Manipulator Kinematics

Lecture 3:
Inverse Kinematics and Differential Relations

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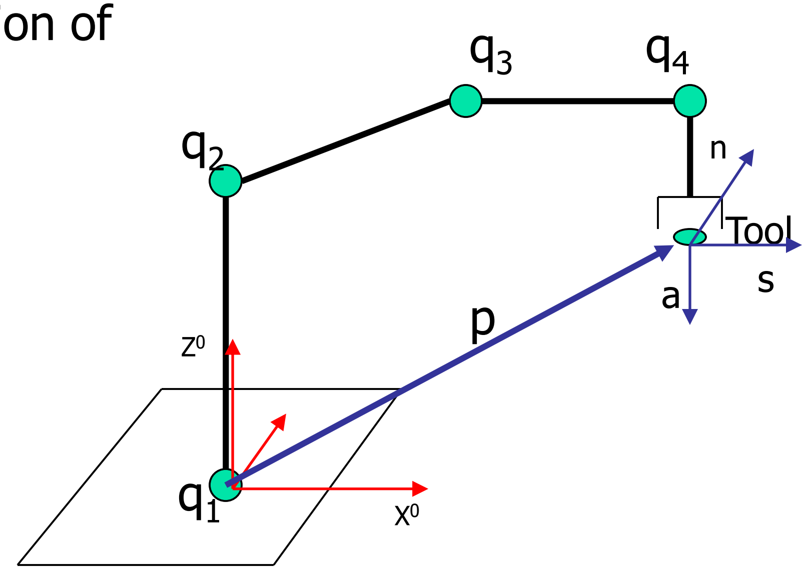
Manipulator tasks are normally formulated in terms of the desired position and orientation.

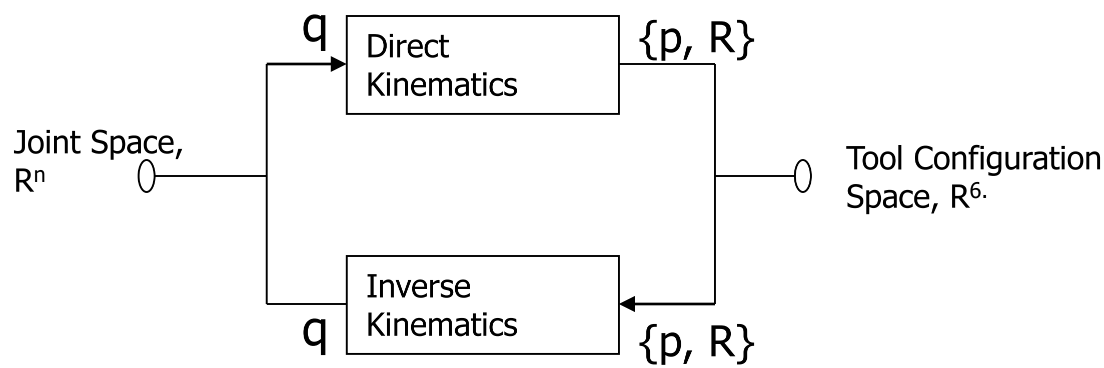
- A systematic closed form solution applicable to robots in general is not available.
- Unique solutions are rare; multiple solutions exist.
- Inverse problem is more difficult than forward problem

The Arm matrix represents the position p and orientation R of the tool in base coordinate frame as a function of joint variable q.

$$T_{base}^{tool}(q) = \left[\begin{array}{ccc|c} & R(q) & & p(q) \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Inverse Kinematics: Given a desired position p and orientation R for the tool, find values for the joint variables q which satisfy the arm equation





Solvability

$$\begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{ccc|c} C_1C_{234}C_5 + S_1S_5 & -C_1C_{234}S_5 + S_1C_5 & -C_1S_{234} & C_1(177.8C_2 + 177.8C_{23} - 129.5S_{234}) \\ S_1C_{234}C_5 - C_1S_5 & -S_1C_{234}S_5 - C_1C_5 & -S_1S_{234} & S_1(177.8C_2 + 177.8C_{23} - 129.5S_{234}) \\ -S_{234}C_5 & S_{234}S_5 & -C_{234} & 215 - 177.8S_2 - 177.8S_{23} - 129.5C_{234} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

- 12 equations and ‘n’ unknowns. (n=6 for 6 axis robot)
- Out of 9 equations from the rotation part, only 3 are independent.
- From the position vector part, there are 3 independent equations.
- 6 independent equations and ‘n’ unknowns
- Nonlinear equations, difficult solve

Existence of Solutions

A manipulator is **solvable** if all the sets of joint variables can be found corresponding to a given end-effector location.

Necessary conditions

•Tool point within the workspace

•n $\geq$6, to have any arbitrary orientation of tool

•Tool orientation is such that none of the joint limitations are violated

Two kinds of solutions:
Closed form solutions: analytical expressions
Numerical solutions: iterative search – time consuming

For closed-form Solution:- Sufficiency Condition
Three adjacent joint axes intersecting or
Three adjacent joint axes parallel to one another

Uniqueness of Solution:- Multiple Solutions

•Kinematically redundant robots

•Elbow-up, Elbow-down solutions

There are two approaches for deriving closed form solutions: **algebraic** vs. **geometric**.
Algebraic approach:
Obtain scalar equations from the matrix form (at most 12 of them)

Trick 1: use trigonometric identities to combine two equations (such as first square them and then add them) and eliminate certain variables.

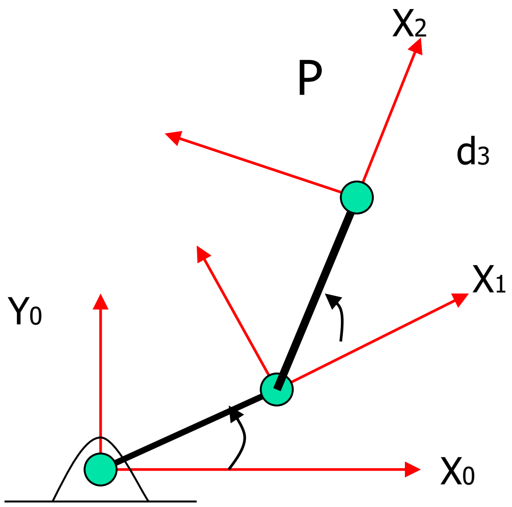
Trick 2: use the following variable substitution

$$u = \tan(\theta/2)$$
$$\cos\theta = (1-u^2)/(1+u^2)$$
$$\sin\theta = 2u/(1+u^2)$$

to convert these equations to polynomial ones, and solve the polynomial ones instead.

