# EE 325 Group 13 - Assignment 3

Manideep Vudayagiri **190070074**
Shreyas Nadkarni **19D170029**
Tarun Kumar **19D070062**
Ashish Pol **190070011**
Rohit Ahirwar **190070053**

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## Contents

# 1 Question 1

$F(x)$ is a valid distribution. Which of the following functions of $F(x)$ are also valid distributions. Provide a proof for your claim.

**SOLUTION :**
Since, $F(x)$ is a cumulative distribution function(cdf), it will have the following properties:

$$\lim_{x \to -\infty} F(x) = 0 \tag{1}$$

$$\lim_{x \to \infty} F(x) = 1 \tag{2}$$

$$F(x) \text{ is a non-decreasing function in the domain of } x \text{ i.e.}$$

$$\frac{d}{dx}(F(x)) \geq 0 \tag{3}$$

**(a)** $aF(x) + (1-a)F(x)$ where $0 \leq a \leq 1$ .

Let us denote this distribution function as $F_1(x)$

$$\begin{aligned} F_1(x) &= aF(x) + (1-a)F(x) \\ &= F(x) \text{ irrespective of 'a'} \end{aligned}$$

This function is the same as the initial function. Hence, it's a valid distribution by default.

**(b)** $F_1(X) = (F(x))^r$

This function has to satisfy the properties of a *cdf* to be a valid distribution.
For $r = 0$, $F_1(x) = 1\ \forall x \in \mathrm{R}$. This doesn't satisfy $F'(-\infty) = 0$. So, we'll check for $r \neq 0$.

If $r < 0$ :

$$\begin{aligned} \lim_{x \to -\infty} F_1(x) &= \lim_{x \to -\infty} (F(x))^r \\ &= \left( \lim_{x \to -\infty} F(x) \right)^r \\ &= (0)^r \\ &\Rightarrow \text{not defined} \end{aligned}$$

If $r > 0$ :

$$\begin{aligned}
\lim_{x\to-\infty} F_1(x) &= \lim_{x\to-\infty} (F(x))^r \\
&= \left(\lim_{x\to-\infty} F(x)\right)^r \\
&= 0 \\
\lim_{x\to\infty} F_1(x) &= \lim_{x\to\infty} (F(x))^r \\
&= \left(\lim_{x\to\infty} F(x)\right)^r \\
&= (1)^r \\
&= 1 \\
\frac{d}{dx}(F_1(x)) &= \frac{d}{dx}(F(x))^r \\
&= r(F(x))^{r-1}\frac{d}{dx}(F(x)) \\
&\geq 0
\end{aligned}$$

Therefore, $F_1(x)$ is a valid distribution for $r > 0$.

We'll check the validity of the remaining two functions in a similar way.

**(c)** $F_1(x) = 1 - (1 - F(x))^r$

Again, $r = 0 \Rightarrow F_1(x) = 1$, not a distribution function.

If $r < 0$ :

$$\begin{aligned}
\lim_{x\to\infty} F_1(x) &= \lim_{x\to\infty} (1 - (1 - F(x))^r) \\
&= 1 - \left(\lim_{x\to\infty} (1 - F(x))\right)^r \\
&= 1 - (1-1)^r \\
&= 1 - (0)^r \\
&\Rightarrow \text{not defined}
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to-\infty} F_1(x) &= \lim_{x\to-\infty} (1-(1-F(x))^r) \\
&= 1-\left(\lim_{x\to-\infty}(1-F(x))\right)^r \\
&= 1-(1-0)^r \\
&= 0 \\
\lim_{x\to\infty} F_1(x) &= \lim_{x\to\infty} (1-(1-F(x))^r) \\
&= 1-\left(\lim_{x\to\infty}(1-F(x))\right)^r \\
&= 1-(1-1)^r \\
&= 1 \\
\frac{d}{dx}(F_1(x)) &= \frac{d}{dx}(1-(1-F(x))^r) \\
&= r(1-F(x))^{r-1}\frac{d}{dx}(F(x)) \\
&\geq 0
\end{aligned}$$

Therefore, $F_1(x)$ is a valid distribution for $r > 0$.

