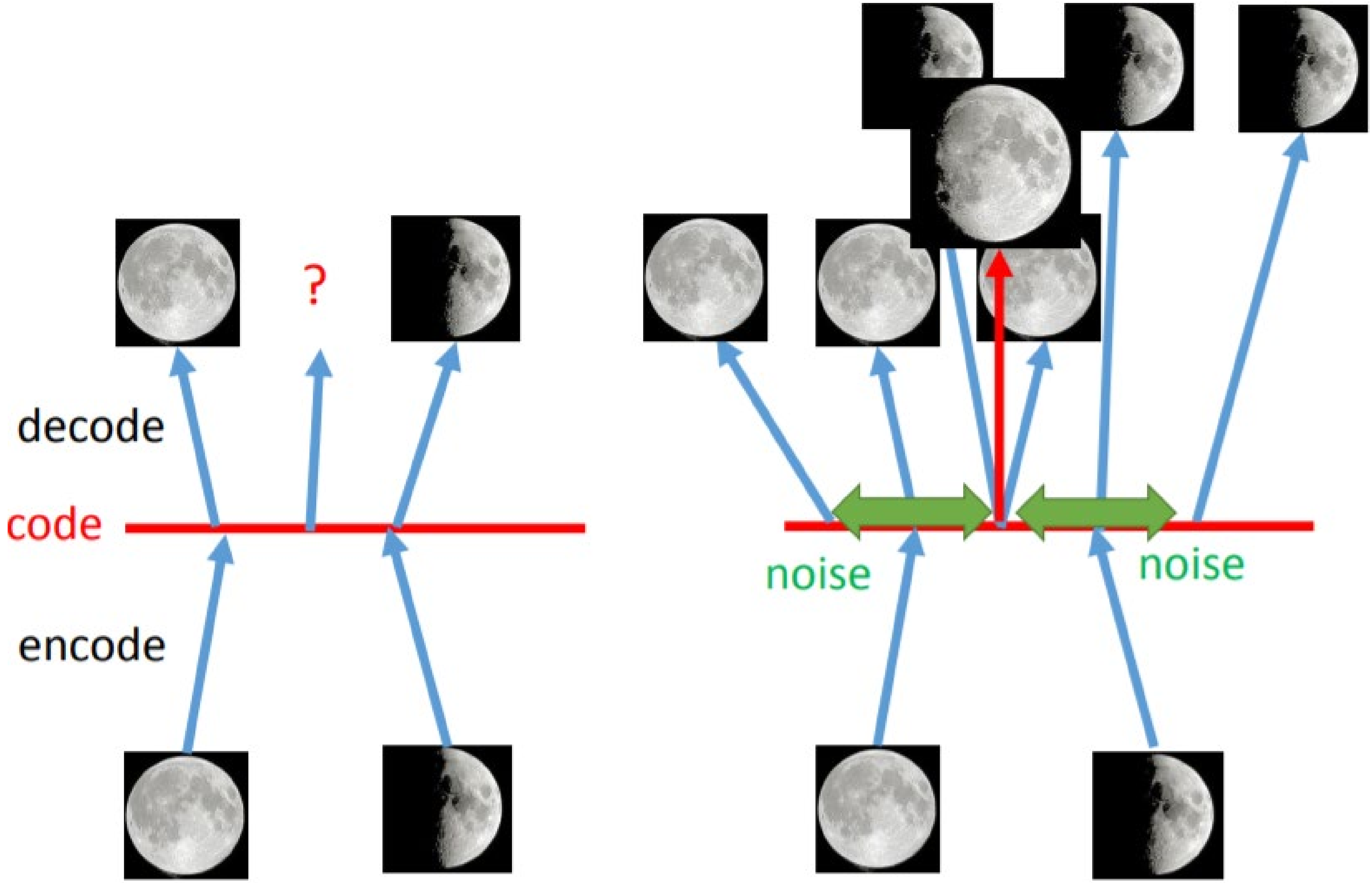
Variational Auto-Encoder (VAE)

Why VAE?

Assume 1-d code

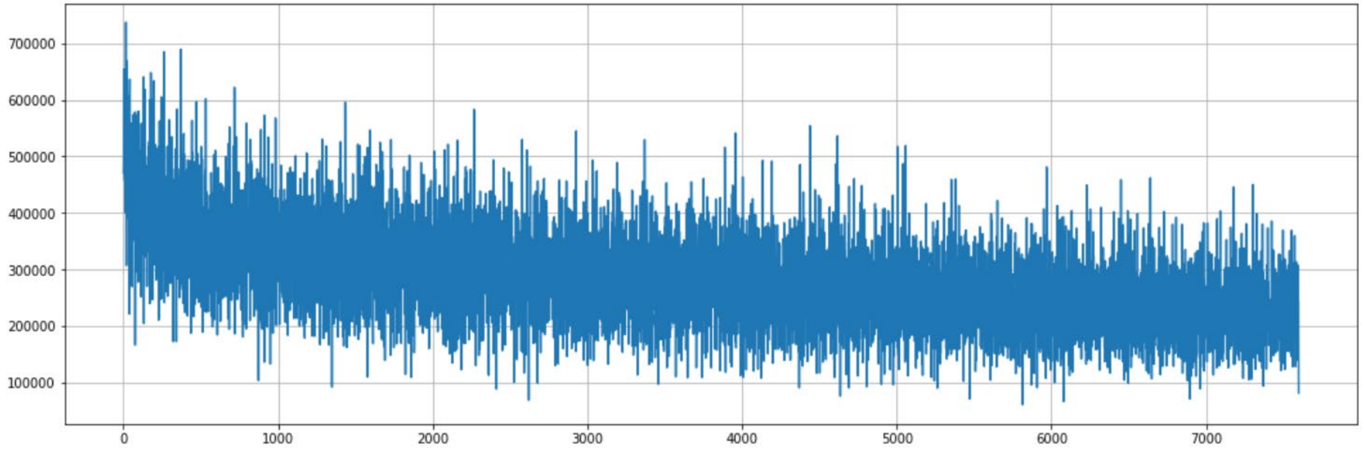


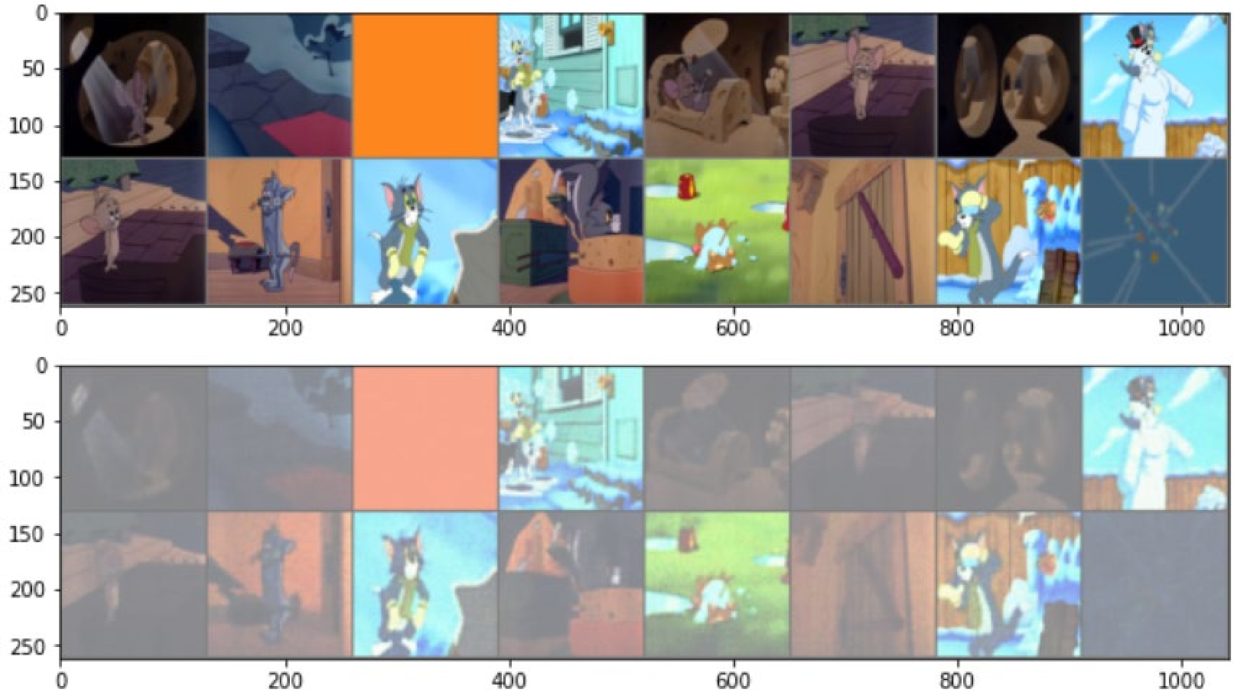
Reference: 李弘毅 ML Lecture 18 https://youtu.be/8zomhgKrsmQ

