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Constructive Methods for Linear Inverse Problems in Transport Theory

BACHELOR THESIS

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Introduction

Linear inverse problems for the transport equation often arise in applied mathematical physics.

This bachelor thesis is devoted to the study of a linear inverse problem for a one-dimensional simple transport equation with absorption with a special choice of the right side of the differential equation. It is required to determine the density of additional sources of the substance, as well as the density of the substance itself within a limited interval on $\mathbb{R}$ for a given incoming flow from given information about the initial and final states of the system. The main goal of our study is to develop a constructive algorithm for solving the problem and to conduct numerical experiments that implement this algorithm.

When studying the problem posed, we use the result of the article [9], which indicates a general algorithm for solving an abstract inverse problem of a special form for an evolution equation with a nilpotent semigroup. In this paper, we present the most complete derivation of the solution of an abstract problem in the language of semigroup theory, and the result obtained is extended to the case of a problem for the transport equation with absorption. The basic concepts and statements from the theory of semigroups and the theory of functional analysis can be found in [1],[6],[8]. Information on the theory of partial differential equations was taken from the books [4],[3],[10].

The paper also presents the results of computer experiments and their detailed analysis. For a practical assessment of the correctness of the algorithm, numerous examples are programmed with various input data, for example, cases with the presence or absence of absorption, examples with zero input flow, etc. The program was prepared using the MATLAB programming language, the documentation of which is presented in the electronic resource [2].

1 Required theoretical information

1.1 Linear operators and semigroups

Let us recall some notions from functional analysis that are necessary for studying the inverse problem posed in terms of semigroup theory. Everywhere below we assume that E – real Banach space. In it, we will consider linear closed operators A with domain of definition $D(A) \subset E$ and bounded linear operators $B\colon E \to E$. Let us formulate exact definitions.

Definition 1. Operator A is *linear* on E, if its domain $D(A)$ is a linear subspace of E, and

$$A(\alpha f + \beta g) = \alpha Af + \beta Ag$$

for all elements $f, g \in D(A)$ and all numbers $\alpha, \beta \in \mathbb{R}$.

Definition 2. Linear operator A with its domain of definition $D(A)$ is *closed* on E, if from relations

$$f_n \in D(A), \;\; f_n \to f_0, \;\; Af_n \to g_0$$

we get that $f_0 \in D(A)$ and $Af_0 = g_0$.

Definition 3. Linear operator B, defined on the entire space E is called *bounded* if the value

$$\|B\| = \sup_{\|f\| \leqslant 1} \|Bf\|,$$

called the norm of the operator B. In this case $\|Bf\| \leqslant \|B\| \cdot \|f\|$ for any element $f \in E$. The boundedness of an operator is equivalent to its continuity at every point in the space E.

Definition 4. A one parameter family of bounded linear operators $U(t)\colon E \to E$ with parameter $t \geqslant 0$ is a *semigroup* (or *semigroup of bounded linear operator*), if

a) $U(0) = I$, where I is the identity operator on E,

b) $U(t_1 + t_2) = U(t_1)U(t_2) = U(t_2)U(t_1)$ for every $t_1, t_2 \geqslant 0$.

Definition 5. Semigroup $U(t)$ is *strongly continuous* on E or C_0*-semigroup*, if

$$\lim_{t \to 0+0} U(t)f = f, \quad \forall f \in E.$$

Definition 6. Operator A is the *infinitesimal generator* of the semigroup $U(t)$, if

$$Af = \lim_{t\to 0+0} \frac{U(t)f - f}{t}$$

on the domain of definition

$$D(A) = \left\{ f \in E \,\middle|\, \exists \lim_{t\to 0+0} \frac{U(t)f - f}{t} \right\}.$$

In these formulas, the limit is understood in terms of the norm of the space E. It is also said that the operator A *generates* the semigroup $U(t)$.

Definition 7. Semigroup $U(t)$ is *nilpotent* on E, if

$$U(t) = 0, \quad \forall t \geqslant t_0 > 0,$$

with a fixed value $t_0 > 0$. This value is also called the *nilpotency index* of the semigroup $U(t)$. Nilpotent semigroups arise when considering the transport equation.

1.2 Properties of generating operators

Statement 1. *Let the operator A generate a C_0-semigroup $U(t)$ of class C_0. Then if $f \in E$, then $\int\limits_0^t U(t)f \in D(A)$ and the equality*

$$A\left(\int\limits_0^t U(s)f\,ds\right) = U(t)f - f, \quad t > 0. \tag{1}$$

Proof. Let $f \in E$ and $h > 0$. Then

$$\frac{U(h) - I}{h}\int\limits_0^t U(s)f\,ds = \frac{1}{h}\int\limits_0^t \left(U(s+h)f - U(s)f\right)\,ds =$$
$$= \frac{1}{h}\int\limits_t^{t+h} U(s)f\,ds - \frac{1}{h}\int\limits_0^h U(s)f\,ds.$$

Getting the limit at $h \to 0+0$, we obtain

$$A\left(\int\limits_0^t U(s)f\,ds\right) = U(t)f - f.$$

□

Statement 2. *Let the operator A generate C_0-semigroup. Then the following equality holds*

$$A\int_{t_1}^{t_2} U(s)f\,ds = U(t_2)f - U(t_1)f, \qquad \forall t_1, t_2 > 0,\ \ \forall f \in E. \tag{2}$$

Proof. Represent the integral in a different form

$$\int_{t_1}^{t_2} U(s)f\,ds = \int_0^{t_2} U(s)f\,ds - \int_0^{t_1} U(s)f\,ds$$

Under the action of the operator A, taking (1) into account, we obtain

$$A\int_{t_1}^{t_2} U(s)f\,ds = U(t_2)f - U(t_1)f.$$

□

Statement 3. *Let the operator A generate a C_0-semigroup $U(t)$. Then if $f \in D(A)$, then $U(t)f \in D(A)$ and the equality*

$$\frac{d}{dt}U(t)f = AU(t)f = U(t)Af, \quad t > 0. \tag{3}$$

Proof. Let $f \in D(A)$ and $h > 0$. Let us first show that

$$\frac{U(t)-I}{h}U(t)f = U(t)\left(\frac{U(t)-I}{h}\right)f \underset{h\to 0}{\longrightarrow} U(t)Af. \tag{4}$$

Then, $U(t)f \in D(A)$ and $AU(t)f = U(t)Af$.

The expression (4) also implies that

$$\frac{d^+}{dt}U(t)f = AU(t)f = U(t)Af.$$

To prove (3), we show that the left derivative $\dfrac{d^-}{dt}U(t)f$ exists and is equal to $U(t)Af$.

$$\lim_{h\to 0}\left[\frac{U(t)f - U(t-h)f}{h} - U(t)Af\right] =$$
$$= \lim_{h\to 0} U(t-h)\left[\frac{U(h)f-f}{h} - Af\right] + \lim_{h\to 0}\Big(U(t-h)Af - U(t)Af\Big).$$

The first term on the right-hand side is equal to 0 because $f \in D(A)$ and $\|U(t-h)\|$ is bounded on the interval $0 \leqslant h \leqslant t$. The second term is also equal to zero due to the strong continuity of the semigroup. That is, the right and left derivatives are equal to $U(t)Af$, which means that the assertion is proved. □

Statement 4. *Let the operator A generate a C_0-semigroup $U(t)$. Then A is a linear closed operator with domain $D(A)$ everywhere dense in E.*

Proof. From the strong continuity of the semigroup it follows that

$$\lim_{t\to 0+0}\frac{1}{t}\int_0^t U(s)f\,ds = f, \quad f \in E.$$

From Statement 1 we know that

$$\int_0^t U(s)f\,ds \in D(A).$$

From these two expressions, we can conclude that the closure of the set $D(A)$ coincides with the Banach space E, that is, the domain of $D(A)$ is everywhere dense in E.

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of elements from $D(A)$ converging to f_0 and such that the corresponding sequence $\{Af_n\}_{n=1}^{\infty}$ converges to the element g_0.

Let us show that $f_0 \in D(A)$ and $Af_0 = g_0$. For $f_n \in D(A)$ we get

$$\frac{1}{t}\int_0^t U(s)Af_n\,ds = \frac{1}{t}\int_0^t \frac{d}{ds}U(s)Af_n\,ds = \frac{1}{t}[U(t)f_n - f_n] = A_t f_n.$$

Hence, getting the limit at $n \to \infty$ and using the estimate, we have

$$\frac{1}{t}\int_0^t U(s)g_0\,ds = A_t f_0.$$

The limit of the left side exists for $t \to 0+0$ and is equal to g_0, then the element f_0 belongs to the domain of $D(A)$, and $g_0 = Af_0$. □

1.3 Inhomogeneous abstract Cauchy problem

In a Banach space E on the segment $[0,T] \subset \mathbb{R}$ consider the abstract Cauchy problem for an evolution equation of the form

$$\frac{du(t)}{dt} = Au(t) + \varphi(t)g, \qquad 0 \leqslant t \leqslant T, \tag{5}$$

$$u(0) = f. \tag{6}$$

We believe that

- A — linear closed operator with dense domain $D(A) \subset E$, while the operator A generates C_0 semigroups$U(t)$,
- $\varphi(t)g$ — inhomogeneous term, product of a scalar function $\varphi(t)$ and element $g \in E$,
- f — given element in E,
- $T > 0$ — duration of measurements,
- $u(t)$ — unknown function.

Definition 8. A function $u : [0,T] \to D(A)$ ia a *classical solution* of (5)–(6) on $[0,T]$, if u is continuous on $[0,T]$, u continuously differentiable on $(0,T)$, $u(t) \in D(A)$ for $t \in (0,T)$ and (5)–(6) is satisfied.

Theorem 1. *The formal solution of the Cauchy problem* (5)–(6) *is written using the Duhamel integral*

$$u(t) = U(t)f + \int\limits_0^t U(t-s)\varphi(s)g\,ds, \quad 0 \leqslant t \leqslant T. \tag{7}$$

Proof. Let $U(t)$ be the C_0 semigroup generated by A and let u be a solution of (5)–(6). Then the function $g(s) = U(t-s)u(s)$ is differentiable for $0 < s < t$. Using (3), we get

$$\begin{aligned} \frac{dg}{ds} &= -AU(t-s)u(s) + U(t-s)u'(s) = \\ &= -AU(t-s)u(s) + U(t-s)Au(s) + U(t-s)\varphi(s)g = \\ &= \quad U(t-s)\varphi(s)g. \end{aligned}$$

Integrating resulting expression from 0 to t yields

$$g(t) - g(0) = \int\limits_0^t U(t-s)\varphi(s)g\,ds.$$

Given that $g(s) = U(t-s)u(s)$, we get the expression (7), which was required to prove

$$u(t) = U(t)f + \int\limits_0^t U(t-s)\varphi(s)g\,ds.$$

$\square$

2 An abstract version of the inverse problem

2.1 Formulation of the problem

Consider the following one-dimensional problem for the evolution equation on a segment, given for a limited time interval

$$u'(t) = Au(t) + \varphi(t)g, \qquad 0 \leqslant t \leqslant T, \tag{8}$$

$$u(0) = u_0, \quad u(T) = u_1. \tag{9}$$

We consider a special case of this problem, when the function $\varphi(t)$ is piecewise constant on the segment $[0, T]$. Here and below, we assume that

- A — linear closed operator with dense domain $D(A) \subset E$, while the operator A generates a nilpotent semigroup $U(t)$ with fixed value $t_0 \geqslant 0$,
- $T > 0$ — duration of measurements,
- $\varphi(t)g$ — product of a scalar function $\varphi(t)$ and element $g \in E$,
- $u_0 \in D(A)$ — initial state of the system,
- $u_1 \in D(A)$ — final state of the system.

