Computer Science 340
Reasoning about Computation
Princeton University, Fall 2020

Assignment 2 Handout
Due 9/30/2020, 9:00PM

Problem 1 [20 points]
Show that

$$\binom{a}{b} \cdot \binom{b}{c} = \binom{a}{c} \cdot \binom{a-c}{b-c},$$

where $a \geq b \geq c \geq 0$, providing two different proofs.
(By definition, $0! = 1$.)
(A): an algebraic proof
(B): a combinatorial proof

Problem 2 [20 points]
There are $n \geq 3$ balls and $m \geq 2$ boxes. The boxes are numbered $1, 2, \ldots, m$. In how many ways can the balls be distributed in the boxes assuming [1]
(A): The balls are identical.
(Hint: Find a representation of any configuration as a bit vector; for example, if you had three boxes with

- 2 balls in the first box
- no balls in the second box
- 3 balls in the third box

you could represent this configuration by $BB||BBB$, where B represents a ball and $|$ the space between two consecutive boxes.)

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[1] Parts **(A)**, **(B)**, **(C)**, **(D)** and **(E)** are unrelated, that is, the restrictions are not accumulated across parts (beyond the ones repeated in different parts of course).

(B): The balls are identical and not all balls must be placed in boxes (i.e. there may be balls that are not placed in boxes).
(C): The balls are identical and the box with number 2 ends up with at least 3 balls.
(D): The balls are numbered $1, 2, \ldots, n$, namely they are pairwise different and the order they are placed in a box matters (first, second, etc).
(E): The balls are numbered $1, 2, \ldots, n$, namely they are pairwise different, the order they are placed in a box matters (first, second, etc) and only the balls $1, 2, 3$ are placed in box with number 1.

Problem 3 [20 points]
Let n and m be positive integers with $n \geq m \geq 1$.
(A): How many different m-word sentences can be constructed using an alphabet of n letters where each of the letters appears exactly once? A sentence is a sequence of words separated by spaces with a period at the end (and there are no other punctuation marks apart from the period character). You may assume that words are comprised of at least one letter and they can be nonsensical.
(B): How many different m-word paragraphs can be constructed using an alphabet of n letters where each of the letters appears exactly once? A paragraph is a sequence of sentences, each containing at least one word. You may also assume that each sentence is comprised of at least one word and is a sequence of words separated by spaces with a period at the end (and there are no other punctuation marks apart from the period character). Finally, words are comprised of at least one letter and they can be nonsensical.

Problem 4 [20 points]
You roll a 6-sided die twice. A friend of yours makes the following claim: "Getting two sixes if you know that the first roll is a six is at least as likely as getting two sixes if you know that at least one of the rolls is a six".
Do you agree with your friend's logic? Prove your answer. Note that you

may not assume that the die is “fair”.

Problem 5 [20 points]
A software company created a diagnostics software for car engines. The software can have only two outcomes:

- The software has detected an issue with the engine (the software determines that the engine malfunctions).
- The software has detected no issues with the engine (the software determines that the engine functions properly).

The software could provide misleading results though. In particular

- The probability that the software determines the engine has an issue given that the engine functions properly is 3%.
- The probability that the software determines the engine functions properly given that the engine malfunctions is 10%.

Assume that 95% of the engines function properly. Furthmore, assume that an engine can either function properly or malfunction.
(A): What is the probability that an engine that tests as functioning properly actually malfunctions?
(B): What is the probability that an engine that tests as it malfunctions actually functions properly?

Problem 6 [20 points]
Consider E_1, E_2 and E_3 mutually independent events, where $\Pr[E_3] > 0$. Show that E_1 and E_2 are conditionally independent given E_3, namely that

$$\Pr[E_1 \cap E_2 | E_3] = \Pr[E_1 | E_3] \cdot \Pr[E_2 | E_3].$$