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Advanced Machine Learning

(COMP 5328)

Sparse Coding and Regularisation

Tongliang Liu

Quiz results

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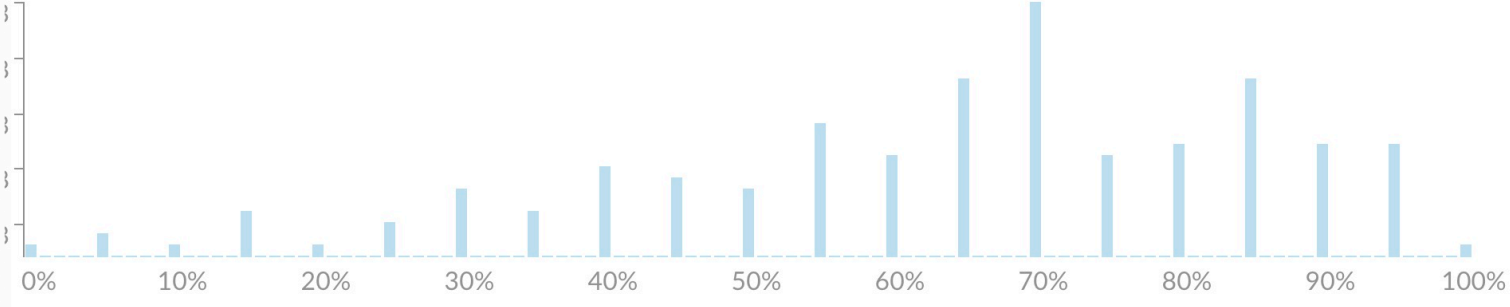
Average Score	High Score	Low Score	Standard Deviation	Average Time
63%	100%	0%	22.28	49:21



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Review

Dictionary learning

What is a dictionary in machine learning?

Let $x \in \mathbb{R}^d$, $D \in \mathbb{R}^{d \times k}$

$$\alpha^* = \arg\min_{\alpha \in \mathbb{R}^k} \|x - D\alpha\|^2.$$

Note that $\|x\| = \sqrt{x^\top x}$ is the ell 2 norm.

Given $x_1, \ldots, x_n \in \mathbb{R}^d$

$$\{D^*, \alpha_1^*, \ldots, \alpha_n^*\} = \arg\min_{D \in \mathbb{R}^{d \times k}, \alpha_1, \ldots, \alpha_n \in \mathbb{R}^k} \frac{1}{n} \sum_{i=1}^{n} \|x_i - D\alpha_i\|^2.$$

Dictionary learning

Note that

$$\frac{1}{n}\sum_{i=1}^{n}\|x_i - D\alpha_i\|^2 = \frac{1}{n}\|X - DR\|_F^2,$$

where $X = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^{d\times n}$,

$R = [\alpha_1, \alpha_2, \ldots, \alpha_n] \in \mathbb{R}^{k\times n}$,

$\|X\|_F = \sqrt{\mathrm{trace}(X^\top X)} = \sqrt{\sum_{i=1}^{d}\sum_{j=1}^{n} X_{i,j}^2}$ is the Frobenius norm of X.

Dictionary learning

Note that

$$\arg \min_{D \in \mathcal{D}, R \in \mathcal{R}} \|X - DR\|_F^2,$$

where $\mathcal{D}$ and $\mathcal{R}$ are some specific domains for D and R.

Optimisation

Objective:

$$\min_{D \in \mathcal{D}, R \in \mathcal{R}} \|X - DR\|_F^2$$

The objective is convex with respect to either R or D but not to both.

Fix R, solve for D

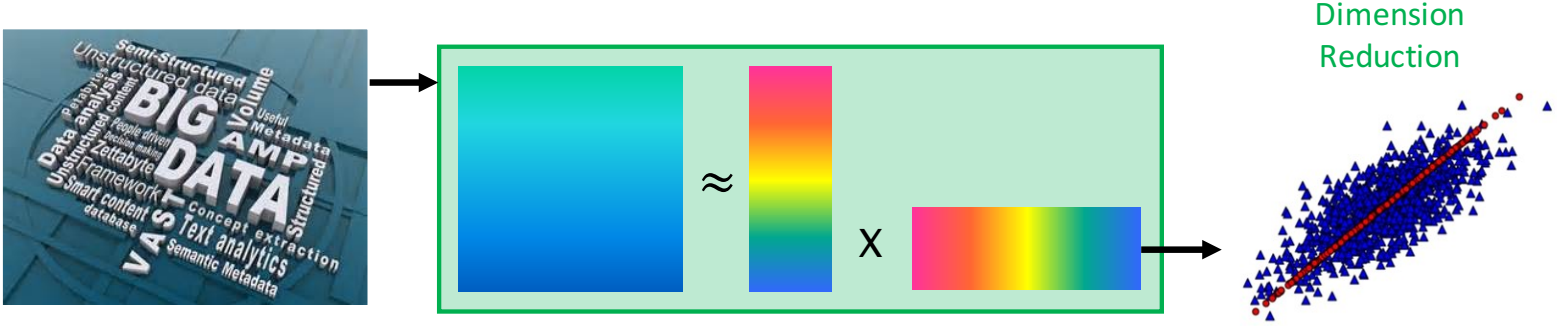
$$\min_{D \in \mathcal{D}} \|X - DR\|_F^2$$

Fix D, solve for R

$$\min_{R \in \mathcal{R}} \|X - DR\|_F^2$$

Engan, Kjersti, Sven Ole Aase, and J. Hakon Husoy. "Method of optimal directions for frame design." Acoustics, Speech, and Signal Processing, 1999. Proceedings., 1999 IEEE International Conference on. Vol. 5. IEEE, 1999.

