Kurt Gödel, completeness, incompleteness

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Abstract. The famous results of Gödel – semantic completeness of classical (predicate) logic and deductive incompleteness of any "reasonable" arithmetic – are briefly presented for a reader able of elementary mathematical thinking.

This is a short paper not assuming familiarity with mathematical logic; the reader is only assumed to be able of elementary mathematical thinking. We shall formulate the basic theorems precisely but without elaborating all used notions and their properties; thus the reader will hopefully get some feeling on what Gödel's famous theorems say.

Kurt Gödel is undoubtedly the greatest mathematical logician of the XX. century. Born 1906 in Brno (Brünn, at that time Austrian-Hungarian monarchy, since 1918 Czechoslovakia, now Czech Republic), he moved in 1924 to Vienna to study at the university. In 1929 he presented his dissertation, containing his completeness theorem and in 1930 announced his famous incompleteness theorem; so he obtained his celebrated theorems rather young.

Before we formulate these theorems we have to sketch some facts on mathematical logic.

1. Mathematical logic is a mathematical theory of consequence of propositions. Propositions are well defined mathematical objects. One defines what it means that a proposition B is a consequence of a proposition A (or of a set T of propositions); A is the assumption (T is the set of assumptions) and B is the conclusion.

2. There are two notions of a consequence: syntactic – a proposition is provable from the assumptions, and semantic – preservation of truth: always when the assumptions are true then the consequence is also true. Provability and truth are mathematical notions concerning a logical calculus.

3. The main question reads: is being true equivalent with being provable? This question has two answers: yes and no - for two different notions of truth. The answers are given by the theorem of completeness and incompleteness.

4. Our discussion concerns the first order predicate calculus (first order logic): its propositions are built from atomic formulas using logical connectives $\rightarrow$ implication, & conjunction, $\vee$ disjunction, $\equiv$ equivalence, $\neg$ negation, and quantifiers $\forall, \exists$. Atomic formulas have the form $P(t_1, \ldots, t_n)$ where P is a predicate of arity (number of arguments) n, and $t_1, \ldots, t_n$ are variables, constants or expressions from them using function symbols. Our most important example:

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formulas of arithmetic use two predicates $=$ and $\leq$ (binary, i.e. having two arguments), one constant $\bar{0}$, one unary (one-argument-) function symbol S for successor (adding 1) and two binary function symbols $+$ (addition) and $*$ (multiplication). For example, $(\exists z)(x+y=z)$ is a formula (read " there exists a z such that $z = x + y$"; in it the variable z is *bound* by the existential quantifier whereas x and y are free. A formula is *closed* if all its variables are bound.

5. To be able to speak on truth or falsity of a formula one has to give a meaning (denotation) to its components. An *interpretation* of a predicate language is a structure $\mathbf{M} = (M, (r_P)_P, (f_F)_F, (m_c)_c)$, where $M \neq 0$ is the domain of objects on which the formula expresses something; for each predicate P having n arguments, r_P is the meaning of the predicate, i.e. a set of n-tuples of elements of M; and for each function symbol F (with n arguments), f_F is an operation assigning to each n-tuple of elements of M an element of M and m_c is just an element of M.

Example: the standard model of arithmetic – the structure of natural numbers. The domain is the set of all natural numbers, the predicates $=, \leq$, functions $S, +, *$ and constant $\bar{0}$ are interpreted as equality, the relation "less-than-or-equal-to", the operations of successor, addition, multiplication of natural numbers respectively and $\bar{0}$ is interpreted as the number 0.

6. There is a natural (Tarskian) definition of the notion "the formula φ is satisfied in the structure $\mathbf{M}$ by the sequence $\boldsymbol{a}$" – notation: $\mathbf{M} \models \varphi[\boldsymbol{a}]$. Further one defines $\mathbf{M} \models \varphi$ (read: φ is *true in* $\mathbf{M}$), meaning that it is satisfied by each sequence $\boldsymbol{a}$.

