Homework 1: Problem 6

William Svoboda (`wsvoboda`)

Collaborators: None

Problem 6:
We wish to prove

$$P(n) : \sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$$

for all integers $n > 0$.

(Base Case) We first show $P(1)$ is true, i.e. that the formula is valid when $n = 1$. If $n = 1$, then the summation becomes

$$\frac{1}{(2 \cdot 1 - 1)(2 \cdot 1 + 1)} = \frac{1}{2 \cdot 1 + 1}$$

$$\frac{1}{1 \cdot 3} = \frac{1}{3}$$

$$\frac{1}{3} = \frac{1}{3}$$

and $P(1)$ holds.

(Inductive Step) Let $k > 0$ and suppose that $P(n)$ is true when $n = k$. It now remains to be shown following our inductive hypothesis that $P(k+1)$ is true.
From our assumption that $P(k)$ is true, we can rewrite the summation for $n = k + 1$ as

$$\sum_{i=1}^{k+1} \frac{1}{(2i-1)(2i+1)} = \frac{k}{2k+1} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

We can now simplify the right-hand side as

$$= \frac{k}{2k+1} + \frac{1}{(2k+2-1)(2k+2+1)}$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k(2k+3)}{(2k+1)(2k+3)} + \frac{1}{(2k+1)(2k+3)}$$

$$
\begin{aligned}
&= \frac{k(2k+3)+1}{(2k+1)(2k+3)} \\
&= \frac{2k^2+3k+1}{(2k+1)(2k+3)} \\
&= \frac{2k^2+k+2k+1}{(2k+1)(2k+3)} \\
&= \frac{k(2k+1)+(2k+1)}{(2k+1)(2k+3)} \\
&= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} \\
&= \frac{k+1}{2k+3}
\end{aligned}
$$

This is what we expect for $\frac{n}{2n+1}$ when $n = k+1$, thus $P(k+1)$ is true. By Weak Induction we have proven that the formula is valid for all integers $n > 0$.