SC 651: Estimation on Lie Groups

Assignment-1

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Contents

1 Problem 1

1.1 Spring Mass Damper

1.1.1 Theory

Given the Spring Mass Damper's differential equation:

$$\ddot{x} + \dot{x} + x = 0, \tag{1}$$

the analytical solution can be expressed as:

$$x(t) = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t)), \tag{2}$$

where $\alpha = -\frac{1}{2}$ and $\beta = -\frac{\sqrt{3}}{2}$ are the real and imaginary parts of the roots r_1 and r_2 of the characteristic equation $r^2 + r + 1 = 0$.

Given initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$, the specific solution is determined as:

$$x(t) = e^{-\frac{1}{2}t}\left(C_1 \cos\left(-\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(-\frac{\sqrt{3}}{2}t\right)\right), \tag{3}$$

where C_1 and C_2 are constants calculated based on the initial conditions.

For example, with initial conditions $x_0 = 4$ and $v_0 = 4\sqrt{3}$, the constants C_1 and C_2 can be determined, leading to a specific form of the solution.

1.1.2 Methods

The Spring-Mass-Damper system, governed by the differential equation

$$\ddot{x} + \dot{x} + x = 0, \tag{4}$$

was simulated using three different numerical methods: Explicit Euler, Implicit Euler, and Symplectic Euler. The parameters for the simulations were as follows:

- Time step (Δt): 0.001 seconds.
- Total simulation time: 25 seconds.
- Initial conditions: $x(0) = 1$, $\dot{x}(0) = 0$.

The analytical solution of the system is given by

$$x(t) = e^{-\frac{1}{2}t}\left(x_0 \cos\left(\frac{\sqrt{3}}{2}t\right) + \left(\frac{v_0 + \frac{1}{2}x_0}{\frac{\sqrt{3}}{2}}\right) \sin\left(\frac{\sqrt{3}}{2}t\right)\right). \tag{5}$$

1.1.3 Results

The results of these simulations, alongside the analytical solution, are presented in Figure 1. The Figure contains the simulation output and error analysis for the Explicit Euler method, Implicit Euler method, and Symplectic Euler method. These error plots highlight the deviation of each numerical method from the analytical solution.

