STA 331 2.0 Stochastic Processes

6. Poisson process (cont.)

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Interarrival times

Inter-arrival Time Distribution for a Poisson Process

Consider a Poisson process with rate λ. Let,

T_1 be the time of the first arrival. Let T_2 be the time elapsed between the first and the second arrival. Further, for $n > 1$, let T_n denote the elapsed time between the $(n-1)st$ and the nth event. The sequence $\{T_n, n = 1, 2, 3, ..\}$ is called the **sequence of interarrival times**.

We shall now determine the distribution of the T_n.

Proposition

Let $N(t)$ be a Poisson process with rate λ, then then the interarrival times T_n, $n = 1, 2, ...$, are independent identically distributed with *Exponential*(λ) distribution.

The exponential distribution is memoryless

If X is exponential with parameter $\lambda > 0$, then X is a **memoryless random variable**, that is $P(X > x + a | X > a) = P(X > x)$, for a, $x \geq 0$.

Proof: In-class

Example

The number of customers arriving at a railway reservation counter follows a Poisson process with intensity $\lambda = 10$ customers per hour. Find the probability that there are 2 customers between 11:00 and 11:20. Find the probability that there are 3 customers between 11:00 and 11:20 and 7 customers between 11:20 and 12noon.

Example

Let $N(t)$ be a Poisson process with parameter $\lambda = 2$, and let $T_1, T_2, ...,$ be the corresponding interarrival times.

i) Find the probability that the first arrival occurs after $t = 0.5$.
ii) Given that we have had no arrivals before $t = 1$, find $\mathrm{P}(T_1 > 3)$.
iii) Given that the third arrival occurred at time $t = 2$, find the probability that the fourth arrival occurs after $t = 4$.
iv) Suppose that we start observing the process at time $t = 10$. Let T_1 be the time of the first arrival after we start observing the process. Find the mean and variance of the random variable T_1.

Arrival times (Waiting time)

Waiting time distribution

Let $N(t)$ is a Poisson process with rate λ. We first generate i.i.d. random variables $T_n, n = 1, 2, ..,$ where T_n $Exponential(\lambda)$.

Another quantity of interest is W_n, **the arrival time of the** n**th event**, also called **the waiting time until the** n**th event**. Then the arrival times are given by

$W_1 = T_1$

$W_2 = T_1 + T_2$

$W_2 = T_1 + T_2 + T_3$

...

Waiting time distribution (cont.)

Since T_n's are independent and each T_n has an exponential distribution with mean $1/\lambda$, W_n has a **Gamma distribution** with shape parameter n and rate parameter λ. The mean waiting time is n/λ.

Example

Suppose that people arrive at a territory in accordance with a Poisson process of rate $\lambda = 10$ per day.

i) What is the expected time until the 100th person arrives?
ii) What is the probability that the elapsed time between the 100th and the 200th arrival exceeds 10 days?

Acknowledgement

The contents in the slides are mainly based on

Introduction to Probability Models by Sheldon M. Ross.

H. Pishro-Nik, “Introduction to probability, statistics, and random processes”, Kappa Research LLC, 2014.