PH307: Introduction to Numerical Analysis
Tutorial Sheet 8

Note: This tutorial sheet deals with the numerical solutions of a partial differential equation, namely, the heat equation in 1+1 dimensions.

1. Consider the heat equation in 1+1 dimensions with the specified boundary conditions

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = f(x)$$

We have to obtain the numerical values of $u(x,t)$ using the method of finite differences as discussed in the class for

$$0 \leq x \leq 1$$
$$0 \leq t \leq t_{max},$$

where t_{max} is user defined. We showed in the lectures that if we adopt the so-called "explicit method", the solution can be obtained using the following recursion relation

$$u(n,m+1) = \sigma u(n+1,m) + (1-2\sigma)u(n,m) + \sigma u(n-1,m), \tag{1}$$

above $\sigma = \frac{k}{h^2}$, where h and k represent the grid sizes for the x and t coordinates, and $u(n,m)$ denotes a 2D array with $0 \leq n \leq N$ and $0 \leq m \leq M$, where N and M are the total number of bins for x and t, respectively. Obviously

$$h = \frac{1}{N}$$
$$k = \frac{t_{max}}{M}.$$

Note that the solutions obtained using Eq. 1 are stable only when $(1-2\sigma) \geq 0$. To initiate the solution, first, you will have to initialize the array $u(n,m)$ to account for the initial conditions at $t=0$ as

$$u(n,0) = f(x_n), \quad \text{for } 0 \leq n \leq N.$$

Consider the case $f(x) = \sin \pi x$, for which the exact solution of this problem is $u(x,t) = \sin \pi x e^{-\pi^2 t}$. Compare your numerical solutions obtained from Eq. 1 to the exact solution for all values of x, t. Also explore the quality of the solution for different values of $(1-2\sigma)$, including the case where $\sigma = 0.5$, so that $(1-2\sigma) = 0$. For example, for $h = 0.1$ and $k = 0.005$, this condition is satisfied.