Dictionary learning application



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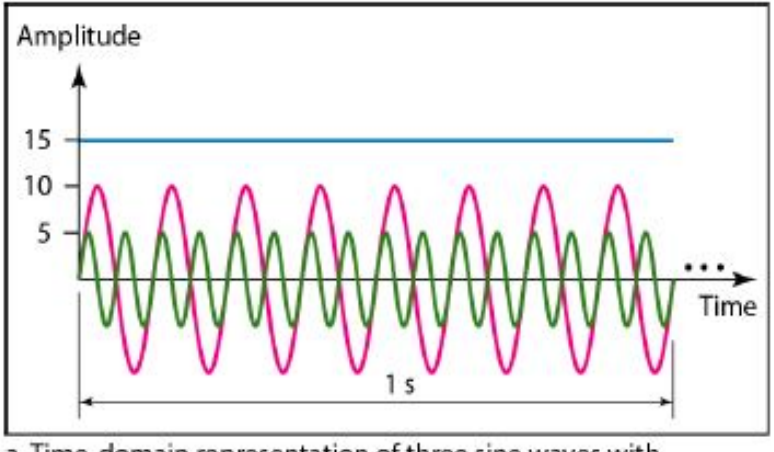
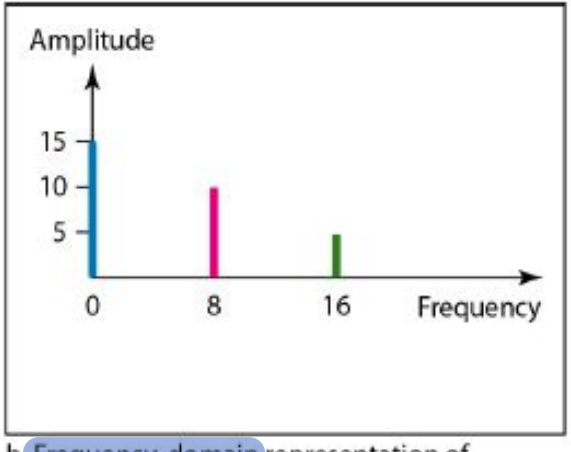
Sparse Coding

Why sparse?

Many signals are sparse in some transform domain.



a. Time-domain representation of three sine waves with frequencies 0, 8, and 16



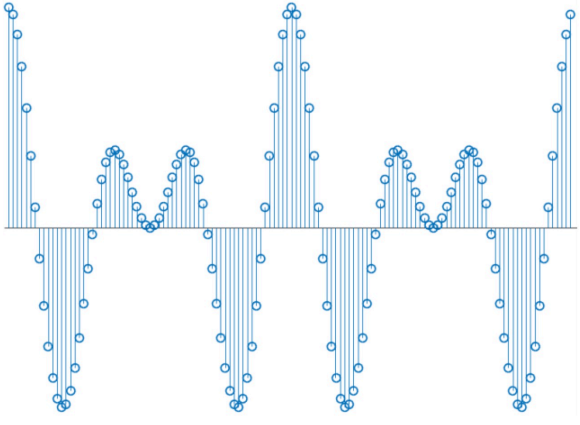
b. Frequency-domain representation of the same three signals

sparse

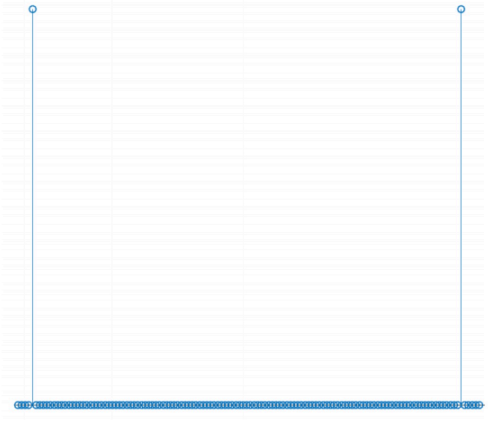
Image credit: https://datacommandnet.blogspot.com/p/periodic-analog-signals.html

Why sparse?

Many signals are sparse in some transform domain.



time-representation of $\boldsymbol{y}_1$

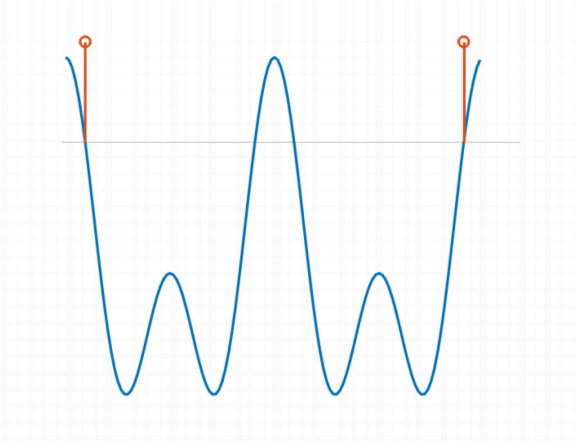


frequency-representation of $\boldsymbol{y}_1$

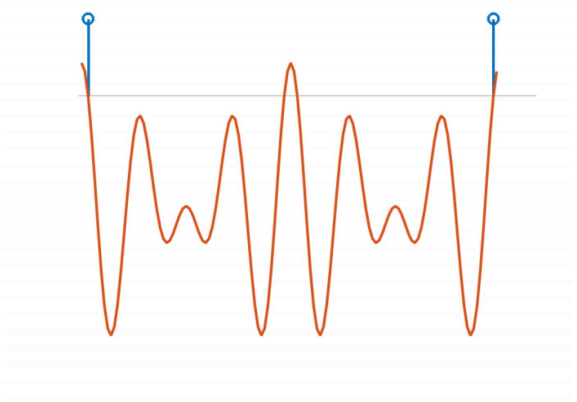
Image credit: http://www.princeton.edu/~yc5/ele538b_sparsity/lectures/sparse_representation.pdf

Why sparse?

Many signals are sparse in some transform domain.



time representation of $\boldsymbol{y}_2$



frequency representation of $\boldsymbol{y}_2$

Image credit: http://www.princeton.edu/~yc5/ele538b_sparsity/lectures/sparse_representation.pdf

Why sparse?

Many signals are sparse in some transform domain.

