1. $D_{in} = D_{out} = k\mathbb{1}_n$ for all vertices
$A = A^T$ as G is undirected } Given

a) Let $\mathbb{1}_n$ be an eigenVector of A

$$A\mathbb{1}_n = \lambda \mathbb{1}_n$$

we know $A\mathbb{1}_n = D_{out}$

$\therefore$ $D_{out} = \lambda \mathbb{1}_n$

$\Rightarrow$ $k\mathbb{1}_n = \lambda \mathbb{1}_n$, since $D_{out} = k\mathbb{1}_n$ is given

$\therefore$ $\lambda = k$ //, $\therefore$ $\mathbb{1}_n$ is an eigen Vector of A as a λ exist.

b) for a symmetric matrix eigen Vectors are orthogonal.

We already know $\mathbb{1}_n$ is an eigen Vector of A.
Let's say v is an eigenVector of A other than $\mathbb{1}_n$
then $v \cdot \mathbb{1}_n = 0$, $v = [v_1 \ v_2 \ \dots \ v_n]$

$\Rightarrow$ $v_1 + v_2 + \dots + v_n = 0 \longrightarrow (1)$

elements of V are $V_1, V_2, \ldots, V_n$

from (1) we can infer that

all the elements of V cannot be strictly positive as the summation of all positive numbers result in a positive value, but our result is Ohm

hence proved.

c) $\max(Ax) = k \cdot x_{max}$ because k edges are contributing towards the vector multiplication of A: and x, Hence k is the leading eigenvalue

We set: $x_1 = x_2 = \cdots = x_n = x$ and $c = 1$

and α as the attenuating factor

we can write

$$x = \alpha k x + 1 \Rightarrow x = \frac{1}{1-\alpha k}$$

$\therefore$ The katz Centrality increases as k increases, All nodes have equal Centrality (both eigenvector and katz) due to symmetry.

2. $\alpha =]0,1[\longrightarrow 0 < \alpha < 1$

$$x_1(k+1) = x_1(k)$$
$$x_2(k+1) = \alpha x_1(k) + (1-\alpha) x_2(k)$$

a) $x(k+1) = A \cdot x(k)$

$$\Rightarrow \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ \alpha & 1-\alpha \end{bmatrix}}_{A} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$A\mathbb{1}_n = \begin{bmatrix} 1 & 0 \\ \alpha & 1-\alpha \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbb{1}_n$$

$\therefore$ A is row - stochastic

b) $|A - \lambda I| = 0$

$$\left| \begin{bmatrix} 1-\lambda & 0 \\ \alpha & 1-\alpha-\lambda \end{bmatrix} \right| = 0$$

$$(1-\lambda)(1-\alpha-\lambda) = 0$$

$\therefore \lambda = 1, 1-\alpha$

eigen - values are 1 & $1-\alpha$

eigen vector for $\lambda = 1$

$$\begin{bmatrix} 1 & 0 \\ \alpha & 1-\alpha \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Scanned with CamScanner

$$x_1 = x_1, \quad \alpha x_1 + (1-\alpha) x_2 = x_2$$

$$\Rightarrow \quad \alpha x_1 = \alpha x_2$$

$$x_1 = x_2$$

eigen vector $= [x_1, x_1] \Rightarrow [1, 1]$

eigen vector for $\lambda = 1-\alpha$

$$\begin{bmatrix} 1 & 0 \\ \alpha & 1-\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (1-\alpha) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1 = (1-\alpha) x_1$$

