Chapter 5: Numerical Solutions of Ordinary Differential Equations

Prof. Alok Shukla

Department of Physics
IIT Bombay, Powai, Mumbai 400076

Course Name: Introduction to Numerical Analysis (PH 307)

Ordinary Differential Equations

- In this chapter we will explore various numerical methods aimed at numerically solving ordinary differential equations (ODEs)
- But, what is an ODE, and how does it differ from a partial differential equation (PDE)?
- A PDE is a differential equation involving more than one variable
- On the other hand, an ODE is a differential equation involving just one variable, taken normally to be x
- An ODE can be considered as a special case of a PDE in which the number of variables is just one.
- Therefore, we will study the numerical solutions of ODEs before tackling the PDEs

ODEs (contd.)

- For the time being we will focus on linear differential equations
- A general n-th order linear differential equation for an unknown function $y(x)$ can be written as

$$\left\{D^n + G_{n-1}(x)D^{n-1} + \cdots + G_1(x)D + G_0(x)\right\} y = Q(x), \quad (1)$$

where $D^n = \frac{d^n}{dx^n}$ and $G_i(x)$, $i = 0, \ldots, n-1$, $Q(x)$ are known functions of the variable x.

- Given the ODE of Eq. 1, and the values of $y(x)$ and/or its derivatives at some values of x, one has to solve for the unknown function $y(x)$ for all values of x for which the ODE is defined.

First order initial value problems

- As far as the numerical methods are concerned, we will show that the methods developed for the first-order ODE ($n = 1$) can be used to solve a large class of higher-order ODEs
- Note that for $n = 1$, Eq. 1, reduces to an intial-value problem

$$\begin{aligned} \frac{dy}{dx} &= F(x, y) \\ y(x_0) &= y_0 \text{ (given)} \end{aligned} \tag{2}$$

  above $F(x, y)$ is a function of both x and y, and the value of the solution $y(x)$ is given to be y_0 at a point $x = x_0$
- y_0 is also called the initial value, hence the name "initial value problem" for such ODEs.

Numerical solution

- Let us assume that we want to numerically determine $y(x)$, for all values of $x \in [a, b]$
- As in earlier cases of interpolation and numerical integration, we divide this interval in N equal bins whose width h is given by

$$h = \frac{b-a}{N}$$

- So that

$$\begin{aligned} x_0 &= a \\ x_N &= b \\ x_{n+1} &= x_n + h \end{aligned}$$

- With the initial condition $y_0 = y(x_0) = y(a)$

Numerical solution: Euler's method

- Euler's method is one of the simplest techniques for solving a first-order ODE numerically
- Because of its simplicity, it is not very accurate.
- Nevertheless, it is worthwhile to start the discussion of the numerical methods from this approach
- As it turns out, it belongs to a class of approaches called Taylor Expansion Approach
- Let us assume that we have determined the solution for the x values $\{x_0, x_1, x_2, \ldots, x_n\}$
- This implies that we know the values $\{y_0, y_1, y_2, \ldots, y_n\}$, where $y_i = y(x_i)$

Euler's Method (contd.)

- Using this we want to compute the value of the function y_{n+1} at the next point x_{n+1}
- We can write

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} y'(x)dx, \tag{3}$$

  where $y'(x) = \frac{dy}{dx}$
- For small values of h, we can approximate

$$\int_{x_n}^{x_{n+1}} y'(x)dx \approx hy'_n$$

- Because our ODE is

$$y'(x) = F(x, y)$$

  we can substitute $y'_n = F(x_n, y_n)$ above leading to the famous Euler's formula

$$y_{n+1} = y_n + hF(x_n, y_n) \tag{4}$$

- Note that this formula has the form of a recursion relation, and it yields the solution at the point $x = x_{n+1}$ in terms of solution at the previous point $x = x_n$
- Because the solution at $x = x_0$ is known, therefore, this formula will yield the solution for all values of x_i, $i = 1, \ldots, N$

Euler's Method: An Example

- Let us consider a simple ODE whose exact solution is known, namely

$$\begin{aligned} \frac{dy}{dx} &= y \\ y_0 &= y(x=0) = 1 \end{aligned} \tag{5}$$

- We know that the exact solution of this initial value problem for $x \in [0, \infty)$ is

$$y(x) = e^x$$

- Let us try to obtain numerical solution of this ODE using the Euler's method in the domain $x \in (0, 1)$
- The bin size h will be

$$\begin{aligned} h &= \frac{b-a}{N} = \frac{1}{N} \\ x_0 &= 0,\ x_N = 1 \\ x_n &= nh = \frac{n}{N} \ \text{ for } 0 \leq n \leq N \end{aligned}$$

Euler's Method: Example (contd.)