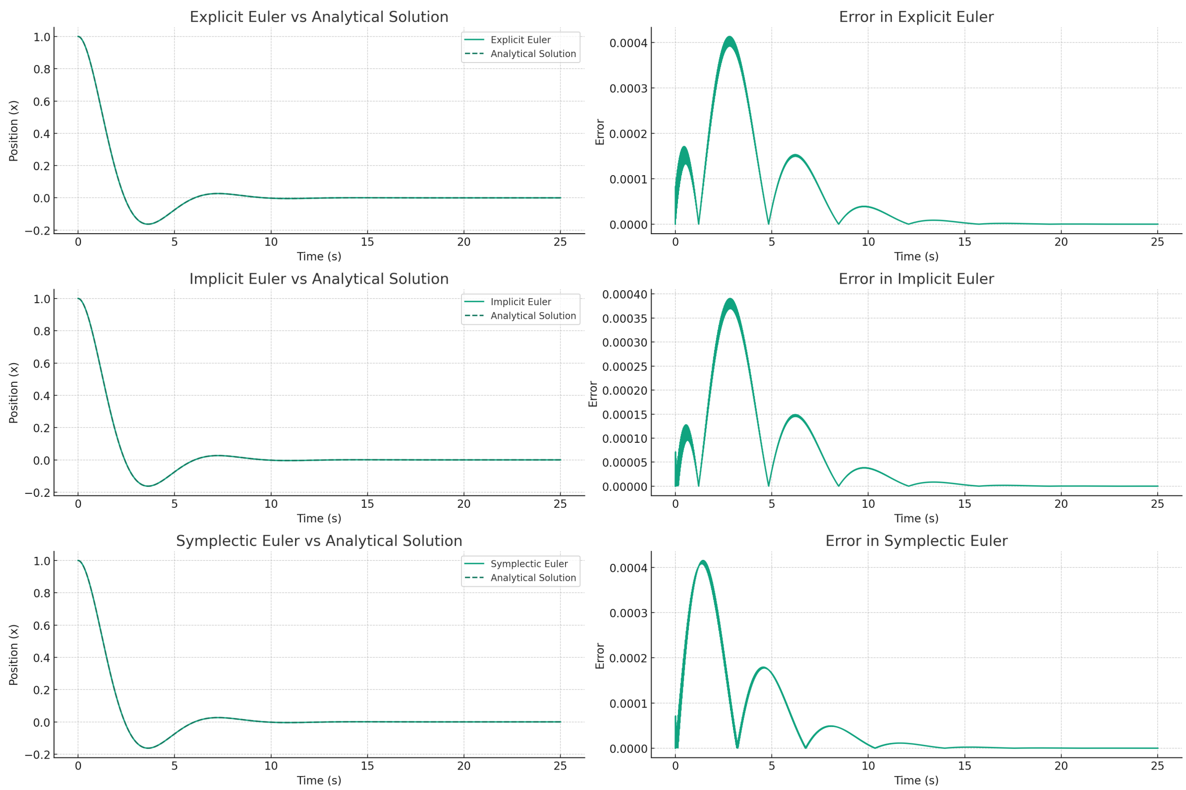

Figure 1: Spring-Mass-Damper system Plots

1.1.4 Conclusion

The Explicit Euler method, although simple to implement, shows a significant deviation from the analytical solution over time, particularly in stiffer systems. The Implicit Euler method offers better stability but requires more computational effort. The Symplectic Euler method, known for its energy-preserving characteristics in conservative systems, exhibits a unique behavior in the damped Spring-Mass-Damper system.

1.2 Planar Pendulum

1.2.1 Model

We analyzed the dynamics of a Planar Pendulum, governed by the non-linear differential equation

$$\ddot{\theta} + \sin(\theta) = 0. \tag{6}$$

Due to the non-linear nature of this equation, finding analytical solutions is not feasible. Therefore, the pendulum's motion was simulated using numerical methods exclusively. The details of the simulations are as follows:

- The time step was varied between 0.01 and 0.0001 to observe the impact on the simulation results.
- Four different numerical methods were employed: Euler, Explicit Euler, Implicit Euler, and Symplectic Euler.