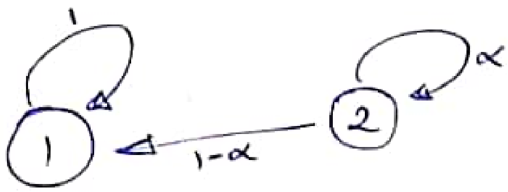
$$\Rightarrow x_1 = 0$$

$$\cancel{\alpha x_1}^{\,0} + x_2 - \alpha x_2 = x_2 - \alpha x_2$$

$$\Rightarrow x_2 = x_2$$

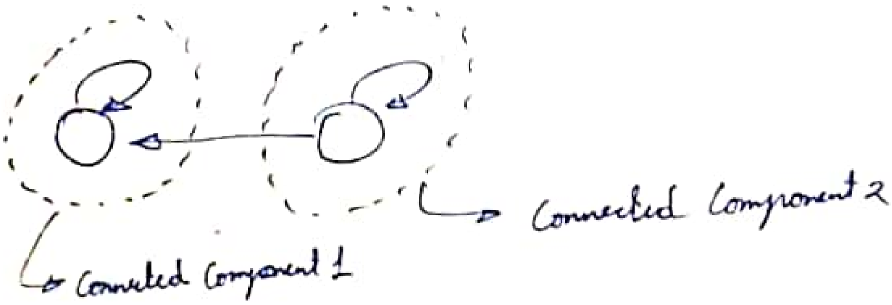
eigen vector $= [0, x_2] \Rightarrow [0, 1]$

c)



The graph is not strongly connected

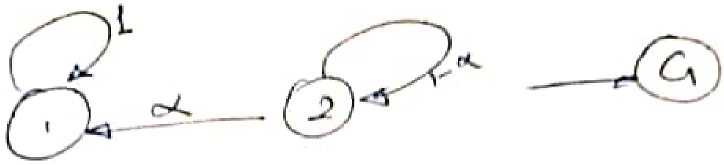
d)



$\Rightarrow$

2. c) $x_1(k+1) = x_1(k)$

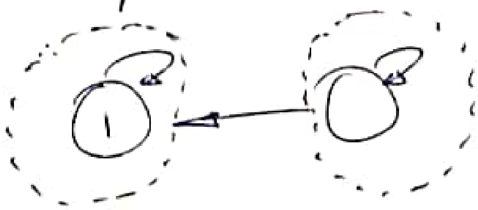
$x_2(k+1) = \alpha x_1(k) + (1-\alpha) x_2(k)$



$$A = \begin{bmatrix} 1 & 0 \\ \alpha & 1-\alpha \end{bmatrix}$$

$\longrightarrow$ row-stochastic matrix

Condensed Graph



sink

Using theorem 5.1

Consensus of row-stochastic matrices with a globally-reachable aperiodic strongly-connected component

then $\lim_{k \to \infty} A^k = \mathbb{1}_n \omega^T$

where ω is the left dominant eigenvector of A

$$\omega^T A = \lambda \omega^T$$

$$\omega^T [A - \lambda I] = 0 \Rightarrow |A - \lambda I| = 0$$

$$\begin{bmatrix} 1 & 0 \\ \alpha & 1-\alpha \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\begin{bmatrix} 1-\lambda & 0 \\ \alpha & 1-\alpha-\lambda \end{bmatrix} = 0$$

Scanned with CamScanner

$$(1-\lambda)(1-\alpha-\lambda) = 0$$

$$\boxed{\lambda = 1,\ 1-\alpha}$$

1 is greater than $1-\alpha$ as $0<\alpha<1$

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha & 1-\alpha \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix}$$

$$\begin{bmatrix} a+b\alpha & b(1-\alpha) \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix}$$

$$a + b\alpha = a \longrightarrow (1)$$

$$b(1-\alpha) = b \longrightarrow (2)$$

$$(2) \Rightarrow b = 0$$

$$(1) \Rightarrow a + 0 = a \Rightarrow a \in \mathbb{R}$$

$$\lim_{k\to\infty} A^k = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\lim_{k\to\infty} x(k) = \lim_{k\to\infty} A^k \cdot x(0)$$

$$= a \cdot x_1(0) + 0 \cdot x_2(0)$$

$$= a \cdot x_1(0)$$

$$\cancel{\lim_{k\to\infty} x(k) = a \begin{bmatrix} x_1(0) \\ 0 \end{bmatrix}}$$