- Clearly, here $F(x,y) = y$, therefore on substituting in Eq. 4, we obtain

$$y_{n+1} = y_n + hy_n = (h+1)y_n \tag{6}$$

- For the present case of the range of solution $x \in [0,1]$, we have

$$y_{n+1} = \left(\frac{N+1}{N}\right) y_n \tag{7}$$

Euler Method: Example...

- If we take $N = 10$, we have

$$y_{n+1} = \left(\frac{11}{10}\right) y_n = 1.1 y_n$$

- This formula readily yields

x_n	$y_n = (1.1)^n$ (Euler)	$y_n = e^{x_n}$ (exact)
0.0	1.000	1.0
0.1	1.100	1.105
0.2	1.210	1.221
$\vdots$	$\vdots$	$\vdots$
0.9	2.358	2.450
1.0	2.594	2.718

Euler Method: Example

- We note that with the increasing values of x, the Euler solution is systematically smaller than the exact one
- Q: Why is that the case?
  - The first reason is that we have used a rather large value of $h = 0.1$. If we were to use $N = 100$, so that $h = 0.01$. Using that in Eq. 7, obtain $y_n = (1.01)^n$ leading to

$$
\begin{aligned}
y(x = 0.1) &= (1.01)^{10} = 1.105 \\
y(x = 0.9) &= (1.01)^{90} = 2.445 \\
y(x = 1.0) &= (1.01)^{100} = 2.705
\end{aligned}
\tag{8}
$$

  - Now we note that there is much better agreement between the exact values and the calculated values
  - But, if we do calculations for larger values of x, even for this value of h, the disagreement will become severe.
  - Therefore, a smaller value of h will lead to a better agreement with the exact values because as we will see that the errors in the Euler's method scale as $\sim h^2$

Problems with the Euler's method

- But, there are some inherent problems in the Euler's approach.
- We will show that it is the lowest order formula obtained from the general Taylor expansion approach for solving the ODEs
- Another problem is that we are using the slope of $y(x)$ at the point $x = x_n$ to compute it at $x = x_{n+1}$
- What if we used its average value over x_n and x_{n+1}

$$y_{n+1} = y_n + \frac{h}{2}\left(y_n^{'} + y_{n+1}^{'}\right) \tag{9}$$

- The problem with this formula is that it is trying to determine y_{n+1} in terms of $y_{n+1}^{'}$, which we don't know!
- However, such formulas are used in what are called the Predictor-Corrector methods which we will discuss later
- Next, we discuss the Taylor expansion approach for solving the first-order ODEs

The Taylor Expansion Approach

- Let us Taylor expand $y_{n+1} = y(x_n + h)$ around the point x_n

$$y_{n+1} = y_n + hy_n^{'} + \frac{h^2}{2}y_n^{''} + \frac{h^3}{6}y_n^{'''} + \frac{h^4}{24}y_n^{''''} + \cdots \qquad (10)$$

- Note that if we discard all the terms except the first order one in h, we obtain the Euler's formula of Eq. 4.
- We note that this approach is systematically improvable by retaining the successively higher-order terms
- However, the problem is how to calculate the second and higher-order derivatives needed in Eq. 10, given $y^{'}(x) = F(x, y)$
- Mathematically, speaking it can be done as follows

$$\begin{aligned} y^{''}(x) = \frac{dy^{'}}{dx} &= \frac{dF(x,y)}{dx} \\ &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = F_x + FF_y \end{aligned}$$

above we have used the notations $F_x = \frac{\partial F}{\partial x}$ and $F_y = \frac{\partial F}{\partial y}$

Taylor Expansion approach...