Trick 3: find out expressions for both *sin* and *cos* of a joint angle θ_i, and express the angle using the function $Atan2(\sin\theta_i, \cos\theta_i)$. With *Atan2* rather than the conventional *arctan*, the angle can be uniquely expressed

2 DoF Planar manipulator



$$P_x^2 + P_y^2 = l_1^2 + l_2^2 + 2l_1l_2C_2$$

$$\Rightarrow C_2 = \frac{P_x^2 + P_y^2 - (l_1^2 + l_2^2)}{2l_1l_2}$$

$$S_2 = \pm\sqrt{1 - C_2^2}$$

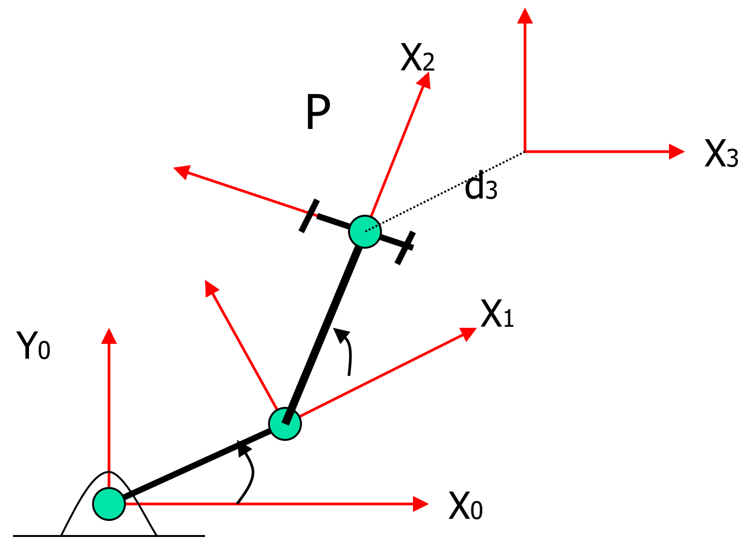
$$\theta_2 = \text{atan2}(S_2, C_2)$$

The function atan2 denotes a four quadrant version of arctan function. It allows us to recover angles over the entire range of [$-\pi$ to π]

if $x = y = 0$, then the result is indefinite,
if $x > 0$ and $y = 0$, then atan2 = 0,
if $x < 0$ and $y = 0$, then atan2 = π, else
if $y < 0$, then $-\pi <$ atan2 < 0,
if $y > 0$, then $0 <$ atan2 $< \pi$.

Example- 3 DoF

$$T_{base}^{tool} = \begin{bmatrix} n_x & s_x & 0 & p_x \\ n_y & s_y & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$T_{base}^{tool} = \begin{bmatrix} C_{123} & -S_{123} & 0 & l_1C_1 + l_2C_{12} \\ S_{123} & C_{123} & 0 & l_1S_1 + l_2S_{12} \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a	d	α	θ
l1	0	0	
l2	0	0	
0	d3	0	

$$n_x = C_{123}; \quad (1) \quad n_y = S_{123} \quad (2)$$

$$s_x = -S_{123}; \quad (3) \quad s_y = C_{123} \quad (4)$$

$$Px = l_1C_1 + l_2C_{12} \quad (5)$$

$$Py = l_1S_1 + l_2S_{12} \quad (6)$$

On squaring and adding (5) and (6), we get

$$P_x^2 + P_y^2 = l_1^2 + l_2^2 + 2l_1l_2C_2$$

$$\Rightarrow C_2 = \frac{P_x^2 + P_y^2 - (l_1^2 + l_2^2)}{2l_1l_2}$$

$$S_2 = \pm\sqrt{1-C_2^2}$$

$$\theta_2 = \text{atan2}(S_2, C_2)$$

The function atan2 denotes a four quadrant version of arctan function. It allows us to recover angles over the entire range of [$-\pi$ to π]

if $x = y = 0$, then the result is indefinite,
if $x > 0$ and $y = 0$, then atan2 = 0,
if $x < 0$ and $y = 0$, then atan2 = π, else
if $y < 0$, then $-\pi <$ atan2 < 0,
if $y > 0$, then $0 <$ atan2 $< \pi$.

$$Solving\ eqn.(5),(6)\ for \theta_1,$$

$$Px = k_1C_1 - k_2S_1$$

$$Py = k_1S_1 + k_2C_1;\ \text{ where } k_1 = l_1 + l_2C_2\ ; \text{k}_2 = l_2S_2$$

Substituting

$$k_1 = r\cos\gamma; k_2 = r\sin\gamma \quad \text{where } r = \sqrt{k_1^2 + k_2^2}; \gamma = a\tan 2(k_2, k_1)$$

We get,

$$\frac{P_x}{r} = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1$$

$$\frac{P_y}{r} = \cos\gamma\sin\theta_1 + \sin\gamma\cos\theta_1$$

$$\Rightarrow \cos(\gamma + \theta_1) = \frac{P_x}{r}$$

$$\sin(\gamma + \theta_1) = \frac{P_y}{r}$$

$$\gamma + \theta_1 = a\tan 2(P_y / r, P_x / r) = a\tan 2(P_y, P_x)$$

$$\therefore \theta_1 = a\tan 2(P_y, P_x) - a\tan 2(k_2, k_1)$$

From eq. (1) and (2), we have

$$a\tan 2(n_y, n_x) = \theta_{123}$$

$$\therefore \theta_3 = \theta_{123} - \theta_1 - \theta_2$$

Commonly used equations and their Solutions

$$Eq.\quad a = b\sin(\theta)$$
$$Solution: \theta = a\tan 2\left(a/b, \sqrt{(1-(a/b)^2}\right)$$

$$Eq.\quad a = b\cos(\theta)$$
$$Solution: \theta = a\tan 2\left(\sqrt{(1-(a/b)^2}, a/b\right)$$

$$Eq.\quad a = b\cos(\theta),\quad c = d\sin\theta$$
$$Solution: \theta = a\tan 2\left(c/d, a/b\right)$$

$$Eq.\quad a\sin(\theta) + b\cos(\theta) = 0$$
$$Solution: \theta = a\tan 2\left(-b, a\right)$$

$$Eq.\quad a\sin(\theta) + b\cos(\theta) = c$$
$$Solution: \theta = a\tan 2\left(a, b\right) + a\tan 2(\pm\sqrt{a^2+b^2-c^2}, c)$$

$$Eq.\quad a\cos(\theta_i+\theta_j) + b\cos(\theta_i) = c$$
$$a\sin(\theta_i+\theta_j) + b\sin(\theta_i) = d$$
$$Solution: \cos(\theta_j) = (c^2+d^2-a^2-b^2)/2ab$$
$$\sin(\theta_j) = \sqrt{1-\cos^2(\theta_j)}$$
$$\theta_j = a\tan 2(\sin(\theta_j), \cos(\theta_j))$$
$$\theta_i = a\tan 2(rd - sc, rc + sd), where$$
$$r = a\cos(\theta_j) + b, s = a\sin(\theta_j)$$