Practice

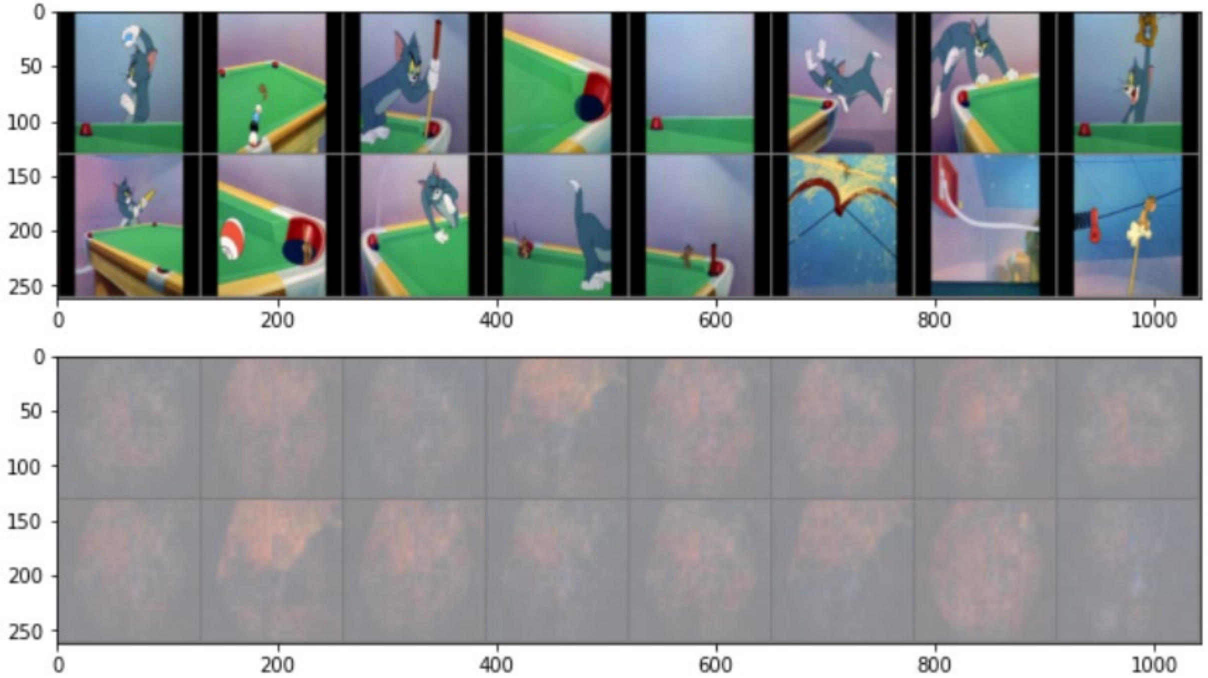
- Run “7.2.Conv_VAE.ipynb"

