PH307: Introduction to Numerical Analysis
Tutorial Sheet 3

You are required to submit the computer program corresponding to problem 1 to your respective TAs.

1. This problem is to give you a feel for the Romberg integration by means of hand computation. Using the computer program of problem 1, write down the trapezoidal rule values of the integral $\int_0^{\pi/2} \sin x \, dx$ computed with 4, 8, 16 and 32 points. Using these values, generate the Romberg array by means of hand computation (i.e. using a calculator) and show that it rapidly converges to the correct value of the integral 1.

2. Using the Rhomberg integration method calculate $T(1,1)$ for the integral

$$I = \int_0^1 \frac{4dx}{(1+x^2)}.$$

   Compute your result accurate up to five decimal places. Exact value of the integral is π.

3. In the Romberg approach $T(n,0)$ denotes an estimate of $\int_a^b f(x)dx$ with intervals of size $h = (b-a)/2^n$. If it were known that

$$\int_a^b f(x)dx = T(n,0) + a_3 h^3 + a_6 h^6 + \cdots,$$

   how should the Romberg method be modified?

4. Consider the integral $I(h) = \int_a^{a+h} f(x)dx$, where h is a small interval. Obtain the leading error terms for each of the following formulas

   (a) $I(h) \approx hf(a+h)$

   (b) $I(h) \approx hf(a) - \frac{1}{2}h^2 f'(a)$

5. Write a computer program to calculate a definite integral using the following approaches:

   (a) Trapezoidal Rule

   (b) Simpson Rule

   (c) Romberg Integration

   Again, program should prompt the user for: (a) which method to use, (b) limits of integration, (c) which function to integrate, and (d) what spacing (h) to use? Write different subroutines for different methods. Your trapezoidal routine should be general enough so that it can be used in the Romberg method as well. Recall that in case of

Romberg integration you have to successively compute the integral by Trapezoidal rule, by halving the value of h as compared to the previous iteration. All this information should then be used to generate the Romberg array, until the result converges to a specified tolerance. Your program should be as general as possible. Apply your programs to compute the following definite integrals

(a) $\int_0^{\pi/2} \sin x \, dx$

(b) $\int_0^2 e^{-x} dx$

(c) $\int_0^1 \frac{dx}{1+x^2}$ (exact value $= \pi/4$)

(d) $\int_1^2 \frac{dx}{x}$ and compare with the exact value

(e) $\int_0^{\pi/2} \sqrt{1 - \frac{1}{4}\sin^2 t} \, dt$ (answer correct to four places 1.4675)

(f) $\int_0^{\pi/2} \frac{dx}{\sin^2 x + \frac{1}{4}\cos^2 x}$ (correct answer π)

The Romberg array satisfies the recursion relation

$$T(k+1, i+1) = T(k+1, i) + \frac{T(k+1, i) - T(k, i)}{2^{2(i+1)} - 1},$$

with $T(k, 0) = I_{trap}(h = h_k)$. Here $I_{trap}(h = h_k)$ is the value of the integral computed using the trapezoidal rule with the spacing h taken as $h_k = (b - a)/2^k$.