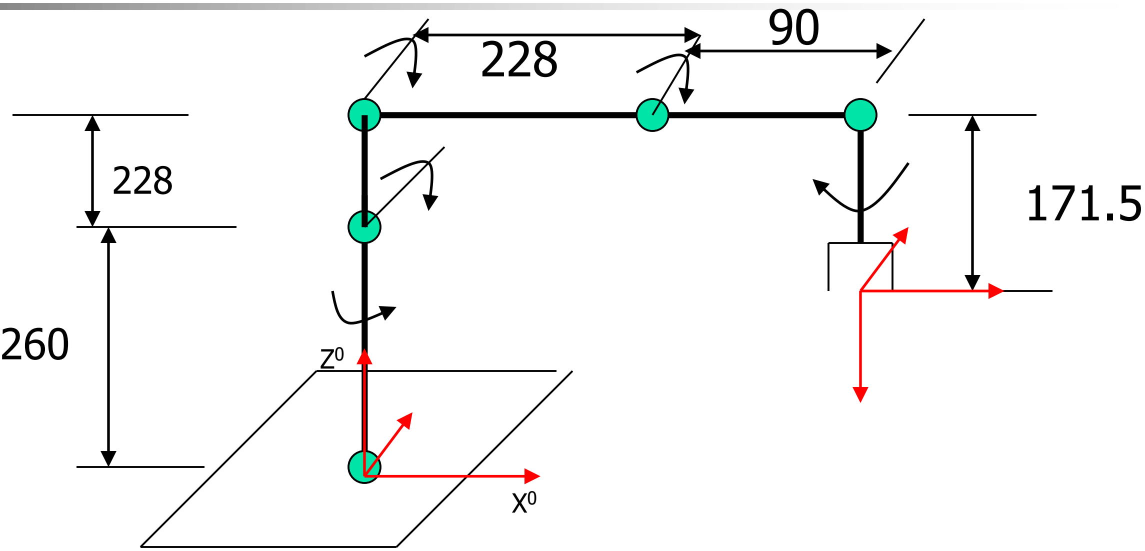
$$Eq.\quad a\cos(\theta_i) + b\cos(\theta_j) = c$$
$$a\sin(\theta_i) + b\sin(\theta_j) = d$$
$$Solution: \theta_i = a\tan 2\left(d, c\right) + a\tan 2(\pm\sqrt{c^2+d^2-s^2}, s)$$
$$where\quad s = (a^2-b^2+c^2+d^2)/2a$$
$$\theta_j = \theta_i + a\tan 2(\pm\sqrt{4a^2b^2-t^2}, t)$$
$$where\quad t = (c^2+d^2-a^2-b^2)$$

$$Px = l_1C_1 + l_2C_{12}$$
$$Py = l_1S_1 + l_2S_{12}$$
$$C_2 = \frac{P_x^2+P_y^2-(l_1^2+l_2^2)}{2l_1l_2}$$

$$Eq.\quad a\sin(\theta) + b\cos(\theta) = c$$
$$a\cos(\theta) - b\sin(\theta) = d$$
$$Solution: \theta = a\tan 2\left(ac - bd, ad + bc\right),\quad also\quad a^2+b^2 = c^2+d^2$$

Example: Rhino XR-3, 5 axis articulated Arm

Axis	θ	d	a	α
1	Θ_1	260	0	-90
2	Θ_2	0	228	0
3	Θ_3	0	228	0
4	Θ_4	0	90	-90
5	Θ_5	171	0	0



$$\begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_x = C_1\left(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}\right)(1)$$

$$P_y = S_1\left(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}\right)(2)$$

$$Pz = d_1 - a_2S_2 - a_3S_{23} - a_4S_{234} - d_5C_{234}\,(3)$$

$$n_x = C_1C_{234}C_5 + S_1S_5; \quad n_y = S_1C_{234}C_5 - C_1S_5$$

$$a_x = -C_1S_{234} \quad \Rightarrow a_y = -S_1S_{234}$$

$$P_x = C_1\left(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}\right)$$
$$P_y = S_1\left(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}\right)$$

$\Rightarrow$

$$P_x / P_y = C_1 / S_1$$
$$\theta_1 = a\tan 2(P_y, P_x)$$

$$a_x = -C_1S_{234} \quad \Rightarrow a_y = -S_1S_{234}; \quad S_{234} = -(a_xC_1 + a_yS_1); \quad C_{234} = -a_z$$
$$\theta_{234} = a\tan 2\left(-(a_xC_1 + a_yS_1), -a_z\right)$$

$$n_x = C_1C_{234}C_5 + S_1S_5; \quad n_y = S_1C_{234}C_5 - C_1S_5$$
$$s_x = -C_1C_{234}S_5 + S_1C_5; \quad s_y = -S_1C_{234}S_5 - C_1C_5$$
$$S_5 = n_xS_1 - n_yC_1$$
$$C_5 = s_xS_1 - s_yC_1 \Rightarrow \theta_5 = a\tan 2(S_5, C_5)$$

$$Eq.1 \Rightarrow a_3C_{23} + a_2C_2 = \frac{P_x}{C_1} + d_5S_{234} - a_4C_{234}$$
$$Eq.3 \Rightarrow a_3S_{23} + a_2S_2 = d_1 - a_4S_{234} - d_5C_{234} - P_z$$

$$Eq. \quad a\cos(\theta_i + \theta_j) + b\cos(\theta_i) = c$$
$$a\sin(\theta_i + \theta_j) + b\sin(\theta_i) = d$$
$$Solution: \cos(\theta_j) = (c^2 + d^2 - a^2 - b^2)/2ab$$
$$\sin(\theta_j) = \sqrt{1 - \cos^2(\theta_j)}$$
$$\theta_j = a\tan 2(\sin(\theta_j), \cos(\theta_j))$$
$$\theta_i = a\tan 2(rd - sc, rc + sd), where$$
$$r = a\cos(\theta_j) + b, s = a\sin(\theta_j)$$

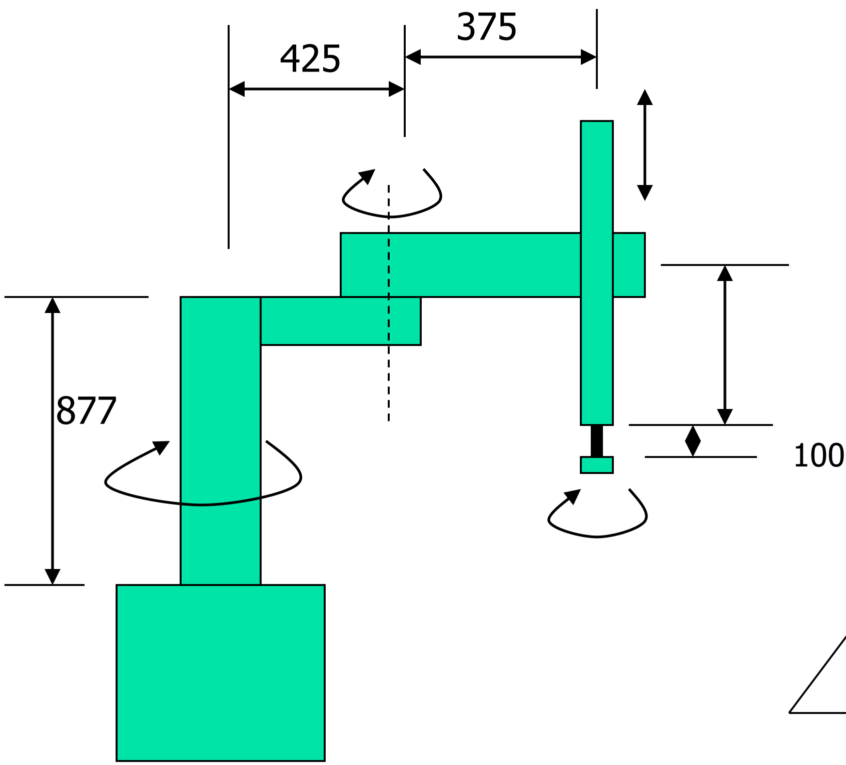
$$\cos(\theta_3) = (c^2 + d^2 - a_3{}^2 - a_2{}^2)/2a_3a_2$$
$$\sin(\theta_3) = \sqrt{1 - \cos^2(\theta_3)}$$

