Chapter 3: Interpolation of Functions

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Introduction

- Suppose the only information we have about a function $f(x)$ are its values for a finite number of values of x.
- That is, the function has been tabulated for $N+1$ values of x, $\{x_0, x_1, x_2, \ldots, x_{N-1}, x_N\}$ as shown below

x	$f(x)$
x_1	$f(x_1)$
x_2	$f(x_2)$
x_3	$f(x_3)$
$\vdots$	$\vdots$
x_{N-2}	$f(x_{N-2})$
x_{N-1}	$f(x_{N-1})$
x_N	$f(x_N)$

- We further assume an arbitrary spacing between successive x values x_i and x_{i+1}

Introduction...

- The case of equally spaced x values will be treated as a special case
- The question is can we obtain a function form of $f(x)$ so that we can obtain its values for those values of x which have not been tabulated
- This is the mathematical problem of interpolation of functions
- This can be posed as: for $x_0 \leq x \leq x_N$, $f(x) = ?$
- We will utilize two approaches for the purpose
  - Newton's divided difference method
  - Lagrangian method
- For the special case of equispaced arguments, Newton's forward and backward difference formulas will follow as the corollaries of the Newton's divided difference formula

Newton's divided difference approach

- First we introduce the notion of divided differences (DDs)
- Divided differences of orders $0, 1, 2, ..., k$ are defined iteratively

$$f[x_0] = f(x_0) \tag{1}$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{2}$$

- and k-th order DD is defined in terms of the $(k-1)$-th order DDs as a recursion relation

$$f[x_0 ... x_k] = \frac{f[x_1 ... x_k] - f[x_0, ..., x_{k-1}]}{x_k - x_0} \tag{3}$$

- To see the explicit form of the higher-order DDs, we compute

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{1}{x_2 - x_0} \left\{ \frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right\}$$

Divided Differences (contd.)

$$= \frac{f(x_2)}{(x_2 - x_1)(x_2 - x_0)} - \frac{f(x_1)}{(x_2 - x_0)} \left\{ \frac{x_1 - x_0 + x_2 - x_1}{(x_2 - x_1)(x_1 - x_0)} \right\} + \frac{f(x_0)}{(x_2 - x_0)(x_1 - x_0)}$$

- Which can be put in a more symmetric form

$$f[x_0, x_1, x_2] = \frac{f[x_0]}{(x_0 - x_1)(x_0 - x_2)} + \frac{f[x_1]}{(x_1 - x_0)(x_1 - x_2)} + \frac{f[x_2]}{(x_2 - x_0)(x_2 - x_1)} \tag{4}$$

  Above we used $f[x_i] = f(x_i)$.

- Using the method of induction we can show that

$$f[x_0, x_1, \ldots, x_k] = \frac{f[x_0]}{(x_0 - x_1)\ldots(x_0 - x_k)} + \frac{f[x_1]}{(x_1 - x_0)\ldots(x_1 - x_k)} + \cdots + \frac{f[x_k]}{(x_k - x_0)(x_k - x_{k-1})} \tag{5}$$

- Or in a more compact form

$$f[x_0, x_1, \ldots, x_k] = \sum_{i=0}^{k} \alpha_i^{(k)} f(x_i) \tag{6}$$

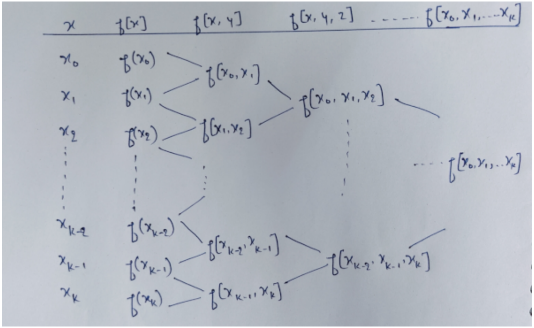
- where

$$\alpha_i^{(k)} = \frac{1}{(x_i - x_0)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_k)} \tag{7}$$

From Eqs. 6 and 7 it is obvious that in any DD, the order of arguments is irrelevant

Divided Difference Table

- A divided different table is constructed in the following manner



- An illustration for the case of k=2 for $f(x) = \sinh(x)$ is as follows:

x	$f[x]$	$f[x,y]$	$f[x,y,z]$
0.0	0.00000		
		1.0067	
0.2	0.20134		0.08367
		1.0318	
0.3	0.30452		

