

# STA 331 2.0 Stochastic Processes

## 11. Birth-and-Death Process: Important results (cont)

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# Birth-and-death process

Consider the stochastic process with  $\{N(t), t \geq 0\}$  with  $N(0) = a(\geq 1)$ , and

$$P[N(t+h) = n+k | N(t) = n] = \begin{cases} 1 - \lambda_n h - \mu_n h + o(h), & k=0 \\ \lambda_n h + o(h), & k=1 \\ \mu_n h + o(h), & k=-1 \\ o(h), & k \geq 2 \text{ or } k \leq -2 \end{cases} \quad (1)$$

is called a birth and death process. Note  $\mu_0 = 0$

# Birth-and-death process

The partial differential-difference equations are

$$P'_0(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t) \text{ and}$$

$$P'_n(t) = -(\mu_n + \lambda_n) P_n(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t) \text{ for } n \geq 1.$$

# Proof

Let  $P_n(t) = P[N(t) = n]$

Then, for  $n \geq 1$ ,

$$\begin{aligned} P_n(t+h) &= P(N(t) = n)P(N(t+h) = n | N(t) = n) + \\ &\quad P(N(t) = n+1)P(N(t+h) = n | N(t) = n+1) + \\ &\quad P(N(t) = n-1)P(N(t+h) = n | N(t) = n-1) + \\ &\quad \sum_{r \neq -1, 0, 1}^{\infty} P(N(t) = n-r)P(N(t+h) = n | N(t) = n-r) \end{aligned}$$

## Proof (cont.)

$$\begin{aligned}P_n(t+h) &= P_n(t)(1 - \mu_n h - \lambda_n h + o(h)) + \\&\quad P_{n+1}(t)(\mu_{n+1} h + o(h)) + \\&\quad P_{n-1}(t)(\lambda_{n-1} h + o(h)) + \\&\quad o(h).\end{aligned}$$

$$\begin{aligned}P_n(t+h) &= P_n(t) - \mu_n P_n(t)h - \lambda_n P_n(t)h + \\&\quad P_{n+1}(t)\mu_{n+1}h + P_{n-1}(t)\lambda_{n-1}h + \\&\quad o(h) \text{ for } n \geq 1.\end{aligned}$$

## Proof (cont.)

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{P_n(t+h) - P_n(t)}{h} &= -\mu_n P_n(t) - \lambda_n P_n(t) + \\ &\quad P_{n+1}(t) \mu_{n+1} + P_{n-1}(t) \lambda_{n-1} + \\ &\quad \lim_{h \rightarrow 0} \frac{o(h)}{h} \text{ for } n \geq 1. \end{aligned}$$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{P_n(t+h) - P_n(t)}{h} &= -(\mu_n + \lambda_n) P_n(t) + \\ &\quad P_{n+1}(t) \mu_{n+1} + P_{n-1}(t) \lambda_{n-1} + \\ &\quad \lim_{h \rightarrow 0} \frac{o(h)}{h} \text{ for } n \geq 1. \end{aligned}$$

$$P'_n(t) = -(\mu_n + \lambda_n) P_n(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t) \text{ for } n \geq 1.$$

## Proof (cont.)

For  $n = 0$

$$\begin{aligned}P_0(t+h) &= P(N(t) = 0)P(N(t+h) = 0|N(t) = 0) + \\&\quad P(N(t) = 1)P(N(t+h) = 0|N(t) = 1) + \\&\quad \sum_{r=2}^{\infty} P(N(t) = n-r)P(N(t+h) = 0|N(t) = n-r)\end{aligned}$$

$$\begin{aligned}P_0(t+h) &= P_0(t)(1 - \mu_0 h - \lambda_0 h + o(h)) + \\&\quad P_1(t)(\mu_1 h + o(h)) + \\&\quad o(h)\end{aligned}$$

## Proof (cont.)

We know that  $\mu_0 = 0$

$$\begin{aligned}P_0(t+h) &= P_0(t)(1 - \lambda_0 h + o(h)) + \\&\quad P_1(t)(\mu_1 h + o(h)) + \\&\quad o(h)\end{aligned}$$

$$\begin{aligned}P_0(t+h) &= P_0(t) - P_0(t)\lambda_0 h + P_0(t)o(h) + \\&\quad P_1(t)\mu_1 h + P_1(t)o(h) + \\&\quad o(h)\end{aligned}$$



## Proof (cont.)

$$\lim_{h \rightarrow 0} \frac{P_0(t+h) - P_0(t)}{h} = -\lambda_0 P_0(t) + P_1(t) \mu_1 +$$
$$\lim_{h \rightarrow 0} \frac{o(h)}{h} \text{ for } n \geq 1.$$

$$P'_0(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t)$$

# Linear Birth and Death Process

When  $\lambda_n = n\lambda$  and  $\mu_n = n\mu$ , i.e when the birth and death rates are linear in the present size of the population, the birth and death process is said to be a linear birth and death process. Let us assume  $N(0) = a(\geq 1)$ .

