Homework 3: Problem 1

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Problem 1:
(A): We wish to find the probability that a particular pigeonhole never has a pigeon of a certain type. Each pigeon picks a pigeonhole at random and independently from one another, so a given hole has a $\frac{1}{n}$ chance of being picked. The chance of a given hole *not* being picked, then, is equal to $1 - \frac{1}{n} = \frac{n-1}{n}$. The probability then that the n pigeons of a particular type never pick the same pigeonhole is

$$(\frac{n-1}{n})^n$$

(B): We wish to find the expected number of pigeons of a particular type that fly away. Let X_i equal 1 if a pigeon flies away from hole i, and 0 if otherwise. By linearity of expectation, we can say that X, the expected number of pigeons of a given type that fly away, is equal to the summation over all the values that X can take. From part A, we know that the probability that the n pigeons of a particular type never pick the same pigeonhole is $(\frac{n-1}{n})^n$. It is given that there are n pigeonholes, so the summation can be given as

$$X = \sum_{x=1}^{n} (\frac{x-1}{x})^x$$

(C): We wish to find the expected number of pigeons that fly away. From part B, we know that the expected number of pigeons of a particular type that fly away is equivalent to $\sum_{x=1}^{n} (\frac{x-1}{x})^x$. It is given that there are n types of pigeons, so this expectation needs to be applied to each type of pigeon to find the total expected number of pigeons that fly away. If X is defined as the total number of pigeons that fly away, then

$$X = n \cdot \sum_{x=1}^{n} (\frac{x-1}{x})^x$$