Newton's Fundamental Formula

- Using the divided difference calculus described above, we can develop an approximate polynomial interpolation formula for $f(x)$
- The property of this polynomial will be that for $N+1$ values of x, $\{x_0, x_1, x_2, \ldots, x_{N-1}, x_N\}$, $f(x)$ reproduces the tabulated values, i.e.,

  $$f(x = x_i) = f(x_i),$$

  where LHS above denotes the value obtained using the polynomial, while the RHS denotes the tabulated value.
- The formula is called Newton's fundamental formula or Newton's divided difference formula
- We derive it next

Derivation of Newton's Fundamental Formula

- Using the definition of DD (Eq. 2), we can write

$$f(x) = f(x_0) + (x - x_0) f[x_0, x] \tag{8}$$

- Using Eq. 3 for $k = 2$, we have

$$f[x_0, x_1, x] = f[x, x_0, x_1] = \frac{f[x_0, x_1] - f[x, x_0]}{x_1 - x}$$

$$f[x_0, x] = f[x_0, x_1] + (x - x_1) f[x_0, x_1, x] \tag{9}$$

- Or, for an arbitrary order i, we can show

$$f[x_0, x_1, \ldots, x_{i-1}, x] = f[x_0, x_1, \ldots, x_i] + (x - x_i) f[x_0, \ldots, x_i, x] \tag{10}$$

- On substituting Eq. 9 in 8, we obtain

$$f(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x]$$

- Repeating this procedure using Eq. 10, we get the famous Newton's divided difference (or fundamental) formula

$$f(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] +$$

$$.. + (x - x_0)(x - x_1)...(x - x_{N-1})f[x_0, x_1, ...x_N] + E(x) \quad (11)$$

- where $E(x)$ is nothing but the next term in the expansion and is interpreted as the error term

$$E(x) = (x - x_0)(x - x_1)...(x - x_N)f[x_0, x_1, ...x_N, x] \qquad (12)$$

- Note that $f(x)$ of Eq. (11) is a polynomial of degree N which takes the tabulated values $f(x_i)$ for $x_i \in (x_{0.}, x_{1,}, .., x_N)$.

Forward, Backward, and Central Difference Operators

- When the function $f(x)$ has been tabulated for $N+1$ equispaced points $\{x_0, x_1, x_2, \ldots, x_N\}$, so that the separation between the successive points is h, i.e.,

$$x_{k+1} = x_k + h \tag{13}$$

- with the function values denoted by $\{f(x_0), f(x_1), f(x_2), \ldots, f(x_N)\}$
- For such a case it is useful to define the following three "differences":
    1. the forward difference Δ of $f(x)$ at point x_k is defined as

    $$\Delta f(x_k) = f(x_k + h) - f(x_k) = f(x_{k+1}) - f(x_k) \tag{14}$$

    2. backward difference ∇ of $f(x)$ at point x_k is defined as

    $$\nabla f(x_k) = f(x_k) - f(x_k - h) = f(x_k) - f(x_{k-1})$$

    3. Central difference δ

    $$\delta f(x_k) = f(x_k + \frac{h}{2}) - f(x_k - \frac{h}{2})$$

Difference operators...

- Note that the the application of the central difference operator leads to untabulated function values
- And, in general

$$\Delta f(x) = f(x+h) - f(x)$$
$$\nabla f(x) = f(x) - f(x-h)$$
$$\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$$

- We can also define higher order difference operators. For example, the second-order operator will be

$$\Delta^2 f(x_k) = \Delta(\Delta f(x_k)) = \Delta f(x_k + h) - \Delta f(x_k)$$

or

$$\begin{aligned} \Delta^2 f(x_k) &= f(x_k + 2h) - f(x_k + h) - f(x_k + h) + f(x_k) \\ &= f(x_k + 2h) - 2f(x_k + h) + f(x_k) \end{aligned} \tag{15}$$

- Similarly

$$\Delta^3 f(x_k) = \Delta(\Delta^2 f(x_k))$$

$$= \Delta f(x_k + 2h) - 2\Delta f(x_k + h) + \Delta f(x_k)$$

or

$$\Delta^3 f(x_k) = f(x_k + 3h) - f(x_k + 2h) - f(x_k + 2h) + f(x_k + h)$$

$$-f(x_k + 2h) + f(x_k + h) + f(x_k + h) - f(x_k)$$

- Or

$$\Delta^3 f(x_k) = f(x_k + 3h) - 3f(x_k + 2h) + 3f(x_k + h) - f(x_k) \quad (16)$$

