

# STA 331 2.0 Stochastic Processes

## 8. Birth and Death Processes

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# Birth and Death Processes

- The birth-and-death process is a subclass of **continuous-time Markov chains**.
- The birth-and-death processes are characterized by the property that whenever a transition occurs from one state to another, then this transition can be to a neighbouring state only.

# Transition types

- a transition occurs from one state to another and this transition can be to a neighbouring state only.
  - Eg: State space  $S = \{0, 1, 2, \dots, i, \dots\}$
  - transition that occurs from state  $i$ , can be only to a neighboring state  $(i - 1)$  or  $(i + 1)$ .

# Birth rate and Death rate

## Birth rate

$\lambda_i$  - birth rate from state  $i$ ,  $i \geq 0$

## Death rate

$\mu_i$  - death rate from state  $i$ ,  $i \geq 0$

# Queueing systems

1. Birth - equivalent to the arrival of a customer.
2. Death - equivalent to the departure of a served customer.

# Notations

A continuous-time Markov chain  $[X(t), t \in T]$  with state space  $S = \{0, 1, 2, \dots\}$  with rates

$$q_{i,i+1} = \lambda_i, \quad i = 0, 1, \dots,$$

$$q_{i,i-1} = \mu_i, \quad i = 1, 2, \dots,$$

$$q_{i,j} = 0, \quad j \neq i \pm 1, \quad j \neq i, \quad i = 0, 1, \dots, \text{ and}$$

$$q_i = (\lambda_i + \mu_i), \quad i = 0, 1, \dots, \text{ and } \mu_0 = 0.$$

# Pure birth process, pure death process, birth-and-death process

- i) a pure birth process if  $\mu_i = 0$  for  $i = 1, 2, \dots$ 
  - No decrements, only increments.
- ii) a pure death process if  $\lambda_i = 0$  for  $i = 1, 2, \dots$ 
  - No increments, only decrements.
- iii) a birth-and-death process if some of the  $\lambda_i$ 's and some of the  $\mu_i$ 's are positive.

# Examples of random phenomena modelled through birth and death processes

- Spread of epidemic disease
- Mutant gene dynamics
- Cell kinetics (proliferation of cancer cells)



# Special cases

1. Linear birth process: Yule-Furry process
2. Linear death process
3. Linear birth and death process
4.  $M/M/1$  queue

# Pure Birth Process

- Special case of a **continuous-time Markov process** and a **generalisation of a Poisson process**.
- Consider a population of individuals where only the appearances of new individuals, which are called “birth” occur.

# General birth processes

Let us consider a birth process whose total number of individuals at time  $t$  is denoted by a discrete random variable  $N(t)$ . As parameter  $t$  varies  $\{N(t) : t \geq 0\}$  represents a stochastic process with a continuous parameter (time) space and a discrete state space.

Let us assume that the birth rate depends on the present size of the population. Further we assume that the births occur according to the following postulates:

$$P[N(t+h) = n+k | N(t) = n] = \begin{cases} \lambda_n h + o(h), & k=1 \\ o(h), & k \geq 2 \\ 1 - \lambda_n h + o(h), & k=0 \end{cases}$$

# General birth processes (cont)

## Condition 1

$$P[N(t+h) = n+k | N(t) = n] = \begin{cases} \lambda_n h + o(h), & k=1 \\ o(h), & k \geq 2 \\ 1 - \lambda_n h + o(h), & k=0 \end{cases}$$

where  $\lambda_n$  is the rate at which the births occur at time  $t$  and  $n$  being the size of the population at time  $t$ .

## Condition 2

$$N(0) > 0$$

## Your turn

Compare the differences in conditions between Poisson process, Non-homogeneous Poisson Process and Birth Process

## Goal: Probability Mass Function of $N(t)$

What is the probability that the population size at a given time,  $t$ , equals  $N(t) = n$ ?

$$P_n(t) = P[N(t) = n] = ?$$

For example,

$$P_0(t) = P[N(t) = 0] = ?$$

$$P_1(t) = P[N(t) = 1] = ?$$

$$P_2(t) = P[N(t) = 2] = ?$$

## Linear Birth Process (Yule-Furry Process)

When,  $\lambda_n = n\lambda$ , i.e. when the birth rate is linear in the present size of the population.

