PH307: Introduction to Numerical Analysis
Tutorial Sheet 6

Note: This tutorial sheet deals with the numerical solutions of first-order initial value problems. The last problem of this sheet constitutes your lab assignment.

1. Consider the differential equation

$$dy/dx = -xy^2, \quad y(0) = 2.$$

It can be solved by elementary methods to get $y(x) = 2/(1 + x^2)$.

With the help of a calculator, obtain numerical solutions of this differential equation from $x = 0$ to $x = 2$ by:

(a) Euler method

(b) Second order Taylor expansion

(c) Second order Runge-Kutta method.

Use $h = 0.5, 0.2, 0.1, 0.01$ and compare your results with the exact ones for some selected values of x.

2. Write a computer program to solve the differential equation $\frac{dy}{dx} = x + y$, with $y(0) = 0$ (exact solution $y(x) = e^x - x - 1$) in the range $0 \leq x \leq x_{max}$. The ODE should be solved for user defined values of h and x_{max}, using the methods:

(a) Euler method

(b) Fourth-order Taylor expansion approach

(c) Fourth-order Runge-Kutta method.

Your program should prompt the user for (a) h, (b) x_{max}, and (c) the method to be used. As the output, your program should print: (a) each value x_i, (b) the value of the numerical solution y_i, and (c) the value of the exact solution $y_{ex}(x_i)$.

Recall that for the differential equation $\frac{dy}{dx} = F(x, y)$, the fourth-order Runge-Kutta formula is

$$\begin{aligned}
k_0 &= hF(x_n, y_n) \\
k_1 &= hF\left(x_n + \frac{h}{2}, y_n + \frac{k_0}{2}\right) \\
k_2 &= hF\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\
k_3 &= hF\left(x_n + h, y_n + k_2\right) \\
y_{n+1} &= y_n + \frac{1}{6}\left(k_0 + 2k_1 + 2k_2 + k_3\right)
\end{aligned}$$