Birth and death process takes the form

$$P'_0(t) = \mu P_1(t)$$

$$P'_n(t) = -n(\mu + \lambda)P_n(t) + (n-1)\lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t)$$

for  $n \geq 1$ .

# Q1

We are going to show that

$$E[N(t)] = ae^{\lambda - \mu}t.$$

## Q1 (proof)

$$P'_0(t) = \mu P_1(t)$$

$$P'_n(t) = -n(\mu + \lambda)P_n(t) + (n-1)\lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t)$$

for  $n \geq 1$ .

Using the same method as in pure birth process, we can show that

$$\frac{\partial}{\partial t} M_{N(t)}(\theta, t) - [\lambda(e^\theta - 1) + \mu(e^{-\theta} - 1)] \frac{\partial}{\partial \theta} M_{N(t)}(\theta, t) = 0$$

(2)

## Q1 (proof)

The auxiliary system is

$$\frac{dt}{1} = \frac{-d\theta}{[\lambda(e^\theta - 1) + \mu(e^{-\theta} - 1)]} = \frac{dM_{N(t)}}{0},$$

$$\frac{dM_{N(t)}}{0} = 0$$

$$\Rightarrow M_{N(t)}(\theta, t) = \text{constant}.$$

## Q1 (proof):

$$dt = \frac{-d\theta}{[\lambda(e^\theta - 1) + \mu(e^{-\theta} - 1)]}$$

$$t = \begin{cases} -\frac{1}{\lambda - \mu} \ln \frac{(e^{-\theta} - 1)}{\lambda e^\theta - \mu} + \text{constant}, & \text{when } \lambda \neq \mu \\ -\frac{1}{\lambda(e^\theta - 1)} + \text{constant}, & \text{when } \lambda = \mu \end{cases} \quad (3)$$

$\Rightarrow$

$$\frac{(e^\theta - 1)e^{(\lambda - \mu)t}}{\lambda e^\theta - \mu} = \text{constant when } \lambda \neq \mu$$

$$\lambda t - \frac{1}{(e^\theta - 1)} = \text{constant when } \lambda = \mu.$$

## Q1 Proof(cont.)

When  $\lambda \neq \mu$ , the general solution of eq 2 is

$$M_{N(t)}(\theta, t) = \Psi \left( \frac{(e^\theta - 1)e^{(\lambda - \mu)t}}{\lambda e^\theta - \mu} \right). \quad (4)$$

Initially there are,  $a$ , individuals ( $N(0) = a$ ). Hence boundary conditions are  $P_a(0) = 1$  and  $P_n(0) = 0$  for  $n \neq a$ . Hence,

$$M_{N(t)}(\theta, 0) = \sum_{n=-\infty}^{\infty} e^{n\theta} P_n(0) = e^{a\theta}.$$

Therefore,

$$M_{N(t)}(\theta, 0) = e^{a\theta} = \Psi \left[ \frac{e^\theta - 1}{\lambda e^\theta - \mu} \right]. \quad (5)$$

## Q1 (proof)

Let,

$$\alpha = \frac{e^{\theta} - 1}{\lambda e^{\theta} - \mu}.$$

Then we get

$$e^{\theta} = \frac{\mu\alpha - 1}{\lambda\alpha - 1}.$$



## Q1 (proof)

Substitute to eq 5, we get

$$\Psi(\alpha) = \left( \frac{\mu\alpha - 1}{\lambda\alpha - 1} \right)^a. \quad (6)$$

From eq 4 we have,

$$M_{N(t)}(\theta, t) = \Psi \left( \frac{(e^\theta - 1)(e^{(\lambda - \mu)t})}{\lambda e^\theta - \mu} \right).$$

Let  $\nu(\theta, t) = \frac{(e^\theta - 1)e^{(\lambda - \mu)t}}{\lambda e^\theta - \mu}$ . Therefore,

$$M_{N(t)}(\theta, t) = \left( \frac{\mu\nu(\theta, t) - 1}{\lambda\nu(\theta, t) - 1} \right)^a. \text{ (similar to eq 6 format)}$$

## Q1 (proof): Your turn

Using the MGF, show that

$$E(N(t)) = ae^{(\lambda - \mu)t}.$$