2. $e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$

$\sum_{k=0}^{\infty} A_k$ Converges if $\sum_{k=0}^{\infty} \|A_k\|$ converges

$$\|e^A\| = \sum_{k=0}^{\infty} \frac{1}{k!} \|A^k\| = I_n + \|A\| + \frac{1}{2!}\|A^2\| + \cdots$$

$$\leq I_n + \|A\| + \frac{1}{2!}\|A\|\cdot\|A\| + \frac{1}{3!}\|A\|\|A\|\|A\| + \cdots$$

$$= I_n + \|A\| + \frac{1}{2!}\|A\|^2 + \frac{1}{3!}\|A\|^3 + \cdots$$

$$= e^{\|A\|}$$

b) $A = \text{diag}(a_1, \dots, a_n)$

$A = Q^* \lambda Q^{-1}$

here $Q = I$

we know that

$A^k = Q \lambda^k Q^{-1}$, here $\lambda$ is the eigen value matrix and $\lambda = \text{diag}(a_1, \dots, a_n)$

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I_n + Q \lambda Q^{-1} + \frac{1}{2!} Q \lambda^2 Q^{-1} + \frac{1}{3!} Q \lambda^3 Q^{-1} + \dots$$

* if $\lambda = \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix}$

we know that $\lambda^k = \begin{bmatrix} a_1^k & & & \\ & a_2^k & & \\ & & \ddots & \\ & & & a_n^k \end{bmatrix}$

$$e^A = I_n + Q\left[\lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots\right] Q^{-1}$$

$$= Q\left[\lambda^0 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots\right] Q^{-1}$$

$$= Q \begin{bmatrix} 1 + a_1 + \frac{a_1}{2!} + \frac{a_1}{3!} + \dots & & & \\ & 1 + a_2 + \frac{a_2}{2!} + \frac{a_2}{3!} + \dots & & \\ & & \ddots & \\ & & & 1 + a_n + \frac{a_n}{2!} + \frac{a_n}{3!} + \dots \end{bmatrix} Q^{-1}$$

$$= Q \begin{bmatrix} e^{a_1} & & & \\ & e^{a_2} & & \\ & & \ddots & \\ & & & e^{a_n} \end{bmatrix} Q^{-1}$$

as $\varphi$ is Identity matrix

$$= \begin{bmatrix} e^{a_1} & & & \\ & e^{a_2} & & \\ & & \ddots & \\ & & & e^{a_n} \end{bmatrix}$$

$$= \operatorname{diag}(e^{a_1}, \dots, e^{a_n})$$

c) $$e^{TAT^{-1}} = \sum_{k=0}^{\infty} \frac{1}{k!} (TAT^{-1})^k$$

$$(TAT^{-1})^k = (TA\cancel{T^{-1}})(\cancel{T}A\cancel{T^{-1}}) \dots (\cancel{T}AT^{-1})$$

$$= TA^kT^{-1}$$

$$e^{TAT^{-1}} = I_n + TAT^{-1} + \frac{1}{2!} TA^2T^{-1} + \frac{1}{3!} TA^3T^{-1} + \dots$$

$$= T\left[A^0 + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots\right]T^{-1}$$

