Homework 2: Problem 4

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Problem 4:
We wish to prove or disprove that rolling two sixes if you know the first roll is a six is at least as likely as rolling two sixes if you know that at least one of the rolls is a six.

Let $P(A)$ be the probability the first roll is a six, and let $P(B)$ be the probability that the second roll is a six. This means the probability of rolling two sixes is $P(A \cap B)$, because intuitively the intersection where both the first roll is a six and the second roll is a six is just where both rolls are sixes. We can also say that the probability of one of the rolls being a six is $P(A \cup B)$ since by the inclusion-exclusion principle the union of two sets is their sum minus their intersection (with the intersection just being where both rolls produce sixes).

We can then write the inequality as

$$P(A \cap B|A) \geq P(A \cap B|A \cup B)$$

Using the formula for conditional probability, we can expand the terms on each side:

$$\frac{P((A \cap B) \cap A)}{P(A)} \geq \frac{P((A \cap B) \cap (A \cup B))}{P(A \cup B)}$$

On the left side, we know that $(A \cap B) \cap A$ is still $A \cap B$ and that on the right side that $(A \cap B) \cap (A \cup B)$ is still $A \cap B$ so we have that

$$\frac{P(A \cap B)}{P(A)} \geq \frac{P(A \cap B))}{P(A \cup B)}$$

$$\frac{1}{P(A)} \geq \frac{1}{P(A \cup B)}$$

We can now see clearly that the left hand event (the probability of rolling two sixes given the first roll is a six) is at least as likely as rolling two sixes given one of the rolls is a six. $A \cup B$ covers the entire area that A and B do, which is larger than A alone. This means that $\frac{1}{P(A)}$ is greater than $\frac{1}{P(A \cup B)}$. At worst, if A and B intersect completely it would simply mean that $\frac{1}{P(A)} = \frac{1}{P(A \cup B)}$.