Robustness

① assumed model has reasonable efficiency (low var)

② small deviations from model assumptions should impair the performance only slightly

③ somewhat larger deviations from model should not cause a catastrophe.

e.g. sample mean

- model misspecified: data is cauchy, but assume normal

- model correctly specified, but extreme values.

M-estimators –

criterion: mean estimator. minimizer: $\min \sum (x_i - a)^2$

Define a criterion function: $\rho$

$$\sum \rho (x_i - a)$$

$$\rho(x) = \begin{cases} \frac{1}{2} x^2 & , \text{ if } |x| \leq k \\ k|x| - \frac{1}{2} k^2 & , \text{ if } |x| > k \end{cases}$$

Define $\psi = \rho'$

score function is then:

$$\sum \psi (x_i - \theta) = 0 \qquad \text{Let score} = 0, \Rightarrow \hat{\theta}$$

类似地，将 Score 泰勒在真值 $\theta_0$ 展开:

$$\underline{\sum \psi(x_i - \theta)} = \sum \psi(x_i - \theta_0) + (\theta - \theta_0) \frac{\sum \psi'(x_i - \theta_0)}{1!} + \cdots$$

$\underset{=0}{}$

$\therefore -(\theta - \theta_0) = \dfrac{\frac{1}{n} \sum \psi(x_i - \theta_0)}{\frac{1}{n} \sum \psi'(x_i - \theta_0)}$ multiply by $\sqrt{n}$ : $\Longrightarrow \sqrt{n}(\theta - \theta_0) = \dfrac{-\frac{1}{\sqrt{n}} (\sum \psi(x_i - \theta_0))}{\frac{1}{n} \sum \psi'(x_i - \theta_0)}$

其中① by CLT $\underline{\sqrt{n}(\frac{1}{n} \sum \psi(x_i - \theta_0))} \sim N( E(\psi(x_i - \theta_0)), \text{ Var}(\psi(x_i - \theta_0)) )$

where $E(\psi(x_i - \theta_0)) = 0$, $\text{Var}(\psi(x_i - \theta_0)) = -E(\psi'(x_i - \theta_0)) = E(\psi(x - \theta_0)^2)$

② $\frac{1}{n} \sum \psi'(x_i - \theta_0)$ by WLLN. $\xrightarrow{P} E(\psi'(x_i - \theta_0))$