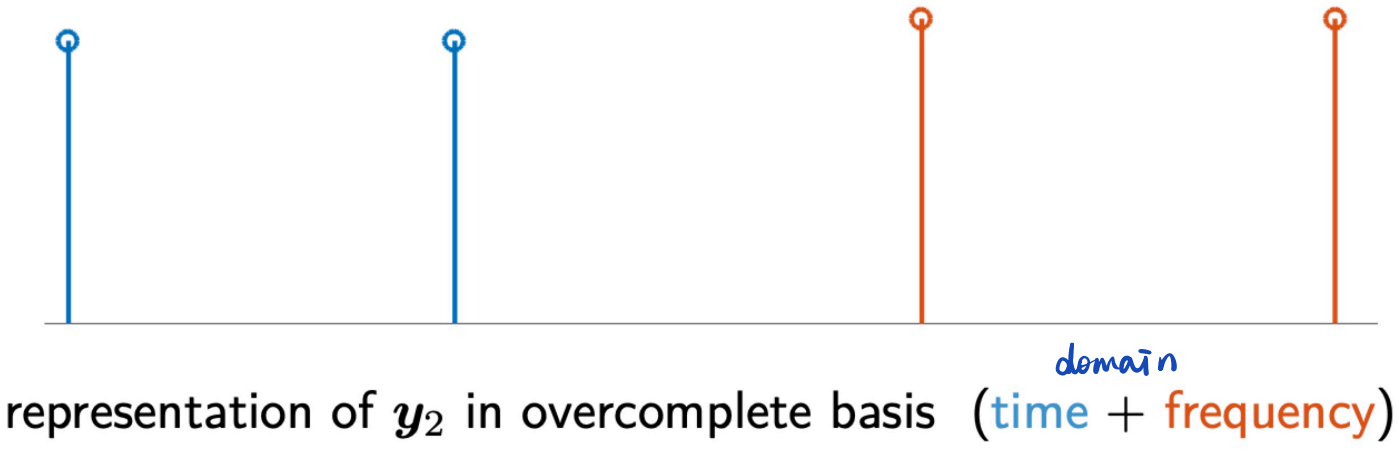
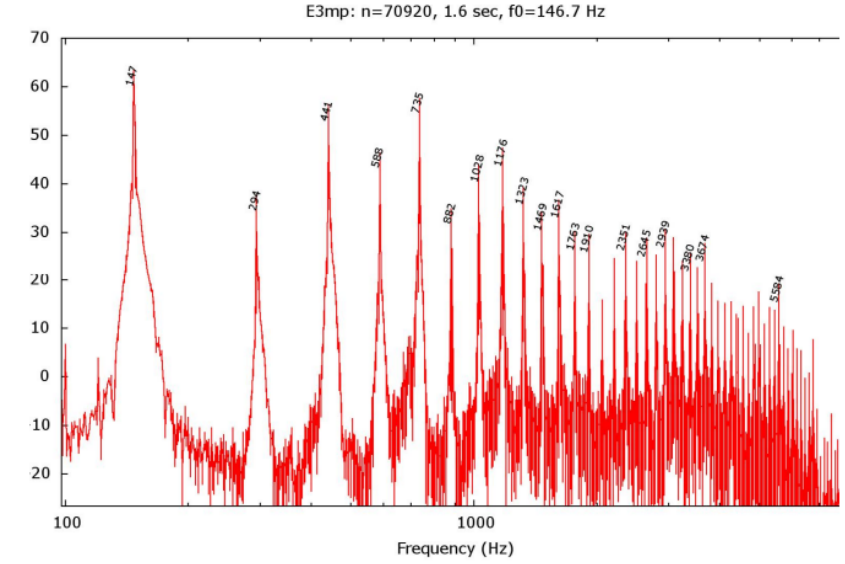


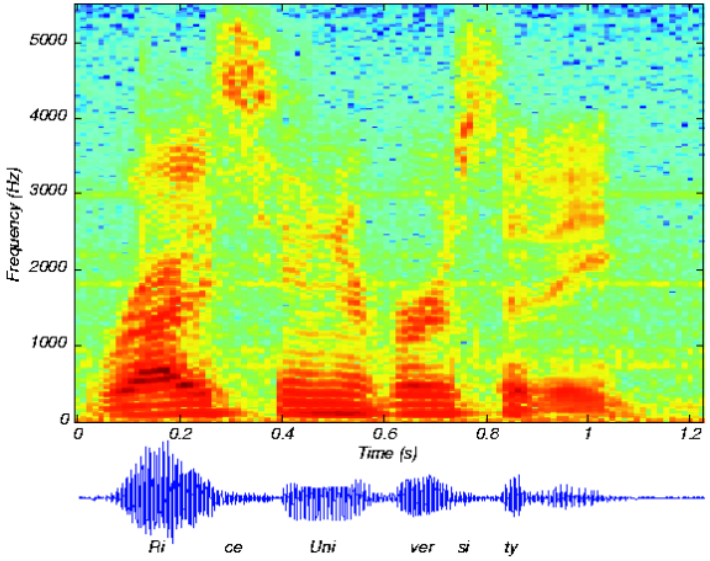
Image credit: http://www.princeton.edu/~yc5/ele538b_sparsity/lectures/sparse_representation.pdf

Why sparse?

Many signals are sparse in some transform domain.



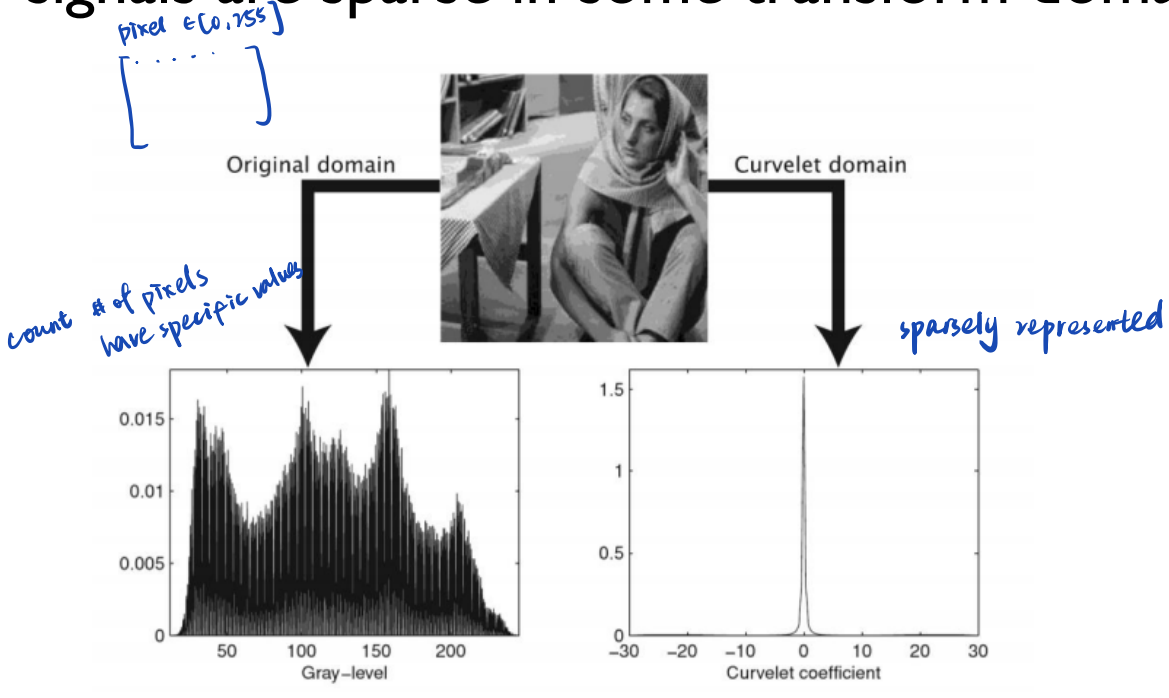
Musical instrument spectrum



Speech spectrogram

Why sparse?

