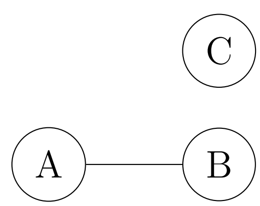
Computer Science 340
Reasoning about Computation
Princeton University, Fall 2020

Assignment 4 Handout

Due 11/11/2020, 9:00PM

Problem 1 [20 points]
Prove that every collection of disjoint intervals (of positive length) on the real line is countable (namely it consists of countably many such intervals). (Hint: Any interval $I \subset \mathbb{R}$ contains a rational number)

Problem 2 [20 points]
An undirected graph $G = (V, E)$ is an interval graph if there exists a family of intervals $\{I_u\}_{u \in V}$ on the real line such that I_u intersects I_v if and only if $(u, v) \in E$. For example, the following graph



is an interval graph since it corresponds to the family of intervals $\{I_a, I_b, I_c\}$ with $I_a = [0, 1]$, $I_b = [0.3, 5]$ and $I_c = [8, 9]$.
Prove that the chromatic number of an interval graph is equal to its clique number (the clique number of a graph is the number of vertices in the largest complete subgraph of the graph.) You can assume without loss of generality that the endpoints of intervals are all distinct.
(Hint: Use a greedy algorithm to color the vertices of the graph.)

Problem 3 [20 points]

(A): Consider the complete graph K_n, where $n \geq 1$. In how many ways can we color the vertices of the graph[1] when provided with k different colors, where $k \geq n$, such that each vertex has a different color than its adjacent vertices?

(Not all k colors have to be used in a coloring of the graph.)

(B): Consider T_n a tree with $n \geq 1$ vertices. Show that the number of ways we can color the vertices of the graph[2] when provided with k, where $k \geq 2$, different colors, such that each vertex has a different color than its adjacent vertices, is $k \cdot (k-1)^{n-1}$.

(Not all k colors have to be used in a coloring of the graph.)

Problem 4 [20 points]

Consider the simple random graph constructed as follows. There are $n \geq 3$ vertices $v_1, v_2, \ldots v_n$ that comprise the vertex set V. Each pair of vertices is adjacent with probability p, where $p \in [0,1]$, independently of other pairs of vertices. Let k be an integer such that $3 \leq k \leq n$.

(A): Show that the expected number of cycles of length k is

$$\frac{n!}{2k \cdot (n-k)!} \cdot p^k$$

(B): Show that the expected number of cycles of length k is at most

$$\frac{n^k \cdot p^k}{2k}$$

(C): Show that for $p = n^{-a}$, with $a > 1$, the limit of the probability that the graph has at least one cycle of length k is 0 as n goes to infinity.

Problem 5 [20 points]

[1] We count two ways to color the graph as different if any vertex of the graph is colored differently between the two ways.

[2] We count two ways to color the graph as different if any vertex of the graph is colored differently between the two ways.

Prove that the number of vertices in a graph is at most the product of the independence number and the chromatic number. The independence number of a graph is the size of the maximum independent set of the graph. As a reminder, an independent set of a graph is a set of vertices of the graph with no edges between any two of them.

Problem 6 [20 points]
Consider a $2n \times 2n$ board (namely a board with $2n$ rows and $2n$ columns for a total of $4n^2$ squares) for $n \geq 1$. We place pebbles on squares (at most one per square). The placement of the pebbles ensures that each column and each row contains exactly n pebbles. Consider the coordinates of a placed pebble to be $x : y$, with $x, y \in \{1, 2, \ldots, 2n\}$, where x is the row number and y is the column number. Show that for any possible valid placement of the pebbles, there is a subset of $2n$ pebbles such that the row numbers are not the same and the column numbers are not the same for any pair of those $2n$ pebbles.