Values of parameter T, elements u_0, u_1 and function $\varphi(t)$ are assumed to be known; the inverse problem is to find g and $u(t)$, that satisfy the conditions of the problem (8)–(9).

For further reasoning, it will be convenient to represent the piecewise constant function $\varphi(t)$ in the form

$$\varphi(t) = \begin{cases} \alpha_1, & \tau_0 \leqslant t < \tau_1, \\ \alpha_2, & \tau_1 \leqslant t < \tau_2, \\ . \quad . \quad . \quad . \quad . & . \quad . \quad . \quad . \quad . \\ \alpha_p, & \tau_{p-1} \leqslant t \leqslant \tau_p, \end{cases} \tag{10}$$

where α_1,α_2,...,α_p are real numbers, while $\alpha_{j+1} \neq \alpha_j$ for $j = 1, 2, ..., p-1$ and $\alpha_p \neq 0$. Values τ_0,τ_1,...,τ_p set the division of the time interval according to the rule $0 = \tau_0 < \tau_1 < ... < \tau_{p-1} < \tau_p = T$.

2.2 General scheme of the study

Formal solution of the Cauchy problem (8)–(9) is written in terms of the Duhamel integral(7)

$$u(t) = U(t)u_0 + \int\limits_0^t U(t-s)\varphi(s)g\,ds, \quad 0 \leqslant t \leqslant T.$$

When $t = T$, we get

$$u(T) = U(T)u_0 + \int\limits_0^T U(T-s)\varphi(s)g\,ds.$$

$$u_1 - U(T)u_0 = \int\limits_0^T U(s)\varphi(T-s)g\,ds.$$

Let us substitute into the resulting equality the representation (10) of the function $\varphi(t)$

$$u_1 - U(T)u_0 = \sum_{j=1}^{p} \alpha_j \int\limits_{T-\tau_j}^{T-\tau_{j-1}} U(s)g\,ds.$$

Let's introduce a new element $h = u_1 - U(T)u_0$, act on it by the operator $(-A)$ and using expression (2) we get operator equation

$$\beta g - Bg = -Ah,$$

where $\beta = \alpha_p \neq 0$ and operator B given by the expression

$$B = \alpha_p U(T-\tau_{p-1}) + \sum_{j=1}^{p-1} \alpha_j (U(T-\tau_{j-1}) - U(T-\tau_j)).$$

Let's introduce $\alpha_0 = 0$ and write an expression for the operator B in a more convenient way

$$B = \sum_{k=1}^{p} (\alpha_k - \alpha_{k-1}) U(T-\tau_{k-1}).$$

From the conditions on an operator A it is known that $U(t)$ is a nilpotent semigroup, thus operator B is also nilpotent. From this we can conclude that $B = 0$ and $B^n = 0$ for all indicators $n \in \mathbb{N}$: $n \geqslant \dfrac{t_0}{t - \tau_{p-1}}$.

Therefore, the expression for the unknown element g can be uniquely found using the formula

$$g = \sum_{n=0}^{N_0} \frac{1}{\beta^{n+1}} B^n(-Ah), \quad N_0 = \left\lceil \frac{t_0}{T - \tau_{p-1}} \right\rceil - 1.$$

3 One-dimensional transport equation with absorption

3.1 Statement of the direct problem

Consider the following one-dimensional problem for the simple transport equation with absorption

$$u_t + au_x + \sigma(x)u = f(x,t), \qquad 0 \leqslant x \leqslant l, \quad 0 \leqslant t \leqslant T, \tag{11}$$

$$u(0,t) = \gamma(t), \tag{12}$$

$$u(x,0) = u_0(x). \tag{13}$$

We believe that

- $a = \text{const} > 0$ — transfer speed,
- $T > 0$ — duration of measurements,
- $l > 0$ — the length of the rod with the substance,
- $\sigma(x)$ — absorption coefficient,
- $\gamma(t)$ — incoming stream
- $f(x,t)$ — density of additional sources of substance,
- $u_0(x)$ — initial state of the system,
- $u(x,t)$ — density of the transferred substance.

Values of parameters a, l, T and functions $\sigma(x), u_0(x), \gamma(t), f(x,t)$ assumed to be known; it is needed to find a function $u(x,t)$, that satisfies the conditions of the problem (11)–(13).

The formulation of the problem in this form may arise in the study of various physical phenomena. For example, when considering the movement of a substance along a tube of length l with a speed of a, when there are any radioactive particles inside the tube (i.e. additional sources of substance), which begin to move along with the substance considered initially. The process of absorption by the medium is characterized by the quantity $\sigma(x)$. And $\gamma(t)$ will denote the amount of substance blown through the left edge of the tube.

3.2 Direct problem solution

Many problems of finding solutions to equations in partial derivatives of the first order are usually solved by the method of characteristics. Let's use it to obtain explicit resolving formulas for solving the (11)–(13) problem. We reduce the equation (11) to the equation

$$\frac{d}{d\tau}u\big(\alpha(\tau),\beta(\tau)\big)+\sigma\big(\alpha(\tau)\big)u\big(\alpha(\tau),\beta(\tau)\big)=f\big(\alpha(\tau),\beta(\tau)\big), \tag{14}$$

where $\big(\alpha(\tau),\beta(\tau)\big)$ is a characteristic.

We use the property of the derivative of a complex function and obtain

$$\frac{\partial u}{\partial\alpha}\frac{\partial\alpha}{\partial\tau}+\frac{\partial u}{\partial\beta}\frac{\partial\beta}{\partial\tau}+\sigma\big(\alpha(\tau)\big)u\big(\alpha(\tau),\beta(\tau)\big)=f\big(\alpha(\tau),\beta(\tau)\big).$$

Let's assume that $\dfrac{\partial\alpha}{\partial\tau}=a$ and $\dfrac{\partial\beta}{\partial\tau}=1$. Hence it follows that

$$\begin{aligned}&\alpha'(\tau)=a \quad\Rightarrow\quad \alpha(\tau)=a\tau+x_0,\\ &\beta'(\tau)=1 \quad\Rightarrow\quad \beta(\tau)=\tau+t_0.\end{aligned}$$

Let us substitute the resulting expressions for $\alpha(\tau)$ and $\beta(\tau)$ into the equation (13)

$$\frac{d}{d\tau}u(x_0+a\tau,t_0+\tau)+\sigma(x_0+a\tau)u(x_0+a\tau,t_0+\tau)=f(x_0+a\tau,t_0+\tau).$$

Replacing the point (x_0,t_0) with an arbitrary point (x,t), we obtain the transport equation in a new form

$$\frac{d}{d\tau}u(x+a\tau,t+\tau)+\sigma(x+a\tau)u(x+a\tau,t+\tau)=f(x+a\tau,t+\tau). \tag{15}$$

We multiply the resulting equation by the integrating factor

$$\mu(\tau)=\exp\Big(\int\limits_0^\tau\sigma(as+x)\,ds\Big).$$

Then we obtain the transport equation with absorption on the characteristics in a partially integrated form

$$\frac{d}{d\tau}\left[u(x+a\tau,t+\tau)\exp\Big(\int\limits_0^{\tau}\sigma(as+x)\,ds\Big)\right]=f(x+a\tau,t+\tau)\exp\Big(\int\limits_0^{\tau}\sigma(as+x)\,ds\Big). \tag{16}$$

1. In the region $x-at>0$, we integrate the expression (16) from $(-t)$ to 0:

$$u(x,t)-u(x-at,0)\exp\Big(\int\limits_0^{-t}\sigma(as+x)\,ds\Big)=\int\limits_{-t}^{0}f(x+a\tau,t+\tau)\exp\Big(\int\limits_0^{\tau}\sigma(as+x)\,ds\Big)d\tau,$$

$$u(x,t)=u_0(x-at)\exp\Big(-\int\limits_0^{t}\sigma(x-as)\,ds\Big)+\int\limits_0^{t}f(x-a\tau,t-\tau)\exp\Big(-\int\limits_0^{\tau}\sigma(x-as)\,ds\Big)d\tau.$$

2. In the region $x-at<0$, we integrate the expression (16) from $(-x/a)$ to 0:

$$u(x,t)-u\Big(0,t-\frac{x}{a}\Big)\exp\Big(\int\limits_0^{-x/a}\sigma(as+x)\,ds\Big)=\int\limits_{-x/a}^{0}f(x+a\tau,t+\tau)\exp\Big(\int\limits_0^{\tau}\sigma(as+x)\,ds\Big)d\tau,$$

$$u(x,t)=\gamma\Big(t-\frac{x}{a}\Big)\exp\Big(-\int\limits_0^{x/a}\sigma(x-as)\,ds\Big)+\int\limits_0^{x/a}f(x-a\tau,t-\tau)\exp\Big(-\int\limits_0^{\tau}\sigma(x-as)\,ds\Big)d\tau.$$

As a result, we obtain a formula for solving the problem (11)–(13)

$$u(x,t)=\begin{cases} u_0(x-at)\exp\Big(-\int\limits_0^{t}\sigma(x-as)\,ds\Big)+ \\ +\int\limits_0^{t}f(x-a\tau,t-\tau)\exp\Big(-\int\limits_0^{\tau}\sigma(x-as)\,ds\Big)d\tau, \quad x-at\geqslant 0, \\ \gamma\Big(t-\frac{x}{a}\Big)\exp\Big(-\int\limits_0^{x/a}\sigma(x-as)\,ds\Big)+ \\ +\int\limits_0^{x/a}f(x-a\tau,t-\tau)\exp\Big(-\int\limits_0^{\tau}\sigma(x-as)\,ds\Big)d\tau, \quad x-at\leqslant 0. \end{cases}$$

After a series of substitutions, we write the expression for the desired function for a fixed t, moving along the segment $[0, l]$ from left to right

$$u(x,t) = \begin{cases} \gamma\Big(t - \dfrac{x}{a}\Big) \exp\Big(- \dfrac{1}{a} \displaystyle\int_0^x \sigma(s)\, ds\Big) + \\ + \dfrac{1}{a} \displaystyle\int_0^x f(\tau, t - \dfrac{x-\tau}{a}) \exp\Big(- \dfrac{1}{a} \int_\tau^x \sigma(s)\, ds\Big) d\tau, & 0 \leqslant x \leqslant at, \\ u_0(x - at) \exp\Big(- \dfrac{1}{a} \displaystyle\int_{x-at}^x \sigma(s)\, ds\Big) + \\ + \dfrac{1}{a} \displaystyle\int_{x-at}^x f(\tau, \tau - \dfrac{x-\tau}{a}) \exp\Big(- \dfrac{1}{a} \int_\tau^x \sigma(s)\, ds\Big) d\tau, & at \leqslant x \leqslant l. \end{cases} \tag{17}$$

If the value of at is greater than l, then only the first part of the (17) formula is used. This agreement is valid in the future for all similar two-part formulas.

