Weighting Methods

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Motivation

- Matching methods for improving covariate balance
- Potential limitations of matching methods:
  1. inefficient $\rightsquigarrow$ it may throw away data
  2. ineffective $\rightsquigarrow$ it may not be able to balance covariates
- Recall that matching is a special case of weighting:

$$
\begin{aligned}
\hat{\tau}_{\text{match}} &= \frac{1}{n_1}\sum_{i=1}^{n} T_i \left(Y_i - \frac{1}{|\mathcal{M}_i|}\sum_{i' \in \mathcal{M}_i} Y_{i'} \right) \\
&= \frac{1}{n_1}\sum_{i:T_i=1} Y_i - \frac{1}{n_0}\sum_{i:T_i=0} \underbrace{\left(\frac{n_0}{n_1}\sum_{i':T_{i'=1}} \frac{\mathbf{1}\{i \in \mathcal{M}_{i'}\}}{|\mathcal{M}_{i'}|} \right)}_{w_i} Y_i
\end{aligned}
$$

- Idea: weight each observation in the control group such that it looks like the treatment group (i.e., good covariate balance)

Inverse Probability-of-Treatment Weighting (IPW)

- Weighting for surveys: down-weight over-sampled respondents
- Sampling weights inversely proportional to samplig probability
- Horvitz-Thompson estimator (1952. *J. Am. Stat. Assoc.*):

$$\widehat{\mathbb{E}(Y_i)} = \frac{1}{N}\sum_{i=1}^{N}\frac{S_i Y_i}{\Pr(S_i = 1)}$$

- Weight by the inverse of propensity score:

$$\begin{aligned}
\widehat{\text{ATE}} &= \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{T_i Y_i}{\hat{\pi}(\mathbf{X}_i)} - \frac{(1-T_i)Y_i}{1-\hat{\pi}(\mathbf{X}_i)}\right\} \\
\widehat{\text{ATT}} &= \frac{1}{n_1}\sum_{i=1}^{n}\left\{T_i Y_i - \frac{\hat{\pi}(\mathbf{X}_i)(1-T_i)Y_i}{1-\hat{\pi}(\mathbf{X}_i)}\right\} \\
\widehat{\text{ATC}} &= \frac{1}{n_0}\sum_{i=1}^{n}\left\{\frac{(1-\hat{\pi}(\mathbf{X}_i))T_i Y_i}{\hat{\pi}(\mathbf{X}_i)} - (1-T_i)Y_i\right\}
\end{aligned}$$

- Identical propensity score $\rightsquigarrow$ difference-in-means estimator

Normalized Weights

- Survey sampling when the population size is unknown
- Hajek Estimator:

$$\widehat{\mathbb{E}(Y_i)} \;=\; \frac{\sum_i S_i Y_i / \Pr(S_i = 1)}{\sum_i S_i / \Pr(S_i = 1)}$$

- Weights are normalized but no longer unbiased
- Normalization of weights may be important when propensity score is estimated

$$\widehat{\text{ATE}} \;=\; \frac{\sum_{i=1}^n T_i Y_i / \hat{\pi}(\mathbf{X}_i)}{\sum_{i=1}^n T_i / \hat{\pi}(\mathbf{X}_i)} - \frac{\sum_{i=1}^n (1 - T_i) Y_i / \{1 - \hat{\pi}(\mathbf{X}_i)\}}{\sum_{i=1}^n (1 - T_i) / \{1 - \hat{\pi}(\mathbf{X}_i)\}}$$

- Weighted least squares gives automatic normalization:

$$(\hat{\alpha}_{\text{wls}}, \hat{\beta}_{\text{wls}}) \;=\; \underset{\alpha,\beta}{\operatorname{argmin}} \sum_{i=1}^n \frac{T_i(1 - \hat{\pi}(\mathbf{X}_i)) + (1 - T_i)\hat{\pi}(\mathbf{X}_i)}{\hat{\pi}(\mathbf{X}_i)\{1 - \hat{\pi}(\mathbf{X}_i)\}} (Y_i - \alpha - \beta T_i)^2$$