- Similarly

$$
\begin{aligned}
y'''(x) = \frac{dy''}{dx} &= \frac{d}{dx}(F_x + FF_y) \\
&= F_{xx} + F_{xy}F + (F_x + FF_y)F_y + FF_{yx} + FF_{yy}F \\
&= F_{xx} + 2FF_{xy} + F^2F_{yy} + F_xF_y + FF_y^2
\end{aligned}
$$

  above we used the fact that $F_{xy} = F_{yx}$
- We note that the calculation of the higher order derivatives becomes quite tedious
- However, if it can be done for some ODE, this approach yields quite accurate results

- Let us again consider the ODE of Eq. 5

$$\frac{dy}{dx} = y$$
$$y_0 = y(x=0) = 1$$

- The Taylor expansion solution for this will be

$$y_{n+1} = (1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} + \cdots)y_n$$

- Let us again consider $h = 0.1$, which leads to

$$\begin{aligned} y_{n+1} &= (1 + 0.1 + 0.005 + 0.00017 + 0.0000042 + \cdots)y_n \\ \implies y_n &= (1.105)^n \text{ (2nd order)} \\ y_n &= (1.1051666...)^n \text{ (3rd order)} \\ y_n &= (1.1051708332..)^n \text{ (4th order)} \end{aligned}$$

- We tabulate the results obtained for various orders of Taylor expansion for some points $x \in [0, 1]$, and compare those to the exact results

Taylor Expansion Approach: example...

-

x_n	y_n (Euler)	y_n (2nd order)	y_n (3rd order)	y_n 4th order)	$y_n = e^{x_n}$ (exact)
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.1000	1.1050	1.1052	1.1052	1.1052
0.2	1.2100	1.2210	1.2214	1.2214	1.2214
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.9	2.3579	2.4562	2.4595	2.4596	2.4596
1.0	2.5937	2.7141	2.7182	2.7183	2.7183

- Note that the 4th-order result is exact up to 10^{-4}, which it should be for $h = 10^{-1}$.

Runge-Kutta Approach

- We say that the Taylor-expansion approach for solving the first order ODEs of the form

$$y' = F(x, y), \quad \text{with } y(x_0) = y_0$$

is based on the formula

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2} y_n'' + \frac{h^3}{6} y_n''' + \frac{h^4}{24} y_n'''' + \cdots$$

- We also briefly discussed that it can be tedious to calculate the higher order derivatives $y^{(r)}(x)$ $(r \geq 2)$ needed for the Taylor expansion
- Runge, Kutta, and a few others proposed an approach to avoid this problem approach is aimed at avoiding this by using a formula of the form

$$y_{n+1} = y_n + \alpha_0 k_0 + \alpha_1 k_1 + \cdots + \alpha_p k_p \tag{11}$$

- where

$$
\begin{aligned}
k_0 &= hF(x_n, y_n) \\
k_1 &= hF(x_n + \mu_1 h, y_n + \lambda_{10} k_0) \\
k_2 &= hF(x_n + \mu_2 h, y_n + \lambda_{20} k_0 + \lambda_{21} k_1) \\
\vdots &= \vdots \\
k_p &= hF(x_n + \mu_p h, y_n + \lambda_{p0} k_0 + \lambda_{p1} k_1 + \cdots + \lambda_{pp-1} k_{p-1})
\end{aligned}
\tag{12}
$$

  the unknown coefficients α_i, μ_i, and λ_{ij} are determined by Taylor expanding the above equations around x_n and y_n, and then comparing the results with Taylor expansion of Eq. 10 to a suitable order.

- Next, we do that for the simple case of $p = 1$

Verification of the second-order Runge-Kutta Formula

- For $p = 1$, the Runge-Kutta (RK) equations are

$$\begin{aligned} y_{n+1} &= y_n + \alpha_0 k_0 + \alpha_1 k_1 \\ k_0 &= hF(x_n, y_n) \\ k_1 &= hF(x_n + \mu h, y_n + \lambda k_0) \end{aligned} \quad (13)$$

- Next, we Taylor expand k_1 of the previous equation around points (x_n, y_n) retaining terms up to $O(h^2)$, using the notations $F = F(x_n, y_n)$, $F_x = \frac{\partial F}{\partial x}$, and $F_y = \frac{\partial F}{\partial y}$,

$$\begin{aligned} k_1 &= h\left[F + h\mu F_x + \lambda k_0 F_y + O(h^2)\right] \\ &= hF + h^2 \mu F_x + \lambda h^2 F F_y + O(h^3) \end{aligned}$$

- Substituting the expressions for k_0 and k_1 in the first of Eq. 13

$$y_{n+1} = y_n + (\alpha_0 + \alpha_1)hF + h^2 \alpha_1 (\mu F_x + \lambda F F_y) + O(h^3) \quad (14)$$

RK2 derivation (contd.)