Train 400 epochs



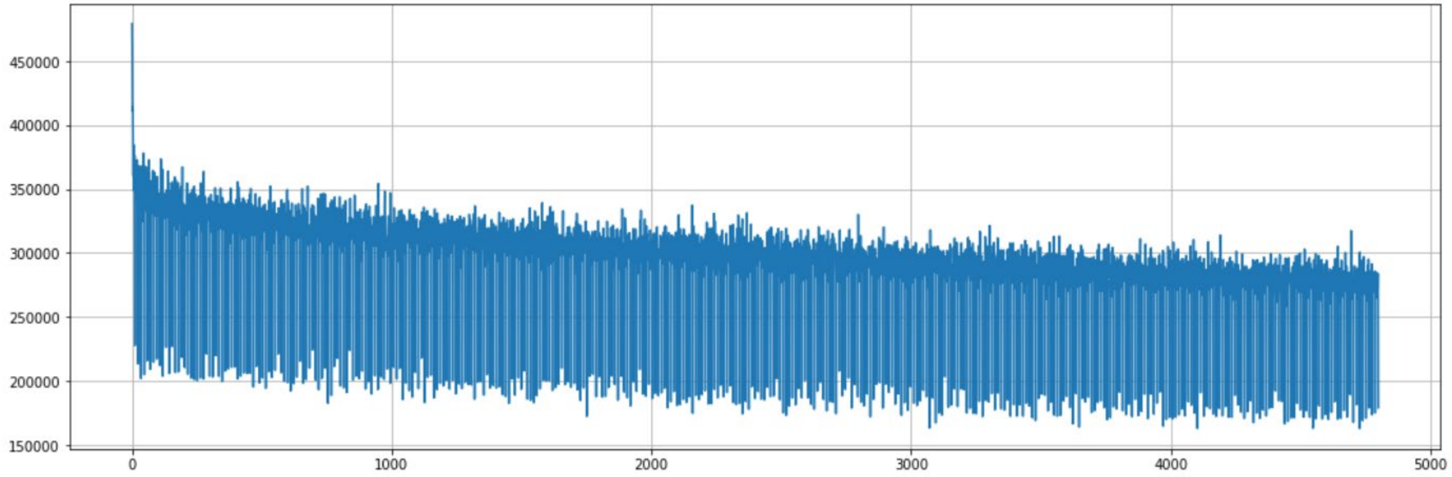


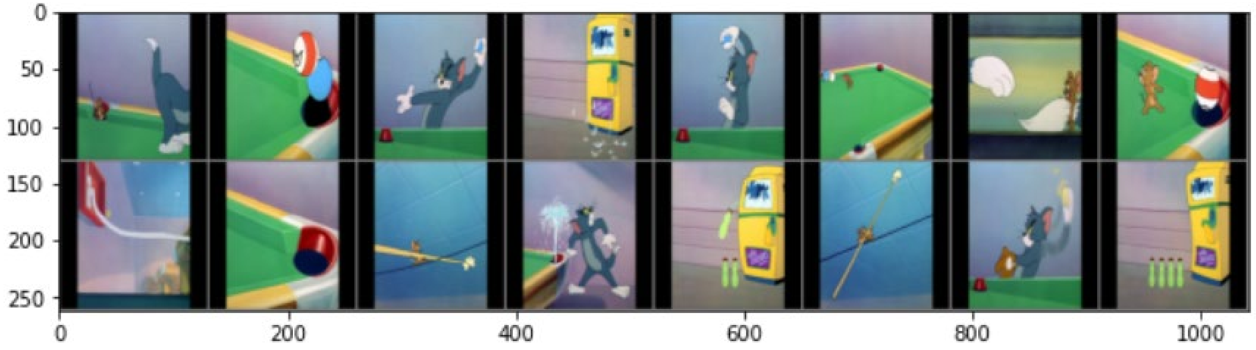
Able to recover the training images

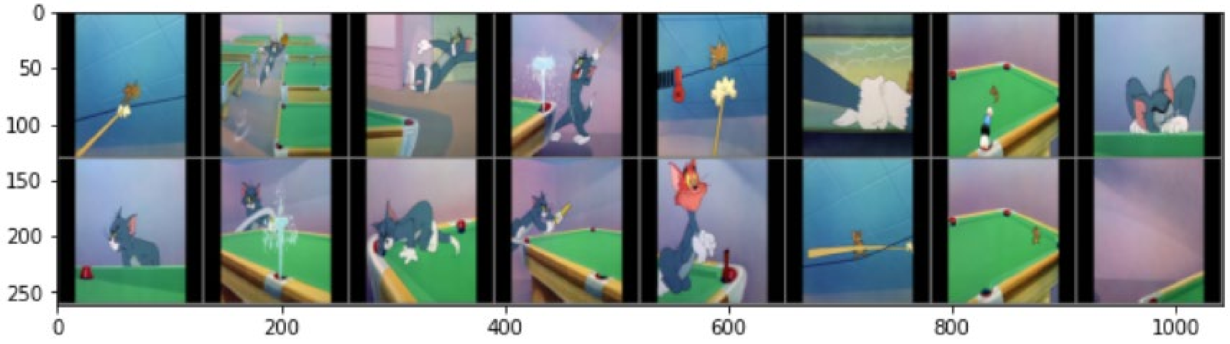


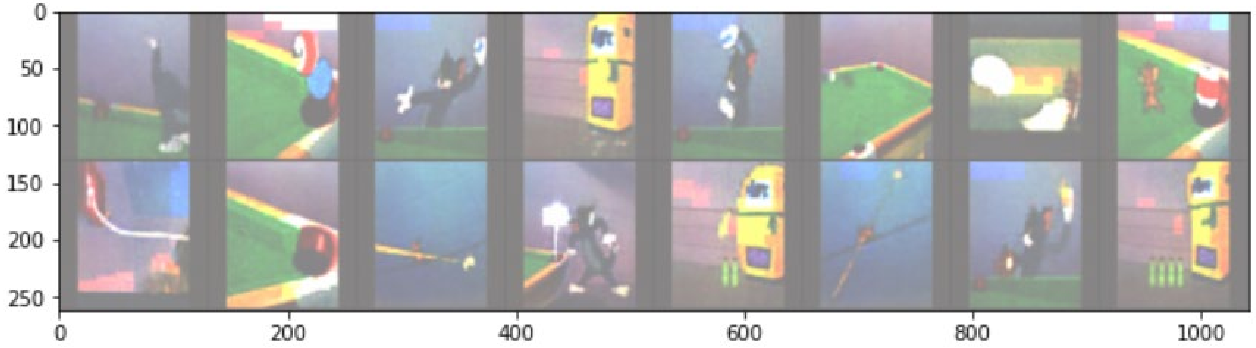
Fails to reconstruct the test images

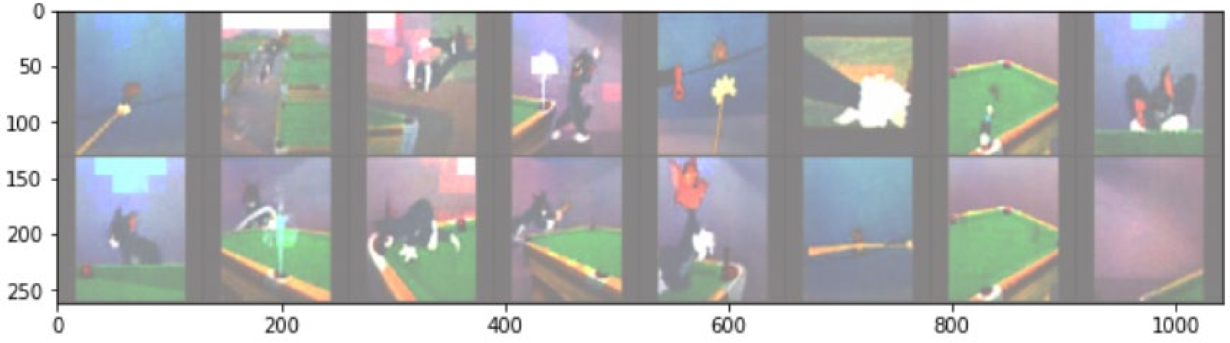
400+400 epochs (total = 800 epochs)





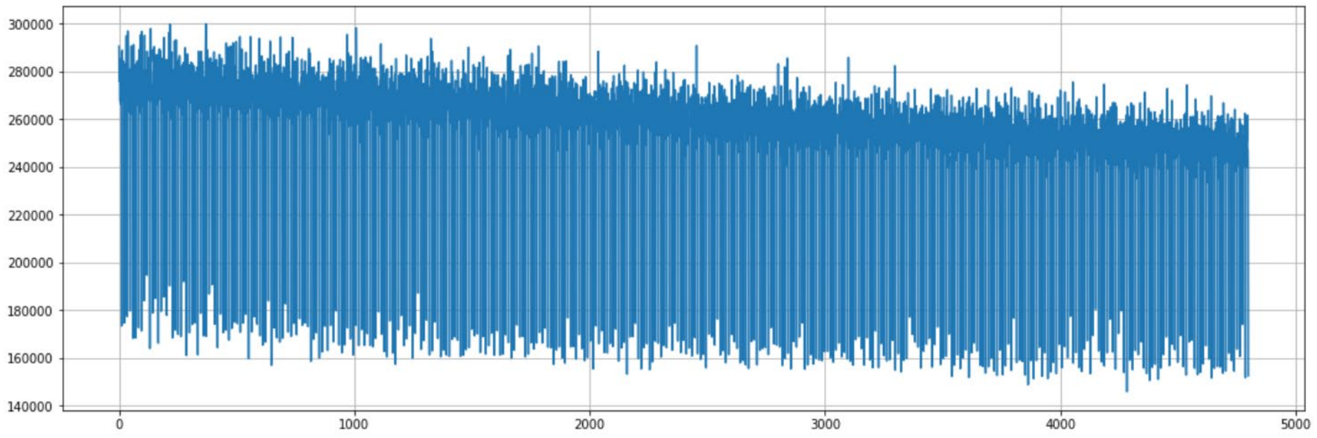


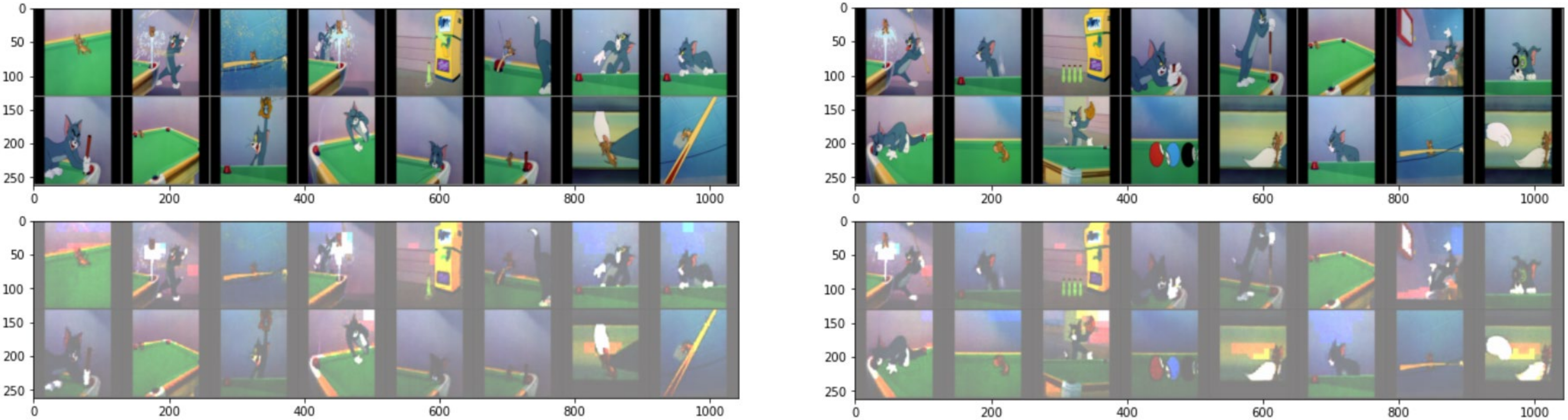




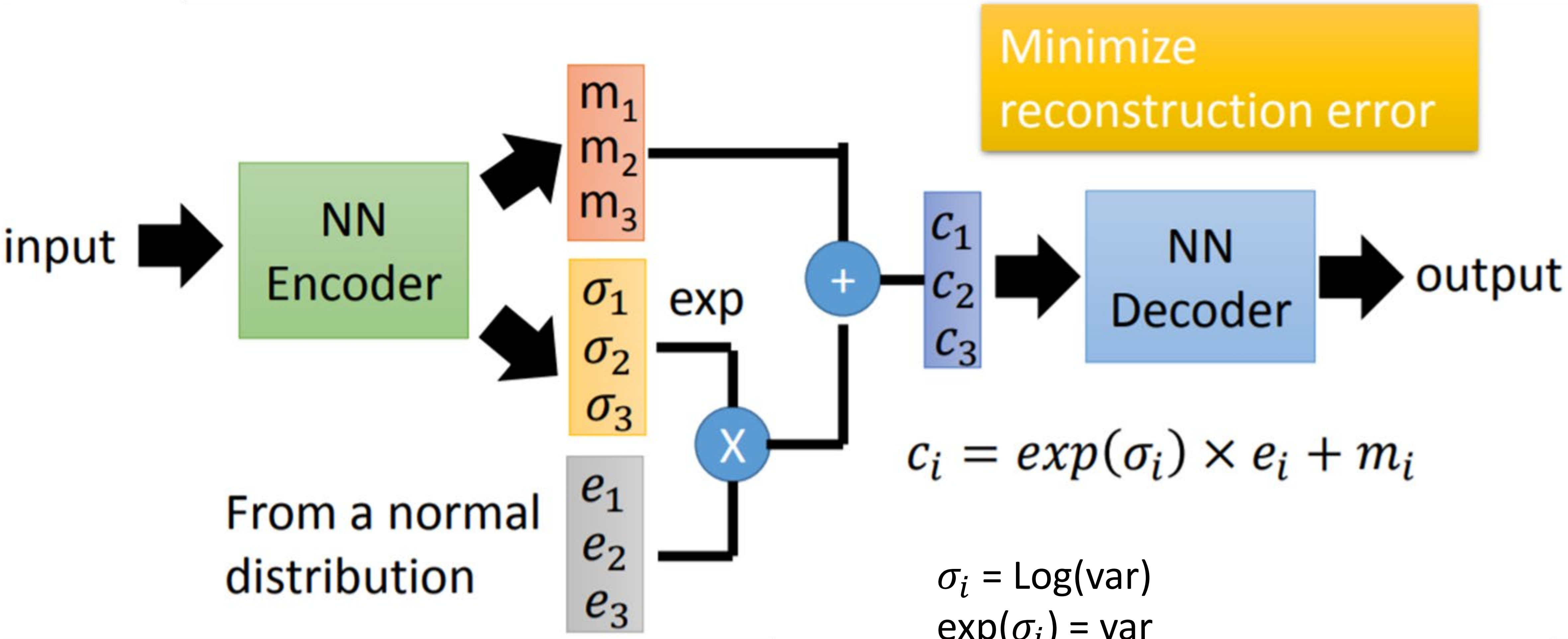
able to reconstruct the test images

400+400+400 epochs (total = 1200 epochs)





