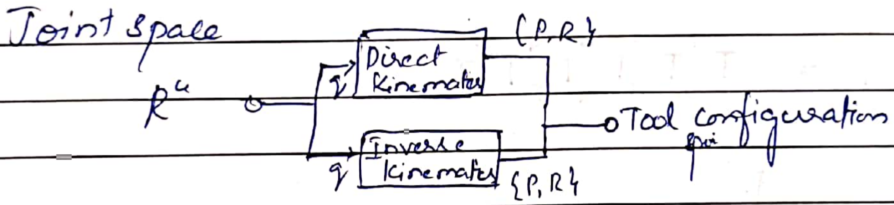
ID 6040 - Inverse Kinematics

Lecture 3.1:

Inverse Kinematics → q (joint variables) given P & R, find q using arm equations
(position) (orientation)

Arm equation ⇒

$$ {}^{base}_{tool}T = \left[\begin{array}{c|c} R(q) & P(q) \\ \hline 0\ 0\ 0 & 1 \end{array}\right] $$



Note: For a general 'T',

$$\begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ 12 equations
unknowns
→ 'n' state variables
(n-joints)

P_x, P_y, P_z → 3 independant equations

Rotation matrix → only 3 independant equations
(logically you require only 3 quantities to define orientation)

∴ 6 independant equations, n variables

(All are non-linear equations)
Difficult to solve!

Existance of Solutions:

A manipulator is solvable if all sets of joint variables can be found corresponding to a given end-effector location.

. Tool point within the work space.

. $n \geq 6$ to have any arbitary orientation of tool

. Total Orientation is such that none of the joint limitations are violated

Solutions

Closed form (analytic expression)

Numerical (iterative search - time consuming)

→ Conditions: Sufficiency condition

Three adjacent joint axes intersecting/parallel

closed-form solutions

algebraic

geometric

Lecture 3.2:

For solving closed form algebraic equations →

Trick 1 → Use trigonometric identities to combine two eq: and eliminate certain values eg: (Sq + adding)

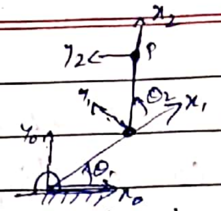
Trick 2 → $u = \tan(\theta/2)$

$\cos\theta = (1-u^2)/(1+u^2)$

$\sin\theta = 2u/1+u^2$

Trick 3 → Express using function Atan2 ($\sin\theta_i, \cos\theta_i$) as the quadrant is mentioned angle is uniquely mentioned.

Example:



$P_x = l_1 c_1 + l_2 c_{12}$

$P_y = l_1 s_1 + l_2 s_{12}$

$$P_x^2 + P_y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 c_2$$

$$\Rightarrow c_2 = \frac{P_x^2 + P_y^2 - (l_1^2 + l_2^2)}{2 l_1 l_2}$$

but in this case we won't get the quadrant.

instead

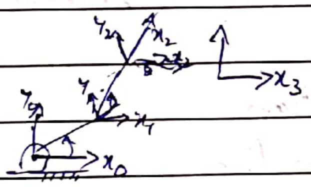
$s_2 = \pm\sqrt{1 - c_2^2}$ atan2 $\rightarrow$ four quadrant version of atan function.

$\theta_2 = \text{atan}^2(s_2, c_2)$ (ie $-\pi$ to π)

[General procedure]

Lecture 3.3:

Example 3 DOF robot: Third axle is projected out of the paper



$$T^{tool}_{base} = \begin{bmatrix} x_x & s_x & 0 & P_x \\ n_y & s_y & 0 & P_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

DH parameters $\rightarrow$

a	d	α	θ
l_1	0	0	θ_1
l_2	0	0	θ_2
0	d_3	0	θ_3

$\hookrightarrow$ equations both of these

first solve for the forward kinematics

$$T^{tool}_{base} = \begin{bmatrix} (c_{123}) & (-s_{123}) & 0 & (l_1 c_1 + l_2 c_2) \\ (s_{123}) & (c_{123}) & 0 & (l_1 s_1 + l_2 s_2) \\ 0 & 0 & 1 & (d_3) \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{gives 6 eq.}$$

$n_x = c_{123}$ $\quad$ $n_y = s_{123}$ $\Rightarrow \theta_1 + \theta_2 + \theta_3$