If the absorption coefficient is $\sigma(x) = 0$, then the formula becomes

$$u(x,t) = \begin{cases} \gamma\Big(t - \dfrac{x}{a}\Big) + \dfrac{1}{a} \displaystyle\int_0^x f\Big(\tau, t - \dfrac{x-\tau}{a}\Big) d\tau, & 0 \leqslant x \leqslant at, \\ u_0(x - at) + \dfrac{1}{a} \displaystyle\int_{x-at}^x f\Big(\tau, \tau - \dfrac{x-\tau}{a}\Big) d\tau, & at \leqslant x \leqslant l. \end{cases}$$

4 Inverse problem for the transport equation

4.1 Statement of the inverse problem

Consider the inverse problem for the one-dimensional equation of simple transport with absorption, the study of which is the main goal of this paper.

$$u_t + au_x + \sigma(x)u = \varphi(t)g(x), \qquad 0 \leqslant x \leqslant l, \quad 0 \leqslant t \leqslant T, \tag{18}$$

$$u(0,t) = \gamma(t), \tag{19}$$

$$u(x,0) = u_0(x), \quad u(x,T) = u_1(x). \tag{20}$$

Here we consider that

- $a = \text{const} > 0$ — transfer speed,
- $T > 0$ — duration of measurements,
- $l > 0$ — the length of the rod with the substance,
- $\sigma(x)$ — absorption coefficient,
- $\varphi(t)$ — piecewise constant function,
- $\gamma(t)$ — incoming stream
- $u_0(x)$ — initial state of the system,
- $u_1(x)$ — final state of the system,
- $u(x,t)$ — density of the transferred substance.

All parameters a, l, T and functions $\sigma(x), u_0(x), u_1(x), \gamma(t)$ are assumed to be known; it is required to find function for the density of additional sources $g(x)$ and function for the density of the initial substance $u(x,t)$ that satisfy the conditions of the problem (18)–(20).

Note that the formulated differential equation (18) is a special case of the equation (8), where $A = -a\dfrac{d}{dx} - \sigma(x)$ is an operator in the space $L_1[0,l]$ on the domain of definition $D(A) = \{f \in AC[0,l] \,:\, f(0) = 0\}$.

The operator of simple translation with absorption A in this case generates a semigroup $U(t)$ acting according to the rule

$$U(t)f(x) = \begin{cases} 0, & 0 \leqslant x \leqslant at, \\ f(x-at)\exp\Big(-\dfrac{1}{a}\int\limits_0^{at}\sigma(x-s)\,ds\Big), & at < x \leqslant l. \end{cases} \tag{21}$$

4.2 Solution of the inverse problem

Let us derive a theoretical scheme for solving the inverse problem for the one-dimensional equation of simple transport with absorption in the special case when the function $\varphi(x)$ is piecewise constant.

Consider an auxiliary problem, for this we introduce a new function $w(x,t)$ and obtain a homogeneous problem

$$w_t + a w_x + \sigma(x) w = 0, \qquad 0 \leqslant x \leqslant l, \quad 0 \leqslant t \leqslant T, \tag{22}$$

$$w(0,t) = \gamma(t), \tag{23}$$

$$w(x,0) = u_0(x). \tag{24}$$

The resulting formulation corresponds to the direct problem, which was studied in detail in the previous chapter. Let's use the formula (17) and write out the resolving formula for solving the problem (22)–(24):

$$w(x,t) = \begin{cases} \gamma\Big(t - \dfrac{x}{a}\Big) \exp\Big(- \dfrac{1}{a} \displaystyle\int\limits_0^x \sigma(s)\, ds\Big), & 0 \leqslant x \leqslant at, \\ u_0(x - at) \exp\Big(- \dfrac{1}{a} \displaystyle\int\limits_{x-at}^x \sigma(s)\, ds\Big), & at \leqslant x \leqslant l. \end{cases} \tag{25}$$

For the function $u(x,t)$ sought in the problem (18)–(20) we make the change

$$u(x,t) = v(x,t) + w(x,t) \tag{26}$$

with the function $w(x,t)$ found by the formula (25).

For a new unknown function $v(x,t)$ we get the problem

$$v_t + a v_x + \sigma(x) v = \varphi(t) g(x), \qquad 0 \leqslant x \leqslant l, \quad 0 \leqslant t \leqslant T, \tag{27}$$

$$v(0,t) = 0, \tag{28}$$

$$v(x,0) = 0, \quad v(x,T) = u_1(x) - w(x,T). \tag{29}$$

Explicit resolving formulas for this problem can be obtained using the formulas from Section 2.2.

We introduce a new function $h(x)$ in the form

$$h(x) = v(x,T) - U(T) v(x,0) = u_1(x) - w(x,T).$$

$$h(x)=\begin{cases} u_1(x)-\gamma\Big(T-\dfrac{x}{a}\Big)\exp\Big(-\dfrac{1}{a}\displaystyle\int\limits_0^x\sigma(s)\,ds\Big), & 0\leqslant x\leqslant aT,\\ u_1(x)-u_0(x-aT)\exp\Big(-\dfrac{1}{a}\displaystyle\int\limits_{x-aT}^x\sigma(s)\,ds\Big), & aT\leqslant x\leqslant l. \end{cases} \tag{30}$$

The statement of the reduced inverse problem (27)–(29) is conveniently presented in an abstract form using the simple translation operator with absorption A

$$\begin{aligned} &v'(t)=Av+\varphi(t)g, && 0\leqslant t\leqslant T,\\ &v(0)=0,\quad v(T)=h. \end{aligned}$$

Since the analysis of the abstract problem was described in detail in Chapter 2, we will carry it out for the problem we have just obtained. Let us act on the element $h(x)$ by the operator $(-A)=a\dfrac{d}{dx}+\sigma(x)$, we obtain

$$f(x)=(-Ah(x))=a\frac{dh}{dx}+\sigma(x)h(x)=ah'(x)+\sigma(x)h(x). \tag{31}$$

The formula for $g(x)$ takes the form:

$$g(x)=\sum_{n=0}^{N_0}\frac{1}{\beta^{n+1}}B^nf(x),\quad N_0=\left\lceil\frac{l}{a(T-\tau_{p-1})}\right\rceil-1, \tag{32}$$

where the operator B acts on $(-Ah(x))$ according to the rule

$$Bf(x)=\sum_{k=1}^{p}(\alpha_k-\alpha_{k-1})U(T-\tau_{k-1})f(x). \tag{33}$$

As a result, having obtained the expression for the required function $g(x)$, we can solve the direct problem with the unknown function $v(x,t)$ using the formula (17)

$$v(x,t)=\begin{cases} \dfrac{1}{a}\displaystyle\int\limits_0^x\varphi\Big(t-\dfrac{x-\tau}{a}\Big)g(\tau)\exp\Big(-\dfrac{1}{a}\int\limits_\tau^x\sigma(s)\,ds\Big)d\tau, & 0\leqslant x\leqslant at,\\ \dfrac{1}{a}\displaystyle\int\limits_{x-at}^x\varphi\Big(t-\dfrac{x-\tau}{a}\Big)g(\tau)\exp\Big(-\dfrac{1}{a}\int\limits_\tau^x\sigma(s)\,ds\Big)d\tau, & at\leqslant x\leqslant l. \end{cases} \tag{34}$$

We use the formula (17) and return to solving the problem (18)–(20), knowing the values for $v(x,t)$ and $w(x,t)$ from the expressions (**??**) and (34) respectively, we get

$$u(x,t) = \begin{cases} \gamma\Big(t - \dfrac{x}{a}\Big) \exp\Big(-\dfrac{1}{a}\int\limits_0^x \sigma(s)\,ds\Big) + \\ +\dfrac{1}{a}\int\limits_0^x \varphi\Big(t - \dfrac{x-\tau}{a}\Big) g(\tau) \exp\Big(-\dfrac{1}{a}\int\limits_\tau^x \sigma(s)\,ds\Big) d\tau, & 0 \leqslant x \leqslant at, \\ u_0(x-at) \exp\Big(-\dfrac{1}{a}\int\limits_{x-at}^x \sigma(s)\,ds\Big) + \\ +\dfrac{1}{a}\int\limits_{x-at}^x \varphi\Big(t - \dfrac{x-\tau}{a}\Big) g(\tau) \exp\Big(-\dfrac{1}{a}\int\limits_\tau^x \sigma(s)\,ds\Big) d\tau, & at \leqslant x \leqslant l. \end{cases} \tag{35}$$

If the absorption coefficient is $\sigma(x) = 0$, then the formula becomes

$$u(x,t) = \begin{cases} \gamma\Big(t - \dfrac{x}{a}\Big) + \dfrac{1}{a}\int\limits_0^x \varphi\Big(\tau - \dfrac{x-\tau}{a}\Big) g(\tau) d\tau, & 0 \leqslant x \leqslant at, \\ u_0(x-at) + \dfrac{1}{a}\int\limits_{x-at}^x \varphi\Big(\tau - \dfrac{x-\tau}{a}\Big) g(\tau) d\tau, & at \leqslant x \leqslant l. \end{cases}$$

4.3 Continuity and smoothness of the solution

In any situation under consideration, the function $u(x,t)$ will turn out to be continuous, but its derivatives may be discontinuous, for example, at the points of discontinuity of the function $\varphi(t)$.

Next, we note the conditions with which the nature of the function $g(x)$ obtained by the formula (32) is connected.

The existence of the function $g(x)$ requires the fulfillment of the equalities

$$\gamma(0) = u_0(0); \quad \gamma(T) = u_1(0). \tag{36}$$

Next, we derive conditions under which the function $g(x)$ will be continuous. For this it is necessary that the function $h(x)$ from the formula (30) be continuously differentiable, and the function $f(x)$ from the formula (31) be equal to 0 for $x = 0$.

We obtain the continuity condition for the function $h(x)$, for this we need to avoid a discontinuity at the point $x = aT$:

$$u_1(aT) - \gamma(0)\exp\Big(-\frac{1}{a}\int\limits_0^{aT}\sigma(s)\,ds\Big) = u_1(aT) - u_0(0)\exp\Big(-\frac{1}{a}\int\limits_{aT-aT}^{aT}\sigma(s)\,ds\Big),$$

Thus, the continuity condition for the function $h(x)$: $\gamma(0) = u_0(0)$.