Many signals are sparse in some transform domain.



Natural image

Image credit: Zaouali, Bouzidi, and Zagrouba: Review of multiscale geometric decompositions in a remote sensing context

Dictionary learning

Note that

$$\arg \min_{D \in \mathcal{D}, R \in \mathcal{R}} \|X - DR\|_F^2,$$

where $\mathcal{D}$ and $\mathcal{R}$ are some specific domains for D and R.

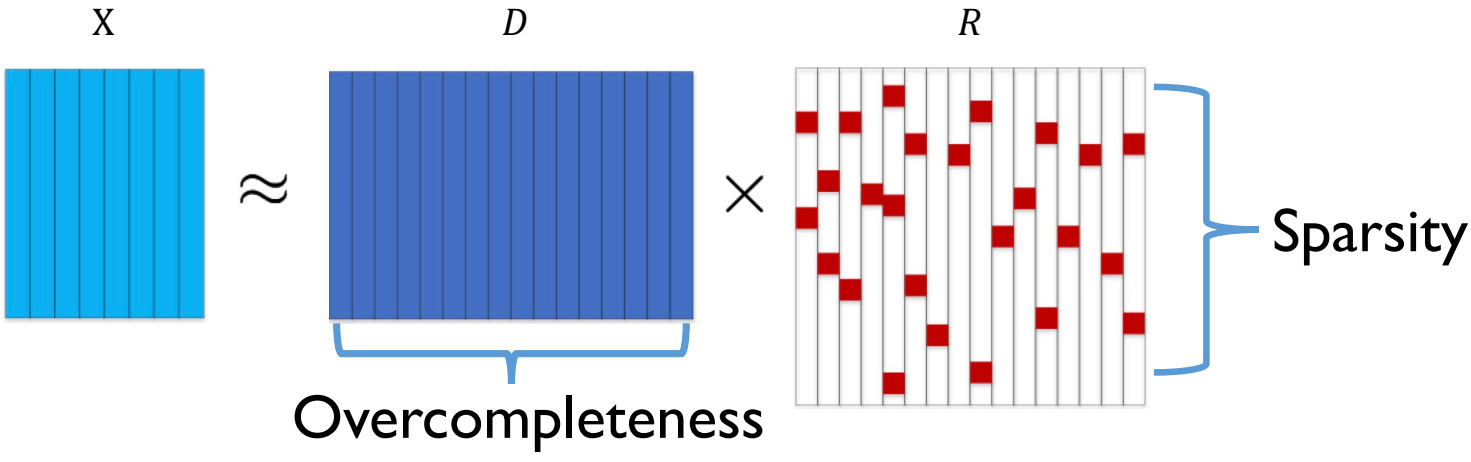
Sparse coding

Note that requires representation to be sparse.

$$\arg \min_{D \in \mathcal{D}, R \in \mathcal{R}} \|X - DR\|_F^2$$

dictionary contains information

majority value is 0 → sparsity



ℓ_p norm

ℓ_p norm: $\|\alpha\|_p = \left(\sum_{j=1}^{k} |\alpha_j|^p \right)^{1/p}$, where $\alpha \in \mathbb{R}^k$.

by default $\|\alpha\| = \|\alpha\|_2$ ℓ_2-norm

ℓ_0-norm, $\|\alpha\|_0$ counts non-zero entries

$\alpha = \begin{pmatrix} 3 \\ -7 \\ 0 \end{pmatrix}$ $\|\alpha\|_0 = 2$.

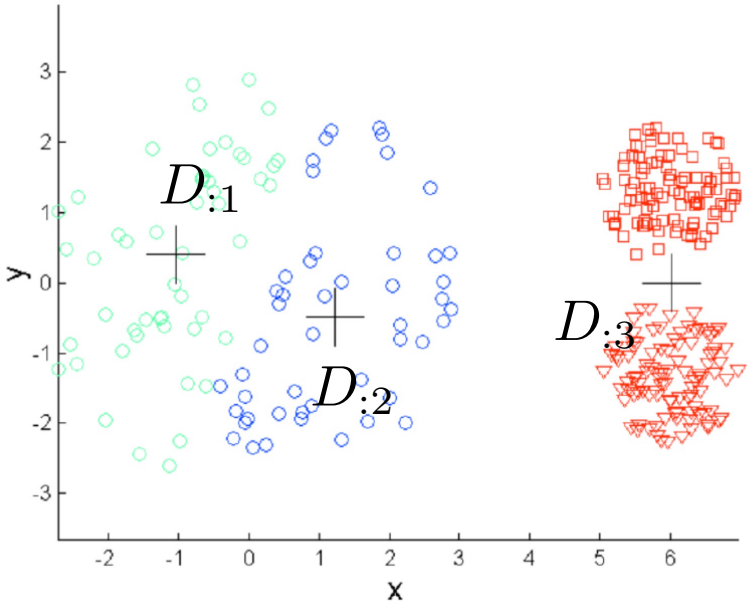
$\|\alpha\|_0 = 5$

In other words, $\|\alpha\|_p^p = \sum_{j=1}^{k} |\alpha_j|^p$.