VAE



$$c_i = exp(\sigma_i) \times e_i + m_i$$

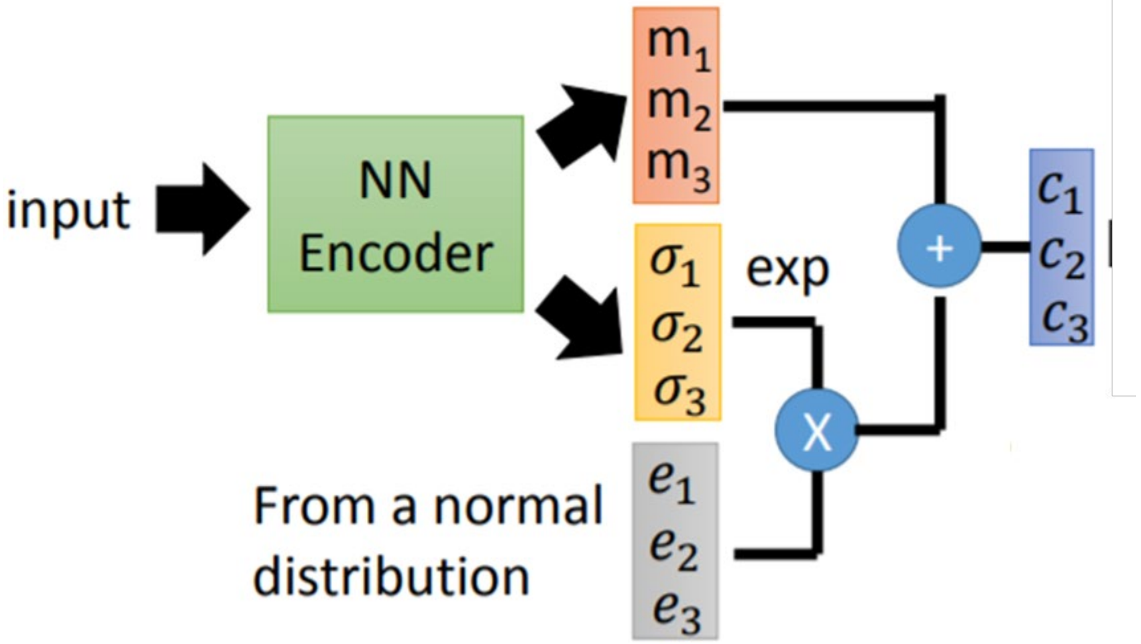
σ_i = Log(var)
exp(σ_i) = var

Encoder

```
[15]: for batchX, _ in loader:
          break;
      print(batchX.shape)
```

```
torch.Size([16, 3, 128, 128])
```

```
(fc1): Linear(in_features=1024, out_features=64,
(fc2): Linear(in_features=1024, out_features=64,
(fc3): Linear(in_features=64, out_features=1024,
```




```
16]: h = model.encoder(batchX.to(device))
     print(h.shape)
```

```
torch.Size([16, 1024])
```

```
[17]: mu=model.fc1(h)
      print(mu.shape)
```

```
torch.Size([16, 64])
```

```
[18]: logvar=model.fc2(h)
      print(logvar.shape)
```

```
torch.Size([16, 64])
```

```
[19]: std = logvar.mul(0.5).exp_()
      print(std.shape)
```

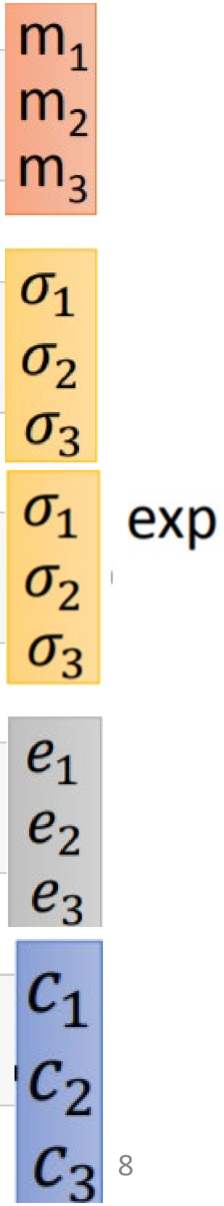
```
torch.Size([16, 64])
```

```
[20]: esp=torch.randn(*mu.size())
      print(esp.shape)
```

```
torch.Size([16, 64])
```

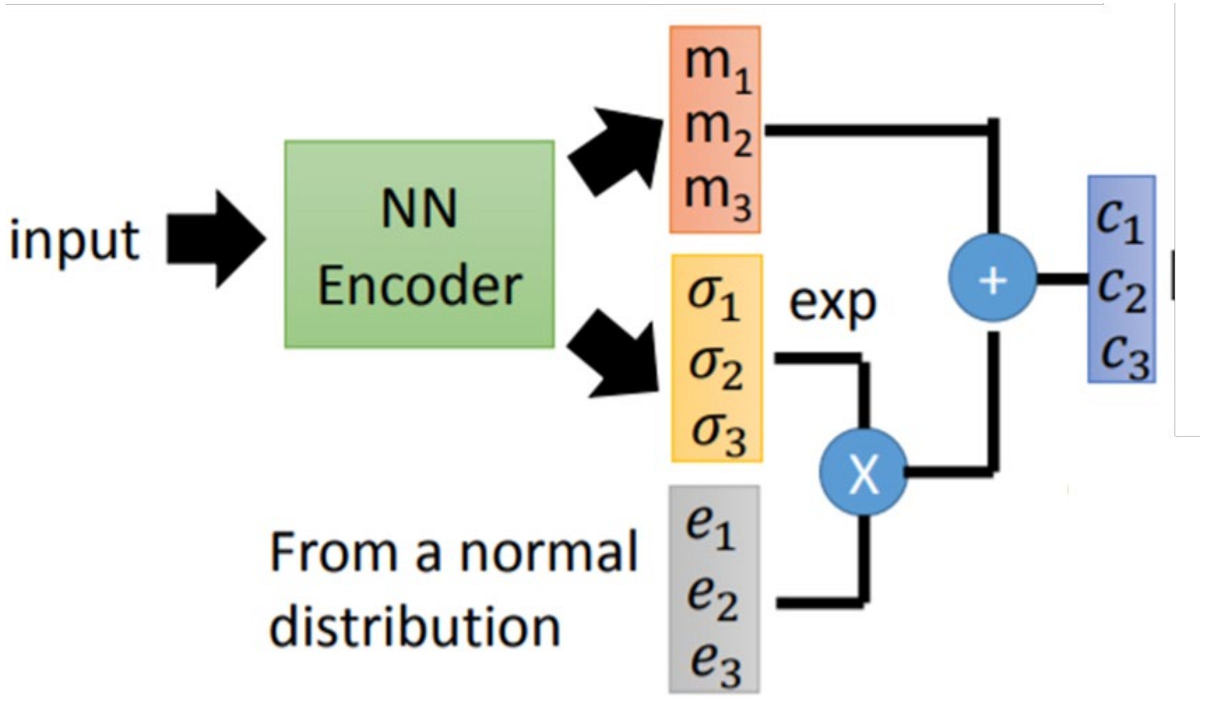
```
[21]: z=mu+std*esp.to(device)
      print(z.shape)
```

```
torch.Size([16, 64])
```



Loss function

σ_i = Log(var)



We want σ_i close to 0
(variance close to 1)

Minimize

$$\sum_{i=1}^{3}\left(\underline{exp(\sigma_i)} - \underline{(1+\sigma_i)} + \underline{(m_i)^2}\right)$$

L2 regularization

Loss function

```
[9]: def loss_fn(recon_x, x, mu, logvar):
       #BCE = F.binary_cross_entropy(recon_x, x, size_average=False).to(device)
       MSE = F.mse_loss(recon_x, x, reduction='sum')
       # see Appendix B from VAE paper:
       # Kingma and Welling. Auto-Encoding Variational Bayes. ICLR, 2014
       # 0.5 * sum(1 + log(sigma^2) - mu^2 - sigma^2)
       KLD = -0.5*torch.mean(1+logvar-mu.pow(2)-logvar.exp()).to(device)
       return MSE+KLD, MSE, KLD
```

Minimize

$$\sum_{i=1}^{3}(exp(\sigma_i) - (1+\sigma_i) + (m_i)^2)$$

L2 regularization

```
[23]: tensorY,mu,logvar = model(batchX.to(device))
      print(tensorY.shape)
```