Next, we derive a condition for the continuously differentiability of the function $h(x)$. The derivative of the function $h(x)$ in the interval $0 \leqslant x \leqslant aT$ is equal to

$$\frac{d}{dx}\Bigg(u_1(x) - \gamma\Big(T - \frac{x}{a}\Big)\exp\Big(-\frac{1}{a}\int\limits_0^{x}\sigma(s)\,ds\Big)\Bigg) =$$

$$= u_1'(x) - \gamma'\Big(T - \frac{x}{a}\Big)\Big(-\frac{1}{a}\Big)\exp\Big(-\frac{1}{a}\int\limits_0^{x}\sigma(s)\,ds\Big) -$$

$$-\gamma\Big(T - \frac{x}{a}\Big)\Big(-\frac{1}{a}\Big)\exp\Big(-\frac{1}{a}\int\limits_0^{x}\sigma(s)\,ds\Big)\sigma(x).$$

The derivative of the function $h(x)$ in the interval $aT \leqslant x \leqslant l$ is equal to

$$\frac{d}{dx}\Bigg(u_1(x) - u_0(x-aT)\exp\Big(-\frac{1}{a}\int\limits_{x-aT}^{x}\sigma(s)\,ds\Big)\Bigg) =$$

$$= u_1'(x) - u_0'(x-aT)\exp\Big(-\frac{1}{a}\int\limits_{x-aT}^{x}\sigma(s)\,ds\Big) -$$

$$-u_0(x-aT)\Big(-\frac{1}{a}\Big)\exp\Big(-\frac{1}{a}\int\limits_{x-aT}^{x}\sigma(s)\,ds\Big)\Big(\sigma(x)-\sigma(x-aT)\Big).$$

To avoid the discontinuity of the function $h'(x)$ at the point $x = aT$ we derive the condition

$$-\frac{1}{a}\gamma'(0)\exp\Big(-\frac{1}{a}\int\limits_{0}^{aT}\sigma(s)\,ds\Big) - \frac{1}{a}\gamma(0)\exp\Big(-\frac{1}{a}\int\limits_{0}^{aT}\sigma(s)\,ds\Big)\sigma(aT) =$$

$$= u_0'(0)\exp\Big(-\frac{1}{a}\int\limits_{0}^{aT}\sigma(s)\,ds\Big) - \frac{1}{a}u_0(0)\exp\Big(-\frac{1}{a}\int\limits_{0}^{aT}\sigma(s)\,ds\Big)\Big(\sigma(aT)-\sigma(0)\Big).$$

Using the condition obtained above for the continuity of $h(x)$, we derive the following condition for the continuous differentiability of $h(x)$:

$$\gamma'(0) + au_0'(0) + \sigma(0)u_0(0) = 0.$$

Finally, we derive the condition under which $f(0) = 0$, where $f(x)$ is defined by the formula (31):

$$f(0) = ah'(0) + \sigma(0)h(0) = \gamma'(T) + au_1'(0) + \sigma(0)u_1(0) = 0.$$

As a result, for the continuity of the function $g(x)$, it is required to add additional conditions to the equalities (36)

$$\gamma'(0) + au_0'(0) + \sigma(0)u_0(0) = 0, \quad \gamma'(T) + au_1'(0) + \sigma(0)u_1(0) = 0. \tag{37}$$

5 Program description

5.1 Short review

The program is written using MATLAB programming language. The following parameters and functions are input

1. l — positive number, the length of $[0, l]$.

2. T — positive number, length of the time segment $[0, T]$.

3. a — a positive number, the transfer speed.

4. $[\tau_1, ..., \tau_p]$ —an array of numbers from the interval $(0, T)$, sorted in ascending order; internal partition points of the segment $[0, T]$.

5. $[\alpha_1, ..., \alpha_p]$ — array of numbers;n values of a piecewise constant function $\varphi(t)$.

6. τ_0 and α_0 are always 0.

7. $\gamma(t)$ — real function corresponding to the boundary condition $u(0, t) = \gamma(t)$.

8. $u_0(x)$ — real function corresponding to the initial condition$u(x, 0) = u_0(x)$.

9. $u_1(x)$ — real function corresponding to the final condition$u(x, T) = u_1(x)$.

Additionally, the number of points of the uniform partition of the segment $[0, l]$ and the segment $[0, T]$ is chosen.

At the output, we get a graph for the function $g(x)$ and a three-dimensional graph for $u(x, t)$.

5.2 Algorithm

Consider the main points of the algorithm for constructing a solution to the (18)–(20) problem, which were implemented in a program in the MATLAB programming language.

1. Search for the function $h(x)$ using the formula (30). The result of the auxiliary task (22)–(24) is automatically taken into account.

2. Search for the function $f(x)$ obtained by the action of the transfer operator on the function $h(x)$ according to the formula (31).

3. Search for the function $Bf(x)$ obtained by the action of the operator B on the function found in the previous paragraph by the formula (33). The function $B^2f(x)$ can be found by applying the operator B to the function $Bf(x)$. All powers of the operator B can be found by successively applying this operator several times.

4. Search for the function $g(x)$ using the formula (32).

5. Search for a three-dimensional function $u(x,t)$ by the formula (35).

We specially emphasize that the algorithm used, due to the peculiarity of the method, gives a solution to the problem in a finite number of steps. In computer calculations, there is only a small error in the answer of the order of 10^{-4} due to the accumulating computational error.

6 Results of numerical experiments

This section will consider examples of solving inverse problems using a program written in the MATLAB language. We will consider the following statement of the problem

$$
\begin{aligned}
& u_t + a u_x + \sigma(x) u = \varphi(t) g(x), && 0 \leqslant x \leqslant l, \quad 0 \leqslant t \leqslant T,\\
& u(0,t) = \gamma(t), && \\
& u(x,0) = u_0(x), \quad u(x,T) = u_1(x). &&
\end{aligned}
$$

As input data, we set the parameters $l > 0,\ T > 0,\ a > 0$, as well as functions

- $\sigma(x)$ — absorption coefficient,
- $\gamma(t)$ — incoming stream
- $u_0(x)$ — initial state,
- $u_1(x)$ — final state
- $\varphi(t)$ — piecewise constant hardware function.

At the output, we obtain the source density function $g(x)$ and the main function $u(x,t)$.

6.1 Example 1

Let

$$
\begin{aligned}
& l = 3, \quad T = 2, \quad a = 1,\\
& \sigma(x) = 0, \quad \gamma(t) = 0, \quad u_0(x) = x^2 + x^3, \quad u_1(x) = \frac{x^3}{3}.
\end{aligned}
$$

Define the function $\varphi(t)$ as follows

$$
\varphi(t) = \begin{cases} -2, & 0 \leqslant t < 1,\\ 2, & 1 \leqslant t \leqslant 2. \end{cases}
$$

The resulting function $g(x)$ in the process of running the program looks like this:

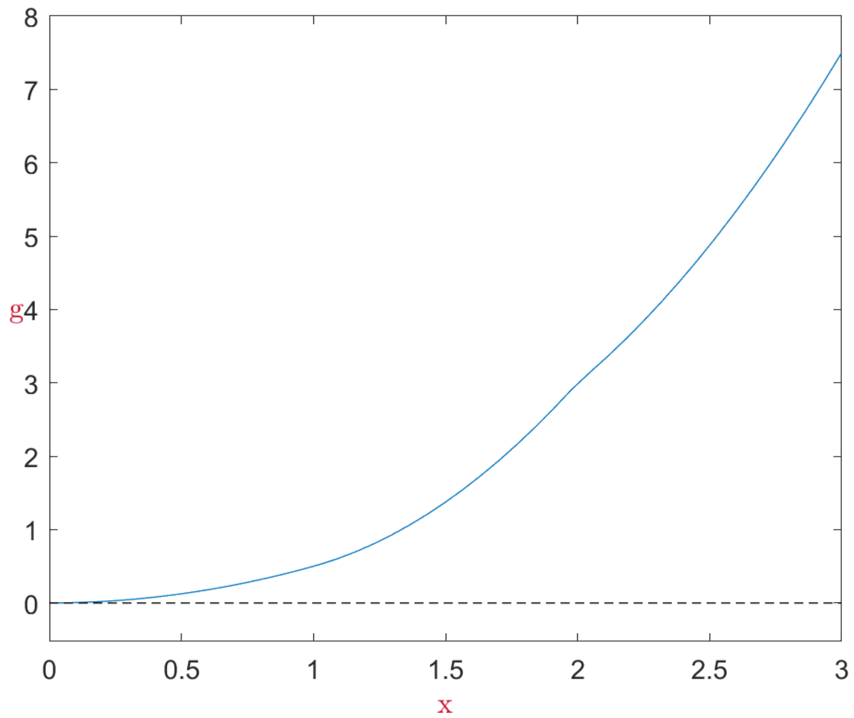

Figure 1: Function $g(x)$ in example 1.

The function $g(x)$ turns out to be continuous, since the matching conditions (36) and (37) are satisfied.

Let us consider in detail how this function $g(x)$ is obtained by the formula (32). In this example, the constant variable N_0 will take on the value:

$$N_0 = \left\lceil \frac{3}{2-1} \right\rceil - 1 = 2.$$

So the function $g(x)$ will be composed of the sum of three terms, namely:

$$g(x) = \frac{1}{\beta} f(x) + \frac{1}{\beta^2} B f(x) + \frac{1}{\beta^3} B^2 f(x), \quad \beta = 2.$$

Next, let's see how functions are compiled by the iteration method from the formula for $g(x)$.

Below are graphs for the functions $h(x)$ and $f(x)$, obtained by formulas (30) and (31) respectively.

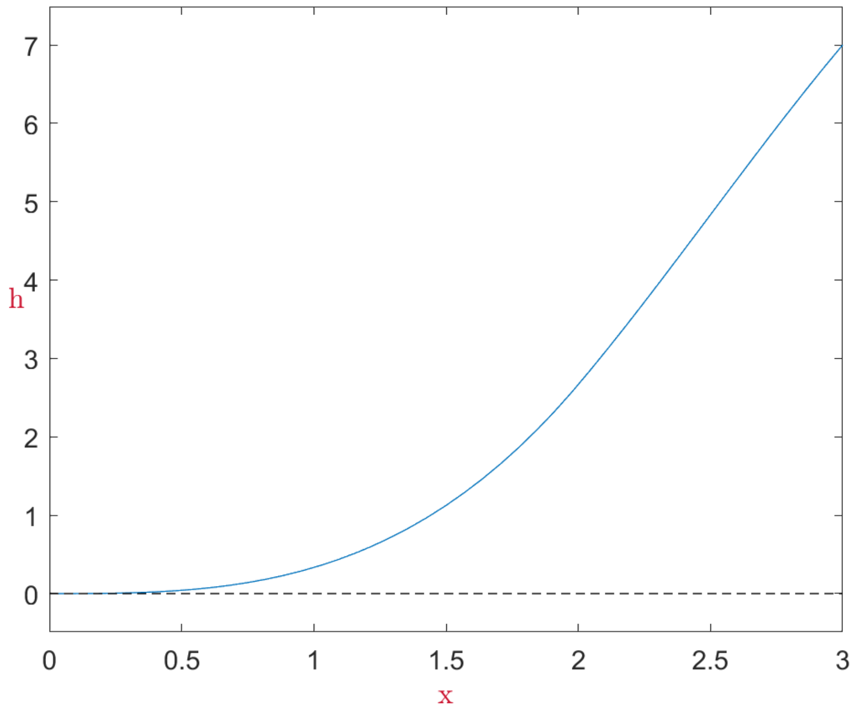

Figure 2: Function $h(x)$ in example 1.