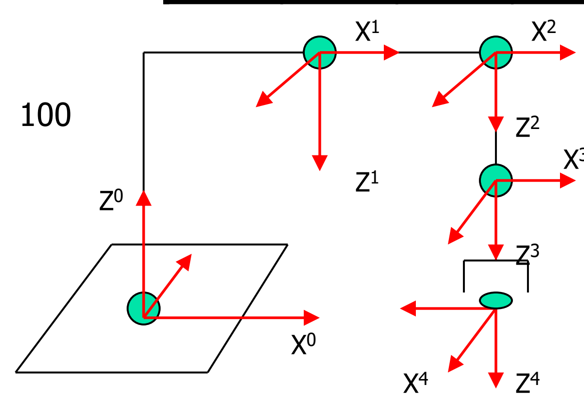
$$\theta_2 = a\tan 2(rd - sc, rc + sd)$$
$$r = a_3\cos(\theta_3) + a_2, \quad s = a_3\sin(\theta_3)$$

Homework : Four Axis SCARA- ADEPT One

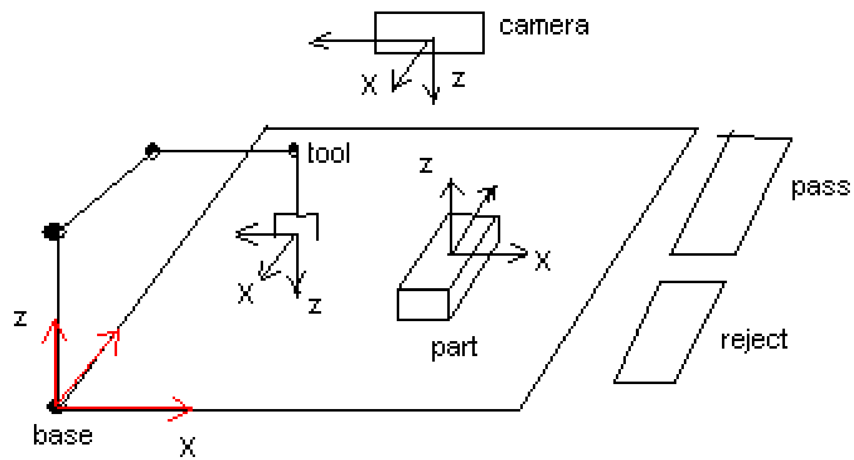
Find the inverse solution for the 4 axis SCARA robot. Write a computer program to get all possible solutions for a given tool configuration

Axis	θ	d	a	α
1	Θ_1	877	425	Π
2	Θ_2	0	375	0
3	0	d3	0	0
4	Θ_4	100	0	0





Robotic Work Cell



$$T_{camera}^{part} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -5 \\ 0 & 0 & -1 & 19 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{camera}^{base} = \begin{bmatrix} 0 & -1 & 0 & 15 \\ -1 & 0 & 0 & 25 \\ 0 & 0 & -1 & 20 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

location of part wrt base: $T_{base}^{part} = T_{base}^{camera} T_{camera}^{part}$

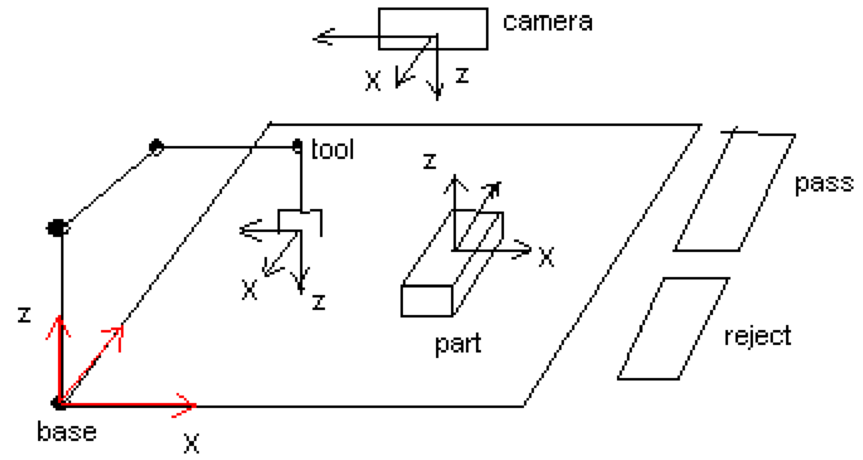
$$= (T_{base}^{camera}) T_{camera}^{part}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 25 \\ -1 & 0 & 0 & 15 \\ 0 & 0 & -1 & 20 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -5 \\ 0 & 0 & -1 & 19 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 30 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Object grasping by the robot:

Tool (gripper) orientation

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$



$$T_{base}^{tool} = \begin{bmatrix} 1 & 0 & 0 & 30 \\ 0 & -1 & 0 & 15 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Suppose the robot is Rhino XR3, then

$$P_x / P_y = C_1 / S_1$$

$$\theta_1 = a\tan 2(P_y, P_x) \Rightarrow a\tan 2(15{,}30) = 26.5$$

$$a_x = -C_1 S_{234} \quad \Rightarrow a_y = -S_1 S_{234};$$

$$S_{234} = -(a_x C_1 + a_y S_1); \quad C_{234} = -a_z$$

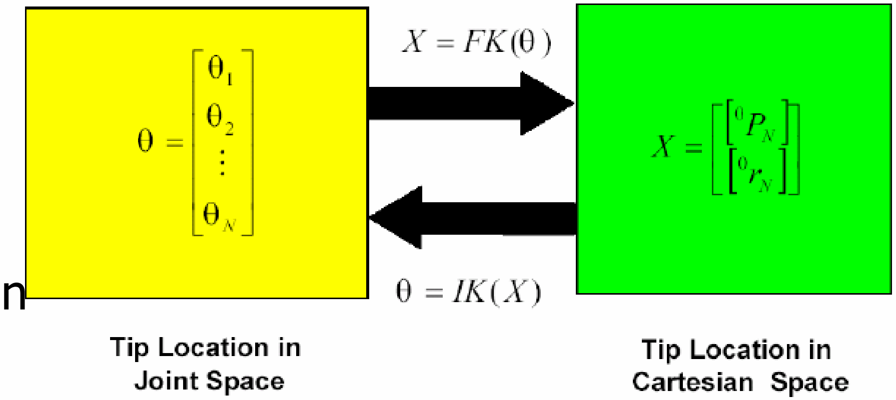
$$\theta_5 = a\tan 2(n_x S_1 - n_y C_1, s_x S_1 - s_y C_1)$$