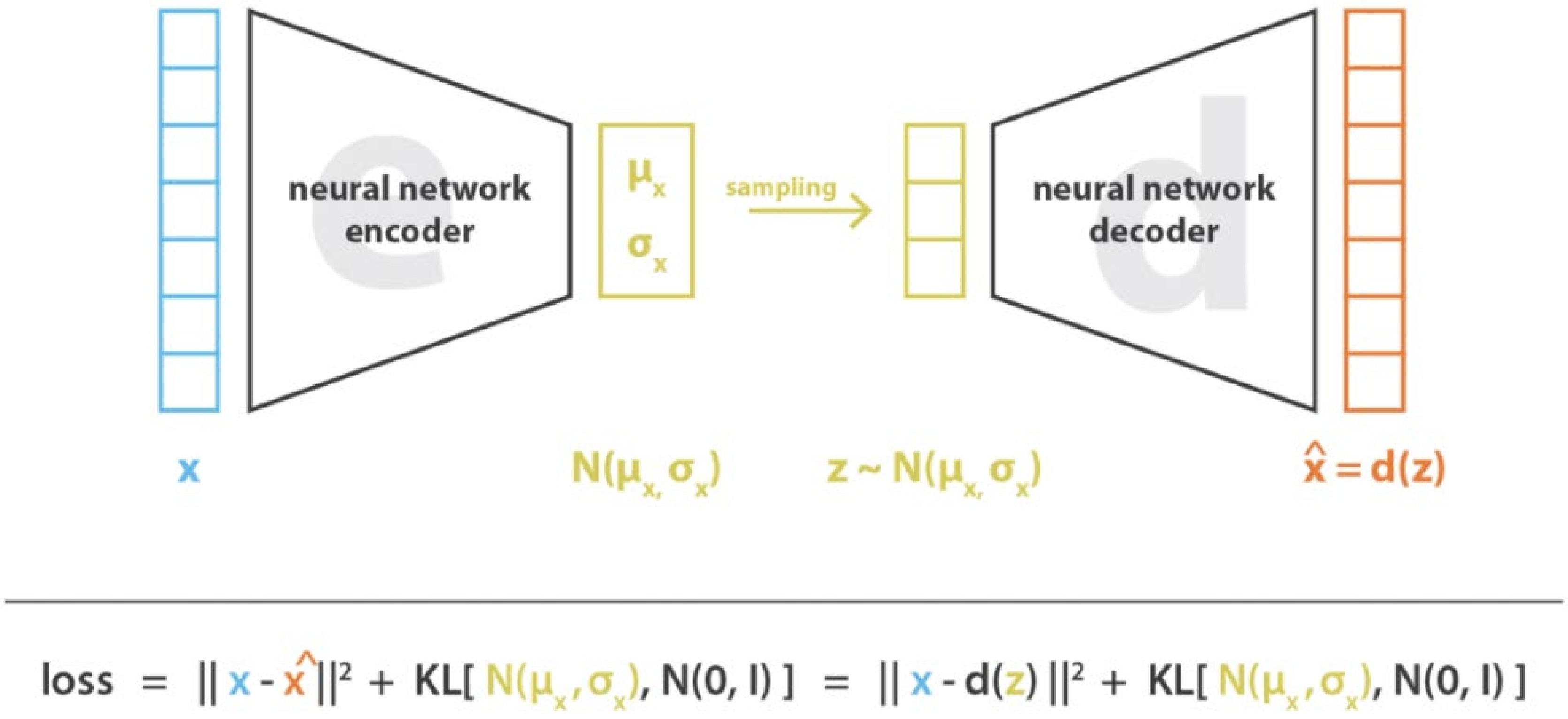
```
torch.Size([16, 3, 128, 128])
```

```
[24]: loss, mse,kld = loss_fn(tensorY, batchX.to(device), mu, logvar)
      print(loss)
```

```
tensor(627375.3750, device='cuda:0', grad_fn=<AddBackward0>)
```

Why loss = $MSE(x, \hat{x}) + KL(q(z|x)||P(z))$?

Source: https://towardsdatascience.com/understanding-variational-autoencoders-vaes-f70510919f73

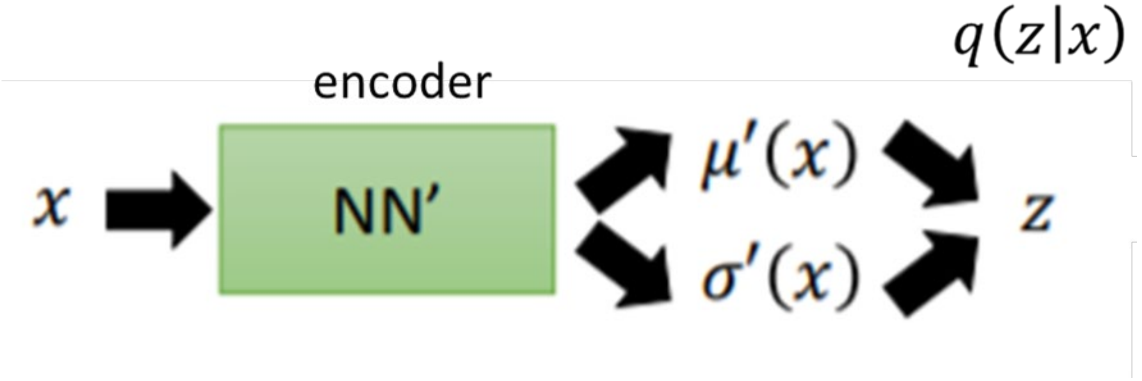


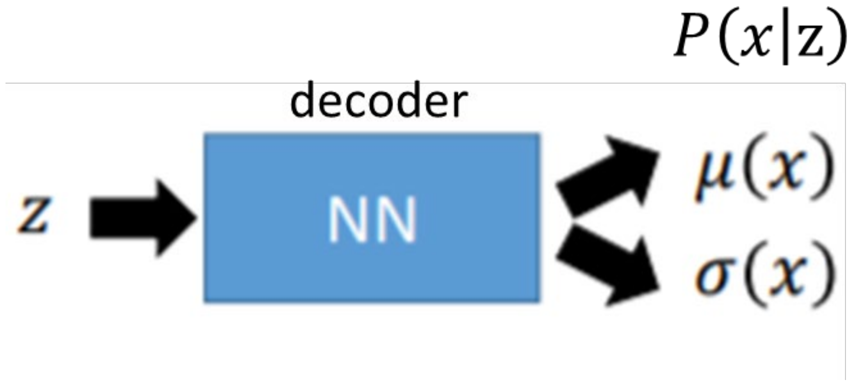
$$D_{KL}(p||q) = \sum_{i=1}^{N} p(x_i) log(\frac{p(x_i)}{q(x_i)})$$

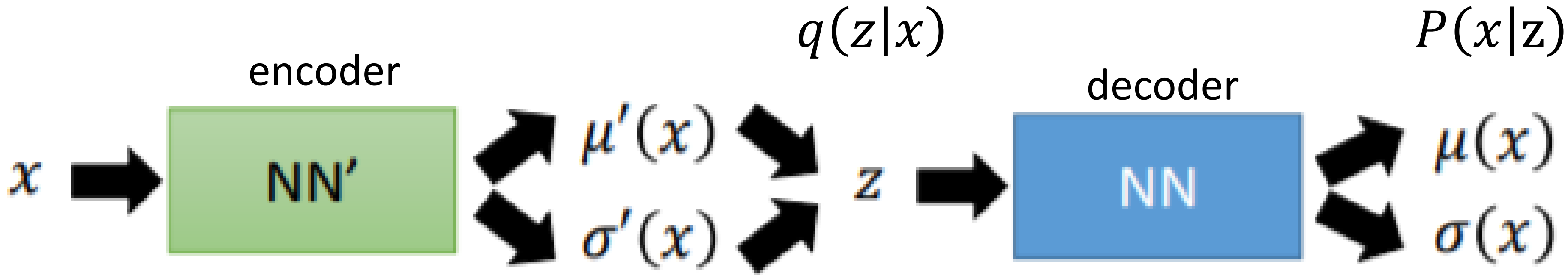
Minimize

$$\boxed{\sum_{i=1}^{3} (exp(\sigma_i) - (1 + \sigma_i) + (m_i)^2)}$$

The decoder and encoder of VAE model two conditional probability distributions







Gaussian mixture model

Gaussian Mixture Model

$$P(x) = \sum_m P(m)P(x|m)$$

How to sample?