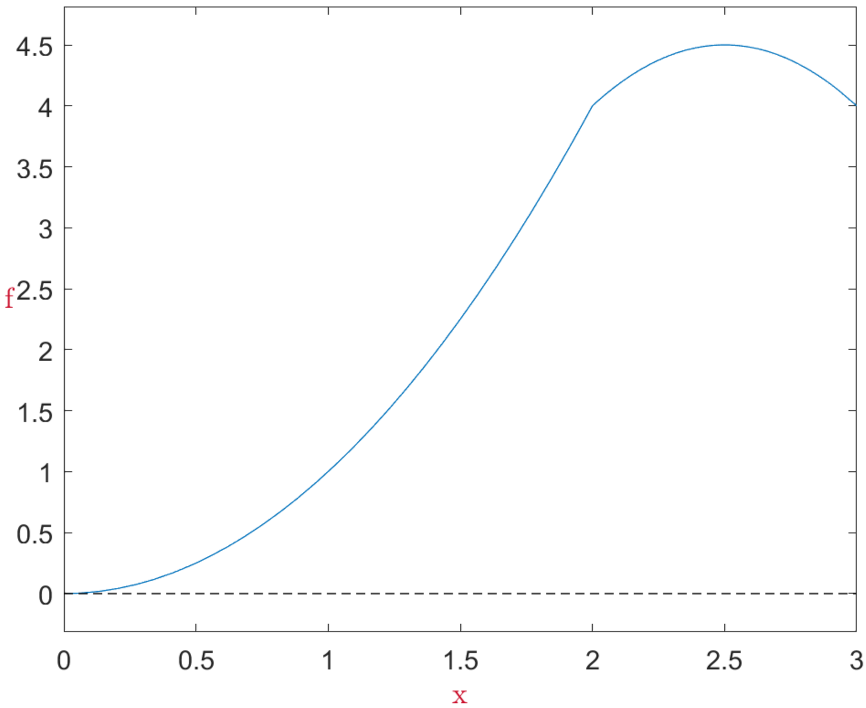

Figure 3: Function $f(x)$ in example 1.

Below are graphs for the functions $Bf(x)$ and $B^2f(x)$, obtained by the formula (33).

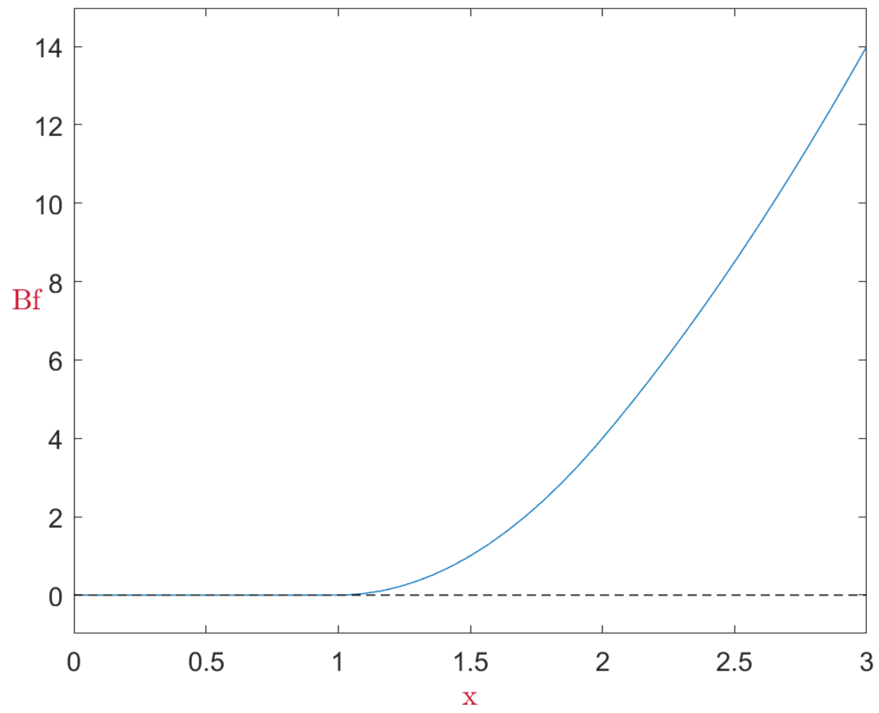

Figure 4: Function $Bf(x)$ in example 1.

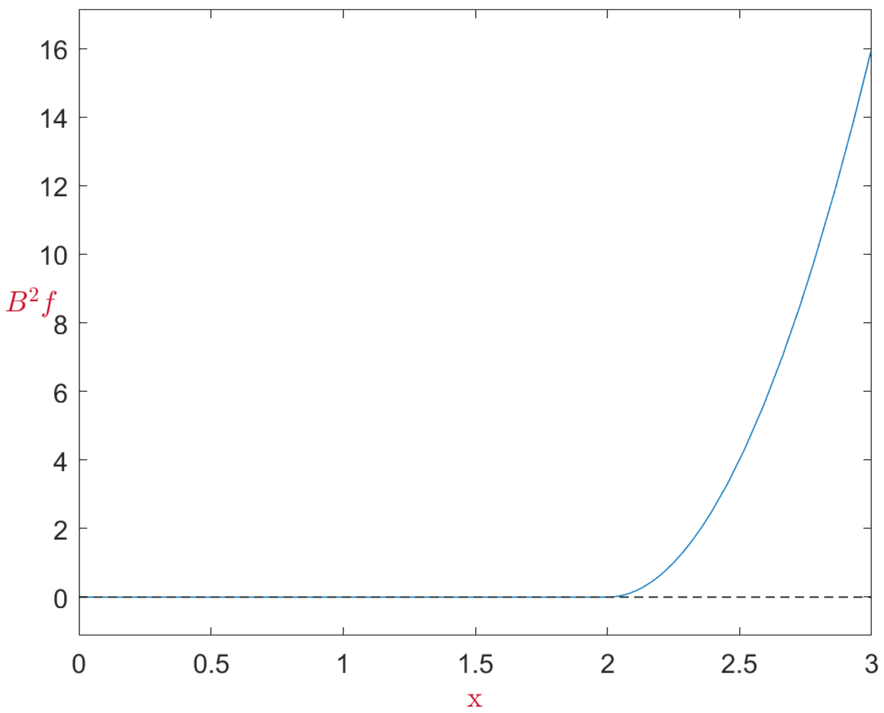

Figure 5: Function $B^2f(x)$ in example 1.

Three-dimensional graph for the function $u(x,t)$ obtained by the formula (35):

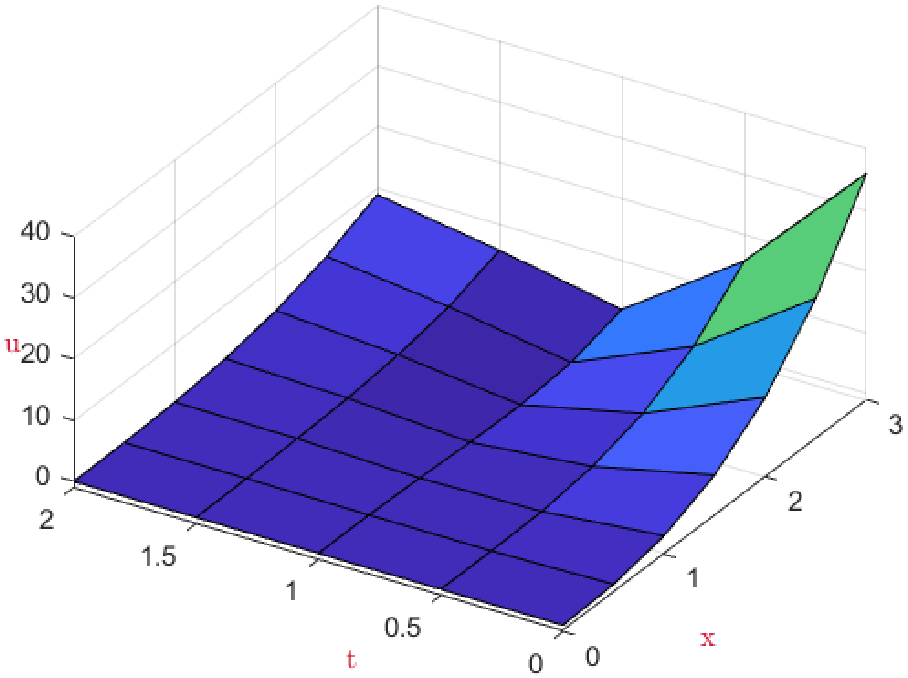

Figure 6: Function $u(x,t)$ in example 1.

6.2 Example 2

Let

$$l = 4, \quad T = 3, \quad a = 1,$$
$$\sigma(x) = 0, \quad \gamma(t) = 0, \quad u_0(x) = 0, \quad u_1(x) = x.$$

Define the function $\varphi(t)$ as follows

$$\varphi(t) = \begin{cases} 1, & 0 \leqslant t < 1, \\ -1, & 1 \leqslant t < 2, \\ 2, & 2 \leqslant t \leqslant 3. \end{cases}$$

The resulting function $g(x)$ in the process of running the program looks like this:

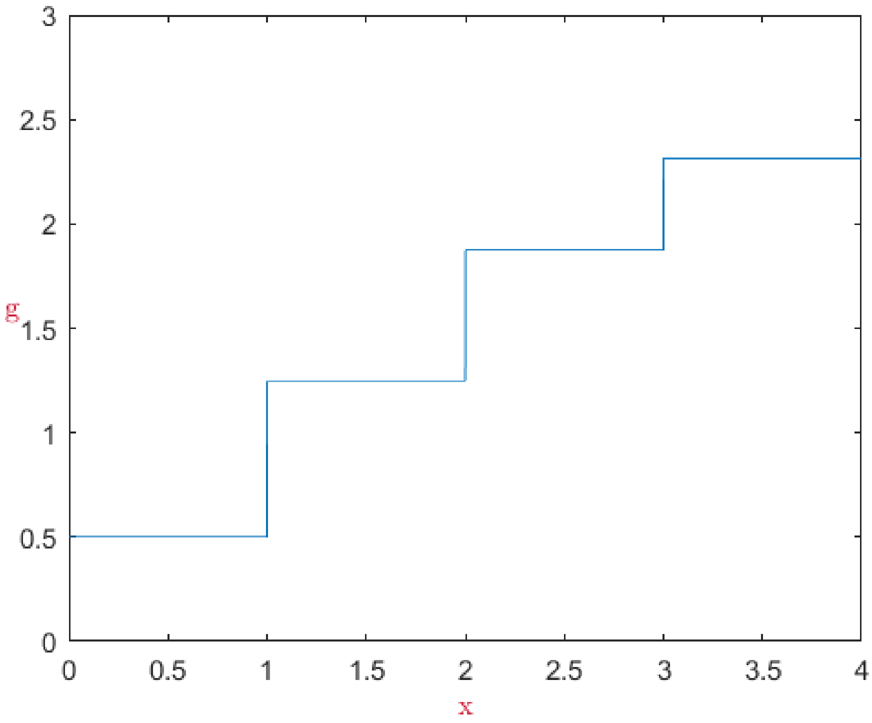

Figure 7: Function $g(x)$ in example 2.

The function $g(x)$ exists, but is not continuous, since the matching conditions (36) are satisfied, but the conditions (37) are not.