$$\lim_{k \to \infty} x(k) = a \begin{bmatrix} x_1(0) \\ x_1(0) \end{bmatrix}, \text{ let } a \text{ be } 1$$

without using theorem 5.1

$$x(1) = \begin{bmatrix} 1 & 0 \\ \alpha & 1-\alpha \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$= \begin{bmatrix} x_1(0) \\ \alpha x_1(0) + (1-\alpha) x_2(0) \end{bmatrix}$$

$$x(2) = \begin{bmatrix} 1 & 0 \\ \alpha & 1-\alpha \end{bmatrix} \begin{bmatrix} x_1(0) \\ \alpha x_1(0) + (1-\alpha) x_2(0) \end{bmatrix}$$

$$= \begin{bmatrix} x_1(0) \\ (\alpha + (1-\alpha)\alpha) x_1(0) + (1-\alpha)^2 x_2(0) \end{bmatrix}$$

$$\lim_{k \to \infty} x(k) = \begin{bmatrix} x_1(0) \\ \alpha[1 + (1-\alpha) + (1-\alpha)^2 + \cdots] x_1(0) + (1-\alpha)^k x_2(0) \end{bmatrix}$$

2nd term

$$\lim_{k \to \infty} (1-\alpha)^k = 0 \quad \text{as } \boxed{0 < 1-\alpha < 1}$$

1st term

$$\lim_{k \to \infty} \alpha[1 + (1-\alpha) + (1-\alpha)^2 + \cdots] = \alpha \cdot \frac{1}{1-(1-\alpha)}$$

$$= \frac{\alpha}{\alpha} = 1$$

$$\therefore \lim_{k \to \infty} x(k) = \begin{bmatrix} x_1(0) \\ x_1(0) \end{bmatrix}$$

3. $x(k+1) = A\,x(k)$

$\sum_j a_{ij} = 1 \longrightarrow$ row - stochastic

a) same weightage to every node's opinion

$$A = \begin{bmatrix} 1/n & \frac{1}{n} & \cdots & \frac{1}{n} \\ 1/n & \ddots & & \\ \vdots & & \ddots & \\ 1/n & \cdots & & \end{bmatrix}_{n \times n}$$

every entry is $1/n$ as we need to satisfy

$$\sum_j a_{ij} = 1$$

b) $k^{th} \longrightarrow$ weights are equal

others $\longrightarrow$ $\frac{1}{2}$ for their opinion, $\frac{1}{2}$ for k^{th} opinion

$$A = \begin{bmatrix} 1/2 & 0 & 0 & \cdots & 1/2 & 0 & 0 & \cdots & 0 \\ 0 & 1/2 & 0 & \cdots & 1/2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 1/n & 1/n & 1/n & \cdots & 1/n & 1/n & 1/n & \cdots & 1/n \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1/2 & 0 & 0 & \cdots & 1/2 \end{bmatrix}_{n \times n}$$

c) 1^{st} node $\longrightarrow$ only values it's own opinion

others $\longrightarrow$ $1/2$ for their opinion,

$1/2$ for the 1^{st} node's opinion

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1/2 & 1/2 & 0 & \cdots & 0 \\ 1/2 & 0 & 1/2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/2 & 0 & 0 & \cdots & 1/2 \end{bmatrix}_{n \times n}$$

Theorem 4.7

3.d) last part

Primitive → strongly connected, aperiodic

Theorem 2.12

$\lambda = 1$ is simple and strictly dominant

for primitive row-stochastic matrix

case (a) :

1. every node is connected to every other node by definition, with self loops.

