COS 445 - PSet 4, Problem 2

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Problem 2: Noisy Optimizers aren't Good Enough

Part a

We wish to prove that for any given $\vec{b}_{-i}$ there exists a price p such that bidder i will either win the item and pay p or not get the item (and pay nothing) no matter what bid they make.

Suppose the auction is run with $\vec{b}_{-1}$, that is without bidder i being considered. Let p be the bid of the resulting winner $j \neq i$. It is already given that each accepted bid is nonnegative. We note that when using $A(\vec{b}_{-1}, -2)$ the price that is output after running A is equivalent to the price output when just running the auction without b_i. In other words, the resulting price that is charged is still p.

Therefore, no matter what b does they will either win and pay p or lose and not pay anything.

Part b

We wish to prove that if all other bidders tell the truth then bidder i's best response is to make a winning bid.

A bidder's utility is equal to $v - p$, where v is the value of the item to a particular bidder and p is the price they pay for that item. If $v_i > v_j$ for all bidders $j \neq i$, then we know that bidder i's utility $v_i - p$ must also be the greatest.

We also know that if bidder i wins their utility must be equal to $v_i - b_{A(\vec{b}_{-1},-2)}$, since it is given that the winner is charged $b_{A(\vec{b}_{-1},-2)}$. Additionally, whatever bid b_i bidder i makes must also be greater than $b_{A(\vec{b}_{-1},-2)}$. Finally, because it is assumed that all other bidders $j \neq i$ tell the truth the price they would pay is equal to $v_{A(\vec{b}_{-1},-2)}$. In other words, they are strictly incentivized to report the true value of the item to them as their bid (and so would end up paying $v_{A(\vec{b}_{-1},-2)}$).

Since $v_i > v_j$ for all bidders $j \neq i$ and $b_i > b_{A(\vec{b}_{-1},-2)}$, a winning bid for bidder i will always be positive. Therefore, it is in bidder i's best interest to make a winning bid.

Part c

We wish to prove that error-prone second-price auction is not incentive compatible by providing an example where the second-highest bidder wins and is strictly worse off despite all voters telling the truth. Let us consider three bidders 1, 2, 3 and their associated bids:

Bid	Amount
b_1	4
b_2	3
b_3	1

Even though b_1 is the largest bid bidder 2 will be selected as the winner because b_2 is exactly one less than b_1. Therefore, b_2 will be declared the winner and will be charged $b_{A(\vec{b}_{-1},-2)}$. We then consider the remaining set of bids

Bid	Amount
b_1	4
b_3	1

The bid b_1 will be selected since it is more than one greater than the second-highest b_3. Therefore, the winner bidder 2 will be charged $b_1 = 4$. If we assume that everyone tells the truth then bidder 2's payoff must be $v_2 - p = 3 - 4 = -1$ since v_2 is taken as the true value of the item to them. Because bidder 2 received a strictly worse payoff even though all bidders told the truth, the auction is not incentive compatible.

Part d

We wish to prove that error-prone second-price auction is not incentive compatible by providing an example where the highest bidder wins but the second-highest bidder would have been strictly happier by lying.

Let us consider three bidders 1, 2, 3 and their associated bids:

Bid	Amount
b_1	4
b_2	3.5
b_3	3

Bidder 1 will be selected as the winner because the next highest bid, b_2, is not exactly one less than b_1. We then consider the remaining set of bids:

Bid	Amount
b_2	3.5
b_3	3

This means that bidder 1 will be charged $b_2 = 3.5$. However we then consider the following alternate bids where bidder 2 chooses to lie:

Bid	Amount
b_1	4
b_2	4.5
b_3	3

Here bidder 2 will be selected as the winner, and will end up paying $b_3 = 3$. Since $b_2 > b_3$, bidder 2 would have been strictly happier by lying.