COS 445 - Warmup PSet

Due online Monday, February 8th at 11:59 pm

Instructions:

- Submit your solution as a single PDF to codePost. If you collaborated with other students, or consulted an outside resource, submit a (very brief) collaboration statement as well.
- This homework is considerably shorter and simpler than all others, and weighted considerably less. It's purpose is to help you gauge the math/proof background expected at the beginning of the course. You may wish to reference this cheatsheet: `http://www.cs.princeton.edu/~smattw/Teaching/cheatsheet445.pdf`.
- Please reference the course collaboration policy here: `http://www.cs.princeton.edu/~smattw/Teaching/infosheet445sp21.pdf`.

Problem 1: Basic Probability I (10 points)

(a) Compute the expectation of the discrete random variable X, where $X = i$ with probability 2^{-i}, for all integers $i \geq 1$.

(b) Compute the expectation of the non-negative, continuous random variable Y with CDF $F_Y(x) = 1 - 1/(x+1)^3$, $x \geq 0$.

(c) Compute the expectation of the random variable $Z = X + Y$.

Problem 2: Basic Probability II (10 points)

Suppose that there are n balls and n bins. Each ball is thrown, independently, into a uniformly random bin.

(a) What is the probability that Bin 1 is empty?

(b) What is the expected number of empty bins?

(c) What is the expected number of bins which contain exactly two balls?

Problem 3: Basic Continuous Optimization (10 points)

(a) Let $f(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2$. Minimize $f(x_1, x_2)$ over all $(x_1, x_2) \in \mathbb{R}^2$ (and prove that it is the minimum).

(b) Let $f(x_1, x_2) = x_1 - x_2 + x_1 x_2$. Maximize $f(x_1, x_2)$ over the range $[-1, 1] \times [-1, 1]$ (and prove that it is the maximum).

Problem 4: Basic Proofs I (10 points)

You're trying to collect all n distinct cards from your favorite trading card game. The only way to obtain new cards is to purchase a sealed pack of one uniformly random card. After you buy a pack, you open it, see the card inside, and add it to your collection. If you have at least one copy of each of the n cards, you stop. Otherwise, you purchase a new pack. Prove that the expected number of packs you purchase is $\Theta(n \log n)$.[1][2][3]

Problem 5: Basic Proofs II (10 points)

Every pair of people in the world are either buddies or not buddies (and if Alice is buddies with Bob, then Bob is buddies with Alice). Say that a set of people is *buddy-full* if everyone is buddies with everyone else. Say that a set is *buddy-free* if no one is buddies with anyone else. Prove that if six people are together in a room, then there is either a buddy-full set of size 3, or a buddy-free set of size 3. Prove that if only five people are in the room, then it's possible that that there is no buddy-full set of size 3, nor a buddy-free set of size 3.

Extra Credit: Fun with Coupling

Recall that extra credit is not directly added to your PSet scores, but will contribute to your participation grade. Some extra credits are **quite** challenging and will contribute significantly.

For both parts of this question, it will take you some time to figure out the right proof approach. Once you figure out a good proof approach, however, there is a short proof which is very easy to follow. For full credit, your proof should be comparably easy to follow.[4]

Hint: For any part where you are proving a claim, you should try to use a coupling argument. If you don't remember what a coupling argument is, you can find a definition in the cheatsheet.

Part a

Let D be any distribution which samples only (strictly) positive numbers. For all integers $i \geq 1$, let X_i denote an independent sample from D. For any $t \geq 0$, let $N(t)$ be such that:

- $\sum_{i=1}^{N(t)} X_i \geq t$.
- $\sum_{i=1}^{N(t)-1} X_i < t$.

That is, $N(t)$ is the smallest index such that the first $N(t)$ random variables sum to $\geq t$. Observe that $N(t)$ is itself a random variable. Finally, define $f(t) := \mathbb{E}[N(t)]$. Prove that, for all $s, t \geq 0$, $f(s) + f(t) \geq f(s + t)$.

[1]Recall that $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. Recall further that $f(n) = O(g(n))$ if there exist absolute constants C, n_0 such that $f(n) \leq C \cdot g(n)$ for all $n \geq n_0$. $f(n) = \Omega(g(n))$ if there exist absolute constants C, n_0 such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$.

[2]You may use without proof the fact that $\ln(n+1) + 1 > \sum_{i=1}^{n} 1/i > \ln(n+1)$.

[3]You may also use the following fact without proof: if a coin is heads with probability p (independently on each flip), then the expected number of flips until seeing a heads is $1/p$.

[4]Thank you to Nick Arnosti for originally suggesting this question.

Part b

Let S be any finite set whose elements are non-negative real numbers, and let $n := |S|$, and let A be the sum of the elements in S. Let $X_1, \ldots, X_n$ be random variables equal to the elements of S in uniformly random order (i.e., sample the elements of S without replacement, one at a time). For any real number $t \in [0, A]$, again define $N(t)$ to be such that

- $\sum_{i=1}^{N(t)} X_i \geq t$.
- $\sum_{i=1}^{N(t)-1} X_i < t$.

That is, $N(t)$ is the smallest index such that the first $N(t)$ random variables sum to $\geq t$. Observe that $N(t)$ is itself a random variable, and it is well-defined for $t \in [0, A]$. Again, define $f(t) := \mathbb{E}[N(t)]$. Prove the following claim, or find (and analyze) a counterexample: for all $s, t \geq 0$, $f(s) + f(t) \geq f(s + t)$.[5]

[5]Observe that if the random variables were sampled *with* replacement, this claim would be a special case of Part a.