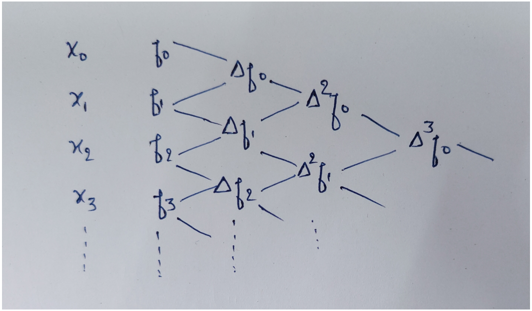
- Or, in general,

$$\Delta^r (f(x_k)) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f(x_{k+r-i}) \quad (17)$$

- A shorthand notation for $f(x_k)$ is $f_k = f(x_k)$

Forward Difference Operator (contd.)

- The relationship between various differences can be exhibited in a different table.



- Note that all the finite differences are written along the forward diagonals.
- And $\Delta^r f_k$ depends upon elements extending up to $(r+k)$-th backward diagonal.

Backward Difference Operator

- When one has to perform calculations on a tabulated function towards the end of the table, it is useful to define the concept of backward differences.

$$\nabla f(x) = f(x) - f(x-h) \tag{18}$$

- So that

$$\nabla f(x) = f(x_k) - f(x_{k-1}) \tag{19}$$

$$\nabla^2 f(x) = \nabla f(x_k) - \nabla f(x_{k-1})$$

$$= f(x_k) - f(x_{k-1}) - f(x_{k-1}) + f(x_{k-2})$$

- or

$$\nabla^2 f(x_k) = f(x_k) - 2f(x_{k-1}) + f(x_{k-2}) \tag{20}$$

Backward Difference Operator (contd.)

- and

$$\nabla^3 f(x_k) = \nabla f(x_k) - 2\nabla f(x_{k-1}) + \nabla f(x_{k-2})$$

$$= f(x_k) - f(x_{k-1}) - 2f(x_{k-1}) + 2f(x_{k-2}) + f(x_{k-2}) - f(x_{k-3})$$

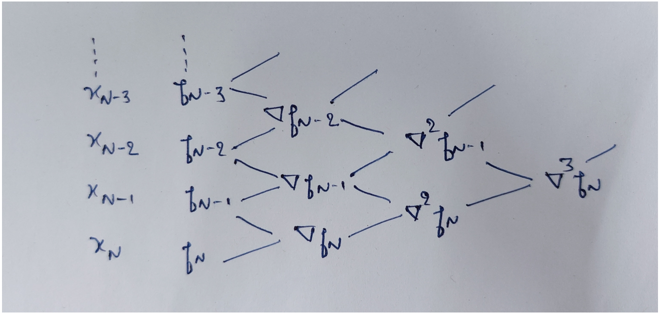
$$\nabla^3 f(x_k) = f(x_k) - 3f(x_{k-1}) + 3f(x_{k-2}) - f(x_{k-3}) \qquad (21)$$

- Thus, in general,

$$\nabla^r (f(x_k)) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f(x_{k-i}) \qquad (22)$$

Backward Difference Operator (contd.)

- The difference table for the backward difference operator is



Central Difference Operator

- For calculations near the center of the table, it is useful to introduce a central difference operator

$$\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2}) \tag{23}$$

- Clearly, $\delta f(x_k) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$ does not involve the tabulated values.
- However, as seen below, the second central difference δ^2 does involve the tabulated values,

$$\delta^2 f_k = \delta f(x_k+\frac{h}{2}) - \delta f(x_k-\frac{h}{2})$$

$$= f(x_k+h) - f(x_k) - f(x_k) + f(x_k-h)$$

- or

$$\delta^2 f_k = f(x_k+h) - 2f(x_k) + f(x_k-h) \tag{24}$$

Central Difference Operator (contd.)

