EE6417

CHELLA THIYAGARAJAN N
ME17B179

Assignment - 2

1). a) Vertices $i = 1, \ldots, n$

in degree of $i = k_i^{in}$

~~$k_{in}(1) =$~~ $k_1^{in} = n-1$

$k_2^{in} = n-2$

$\vdots$

$k_r^{in} = n-r$

$$\sum_{i=1}^{r} k_i^{in} = rn - \frac{r(r+1)}{2}$$

out degree of $i = k_i^{out}$

$k_1^{out} = 0$

$k_2^{out} = 1$

$\vdots$

$k_r^{out} = r-1$

$$\sum_{i=1}^{r} k_i^{out} = \frac{r(r-1)}{2}$$

Total number of edges incoming to vertices 1 to r $= \sum_{i=1}^{r} k_i^{in}$

Total number of edges outgoing from vertices 1 to r $= \sum_{i=1}^{r} k_i^{out}$

b) Vertices $i = \underbrace{1, \ldots, r}_{r \text{ vertices}}, \underbrace{r+1, \ldots, n}_{n-r \text{ vertices}}$

1 to r has incomings from all the vertices that are from $r+1$ to n

$\therefore$ Total number of edges $= r \cdot (n-r)$

$$r \cdot (n-r) = rn - r^2 = rn - \frac{r[r+1-1+r]}{2}$$

$$= \underbrace{rn - \frac{r(r+1)}{2}}_{} - \underbrace{\frac{r(r-1)}{2}}_{}$$

$$= \sum_{i=1}^{r} k_i^{in} - \sum_{i=1}^{r} k_i^{out}$$

$$\boxed{\#\text{of edges} = \sum_{i=1}^{r} k_i^{in} - \sum_{i=1}^{r} k_i^{out}}$$

2.) order = number of nodes of a graph
size = number of edges of a graph

a) order = 8 , size = 15

Let x vertices have 5 neighbours

$\therefore$ $(8-x)$ vertices will have 3 neighbours

$$\#\text{ of edges} = \frac{x \cdot 5 + (8-x) \cdot 3}{2}$$

$$= \frac{5x - 3x + 24}{2}$$

$$= \frac{2x + 24}{2} = x + 12 \longrightarrow (1)$$

$$\text{size} = 15 \longrightarrow (2)$$

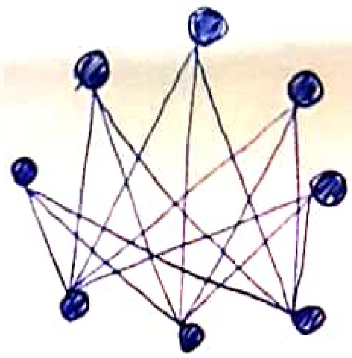
$$(1) = (2)$$

$$x + 12 = 15$$

$$x = 3$$

$\therefore$ 3 vertices have degree 5 & other 5 vertices have degree 3

Sample graph



b) $m = 2n - 3$

m = size

n = order

size = 2 . order − 3

It is well known that the necessary and sufficient conditions of a k regular graph of order n to exist are that $n > k+1$ and that nk is even.

Also note that if any regular graph has order n then the number of edges are $\frac{nk}{2}$ so nk has to be even

Let order $= n$, $m = \frac{n.k}{2}$ = size

$$\frac{nk}{2} = 2n - 3$$

$$4n - kn = 6$$

$$n = \frac{6}{4-k}$$

for $k = 0$, n is a fraction $\therefore$ it is not valid

✓ for $k = 1$, $n = 2$, $m = 1$, $k = 1$, Valid

✓ For $k = 2$, $n = 3$, $m = 3$, $k = 2$, Valid

✓ For $k = 3$, $n = 6$, $m = 9$, $k = 3$, Valid

for $k = 4$, n is negative, not Valid

for $k > 4$, n is negative, not Valid

3. n - nodes

. m - edges without self - loop

a) $\Rightarrow$ b)

G is a tree,

tree is undirected, acyclic, only one simple path between two nodes

Existence of only one simple path between any two nodes, supports ~~that~~ the fact that there is always existence of a path between 2 nodes in this graph this proves G is Connected

The least number of edges needed to Connect n nodes undirectedly ~~requires~~ is (n-1) edges. The least number of nodes ensured acyclic property and connected property.

$\therefore \; m = n-1$

b) $\Rightarrow$ c)

m = n-1, where n-1 is the least number of edges required to fully Connect G of n nodes.

$\therefore$ this ensures acyclic and a simple path between 2 nodes in G

and it is given m = n-1

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4.) Two graphs are isomorphic if and only if there is a permutation matrix P relating their adjacency matrices such that

$$A' = PAP^{-1}$$

where A & A' are the adjacency matrices of 2 graphs. More compactly, 2 graphs are isomorphic if and only if their adjacency matrices are similar through a permutation matrix.

Suppose σ be a permutation of the vertices, let P be the corresponding permutation matrix, the adjacency matrices are related by

$$a'_{ij} = a_{\sigma(i),\sigma(j)}$$

since the entry $a'_{i,j}$ gives the connectivity condition between vertices i and j, this is precisely the isomorphism condition.

Note: PAP^T is the similarity transformation of A defined by P. because the permutation matrix P satisfies $P^{-1} = P^T$, Moreover PAP^T is simply a reordering of rows and columns of A.

