COS 445 - PSet 1, Problem 1

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Problem 1: Instances with many stable matchings

We wish to prove that for all even $n \geq 2$ there exists a stable matching instance with n students and n universities with one slot each, such that there are at least $2^{\frac{n}{2}}$ distinct stable matchings.

We start by proving the claim for the base case. When $n = 2$ there are only two students, S_1 and S_2, and two universities U_1 and U_2. Let us assign the preferences so that S_1 prefers U_1 to U_2, S_2 prefers U_2 to U_1, U_1 prefers S_2 to S_1, and U_2 prefers S_1 to S_2. This means there are only two distinct stable matchings. Either the pairs are (S_1, U_1) and (S_2, U_2) or they are (S_1, U_2) and (S_2, U_1). The two matchings are stable, because changing one of them would not strictly enhance the happiness of all elements in the new pairs. In the first possible matching, for example, one might change the pairings such that U_2 gets its first choice (S_1) and U_1 gets its first choice (S_2). However, this would mean that the happiness of S_1 would decrease since U_1 was their first choice and was originally paired with them. Since there are $2^{\frac{2}{2}} = 2$ distinct stable matchings when $n = 2$, we have proven the claim for the base case.

We will then prove the claim for all even $k \geq 2$ through induction, by assuming that the claim is true for $k \geq 2$. It remains to be shown that the claim holds for $k + 2$. We can consider a pair of students (S_i, S_{i+1}) and a pair of universities (U_i, U_{i+1}) in sequence. Let us assign the preferences to mirror those seen when $n = 2$:

Element	Preferences
S_i	$U_i > U_{i+1}$
S_{i+1}	$U_{i+1} > U_i$
U_i	$S_{i+1} > S_i$
U_{i+1}	$S_i > S_{i+1}$

This means that, as in the base case, there will never be a blocking pair that could exist between S_i, S_{i+1}, U_i and U_{i+1}. If each pair of students and each pair of universities is considered in sequence, there are $\frac{n}{2}$ total pairs (since n is always even). However, it might still be possible to have blocking pairs formed between students and universities that are not in sequence. To prevent this, let S_i, S_{i+1}, U_i and U_{i+1} prefer each each other more than the other $n - 2$ students and universities. This way, the matchings formed will always follow the pattern seen in the base case. Therefore, there must be at least $2^{\frac{n}{2}}$ distinct stable matchings for all even $n \geq 2$.