- And

$$\delta^3 f_k = \delta f_{k+1} - 2\delta f_k + \delta f_{k-1}$$

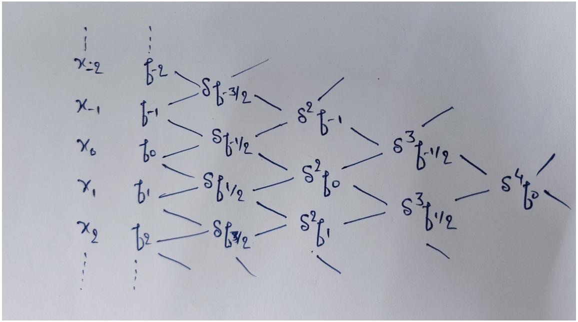
$$= f_{k+\frac{3}{2}} - f_{k+\frac{1}{2}} - 2f_{k+\frac{1}{2}} + 2f_{k-\frac{1}{2}} + f_{k-\frac{1}{2}} - f_{k-\frac{3}{2}}$$

$$\delta^3 f_k = f_{k+\frac{3}{2}} - 3f_{k+\frac{1}{2}} + 3f_{k-\frac{1}{2}} - f_{k-\frac{3}{2}} \tag{25}$$

- Or, in general,

$$\delta^m f_k = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} f(x_{k+\frac{m}{2}-i}) \tag{26}$$

- Note that, in general, $\delta^{2m} f_k$ and $\delta^{2m+1} f_{k\pm\frac{1}{2}}$ will involve tabulated function values, whole the differences $\delta^{2m+1} f_k$ and $\delta^{2m} f_{k\pm\frac{1}{2}}$ will not.
- Clearly, the difference table, in the neighborhood of an interior tabulated point x_0, is



Relationship with the divided differences

- We will now establish a relationship between the divided differences and the finite differences when the function is tabulated for equally spaced arguments.
- Clearly,

$$\Delta^0 f(x_i) = f[x_i] = f(x_i) \tag{27}$$

$$\Delta f(x_i) = f(x_{i+1}) - f(x_i) = hf[x_i, x_{i+1}] \tag{28}$$

$$\Delta^2 f(x_i) = \Delta f(x_{i+1}) - \Delta f(x_i) = f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)$$

$$= 2h^2\{\frac{f(x_i)}{(x_i - x_{i+1})(x_i - x_{i+2})} + \frac{f(x_{i+1})}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})}$$

$$+ \frac{f(x_{i+2})}{(x_{i+2} - x_i)(x_{i+1} - x_i)}\}$$

Relationship with divided differences (contd.)

- Or
$$\Delta^2 f(x_i) = 2h^2 f[x_i, x_{i+1}, x_{i+2}] \tag{29}$$
- One can prove, in general
$$\Delta^r f[x_i] = r!h^r f[x_i, x_{i+1}, ..., x_{i+r}] \tag{30}$$
- With backward differences, one has the relationship
$$\nabla^0 f(x_i) = f[x_i] = f(x_i) \tag{31}$$
$$\nabla f(x_i) = f(x_i) - f(x_{i-1}) = hf[x_i, x_{i-1}] \tag{32}$$
$$\nabla^2 f(x_i) = \nabla f(x_i) - \nabla f(x_{i-1}) = f(x_i) - 2f(x_{i-1}) + f(x_{i-2})$$
$$= 2h^2\{\frac{f(x_i)}{(x_i - x_{i-1})(x_i - x_{i-2})} + \frac{f(x_{i-1})}{(x_{i-1} - x_i)(x_{i-1} - x_{i+2})}$$
$$+ \frac{f(x_{i-2})}{(x_{i-2} - x_i)(x_{i-2} - x_{i-1})}\}$$

Relationship with divided differences (contd.)

- Or
$$\nabla^2 f(x_i) = 2h^2 f[x_i, x_{i-1}, x_{i-2}] \tag{33}$$
- In general,
$$\nabla^r f[x_i] = r!h^r f[x_i, x_{i-1}, ..., x_{i-r}] \tag{34}$$
- Similarly, the following relationships with the central difference operator can be established

$$\delta f_{1/2} = f_1 - f_0 = hf[x_0, x_1]$$

$$\delta f_{-1/2} = f_0 - f_{-1} = hf[x_0, x_{-1}]$$

$$\delta^2 f_1 = \delta f_{3/2} - \delta f_{1/2} = f_2 - 2f_1 + f_0$$

$$\delta^2 f_1 = 2!h^2 f[x_0, x_1, x_2] \tag{35}$$

Relationship with divided differences (contd.)