Example: $\mathbf{N} \models x_1 = x_2 + x_3, [7,4,3]$
$\mathbf{N} \not\models x_1 = x_2 + x_3, [3,3,3,]$
$\mathbf{N} \models x_1 + x_2 = x_2 + x_1$
$\mathbf{N} \models (\forall x_1)(\forall x_2)(\exists x_3)(x_1 = x_2 + x_3 \vee x_2 = x_1 + x_3)$

7. φ is a *tautology* if $\mathbf{M} \models \varphi$ for each $\mathbf{M}$. Certain tautologies are declared to be logical axioms. To be a logical axiom is algorithmically decidable. There are two *deduction rules*: modus ponens (from φ and $\varphi \to \psi$ infer ψ) and generalization (from φ infer $(\forall x)\varphi$). A logical *proof* is a sequence of formulas, each being an axiom or following from some preceding formulas by a deduction rule. A formula is *provable* if it is the last member of a proof. The calculus is *sound:* each provable formula is a tautology.

Let us list the *axioms* of the predicate calculus: for any formulas φ, ψ, χ of the predicate calculus,
$\varphi \to (\psi \to \varphi)$,
$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$,
$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$,
(axioms of propositional calculus) and further
$(\forall 1)$ $\quad (\forall x)\varphi(x) \to \varphi(y)$,
if y is "substitutable" for x into φ , (some technical condition)
$(\forall 2)$ $\quad (\forall x)(\nu \to \varphi) \to (\nu \to (\forall x)\varphi)$,
where x is not free in ν.

8. Gödel's theorem on the (semantic) completeness of the predicate logic says: *Each formula (of predicate calculus) is logically provable* **if and only if** *it is a tautology.* (Thus being logically provable = being true in all interpretations.)

This can be made stronger (and was also by Gödel in his dissertation). The stronger form deals with provability in theories over the predicate calculus.

9. A *theory* is a set of formulas (special axioms of the theory). $\mathbf{M}$ is a model of the theory T

if all $\varphi \in T$ are true in $\mathbf{M}$. The following theory is our important example (in the language of arithmetic) – *Robinson's arithmetic* Q :

$(Q1)$ $S(x) \neq \bar{0}$
$(Q2)$ $S(x) = S(y) \rightarrow x = y$
$(Q3)$ $x \neq 0 \rightarrow (\exists y)(x = S(y))$
$(Q4)$ $x + \bar{0} = x$
$(Q5)$ $x + S(y) = S(x + y)$
$(Q6)$ $x * \bar{0} = \bar{0}$
$(Q7)$ $x * S(y) = (x * y) + x$
$(Q8)$ $x \leq y \equiv (\exists z)(z + x = y)$

(All these axioms are true in the standard model $\mathbf{N}$.) Our second example is Peano arithmetic PA : its axioms are the axioms$(Q1) - (Q8)$ of Robinson arithmetic and, for each formula A (of the language of arithmetic), the induction axiom
$(A(\bar{0})\&(\forall x)(A(x) \rightarrow A(S(x)))) \rightarrow (\forall x)A(x)$.

A *proof in a theory* T is a sequence of formulas, each being a logical axiom, an axiom of the theory T or follows from some preceding formulas by a deduction rule. Notation: $T \vdash \varphi$ (read: φ has a proof in T).

10. Gödel's strong completeness theorem: *For each theory T and formula φ, $T \vdash \varphi$ if and only if φ is true in each model of the theory T.*

Thus provable in a theory = true in all models of the theory. Not that this is proved using the following lemma: Each consistent theory has a model. (T is inconsistent if $T \vdash \varphi \,\&\, \neg\varphi$).

11. This was the semantic completeness of the first order predicate logic. The completeness theorem was expected. There were attempts to give an algorithm (method) to decide each mathematical question (in the spirit of Hilbert famous saying *Wir müssen wissen, wir werden wissen*); Gödel: tried to get partial results in this direction, **but**, to a big surprise, he showed that it is not possible.