$s_x = -s_{123}$ $\quad$ $s_y = c_{123}$

$P_x = l_1 c_1 + l_2 c_2$

$P_y = l_1 s_1 + l_2 s_2$ $\quad \Rightarrow c_2 = \dfrac{P_x^2 + P_y^2 - (l_1^2 + l_2^2)}{2 l_1 l_2}$

$S_2 = \pm\sqrt{1-c_2^2}$

$\theta_2 = atan^2(S_2, C_2)$ also quadrant is obtained

Solve (5) & (6) for θ_1

$P_x = k_1 C_1 - k_2 S_1$ where $k_1 = l_1 + l_2 C_2$

$P_y = k_1 S_1 + k_2 C_1$ $k_2 = l_2 S_2$

let eq: $r\cos\gamma$, $k_2 = r\sin\gamma$

$\frac{P_x}{r} = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1$

$P_y/r = \cos\gamma\sin\theta_1 + \sin\gamma\cos\theta_1$

$\Rightarrow \cos(\gamma+\theta_1) = P_x/r$

$\sin(\gamma+\theta_1) = P_y/r$

to find θ_1

$\gamma + \theta_1 = atan^2(P_y/r, P_x/r) = atan2(P_y, P_x)$

$$\boxed{\therefore \theta_1 = atan2(P_y, P_x) - atan2(k_2, k_1)}$$

$\therefore \theta_{123} = atan2(n_y, n_x)$

$$\boxed{\therefore \theta_3 = \theta_{123} - \theta_1 - \theta_2}$$

General Solutions ->

* $a = b\sin\theta$

$\theta = atan2(a/b, \sqrt{1-a/b^2})$

* $\boxed{a = b\cos\theta}$

$\theta = atan^2 \sqrt{1-(a/b)^2}$

* $a = b\cos\theta$ $c = d\sin\theta$

$\theta = atan^2(c/d, a/b)$

$a\sin\theta + b\cos\theta = 0$

$\theta = atan^2(-b, a)$

Eq: $a\sin\theta + b\cos\theta = c$

Solution: $\theta = atan2(a, b) + atan2(\pm\sqrt{a^2+b^2-c^2}, c)$

Eq: $a\cos(\theta_i) + b\cos(\theta_j) = c$

$a\sin(\theta_i) + b\sin(\theta_j) = d$

Eq: $a\cos(\theta_i + \theta_j) + b\cos(\theta_i) = c$

$a\sin(\theta_i + \theta_j) + b\sin(\theta_i) = d$

Solution

Where $S = (a^2 - b^2 + c^2 + d^2)/2a$

$\theta_j = atan2(\sin(\theta_j), \cos(\theta_j))$

$\theta_j = \theta_i + \text{atan2}(\pm\sqrt{4a^2b^2 - t^2},\ t)$ $\quad \theta_i = \text{atan2}(rd - sc,\ rc + sd)$

where $t = (c^2 + d^2 - a^2 - b^2)$ $\quad r = a\cos(\theta_j) + b,\ s = a\sin(\theta_j)$

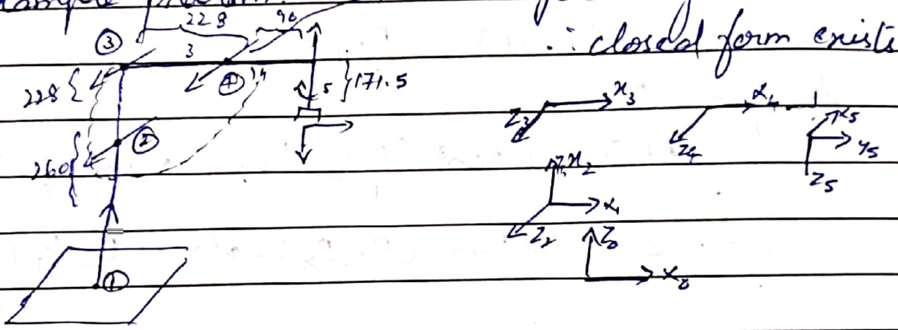
$$\boxed{\begin{aligned} \text{Eq: } & a\sin\theta + b\cos\theta = c \\ & a\cos\theta - b\sin\theta = d \end{aligned}}$$