Then the pure birth process is said to a **Linear Birth Process** or **Yule-Furry Process**.

Let is assume that **there is only one individual in the population initially**,  $N(0) = 1$ . It can be shown that for any  $t > 0$ .

$$P(N(t) = 0) = 0$$

$$P(N(t) = n) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}, n \geq 1.$$

## Proof (general situation):

For  $n = 0$

$$P_0(t+h) = P(N(t) = 0)P(N(t+h) - N(t) = 0 | N(t) = 0)$$

$$P_0(t+h) = P_0(t)(1 - \lambda_0 h + o(h))$$

i.e.

$$P_0(t+h) = P_0(t) - \lambda_0 h P_0(t) + o(h)P_0(t)$$

$$\lim_{h \rightarrow 0} \frac{P_0(t+h) - P_0(t)}{h} = -\lim_{h \rightarrow 0} \lambda_0 P_0(t) + \lim_{h \rightarrow 0} \frac{o(h)}{h} P_0(t)$$

i.e.

$$P'_0(t) = -\lambda_0 P_0(t).$$

We assume that **there is only one individual in the population initially**,  $N(0) = 1$ . Hence,  $P[N(t) = 0] = 0$ .

That is  $P_0(t) = 0$ .



## Proof: (cont)

For  $n \geq 1$

$$\begin{aligned} P_n(t+h) &= P(N(t) = n)P(N(t+h) - N(t) = 0 | N(t) = n) + \\ &\quad P(N(t) = n-1)P(N(t+h) - N(t) = 1 | N(t) = n-1) + \\ &\quad \sum_{r=2}^{n-1} P(N(t) = n-r)P(N(t+h) - N(t) = r | N(t) = n-r) \end{aligned}$$

i.e

$$\begin{aligned} P_n(t+h) &= P_n(t)(1 - \lambda_n h + o(h)) + \\ &\quad P_{n-1}(t)(\lambda_{n-1} h + o(h)) + \\ &\quad o(h) \end{aligned}$$

## Proof: (cont)

$$P_n(t+h) = P_n(t) - \lambda_n h P_n(t) + \lambda_{n-1} h P_{n-1}(t) + o(h) \text{ for } n \geq 1$$

$$\lim_{h \rightarrow 0} \frac{P_n(t+h) - P_n(t)}{h} = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) + \lim_{h \rightarrow 0} \frac{o(h)}{h}$$

i.e.

$$P'_n(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) \text{ for } n \geq 1.$$

Therefore the partial differential-difference equations is

$$\text{For } n \geq 1, P'_n(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t).$$

When  $n = 1$

$$P_1'(t) = -\lambda_1 P_1(t),$$
$$\int \frac{P_1'(t)}{P_1(t)} dt = -\lambda_1 \int dt,$$

$$\ln P_1(t) = -\lambda_1 t + c$$

$$P_1(t) = c_1 e^{-\lambda_1 t}$$

When  $t = 0$ ,  $c_1 = 1$

$$P_1(t) = e^{-\lambda_1 t}$$

When  $n = 2$

$$P_2'(t) = -\lambda_2 P_2(t) + \lambda_1 P_1(t),$$

$$P_2'(t) + \lambda_2 P_2(t) = \lambda_1 e^{-\lambda_1 t},$$

Multiply by  $e^{\lambda_2 t}$

$$P_2'(t)e^{\lambda_2 t} + \lambda_2 P_2(t)e^{\lambda_2 t} = \lambda_1 e^{-\lambda_1 t} e^{\lambda_2 t},$$

$$\int \frac{d}{dt}[e^{\lambda_2 t} P_2(t)] dt = \int \lambda_1 e^{(\lambda_2 - \lambda_1)t} dt,$$

$$e^{\lambda_2 t} P_2(t) = \frac{\lambda_1 e^{(\lambda_2 - \lambda_1)t}}{\lambda_2 - \lambda_1} + c$$

When  $t = 0$ ,

We know that  $P_2(0) = 0$ . hence,

$$c = -\frac{\lambda_1}{\lambda_2 - \lambda_1}.$$

Hence,

$$P_2(t) = \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t} [e^{(\lambda_2 - \lambda_1)t} - 1]$$

# Linear birth process (Yule-Furry Process)

When,

$$\lambda_n = n\lambda.$$

That is the birth rate is linear in the present size of the population.

