COS 445 - PSet 2, Problem 2

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Problem 2: Find the Bug!

Part a

We wish to prove that F is Equivalent, unanimous, not a dictatorship, and not a Condorcet extension when there are only $m = 2$ alternatives and any number $n \geq 3$ voters.

Equivalent

When there are only $m = 2$ candidates, each of the $n \geq 3$ voters can only have two possible preferences lists: $a_1 > a_2$ or $a_2 > a_1$. This means that all voters who prefer a_1 will have the same preference lists (the same is true among voters who prefer a_2). Since a $\succ$ that prefers the same candidate will always be a-equivalent to a $\succ$ that prefers the same candidate, F will always be Equivalent.

Unanimous

F is unanimous if whenever a is everyone's favorite candidate, the rule selects a.

Let us consider the scenario when all prefer a_1 to a_2. In this situation, a_2 is dominated by a_1 and a_1 is not dominated by the only other candidate. F selects the minimum candidate not dominated by any other candidate, so the output will be a_1. Since a_1 is also everyone's favorite candidate, a_1 is chosen unanimously. Likewise, when a_2 is preferred by everyone F will be unanimous since a_2 will be the minimum candidate not dominated by any other candidate.

No other possibility exists where a candidate is everyone's favorite. Therefore, F is unanimous.

Not a dictatorship

F is a dictatorship if there exists a voter i such that F always outputs i's favorite candidate.

Let us assume for contradiction that a voter i does exist such that F always outputs i's favorite candidate. Let i prefer a_2 to a_1. If F is indeed a dictatorship, then a_2 will always be selected. However, if at least one other voter prefers a_1 then neither candidate is dominated. This means that F will output a_1 since it is the minimum candidate not dominated by any other candidate.

This is a contradiction, since i's preference was a_2 not a_1. Therefore, F is not a dictatorship.

Not a Condorcet extension

F is a Condorcet extension if it always selects a Condorcet winner, when one exists.

Let us assume for contradiction that F is a Condorcet extension. Let us also assume that among the n voters, $n-1$ vote for a_2 and 1 votes for a_1. The Condorcet winner is a_2, since a strict majority ($n-1$ vs 1) of voters prefer a_2 to a_1. If F is a Condorcet extension, it must then select a_2. However, it is given that F outputs the minimum candidate not dominated by any other candidate. Since a_1 has one vote in this scenario, it is the minimum candidate that is not dominated and will thus be selected.

Because this is a contradiction, F is therefore not a Condorcet extension.

Part b

We wish to prove that F is not Equivalent when $m \geq 3$. Let us assume there are two voters, v_1 and v_2, and three candidates a_1, a_2, and a_3 where the preference lists are as follows:

Voter	Preferences
v_1	$a_2 > a_3 > a_1$
v_2	$a_3 > a_1 > a_2$

In this situation, a_3 dominates a_1 since all voters prefer a_3 to a_1. However, a_2 still wins since it is both not dominated (it is v_1's first choice) and it is the minimum candidate that is not dominated. However, let us define a new preference list v_1' such that the preference lists are as follows:

Voter	Preferences
v_1'	$a_2 > a_1 > a_3$
v_2	$a_3 > a_1 > a_2$

Now a_1 is no longer dominated, and so wins because it is now the minimum candidate that is not dominated. If F is equivalent, then altering the preferences below a_2 for v_1 should not make a difference in the selected candidate. However, we observe that v_1', which swaps the two lower preferences, changes the output.

Because this is a contradiction, F is therefore not Equivalent.

Part c

We wish to find the specific line in the proof that is incorrect and prove that is false.

In this proof, the sentence "Every voter prefers candidate a_i to candidate a_j, for all $j < i$" is incorrect. It is not true that the winner selected by F is necessarily the one you prefer. As a counterexample, we consider the scenario when there are two voters, v_1 and v_2, and three candidates a_k, a_i, and a_j where $k > i > j$. Let us assume preference lists are as follows:

Voter	Preferences
v_1	$a_i > a_k > a_j$
v_2	$a_k > a_j > a_i$

We observe that all voters prefer a_k to a_j, but that v_2 actually prefers a_j to a_i. However, a_i still wins because a_j is dominated by a_k. This means that when a_i is selected, it does not mean that every voter always prefers a_i to a_j. This contradicts the statement that "every voter prefers candidate a_i to candidate a_j, for all $j < i$" when a_i is selected. Therefore, the statement is false.