1.2.2 Results

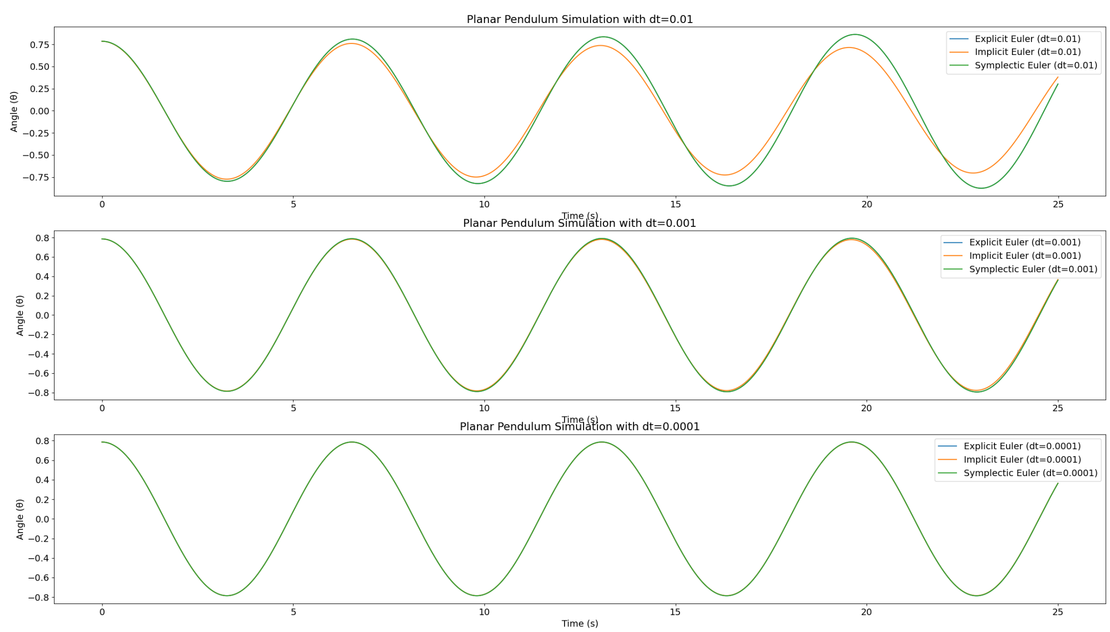

Figure 2: Planar Pendulum Plots(you might have to zoom in)

In the conducted simulations of the Planar Pendulum system using different time steps (0.01, 0.001, and 0.0001), the results highlight the distinct behaviors of the Explicit Euler, Implicit Euler, and Symplectic Euler methods.

For the larger time step of 0.01, the Explicit Euler method exhibited significant numerical instability, particularly noticeable in the amplitude of the pendulum's oscillation. The Implicit Euler method, while more stable, showed a slight phase shift compared to the other methods. The Symplectic Euler method, known for its conservation properties in Hamiltonian systems, maintained stability but at the expense of accuracy in the angle measurements.

Reducing the time step to 0.001 and 0.0001 gradually improved the accuracy and stability of all methods. The discrepancies between the methods became less pronounced, especially in the case of the Symplectic Euler method, which showed improved angle accuracy while still preserving the system's overall dynamics.

1.2.3 Conclusion

The simulation results demonstrate the varying strengths and weaknesses of the Euler methods in solving non-linear differential equations like those governing the Planar Pendulum system. The Explicit Euler method, while simple to implement, can suffer from stability issues at larger time steps. The Implicit Euler method offers improved stability but may introduce phase errors. The Symplectic Euler method, particularly effective in conserving the energy of the system, can be a good choice for long-term simulations where maintaining the system's physical properties is crucial.

These findings underscore the importance of method selection based on the specific requirements of the physical system being modeled. For systems where accuracy in the short term is paramount, a method like the Implicit Euler might be preferred. In contrast, for systems where long-term behavior and energy conservation are more critical, the Symplectic Euler method could be more appropriate. Additionally, the choice of time step is crucial, with smaller time steps generally providing better accuracy at the cost of increased computational effort.

1.3 Analysis of Integrator Fidelity and System Characteristics

The fidelity of an integrator in numerical simulations is crucial for ensuring accurate and reliable results. In the context of the Spring-Mass-Damper system and the Planar Pendulum system, the integrators' fidelity can be assessed by how well they maintain the physical properties and dynamics of these systems over time.

1.3.1 Fidelity of Integrators

In the Spring-Mass-Damper system, the variable x, representing the displacement, evolves in a linear state space. The fidelity of the integrators for this system is predominantly measured by their ability to accurately replicate the damping and oscillatory behavior inherent in the system's linear dynamics.

On the other hand, the Planar Pendulum system, with θ representing the angular displacement, involves non-linear dynamics. The pendulum's motion evolves on a circular trajectory, which is a non-linear manifold. The fidelity of the integrators in this case is assessed based on their ability to handle the non-linearities effectively, particularly in preserving the angular nature of the system's evolution.

1.3.2 Underlying Differences Between the Systems

The fundamental difference between the Spring-Mass-Damper and the Planar Pendulum systems lies in their state space and the nature of their dynamics. The Spring-Mass-Damper system operates in a linear state space, where the force is proportional to the displacement and velocity. This linearity leads to a predictable, harmonic oscillatory motion, which is simpler to simulate accurately.