Let us assume that there is **only one individual in the population** initially. That is  $N(0) = 1$ .

Then the difference-differential equations of the linear birth process takes the form

$P'_n(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)$  for  $n \geq 1$  with the initial conditions  $P_1(0) = 1$  and  $P_n(0) = 0$  for  $n \geq 2$ .

## Linear birth process (Yule-Furry Process) (cont)

$P'_n(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)$  for  $n \geq 1$  with the initial conditions  $P_1(0) = 1$  and  $P_n(0) = 0$  for  $n \geq 2$ .

Multiplying the equation for  $n$  by  $z^n$  and summing over all  $n$  we obtain

$$\frac{\partial}{\partial t} \sum_{n=1}^{\infty} P_n(t) z^n = -\lambda z \frac{\partial}{\partial z} \sum_{n=1}^{\infty} P_n(t) z^n + \lambda z^2 \frac{\partial}{\partial z} \sum_{n=1}^{\infty} P_{n-1}(t) z^{n-1}$$

Let  $\Pi(z, t) = \sum_{n=1}^{\infty} P_n(t) z^n$ . Then the above equations becomes

$$\frac{\partial \Pi(z, t)}{\partial t} = -\lambda z \frac{\partial \Pi(z, t)}{\partial z} + \lambda z^2 \frac{\partial \Pi(z, t)}{\partial z}$$

## Linear birth process (Yule-Furry Process) (cont)

$$\text{i.e. } \frac{\partial \Pi(z,t)}{\partial t} = \lambda z(z-1) \frac{\partial \Pi(z,t)}{\partial z}$$

$$\frac{\partial \Pi(z,t)}{\partial t} - \lambda z(z-1) \frac{\partial \Pi(z,t)}{\partial z} = 0$$

Subsidiary equations take the form

$$\frac{dt}{1} = \frac{dz}{-\lambda z(z-1)} = \frac{d\Pi}{0}$$

Two independent solutions can be obtained one from  $d\Pi = 0$  and the other from  $-\lambda dt = \frac{dz}{z(z-1)}$ .

$$d\Pi = 0 \Rightarrow \Pi(z, t) = \text{constant.}$$

$$-\lambda dt = \frac{dz}{z(z-1)} \Rightarrow \frac{z}{z-1} e^{-\lambda t} = \text{constant.}$$



## Linear birth process (Yule-Furry Process) (cont)

The general solution can be written as

$\Pi(z, t) = f\left(\frac{z}{z-1}e^{-\lambda t}\right)$  where  $f$  is an arbitrary function.

The initial conditions  $P_1(0) = 1$  and  $P_n(0) = 0$  for  $n \geq 2$  imply that  $\Pi(z, 0) = z$ .

$$\therefore \Pi(z, 0) = f\left(\frac{z}{z-1}\right) = z.$$

Let  $\omega = \frac{z}{z-1} \Rightarrow z = \frac{\omega}{\omega-1}$  and hence we obtain  $f(\omega) = \frac{\omega}{\omega-1}$ .

## Linear birth process (Yule-Furry Process) (cont)

$$\therefore \Pi(z, t) = \frac{\frac{z}{z-1}e^{-\lambda t}}{\frac{z}{z-1}e^{-\lambda t} - 1} = \frac{ze^{-\lambda t}}{ze^{-\lambda t} - (z-1)} = \left(1 - \frac{z-1}{z}e^{-\lambda t}\right)^{-1}$$

Considering coefficients of  $z^n$  we have

$$P_n(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1} \text{ for } n \geq 1.$$

In proving the above results we assume that initially there is only one individual in the population. That is  $N(0)=1$ .

Now let's prove for the case  $N(0) = a, a \geq 1$ . For that we use moment generating functions.

