PH307: Introduction to Numerical Analysis
Tutorial Sheet 2

The problem of this sheet constitutes the lab assignment.

1. The central difference operator δ is defined as $\delta f_k = f_{k+1/2}+f_{k-1/2}$. Prove the following identities

$$\begin{aligned}
\delta^{2m+1} f_{k+1/2} &= h^{2m+1}(2m+1)! f[x_{k-m}, \ldots x_k, \ldots, x_{k+m}, x_{k+m+1}] \\
\delta^{2m+1} f_{k-1/2} &= h^{2m+1}(2m+1)! f[x_{k-m-1}, x_{k-m}, \ldots x_k, \ldots, x_{k+m}] \\
\delta^{2m} f_k &= h^{2m}(2m)! f[x_{k-m}, \ldots x_k, \ldots, x_{k+m}]
\end{aligned}$$

above k and m are integers, and $h = x_{i+1} - x_i$.

2. Assume that the arguments and the function values are labeled as $\{x_{-n}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n\}$, and $\{f_{-n}, \ldots, f_{-1}, f_0, f_1, \ldots, f_n\}$, respectively. Prove that the interpolating polynomial of degree $2n$ is given by the Stirling formula

$$\begin{aligned}
f(x) &= f_0 + \frac{(x-x_0)}{h}\mu\delta f_0 + \frac{(x-x_0)^2}{2!h^2}\delta^2 f_0 + \frac{(x-x_0)(x-x_1)(x-x_{-1})}{3!h^3}\mu\delta^3 f_0 \\
&\quad + \frac{(x-x_0)^2(x-x_1)(x-x_{-1})}{4!h^4}\delta^4 f_0 + \cdots \\
&\quad + \frac{(x-x_0)^2(x-x_1)(x-x_{-1})\ldots(x-x_{n-1})(x-x_{-n+1})}{2n!h^{2m}}\delta^{2n} f_0 \\
&\quad + E,
\end{aligned}$$

with error E given by

$$E = \frac{h^{2n+1}(x-x_0)(x-x_1)(x-x_{-1})\ldots(x-x_n)(x-x_{-n})}{(2n+1)!} f^{(2n+1)}(\xi),$$

and μ being the averaging operator defined as $\mu f_i = \frac{1}{2}(f_{i+1/2} + f_{i-1/2})$.

3. For the values of $x = \{0, 1, 2, 4\}$, we have $f(x) = \{1, 1, 2, 5\}$. Obtain interpolation polynomial of degree three for this case using:

   (a) Newton's fundamental formula

   (b) Lagrange interpolation method

   (c) Aitken's interpolation scheme.

   Note that all the three methods should yield the same interpolation polynomial.

4. Obtain the interpolating polynomial of degree three using the Newton's forward difference formula when, for the values of the argument $x = \{4, 6, 8, 10\}$, the unknown function takes the values $f(x) = \{1, 3, 8, 20\}$.

5. Demonstrate that if you apply the Newton backward-difference formula to the data of the previous problem, the polynomial obtained will be identical to the one obtained by the forward-difference formula.

6. Write a computer program to perform polynomial interpolation of arbitrary order (n) using the following methods:

   (a) Newton's Fundamental Formula

   (b) Lagrange Interpolation scheme

   (c) Aitken's iterative interpolation scheme.

   Your program should prompt the user for: (a) which method to use, (b) order of the interpolation polynomial (n), and (c) value of x at which interpolation needs to be performed. After that either it should read $\{x_0, \dots x_n\}$ and $\{f_{0,} \dots, f_n\}$ from the input, or generate these quantities in a separate subroutine for the specifed function. Your task will be simplified if you store these values in arrays. To allow for an arbitrary value of n, use dynamic memory allocation for these, and, other arrays. Your interpolation subroutines should be fully general so that they allow interpolation of any function, and not just a particular function. Try your program for the function $f(x) = \cos(x)$, with $0 \leq x \leq 1$, and $x_i = ih$ $(0 \leq i \leq n)$, for various values of n and x. Note, here we are assuming equispaced arguments with $h = (x_n - x_0)/n$. Also print out the value of $\cos(x)$ computed by the corresponding inbuilt function. A check of your code is that for a given value of x and n, all the methods should yield the same result.

Coding Hints: For method (a) you need the divided differences of the form $f[x_i, x_{i+1}, \dots x_{i+k}]$ with $0 \leq i \leq n$ and $0 \leq k \leq n$. It is most convenient to store them in a two dimensional array (Fortran 90 notation) diff$(0 : n, 0 : n)$, as diff(i, k). Note that then they can be generated recursively using the inititalization diff$(i, 0) = f_i$ and the recursion relation diff$(i, k) = ($diff$(i + 1, k - 1) -$ diff$(i, k - 1))/(x_{k+i} - x_i)$. Note that a similar scheme can be used to store and generate the Aitken's array elements $p_{i,i+1,i+2,\dots i+k}$.