2. due to self loops, our graph is aperiodic

3. our graph is strongly connected

$\therefore \lambda = 1$ is dominant

$$\left[\begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots \\ \vdots & & \\ \vdots & & \end{bmatrix} \right. = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}$$

$$\begin{bmatrix} \frac{w_1 + \cdots w_n}{n} & \frac{w_1 + \cdots + w_n}{n} & \cdots & \frac{w_1 + \cdots + w_n}{n} \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}$$

$\therefore w_1 = w_2 \cdots\cdots = w_n = 1$!

$$\lim_{k\to\infty} A^k = 1_n \omega^T$$

$$= \begin{bmatrix} 1 \\ \vdots \\ \vdots \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & \cdots & \cdot \\ 1 & \cdot & \cdots & \cdot \\ \vdots & \vdots & & \end{bmatrix}$$

Case (b):

~~k^{th} node is the sink~~

1. we can go to any node from k^{th} node ~~and~~
2. from any node we can go to k^{th} node
3. all nodes have self loops

1, 2, 3 are from definition

3 proves aperiodicity

1 & 2 proves strongly connected

$\therefore \lambda = 1$ is dominant

$$\begin{bmatrix} \omega_1 & \omega_2 & \cdots & \omega_n \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \cdots & \frac{1}{2} & 0 & \cdots \\ 0 & \frac{1}{2} & \cdots & \frac{1}{2} & 0 & \cdots \\ \vdots & & & & & \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & & \cdots \\ 0 & \cdots & & -\frac{1}{2} & \cdots & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \omega_1 & \omega_2 & \cdots & \omega_n \end{bmatrix}$$

all nodes except k^{th} node

$$\frac{1}{2}\omega_i + \omega_k \cdot \frac{1}{n} = \omega_i$$

$$\frac{\omega_k}{n} = \frac{\omega_i}{2} \longrightarrow (1)$$

k^{th} ~~node term~~ column

$$\frac{\omega_1 + \cdots \omega_n}{2} + \frac{\omega_k}{n} = \omega_k \longrightarrow (2)$$

from (1)

$$(2) \Rightarrow \frac{(n-1)\omega_k}{n} + \frac{\omega_k}{n} = \omega_k$$

$$\omega_k = \omega_k$$

$$\therefore \omega^T = \left[\frac{2\omega_k}{n} \quad \frac{2\omega_k}{n} \quad \cdots \quad \underbrace{\omega_k}_{k\text{th term}} \quad \cdots \quad \frac{2\omega_k}{n}\right]$$

$$\lim_{k \to \infty} A^k = \mathbb{1}_n \omega^T$$

$$= \begin{bmatrix} 1 \\ \vdots \\ \vdots \end{bmatrix} \left[\frac{2\omega_k}{n} \quad \frac{2\omega_k}{n} \quad \cdots \quad \omega_k \quad \cdots \quad \frac{2\omega_k}{n}\right]$$

$$= \omega_k \begin{bmatrix} \frac{2}{n} & \frac{2}{n} & \cdots & 1 & \cdots & \frac{2}{n} \\ 2/n & \frac{2}{n} & \cdots & 1 & \cdots & \frac{2}{n} \\ \vdots & & & & & \\ 2/n & \frac{2}{n} & \cdots & 1 & \cdots & \frac{2}{n} \end{bmatrix}, \omega_k \in \mathbb{R}$$

Case C :

1. node 1 has self loop and it's a sink
2. all ~~other~~ nodes can reach 1^{st} node
3. all nodes have self loop

using Theorem 5.1

1. A is row stochastic
2. 1^{st} node is globally reachable
3. all nodes have self loops, therefore aperiodic

$\therefore$ A is semi-convergent & $\lim_{k \to \infty} A^k = \mathbb{1}_n \omega^T$

dominant eigen value $= 1$

$$[\omega_1 \;\; \omega_2 \cdots \omega_n] \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 1/2 & 1/2 & \cdots & \cdots & 0 \\ 1/2 & 0 & 1/2 & \cdots & 0 \\ 1/2 & \cdots & \cdots & \cdots & 1/2 \end{bmatrix} = [\omega_1 \cdots \omega_n]$$

at 1^{st} term

$$\omega_1 + \frac{\omega_2 + \cdots \omega_n}{2} = \omega_1$$

$$\Rightarrow \omega_2 + \cdots + \omega_n = 0$$

from 2^{nd} to n^{th}

$$\frac{1}{2} \cdot \omega_i = \omega_i$$

$$\therefore \omega_i = 0 \quad \text{for } i = 2, \ldots, n$$

$$\omega^T = [\omega_1 \; 0 \cdots 0]$$

$$\lim_{k \to \infty} A^k = \mathbb{1}_n \omega^T$$

$$= \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \omega_1 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \omega_1 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