$$= T e^A T^{-1}$$

AABABB
AAABBB

d) $AB = BA$

$$e^{AB} = I_n + AB + \frac{(AB)^2}{2!} + \frac{(AB)^3}{3!} + \dots$$

$$= I_n + AB + \frac{ABAB}{2!} + \frac{ABABAB}{3!} + \dots$$

if $AB = BA$, then $(AB)^k = A^kB^k$

$$e^{AB} = I_n + AB + \frac{A^2B^2}{2!} + \frac{A^3B^3}{3!} + \dots$$

$\longrightarrow$ 1

if $AB = BA$, then $(BA)^k = B^k A^k$ ~~$(AB)^k$~~

$$e^{BA} = I_n + BA + \frac{(BA)^2}{2!} + \cdots$$

$$= I_n + BA + \frac{B^2A^2}{2!} + \frac{B^3A^3}{3!} + \cdots$$

$\to$ 2

if $AB = BA$, we know that $(AB)^k = (BA)^k$

bcz if $AB = BA = C$, then $C^k = C^k$

$\therefore (AB)^k = A^k B^k = (BA)^k = B^k A^k$

$\therefore$ 1 $=$ 2

Hence proved

$$e^{AB} = e^{BA}$$

e) Commutative law doesn't work for matrices

$AB \neq BA$

In the $d^{th}$ part we proved

$AB = BA$ implies $e^{AB} = e^{BA}$

~~base~~ on the contradictory here we have $AB \neq BA$

$\therefore \quad e^{AB} \neq e^{BA}$

f) $e^{tJ} = \sum_{k=0}^{\infty} \frac{1}{k!}(tJ)^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} J^k$

$$J(\lambda)^k = (\lambda I + N)^k = \sum_{r=0}^{k} {}^kC_r \lambda^{k-r} N^r = \sum_{r=0}^{\min(k,\,l-1)} {}^kC_r \lambda^{k-r} N^r$$

here $N$ denote the nilpotent matrix whose super diagonal contains ones, and others 0.

$N^{l} = 0$

$$\therefore e^{tJ} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left( \sum_{r=0}^{\min(k, l-1)} {}^{k}C_r \lambda^{k-r} N^r \right)$$

1). $\mathbb{R}^{n \times m}$

$$Vec(x) = [x_{11}, \dots, x_{n1}, x_{12}, \dots, x_{n2}, \dots, x_{1m}, \dots x_{nm}]$$

a) $x = [x_1, \dots, x_n]$

$y = [y_1, \dots, y_m]$

$$xy^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} [y_1, \dots, y_m]$$

$$= \begin{bmatrix} x_1 y_1, & \dots & x_1 y_m \\ \vdots & & \vdots \\ x_n y_1, & \dots & x_n y_m \end{bmatrix}$$

$$Vec(xy^T) = [x_1 y_1, \dots, x_n y_1, x_1 y_2, \dots, x_n y_2, x_1 y_m, \dots, x_n y_m] \rightarrow ①$$

$\otimes \longrightarrow$ kronecker product

$y \rightarrow m \times 1$, $x \longrightarrow n \times 1$, $y \otimes x \longrightarrow mn \times 1$

$$y \otimes x = \begin{bmatrix} y_1 \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ y_2 \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ \vdots \\ y_m \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \end{bmatrix}$$

$$y \otimes x = [y_1x_1, \ldots, y_1x_n, y_2x_1, \ldots, y_2x_n, \ldots, y_mx_1, \ldots, y_mx_n] \longrightarrow (2)$$

$\therefore$ we can see that ~~vec~~

$$vec(xy^T) = y \otimes x \quad , \text{ bcz } (1) = (2)$$

b) Given:

$$vec(ABC) = (C^T \otimes A)\, vec(B) \longrightarrow (3)$$

$$A \to \mathbb{R}^{n \times n}, \quad B \to \mathbb{R}^{m \times m}, \quad C \to \mathbb{R}^{n \times m}$$

$$AX + XB = C \longrightarrow \text{Sylvester equation}$$

$$X \to \mathbb{R}^{n \times m}$$

a) $$I_m \otimes A = \begin{bmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A \end{bmatrix} \in \mathbb{R}^{mn \times mn}$$

$$B^T \otimes I_n = \begin{bmatrix} b_{11} I_n & \cdots & b_{m1} I_n \\ \vdots & \ddots & \vdots \\ b_{1m} I_n & \cdots & b_{mm} I_n \end{bmatrix} \in \mathbb{R}^{mn \times mn}$$

using (3)

$$vec(AXI_m) = (I_m^T \otimes A)\, Vec(X)$$
$$= (I_m \otimes A)\, Vec(X) \longrightarrow (4)$$