Let T be an arithmetic (a theory such that natural numbers are its model, e.g. PA). We know: $T \vdash \varphi$ if and only if φ is true in each model of T. We may ask: Does T have also models essentially different from the standard model $\mathbf{N}$? This would mean that for some sentence (closed formula) φ, both $\mathbf{M}$ and $\mathbf{N}$ are models of T, and $\mathbf{N} \models \varphi$, but $\mathbf{M} \models \neg\varphi$ – then T is an *incomplete theory* - it proves neither φ nor $\neg\varphi$.

We are interested only in algorithmically (recursively, computably) axiomatized theories. *Caution:* For such theories we may algorithmically decide whether a sequence of formulas is a proof but not necessarily decide whether a given formula has a proof (is provable).

12. First Gödel's incompleteness theorem (in one possible formulation): *Each recursively axiomatized theory T containing some arithmetic (Robinson's Q) and true in* $\mathbf{N}$ *(standard model) is incomplete, there are independent formulas.*

Thus $T \not\vdash \varphi$, $T \not\vdash \neg\varphi$. To prove his theorem Gödel invented the method of arithmetization of syntax: Coding formulas, proofs etc. by natural numbers such that the relation "d is a proof of φ (in T)" is definable by a (suitably chosen,syntactically simple) formula $Prf(x, y)$ of arithmetic in such a way that [d is a proof of φ] iff $\mathbf{N} \models Prf(\bar{d}, \bar{\varphi})$ iff $T \vdash Prf(\bar{d}, \bar{\varphi})$ (and Prf has some other desirable properties; $\bar{d}$ is the code of d, similarly for $\bar{\varphi}$).

Using this Gödel defined his diagonal formula (call it λ) such that: $T \vdash \lambda \equiv \neg(\exists x)Prf(x, \bar{\lambda})$ (which can be interpreted as saying "I am not provable"). And for this formula one can show that it is neither provable nor refutable in T (is an independent formula). This resembles the famous liar's paradox ("What I am just saying is false") but Gödel's formula speaks on unprovability, not falsity).

13. The first Gödel's incompleteness theorem has the following important*Corollary*: The set of formulas provable in T is not algorithmically decidable. (If it were we could construct a complete recursively axiomatized consistent extension of our T, violating the first incompleteness theorem.)

13. The method of arithmetization yields also a formal expression of consistence of our theory. Con is the formula $\neg(\exists x)Prf(x, \overline{0=1})$. The second Gödel's incompleteness theorem says: *For each recursively axiomatizable theory T containing Peano arithmetic and (suitably chosen formal) definition of our theory of T, the formula Con (consistency statement) is unprovable in T.*

This is because under reasonable assumptions (satisfied e.g. by Peano arithmetic) the formula Con is equivalent to Gödel's diagonal formula λ.

14. Till now our formulations have dealt with arithmetics true in $\mathbf{N}$. More generally: The incompleteness theorems are valid for each theory that is recursively axiomatized, consistent, contains the notion of natural numbers and proves for them axioms of "reasonable" arithmetic (and for suitably chosen formula Prf).

For example set theory: if it is consistent then it is incomplete and does not prove its own consistency. In particular: No "reasonable" mathematical theory proves all truths on natural numbers.

15. What are philosophical consequences?? I am not competent to discuss this. Godel's interest in philosophy is well known. But be careful in taking philosophical consequences: Gödel's theorems are a part of exact mathematics.

Let us repeat: Gödel gave us double answer to the question whether true = provable: For each theory T (over first order logic), provability of each formula is equivalent to truth of this formula in *all models* of T. (This is the semantical completeness of logic.)

But each recursively axiomatizable consistent theory T containing a few axioms of arithmetic is deductively incomplete, there are sentences neither provable nor refutable in T. Thus truth in *the (unique) standard model of arithmetic* is NOT equivalent to provability in any recursively axiomatized theory.

Gödel's completeness theorem and both incompleteness theorems are absolutely fundamental for mathematical logic. Needless to say, these theorems are by far not the only results of Gödel; let us mention at least results on the (relative) consistency of the axiom of choice and of the hypothesis of continuum with the axioms of set theory. His completeness/incompleteness theorems are presented in each good monograph devoted to the introduction to mathematical logic; we refer below just to Gödel's posthumously published collected works with detailed comments of other authors and to two monographs.

References

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