Coxeter Graphs

Sisam Khanal

July 2, 2024

Introduction

There are infinitely many finite Coxeter systems. Our goal is to classify them (up to isomorphism), reduce them into irreducible components and study them.

1 Group Theory

Theorem 1.1. *Let G be a group with normal subgroups H and K satisfying the following conditions:*

- $G = HK = \{hk : h \in H,\ k \in K\}$,
- $H \cap K = \{e\}$,

then $G \cong H \times K$

Proof. Let $\phi : G \to H \times K;\ g \mapsto (h, k)$. To show that ϕ is an isomorphism, we need to show ϕ is well defined, preserves group operation and is bijective.

Well defined. Since $G = HK$ let $g = hk$ and $g = h'k'$ where $g \in G,\ h, h' \in H$ and $k, k' \in K$. $\implies hk = h'k' \implies \underbrace{h'^{-1}h}_{\in H} = \underbrace{k'k^{-1}}_{\in K}$ But since $H \cap K = \{e\}$ it follows that $h'^{-1}h = e \implies h = h'$ likewise $k = k'$. With that ϕ is well defined.

Preserves group operation. Consider the product $hkh^{-1}k^{-1}$

$$h\underbrace{\left(kh^{-1}k^{-1}\right)}_{\in H} \in H \quad \text{and} \quad \underbrace{\left(hkh^{-1}\right)}_{\in K}k^{-1} \in K$$

$\implies hkh^{-1}k^{-1} = e \implies hk = kh$. The subgroups H and K commute.
And now

$$\begin{aligned}
\phi(g_1g_2) &= \phi\left(h_1k_1h_2k_2\right) \\
&= \phi\left(h_1h_2k_1k_2\right) && \text{since } h_2 \text{ and } k_1 \text{ commute} \\
&= (h_1h_2, k_1k_2) \\
&= (h_1, k_1)(h_2, k_2) \\
&= \phi(g_1)\phi(g_2)
\end{aligned}$$

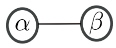
Bijection. It is clear that $|H \times K| = |\phi(G)| = |G|$. Thus bijection. □

2 Coxeter Graphs

W is determined up to isomorphism by the set of integers $m(\alpha,\beta)$ where $\alpha,\beta\in\Delta$. We construct a weighted graph whose vertices are the element of Δ. We lable the edge between the vertex α and $\beta\in\Delta$ with $m(\alpha,\beta)$, if $m(\alpha,\beta)\geq 3$. Since the weight 3 occurs frequently, we shall omit it while drawing the graph. For a pair of vertices not joined by an edge, it is understood that $m(\alpha,\beta)=2$. Such a weighted graph is called **Coxeter Graph** and denoted by Γ.

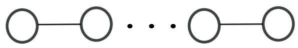
Examples.

- For a coxeter system (W,S), subject only to $m(\alpha,\beta)=2$ for $\alpha,\beta\in\Delta$ we have the following graph:

  This is the coxeter graph consisting of the orthogonal transformations which preservers the symmetries a triangle.

- When S_n acts on $\mathbb{R}^n$, it permutes the basis vectors resulting in reflections. As an example, S_2 acting on $\mathbb{R}^2$ it switches the y-axis with the x-axis resulting a reflection about $H_{(1,-1)}$. Every element of S_n can be represented with $n-1$ transpositions, which act like reflections. Hence the coxeter graph for S_n contains $n-1$ vertices.

3 Isomorphism

Here we give a more precise criterion for reflection groups to be isomorphic.

Theorem 3.1. *For $i=1,2$ let W_i be finite reflection groups acting on the euclidean space V_i. Assume W_i is essential, meaning it reflects every non-zero point in V_i. If W_1 and W_2 have the same Coxeter graph, then there is an isometry (distance and angle preserving transformation) of V_1 onto V_2 inducing an isomorphism of W_1 onto W_2. In particular, if $V_1=V_2$, the subgroups W_1 and W_2 are conjugate in $O(V)$*

Proof. First we fix a simple system Δ_i for W_i. Δ_i is a basis of V_i and all roots are of unit length. Let $\varphi:\Delta_1\to\Delta_2$, such that the Coxeter graph remains unchanged. This means that $m(\alpha,\beta)=m(\varphi(\alpha),\varphi(\beta))$. φ then extends to a vector space isomorphism of V_1 onto V_2. Let $\alpha\neq\beta,\alpha\in\Delta$ then the angel between them is $\theta=\pi-\frac{\pi}{m(\alpha,\beta)}$, because $(\alpha,\beta)\leq 0$. Then

$$(\alpha,\beta)=\cos\theta=\cos\left(\pi-\frac{\pi}{m(\alpha,\beta)}\right)=-\cos\left(\frac{\pi}{m(\alpha,\beta)}\right)$$

The same calculations applies to the scalar product of the roots in Δ_2 corresponding to α,β since the same $m(\alpha,\beta)$. The scalar product is preserved, hence making φ an isometry. Which induces an isomorphism of W_1 onto W_2.
When $V_1=V_2$ we get $\varphi w_1\varphi^{-1}\in W_2$ where $\varphi\in O(V)$ $\square$

4 Irreducible components

We say that the Coxeter system (W, S) is **irreducible** if the Coxeter graph Γ is connected. We also say that the root system Φ is irreducible in this case. Let Γ be a Coxeter graph and $\Gamma_1, \ldots, \Gamma_r$ it's connected (to themselves) components, and let Δ_i, S_i be the corresponding set of simple roots and simple reflections.