5. b) $\text{rank}(L) = n - n_s$

n_s = the number of connected components of the graph.

Proof: Let the incident matrix of G be B

we know the $\dim(\text{nullspace}(B)) = n_s$

Let $x^T B = 0$ where x is a vector

this implies that x takes the same value on all the vertices of the same connected component.

$\therefore \text{rank}(B) = n - n_s$

$L = BB^T$, multiply both sides with v

$$LV = BB^T V$$

if $V^T BB^T V = 0$,

Let $M^T = V^T BB^T$, i.e., $\| M^T V \| = 0$

$\therefore M^T V = 0 \implies MM^T V = 0$

Using Nullity theorem: $\text{rank}(L) = \text{rank}(M)$

& $\text{rank}(M) = \text{rank}(B) = n - n_s$

c). undirected ~~matrix~~ graph is a subset of directed graph, therefore $L = L^T$ which is symmetric

By Lemma 6.2 we know at least one of the eigen values of $L = 0$

By Lemma 6.5, we know all eigen values are non-negative

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

we know $\lambda_1 = 0$ with eigen vector $\mathbb{1}_n$

if $\lambda_2 = 0$, we need to have a eigen vector that is orthogonal to $\mathbb{1}_n$ as L is symmetric

Let the eigen vector of λ_2 be x

$$x \cdot \mathbb{1}_n = 0$$

$$\Rightarrow \quad x_1 + x_2 + \cdots + x_n = 0$$

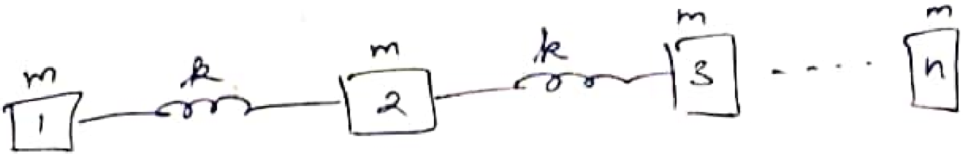
and $\quad L x = 0_n$

$L x$ is 0_n when x is a scalar multiple of $\mathbb{1}_n$ and when G ~~is~~ of L is connected

here x is not a multiple of $\mathbb{1}_n$, therefore we proove that L is not connected.

$\therefore$ undirected G associated to L is connected when $\lambda_2 > 0$

1. Free Network



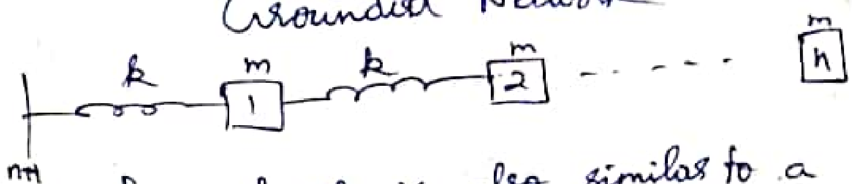
we know that our system is similar to a typical path graph and we know that a typical adjacency matrix to be

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 \cdots & 0 & 0 \\ \vdots & \vdots & & & & \\ 0 & 0 & \cdots & & 1 & 0 \end{bmatrix}_{n \times n}$$

$$\therefore \quad L = D_{out} - A$$

$$\mathcal{L}_{free,n} = \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ \cdot & -1 & 2 & -1 & & \\ & & & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}_{n \times n}$$