- Or, in general,

$$\delta^{2m+1} f_{k+1/2} = (2m+1)! h^{(2m+1)} f[x_{k-m}, \ldots, x_k, \ldots, x_{k+m}, x_{k+m+1}]$$

$$\delta^{2m+1} f_{k-1/2} = (2m+1)! h^{(2m+1)} f[x_{k-m-1}, x_{k-m}, \ldots, x_k, \ldots, x_{k+m}] \tag{36}$$

- And,

$$\delta^{2m} f_k = (2m)! h^{2m} f[x_{k-m}, \ldots, x_k, \ldots, x_{k+m}] \tag{37}$$

Newton's Forward Difference Formula

- By substituting the relationship between the forward differences and the divided differences (Eq. 30) in the Newton's fundamental formula (Eq. 11) we obtain the Newton's Forward Difference formula for $x_0 \leq x \leq x_N$

$$f(x) = f_0 + (x - x_0)\frac{\Delta f_0}{1!h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!h^2}$$

$$+ \cdots + (x - x_0)(x - x_1)...(x - x_{N-1})\frac{\Delta^N f_0}{N!h^N} + E(x) \qquad (38)$$

- where

$$E(x) = \frac{(x - x_0)...(x - x_N)f^{(N+1)}(\xi)}{(N+1)!} \qquad (39)$$

- Similarly, by treating the x values in the reverse order $\{x_N, x_{N-1}, \ldots, x_1, x_0\}$, and using the relationship between the backward differences and the divided differences (Eq. 34) in the Newton's fundamental formula (Eq. 11), we obtain the Newton's Backward Difference formula for $x_0 \leq x \leq x_N$

$$f(x) = f_N + (x - x_N)\frac{\nabla f_N}{1!h} + \frac{(x - x_N)(x - x_{N-1})\nabla^2 f_N}{2!h^2} +$$

$$\ldots \frac{(x - x_N)(x - x_{N-1})(x - x_{N-2})\nabla^N f_N}{N!h^N} + E \qquad (40)$$

where

$$E = \frac{h^{N+1}}{(N+1)!}(x - x_N)(x - x_{N-1}).....(x - x_0)f^{N+1}(\xi) \qquad (41)$$

- Sometimes it is more advantageous to express the interpolating polynomial directly in terms of the tabulated function values rather than in terms of finite differences or divided differences.
- This is the essence of Lagrangian approach described next.

Lagrangian approach:

- Let us suppose that a function has been tabulated on n+1 arguments $(x_{0.}, x_{1,}, .., x_n)$ with values $(f_{0.}, f_{1,}, .., f_n)$.
- Our aim is to find the polynomial of degree n and which agrees with the tabulated function values on all n+1 points.

Lagrangian Interpolation (contd.)

- Let us assume that the polynomial is of the form

$$y(x) = A_0 + A_1 x + \ldots + A_n x^n = \sum_{k=0}^{n} A_k x^k \tag{42}$$

- and it should also satisfy

$$\begin{aligned} f(x_0) &= A_0 + A_1 x_0 + \ldots + A_n x_0^n \\ f(x_1) &= A_0 + A_1 x_1 + \ldots + A_n x_1^n \\ \vdots \quad & \quad \vdots \qquad\quad \vdots \\ f(x_n) &= A_0 + A_1 x_n + \ldots + A_n x_n^n \end{aligned} \tag{43}$$

- The condition that Eqs. 42 and 43 are simultaneously satisfied is that the determinant of these equations vanishes, i.e.,

$$\begin{vmatrix} y & 1 & x & x^2 & \dots & x^n \\ f_0 & 1 & x_0 & x_0^2 & \dots & x_0^n \\ f_1 & 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & 1 & \vdots & \vdots & \dots & \vdots \\ f_n & 1 & x_n & x_n^2 & \dots & x_n^n \end{vmatrix} = 0 \tag{44}$$

- which when expanded will give us the required polynomial.
- Alternatively, we would write y(x) directly in the required form

$$y(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + \dots l_n(x)f(x_n) = \sum_{k=0}^{n} l_k(x)f(x_k) \tag{45}$$

Lagrangian Interpolation (contd.)

- where $l_0(x), ...l_n(x)$ are all polynomials of degree n or less.
- Eq. 45 should take the value

$$y(x_j) = f(x_j); j = 0, 1, 2, ...k$$

i.e.

$$\sum_{k=0}^{n} l_k(x_j) f(x_k) = f(x_j) \quad (46)$$

$$\Rightarrow l_k(x_j) = \delta_{kj} \quad (47)$$

- which immediately leads to the conjecture

$$l_k(x) = c_k[(x - x_0)(x - x_1)...(x - x_{k-1})(x - x_{k+1})...(x - x_n)] \quad (48)$$

- which clearly is a polynomial of degree n.

Lagrangian Interpolation (contd.)