K-means

K-means clustering:

$$\min_{D \in \mathcal{D}, R \in \mathcal{R}} \|X - DR\|_F^2$$



Special requirement: each column of R is an one-hot vector, i.e., $\|R_i\|_0 = 1 \ \& \ \|R_i\|_1 = 1.$

only 1 entries are non-zero, the non-zero entry's value = 1

K-SVD

K-SVD: $$\min_{D \in \mathcal{D}, R \in \mathcal{R}} \|X - DR\|^2$$

Special requirement: each column of R is a sparse vector, i.e., $\|R_i\|_0 \leq k' << k$.

k is small

& also new representation to be sparse

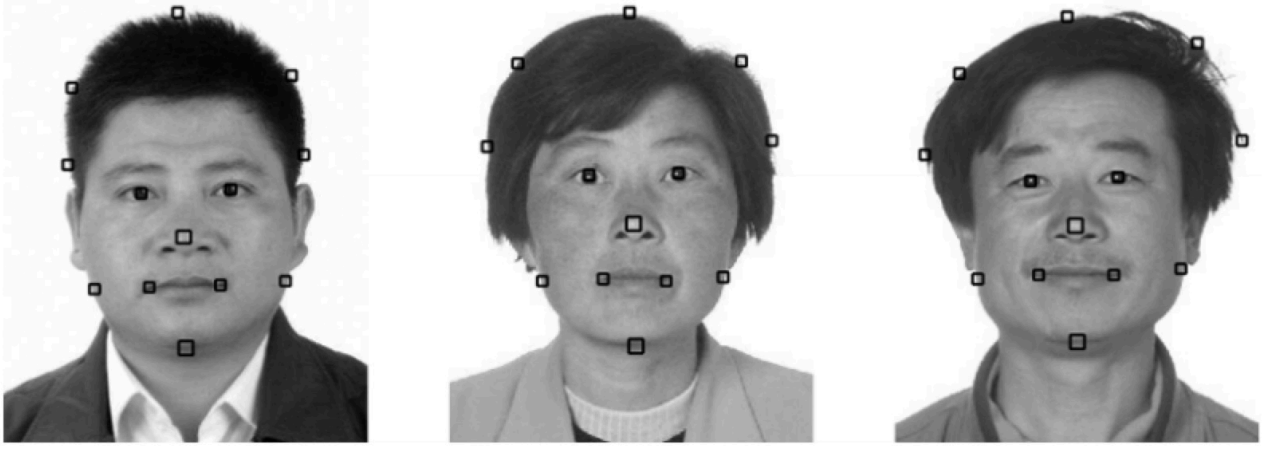
Sparse coding applications

Image Compression: Results for 820 bytes per image

PCA
just keep eigenvectors with large eigenvalue
x
project data to eigenvector space

Image credit: Compression of facial images using the K-SVD algorithm, J. Vis. Comun. Image Represent.

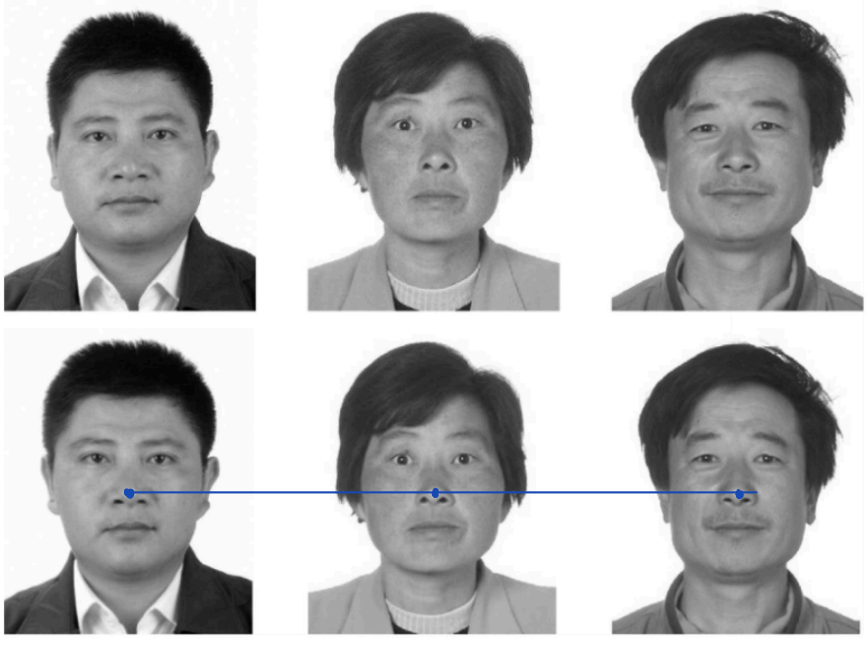
Data pre-processing for K-SVD



Facial feature points. → these feature points have specific location in the human face.

Image credit: CLow Bit-Rate Compression of Facial Images, IEEE Transactions on Image Processing

Data pre-processing for K-SVD

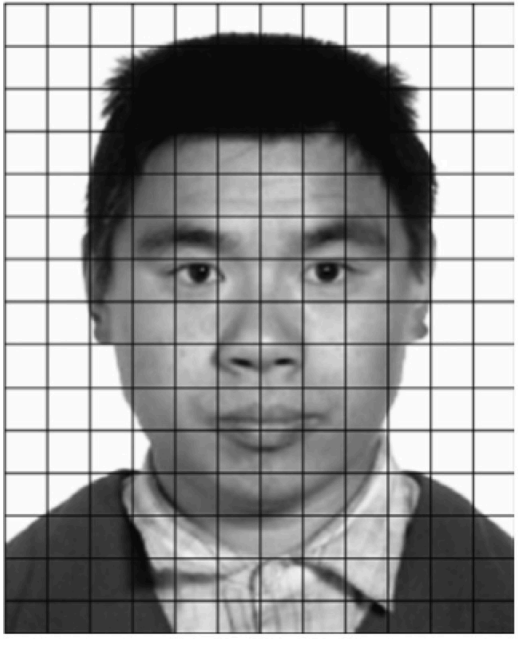


align feature point

(Top) Input images and their (canonical) aligned (bottom) versions.

Image credit: CLow Bit-Rate Compression of Facial Images, IEEE Transactions on Image Processing

Data pre-processing for K-SVD



Uniform slicing

Image credit: Compression of facial images using the K-SVD algorithm, J. Vis. Comun. Image Represent.

Sparse coding applications

Dictionary D can be learned based on { clear dataset / a mixture of clear and polluted data

Inpainting:

⇒ to restore the polluted data

Image credit: http://www.cs.technion.ac.il/~ronrubin/Talks/K-SVD.ppt

Sparse coding applications

Inpainting:

Image credit: http://www.cs.technion.ac.il/~ronrubin/Talks/K-SVD.ppt

Sparse coding applications

Inpainting for text removal:

Image credit: Sparse representation for color image restoration. IEEE Transactions on Image Processing.