Grounded Network



this network is also similar to a Path graph except that 1^{st} noded is connected to the $n+1^{th}$ node instead of the n^{th} node, therefore we may expect little changes on the L of the graph.

$$L_{free,n+1} = \begin{bmatrix} 2 & -1 & 0 & \cdots & & 0 & -1 \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & & \ddots & & & \\ & & & & 2 & -1 & 0 \\ & & & & -1 & 1 & 0 \\ -1 & & & & 0 & 0 & 1 \end{bmatrix}_{n \times n}$$

for $L_{grounded,n}$ we will remove the last row & column

$$L_{grounded,n} = \begin{bmatrix} 2 & -1 & 0 & \cdots & & 0 \\ -1 & 2 & -1 & & & \vdots \\ 0 & -1 & 2 & -1 & & \vdots \\ \vdots & & & \ddots & & \\ \vdots & & & & 2 & -1 \\ 0 & \cdots & & & -1 & 1 \end{bmatrix}_{n \times n}$$

a) $$F_{load} = L_{free,n}\, \underbrace{x}_{\text{displacement}} \longrightarrow (1)$$

we can observe that in $L_{free,n}$ ~~n is column of~~ the summation of the elements of each column is 0

$\therefore \quad \mathbb{1}_n^T (L_{free,n}\, x) = 0$

for existence of equilibrium, we know that the summation of all the elements of the vector

$F_{load} = 0.$

which is same as $\mathbb{1}_n^T (L_{free,n}\, x) = 0.$

$\therefore$ Equilibrium exists for an arbitrary x

∴ which implies that for summation of all the elements of Fload to be 0, we cannot have arbitrary loads.

b) in part a, we proved that

$$1_n^T (L_{free,n}\, x) = 0 \text{ for any displacement } x$$

in part b, given: $1_n F_{load} = 0$

Let's left multiply 1_n^T on both sides of 1

$$1_n^T F_{load} = 1_n^T (L_{free,n}\, x) = 0$$

∴ The resulting equilibrium displacement is not unique, as proved in part a

c) $x = \{[x_1, x_2, \dots, x_n] \mid \text{where } x_i \in \mathbb{R}, i \forall 1, \dots n\}$

d) $$L_{grounded,n} = L_{free} + \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{bmatrix}_{n \times n}$$

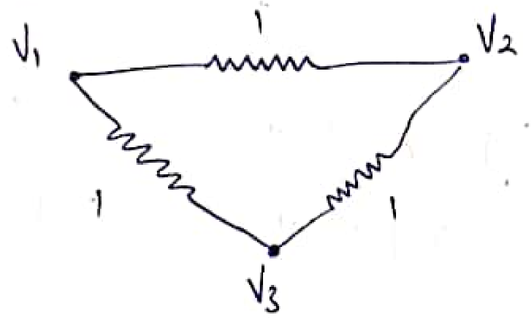
e) we know that $L_{free,n}$ is not invertible as the sum of elements of each of it's column is 0. In $L_{free,n}$ row 1 is -ve of the sum of the all other rows of $L_{free,n}$, this proves linear dependency. In order to ~~mo~~ break this linear dependency, we add 1 to the 1st equation i.e., the 1st row of the matrix. Therefore, now our matrix is Linearly independent.

Our new matrix is $L_{grounded,n}$

due to it's Linear Independency, an inverse exist.

f) $x = L_{grounded}^{-1} F_{load}$

8.