using (3)

$$Vec(I_n X B) = (B^T \otimes I_n)\, Vec(X) \longrightarrow (5)$$

4 + 5

$$\text{Vec}(AX I_m) + \text{Vec}(I_n X B) = (I_m \otimes A)\,\text{Vec}(X) + (B^T \otimes I_n)\,\text{Vec}(X)$$

$$\text{Vec}(AX) + \text{Vec}(XB) = (I_m \otimes A)\,\text{Vec}(X) + (B^T \otimes I_n)\,\text{Vec}(X)$$

$$\text{Vec}(AX + XB) = \left[(I_m \otimes A) + (B^T \otimes I_n)\right] \text{Vec}(X) \quad \to (6)$$

from sylvester equation we know

$$AX + XB = C$$

$$\therefore (6) \Rightarrow \text{Vec}(C) = \left[(I_m \otimes A) + (B^T \otimes I_n)\right] \text{Vec}(X)$$

Hence proved //

b) $AX - XB$ is uniquely solvable for any $C$, if and only if the homogenous equation $AX - XB = 0$ admits only trivial solution

Assume : A and B do not share any eigen value

Let $X'$ be a solution to $AX - XB = 0$

then $AX' = X'B$, using mathematical Induction

$$A^k X' = X' B^k \quad \text{for each } k \geq 0$$

$\therefore$ $p(A)X = X p(B)$ for any polynomial $p$.

Let $p$ be a characteristic polynomial of $A$, then

$p(A) = 0$ due to Cayley hamilton theorem.

using spectral mapping Theorem

$$\sigma(p(B)) = p(\sigma(B))$$

$\hookrightarrow$ spectrum of a matrix

using the assumption that A and B do not share any eigenvalues, $p(\sigma(B))$ does not contain 0, therefore $p(B)$ is non singular, therefore $X = 0$ is the solution

Now Assume: A and B share an eigen value $\lambda$

Let $u$ and $v$ be the right and left eigenvectors of A and B respectively.

$$\text{Let } X = uv^*, \therefore X \neq 0$$

$$AX - XB = A(uv^*) - (uv^*B) = \lambda uv^* - \lambda uv^*$$

$$= 0$$

Hence, X is non trivial solution to $AX = XB$

$\therefore$ $AX + (X(-B)) = C$,

has a unique solution for all C if and only if A and B have no Common eigen values

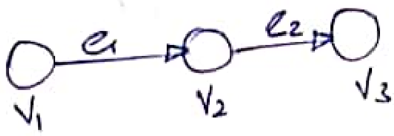
It can be also written as:

$AX + XB = C$,

has a unique solution for all C if and only if A and $-B$ have no Common eigen values.

3)

a) A line graph with three nodes




$$B = \begin{array}{c|cc} & e_1 & e_2 \\ \hline V_1 & 1 & 0 \\ V_2 & -1 & 1 \\ V_3 & 0 & -1 \end{array}$$

$$L_{edge} = B^T B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$L = BB^T = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

a) for all $x \in kernel(B)$

$$Bx = 0$$

multiply by $B^T$

$$B^T B x = B^T 0$$

$$L_{edge}\, x = 0$$

$\therefore$ $\boxed{kernel(B) = kernel(L_{edge})}$ $\longrightarrow$ ①

for all $x \in kernel(L_{edge})$

$$B^T B x = 0$$

multiply by $x^T$

$$x^T B^T B x = 0$$

$$(Bx)^T Bx = 0$$

$$\Rightarrow \quad Bx = 0$$

$$\therefore \boxed{\text{kernel}(L_{edge}) = \text{kernel}(B)} \longrightarrow ②$$