Sparse coding applications

Inpainting for text removal:

Image credit: Sparse representation for color image restoration. IEEE Transactions on Image Processing.

Measure of Sparsity

The objective function:

soft constraint: regularisation should be small, but not smaller than some specific value

a smaller value of $\psi(R)$ means R is more sparse

$$\min_{D \in \mathcal{D}, R \in \mathcal{R}} \|X - DR\|_F^2 + \lambda \psi(R)$$

ALSO ψ should be differentiable

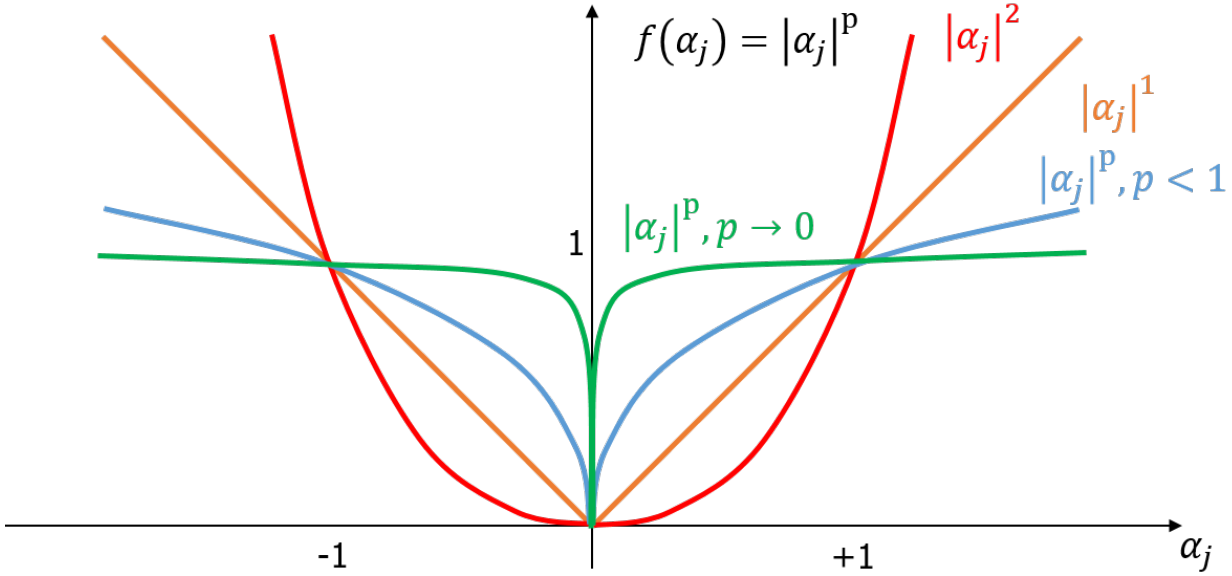
\psi

Data fitting Sparse regularisation

Question: how can we design the regularisation to make *R* to be sparse?

Measure of Sparsity: ℓ_0 norm

$$\|\alpha\|_p^p = \sum_{j=1}^{k} |\alpha_j|^p.$$



As $p \to 0$, we get a count of the non-zeros in the vector.

So we can employ $\|\alpha\|_0$ to measure sparsity.

Measure of Sparsity: ℓ_0 norm

As $p \rightarrow 0$, we get a count of the non-zeros in the vector.
So we can employ $\|\alpha\|_0$ to measure sparsity.

However, the ℓ_0 minimisation is not easy. How to do?

Measure of Sparsity ℓ_1 norm

2D example (compared with ℓ_2-norm):

good replacement for l_0 norm

$$\min_{\alpha} \frac{1}{2}\|x-\alpha\|_2^2 \;\; s.t. \|\alpha\|_1 \leq \mu$$

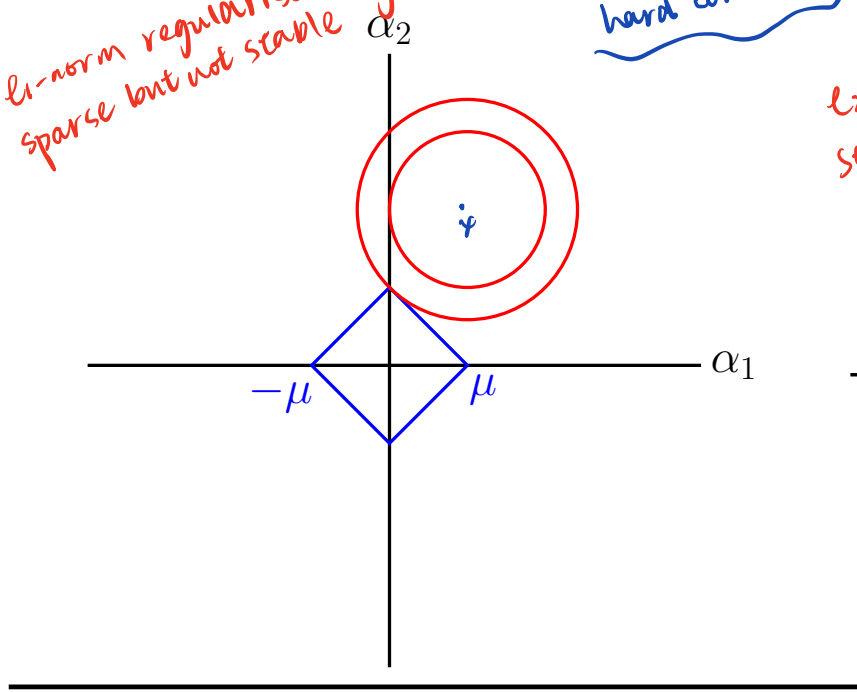
$$\min_{\alpha} \frac{1}{2}\|x-\alpha\|_2^2 \;\; s.t. \|\alpha\|_2 \leq \mu$$