$$P_{dissipated} = V^T L V$$

$$\|V_2\| = 1$$

$$\sqrt{V_1^2 + V_2^2 + V_3^2} = 1$$

$$\Rightarrow V_1^2 + V_2^2 + V_3^2 = 1$$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2V_1 - V_2 - V_3 & 2V_2 - V_1 - V_3 & 2V_3 - V_1 - V_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$= 2V_1^2 - V_2V_1 - V_3V_1 + 2V_2^2 - V_1V_2 - V_3V_2 + 2V_3^2 - V_1V_3 - V_2V_3$$

$$= 2\left[V_1^2 + V_2^2 + V_3^2\right] - 2V_1V_2 - 2V_3V_2 - 2V_1V_3$$

$$= 2 - 2\left[V_1V_2 + V_3V_2 + V_1V_3\right] \longrightarrow (1)$$

we want $(V_1 + V_2 + V_3)^2$ to be minimum and

$$V_1^2 + V_2^2 + V_3^2 = 1$$

$$V_1V_2 + V_2V_3 + V_3V_1 = \frac{1}{2}\left[V_1 + V_2 + V_3\right]^2 - \frac{1}{2}\left[V_1^2 + V_2^2 + V_3^2\right]$$

$$= \frac{1}{2}\left[V_1 + V_2 + V_3\right]^2 - \frac{1}{2}$$

since $\frac{1}{2}\left[V_1 + V_2 + V_3\right]^2 \geq 0$, it's smallest when

$$V_1 + V_2 + V_3 = 0$$

$$\therefore \quad V_1V_2 + V_2V_3 + V_3V_1 \geq -\frac{1}{2}$$

from (1)

$$(1) \Rightarrow 2 - 2\left[-\frac{1}{2}\right] = 2 + 1 = 3$$

$$P_{max} = 3W$$

6. a) Answer understood from "Distributed Control of Robots by F. Bullo".

$$\sum_{j=1}^{n} l_{ij} = d_{out}(v_i) - d_{in}(v_i) \quad \forall \quad i \in \{1, 2, \dots, n\}$$

$$\mathbb{1}_n^T L = \mathbb{0}_n$$

when $D_{out} = D_{in}$

Let $L^T \mathbb{1}_n = \mathbb{0}_n^T$, Consider the system $\dot{v}(t) = -L(G)v$.
& $v(0) = x_0$

together with the +ve definite function $V : \mathbb{R}^n \rightarrow \mathbb{R}$
defined by $V(x) = x^T x$.

The ~~de~~ derivative of V along $x \rightarrow -L(G)x$ as

$\dot{V}(x) = -2x^T L(G) x$, we know $\mathbb{1}_n^T L = \mathbb{0}_n^T$

$$\therefore \quad L\mathbb{1}_n = \mathbb{0}_n$$

Let $\{P_\alpha\}$ be set of $n \times n$ permutation matrices, then there exists time-dependent convex combination coefficients $\sum_\alpha \lambda_\alpha(t) = 1$, $\lambda_\alpha(t) \geq 0$ so that

$$e^{(-Lt)} = \sum_\alpha \lambda_\alpha(t) P_\alpha$$

$$V(e^{-L(G)t}x) = V\left(\sum_\alpha \lambda_\alpha(t) P_\alpha x\right) \leq \sum_\alpha \lambda_\alpha(t) V(P_\alpha x)$$

$$\Rightarrow \sum_\alpha \lambda_\alpha(t) V(x) = V(x)$$

$\therefore \; V(e^{(-Lt)x}) \leq V(x)$

$\Rightarrow \dot{V}(x) \leq 0 \quad \forall x \in R^n$

$-x^T (L+L^T) x = -2x^T L x < 0$

$L+L^T$ is semi-definite

which proves that G is weight balanced.

b) (i) $L = D_{out} - A \longrightarrow (1)$

take transpose on both sides

$L^T = D_{out} - A^T \longrightarrow (2)$

$(1) + (2) \Rightarrow L + L^T = 2D_{out} - (A + A^T) \longrightarrow (3)$

(3) proves that $\cancel{L}+\cancel{L}^T$ is the laplacian matrix which is associated to a graph G that has $A+A^T$ as it's adjacency matrix

~~as~~ (iii) as $L+L^T$ is a laplacian associated to $A+A^T$ adjacency matrix

by using the property of Laplacian matrix

$(L+L^T)\mathbb{1}_n = 0$ as laplacian's row sum $= 0$