STA 331 2.0 Stochastic Processes

3. Markov Chains - Classification of States

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November 16, 2021

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Example 1

Find the equivalence classes.

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem

The relation of communication partitions the state space into mutually exclusive and exhaustive classes. (The states in a given class communicate with each other. But states in different classes do not communicate with each other.)

Definition

Let f_i denote the probability that, starting in state i, the process will ever re-enters state i, i.e,

$f_i = P(X_n = i \text{ for some } n \geq 1 | X_0 = i)$

Example 2

Consider the Markov chain consisting of the states 0, 1, 2, 3 with the transition probability matrix,

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find f_0, f_1, f_2, f_3.

Recurrent and transient states

Let f_i be the probability that, starting in state i, the process will ever re-enter state i. State i is said to be recurrent if $f_i = 1$ and transient if $f_i < 1$.

Example 3

Consider the Markov chain consisting of the states 0,1,2 with the transition probability matrix

$$\mathbf{P}= \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Determine which states are transient and which are recurrent.

Example 4

Consider the Markov chain consisting of the states 0, 1, 2, 3 with the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Determine which states are transient and which are recurrent.

Example 5

Consider the Markov chain consisting of the states 0, 1, 2, 3, 4 with the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Determine which states are transient and which are recurrent.

Theorem

if state i is recurrent then, starting in state i, the process will re-enter state i again and again and again—in fact, infinitely often.

Theorem

For any state i, let f_i denote the probability that, starting in state i, the process will ever re-enter state i. If state i is transient then, starting in state i, the number of time periods that the process will be in state i has a geometric distribution with finite mean $\frac{1}{1-f_i}$.

Proof: In-class

Theorem

State i is

$$\text{recurrent if } \sum_{n=1}^{\infty} P_{ii}^{n} = \infty,$$

$$\text{transient if } \sum_{n=1}^{\infty} P_{ii}^{n} < \infty,$$

Proof: In-class

Corollary 1

If state i is recurrent, and state i communicates with state j ($i \leftrightarrow j$), then state j is recurrent.

Proof: In-class

Corollary 2

In a Markov Chain with a finite number of states not all of the states can be transient (There should be at least one recurrent state).

Proof: In-class

Corollary 3

If one state in an equivalent class is transient, then all other states in that class are also transient.

Proof: In-class

Corollary 4

Not all states in a finite Markov chain can be transient. This leads to the conclusion that **all states of a finite irreducible Markov chain are recurrent**.

Proof: In-class

Acknowledgement

The contents in the slides are mainly based on Introduction to Probability Models by Sheldon M. Ross.