Homework 4: Problem 4

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Problem 4:
(A): We wish to show that the expected number of cycles of length k will be $\frac{n!}{2k\cdot(n-k)!} \cdot p^k$.

Let X be a random indicator variable that is 1 when a cycle exists of length k and 0 when a cycle of that length does not exist. It is given that each pair of vertices is adjacent with probability p, so the probability that a cycle of length k exists is p^k. By linearity of expectation, the expected number of cycles of length k is equal to the sum over all the values that X can take. Because we want to know the number of sequences of length k that exists, order is important and repetitions are not desired. Therefore, the number of sequences of length k is equal to

$$\frac{n!}{(n-k)!}$$

Through linearity of expectation, we know that the product of the number of sequences times the probability of getting a cycle of length k will be equal to the expected number of cycles of length k:

$$E(x) = \frac{n!}{(n-k)!} \cdot p^k$$

However, because of the presence of cycles, we need to adjust this expression to avoid over-counting. These cycles can be walked through in $2k$ different ways. There are k ways to traverse the cycle in a "forward" direction, such as from v_1 to v_2 to v_3 to v_1 when $n = 3$, because it is possible to use any of the vertices as the starting point. Likewise, there are k ways to traverse the cycle in a "backwards" direction since the same paths can be taken in the opposite order. Therefore, the expected number of cycles of length k will be

$$\frac{n!}{2k \cdot (n-k)!} \cdot p^k$$

(B): We wish to show that the expected number of cycles of length k will be at most $\frac{n^k \cdot p^k}{2k}$

This is the same as showing that

$$\frac{n!}{2k \cdot (n-k)!} \cdot p^k \leq \frac{n^k \cdot p^k}{2k}$$

After dividing common terms we are left with the inequality

$$\frac{n!}{(n-k)!} \leq n^k$$

Note that the right-hand term consists of k multiplications of n. While the numerator of the left-hand term is a factorial that becomes n terms once expanded, it is divided by $(n-k)!$ which ensures that there are also k multiplications on the left-hand side. This ensures that $\frac{n^k \cdot p^k}{2k}$ holds.

(C): We wish to show that for $p = n^{-a}$, with $a > 1$, the limit of the probability that the graph has at least one cycle of length k is 0 as n goes to infinity.

By taking the limit of p as n approaches infinity, we can determine the behavior of the function:

$$\lim_{n \to +\infty} \frac{1}{n^a} = 0$$

We know that $a > 1$, so as n approaches infinity the denominator grows larger and larger and $\frac{1}{n^a}$ approaches 0. Since p is a multiplicand of the expression for the probability that the graph has at least one cycle of length k, this probability goes to 0 as well.