Homework 1: Problem 2

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Collaborators: None

Problem 2:
We wish to prove

$$P(n) : (1 + a)^n \geq 1 + an$$

where $n \in \mathbb{N}$ and $a \geq -1$ is a real number.

(Base Case) We first show $P(0)$ is true, i.e. that the formula is valid when $n = 0$. If $n = 0$, then

$$(1 + a)^0 \geq 1 + a \cdot 0$$
$$1 \geq 1$$

and $P(0)$ holds.

(Inductive Step) Let $k \geq 0$ and suppose that $P(n)$ is true when $n = k$. It now remains to be shown following our inductive hypothesis that $P(k + 1)$ is true.
From the inequality given to us we can write $P(k + 1)$ as

$$(1 + a)^{k+1} \geq 1 + a(k + 1)$$
$$(1 + a)^k \cdot (1 + a)^1 \geq 1 + a(k + 1)$$
$$(1 + a)^k \geq \frac{1 + a(k + 1)}{(1 + a)}$$

Now the right-hand side of the inequality can be simplified as

$$= \frac{1 + ak + a}{1 + a}$$
$$= \frac{1 + a}{1 + a} + \frac{ak}{1 + a}$$
$$= 1 + \frac{ak}{1 + a}$$

We now have the full inequality written as

$$(1 + a)^k \geq 1 + ak$$

Thus $P(k+1)$ is true. By Weak Induction we have proven that the inequality is valid where $n \in \mathbb{N}$ and $a \geq -1$ is a real number.