Let us consider in detail how this function $g(x)$ is obtained by the formula (32). In this example, the constant variable N_0 will take on the value:

$$N_0 = \left\lceil \frac{4}{3-2} \right\rceil - 1 = 3.$$

So the function $g(x)$ will be composed of the sum of three terms, namely:

$$g(x) = \frac{1}{\beta} f(x) + \frac{1}{\beta^2} B f(x) + \frac{1}{\beta^3} B^2 f(x) + \frac{1}{\beta^4} B^3 f(x), \quad \beta = 2.$$

Next, let's see how functions are compiled by the iteration method from the formula for $g(x)$.

Below are graphs for the functions $h(x)$ and $f(x)$, obtained by the formulas (30) and (31), respectively.

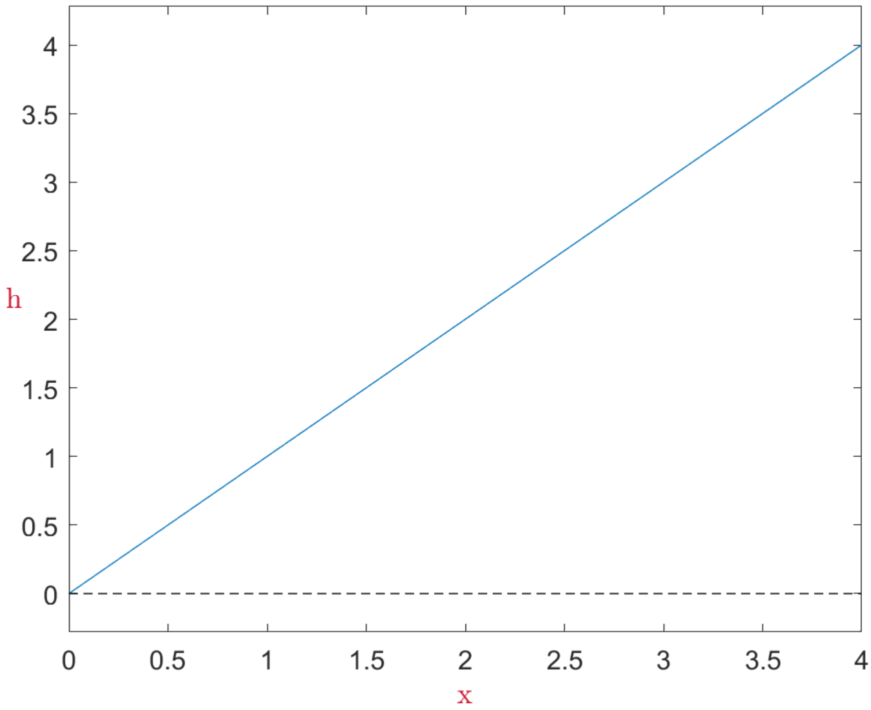

Figure 8: Function $h(x)$ in example 2.

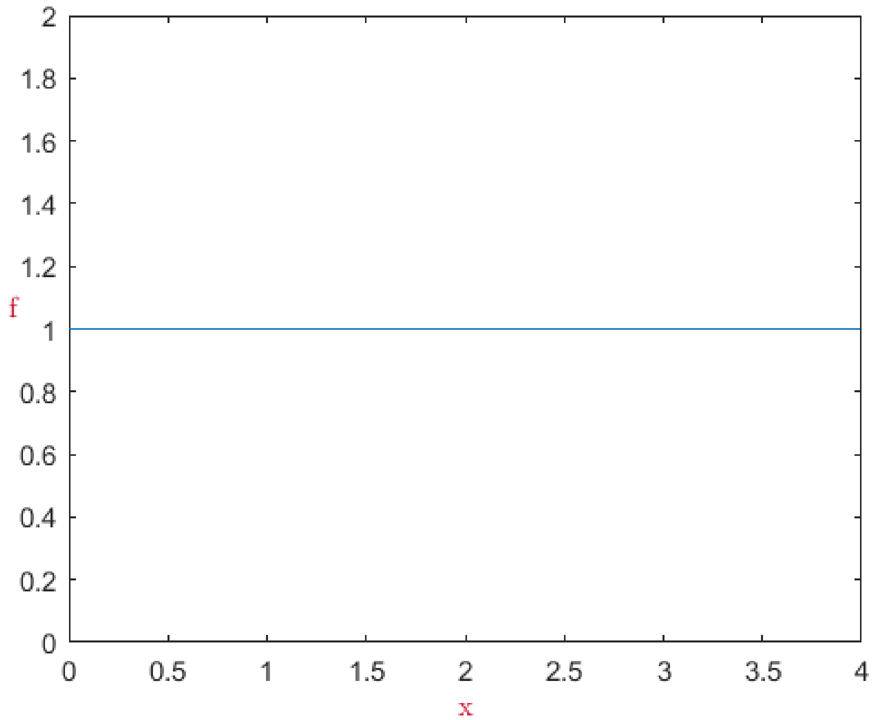

Figure 9: Function $f(x)$ in example 2.

Below are graphs for the functions $Bf(x)$ and $B^2f(x)$, obtained by the formula (33).

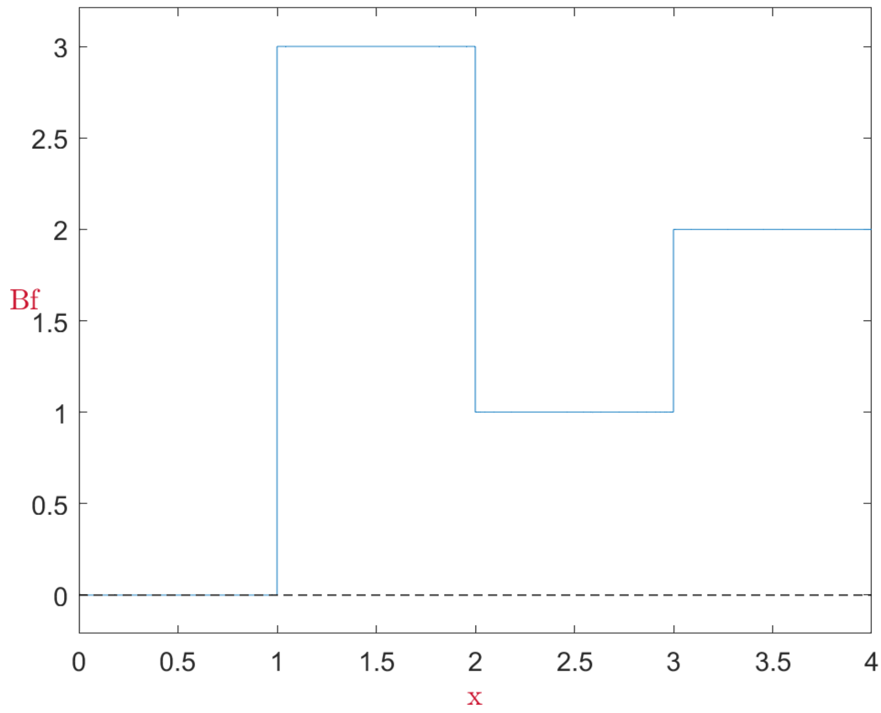

Figure 10: Function $Bf(x)$ in example 2.

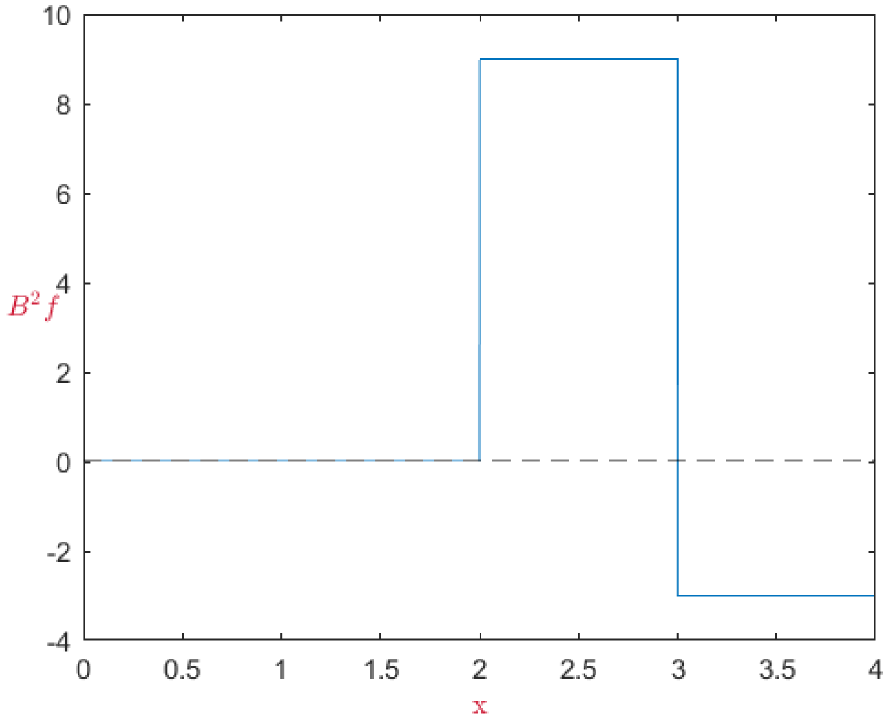

Figure 11: Function $B^2f(x)$ in example 2.

Below are graphs for the functions $B^3f(x)$ and $u(x,t)$, obtained by the formulas (33) and (35), respectively.

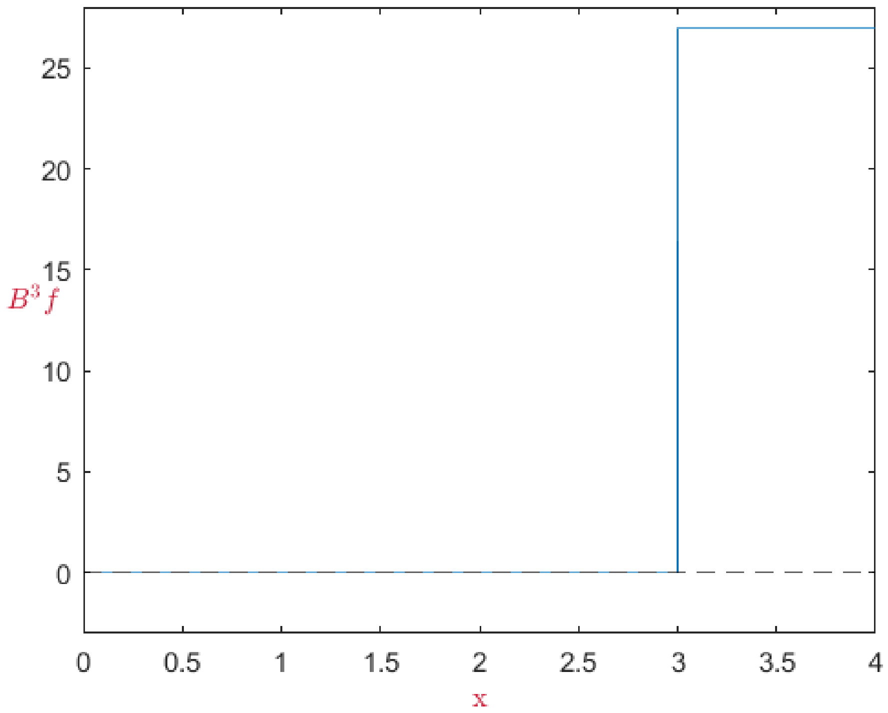

Figure 12: Function $B^3f(x)$ in example 2.