In contrast, the Planar Pendulum system is governed by non-linear equations of motion. Its state evolves on a circular path, representing a constrained motion on a non-linear manifold. This non-linearity introduces complexities in the simulation, such as sensitivity to initial conditions and the potential for chaotic behavior at higher energies or larger angular displacements.

1.3.3 Implications for Integrator Selection

The choice of an integrator for a given system should consider these underlying dynamics. For linear systems like the Spring-Mass-Damper, methods like the Explicit and Implicit Euler can provide reasonable accuracy and stability, especially with smaller time steps. However, for systems with non-linear dynamics like the Planar Pendulum, integrators that can handle non-linearities, such as the Symplectic Euler method, are often more suitable. This method, in particular, is adept at conserving the Hamiltonian of the system, which is crucial for accurately capturing the dynamics of conservative non-linear systems over long periods.

1.4 Comparison of Euler Methods with RK4, RK5, and ODE45

When comparing the Euler methods with higher-order methods like Runge-Kutta 4 (RK4), Runge-Kutta 5 (RK5), and MATLAB's ODE45, several key aspects come into play: accuracy, computational complexity, and suitability for different types of problems.

1.4.1 Accuracy and Stability

Euler methods are first-order (Explicit and Implicit Euler) and second-order (Symplectic Euler) methods, which generally offer lower accuracy, especially for larger time steps. These methods can exhibit significant numerical errors in systems with complex dynamics or stiff equations. In contrast, RK4 and RK5 are higher-order methods providing greater accuracy. They are particularly effective in handling non-linear differential equations, as they reduce the approximation error significantly with each step. ODE45, which is based on an adaptive step-size RK method, combines the Dormand-Prince pair of RK4 and RK5, offering both accuracy and efficiency. ODE45 automatically adjusts the step size to balance error and computational load, making it highly efficient for a wide range of problems.

1.4.2 Computational Complexity

Higher-order methods like RK4 and RK5 involve more computations per step compared to Euler methods. This increased complexity can lead to longer computational times, especially for simulations requiring a large number of time steps. However, the trade-off often favors higher-order methods because their increased accuracy can allow for larger time steps without sacrificing fidelity. ODE45, with its adaptive step-size algorithm, dynamically optimizes computational effort, often outperforming fixed-step methods in terms of overall efficiency.

1.4.3 Applicability to Different Problems

Euler methods, due to their simplicity, are often used for educational purposes and in scenarios where computational simplicity is paramount. They can be effective for linear systems or non-stiff non-linear systems when used with small time steps. However, for complex or stiff systems, higher-order methods like RK4, RK5, and adaptive methods like ODE45 are more suitable. They provide the accuracy and stability necessary to capture the dynamics of such systems reliably.

1.4.4 Conclusion

In conclusion, while Euler methods offer simplicity and ease of implementation, higher-order methods like RK4, RK5, and adaptive methods like ODE45 are generally preferred for solving more complex and stiff differential equations due to their superior accuracy and stability. The choice of method should be guided by the specific requirements of the problem, considering the trade-offs between computational efficiency and accuracy.

2 Problem 3

2.1 Simulation of Kalman Filter

2.2 Results/Plots

2.3 Conclusions

3 Problem 4

Given the state-space model

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}$$

where A is an $n \times n$ real matrix and C is a $p \times n$ real matrix, we consider the estimation of the state $\hat{x}$. The estimate $\hat{x}$ satisfies the following differential equation:

$$\dot{\hat{x}} = A\hat{x} + L(C\hat{x} - Cx) \tag{7}$$

Now, subtracting the equation $\dot{x} = Ax$ from the equation for $\dot{\hat{x}}$ and introducing the estimation error $e := \hat{x} - x$, we obtain:

$$\dot{e} = \dot{\hat{x}} - \dot{x} = (A\hat{x} + L(C\hat{x} - Cx)) - Ax = (A + LC)e \tag{8}$$

Here, e will converge to zero if the eigenvalues of the matrix $A + LC$ are negative, leading to the system being exponentially stable.