5.) $s \longrightarrow t$ path weight $= \alpha^{\ell}$

where $\ell \longrightarrow$ length of the path

a) a_{ij}^{k} represents the # of paths from i to j of length k in A^k matrix.

$$a_{ij}^{k} = \alpha^k A^k$$

$$Z = \sum_{k=1}^{\infty} \alpha^k A^k = \frac{1}{I - \alpha A} = (I - \alpha A)^{-1}$$

where $1 + a + a^2 + \dots \infty$ where $0 < a < 1$

$= \boxed{\frac{1}{1-a}} \longrightarrow$ infinite GP sum formula

b) $$Z_t = \sum_{k=1}^{t} \alpha^t A^t$$

$$Z_{t+1} = \sum_{k=1}^{t+1} \alpha^{t+1} A^{t+1}$$

$$Z_{t+1} - Z_t = \alpha^{t+1} A^{t+1}$$

take $\lim_{t \to \infty}$ on both sides

$$\lim_{t \to \infty} (Z_{t+1} - Z_t) = \lim_{t \to \infty} \alpha^{t+1} A^{t+1}$$

we know ~~as~~ Z converges to $(I - \alpha A)^{-1}$

$\therefore \lim_{t \to \infty} (Z_{t+1} - Z_t) = 0$

$\therefore$ L.H.S = R.H.S $\Rightarrow \lim_{t \to \infty} \alpha^t A^t = 0$, $\alpha < 1, \alpha > 0$

6.) $A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$

to make it upper block triangular

1. swap column 1 and 3

2. swap row 1 and 3

$\therefore$ The permutation matrix P

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$PAP^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} = \text{block triangular form}$$

$\therefore$ A is reducible

bcz there exists a permutation matrix P such that PAP^T is block triangular

1.) Given : A $\longrightarrow$ non-negative - irreducible matrix

a) Given : $\min(A\mathbb{1}_n) \le \rho(A)$

Proove : $\min(A\mathbb{1}_n) \le \rho(A) \le \max(A\mathbb{1}_n)$

A is row substochastic matrix

& $r_1 = \min(A\mathbb{1}_n)$, $r_2 = \max(A\mathbb{1}_n)$

for lower bound, let $B = \frac{r_1 a_{ij}}{\Sigma a_k}$ $\quad A \ge \frac{r_1 a_{ij}}{\sum_{k=1}^{n} a_{ik}} = 0, \rho(A) \ge \rho(B)$

$\sum_{j=1}^{n} \frac{r_1 a_{ij}}{\sum_{k=1}^{n} a_{ik}} = r_1$, for such a matrix

we get $r_1 \le \rho(A)$

for R.H.S :

$\rho(A) \le \|A\|_\infty = \max \sum_{j=1}^{n} a_{ij} \;\forall\, i = 1, \dots, n$

$\therefore$ we get $\min(A\mathbb{1}_n) \le \rho(A) \le \max(A\mathbb{1}_n)$

b) Let's define a maximum row-sum at k by

$\gamma = \max_{i \neq n} \sum_{j=1}^{n} (A^k)_{ij} < 1$

Claims we know already :

a) $e_i^T A^h \mathbb{1}_n < 1$ for $h \in \mathbb{N}$, then $e_i^T A^{h+1} \mathbb{1}_n < 1$

b) if i has out neighbour j, with $e_j^T A^h \mathbb{1}_n < 1$, then $e_i^T A^{h+1} \mathbb{1}_n < 1$

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c) k, such that $A^k \mathbb{1}_n < \mathbb{1}_n$

d) $\rho(A) < 1$

Given a natural number $k^* > k$, we can write $k^* = ak + b$ with a positive integer and $b \in \{0, \dots, k-1\}$

$$A^{k^*}\mathbb{1}_n \leq A^{ak}\mathbb{1}_n \leq \gamma^a \mathbb{1}_n$$

This inequality implies that, as $k^* \to \infty$, i.e. $a \to \infty$, the sequence A^{k^*} converges to 0, this proves $\rho(A) < 1$

$\therefore \min(A\mathbb{1}_n) < \max(A\mathbb{1}_n)$

$\therefore \min(A\mathbb{1}_n) < \rho(A) < \max(A\mathbb{1}_n)$

8.)

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) ST. Gallen (2) $\xrightarrow{1}$ Chur (3) $\xrightarrow{1}$ Zurich (1) $\xrightarrow{1}$ Interloken (4) $\xrightarrow{1}$ Lousanne (7) $\xrightarrow{1}$ Zermatt (8)

\# of links of the shortest path = 5

from matrix $A'_{28} = 0$, $A^2_{28} = 0$, $A^3_{28} = 0$, $A^4_{28} = 0$,

$A^5_{28} > 0$

b) Bern (6) $\xrightarrow{1}$ Basel (5) $\xrightarrow{1}$ Zurich (1) $\xrightarrow{1}$ St. Gallen (2) $\xrightarrow{1}$ Chur (3)

It is possible to go from Bern to Chur using 4 links

It is no possible to go from been to chue in 5 links.

In Matrices :

$A^5_{63} = 0$, $A^4_{63} > 0$

No path exist of length 5

at least a path exist of length 4

c) ① $\xrightarrow{\text{path length} < 9}$ ⑦

$\# \text{ of paths} = \sum_{i=1}^{8} A^k_{17}$,

using python code to calculate the values

$\# \text{ of paths} = 1 + 2 + 3 + 6 + 11 + 20 + 36$

$= 79$ //

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