STA 331 2.0 Stochastic Processes

2. Markov Chains

Dr Thiyanga S. Talagala

Department of Statistics, University of Sri Jayewardenepura

n-step transition probabilities - P_{ij}^n

P_{ij} - One step transition probabilities

P_{ij}^n - n - step transition probabilities

Probability that a process in state i will be in state j after n additional transitions. That is,

$$P_{ij}^n = P(X_{n+k} = j | X_k = i), \;\; n \geq 0, \;\; i, j \geq 0.$$

Chapman-Kolmogrov Equations

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj}^{m} \text{ for all n, m} \geq 0, \text{ all i, j,}$$

where, $P_{ik}^{n} P_{kj}^{m}$ represents the probability that starting in i the process will go to state j in $n + m$ with an intermediate stop in state k after n steps.

In-class

This can be used to compute n-step transition probabilities

In-class

In-class

$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj}^{m}$ for all n, m ≥ 0, all i, j.

Proof:

n - step transition matrix

The n-step transition matrix is

$$\mathbf{P}^{(n)} = \begin{bmatrix} P_{00}^{(n)} & P_{01}^{(n)} & P_{02}^{(n)} & \dots \\ P_{10}^{(n)} & P_{11}^{(n)} & P_{12}^{(n)} & \dots \\ . & . & . & \dots \\ . & . & . & \dots \\ . & . & . & \dots \end{bmatrix}$$

n - step transition matrix (cont.)

The Chapman-Kolmogrov equations imply

$$\mathbf{P}^{(n+m)} = P^{(n)}P^{(m)}.$$

In particular,

$$\mathbf{P}^{(2)} = \mathbf{P}^{(1)}\mathbf{P}^{(1)} = \mathbf{P}\mathbf{P} = \mathbf{P}^2.$$

By induction,

$$\mathbf{P}^{(n)} = \mathbf{P}^{(n-1+1)} = \mathbf{P}^{n-1}\mathbf{P} = \mathbf{P}^n.$$

n - step transition matrix

Proposition

$$P^{(n)} = P^n = P \times P \times P \times ... \times P, \ n \geq 1.$$

That is, $P^{(n)}$ is equal to P multiplied by itself n times.

Example 1

Let $X_i = 0$ if it rains on day i; otherwise $X_i = 1$. Suppose $P_{00} = 0.7$ and $P_{10} = 0.4$. Suppose it rains on Monday. Then, what is the probability that it rains on Friday.

Example 1 - using R

```r
p <- matrix(c(0.7, 0.4, 0.3, 0.6), nrow = 2); p
```

```
     [,1] [,2]
[1,]  0.7  0.3
[2,]  0.4  0.6
```

```r
p%*%p%*%p%*%p
```

```
       [,1]   [,2]
[1,] 0.5749 0.4251
[2,] 0.5668 0.4332
```

So that $P_{00}^{(4)} = 0.5749$

Example 2

Recall the example from class in which the weather today depends on the weather for the previous two days.

Sate	Yesterday	Today	Tomorrow	Probability
0-RR	1	1	1	0.7
1-SR	0	1	1	0.5
2-RS	1	0	1	0.4
3-SS	0	0	1	0.2

$$\mathbf{P} = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

Now suppose that it was sunny both yesterday and the day before yesterday. What's the probability that it will rain tomorrow?

Example 2 (cont.)

```
p <- matrix(c(0.7, 0.5, 0, 0, 0, 0, 0.4, 0.2,
              0.3, 0.5, 0, 0, 0, 0, 0.6, 0.8), ncol=4)
p%*%p
```

```
     [,1] [,2] [,3] [,4]
[1,] 0.49 0.12 0.21 0.18
[2,] 0.35 0.20 0.15 0.30
[3,] 0.20 0.12 0.20 0.48
[4,] 0.10 0.16 0.10 0.64
```

Unconditional Probabilities

Suppose we know the initial probabilities,

$\alpha_i = P(X_0 = i), \quad , \; i = 0, 1, 2, \ldots$

and $\sum_i \alpha_i = 1$.

According to the Law of total probability

$$
\begin{aligned}
P(X_n = j) &= \sum_{i=0}^{\infty} P(X_n = j \cap X_0 = i) \\
&= \sum_{i=0}^{\infty} P(X_n = j | X_0 = i) P(X_0 = i) \\
&= \sum_{i=0}^{\infty} P_{ij}^{(n)} \alpha_i
\end{aligned}
$$

Example 3 (based on Example 1)

Let $X_i = 0$ if it rains on day i; otherwise $X_i = 1$. Suppose $P_{00} = 0.7$ and $P_{10} = 0.4$. Suppose it rains on Monday. Suppose $P(X_0 = 0) = 0.4$ and $P(X_0 = 1) = 0.6$. What is the probability that it will not rain on the 4th day after we start keeping records?

Example 3 (cont.)