- If we apply, the 2nd-order Taylor expansion approach, we have

$$y_{n+1} = y_n + hy_n^{'} + \frac{h^2}{2} y_n^{''} + O(h^3).$$

- Using the fact

$$\begin{aligned} y_n^{'} &= F = F(x_n, y_n) \\ y_n^{''} &= F_x + FF_y \end{aligned}$$

substituting these above, we obtain

$$y_{n+1} = y_n + hF + \frac{h^2}{2}(F_x + FF_y) + O(h^3) \qquad (15)$$

- Comparing different powers of h in Eqs. 14 and 15, we obtain

$$\begin{aligned} \alpha_0 + \alpha_1 &= 1 \\ \mu \alpha_1 &= \frac{1}{2} \\ \lambda \alpha_1 &= \frac{1}{2} \end{aligned}$$

- These equations can be satisfied for any constant c

$$\begin{aligned} \alpha_0 &= 1 - c \\ \alpha_1 &= c \\ \mu &= \frac{1}{2c} \\ \lambda &= \frac{1}{2c} \end{aligned}$$

RK2 Final Formula

- A popular choice is $c = \frac{1}{2}$, leading to the famous symmetric RK2 formula

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{2}(k_0 + k_1) \\ k_0 &= hF(x_n, y_n) \\ k_1 &= hF(x_n + h, y_n + k_0) \end{aligned} \tag{16}$$

Higher Order RK Formulas:RK-3

- One can similarly derive higher-order RK formulas
- Several RK-3 (third order) formulas are possible, while one popular choice is

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{1}{6}(k_0 + 4k_1 + k_2) + O(h^4) \\
k_0 &= hF(x_n, y_n) \\
k_1 &= hF\left(x_n + \frac{h}{2}, y_n + \frac{k_0}{2}\right) \\
k_2 &= hF(x_n + h, y_n + 2k_1 - k_0)
\end{aligned}
\tag{17}
$$

Fourth-Order RK Formula: RK-4

- A fourth-order Runge-Kutta formula is given by

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{1}{6}\left(k_0 + 2k_1 + 2k_2 + k_3\right) + O(h^5) \\
k_0 &= hF(x_n, y_n) \\
k_1 &= hF\left(x_n + \frac{h}{2}, y_n + \frac{k_0}{2}\right) \\
k_2 &= hF\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\
k_3 &= hF\left(x_n + h, y_n + k_2\right)
\end{aligned}
\tag{18}
$$

Predictor-Corrector Methods

- In this course we will not be going deep into the topic of Predictor-Corrector (PC) methods,
- This is because more accurate results can be obtained using simpler algorithms by adopting methods like Taylor series expansion and RK-n
- However, we will briefly mention the lowest order PC method namely Euler PC approach
- In the PC approach, one predicts the value of the solution at a point x_n using the predictor formula
- Next, it is corrected using the so-called corrector formula
- This cycle is repeated for each point x_n until the results of predictor and corrector agree with each other

Euler Predictor-Corrector Method

- For the Euler PC approach, the relevant formulas are

$$\begin{aligned} y_{n+1} &= y_n + h y_n^{'} \qquad \text{Predictor Formula} \\ y_{n+1} &= y_n + \frac{h}{2}\left(y_{n+1}^{'} + y_n^{'}\right) \qquad \text{Corrector Formula} \end{aligned} \tag{19}$$

- Note that the value of $y_{n+1}^{'}$ needed in the corrector formula is obtained using the ODE $y_{n+1}^{'} = F(x_{n+1}, y_{n+1})$, where the value of y_{n+1} obtained from the predictor formula is used
- This cycle is repeated for each x_n until the values of y_{n+1} obtained from the predictor and the corrector formulas agree with each other

Systems of Ordinary Differential Equations

- Let us consider a system of p first-order ODEs

$$\begin{aligned} y_i' &= F_i(x, y_1, y_2, \ldots, y_p) \\ y_i(x_0) &= y_{i0} \end{aligned} \qquad i = 1, 2, 3, \ldots, p$$

- Let us adopt the vector notations

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_p \end{pmatrix}, \quad \tilde{F} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_p \end{pmatrix}$$

