility function, symmetric across periods, and **no interest rate or discounting** optimal consumption is equal across periods:

Let $c_1 = c_2 = c_3 = c$, then:

$$3c = 56 \Rightarrow c = \frac{56}{3} \approx 18.67$$

Answer: - $c_1 = c_2 = c_3 \approx 18.67$

1.2.3 Government announces in period 1 that a 20% income tax will be levied in periods 2 and 3

Adjusted income: - $y_1 = 20$ - $y_2 = y_3 = 20 \times (1 - 0.2) = 16$

Budget constraint:

$$c_1 + c_2 + c_3 = 20 + 16 + 16 = 52$$

Again, utility is symmetric, so optimal consumption is equal across all periods:

$$3c = 52 \Rightarrow c = \frac{52}{3} \approx 17.33$$

Answer: - $c_1 = c_2 = c_3 \approx 17.33$

1.2.4 No announcement, tax 20% unexpectedly applied in periods 2 and 3

In this case, the consumer **does not anticipate the tax** when choosing c_1. So she expects full income in all periods: - Expected income: $y_1 = y_2 = y_3 = 20 \Rightarrow \text{total} = 60$ - Optimal planned consumption: $c_1 = c_2 = c_3 = 60/3 = 20$

But in periods 2 and 3, actual income is only 16 each (due to tax), so consumer faces a **budget shortfall**:

- $c_1 = 20$
- Income in t=2,3 is 16; consumption was planned as 20 needs to adjust

Let's assume the consumer **re-optimizes consumption** in periods 2 and 3 under new constraint:

Remaining resources in t=2,3: $16 + 16 = 32$

She wants to equalize consumption in periods 2 and 3:

$$2c = 32 \Rightarrow c = 16$$

So actual consumption becomes: - $c_1 = 20$ - $c_2 = c_3 = 16$

Answer: - $c_1 = 20$ - $c_2 = c_3 = 16$

Summary Table

Scenario	c_1	c_2	c_3
(1) Given: $c_1 = 15, c_2 = 20, c_3 = 20$	15	20	20
(2) Tax in period 1 only	18.67	18.67	18.67
(3) Tax announced in t=1 for t=2 and t=3	17.33	17.33	17.33
(4) Tax applied in t=2 and t=3 unexpectedly (no announcement)	20	16	16

1.3 Task 3: Data Economy

We are given an economy with **two identical firms** using the production technology:

$$Y_i = A \cdot K_i, \quad \text{for } i = 1, 2$$

Common Technology: Both firms share a common productivity factor:

$$A = d^\eta = (0.5Y_1 + 0.5Y_2)^\eta, \quad \eta = 0.5$$

Since $Y_i = A \cdot K_i$, we substitute:

$$A = (0.5AK_1 + 0.5AK_2)^\eta = (0.5A(K_1 + K_2))^\eta$$

Let's denote $K = K_1 + K_2$, then:

$$A = (0.5A \cdot K)^\eta$$

1.3.1 Express A as a function of K_1 and K_2

From:

$$A = (0.5A(K_1 + K_2))^\eta$$

Raise both sides to the power $1/\eta$:

$$A^{1/\eta} = 0.5A(K_1 + K_2)$$

Substitute $\eta = 0.5 \Rightarrow 1/\eta = 2$:

$$A^2 = 0.5A(K_1 + K_2)$$

Divide both sides by A (assuming $A > 0$):

$$A = 0.5(K_1 + K_2)$$

Answer (1.3.1):

$$A = 0.5(K_1 + K_2)$$

1.3.2 Optimization Problem

Firm i chooses K_i to maximize:

$$\pi_i = A \cdot K_i - R \cdot K_i$$

From (1), recall:

$$A = 0.5(K_1 + K_2)$$

Substitute into the profit function:

$$\pi_1 = 0.5(K_1 + K_2) \cdot K_1 - 0.2 \cdot K_1$$

Take first-order condition (FOC) with respect to K_1:

$$\frac{d\pi_1}{dK_1} = 0.5K_1 + 0.5K_2 - 0.2 = 0 \Rightarrow K_1 + K_2 = 0.4$$

By symmetry (identical firms): $K_1 = K_2 = K$

So:

$$2K = 0.4 \Rightarrow K = 0.2$$

Answer (1.3.2):

$$K_1 = K_2 = 0.2$$

Final Summary

- (1) Productivity is:

$$A = 0.5(K_1 + K_2)$$

- (2) Optimal capital allocations:

$$K_1 = K_2 = 0.2, \quad A = 0.5(0.2 + 0.2) = 0.2$$

Profit for each firm:

$$\pi_i = A \cdot K_i - R \cdot K_i = 0.2 \cdot 0.2 - 0.2 \cdot 0.2 = 0$$

Each firm earns **zero profit** at the optimum under this technology specification and capital cost.