from ① and ②

$$\text{kernel}(L_{edge}) = \text{kernel}(B)$$

for an arbitrary undirected graph

b) $\text{kernel}(B) = \text{kernel}(L_{edge})$ , kernell = Null

$$\dim(N(B)) = \dim(N(L_{edge})) \longrightarrow ③$$

equation ③ implies that

$$\text{rank}(B) = \text{rank}(L_{edge}) \longrightarrow ④$$

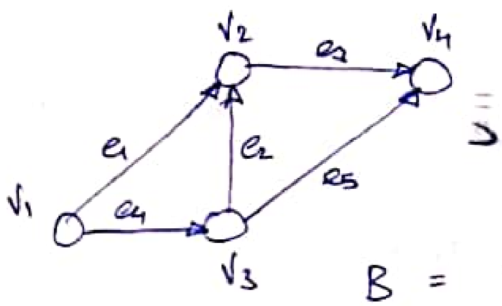
we already know that

$$\text{rank}(B) = \text{rank}(L) \longrightarrow ⑤$$

from ④ and ⑤

$$\text{rank}(L) = \text{rank}(L_{edge})$$

b)




$$B = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{array}$$

Scanned with CamScanner

$$L_{edge} = B^T B = \begin{bmatrix} 2 & 1 & -1 & 1 & 0 \\ 1 & 2 & -1 & -1 & 1 \\ -1 & -1 & 2 & 0 & 1 \\ 1 & -1 & 0 & 2 & -1 \\ 0 & 1 & 1 & -1 & 2 \end{bmatrix}$$

$$L = BB^T = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

c) $\text{rank}(B) = n - C$

$C \rightarrow$ Connected Components

Given : $C = 0$

$\text{rank}(B) = n$

$\dim(N(B)) = \dim(N(L_{edge})) = 0$

$\therefore$ $L_{edge}$ is full rank matrix, so it is non - singular

d) We know that for a square matrix $A$, eigenvalues of $A$ is equal to the eigen values of $A^T$

We also know that for any rectangular matrix $A$, $A^TA$ or $AA^T$ is a square matrix

$$Lq = \lambda q$$

$$BB^Tq = \lambda q$$

$$B^T BB^Tq = \lambda B^Tq$$

$$(B^TB)(B^Tq) = \lambda(B^Tq)$$

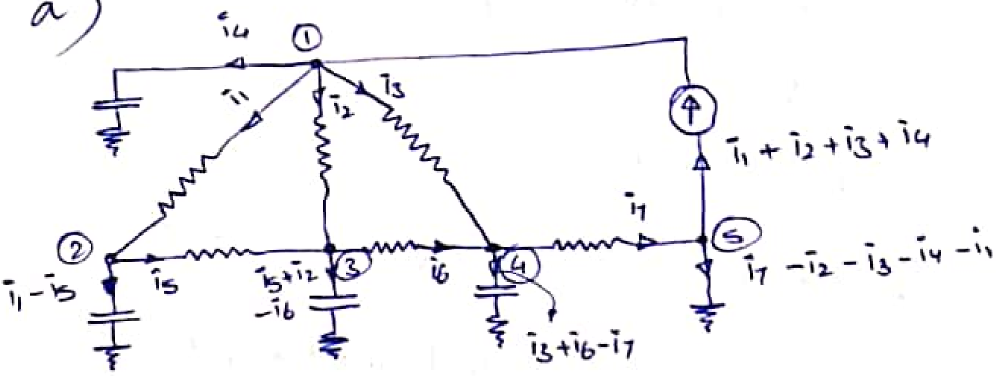
$B^T$ is a full Column rank matrix, therefore $B^Tq$ represents any Vector $x$

$$(B^TB)\,x = \lambda x$$

$$L_{edge}\, x = \lambda x$$

$\therefore$ Non-Zero eigen Values of $L_{edge}$ ~~is~~ are equal to the non-Zero eigen Values of $L$.