System of ODEs: An Example

- Using the vector notation, the system of ODEs can be written in the compact form
$$Y^{'} = \tilde{F},$$
subect to the initial conditions
$$Y_0 = Y(x_0) = \begin{pmatrix} y_1(x_0) \\ y_2(x_0) \\ y_3(x_0) \\ \vdots \\ y_p(x_0) \end{pmatrix}$$
- Let us consider a simple example with $p = 2$
$$\begin{aligned} y_1^{'} &= -y_1 - y_2 = F_1(x, y_1, y_2) \\ y_2^{'} &= y_1 - y_2 = F_2(x, y_1, y_2) \end{aligned} \qquad (20)$$
with the initial conditions $y_1(0) = 1$, $y_2(0) = 0$

A system of ODEs: Taylor Expansion Approach

- Clearly, this equations can be put in the matrix form

$$Y' = AY,$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad \text{and } A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

- Let us set up the Taylor Expansion approach for this system of ODEs
- We will have

$$\begin{aligned} y_1^{n+1} &= y_1(x_n+h) = y_1^n + hy_1'^n + \frac{h^2}{2}y_1''^n + \frac{h^3}{6}y_1'''^n + \frac{h^4}{24}y_1''''^n + \cdots \\ y_2^{n+1} &= y_2(x_n+h) = y_2^n + hy_2'^n + \frac{h^2}{2}y_2''^n + \frac{h^3}{6}y_2'''^n + \frac{h^4}{24}y_2''''^n + \cdots \end{aligned} \tag{21}$$

- Using, Eqs. 20, Let us calculate the higher order derivatives

$$y_1^{''} = \frac{dy_1^{'}}{dx} = \frac{dF_1(x, y_1, y_2)}{dx} = F_{1x} + F_{1y_1}y_1^{'} + F_{1y_2}y_2^{'}$$
$$= F_{1x} + F_{1y_1}F_1 + F_{1y_2}F_2 = 0 - (-y_1 - y_2) - (y_1 - y_2)$$
$$= 2y_2$$
$$y_1^{'''} = \frac{dy_1^{''}}{dx} = 2y_2^{'} = 2y_1 - 2y_2$$

- Similarly

$$y_2^{''} = \frac{dy_2^{'}}{dx} = \frac{dF_2(x, y_1, y_2)}{dx} = F_{2x} + F_{2y_1}y_1^{'} + F_{2y_2}y_2^{'}$$
$$= F_{2x} + F_{2y_1}F_1 + F_{2y_2}F_2 = 0 + (-y_1 - y_2) - (y_1 - y_2)$$
$$= -2y_1$$
$$y_2^{'''} = \frac{dy_2^{''}}{dx} = -2y_1^{'} = -2(-y_1 - y_2) = 2y_1 + 2y_2$$

- Therefore, using Eqs 21, and the expressions for derivatives of y_1/y_2, the 3rd-order Taylor expansion solution for this system can be written as

$$y_1^{n+1} = y_1^n - hy_1^n - hy_2^n + h^2 y_2^n + \frac{h^3}{3} y_1^n - \frac{h^3}{3} y_2^n + \cdots$$
$$y_2^{n+1} = y_2^n + hy_1^n - hy_2^n - h^2 y_1^n + \frac{h^3}{3} y_1^n + \frac{h^3}{3} y_2^n + \cdots$$

- The values of y_1^{n+1}/y_2^{n+1} can be easily generated from these expansions, starting with $n = 0$.
- For this particular case, extension to higher orders of expansion is quite trivial

System of ODEs: Runge-Kutta Approach

- Extension of the Runge-Kutta approach to a system of equations is almost trivial
- It basically amounts to applying the RK method to each ODE of the system, individually
- For the general case of p coupled ODEs, the RK-4 formulas will be, for $i = 1, 2, 3, \ldots, p$

$$
\begin{aligned}
y_i^{n+1} &= y_i^n + \frac{1}{6}\left(k_0^i + 2k_1^i + 2k_2^i + k_3^i\right) \\
k_0^i &= hF_i(x_n, y_1^n, y_2^n, \ldots, y_p^n) \\
k_1^i &= hF_i\left(x_n + \frac{h}{2}, y_1^n + \frac{k_0^1}{2}, y_2^n + \frac{k_0^2}{2}, \ldots y_p^n + \frac{k_0^p}{2}\right) \\
k_2^i &= hF_i\left(x_n + \frac{h}{2}, y_1^n + \frac{k_1^1}{2}, y_2^n + \frac{k_1^2}{2}, \ldots, y_p^n + \frac{k_1^p}{2}\right) \\
k_3^i &= hF_i\left(x_n + h, y_1^n + k_2^1, y_2^n + k_2^2, \ldots, y_p^n + k_2^p\right)
\end{aligned}
\qquad (22)
$$