$\theta = \text{atan2}(ac - bd,\ ad + bc)$

Also $a^2 + b^2 = c^2 + d^2$

Example problem: $\rightarrow$ Three adjacent joint axes are parallel

$\therefore$ closed form exists



DH Parameter table

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	260	θ_1
2	90°	0	0	θ_2
3	0	228	0	θ_3
4	0	228	0	θ_4
5	90°	70	171	θ_5

$${}^0_5T = \begin{bmatrix} s_1 s_5 + c_1 c_5 c_{234} & c_5 s_1 - c_1 s_5 c_{234} & c_1 s_{234} & 3\ 476 c_{23} + 76 c_2 + 3\sqrt{461}\ \sin(\tan^{-1}(10/11) + t_2 + t_3 + t_4) \\ c_5 s_1 c_{234} - c_1 s_5 & -c_1 c_5 - s_1 s_5 c_{234} & s_1 s_{234} & 3 s_1 76 c_{23} + 76 c_2 + 3\sqrt{461}\ \sin(\tan^{-1}(10/11) + t_2 + t_3 + t_4) \\ c_5 s_{234} & -s_5 s_{234} & -c_{234} & 228^\circ s_{23} + 228^\circ s_2 - 9\sqrt{461}\ \cos(\tan^{-1}(10/11) + t_2 + t_3 + t_4) + 260 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

According to convention on video

$$\begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$p_x = C_1(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234})$

$p_y = S_1(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234})$

$p_z = d_1 - a_2S_2 - a_3S_{23} - a_4S_{234} - d_5C_{234}$

$n_x = C_1C_{234}C_5 + S_1S_5 \; ; \; n_y = S_1C_{234}C_5 - C_1C_5$

$a_x = -C_1S_{234}, \; a_y = -S_1S_{234}, \; a_z = C_{234}$

$p_x/p_y \Rightarrow C_1/S_1$

$\therefore \theta_1 = atan2(p_y, p_x)$

$C_1 S_1$ known

$\therefore \theta_{234} = atan2(-(a_xC_1 + a_yS_1), -a_z)$

i.e. approach vector only depends on 2, 3 & 4

$S_5 = n_xS_1 - n_yC_1$

$C_5 = S_xS_1 - S_yC_1 \Rightarrow \theta_5 = atan2(S_5, C_5)$

$a_3C_{23} + a_2C_2 = \frac{p_x}{C_1} + d_5S_{234} - a_4C_{234}$

$a_3S_{23} + a_2S_2 = d_1 - a_4S_{234} - d_5C_{234} - p_z$

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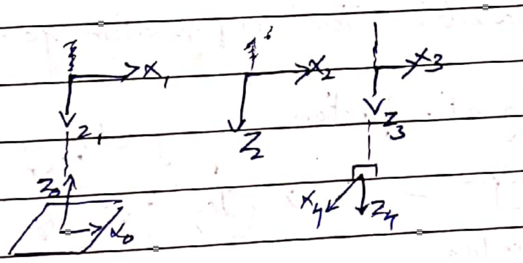
5 equations - $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$



DH Parameter table

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	180	0	-877	θ_1
2	425	0	0	θ_2
3	375	0	d_3	0
4	0	0	100	θ_4

(moodle assignment)

Lecture 3.5

Case Study

∴ using image processing, we can calculate the transform of object wrt camera

(i) Part $^{Camera}T = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & -1 & 19 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Then since camera's position is also known we can define transformation.

From camera frame to base frame (base wrt camera)

$(T_2)\; ^{Camera}_{base}T = \begin{bmatrix} 0 & -1 & 0 & 15 \\ -1 & 0 & 0 & 25 \\ 0 & 0 & -1 & 20 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Now, ${}^{base}_{part}T = {}^{base}_{Camera}T \times {}^{Camera}_{part}T = (T_2^{-1}) * (T_1) = \begin{bmatrix} 1 & 0 & 0 & 30 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\boxed{{}^{B}_{A}T = {}^{A}_{B}T^{-1} = \left[\begin{array}{c|c} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ \hline 0\;0\;0 & 1 \end{array}\right]}$$

{Gripper} = Orientation: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

to grab the goal → the final orientation should be

$\Rightarrow {}^{base}_{gripper}T = \begin{bmatrix} 1 & 0 & 0 & 30 \\ 0 & -1 & 0 & 15 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

If the robot is RhinoXR3, then

$P_x / P_y = C_1/S_1$, $\theta = atan2(P_y, P_x) \Rightarrow atan2(15, 30) = 26.5$

