Theorem

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State i is

$$\text{recurrent if } \sum_{n=1}^{\infty} P_{ii}^{n} = \infty,$$

$$\text{transient if } \sum_{n=1}^{\infty} P_{ii}^{n} < \infty,$$

Proof

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Let

$$I_n = \begin{cases} 1 & \textit{if } X_n = i, \\ 0 & \textit{if } X_n \neq i. \end{cases}$$

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Then what about $\sum_{n=0}^{\infty} I_n$?

Then $\sum_{n=0}^{\infty} I_n$ represents the number of times that the process is in state i.

$$E\left[\sum_{n=0}^{\infty} I_n | X_0 = i\right]$$

$$E\left[\sum_{n=0}^{\infty} I_n | X_0 = i\right]$$

represents theexpected number of times the process is in state i, given the process was in i at the beginning.

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Given that the process starts at state i, what is the expected number of times that the process will be in state i.

$$E\left[\sum_{n=0}^{\infty} I_n | X_0 = i\right]$$

$$= \sum_{n=o}^{\infty} E[I_n|X_0 = i]$$

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$$= \sum_{n=o}^{\infty} E[I_n|X_0 = i]$$

$$= \sum_{n=o}^{\infty} (1P(I_n = 1|X_0 = i) + 0P(I_n = 1|X_0 = i))$$

$$= \sum_{n=o}^{\infty} P(I_n = 1|X_0 = i)$$

$$= \sum_{n=o}^{\infty} P(X_n = i | X_0 = i)$$

$$= \sum_{n=o}^{\infty} P_{ii}^{n}$$

Corollary 1

If state i is recurrent, and state i communicates with state j ($i \leftrightarrow j$), then state j is recurrent.

Since state i communicates with state j, there exists k and m such that

$$P_{ij}^{(k)} > 0$$

$$P_{ji}^{(m)} > 0$$

$$P_{jj}^{(m+n+k)} \geq P_{ji}^{(m)} P_{ii}^{(n)} P_{ij}^{(k)}$$

$$\sum_{n=0}^{\infty} P_{jj}^{(m+n+k)} \geq \sum_{n=0}^{\infty} P_{ji}^{(m)} P_{ii}^{(n)} P_{ij}^{(k)}$$

$$\sum_{n=0}^{\infty} P_{jj}^{(m+n+k)} \geq P_{ji}^{(m)} P_{ij}^{(k)} \sum_{n=0}^{\infty} P_{ii}^{(n)}$$

Corollary 2

In a Markov Chain with a finite number of states not all of the states can be transient (There should be at least one recurrent state).

Proof:

Suppose not. Then the MC will runput of states not to go to an infinite number of times. This is a contradiction.

Corollary 3

If one state in an equivalent class is transient, then all other states in that class are also transient.

Proof:

Suppose not, i.e, suppose there is a recurrent state. Since all states in the equivalent class communicate, corollary 1 implies all states are recurrent. This is a contradiction.