Variance

- IPW estimator as the method of moments estimator:
  1. moment condition from the propensity score model (e.g., score)

$$\sum_{i=1}^{n} \left\{ \frac{T_i}{\pi_\theta(\mathbf{X}_i)} - \frac{1-T_i}{1-\pi_\theta(\mathbf{X}_i)} \right\} \frac{\partial}{\partial\theta}\pi_\theta(\mathbf{X}_i) \;=\; 0$$

  2. moment conditions from the weighting estimator

$$\textit{Horvitz/Thompson}: \; \frac{1}{n}\sum_{i=1}^{n} \frac{T_i Y_i}{\pi_\theta(\mathbf{X}_i)} - \mu_1 = \frac{1}{n}\sum_{i=1}^{n} \frac{(1-T_i)Y_i}{1-\pi_\theta(\mathbf{X}_i)} - \mu_0 = 0$$

$$\textit{Hajek}: \; \frac{1}{n}\sum_{i=1}^{n} \frac{T_i(Y_i-\mu_1)}{\pi_\theta(\mathbf{X}_i)} \;=\; \frac{1}{n}\sum_{i=1}^{n} \frac{(1-T_i)(Y_i-\mu_0)}{1-\pi_\theta(\mathbf{X}_i)} \;=\; 0$$

  $\rightsquigarrow$ large sample variances are readily available

- If the propensity score model is correctly specified, these variances are smaller than those with the true propensity score

Doubly Robust Estimator (Robins et al. 1994. *J. Am. Stat. Assoc.*)

- Augmented IPW (AIPW) estimator:

$$
\begin{aligned}
\hat{\tau}_{\mathrm{DR}} &= \frac{1}{n}\sum_{i=1}^{n}\left[\left\{\frac{T_i Y_i}{\hat{\pi}(\mathbf{X}_i)} - \frac{T_i - \hat{\pi}(\mathbf{X}_i)}{\hat{\pi}(\mathbf{X}_i)}\hat{\mu}_1(\mathbf{X}_i)\right\}\right. \\
&\qquad\qquad \left. - \left\{\frac{(1-T_i)Y_i}{1-\hat{\pi}(\mathbf{X}_i)} - \frac{T_i - \hat{\pi}(\mathbf{X}_i)}{1-\hat{\pi}(\mathbf{X}_i)}\hat{\mu}_0(\mathbf{X}_i)\right\}\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\left\{\hat{\mu}_1(\mathbf{X}_i) + \frac{T_i(Y_i - \hat{\mu}_1(\mathbf{X}_i))}{\hat{\pi}(\mathbf{X}_i)}\right\}\right. \\
&\qquad\qquad \left. - \left\{\hat{\mu}_0(\mathbf{X}_i) + \frac{(1-T_i)(Y_i - \hat{\mu}_0(\mathbf{X}_i))}{1-\hat{\pi}(\mathbf{X}_i)}\right\}\right]
\end{aligned}
$$

- Consistent if either the propensity score model or the outcome model is correct $\rightsquigarrow$ you get two chances to be correct
- Efficient: smallest asymptotic variance among estimators that are consistent when the propensity score model is correct

A Simulation Study (Kang and Schafer. 2007. *Statistical Science*)

- The deteriorating performance of propensity score weighting methods when the model is misspecified
- Led to improvements of doubly robust estimators $\rightsquigarrow$ Cao et al. (2009), Tan (2010), Rotnitzky et al. (2012), Han and Wang (2013) *Biometrika*. etc.
- Setup:
  - 4 covariates X_i^*: all are *i.i.d.* standard normal
  - Outcome model: linear model
  - Propensity score model: logistic model with linear predictors
  - Misspecification induced by measurement error:
    - $X_{i1} = \exp(X_{i1}^*/2)$
    - $X_{i2} = X_{i2}^*/(1 + \exp(X_{1i}^*) + 10)$
    - $X_{i3} = (X_{i1}^* X_{i3}^*/25 + 0.6)^3$
    - $X_{i4} = (X_{i1}^* + X_{i4}^* + 20)^2$
- Weighting estimators to be evaluated:
  1. Horvitz-Thompson
  2. Inverse-probability weighting with normalized weights
  3. Weighted least squares regression with covariates
  4. Doubly-robust least squares regression with covariates