- From Eq. 47 we have $l_k(x_k) = 1$, which combined with Eq. 48 leads to

$$c_k = \frac{1}{(x_k - x_0)(x_k - x_1)...(x_k - x_{k-1})(x_k - x_{k+1})....(x_k - x_n)} \tag{49}$$

- In order to express Eq. 45 in a compact form, we define

$$\pi(x) = (x - x_0)(x - x_1)....(x - x_n) \tag{50}$$

and

$$\pi'(x_i) = (x_i - x_0)...(x_i - x_{i-1})(x_i - x_{i+1})...(x_i - x_n) = \frac{1}{c_i} \tag{51}$$

- we get

$$y(x) = \sum_{k=0}^{n} \frac{\pi(x)}{(x - x_k)\pi'(x_k)} f(x_k) = \sum_{k=0}^{n} l_k(x) f(x_k) \tag{52}$$

Lagrangian Interpolation (contd.)

- where $l_i(x) = \frac{\pi(x)}{(x-x_k)\pi'(x_k)}$
- Or

$$l_i(x) = \frac{(x-x_0)(x-x_1)....(x-x_{i-1})(x-x_{i+1})....(x-x_n)}{(x_i-x_0)(x_i-x_1)....(x_i-x_{i-1})(x_i-x_{i+1})....(x_i-x_n)} \tag{53}$$

- and the error in the approximation

$$f(x) = \sum_{k=0}^{n} l_k(x)f(x_k) + E(x) \tag{54}$$

- where

$$E(x) = \pi(x)f[x_0,...x_n,x] = \pi(x)\frac{f^{n+1}(\xi)}{(n+1)!} \tag{55}$$

Iterative Interpolation (Aitkin's Interpolation)

- This method uses the linear interpolation iteratively to achieve interpolation to the arbitrary order.
- Because of the recursive nature of the approach, it is suitable for machine implementation.
- If we truncate the Newton's fundamental formula to first order, we have

$$f(x) \approx f(x_0) + (x - x_0) f[x_0, x_1]$$

- Or

$$f(x) \approx f(x_0) + \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}(x - x_0)$$

- Or

$$f(x) \approx \frac{(x_1 - x)}{(x_1 - x_0)} f(x_0) - \frac{(x_0 - x)}{(x_1 - x_0)} f(x_1) \tag{56}$$

- Clearly this formula is exact for $x = x_0$ and $x = x_1$.
- We can write Eq. 56 in the following determinantal form:

$$f(x) \approx \frac{1}{(x_1 - x_0)} \begin{vmatrix} f(x_0) & x_0 - x \\ f(x_1) & x_1 - x \end{vmatrix} \tag{57}$$

- We introduce the notation $p_0(x) = f(x_0)$ and $p_1(x) = f(x_1)$
- And

$$p_{0,1}(x) \equiv f(x_0) + (x - x_0) f[x_0, x_1] \tag{58}$$

- So that Eq. 57 can be written as

$$p_{0,1}(x) = \frac{1}{x_1 - x_0} \begin{vmatrix} p_0(x) & x_0 - x \\ p_1(x) & x_1 - x \end{vmatrix} \tag{59}$$

Iterative Interpolation (Aitkin's Interpolation) (contd.)

- According to the Aitkin scheme the next order, i.e. cubic approximation will be given by

$$p_{0,1,2}(x) = \frac{1}{x_2 - x_0} \begin{vmatrix} p_{0,1}(x) & x_0 - x \\ p_{1,2}(x) & x_2 - x \end{vmatrix} \tag{60}$$

- Easy to verify that $p_{0,1,2}(x)$ takes value f_0, f_1, f_2 at x_0, x_1, x_2.
- And to arbitrary order k

$$p_{0,1,2,...k}(x) = \frac{1}{x_k - x_0} \begin{vmatrix} p_{0,1,2,..k-1}(x) & x_0 - x \\ p_{1,2,...k}(x) & x_k - x \end{vmatrix} \tag{61}$$

- This can be easily programmed and also written in the form of a difference table as shown next.

Iterative Interpolation (Aitkin's Interpolation) (contd.)

$$
\begin{array}{lllll}
x_0 & p_0 & & & \\
x_1 & p_1 & p_{0,1} & & \\
x_2 & p_2 & p_{1,2} & p_{0,1,2} & \\
x_3 & p_3 & p_{2,3} & p_{1,2,3} & p_{0,1,2,3}
\end{array}
$$

- Note that as in all differences table, this one also is recursive in nature, i.e. each element is related to elements in the previous column and row.