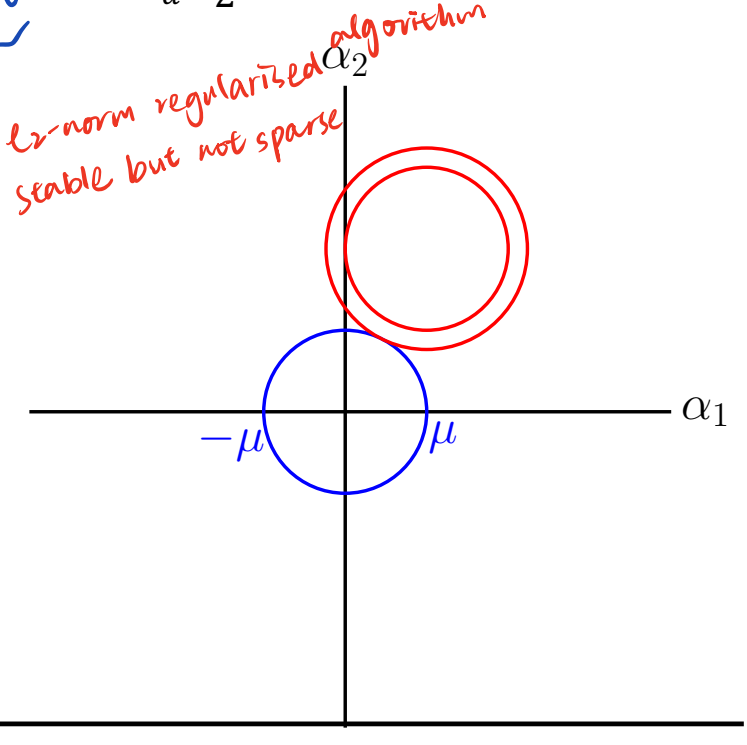
Measure of Sparsity ℓ_1 norm

2D example (compared with ℓ_2-norm):

$$\min_{\alpha} \frac{1}{2}\|x-\alpha\|_2^2 \;\; s.t. \; \|\alpha\|_1 \leq \mu$$

$$\min_{\alpha} \frac{1}{2}\|x-\alpha\|_2^2 \;\; s.t. \; \|\alpha\|_2 \leq \mu$$





Sparse coding learning algorithms

The ℓ_0 norm based approaches:

- $\min_{\alpha} \|X - D\alpha\|_F^2 \ \ s.t. \forall\, i, \|\alpha\|_0 < L.$

- $\min_{\alpha} \|\alpha\|_0 \ \ \text{s.t.} \ \|X - D\alpha\|_F^2 \leq \epsilon.$

Greedy algorithms:

- OMP (Y. Pati, et al. 1993; J. Tropp 2004).
- Subspace pursuit (SP) (W. Dai and O. Milenkovic 2009), CosaMP (D. Needell and J. Tropp 2009).
- IHT (T. Blumensath and M. Davies 2009).

Sparse coding learning algorithms

The ℓ_1 norm based approaches:

- $\min_{\alpha} \|\alpha\|_1 \ \text{s.t.} \ \|X - D\alpha\|_F^2 \leq \epsilon.$
- $\min_{\alpha} \|X - D\alpha\|_F^2 + \lambda \|\alpha\|_1.$

Bayesian approach:

- Relevance vector machine (RVM) (M. Tipping 2001).
- Bayesian compressed sensing (BCS) (S. Ji, et al. 2008).

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Regularisation and algorithmic stability

No-Free-Lunch Theorem

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- Sparse algorithms are not stable!
- A learning algorithm is said to be stable if slight perturbations in the training data result in small changes in the output of the algorithm, and these changes vanish as the data set grows bigger and bigger.

Xu, H., Caramanis, C., & Mannor, S. (2012). Sparse algorithms are not stable: A no-free-lunch theorem. IEEE transactions on pattern analysis and machine intelligence, 34(1), 187-193.

Algorithmic Stability

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We have two different training samples:

only difference

$$S = \{(X_1, Y_1), \ldots, (X_{i-1}, Y_{i-1}), (X_i, Y_i), (X_{i+1}, Y_{i+1}), \ldots, (X_n, Y_n)\}$$

$$S^i = \{(X_1, Y_1), \ldots, (X_{i-1}, Y_{i-1}), (X_i', Y_i'), (X_{i+1}, Y_{i+1}), \ldots, (X_n, Y_n)\}$$

They are different because of only one training example.

An algorithm is uniformly stable if for any example (X, Y)

$$|\ell(X, Y, h_S) - \ell(X, Y, h_{S^i})| \leq \epsilon(n).$$

Note that $\epsilon(n)$ will vanish as n goes to infinity.

$n \to \infty$

$\epsilon(n) \to 0$

Generalisation error

$$
\begin{aligned}
&R(h_S) - \min_{h \in H} R(h) = R(h_S) - R(h^*) \\
&= R(h_S) - R_S(h_S) + R_S(h_S) - R_S(h^*) + R_S(h^*) - R(h^*) \\
&\leq R(h_S) - R_S(h_S) + R_S(h^*) - R(h^*) \\
&\leq |R(h_S) - R_S(h_S)| + |R(h^*) - R_S(h^*)| \\
&\leq \sup_{h \in H} |R(h) - R_S(h)| + \sup_{h \in H} |R(h) - R_S(h)| \\
&= 2 \sup_{h \in H} |R(h) - R_S(h)|.
\end{aligned}
$$

$$
R(h_S) - R_S(h_S) \leq \sup_{h \in H} |R(h) - R_S(h)|.
$$

Algorithmic Stability

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A good stable algorithm will have a good generalisation ability:

$$
\begin{aligned}
&\mathbb{E}[R(h_S) - R_S(h_S)] \\
&= \mathbb{E}_S \left[\mathbb{E}_{X,Y}[\ell(X, Y, h_S)] - \frac{1}{n} \sum_{i=1}^{n} \ell(X_i, Y_i, h_S) \right] \quad \checkmark \\
&= \mathbb{E}_S \left[\mathbb{E}_{S'} \left[\frac{1}{n} \sum_{i=1}^{n} \ell(X'_i, Y'_i, h_S) \right] - \frac{1}{n} \sum_{i=1}^{n} \ell(X_i, Y_i, h_S) \right] \\
&= \mathbb{E}_{S,S'} \left[\frac{1}{n} \sum_{i=1}^{n} (\ell(X'_i, Y'_i, h_S) - \ell(X_i, Y_i, h_S)) \right] \\
&= \mathbb{E}_{S,S'} \left[\frac{1}{n} \sum_{i=1}^{n} (\ell(X'_i, Y'_i, h_S) - \ell(X'_i, Y'_i, h_{S^i})) \right] \\
&\leq \epsilon'(n)
\end{aligned}
$$

$\epsilon'(n)$ would be small if the learning algorithm is stable (because h_S and h_{S^i} are similar). This implies that stable algorithms will have small expected generalisation errors.

where $S' = \{(X'_1, Y'_1), \ldots, (X'_n, Y'_n)\}$.