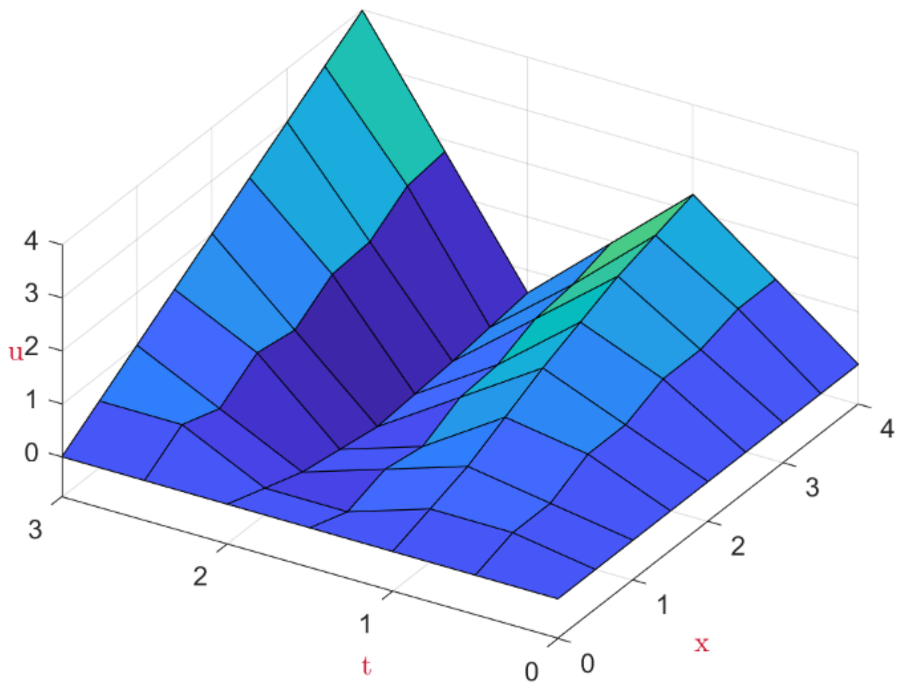

Figure 13: Function $u(x,t)$ in example 2.

Further examples will be given without details, as in Example 1 and Example 2.

6.3 Example 3

Let

$$
\begin{aligned}
&l = 3, \quad T = 2, \quad a = 1, \\
&\sigma(x) = 0, \quad \gamma(t) = 2 + \sin \pi t, \quad u_0(x) = -\pi x + 2, \quad u_1(x) = \frac{x^2}{2} - \pi x + 2.
\end{aligned}
$$

Define the function $\varphi(t)$ as follows

$$
\varphi(t) = \begin{cases} -1, & 0 \leqslant t < 1, \\ 1, & 1 \leqslant t \leqslant 2. \end{cases}
$$

The resulting $g(x)$ function looks like this:

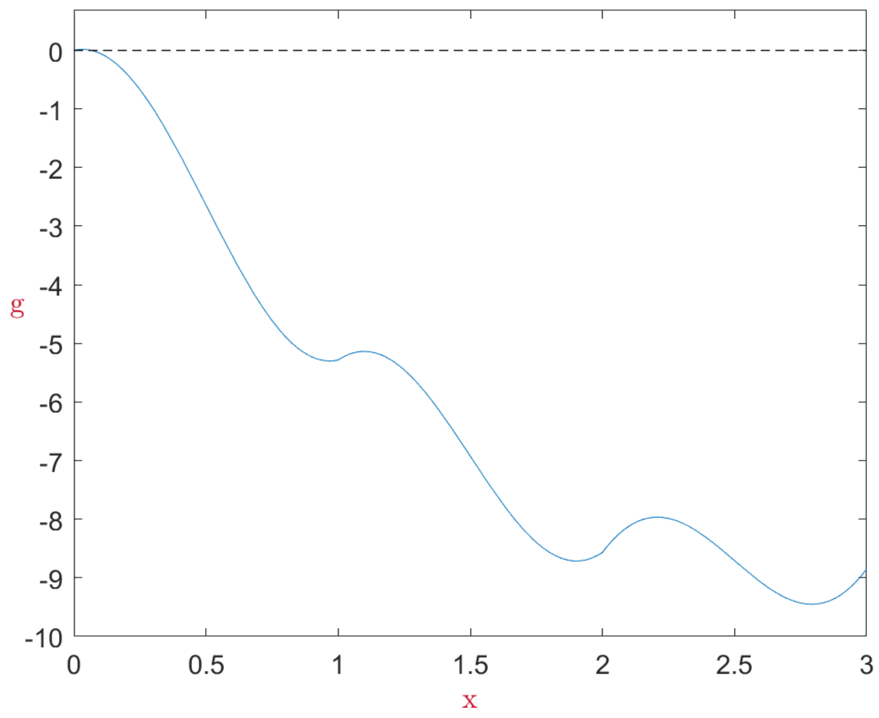

Figure 14: Function $g(x)$ in example 3.

The function $g(x)$ is continuous since the matching conditions (36) and (37) are satisfied.

Next, a three-dimensional graph for the function $u(x,t)$:

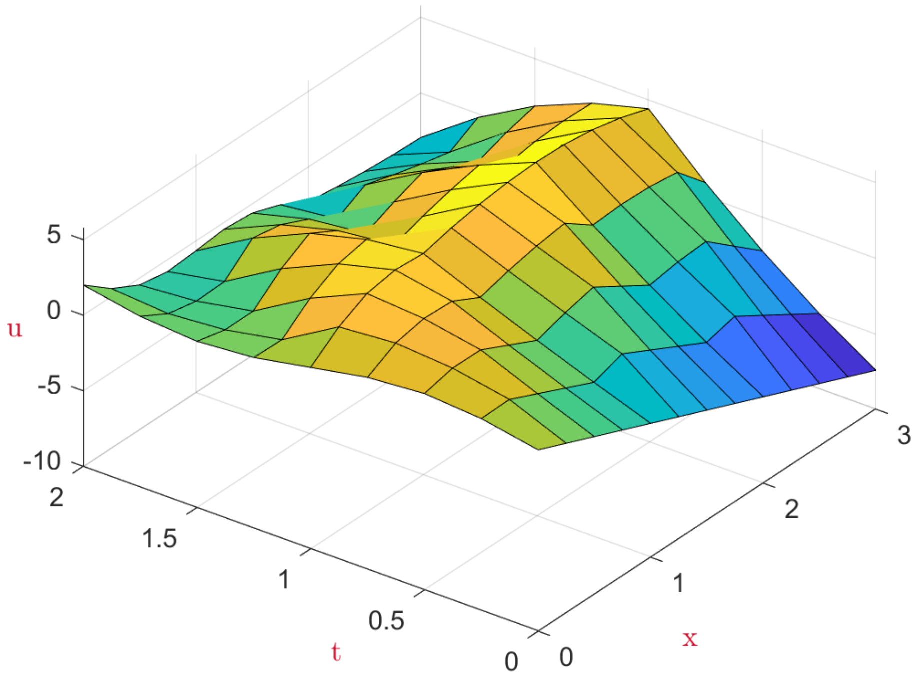

Figure 15: Function $u(x,t)$ in example 3.

6.4 Example 4

Let

$$l = 3, \quad T = 2, \quad a = 1,$$
$$\sigma(x) = -1, \quad \gamma(t) = 1, \quad u_0(x) = e^x, \quad u_1(x) = x + 1.$$

Define the function $\varphi(t)$ as follows

$$\varphi(t) = \begin{cases} 1, & 0 \leqslant t \leqslant 1, \\ 2, & 1 \leqslant t \leqslant 2. \end{cases}$$

The matching conditions (36) and (37) are satisfied, which means that the function $g(x)$ is continuous.

Below are the functions $g(x)$ and $u(x,t)$ obtained in the course of program operation.

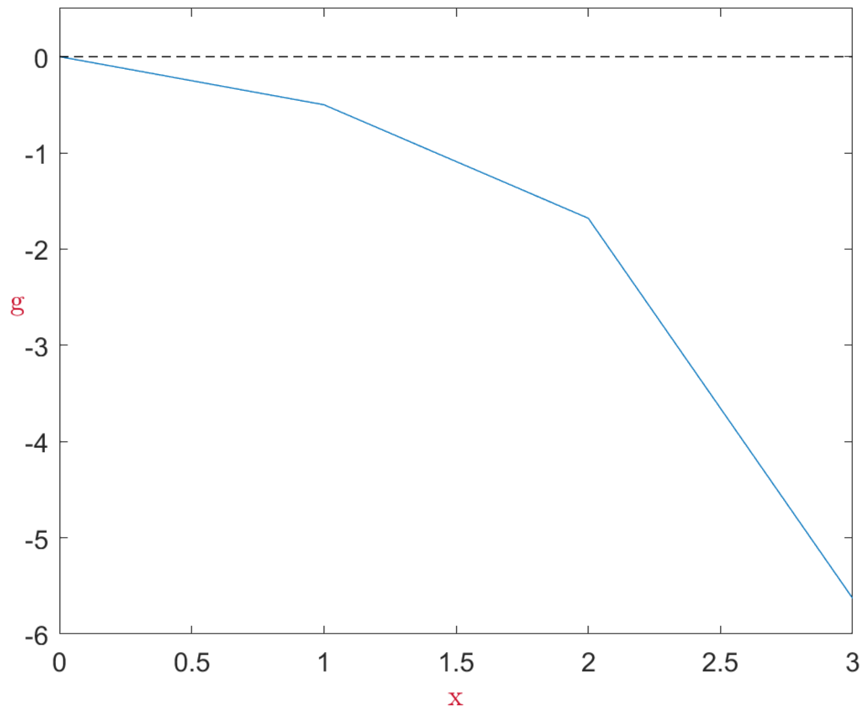

Figure 16: Function $g(x)$ in example 4.

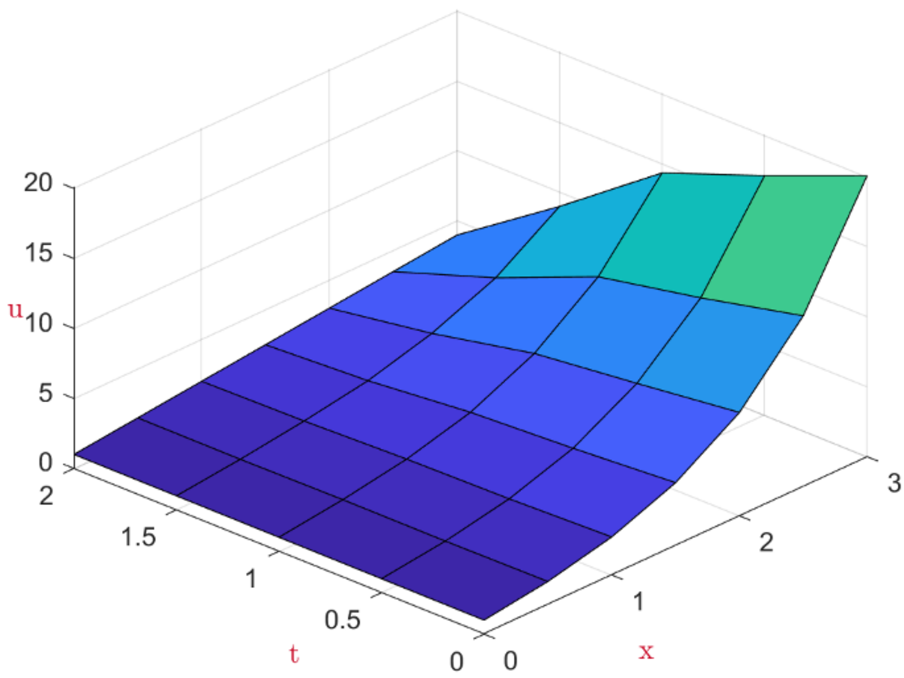

Figure 17: Function $u(x,t)$ in example 4.