If two roots $\alpha \in \Delta_i, \beta \in \Delta_j$, since we assumed that $\Delta_i \in S_i, \Delta_j \in S_j$ to be irreducible, we have $m(\alpha, \beta) = 2$ and therefore $s_\alpha s_\beta = s_\beta s_\alpha$. The theorem below tells that, all Coxeter systems can be broken down into irreducible components and studied independently.

Theorem 4.1. *Let (W, S) have Coxeter graph Γ, with connected components $\Gamma_1, \ldots, \Gamma_r$ and let $S_1, \ldots, S_r$ be the corresponding subsets of S. Then W is the direct product of the parabolic subgroups $W_{S_1}, \ldots, W_{S_r}$ and each Coxeter system (W_{S_i}, S_i) is irreducible.*

Proof. We divide the proof into 2 parts. In the first part (I), we prove the theorem for the Coxeter graph Γ to the Coxeter system (W, S) containing two components namely Γ_1 and Γ_2, for which S_1 and $S_2 \subset S$. Then in the second part (II), using the result of the first part we induce on r.

I. As we have seen before, the elements of S_1 and S_2 commute. $\forall \alpha \in S_1, \forall \beta \in S_2$:

$$s_\alpha s_\beta = s_\beta s_\alpha \iff s_\alpha s_\beta s_\alpha^{-1} = s_\beta \implies s_\alpha W_{S_2} s_\alpha^{-1} = W_{S_2}$$

Making W_{S_1} and W_{S_2} normal in W. Moreover the $S = S_1 S_2 = \{s_1 s_2 | \, s_1 \in S_1, s_2 \in S_2\}$. As S_1 is the generator of W_{S_1} and S_2 of W_{S_2}, we get $W = W_{S_1} W_{S_2}$. Since the graph has two components Γ_1 and Γ_2 corresponding to W_{S_1} and W_{S_2}, making their intersection trivial. $W_{S_1} \cap W_{S_2} = \{e\}$. The subgroups W_{S_1} and W_{S_2} fulfill all the conditions of the theorem 1.1, making W isomorphic to the direct product of W_{S_1} and W_{S_2}.

$$W \cong W_{S_1} \times W_{S_2}$$

II Inducing on r. $r-1 \to r$: For $i = 1, \cdots, r-1$ we have, S_i commutes with $S_r \implies W_i$ normal to W_r and $W = (W_1 W_2 \ldots_{r-1})\, W_r$ and their intersection is trivial. Resulting in:

$$W \cong W_1 \times W_2 \times \cdots \times W_r$$

$\square$

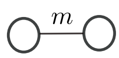
5 Some positive definite Coxeter graphs

To every Coxeter graph Γ we define a Matrix $n \times n$ symmetric Matrix A called *schläfli matrix* with the following entries:

$$a(s, s') := (s, s') = -\cos \frac{\pi}{m(s, s')}$$

We call Γ positive definite or positive semidefinite when the associated schläfli matrix A has corresponding property. When Γ represents a finite reflection group W the associated matrix A is positive definite as we will see in the examples following.

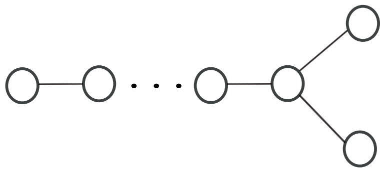
$I_2(m)$:



A_n (S_{n-1})**:**

B_n $(n \geq 2)$**:**

D_n $(n \geq 4)$**:**



$I_2(m)$**:** The corresponding schläfli matrix is

$$2A = \begin{pmatrix} 2 & -2\cos\left(\frac{\pi}{m}\right) \\ -2\cos\left(\frac{\pi}{m}\right) & 2 \end{pmatrix}$$

The determinant: $\det 2A = 4\sin^2\left(\frac{\pi}{m}\right) \geq 0 \implies$ positive definite.

A_n**:** The corresponding schläfli Matrix is

$$2A = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & 1 & \cdots & 0 \\ \vdots & & & \ddots & & 0 \\ 0 & \cdots & & -1 & 2 & -1 \end{pmatrix}$$

$\det 2A = 2d_{n-1} - d_{n-2} = n + 1$