$$a\tan 2(S_1, C_1) = 26.5$$

Differential Motion and Statics

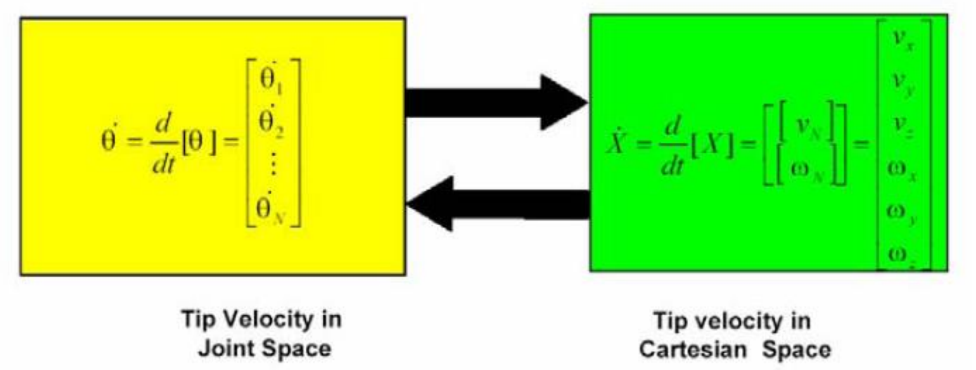
- Tool configuration and Joint Space velocity
- Jacobian
  - Tool configuration Jacobian
  - Manipulator Jacobian
- Singularity
  - Boundary Singularity
  - Interior Singularity
- Generalised inverse
- Pseudo Inverse
- Statics
- Examples

Differential relationship



- Robot path planning problem is formulated in tool-configuration space
- Robot motion is controlled at the joint space

$$x = w(q); \quad x = \text{tool configuration vector,}$$
$$\text{w = tool-configuration function and q = joint variables}$$



Differential relationship

$\dot{x} = J(q)\dot{q}; \quad J(q)$ is a 6xn matrix and is called the Jacobian matrix or Jacobian

$$J_{kj}(q) = \frac{\partial w_k(q)}{\partial q_j} \quad 1 \le k \le 6, \quad 1 \le j \le n$$

$$X = wq \Rightarrow$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \frac{\partial w_1}{\partial q_1} & \frac{\partial w_1}{\partial q_2} & \frac{\partial w_1}{\partial q_3} & . & \frac{\partial w_1}{\partial q_n} \\ \frac{\partial w_2}{\partial q_1} & \frac{\partial w_2}{\partial q_2} & \frac{\partial w_2}{\partial q_3} & . & \frac{\partial w_2}{\partial q_n} \\ \frac{\partial w_3}{\partial q_1} & \frac{\partial w_3}{\partial q_2} & \frac{\partial w_3}{\partial q_3} & . & \frac{\partial w_3}{\partial q_n} \\ \frac{\partial w_4}{\partial q_1} & \frac{\partial w_4}{\partial q_2} & \frac{\partial w_4}{\partial q_3} & . & \frac{\partial w_4}{\partial q_n} \\ \frac{\partial w_5}{\partial q_1} & \frac{\partial w_5}{\partial q_2} & \frac{\partial w_5}{\partial q_3} & . & \frac{\partial w_5}{\partial q_n} \\ \frac{\partial w_6}{\partial q_1} & \frac{\partial w_6}{\partial q_2} & \frac{\partial w_6}{\partial q_3} & . & \frac{\partial w_6}{\partial q_n} \end{bmatrix} . \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{.} \\ \dot{q}_n \end{bmatrix}$$

$$6 \; X \; 1 \qquad\qquad 6 \; X \; n \qquad\qquad n \; X \; 1$$

For a rotary manipulator,

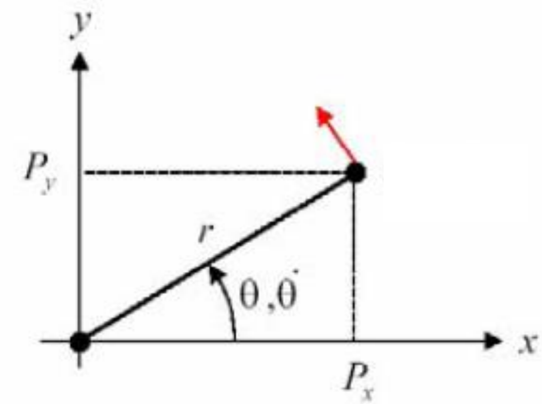
$$\dot{x} = [J(\theta)]\dot{\theta}$$

$$\dot{\theta} = [J(\theta)]^{-1}\dot{x}$$

Calculation of Jacobian

- Get the Forward Kinematics relationship, **X**= w(θ)
- Differentiate **X** wrt θ

<u>Example: Planar 1R robot</u>



The end effector position is given by

$$P_x = r\cos\theta$$

$$P_y = r\sin\theta$$

End effector velocity is given by

$$\dot{P}_x = \dot{x} = -r\sin\theta.\dot{\theta}$$

$$\dot{P}_y = \dot{y} = r\cos\theta.\dot{\theta}$$

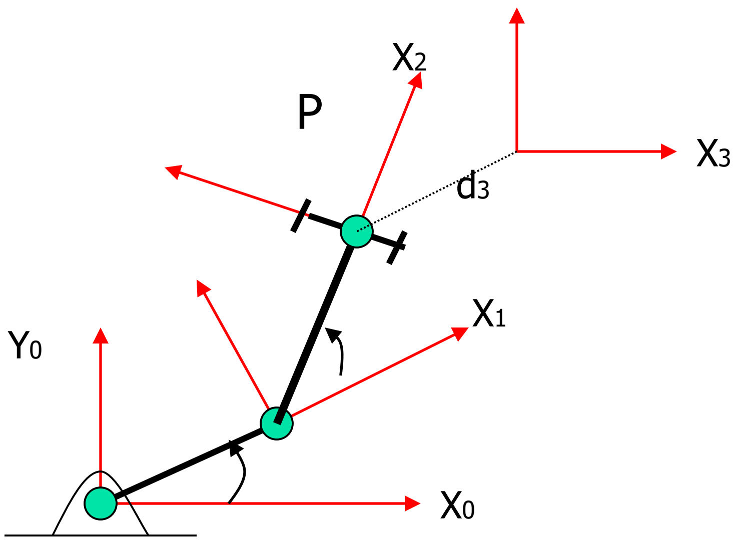
$$\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\theta} \end{bmatrix}$$

Example: 3R Planar Manipulator

$$Px = l_1C_1 + l_2C_{12}$$

$$Py = l_1S_1 + l_2S_{12}$$

$$Pz = d_3$$



$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_x}{\partial \theta_2} & \frac{\partial p_x}{\partial \theta_3} \\ \frac{\partial p_y}{\partial \theta_1} & \frac{\partial p_y}{\partial \theta_2} & \frac{\partial p_y}{\partial \theta_3} \\ \frac{\partial p_z}{\partial \theta_1} & \frac{\partial p_z}{\partial \theta_2} & \frac{\partial p_z}{\partial \theta_3} \end{bmatrix} \cdot \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

$3\,X\,1 \qquad 3\,X\,3 \qquad 3\,X\,1$

$$T_{base}^{tool} = \begin{bmatrix} C_{123} & -S_{123} & 0 & l_1C_1 + l_2C_{12} \\ S_{123} & C_{123} & 0 & l_1S_1 + l_2S_{12} \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

