Homework 1: Problem 1

William Svoboda (`wsvoboda`)

Collaborators: None

Problem 1:
We wish to prove

$$P(n) : T_n \leq 2^n$$

for any nonnegative integer n.

(Base Case) We first show $P(0)$, $P(1)$, and $P(2)$ are true, i.e. that the formula is valid when $n = 0$, $n = 1$, and $n = 2$. If $n = 0$, then

$$T_0 \leq 2^0$$

$$T_0 \leq 1$$

By definition we are given $T_0 = 1$, so

$$1 \leq 1$$

And $P(0)$ holds. If $n = 1$, then

$$T_1 \leq 2^1$$

$$T_1 \leq 2$$

By definition we are given $T_1 = 1$, so

$$1 \leq 2$$

And $P(1)$ holds. If $n = 2$, then

$$T_2 \leq 2^2$$

$$T_2 \leq 4$$

By definition we are given $T_2 = 1$, so

$$1 \leq 4$$

And $P(2)$ holds.

(Inductive Step) let $k \geq 3$ and suppose that that $P(j)$ is true for all $0 \leq j \leq k$. Since $k \geq 3$, then $k - 3 \geq 0$ and so we are assuming in particular that $P(k)$, $P(k-1)$, and $P(k-2)$ are all true. It now remains to be shown following the inductive hypothesis that $P(k+1)$ is true.
From the definition of the sequence we can write

$$T_{k+1} = T_k + T_{k-1} + T_{k-2}$$

Then following from our inductive hypothesis we can write

$$T_{k+1} = T_k + T_{k-1} + T_{k-2} \leq 2^k + 2^{k-1} + 2^{k-2}$$

Now the right-hand side of the inequality can be simplified as

$$= \frac{1}{2} \cdot 2^{k+1} + \frac{1}{4} \cdot 2^{k+1} + \frac{1}{8} \cdot 2^{k+1}$$

$$= 2^{k+1} \cdot (\frac{1}{2} + \frac{1}{4} + \frac{1}{8})$$

$$= 2^{k+1} \cdot (\frac{7}{8})$$

We now have the full inequality written as

$$T_{k+1} \leq 2^{k+1}$$

Thus $P(k+1)$ is true. By Strong Induction we have proven that the inequality is valid for any nonnegative integer n.