Homework 5: Problem 6

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Problem 6:
We wish to show that the HITTING-SET decision problem is NP-complete. To do so, it is necessary to show that it is both in NP and that it is NP-hard.

1. To show that the HITTING-SET decision problem is in NP, we can give a polynomial time verification algorithm. For each of the k elements in the candidate hitting set H, we can go through each of the m subsets of A. If a match is found (that is an intersection with one of the elements in the current subset), that subset is marked as intersecting. If the number of matches equals m, then we know that H is a valid hitting set. Because each of the subsets of A can up to n elements long, the time complexity is $O(k \cdot n \cdot m)$.

2. To show that the HITTING-SET decision problem is NP-hard, we can show the reduction VERTEX-COVER$\leq_P$HITTING-SET. We start by considering an undirected graph $G = (V, E)$. For each edge in G, we make a subset containing the two vertices connected by that edge. G only has a vertex cover of size l if and only if each of the edge subsets in G has a non-zero intersection with the set of vertices that make up the proposed vertex cover. This is equivalent to saying that G has a hitting set of size l. We have successfully transformed the inputs of the VERTEX-COVER decision problem to those of the HITTING-SET decision problem. Because the VERTEX-COVER decision problem is NP-complete, we know that the HITTING-SET decision problem must then be NP-hard.

Likewise, if we had a hitting set of size k for a collection of subsets, with each subset representing an edge between two vertices, we could construct a graph by adding an edge from each vertex in the hitting set to its intersection with a vertex in the edge subsets. This would give us a valid vertex cover.

Because we have shown that the HITTING-SET problem is in NP and is also NP-hard, we know that it is NP-complete.