# Moment generating function of $N(t)$

Let

$$M_{N(t)}(\theta, t) = E[e^{N(t)\theta}],$$

be the moment generating function of  $N(t)$ . Then, for  $t > 0$ ,

$$\begin{aligned} M_{N(t)}(\theta, t) &= \sum_{n=0}^{\infty} e^{n\theta} P(N(t) = n) \\ &= \sum_{n=0}^{\infty} e^{n\theta} P_n(t). \end{aligned} \tag{1}$$

## Moment generating function of $N(t)$ (cont.)

We assume that  $N(0) = a > 0$ . Hence,  $P_n(t) = 0$  for all  $n < a$ .  
Hence,

$$M_{N(t)}(\theta, t) = \sum_{n=a}^{\infty} e^{n\theta} P_n(t). \quad (2)$$

## Moment generating function of $N(t)$ (cont.)

Now we take derivative w.r.t  $\theta$ . Then we get,

$$\frac{\partial}{\partial \theta} M_{N(t)}(\theta, t) = \sum_{n=a}^{\infty} n e^{n\theta} P_n(t).$$

The derivative w.r.t  $t$  is

$$\begin{aligned} \frac{\partial}{\partial t} M_{N(t)}(\theta, t) &= \sum_{n=a}^{\infty} e^{n\theta} P'_n(t) \\ &= \sum_{n=a}^{\infty} e^{n\theta} [-n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)] \\ &= - \sum_{n=a}^{\infty} n e^{n\theta} \lambda P_n(t) + \sum_{n=a}^{\infty} (n-1) e^{n\theta} \lambda P_{n-1}(t) \end{aligned}$$

## Moment generating function of $N(t)$ (cont.)

Since  $P_{a-1}(t) = 0$ , the second summation starts at  $a + 1$ .

Hence,

$$\begin{aligned}\frac{\partial}{\partial t} M_{N(t)}(\theta, t) &= - \sum_{n=a}^{\infty} n e^{n\theta} \lambda P_n(t) + \sum_{n=a+1}^{\infty} (n-1) e^{n\theta} \lambda P_{n-1}(t) \\&= - \sum_{n=a}^{\infty} n e^{n\theta} \lambda P_n(t) + \sum_{m=a}^{\infty} m e^{(m+1)\theta} \lambda P_m(t) \\&= -\lambda \sum_{n=a}^{\infty} n e^{n\theta} P_n(t) + \lambda e^{\theta} \sum_{m=a}^{\infty} m e^{m\theta} P_m(t) \\&= -\lambda \frac{\partial}{\partial \theta} M_{N(t)}(\theta, t) + \lambda e^{\theta} \frac{\partial}{\partial \theta} M_{N(t)}(\theta, t) \\&= \lambda(e^{\theta} - 1) \frac{\partial}{\partial \theta} M_{N(t)}(\theta, t)\end{aligned}$$

## Moment generating function of $N(t)$ (cont.)

$$\frac{\partial}{\partial t} M_{N(t)}(\theta, t) - \lambda(e^\theta - 1) \frac{\partial}{\partial \theta} M_{N(t)}(\theta, t) = 0. \quad (5)$$

**Note:**

A partial differential equation (PDE) for a function  $z(x, y)$  is Lagrange type if it takes the form (General form of first-order quasilinear PDE)

$$P(x, y, z) \frac{\partial z}{\partial x} + Q(x, y, z) \frac{\partial z}{\partial y} = R(x, y, z). \quad (6)$$

The associated characteristic system of ordinary differential equations.

**Note (cont)**

$$\frac{dx}{P(x, y, z)} = \frac{dy}{Q(x, y, z)} = \frac{dz}{R(x, y, z)}. \quad (7)$$

is known as the characteristic (auxiliary) system of equation (5). Suppose that two independent particular solutions of this system have been found in the form

$u(x, y, z) = C_1$  and  $v(x, y, z) = C_2$ , where  $C_1$  and  $C_2$  are arbitrary constants.

Then the general solution to equation (5) can be written as

$$\phi(u, v) = 0 \quad (8)$$

where  $\phi$  is an arbitrary function of two variables.



**Note (cont.)**

With equation (6) solved for  $v$ , one often specifies the general solution in the form  $v = \psi(u)$ , where  $\psi(u)$  is an arbitrary function of one variable. The  $\psi$  can be determined using the boundary conditions.