$\theta_5 = atan2(n_x S_1 - n_y C_1, s_x S_1 - s_y C_1)$ $a_x = -C_1 S_{234} \Rightarrow a_y = -S_1 S_{234}$;

$atan2(S_1, C_1) = 26.5$

$S_{234} = -(a_x C_1 + a_y S_1)$;

$C_{234} = -a_z$

Differential Relations & Statics:

Tool configuration $x = w(q)$ → joint variables

→ tool-configuration function

Tool configuration: joint space velocity

Differential relationship ⟶ $x = J_{cq}\,\dot{q} \Rightarrow J_{kj}(q) = \dfrac{\partial w_k(q)}{\partial q_j}$

jacobian matrix $(6 \times n)$

Here k → no. of joint velocities $(1 \le k \le 6, 1 \le j \le n)$

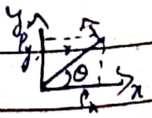
j → no. of degrees of freedom

$X = wq \Rightarrow$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_{6\times1} = \begin{bmatrix} \frac{\partial w_1}{\partial q_1} & \frac{\partial w_1}{\partial q_2} & \cdots & \frac{\partial w_1}{\partial q_n} \\ \frac{\partial w_2}{\partial q_1} & & & \\ \frac{\partial w_3}{\partial q_1} & & & \\ \frac{\partial w_4}{\partial q_1} & & & \\ \frac{\partial w_5}{\partial q_1} & & & \\ \frac{\partial w_6}{\partial q_1} & \cdots & \cdots & \frac{\partial w_6}{\partial q_n} \end{bmatrix}_{6\times n} \cdot \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times1}$$

For a rotatory manipulator ⇒ $\dot{x} = [J(\theta)]\dot{\theta}$

$\therefore \dot{\theta} = [J(\theta)]^{-1}\dot{x}$



Example

① $P_x = r\cos\theta$ $\quad \dot{P}_x = \dot{x} = -r\sin\theta.\dot{\theta}$ $\Rightarrow \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}-r\sin\theta \\ r\cos\theta\end{bmatrix}\begin{bmatrix}\dot{\theta}\end{bmatrix}$

$P_y = r\sin\theta$ $\quad \dot{P}_y = \dot{y} = r\cos\theta.\dot{\theta}$

② $P_x = l_1c_1 + l_2c_2$

$P_y = l_1s_1 + l_2s_2$

$P_z = d_z$

$$\therefore \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} = \begin{bmatrix}\frac{\partial P_x}{\partial \theta_1} & \frac{\partial P_x}{\partial \theta_2} & \frac{\partial P_x}{\partial \theta_3}\\ \frac{\partial P_y}{\partial \theta_1} & \frac{\partial P_y}{\partial \theta_2} & \frac{\partial P_y}{\partial \theta_3}\\ \frac{\partial P_z}{\partial \theta_1} & \frac{\partial P_z}{\partial \theta_2} & \frac{\partial P_z}{\partial \theta_3}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\end{bmatrix} \quad \therefore J = \begin{bmatrix}-l_1s_1-l_2s_2 & -l_2s_2 & 0\\ l_1c_1+l_2c_2 & l_2c_2 & 0\\ 0 & 0 & 0\end{bmatrix}$$

3 DOF

③ 4 Axis SCARA $(\dot{\theta}_1, \dot{\theta}_2, \dot{q}_3, \dot{\theta}_4)$

$$J = \begin{bmatrix}-l_1s_1 - l_2s_{12} & -l_2s_{1-2} & 0 & 0\\ l_1c_1 + l_2c_{1-2} & l_2c_{1-2} & 0 & 0\\ 0 & 0 & -1 & 0\end{bmatrix}$$

$P_x = l_1c_1 + l_2c_{1-2}$

$P_y = l_1s_1 + l_2s_{1-2}$

$P_z = d_1 - q_3 - d_4$

Singularities: $\dot{\theta} = [J(\theta)^{-1}]\dot{x}$

Now, $J(\theta)$ may not be always invertible

(non invertible at some value of θ_2)

At some value of "θ", the jacobian might lose its full rank and the phenomenon this condition is called singularity

The points at which the Jacobean loses rank are known as joint space singularity

Note: The Jacobean Matrix $J(q)$ is full of rank as long as q is not a Joint space singulality

Physically → For a particular θ, we won't be able to drive the manipulator with the desired velocity for any value of Joint space velocities