- We can very easily transform and p-th order initial value problem ODE into a set of p first order coupled first-order ODEs
- Then the pth-order ODE can be solved using the numerical methods developed in the previous section for solving the system of first-order ODEs
- At the beginning of this chapter, we considered a general nth-order ODE

$$\left\{y^{(p)} + G_{n-1}(x)y^{(p-1)} + \cdots + G_1(x)y^{(1)} + G_0(x)y\right\} = Q(x), \tag{23}$$

where $y^{(i)} = \frac{d^i y}{dx^i}$, etc., where the initial conditions $y(x_0) = y_0$, $y^{(1)}(x_0) = y_0^1$, $y^{(2)}(x_0) = y_0^2, \ldots, y^{(p-1)}(x_0) = y_0^p$ are specified.
- We can rewrite Eq. 23 as

$$y^{(p)}(x) = F\left(x, y, y^{(1)}, y^{(2)}, \ldots, y^{(p-1)}\right), \tag{24}$$

where the function F will depend on the exact form of the Eq. 23.

Higher order ODEs...

- Next, we define the following transformations

$$\begin{aligned} y_1 &= y(x) \\ y_2 &= y^{(1)}(x) \\ y_3 &= y^{(2)}(x) \\ \vdots &= \vdots \\ y_{p-1} &= y^{(p-2)}(x) \\ y_p &= y^{(p-1)}(x) \end{aligned} \tag{25}$$

- Using these definitions, we can write Eq. 24 as

$$y^{p'}(x) = F\left(x, y_1, y_2, \ldots, y_p\right), \tag{26}$$

Higher order ODEs...

- We can combine Eqs. 25 and 26 to define the set of n coupled first-order ODEs

$$
\begin{aligned}
y_1^{'}(x) &= f_1(x, y_1, y_2, \ldots, y_p) \\
y^{2'}(x) &= f_2(x, y_1, y_2, \ldots, y_p) \\
\vdots &= \vdots \\
y_{p-1}^{'}(x) &= f_{p-1}(x, y_1, y_2, \ldots, y_p) \\
y_p^{'}(x) &= f_p(x, y_1, y_2, \ldots, y_p)
\end{aligned}
\tag{27}
$$

where

$$
\begin{aligned}
f_i(x, y_1, y_2, \ldots, y_p) &= y_{i+1}, \qquad \text{for } i = 1, 2, 3, \ldots, n-1 \\
f_p(x, y_1, y_2, \ldots, y_p) &= F(x, y_1, y_2, \ldots, y_p)
\end{aligned}
\tag{28}
$$

Second Order ODE: Examples

- Let us first consider the differential equation of a simple-harmonic oscillator of unit angular frequency

$$\begin{aligned} \frac{d^2x}{dt^2} + x &= 0 \\ x(0) &= 0 \\ x'(0) &= 1, \end{aligned} \tag{29}$$

  we know the exact solution of this initial value problem is $x(t) = \sin t$. If we instead change the initial conditions to $x(0) = 1$ and $x'(0) = 0$, the exact solution becomes $x(t) = \cos t$.

- We can easily convert Eq. 29 into two coupled linear ODEs using the transformations

$$\begin{aligned} x_1 &= x \\ x_2 &= \frac{dx}{dt} \end{aligned}$$

Second-order ODEs: examples

- With this, the pair of coupled equations becomes

$$
\begin{aligned}
x_1^{'} &= f_1(t, x_1, x_2) = x_2 \\
x_2^{'} &= f_2(t, x_1, x_2) = -x_1
\end{aligned}
$$

  with the initial conditions $x_1(0) = 0$, $x_2(0) = 1$.
- This equation system can be solved quite easily using the methods described in the previous section
- Next, consider the second-order differential equation

$$
y'' + y' + y^2 = x, \; y(0) = 1, \; y'(0) = 0
$$

- We make the transformation

$$
\begin{aligned}
y_1 &= y \\
y_2 &= y^{'}
\end{aligned}
$$

Second-Order ODEs: Examples...

- With this, our coupled equations become

$$
\begin{aligned}
y_1' &= f_1(x, y_1, y_2) = y_2 \\
y_2' &= f_2(x, y_1, y_2) = x - y_1^2 - y_2
\end{aligned}
$$

which can be solved using the initial conditions
$y_1(0) = 1, \quad y_2(0) = 0$