Weighting Estimators Do Fine If the Model is Correct

Sample size	Estimator	Bias logit	Bias True	RMSE logit	RMSE True
(1) Both models correct					
$n = 200$	HT	0.33	1.19	12.61	23.93
	IPW	-0.13	-0.13	3.98	5.03
	WLS	-0.04	-0.04	2.58	2.58
	DR	-0.04	-0.04	2.58	2.58
$n = 1000$	HT	0.01	-0.18	4.92	10.47
	IPW	0.01	-0.05	1.75	2.22
	WLS	0.01	0.01	1.14	1.14
	DR	0.01	0.01	1.14	1.14
(2) Propensity score model correct					
$n = 200$	HT	-0.05	-0.14	14.39	24.28
	IPW	-0.13	-0.18	4.08	4.97
	WLS	0.04	0.04	2.51	2.51
	DR	0.04	0.04	2.51	2.51
$n = 1000$	HT	-0.02	0.29	4.85	10.62
	IPW	0.02	-0.03	1.75	2.27
	WLS	0.04	0.04	1.14	1.14
	DR	0.04	0.04	1.14	1.14

Weighting Estimators are Sensitive to Misspecification

		Bias		RMSE	
Sample size	Estimator	logit	True	logit	True
(3) Outcome model correct					
$n = 200$	HT	24.25	-0.18	194.58	23.24
	IPW	1.70	-0.26	9.75	4.93
	WLS	-2.29	0.41	4.03	3.31
	DR	-0.08	-0.10	2.67	2.58
$n = 1000$	HT	41.14	-0.23	238.14	10.42
	IPW	4.93	-0.02	11.44	2.21
	WLS	-2.94	0.20	3.29	1.47
	DR	0.02	0.01	1.89	1.13
(4) Both models incorrect					
$n = 200$	HT	30.32	-0.38	266.30	23.86
	IPW	1.93	-0.09	10.50	5.08
	WLS	-2.13	0.55	3.87	3.29
	DR	-7.46	0.37	50.30	3.74
$n = 1000$	HT	101.47	0.01	2371.18	10.53
	IPW	5.16	0.02	12.71	2.25
	WLS	-2.95	0.37	3.30	1.47
	DR	-48.66	0.08	1370.91	1.81

Covariate Balancing Propensity Score (CBPS)

(Imai and Ratkovic. 2015. *J. Royal Stat. Soc. B.*)

- How can we improve the estimation of propensity score?
- Estimate the propensity score such that covariates are balanced
- Covariate balancing conditions:

$$\mathbb{E}\left\{\frac{T_i}{\pi_\theta(\mathbf{X}_i)} - \frac{1-T_i}{1-\pi_\theta(\mathbf{X}_i)}\right\} f(\mathbf{X}_i) \quad = \quad 0$$

  - Usual score condition: $f(\mathbf{X}_i) = \frac{\partial}{\partial\theta}\pi_\theta(\mathbf{X}_i)$
  - Balancing intercept $\rightsquigarrow$ normalization of weights, sample boundedness

- Optimal choice (Fan et al. 2016. *Working Paper*):

$$f(\mathbf{X}_i) \ = \ \pi_\theta(\mathbf{X}_i)\mu_0(\mathbf{X}_i) + \{1-\pi_\theta(\mathbf{X}_i)\}\mu_1(\mathbf{X}_i)$$

  1. double robustness
  2. smallest asymptotic variance when the propensity score is correct

- Estimation via the (generalized) method of moments

More Robust Weighting Methods

Sample size	Estimator	Bias GLM	Bias CBPS1	Bias CBPS2	Bias True	RMSE GLM	RMSE CBPS1	RMSE CBPS2	RMSE True
(3) Outcome model correct									
$n = 200$	HT	24.25	1.09	−5.42	−0.18	194.58	5.04	10.71	23.24
	IPW	1.70	−1.37	−2.84	−0.26	9.75	3.42	4.74	4.93
	WLS	−2.29	−2.37	−2.19	0.41	4.03	4.06	3.96	3.31
	DR	−0.08	−0.10	−0.10	−0.10	2.67	2.58	2.58	2.58
$n = 1000$	HT	41.14	−2.02	2.08	−0.23	238.14	2.97	6.65	10.42
	IPW	4.93	−1.39	−0.82	−0.02	11.44	2.01	2.26	2.21
	WLS	−2.94	−2.99	−2.95	0.20	3.29	3.37	3.33	1.47
	DR	0.02	0.01	0.01	0.01	1.89	1.13	1.13	1.13
(4) Both models incorrect									
$n = 200$	HT	30.32	1.27	−5.31	−0.38	266.30	5.20	10.62	23.86
	IPW	1.93	−1.26	−2.77	−0.09	10.50	3.37	4.67	5.08
	WLS	−2.13	−2.20	−2.04	0.55	3.87	3.91	3.81	3.29
	DR	−7.46	−2.59	−2.13	0.37	50.30	4.27	3.99	3.74
$n = 1000$	HT	101.47	−2.05	1.90	0.01	2371.18	3.02	6.75	10.53
	IPW	5.16	−1.44	−0.92	0.02	12.71	2.06	2.39	2.25
	WLS	−2.95	−3.01	−2.98	0.19	3.30	3.40	3.36	1.47
	DR	−48.66	−3.59	−3.79	0.08	1370.91	4.02	4.25	1.81