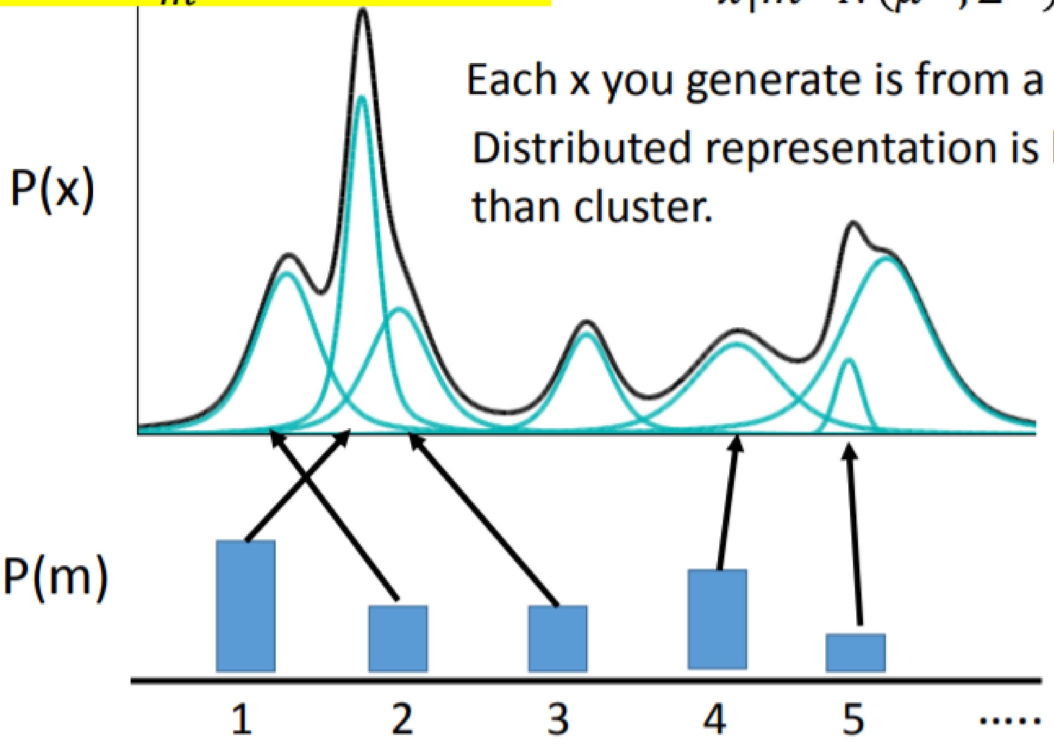
$m \sim P(m)$ (multinomial)

m is an integer

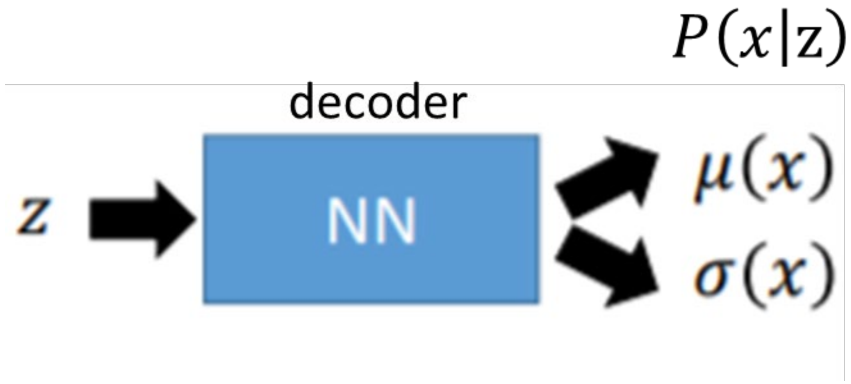
$x|m \sim N(\mu^m, \Sigma^m)$

Each x you generate is from a mixture

Distributed representation is better than cluster.

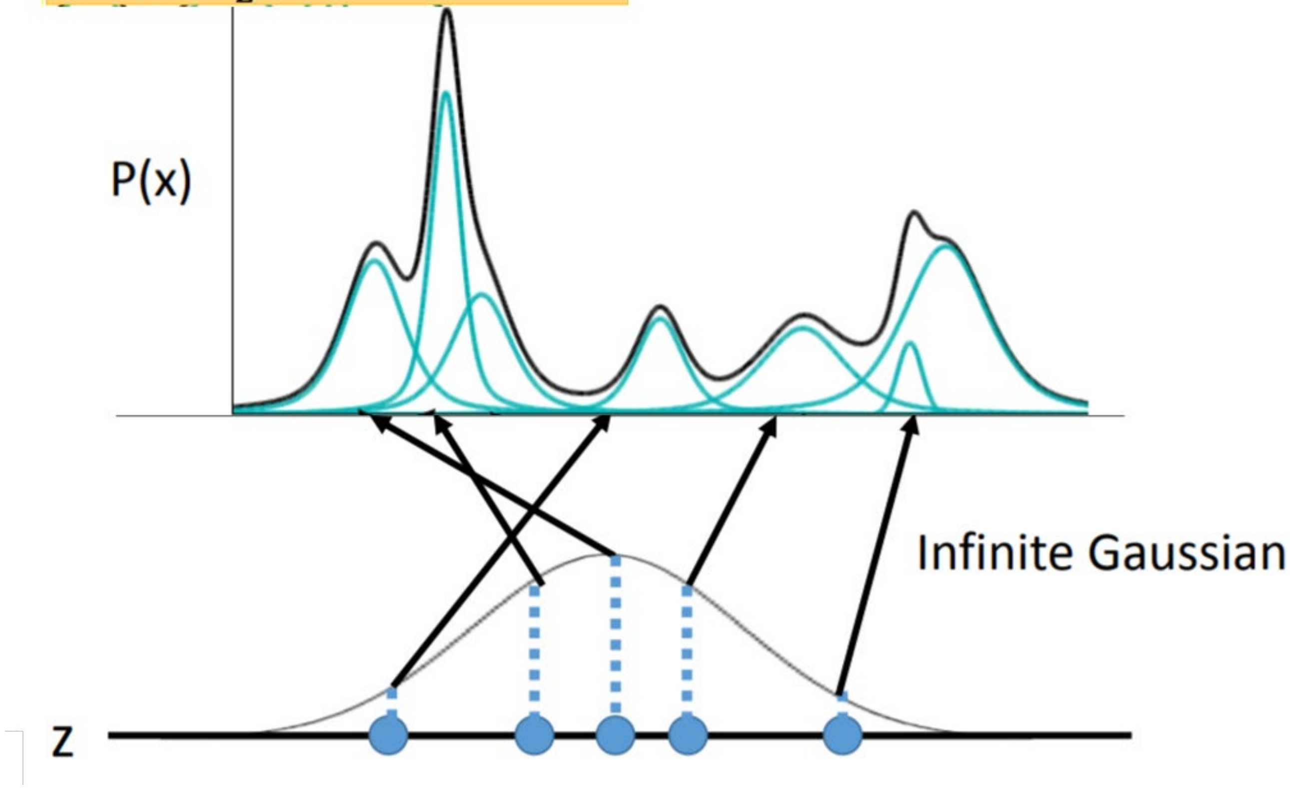


The probability of sampling an output image *x* from latent vector space *z* can be modelled as a Gaussian mixture model



Gaussian Mixture Model

$$P(x) = \int_z P(z)P(x|z)dz$$



We want to train a decoder NN that can maximize the likelihood of observing the training images

Maximizing Likelihood

$$P(x) = \int_z P(z)P(x|z)dz$$

$$L = \sum_x logP(x)$$

Maximizing the likelihood of the observed x

P(z) is normal distribution

$x|z \sim N(\mu(z), \sigma(z))$

$\mu(z), \sigma(z)$ is going to be estimated

$$L = p(x^1) \times p(x^2) \times p(x^3) \times \cdots p(x^m) = \prod_{i=1,\dots,m} P(x^i)$$

Reference: 李弘毅 ML Lecture 18 https://youtu.be/8zomhgKrsmQ

Recap: maximize the likelihood of observing the classification of the training data

Training Data	x^1	x^2	x^3	……	x^N
	C_1	C_1	C_2		C_1

max $\quad L(w,b) = f_{w,b}(x^1) f_{w,b}(x^2) \left(1 - f_{w,b}(x^3)\right) \cdots f_{w,b}(x^N)$

min $\quad -lnL(w,b) = -lnf_{w,b}(x^1) - lnf_{w,b}(x^2) - ln\left(1 - f_{w,b}(x^3)\right) \cdots$

$\hat{y}^n$: 1 for class 1, 0 for class 2

$$= \sum_n -\underbrace{\left[\hat{y}^n lnf_{w,b}(x^n) + (1 - \hat{y}^n) ln\left(1 - f_{w,b}(x^n)\right)\right]}_{\text{Cross entropy between two Bernoulli distribution}}$$

Rewrite the maximum likelihood item $\log P(x)$ as the summation of a lower bound L_b and KL divergence

$$logP(x) = \int_z q(z|x)logP(x)dz$$

q(z|x) can be any distribution

$$= \int_z q(z|x)log\left(\frac{P(z,x)}{P(z|x)}\right)dz = \int_z q(z|x)log\left(\frac{P(z,x)}{q(z|x)}\frac{q(z|x)}{P(z|x)}\right)dz$$

$$= \int_z q(z|x)log\left(\frac{P(z,x)}{q(z|x)}\right)dz + \underbrace{\int_z q(z|x)log\left(\frac{q(z|x)}{P(z|x)}\right)dz}_{KL(q(z|x)||P(z|x))}$$