6.5 Example 5

Let

$$
\begin{aligned}
&l = 3, \quad T = 3, \quad a = 1, \\
&\sigma(x) = x, \quad \gamma(t) = 0, \quad u_0(x) = x, \quad u_1(x) = \sin(x).
\end{aligned}
$$

Define the function $\varphi(t)$ as follows

$$
\varphi(t) = \begin{cases} 1, & 0 \leqslant t < 1, \\ -1, & 1 \leqslant t < 2, \\ 2, & 2 \leqslant t \leqslant 3. \end{cases}
$$

The resulting $g(x)$ function looks like this:

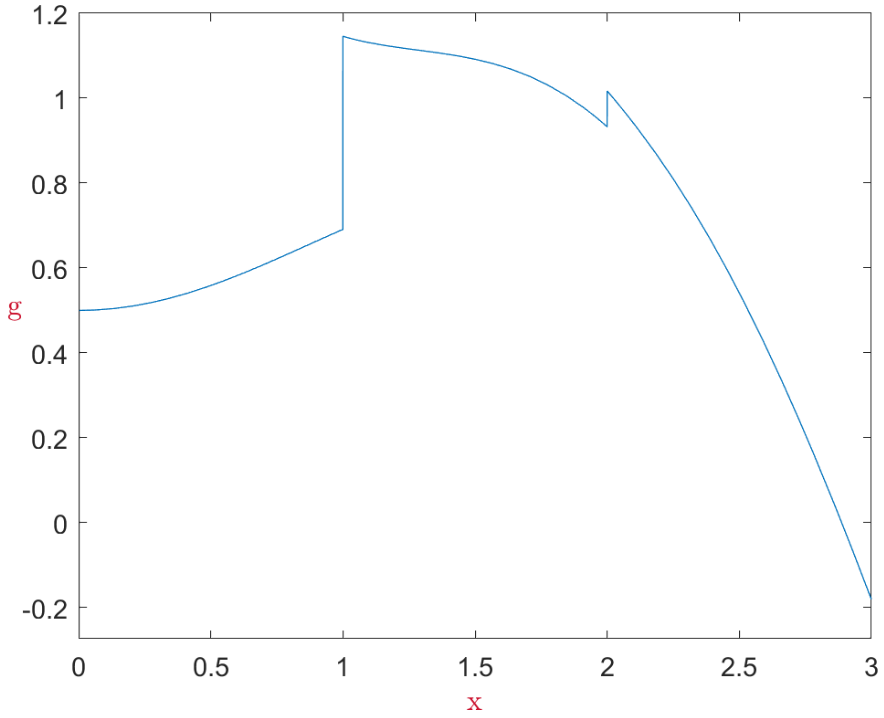

Figure 18: Function $g(x)$ in example 5.

The function $g(x)$ exists, but is not continuous, since the matching conditions (36) are met, but the conditions (37) are not.

Based on the obtained function $g(x)$, we get the graph of the solution $u(x,t)$:

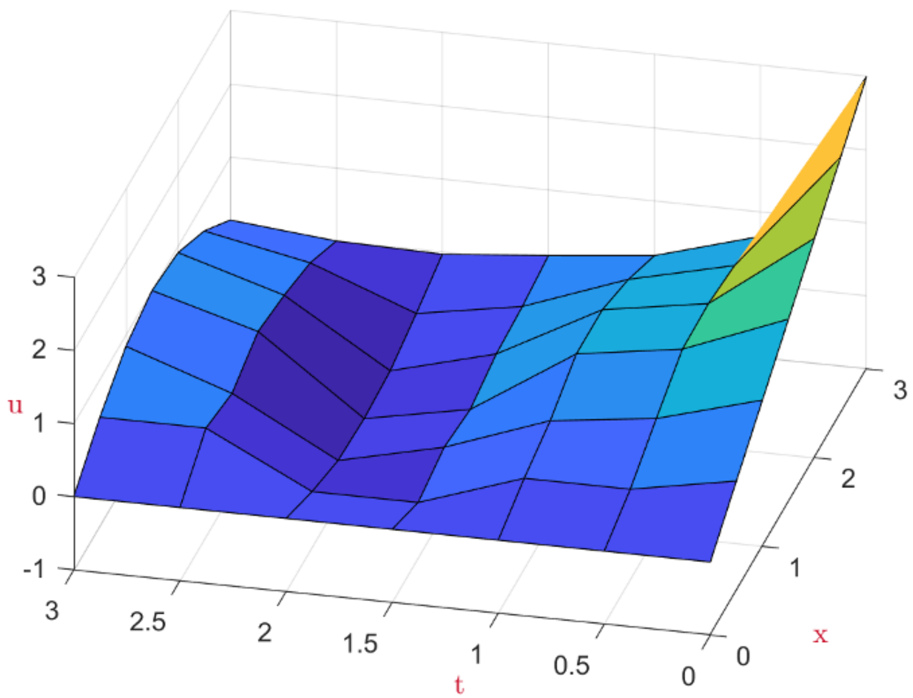

Figure 19: Function $u(x,t)$ in example 5.

6.6 Example 6

Let

$$l = 2, \quad T = 2, \quad a = 1,$$
$$\sigma(x) = 2x^{10}(2-x)^{10}, \quad \gamma(t) = 0, \quad u_0(x) = x^2, \quad u_1(x) = \frac{x^2}{2}, \quad \varphi(t) = 1.$$

The function $g(x)$ will turn out to be continuous, since the matching conditions (36) and (37) are satisfied.

Below are the functions $g(x)$ and $u(x,t)$ obtained in the course of program operation.

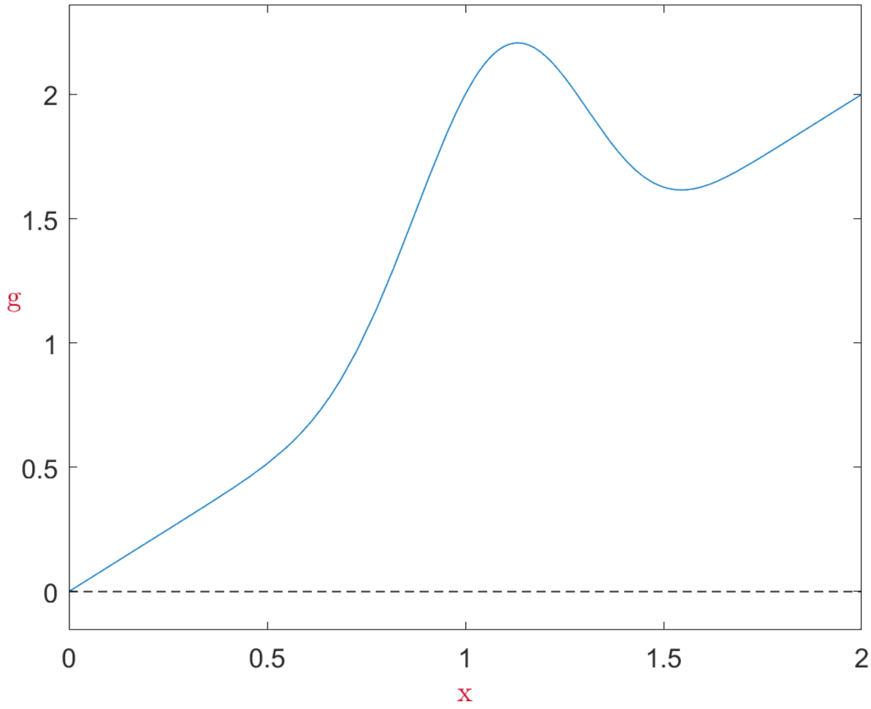

Figure 20: Function $g(x)$ in example 6.

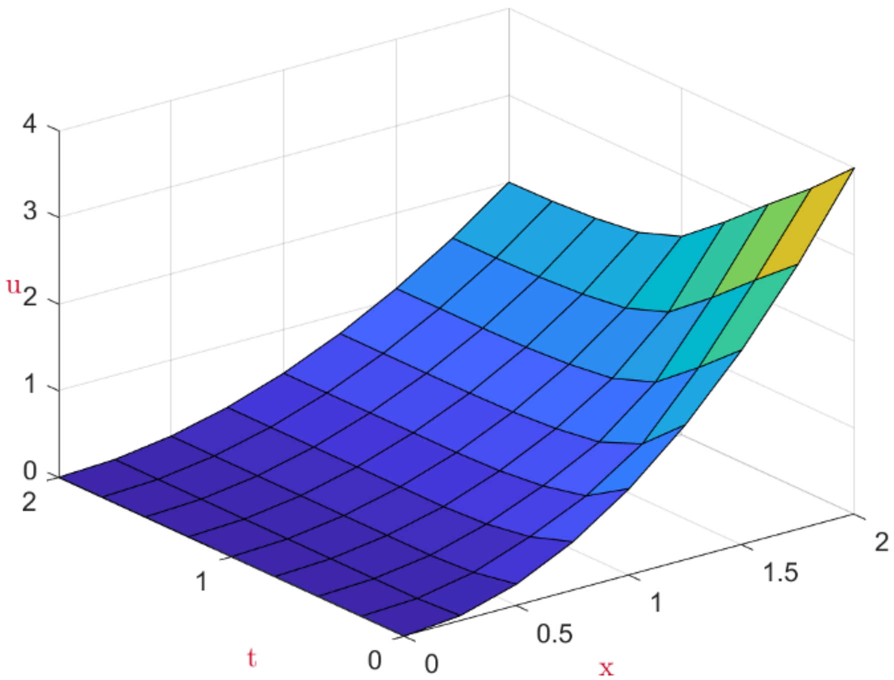

Figure 21: Function $u(x,t)$ in example 6.

Conclusion

In the present work, we study a special case of the inverse problem for the one-dimensional equation of simple transport with absorption. During the research:

1. A detailed derivation of the solution of an abstract problem for an evolution equation with a nilpotent semigroup is obtained.

2. A theoretical scheme for solving the inverse problem is developed and explicit resolving formulas are obtained.

3. Based on the obtained formulas, an algorithm for solving the inverse problem was obtained.

4. The question of continuity and smoothness of the obtained solutions is investigated.

5. A computer program has been written that implements theoretical algorithms, visualizes calculations and exports data.

6. A series of computational experiments was carried out for various values of the input data, which confirmed the high reliability of the algorithm.

As a result of the work done, all the goals set were achieved and the study can be considered completed.

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