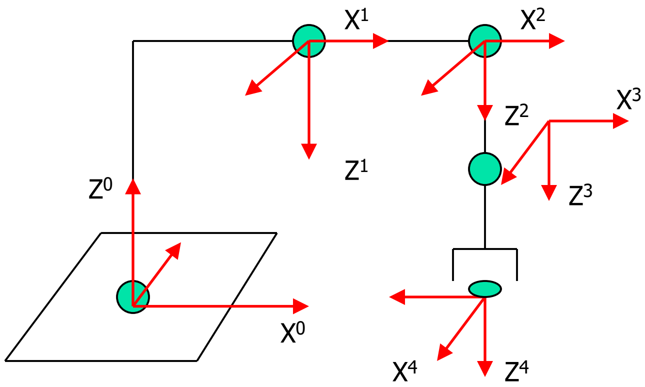
$$J = \begin{bmatrix} -l_1S_1 - l_2S_{12} & -l_2S_{12} & 0 \\ l_1C_1 + l_2C_{12} & l_2C_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example: Four Axis SCARA Manipulator (Adept 1)

$$Px = l_1 C_1 + l_2 C_{1-2}$$

$$Py = l_1 S_1 + l_2 S_{1-2}$$

$$Pz = d_1 - q_3 - d_4$$



$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_x}{\partial \theta_2} & \frac{\partial p_x}{\partial q_3} & \frac{\partial p_x}{\partial \theta_4} \\ \frac{\partial p_y}{\partial \theta_1} & \frac{\partial p_y}{\partial \theta_2} & \frac{\partial p_y}{\partial q_3} & \frac{\partial p_y}{\partial \theta_4} \\ \frac{\partial p_z}{\partial \theta_1} & \frac{\partial p_z}{\partial \theta_2} & \frac{\partial p_z}{\partial q_3} & \frac{\partial p_z}{\partial \theta_4} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{q}_3 \\ \dot{\theta}_4 \end{bmatrix}$$

$3\,X\,1 \qquad 3\,X\,4 \qquad 4\,X\,1$

$$J = \begin{bmatrix} -l_1 S_1 - l_2 S_{1-2} & -l_2 S_{1-2} & 0 & 0 \\ l_1 C_1 + l_2 C_{1-2} & l_2 C_{1-2} & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Singularities

The joint space velocity is given as $\dot{\theta} = [J(\theta)]^{-1} \dot{x}$

One of the potential problems with solving for joint space velocity is the non-existence of inverse. The Jacobian may not be invertible for all the values of θ.

At certain points in joint space, Jacobian loses its rank; i.e. there is a reduction in no. of independent rows and columns. The points at which the Jacobian loses rank are called *Joint space Singularities*.

NOTE: The Jacobian Matrix J(q) is of full rank as long as q is not a joint space singularity.

Manipulator dexterity, dex(q)=det[$\mathbf{J}^T\mathbf{J}$] n<=6

For the general case n<=6, the tool Jacobian matrix is less than full rank if and only if the nXn matrix $\mathbf{J}^T\mathbf{J}$ is singular.

For redundant manipulators (n>6), determinant of the 6X6 matrix, $\mathbf{J}\mathbf{J}^T$ must be used.

A manipulator is at joint space singularity if and only if dex(q)=0.

Boundary singularity occurs when the tool tip is on the surface of the work envelope.

Boundary Singularity of SCARA

$$J = \begin{bmatrix} -l_1S_1 - l_2S_{1-2} & -l_2S_{1-2} & 0 & 0 \\ l_1C_1 + l_2C_{1-2} & l_2C_{1-2} & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\begin{aligned} dex = \det(J^T J) &= (-l_1S_1 - l_2S_{1-2}).(-l_2C_{1-2}) - l_2S_{1-2}(l_1C_1 + l_2C_{1-2}) \\ &= l_1l_2(S_1C_{1-2} - C_1S_{1-2}) \\ &= l_1l_2S_2 \end{aligned}$$

dex() =0 iff $S_2=0$; $\Longrightarrow$ $\theta_2=0,\pi$

When $\theta_2=0$, the arm is fully stretched and the tip is on the surface of the work envelope.

Interior Singularity: Potentially troublesome; formed when two or more axes form a straight line. The effects of rotation about one axis may be cancelled due to a counteracting rotation about the other axis.

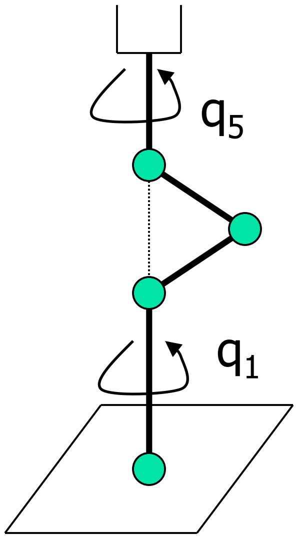
Tool configuration may remain the same even though the robot moves in joint space.

Example: Microbot-AlphaII

Consider the following locus of points in Joint space

$q(\beta)=[q_1,-\beta,\ 2\beta-\pi,\ -\beta,\ q_5]$ $\quad 0<\beta<\pi/2$

If $a_3=a_2$ and $a_4=0$, then $J(q)$ loses full rank along the line $q=q(\beta)$ and $q(\beta)$ represents interior singularities for the articulated robot.



<u>Exercise:</u> For the 3 axis planar robot, show that if a2=a1, then q=[q1,π,q3] is a locus of singularities. Which axes are collinear in this case?

Generalised Inverses

$$\dot{\theta} = [J(\theta)]^{-1} \dot{x}$$

Inverse is not defined when the no. of axes n is arbitrary.

Generalised Inverse: If A is an mxn matrix, then an nxm matrix X is a generalised inverse of A if and only if it satisfies at least property 1 or 2 of the following list of properties:

1. AXA=A
2. XAX=X
3. $(AX)^T=AX$
4. $(XA)^T=XA$

Most well known generalised Inverse is Moore-Penrose Inverse or **Pseudo Inverse (A^+) which satisfies all 4 properties. If A is of full rank then,**

$$A^+ = A^T(AA^T)^{-1} \quad m \le n$$

$$= A^{-1} \quad m = n$$

$$= (A^T A)^{-1} A^T \quad m \ge n$$

Example:

Find the pseudo Inverse of

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

The rank of A is 2.

$$A^{+} = A^{T}(AA^{T})^{-1} \longrightarrow A^{+} = \frac{1}{9}\begin{bmatrix} 1 & 4 \\ 1 & -5 \\ 4 & -2 \end{bmatrix}$$