Calibration Methods

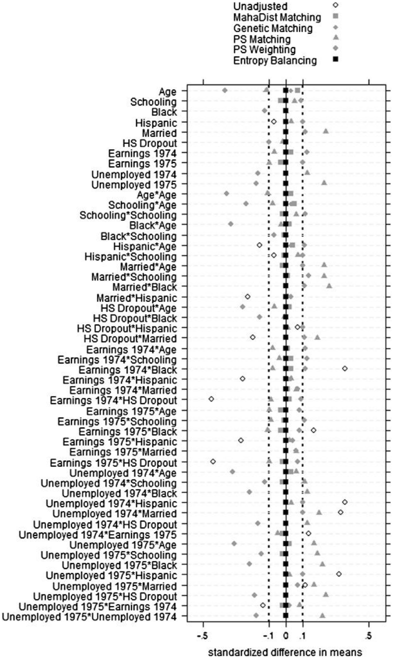
- Avoid modeling the propensity score $\rightsquigarrow$ just balance covariates
  - no modeling assumption about treatment assignment
  - but implicit assumptions about the outome model
  - in theory, propensity score balances the entire distributions
  - validation and interpretation are more difficult
- Entropy balancing (Hainmueller. 2012. *Political Anal.*)

$$\{w_1^*, w_2^*, \ldots, w_{n_0}\} = \operatorname*{argmin}_{w} \sum_{i:T_i=0} w_i \log(w_i/q_i)$$

subject to

$$w_i \geq 0, \quad \sum_{i:T_i=0} w_i = 1, \quad \sum_{i:T_i=0} w_i f(\mathbf{X}_i) = \frac{1}{n_1} \sum_{i:T_i=1} f(\mathbf{X}_i)$$

  - convex optimization problem
  - exact balance in moments
  - possibly extreme weights



- check covariate balance
- check distribution of weights

- Stable weights (Zubizarreta. 2015. *J. Am. Stat. Assoc.*)
  - minimize the variance of weights while keeping a pre-specified level of covariate balance

$$\underset{w}{\text{minimize}} \; \|\mathbf{w} - \bar{\mathbf{w}}\|_2^2$$

  subject to

$$w_i \;\geq\; 0, \quad \sum_{i: T_i = 0} w_i \;=\; 1, \quad \left| \sum_{i: T_i = 0} w_i X_{ij} - \frac{1}{n_1} \sum_{i: T_i = 1} X_{ij} \right| \;\leq\; \delta_j$$

  - quadratic convex programming problem
  - choice of δ_j

- Recent developments on weighting methods:
  - more flexible functions (Wong and Chan. 2018. *Biometrika*; Hazlett. Forthcoming. *Stat. Sinica.*)
  - many covariates (Athey *et al.* 2018. *J. Royal. Stat. Soc. B*; Ning *et al.* Forthcoming. *Biometrika*; Tan. Forthcoming. *Biometrika*)

Summary

- Weighting methods as a generalization of matching methods
  - more efficient and possibly more effective
  - be careful about extreme weights

- Propensity score weighting
- Doubly robust estimation: combine weighting and regression
- Robust estimation of propensity score for balancing covariates
- Calibration methods

- Recommended readings:
  - Imbens and Rubin. Chapter 17 (Section 8)
  - Lunceford and Davidian. 2004. “Stratification and weighting via the propensity score in estimation of causal treatment effects: a comparative study.” *Statistics in Medicine*