Regularisation and Stability

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ℓ_2 norm regularisation will make learning algorithms stable if the employed surrogate loss function is convex.

$$h_S = \arg\min_{h \in H} \frac{1}{n} \sum_{i=1}^{n} \ell(X_i, Y_i, h) + \lambda \|h\|_2^2$$

If the convex surrogate loss function is L-Lipschitz continuous w.r.t. h, $\|X\|_2 \leq B$, we have

$$|\ell(X, Y, h_S) - \ell(X, Y, h_{S^i})| \leq \frac{2L^2B^2}{\lambda n}.$$

Proof (optional)

A surrogate loss function is L-Lipschitz continuous w.r.t. h if for any (X, Y) in the domain and any $h, h' \in H$

$$|\ell(X, Y, h) - \ell(X, Y, h')| \leq L|h(X) - h'(X)|.$$

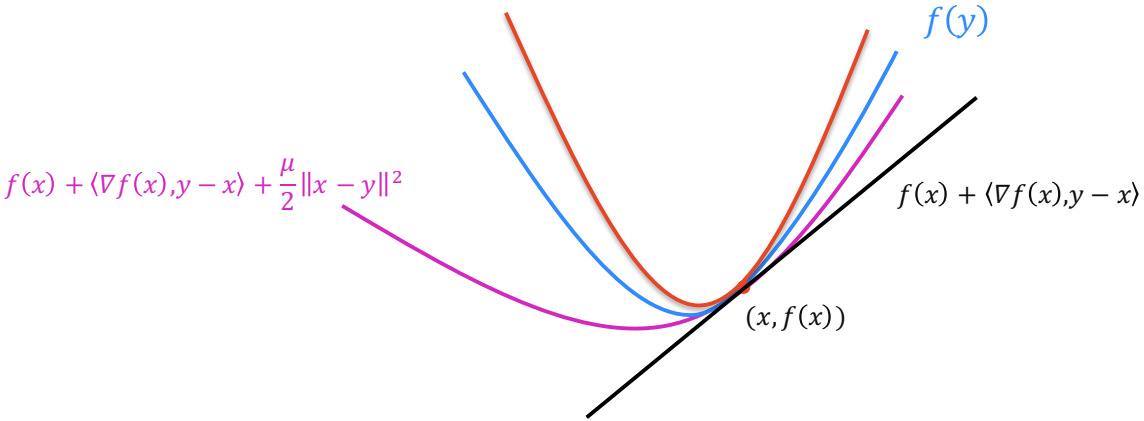
Proof (optional)

A function is μ-strongly convex:

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2}\|x - y\|^2, \forall x, y,$$

$$\Leftrightarrow$$

$$\mu \boldsymbol{I} \preccurlyeq \nabla^2 f(x), \forall x$$



Proof (optional)

Note that the function $\ell(X, Y, h) + \lambda\|h\|^2$ is always strongly convex w.r.t. h .

Let

$$R_{S,\lambda}(h) = \frac{1}{n}\sum_{i=1}^{n} \ell(X_i, Y_i, h) + \lambda\|h\|^2.$$

We have

$$\begin{aligned} R_{S,\lambda}(h_{S^i}) &\geq R_{S,\lambda}(h_S) + \langle \nabla R_{S,\lambda}(h_S), h_{S^i} - h_S \rangle + \frac{\lambda}{2}\|h_{S^i} - h_S\|^2 \\ &= R_{S,\lambda}(h_S) + \frac{\lambda}{2}\|h_{S^i} - h_S\|^2 \\ R_{S^i,\lambda}(h_S) &\geq R_{S^i,\lambda}(h_{S^i}) + \left\langle \nabla R_{S^i,\lambda}(h_{S^i}), h_S - h_{S^i} \right\rangle + \frac{\lambda}{2}\|h_S - h_{S^i}\|^2 \\ &= R_{S^i,\lambda}(h_{S^i}) + \frac{\lambda}{2}\|h_S - h_{S^i}\|^2. \end{aligned}$$

Proof (optional)

We have

$$
\begin{aligned}
\lambda \|h_{S^i} - h_S\|^2 &\leq R_{S^i,\lambda}(h_S) - R_{S,\lambda}(h_S) + R_{S,\lambda}(h_{S^i}) - R_{S^i,\lambda}(h_{S^i}) \\
&\leq \frac{1}{n}|\ell(X_i', Y_i', h_S) - \ell(X_i', Y_i', h_{S^i})| + \frac{1}{n}|\ell(X_i, Y_i, h_S) - \ell(X_i, Y_i, h_{S^i})| \\
&\leq \frac{L}{n}|h_S(X_i') - h_{S^i}(X_i')| + \frac{L}{n}|h_S(X_i) - h_{S^i}(X_i)| \\
\text{aumme } h(X) = \langle h, X \rangle \quad &= \frac{L}{n}|\left\langle h_S - h_{S^i}, X_i' \right\rangle| + \frac{L}{n}|\left\langle h_S - h_{S^i}, X_i \right\rangle| \\
&\leq \frac{L}{n}\|h_S - h_{S^i}\|\|X_i'\| + \frac{L}{n}\|h_S - h_{S^i}\|\|X_i\| \\
&\leq \frac{2LB}{n}\|h_S - h_{S^i}\|,
\end{aligned}
$$

where the fifth inequality holds because of Cauchy-Schwartz inequality: $\langle a, b \rangle \leq \|a\|\|b\|$.

Proof (optional)

Thus,

$$\|h_{S^i} - h_S\| \leq \frac{2LB}{\lambda n}.$$

We then have

$$\begin{aligned} |\ell(X, Y, h_S) - \ell(X, Y, h_{S^i})| &\leq L|h_S(X) - h_{S^i}(X)| \\ &\leq L\|h_S - h_{S^i}\|\|X\| \\ &\leq \frac{2L^2B^2}{\lambda n}. \end{aligned}$$