Computer Science 340
Reasoning about Computation
Princeton University, Fall 2020

Assignment 3 Handout
Due 10/28/2020, 9:00PM

Problem 1 [20 points]
Consider $n \geq 1$ pigeonholes and n^2 pigeons. The pigeons are of n different types, each type consisting of n pigeons.
The pigeons are initially placed in random order on a branch of a tree. Then, one after the other, the pigeons pick a pigeonhole at random (and independently from each other) and

- if there is no other pigeon of the same type in the pigeonhole, the pigeon rests in that pigeonhole,
- otherwise the pigeon flies away and never comes back.

The experiment ends once all pigeons have left the branch.
(A): Consider a particular pigeonhole and the n pigeons of a particular type. What is the probability that that pigeonhole never has a pigeon of that type?
(B): What is the expected number of pigeons of a particular type that fly away?
(C): What is the expected number of pigeons that fly away?

Problem 2 [20 points]
Consider two random variables X, Y, that take values from the set $\{0, 1\}$. For those variables you also know

$$\begin{aligned}
\Pr[X = 1 \text{ and } Y = 1] &= 0.32 \\
\Pr[X = 1 \text{ and } Y = 0] &= 0.08 \\
\Pr[X = 0 \text{ and } Y = 1] &= 0.48
\end{aligned}$$

Are X and Y independent?

Problem 3 [20 points]
Show that any simple graph $G = (V, E)$ with $\delta(G) > \frac{1}{2} \cdot (|V| - 2)$ is connected. Note that $\delta(G)$ is defined as the minimum degree of the vertices of the graph G, namely $\delta(G) := \min\{d(v) : v \in V\}$.
(Hint: You can use a proof by contradiction. Consider two vertices in different connected components and consider the sets of the neighbors of each of the vertices; you may find the inclusion-exclusion principle for two sets helpful.)

Problem 4 [20 points]
An ℓ-regular graph, $\ell \geq 1$, is a simple graph where all of its vertices have degree ℓ.
Let ℓ be an odd positive integer. Is there an ℓ-regular graph $G = (V, E)$ with $|V|$ an odd positive integer?

Problem 5 [20 points]
Suppose that G is a simple graph with $2n$ nodes, for $n \geq 1$, and no triangles (ie, no cycles of length 3). Prove that G has at most n^2 edges.

Problem 6 [20 points]
A graph is k-edge-connected if there is no set of at most $k - 1$ edges of the graph whose removal disconnects the graph.
Show that a simple graph is 2-edge-connected if and only if it is connected and every edge of the graph is traversed by a cycle.