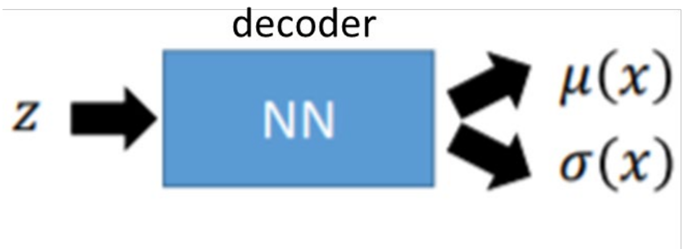
≥ 0

$$\geq \int_z q(z|x)log\left(\frac{P(x|z)P(z)}{q(z|x)}\right)dz$$

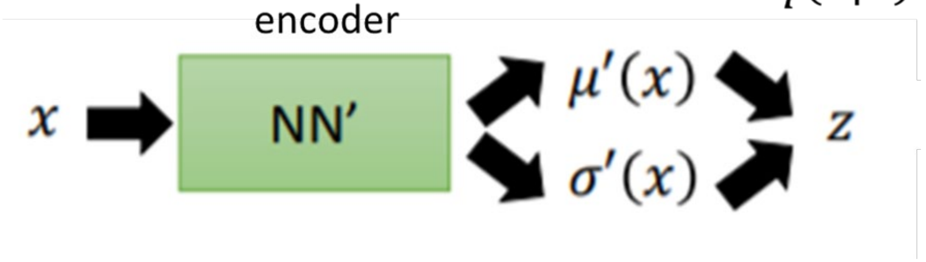
lower bound L_b

$$\int_z q(z|x)dz = 1$$

$P(x|z)$



$q(z|x)$

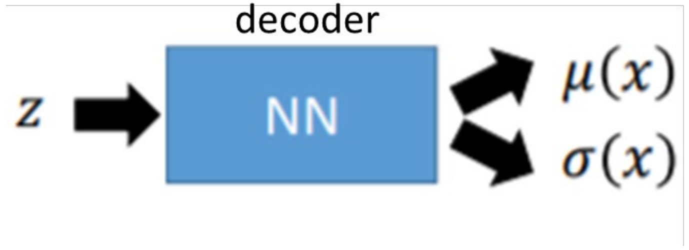


$$D_{KL}(q||p) = \sum_{i=1}^{N} q(x_i)\log(\frac{q(x_i)}{p(x_i)})$$

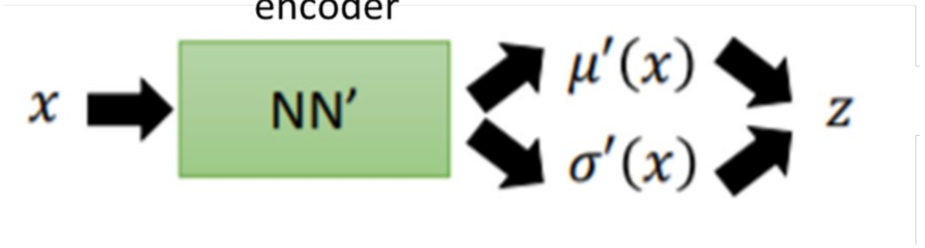
Reference: 李弘毅 ML Lecture 18 https://youtu.be/8zomhgKrsmQ

If we maximum L_b by adjusting $P(x|z)$ and $q(z|x)$ simutaneously, then we can maximum L_b and at the same time minimize the KL distance

$P(x|z)$



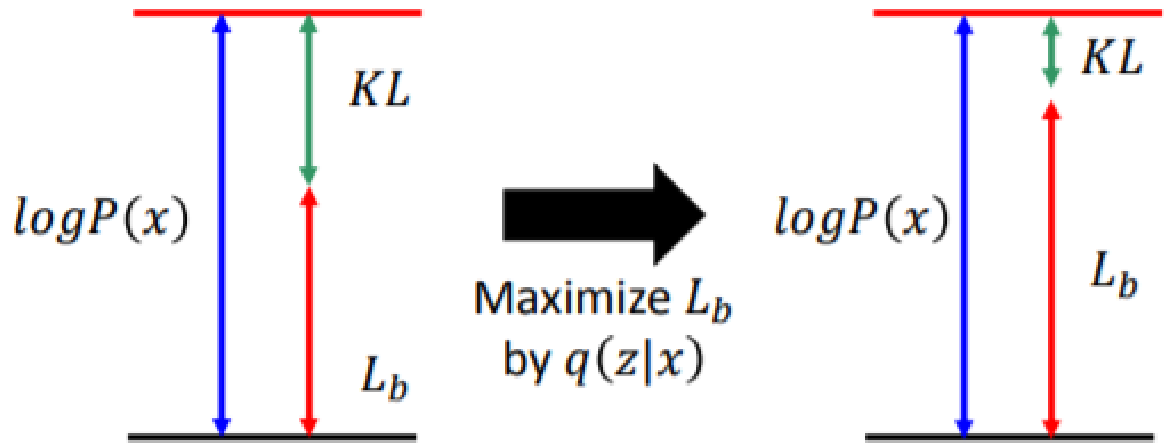
$q(z|x)$



$$logP(x) = L_b + KL(q(z|x)||P(z|x))$$

$$L_b = \int_z q(z|x)log\left(\frac{P(x|z)P(z)}{q(z|x)}\right)dz$$

Find $P(x|z)$ and $q(z|x)$ maximizing L_b



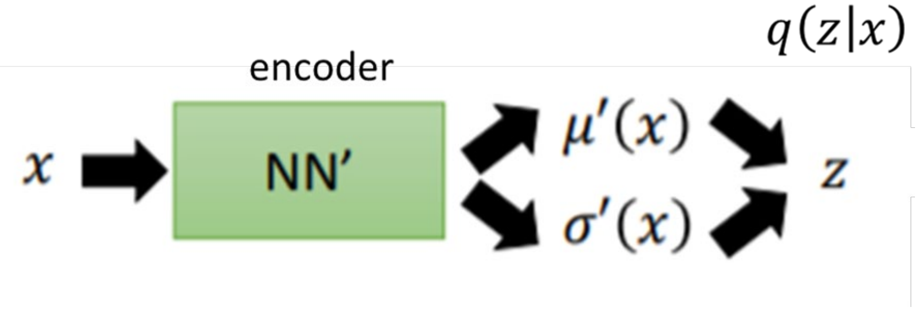
$q(z|x)$ will be an approximation of $p(z|x)$ in the end

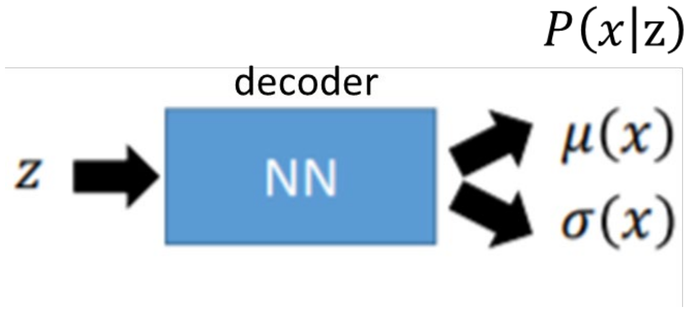
Reference: 李弘毅 ML Lecture 18 https://youtu.be/8zomhgKrsmQ

Rewrite the lower bound L_b as the summation of two terms: $-KL(q(z|x)||P(z))$, and $\int_z q(z|x)logP(x|z)dz$

max
$$L_b = \int_z q(z|x)log\left(\frac{P(z,x)}{q(z|x)}\right)dz = \int_z q(z|x)log\left(\frac{P(x|z)P(z)}{q(z|x)}\right)dz$$

$$= \underbrace{\int_z \boxed{q(z|x)}log\left(\frac{P(z)}{\boxed{q(z|x)}}\right)dz}_{-KL(q(z|x)||P(z))} + \int_z q(z|x)logP(x|z)dz$$





Reference: 李弘毅 ML Lecture 18 https://youtu.be/8zomhgKrsmQ

max. L_b can be done by min. $KL(q(z|x)||P(z))$ and max. $\int_z q(z|x)logP(x|z)dz$. That is why loss = KLD + MSE (x, $\hat{x}$)