Let $X_i = 0$ if it rains on day i; otherwise $X_i = 1$. Suppose $P_{00} = 0.7$ and $P_{01} = 0.4$. Suppose it rains on Monday. Suppose $P(X_0 = 0) = 0.4$ and $P(X_0 = 1) = 0.6$. What is the probability that it will not rain on the 4th day after we start keeping records?

```
p <- matrix(c(0.7, 0.4, 0.3, 0.6), nrow = 2)
p%*%p%*%p%*%p
```

```
       [,1]   [,2]
[1,] 0.5749 0.4251
[2,] 0.5668 0.4332
```

Example 4

Suppose that a taxi driver operates between Wijerama and Nugegoda. If the driver is in Wijerama the probability that he gets a trip to Nugegoda from one passenger or a group of travelling together is 0.2 and that for him to get a trip nearby Wijerama is 0.8. If the driver is in Nugegoda he has equal chance of getting a trip to Wijerama or nearby Nugegoda. The behaviour of the driver evolves over time in a probabilistic manner.

0 - Wijerama, 1 - Nugegoda

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix}$$

Example 4 (cont.)

i) If the driver is currently at Wijerama, what is the probability that he will be back at Wijerama after three trips?

Example 4 (cont.)

i) If the driver is currently at Wijerama, what is the probability that he will be back at Wijerama after three trips?

```
p <- matrix(c(0.8, 0.5, 0.2, 0.5), ncol=2)
p%*%p%*%p
```

```
      [,1]  [,2]
[1,] 0.722 0.278
[2,] 0.695 0.305
```

Example 4 (cont.)

ii) If the driver is at Nugegoda, how many trips on the average will be in Nugegoda before he next goes to Wijerama?

Example 4 (cont.): In-class

Example 4 (cont.): In-class

Suppose $P^{(0)} = (0.5, 0.5)$, equal chance for driver be in either Wijerama or Nugegoda. What is the probability he will be in Wijerama after the first trip.

In-class: Method 1

Probability after n-th step

$$\mathbf{P}^{(n)} = \mathbf{P}^{(0)}\mathbf{P}^{n}$$

In-class: Method 2

Types of States

Definition: If $P_{ij}^{(n)} > 0$ for some $n \geq 0$, state j is **accessible** from i.

Notation: $i \to j$.

Definition: If $i \to j$ and $j \to i$, then i and j **communicate**.

Notation: $i \leftrightarrow j$.

Theorem:

Communication is an equivalence relation:

(i) $i \leftrightarrow i$ for all i (reflexive).

(ii) $i \leftrightarrow j$ implies $j \leftrightarrow i$ (symmetric).

(iii) $i \leftrightarrow j$ and $j \leftrightarrow k$ imply $i \leftrightarrow k$ (transitive).

In-class: Proof

(i) $i \leftrightarrow i$ for all i (reflexive).

In-class: Proof

(ii) $i \leftrightarrow j$ implies $j \leftrightarrow i$ (symmetric).

In-class: Proof

(iii) $i \leftrightarrow j$ and $j \leftrightarrow k$ imply $i \leftrightarrow k$ (transitive).

In-class: Proof

Note:

- Two states that communicate are said to be in the same **class**.
- The concept of communication divides the state space up into a number of separate classes.

In-class: demonstration

Theorem (cont.)

Definition: An equivalence class consists of all states that communicate with each other.

Remark: Easy to see that two equivalence classes are disjoint.

Example: The following P has equivalence classes $\{0, 1\}$ and $\{2, 3\}$

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.75 & 0.25 \\ 0 & 0 & 0.25 & 0.75 \end{bmatrix}$$

Equivalence class (cont.)

What about this?

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.3 & 0.2 & 0 \\ 0 & 0 & 0.75 & 0.25 \\ 0 & 0 & 0.25 & 0.75 \end{bmatrix}$$

Irreducible

Definition: A MC is irreducible if there is only one equivalence class (i.e., if all states communicate with each other).

What about these?

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.75 & 0.25 \\ 0 & 0 & 0.25 & 0.75 \end{bmatrix} \qquad \mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.3 & 0.2 & 0 \\ 0 & 0 & 0.75 & 0.25 \\ 0 & 0 & 0.25 & 0.75 \end{bmatrix}$$

Irreducible (cont.)

What about these?

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 0.25 & 0 & 0.75 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

Identify the equivalence classes

Consider a Markov chain with a state space $S = \{0, 1, 2, 3, 4\}$ and having the following one-step transition probability matrix.

$$\mathbf{P} = \begin{bmatrix} 0.4 & 0.2 & 0 & 0.4 & 0 \\ 0.2 & 0.4 & 0.1 & 0.3 & 0 \\ 0.1 & 0.2 & 0.5 & 0.1 & 0.1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problems [1]

Example 4.10

Example 4.11

Example 4.12

[1] Introduction to Probability Models, Sheldon M. Ross

Classification of States - next week

Reading Section 4.3: Classification of States[2]

[2] Introduction to Probability Models, Sheldon M. Ross