<u>Resolved Motion Rate Control:</u>

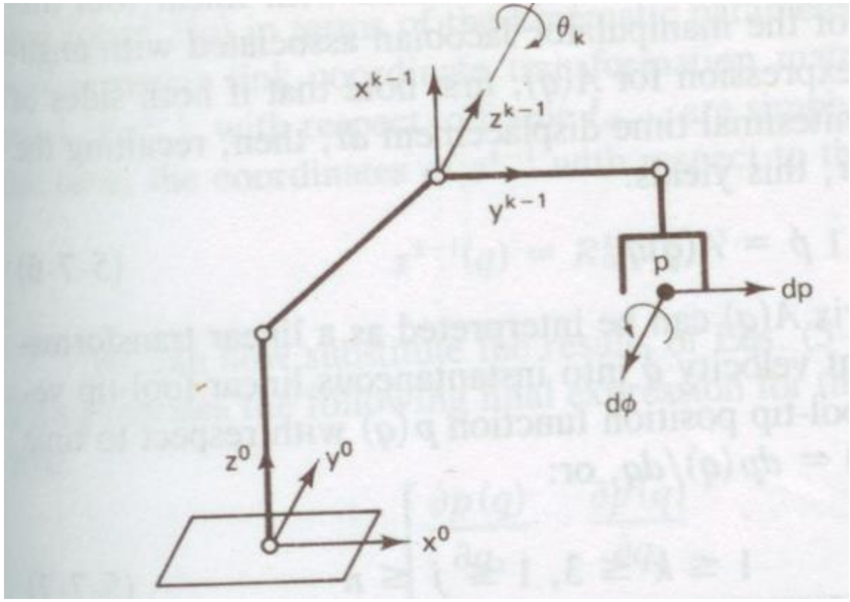
If x(t) be a differentiable tool configuration trajectory which lies inside the work envelope and which does not go through any workspace singularities, and J(q) is the 6xn tool-configuration Jacobian matrix where n<=6, then the joint space trajectory q(t) corresponding to x(t) can be obtained by solving the following non-linear differential equation:

$$\dot{q} = [J(q)^{T} J(q)]^{-1} J^{T}(q)\,\dot{x}$$

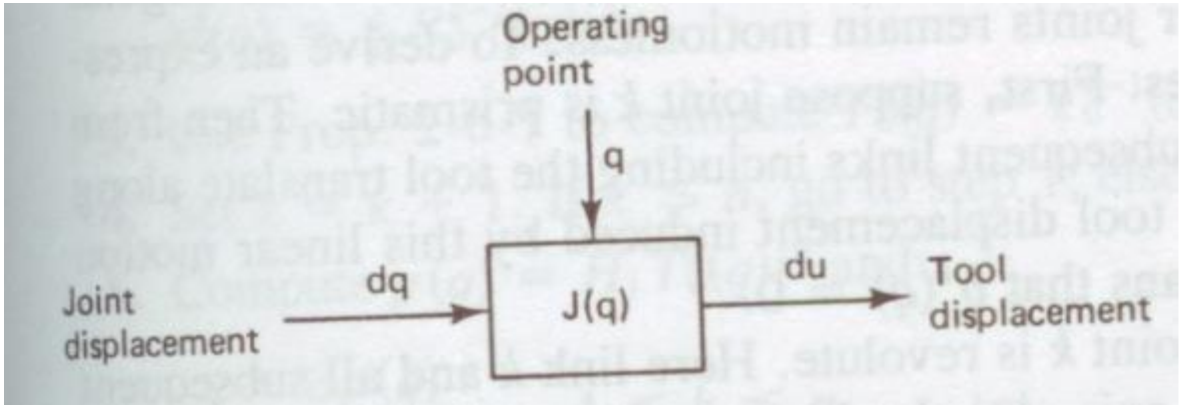
$$= J(q)^{+}\,\dot{x}$$

Introduced by Whitney in 1969, this method is known as Resolved motion rate Control . The motion in tool-configuration is resolved into joint space components

- Analysis of manipulator in static equilibrium



Manipulator Jacobian



Computational Method for Jacobian

$$J(q) = \begin{bmatrix} A(q) \\ \hdashline B(q) \end{bmatrix}$$

A(q) : Jacobian associated with linear tool displacement

B(q) : Jacobian associated with angular tool displacement

$$\mathrm{dp} = \mathrm{A(q)dq}$$
$$\mathrm{d}\phi = \mathrm{B(q)dq}$$

$$A_{kj}(q) = \frac{\partial p_k(q)}{\partial q_j} \qquad 1 \le \mathrm{k} \le 3, 1 \le j \le \mathrm{n}$$

$$\text{The } \mathrm{k}^{\text{th}} \text{ column of } \mathrm{B(q)} = b^k(q) = \xi\, z^{k-1}(q);$$

$$\xi = 0 \text{ if joint k is prismatic, 1 if k is rotary}$$

$$\mathrm{Z}^{\text{k-1}} = R_0^{k-1}(q) i^3; \quad 1 \le k \le n$$

$$J(q) = \left[\begin{array}{cccc} \dfrac{\partial p(q)}{\partial q_1} & \dfrac{\partial p(q)}{\partial q_2} & . & \dfrac{\partial p(q)}{\partial q_n} \\ \hdashline \xi_1 z^0(q) & \xi_2 z^1(q) & . & \xi_n z^{n-1}(q) \end{array}\right] = \begin{bmatrix} A(q) \\ \hdashline B(q) \end{bmatrix}$$

Algorithm:

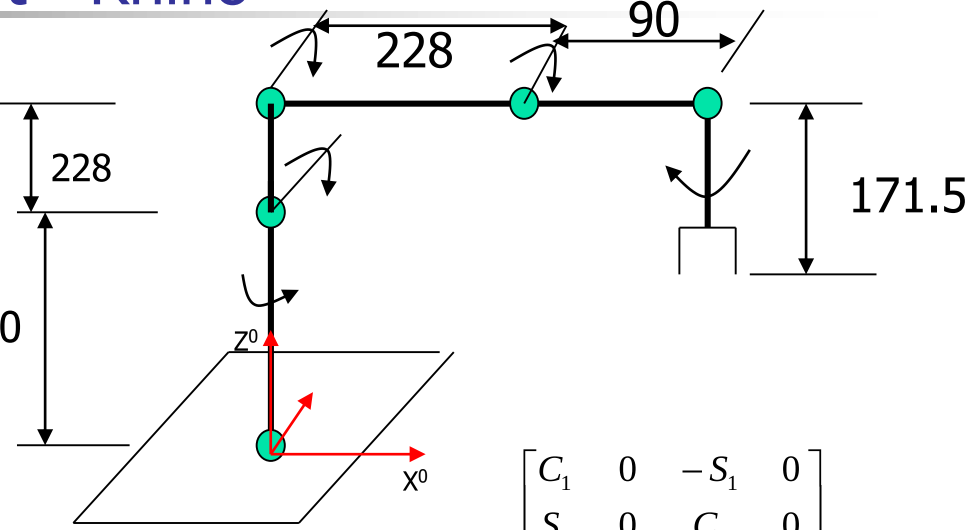
- Set T_0^0=I, k=1
- Compute $b^k(q)$= $\xi_k R_0^{k-1}(q) i^3$
- Compute $T_0^k(q)$
- Set k=k+1. If k<=n, go to step 1; else continue
- Compute p(q) and

$$a^k = \frac{\partial p(q)}{\partial q_k} \quad 1 \le k \le n$$

- Form J(q)

Example: Five Axis Robot - Rhino

Compute $T_0^{\ i}$ for i=1,..5, get $R_0^{\ i}$



$$P_x = C_1\left(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}\right)(1)$$

$$P_y = S_1\left(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}\right)(2)$$

$$Pz = d_1 - a_2S_2 - a_3S_{23} - a_4S_{234} - d_5C_{234}(3)$$

$$T_0^1 = \begin{bmatrix} C_1 & 0 & -S_1 & 0 \\ S_1 & 0 & C_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$b^1(q) = R_0^{\ 0}\ i^3 = i^3$