**(d)** $F_1(x) = F(x) + (1-F(x))\log(1-F(x))$

$$\begin{aligned}
\lim_{x\to-\infty} F_1(x) &= \lim_{x\to-\infty} \Big[F(x) + (1-F(x))\log(1-F(x))\Big] \\
&= \lim_{x\to-\infty} F(x) + \lim_{x\to-\infty} \Big[(1-F(x))\log(1-F(x))\Big] \\
&\text{(Let, } F(x) = t) \\
&= 0 + \lim_{t\to0}[(1-t)\log(1-t)] \\
&= 0+0 \\
&= 0 \\
\lim_{x\to\infty} F_1(x) &= \lim_{x\to\infty} \Big[F(x) + (1-F(x))\log(1-F(x))\Big] \\
&= \lim_{x\to\infty} F(x) + \lim_{x\to\infty} \Big[(1-F(x))\log(1-F(x))\Big] \\
&\text{(Let, } F(x) = t) \\
&= 1 + \lim_{t\to0}[(1-t)\log(1-t)] \\
&= 1+0 \\
&= 1
\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(F_1(x)) &= \frac{d}{dx}[F(x) + (1 - F(x))\log(1 - F(x))] \\ &= \frac{d}{dx}(F(x)) + \Big( - \frac{d}{dx}(\log(1 - F(x))\Big) + \frac{-(1 - F(x))}{1 - F(x)}\Big(\frac{d}{dx}F(x)\Big) \\ &= -\frac{d}{dx}(F(x))\log(1 - F(x))\end{aligned}$$

$$\frac{d}{dx}(F(x)) \geq 0 \text{ and } \log(1 - F(x)) \leq 0 \text{ since } F(x) \in [0, 1]$$

$$\Rightarrow \frac{d}{dx}(F_1(x)) \geq 0$$

Therefore, $F_1(x)$ is a valid distribution.

## 2 Question 2

There are two urns—A containing $n$ black balls and B containing $n$ brown balls. At each step, one ball is chosen at random from both urns and swapped, i.e., the one from $A$ is put into $B$ and vice versa. Let $X_m$ be the number of black balls in urn A after $m$ steps. Observe that this determines the state of the system after $m$ steps, i.e., knowing $X_m$ describes the composition of both the urns. Obtain the pmf of $X_m$. This is a model for diffusion

**SOLUTION :**

## 3 Question 3

Recall the 'capture-release-recapture' problem: Catch m fish, mark them and release them back into the lake. Allow the fish to mix well and then you catch m fish. Of these p are those that were marked before. Assume that the actual fish population in the lakes is n and has not changed between the catches. Let $P_{m,p}(n)$ be the probability of the event (for a fixed p re-catches out of m) coming from n fish in the lake. Generate a plot for $P_{m,p}(n)$ as a function of n for the following values of m and p : m = 100 and p = 10, 20, 50, 75. For each of these p, use the plots to estimate (educated guess) the actual value of n. Call these four estimates $n_1, n_2, n_3, n_4$.

**SOLUTION :**

## 4 Question 5

A fair die is rolled $6n$ times. Let $\rho_n$ be the probability that there $n$ 6s in the $6n$ rolls. Is $\rho_n$ a monotonic function. Prove your statement.

**SOLUTION :**

Let A be the event where a die is rolled and the outcome is 6.

$$P(A) = \frac{1}{6} = \alpha$$

and

$$P(\overline{A}) = \frac{5}{6} = 1 - \alpha$$

Let B be the event where there are n 6s are in 6n rolls. The probability of event B is given by the binomial distribution :

$$\begin{aligned} P(B) &= \binom{6n}{n} \times P(A)^n \times P(\overline{A})^{6n-n} \\ \rho_n &= \frac{6n!}{5n! \times n!} \times \alpha^n \times (1-\alpha)^{5n} \\ &= \frac{6n!}{5n! \times n!} \times \left(\frac{1}{6}\right)^n \times \left(\frac{5}{6}\right)^{5n} \\ &= \frac{6n!}{5n! \times n!} \times \frac{(5)^{5n}}{(6)^{6n}} \end{aligned}$$

For proving that $\rho_n$ is a monotonic function:

$$\begin{aligned} \frac{\rho_{n+1}}{\rho_n} &= \frac{\frac{(6(n+1))!}{(5(n+1))! \times (n+1)!} \times \frac{5^{5(n+1)}}{6^{6(n+1)}}}{\frac{6n!}{5n! \times n!} \times \frac{5^{5n}}{6^{6n}}} \\ &= \frac{(6n+6)(6n+5)(6n+4)(6n+3)(6n+2)(6n+1)}{(5n+5)(5n+4)(5n+3)(5n+2)(5n+1)(n+1)} \times \frac{5^5}{6^6} \\ &= \frac{(n+\frac{5}{6})(n+\frac{4}{6})(n+\frac{3}{6})(n+\frac{2}{6})(n+\frac{1}{6})}{(n+\frac{5}{5})(n+\frac{4}{5})(n+\frac{3}{5})(n+\frac{2}{5})(n+\frac{1}{5})} \\ &< 1 \qquad (\forall n \geqslant 1) \end{aligned}$$

Hence $\rho_n$ is a monotonically decreasing function.