PH307: Introduction to Numerical Analysis
Tutorial Sheet 4

This tutorial sheet deals with the topic of Gaussian quadrature. The last problem of this sheet is the coding problem to be done and submitted during the lab hours.

1. Using the methods of Gaussian quadrature derive the following formulas (i.e. determine x_i, w_i). Note that in the first integral, the weight function $\ln x$ is negative in the range of integration.

   (a) $\int_0^1 \ln x f(x)dx = w_1 f(x_1) + w_2 f(x_2)$

   (b) $\int_{-\pi/2}^{\pi/2} \cos x f(x)dx = w_1 f(x_1) + w_2 f(x_2)$

2. Use two-point Gauss-Legendre quadrature to compute the integral $\int_0^{\pi/2} \sin x \, dx$. Do this problem using a calculator. No computer program is needed.

3. Use two-point Gauss-Laguerre quadrature to compute the integral $\int_0^{\infty} e^{-x} \sin \pi x \, dx$. Do this problem using a calculator. Compare your answer to the exact result. No computer program is needed.

4. Again using a calculator, compute the integral $\int_{-\infty}^{\infty} e^{-x^2} x^2 dx$ using two-point Gauss-Hermite quadrature. How does your result compare with the exact result of $\frac{\sqrt{\pi}}{2}$?

5. Programming assignment: The N-point Gaussian quadrature is defined by the general formula

$$\int_a^b K(x)f(x)dx = \sum_{i=1}^{N} w_i f(x_i),$$

   where $K(x)$ is a weight function, x_i are data points that are roots of a polynomial of degree N, and w_is are the weights. A Gaussian quadrature function takes as input the number N, and in the output gives the roots x_i and the weights w_i corresponding to that quadrature. Write a computer program that will compute the following integrals using the mentioned quadrature. For all the integrals, plot the integral value as a function of N to study the convergence behavior.

(a)

$$I = \int_{-1}^{1} e^{-x^2} dx$$

using the Gauss-Legendre quadrature. For what value of N do you get a result converged up to six decimal places?

(b)

$$I = \int_{0}^{\pi/2} \ln(1+x) dx$$

using the Gauss-Legendre quadrature. First you should carry out a change of variables so that the limits of integration become ∓ 1. The value of this integral correct to six decimal places is $I = 0.856590$. For what value of N do you get this result?

(c)

$$I = \int_{0}^{\infty} e^{-x} \sin x dx$$

using the Gauss-Laguerre quadrature. The exact value of this integral is $I = 0.5$; for what value of N do you get a result accurate to six places of decimal?

(d)

$$I = \int_{0}^{\infty} \frac{e^{-x}\sqrt{x}}{x+4} dx$$

using the Gauss-Laguerre quadrature. Your converged value should be close to 0.16776.

(e)

$$I = \int_{-\infty}^{\infty} e^{-x^2} \sin^2 x dx$$

using the Gauss-Hermite quadrature. For $N = 8$, you should get the value $I \approx 0.560202$, accurate up to six places of decimal.

(f)

$$I = \int_{-\infty}^{\infty} \frac{e^{-x^2}}{\sqrt{1+x^2}} dx$$

also using Gauss-Hermite quadrature. For $N = 12$, you should get $I \approx 0.15239$

Important Notes:

1. You can use the following numpy functions for various Gaussian quadratures:

   (a) polynomial.legendre.leggauss(deg), for generating weights and roots for Gauss-Legendre quadrature

   (b) numpy.polynomial.laguerre.laggauss(deg), for generating weights and roots for Gauss-Laguerre quadrature

   (c) polynomial.hermite.hermgauss(deg), for generating weights and roots for Gauss-Hermite quadrature

   For all the cases above, deg is nothing but N.

2. If in any computer program you need the numerical value of π, calculate it using the formula $\pi = 4.0 \tan^{-1}(1.0)$. Note that in python, the function for computing $\tan^{-1} x$ is math.atan(x).