From $T_0^{\ 1}$,

$$j^1(q) = \begin{bmatrix} -S_1\left(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}\right) \\ C_1\left(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}\right) \\ 0 \\ \hdashline 0 \\ 0 \\ 1 \end{bmatrix}$$

$$j^2(q) = \begin{bmatrix} -C_1\left(a_2S_2 + a_3S_{23} + a_4S_{234} + d_5C_{234}\right) \\ -S_1\left(a_2S_2 + a_3S_{23} + a_4S_{234} + d_5C_{234}\right) \\ -a_2C_2 - a_3C_{23} - a_4C_{234} + d_5S_{234} \\ -S_1 \\ C_1 \\ 0 \end{bmatrix}$$

From T_0^2,

$$j^3(q)=\begin{bmatrix}-C_1\left(a_3S_{23}+a_4S_{234}+d_5C_{234}\right)\\-S_1\left(a_3S_{23}+a_4S_{234}+d_5C_{234}\right)\\-a_3C_{23}-a_4C_{234}+d_5S_{234}\\-S_1\\C_1\\0\end{bmatrix}$$

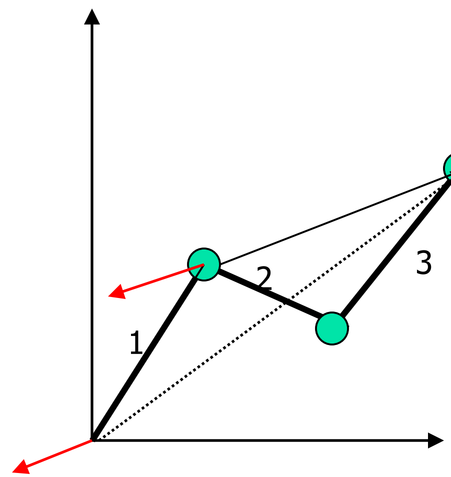
$$j^4(q)=\begin{bmatrix}-C_1\left(a_4S_{234}+d_5C_{234}\right)\\-S_1\left(a_4S_{234}+d_5C_{234}\right)\\-a_4C_{234}+d_5S_{234}\\-S_1\\C_1\\0\end{bmatrix}$$

$$j^5(q)=\begin{bmatrix}0\\0\\0\\-C_1S_{234}\\-S_1S_{234}\\-C_{234}\end{bmatrix}$$

$$J(q)=\left[j^1(q)\quad j^2(q)\quad j^3(q)\quad j^4(q)\quad j^5(q)\right]$$

Numerical computation of Jacobian

Jacobian can be calculated as the cross product of the joint axis vector and the position vector of the joint to end effector



$$J = \begin{bmatrix} J_1 & J_2 & .. & J_n \end{bmatrix}$$

$$J_i = \begin{bmatrix} b^{i-1} \text{ x } \left[P_0^n - P_0^{i-1} \right] \\ b^{i-1} \end{bmatrix}$$

$$\text{where} \quad b^i = R_0^i [0 \quad 0 \quad 1]^T$$

For 3 dof manipulator

$$J_1 = \begin{bmatrix} b^0 \text{ x } \left[P_0^3 \right] \\ b^0 \end{bmatrix}$$

$$\text{where} \quad b^0 = R_0^0 [0 \quad 0 \quad 1]^T$$

$$J_2 = \begin{bmatrix} b^1 \text{ x } \left[P_0^3 - P_0^1 \right] \\ b^1 \end{bmatrix}$$

$$\text{where} \quad b^1 = R_0^1 [0 \quad 0 \quad 1]^T$$

Home work: Write a program for numerical computation of Jacobian for an n-dof manipulator

- The relationship between the robot joint torques and the forces and moments at the robot end effector (static conditions):

- This relationship is given by:

$$\underline{\tau} = J(\underline{\theta})^T \underline{F}$$

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- This expression can be expanded to:

$$\begin{array}{ccc} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \cdots \\ \tau_N \end{bmatrix} = & \begin{bmatrix} \\ & J(\underline{\theta}) & \\ \\ \end{bmatrix}^T & \begin{bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{bmatrix} \\ \textbf{Nx1} & \textbf{Nx6} & \textbf{6x1} \end{array}$$

Force Mapping and Singularities

- The relationship between joint torque and end effector force and moments is given by:

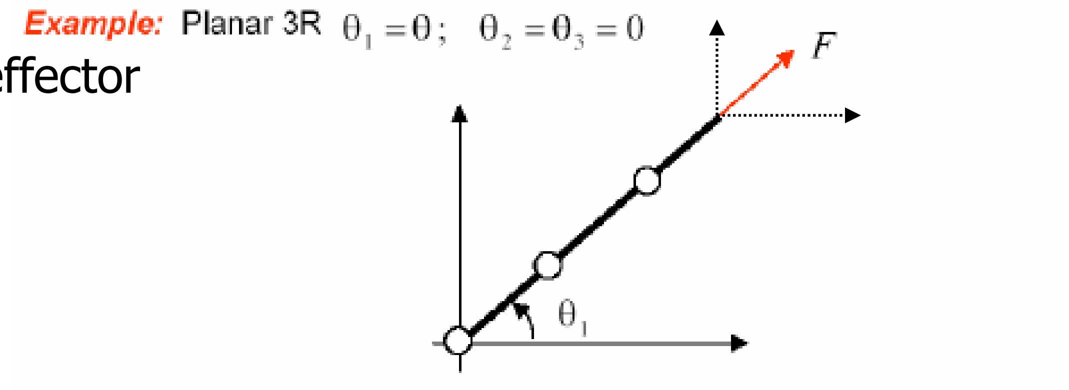
$$\underline{\tau} = J(\underline{\theta})^T \underline{F}$$

- The rank of $J(\theta)^T$ is equal to the rank of $J(\theta)$
- At a singular configuration there exists a non trivial force ***F*** such that

$$J(\underline{\theta})^T \underline{F} = 0$$

- In other words, a finite force can be applied to the end effector that produces no torque at the robot's joints. In the singular configuration, the manipulator can "lock up."

The force acting on the end effector relative to the base frame is



$$F_0 = \begin{bmatrix} FC_1 \\ FS_1 \\ 0 \end{bmatrix}$$

$$\tau_0 = J^T(\theta)F_0 = \begin{bmatrix} -L_1S_1 - L_2S_1 - L_3S_1 & L_1C_1 + L_2C_1 + L_3C_1 & 1 \\ -L_2S_1 - L_3S_1 & L_2C_1 + L_3C_1 & 1 \\ -L_3S_1 & L_3C_1 & 1 \end{bmatrix} \begin{bmatrix} FC_1 \\ FS_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -FS_1C_1(L_1 + L_2 + L_3) + FS_1C_1(L_1 + L_2 + L_3) \\ -FS_1C_1(L_2 + L_3) + FS_1C_1(L_2 + L_3) \\ -FC_1L_3S_1 + FS_1L_3C_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Summary

- Inverse Kinematics
- Tool configuration and Joint Space velocity
- Jacobian
- Singularity
  - Boundary Singularity
  - Interior Singularity
- Generalised inverse
- Pseudo Inverse
- Statics