## Moment generating function of $N(t)$ (cont.)

Revisit equation 4,

$$\frac{\partial}{\partial \theta} M_{N(t)}(\theta, t) - \lambda(e^\theta - 1) \frac{\partial}{\partial \theta} M_{N(t)}(\theta, t) = 0. \quad (9)$$

According to the auxiliary system of equation in (6),

$$\frac{dt}{1} = \frac{d\theta}{-\lambda(e^\theta - 1)} = \frac{M_{N(t)}}{0}$$

$$\frac{dt}{1} = \frac{dM_{N(t)}}{0}$$

$$\frac{dM_{N(t)}}{dt} = 0 \Rightarrow M_{N(t)}(\theta, t) = \text{constant}.$$

## Moment generating function of $N(t)$ (cont.)

Furthermore consider,

$$\begin{aligned}\frac{dt}{1} &= \frac{d\theta}{-\lambda(e^\theta - 1)} \\ \lambda dt &= -\frac{1}{(e^\theta - 1)} d\theta \\ &= \frac{-e^{-\theta}}{1 - e^{-\theta}} d\theta\end{aligned}\tag{10}$$

From equation (9) we can write

$$\lambda t = -\ln(1 - e^{-\theta}) + c$$

## Moment generating function of $N(t)$ (cont.)

Furthermore

$$\ln(e^{\lambda t}) + \ln(1 - e^{-\theta}) = c.$$

Hence,

$$e^{\lambda t}(1 - e^{-\theta}) = \text{constant}.$$

Hence, the general solution for eq(8) is

$$M_{N(t)}(\theta, t) = \Psi[e^{\lambda t}(1 - e^{-\theta})].$$

## Moment generating function of $N(t)$ (cont.)

The boundary conditions  $P_a(0) = 1$ , and  $P_n(0)$  for  $n \neq a$ , imply that  $M_{N(t)}(\theta, 0) = \sum_{n=a}^{\infty} e^{n\theta} P_n(0) = e^{a\theta}$ ,

$$M_{N(t)}(\theta, 0) = e^{a\theta} = \Psi(1 - e^{-\theta}).$$

Let  $\alpha = 1 - e^{-\theta}$ . Then,  $e^{\theta} = (1 - \alpha)^{-1}$ . Hence,

$$e^{a\theta} = \Psi(\alpha) = (1 - \alpha)^{-a}.$$

## Moment generating function of $N(t)$ (cont.)

Therefore,

$$M_{N(t)}(\theta, t) = \Psi[e^{\lambda t}(1 - e^{-\theta})] = [1 - e^{\lambda t}(1 - e^{-\theta})]^{-a}.$$

Let  $p = e^{-\lambda t}$  and  $p + q = 1$ . Then,

$$M_{N(t)}(\theta, t) = [1 - p^{-1}(1 - e^{-\theta})]^{-a} = \left[ \frac{p - 1 + e^{-\theta}}{p} \right]^{-a} = \left( \frac{p}{e^{-\theta} - q} \right)^a.$$

Now from this MGF, we can derive the moments of  $N(t)$ .

## Moment generating function of $N(t)$ (cont.)

It can be shown that

$$E(N(t)) = a/p = ae^{\lambda t} \text{ and}$$

$$V[N(t)] = a(1 - p)/p^2 = a(1 - e^{-\lambda t})e^{2\lambda t}.$$

Furthermore, we recognize the above MGF is in the form of the MGF of a negative binomial random variable  $Y$ , with probability mass function

$$P(Y = y) = {}^{y-1}C_{a-1} p^{a-1} q^{y-1-(a-1)} p = {}^{y-1}C_{a-1} p^a q^{y-a}, \text{ for } y = a, a+1, \dots$$

Hence,

$$P(N(t) = n) = {}^{n-1}C_{a-1} p^a q^{n-a} = {}^{n-1}C_{a-1} e^{-\lambda t a} (1 - e^{-\lambda t})^{n-a} \text{ for } n = a, a+1, \dots$$

# Linear birth process (Yule-Furry Process)

## Summary:

**When,  $N(0) = 1$**

$$P(N(t) = 0) = 0$$

$$P(N(t) = n) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}, n \geq 1.$$

**When,  $N(0) = a$**

$$P(N(t) = n) = \binom{n-1}{a-1} p^a q^{n-a} = \binom{n-1}{a-1} e^{-\lambda t a} (1 - e^{-\lambda t})^{n-a} \text{ for } n = a, a+1, \dots$$



## Exercise

Consider a pure birth process on the states  $\{0, 1, \dots, N\}$  for which  $\lambda_k = (N - k)\lambda$  for  $k = 0, 1, \dots, N$ . Suppose  $N(0) = 0$ . Find  $P_n(t) = P(X(t) = n)$  for  $n = 0, 1$  and  $2$ .