Homework 2: Problem 1

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Problem 1:
We wish to show that

$$\binom{a}{b} \cdot \binom{b}{c} = \binom{a}{c} \cdot \binom{a-c}{b-c}$$

where $a \geq b \geq c \geq 0$ using an algebraic proof and a combinatorial proof.

(A):
For the algebraic proof we can use the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

to expand each of the binomial coefficients and then cancel out matching terms:

$$\begin{aligned}
\frac{a!}{b!(a-b)!} \cdot \frac{b!}{c!(b-c)!} &= \frac{a!}{c!(a-c)!} \cdot \frac{(a-c)!}{(b-c)! \cdot ((a-c)-(b-c))!} \\
\frac{a!}{b!(a-b)!} \cdot \frac{b!}{c!(b-c)!} &= \frac{a!}{c!(a-c)!} \cdot \frac{(a-c)!}{(b-c)! \cdot (a-b)!} \\
\frac{\cancel{a!}}{\cancel{b!}\cancel{(a-b)!}} \cdot \frac{\cancel{b!}}{\cancel{c!}\cancel{(b-c)!}} &= \frac{\cancel{a!}}{\cancel{c!}\cancel{(a-c)!}} \cdot \frac{\cancel{(a-c)!}}{\cancel{(b-c)!} \cdot \cancel{(a-b)!}} \\
1 &= 1
\end{aligned}$$

We observe that both sides are equal so we have proven the equation algebraically.

(B):
For the combinatorial proof, we need to define a question that each side of the equation can separately answer. We can ask:

You have a balls. You wish to separate them into three piles of sizes $a-b$, $b-c$, and c such that there are no balls left over. How many ways can you do this?

1. Choose b balls from a leaving a pile of size $a-b$ and a pile of size b. Now choose c balls from the pile of size b, separating it into a pile of size $b-c$ and a pile of size c. You now have three piles of sizes $a-b$, $b-c$, and c. The first action was done in $\binom{a}{b}$ ways while the second was done in $\binom{b}{c}$ ways, so you have the the left side of the equation in $\binom{a}{b} \cdot \binom{b}{c}$ ways.

2. Choose c balls from a leaving a pile of size $a-c$ and a pile of size c. Now choose $b-c$ balls from the pile of size $a-c$, separating it into a pile of size $b-c$ and a pile of size $a-c-(b-c) = a-b$ balls. You now have three piles of sizes $a-b$, $b-c$, and c. The first action was done in $\binom{a}{c}$ ways while the second was done in $\binom{a-c}{b-c}$ ways, so you have the right side of the equation in $\binom{a}{c} \cdot \binom{a-c}{b-c}$ ways.

Both scenarios were able to answer the same question, so they are equal and we have proven the equation through combinatorial analysis.