Introduction to Online Convex Optimization

MMDM 2023

HSE University

Data Science MSc

HOME ASSIGNMENT 1 :

Hedge algorithm for linear losses on the simplex

Consider the OCO problem with

→ **Decision Space :** $K = \Delta_d$

where Δ_d is the d-dimensional probability simplex

$$\Delta_d = \left\{ x = (x(1), \dots, x(d)) \in \mathbb{R}^d : \forall i,\ x(i) \in [0,1] \text{ and } \sum_{i=1}^{d} x(i) = 1 \right\}.$$

→ **Loss Class :**

$$\mathcal{F} = \left\{ x \in \Delta_d \mapsto \langle x, \ell \rangle : \ell = (\ell(1), \dots, \ell(d)) \in [0,1]^d \right\}.$$

In Lecture 1, we've shown how this version of the OCO problem was connected to the important problem of "prediction with expert advice".

We've also introduced a simple version of the "Hedge" algorithm.

The main drawback of the version presented in Lecture 1 was that the choice of the tuning parameter "ε" (see Lecture 1) had to be adapted to a known time horizon T.

The goal of this home assignment is to analyse a more general version of the Hedge algorithm that involves a time varying parameter "ε_t", independent of a prescibed time horizon, and guaranteed to have low regret at any time.

The algorithm we want to analyse is as follows

Algorithm : General Hedge

Require : $(\varepsilon_t)_{t\geq 1}$

(Tuning parameters, $\varepsilon_t > 0$)

Initialization : $L_0 = (L_0(1), \dots, L_0(d)) = (0, \dots, 0)$

(Cumulative loss function at time 0)

Iterate : For $t \geqslant 1$

- Play $x_t = (x_t(1), \dots, x_t(d)) \in \Delta_d$ where

$$x_t(i) := \frac{e^{-\varepsilon_t L_{t-1}(i)}}{\sum_{j=1}^d e^{-\varepsilon_t L_{t-1}(j)}}$$

- Receive loss $\ell_t = (\ell_t(1), \dots, \ell_t(d)) \in [0,1]^d$ and update

$$L_t := L_{t-1} + \ell_t .$$

Remark : To connect this version of the algorithm with the one discussed in Lecture 1, note that the above algorithm can be equivalently described as follows :

→ Initialization : $w_1 = (w_1(1), \dots, w_1(d)) = (1, \dots, 1)$

→ For $t \geqslant 1$:

- Play x_t , where $x_t(i) = \dfrac{w_t(i)}{\sum_{j=1}^d w_t(j)}$
- Receive ℓ_t

  and update

$$w_{t+1}(i) = \left(w_t(i) \right)^{\frac{\varepsilon_{t+1}}{\varepsilon_t}} . \, e^{-\varepsilon_{t+1} \ell_t(i)}$$

$$w_t(i) = e^{-\varepsilon_t L_{t-1}(i)}$$

Note that we recover the simple Hedge algorithm whenever $\varepsilon_t = \varepsilon$, $\forall t \geq 1$. $\left(w_{t+1}(i) = w_t(i)\, e^{-\varepsilon \ell_t(i)} \right)$.

The goal of the home assignement is to show the following result.

Theorem.

i) Suppose that $0 < \varepsilon_{t+1} \leq \varepsilon_t$, $\forall t \geq 1$. Then, $\forall T \geq 1$, the Hedge algorithm with time varying parameter $(\varepsilon_t)_{t \geq 1}$ satisfies, $\forall T \geq 1$:

$$R_T \leq \frac{1}{8} \sum_{t=1}^{T} \varepsilon_t + \frac{\log d}{\varepsilon_{T+1}}$$

ii) In particular, chosing

$$\varepsilon_t := \sqrt{\frac{8 \log d}{t}}$$

implies that, $\forall T \geq 1$,

$$R_T \leq \sqrt{2T \log d}$$

We divide the proof in 5 steps:

1. Define $\displaystyle W_t := \frac{1}{d} \sum_{i=1}^{d} e^{-\varepsilon_t L_{t-1}(i)}$, $\forall t \geq 1$.

Show that $\forall T \geq 1$

$$\frac{\log W_{T+1}}{\varepsilon_{T+1}} - \frac{\log W_1}{\varepsilon_1} \geq - \inf_{x \in \Delta_d} \sum_{t=1}^{T} \langle x, \ell_t \rangle - \frac{\log d}{\varepsilon_{T+1}}.$$

2. Show that $\forall T \geqslant 1$

$$\frac{\log W_{T+1}}{\varepsilon_{T+1}} - \frac{\log W_1}{\varepsilon_1} = \sum_{t=1}^{T} (a_t + b_t)$$

where $\forall t \geqslant 1$:

$\rightarrow$ $$a_t := \frac{1}{\varepsilon_t} \log\left(\frac{W_{t+1}^{\frac{\varepsilon_t}{\varepsilon_{t+1}}}}{\widetilde{W}_{t+1}} \right),$$

$\rightarrow$ $$b_t := \frac{1}{\varepsilon_t} \log\left(\frac{\widetilde{W}_{t+1}}{W_t} \right),$$

$\rightarrow$ $$\widetilde{W}_{t+1} := \frac{1}{d} \sum_{i=1}^{d} e^{-\varepsilon_t L_t(i)}.$$

3. Show that, $\forall t \geqslant 1$, $a_t \leqslant 0$.

Hint: Use Jensen's inequality, noticing that the function $u \in \mathbb{R} \mapsto u^\alpha$ is convex provided $\alpha \geqslant 1$.

4. Show that, $\forall t \geqslant 1$,

$$b_t \leqslant \frac{\varepsilon_t}{8} - \langle x_t, \ell_t \rangle.$$

Hint: Use Hoeffding's lemma, stating that for any $[a, b]$-valued random variable X, $\forall \lambda \in \mathbb{R}$,

$$\log \mathbb{E}[e^{\lambda X}] \leqslant \frac{\lambda^2 (b-a)^2}{8} + \lambda \mathbb{E}[X].$$

5. Show that statements i) and ii) hold true by combining the 4 previous results.

Instructions

→ Return your solutions before Feb. 20, 2023

→ Options for returning your work
- Drop it in person at my office (S943)
- Upload on a Google Drive folder, (Link will be sent) in PDF format, with file name "mmdm_ha1_familyname(s)"

Preferred

→ You can either type your answers (LateX : preferred) or write them on a tablet. You can also hand write & scan your answers but only if your hand writing and scan quality are good.

→ Finally, you can work in groups of up to 3. In this case, mention clearly the names of all members of the group. Of course, it is understood that all members of the group will get the same grade.