Minimizing $KL(q(z|x)||P(z))$

Minimize

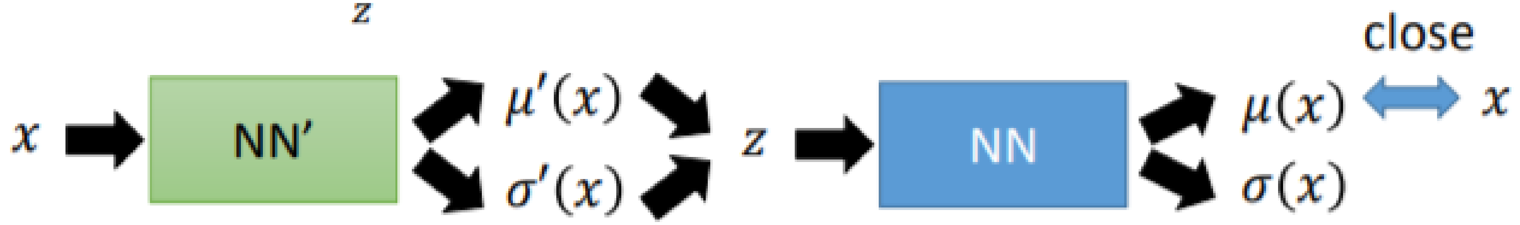
$$\sum_{i=1}^{3}(exp(\sigma_i) - (1 + \sigma_i) + (m_i)^2)$$

(Refer to the Appendix B of the original VAE paper)

Maximizing

$$\int_z q(z|x)logP(x|z)dz = E_{q(z|x)}[logP(x|z)]$$

by min. $MSE(x, \hat{x})$

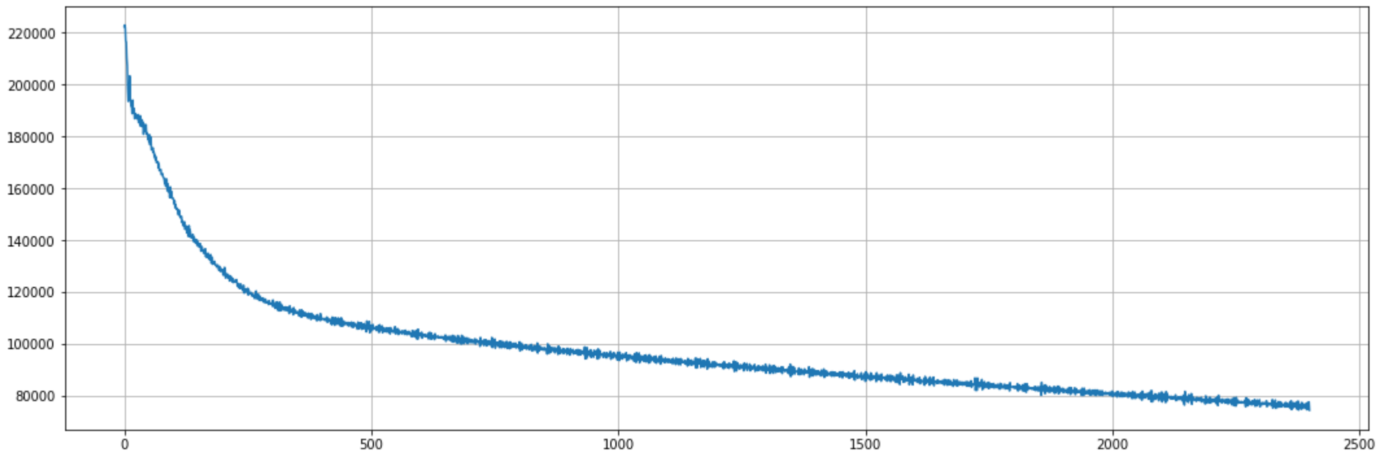


Reference: 李弘毅 ML Lecture 18 https://youtu.be/8zomhgKrsmQ

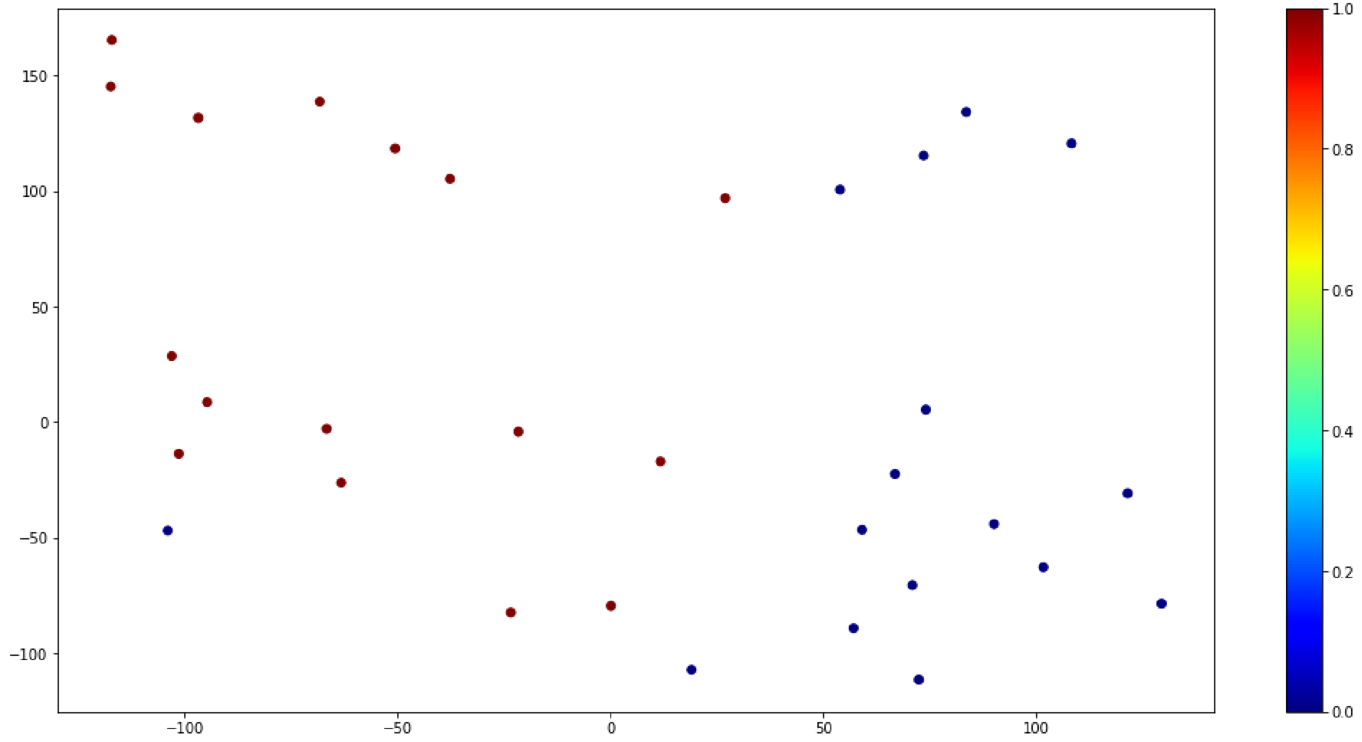
HW6 (2)

- Train an VAE to learn a compact representation (try latent vector of size 20, 30, 50) of your facial expression. Test with 10 happy and 10 angry faces.
- Show the recovered image.
- Send the latent vectors to *t*-SNE to see whether they form clusters.

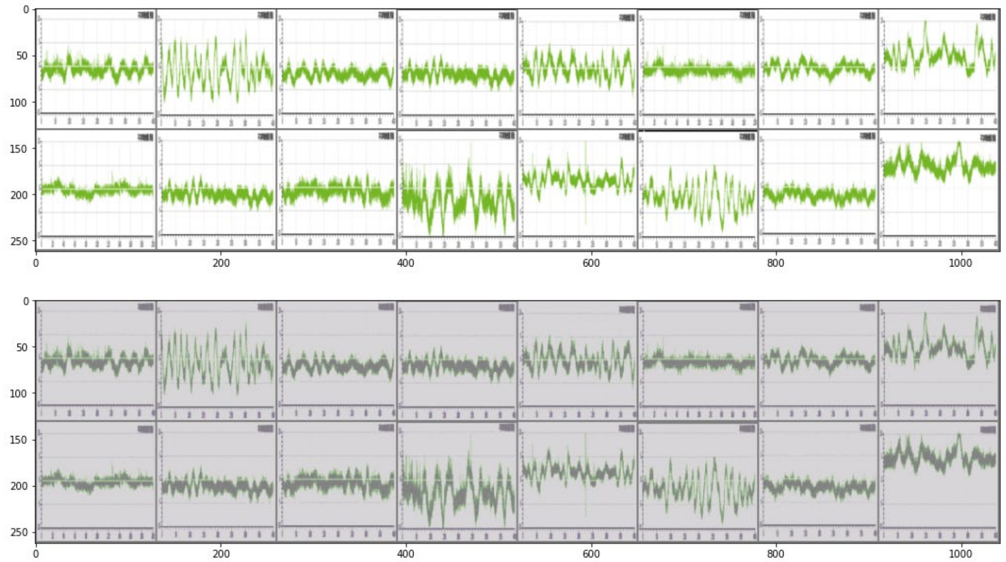
Normal = 16, Abnormal = 16, Latent vector size = 32, 1200 epochs



t-SNE (perplexity=?) results of training images



Recovered training images



Recovered un-seen test images