4. a)




$$V_1 - V_2 = i_1 R_{12}$$
$$V_2 - V_3 = i_5 R_{23}$$
$$V_3 - V_4 = i_6 R_{34}$$
$$V_1 - V_4 = i_3 R_{14}$$
$$V_4 - V_5 = i_7 R_{45}$$

$C_{inj}$ at node ① $= i_1 + i_2 + i_3$

$C_{inj}$ at node ② $= i_1$

$C_{inj}$ at node ③ $= i_5 + i_2$

$C_{inj}$ at node ④ $= i_6 + i_3$

$C_{inj}$ at node ⑤ $= i_7$

$V_5 = 0$ (Grounded)

$$C_{inj} = L\, V$$

$C_{inj} = [C_{inj}$ at ①, $C_{inj}$ at ②, $C_{inj}$ at ③, $C_{inj}$ at ④, $C_{inj}$ at ⑤$]$

$$V = [V_1, V_2, V_3, V_4, V_5]$$

$$L = D - A$$

$$A = \begin{bmatrix} 0 & \frac{1}{R_{12}} & \frac{1}{R_{13}} & \frac{1}{R_{14}} & 0 \\ 1/R_{12} & 0 & \frac{1}{R_{23}} & 0 & 0 \\ 1/R_{13} & 1/R_{23} & 0 & 1/R_{34} & 0 \\ 1/R_{14} & 0 & 1/R_{34} & 0 & 1/R_{45} \\ 0 & 0 & 0 & 1/R_{45} & 0 \end{bmatrix}$$

$$D = \mathrm{diag}\left(\begin{bmatrix} \frac{1}{R_{12}} + \frac{1}{R_{13}} + \frac{1}{R_{14}} \\ \frac{1}{R_{12}} + \frac{1}{R_{23}} \\ \frac{1}{R_{13}} + \frac{1}{R_{23}} + \frac{1}{R_{34}} \\ \frac{1}{R_{14}} + \frac{1}{R_{34}} + \frac{1}{R_{45}} \\ \frac{1}{R_{45}} \end{bmatrix}\right)$$

~~Inflow~~ Conservative quantity is Current here.

By using kirchoff Current Law, I am Conserving current at each node to come up with the $L$ matrix

$\therefore$ current is the Conserved quantity inflow Vector $= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, since we are injecting some current at only node 1 as given in our figure

we have $C_i \frac{d}{dt} V_i = -C_{\text{injected at } i}$

Outflow current, which goes to the ground passes through the Capacitor, the current that the Capacitor takes at each node is $-C_{\text{injected at } i}$

$\therefore$ outflow rate $= f_o = \begin{bmatrix} C_1\, dV_1/dt \\ C_2\, dV_2/dt \\ C_3\, dV_3/dt \\ C_4\, dV_4/dt \\ 0 \end{bmatrix}$

Compartmental matrix $= C = -L^T - \text{diag}(f_o)$

$$C = -(D-A)^T - \text{diag}(f_o) = A^T - D^T - \text{diag}(f_o)$$

$$= \begin{bmatrix} -\left(\frac{1}{R_{12}}+\frac{1}{R_{13}}+\frac{1}{R_{14}}\right) - C_1 dV_1/dt & 1/R_{12} & 1/R_{13} & \frac{1}{R_{14}} & 0 \\ \frac{1}{R_{12}} & -\left(\frac{1}{R_{12}}+\frac{1}{R_{23}}\right) - C_2\frac{dV_2}{dt} & \frac{1}{R_{23}} & 0 & 0 \\ \frac{1}{R_{13}} & 1/R_{23} & -\left(\frac{1}{R_{13}}+\frac{1}{R_{23}}+\frac{1}{R_{34}}\right) - C_3\frac{dV_3}{dt} & \frac{1}{R_{34}} & 0 \\ 1/R_{14} & 0 & 1/R_{34} & -\left(\frac{1}{R_{14}}+\frac{1}{R_{34}}+\frac{1}{R_{45}}\right) - C_4\frac{dV_4}{dt} & \frac{1}{R_{45}} \\ 0 & 0 & 0 & 1/R_{45} & -\frac{1}{R_{45}} \end{bmatrix}$$