

# STA 331 2.0 Stochastic Processes

## 2. Markov Chains

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## **$n$ -step transition probabilities - $P_{ij}^n$**

$P_{ij}$  - One step transition probabilities

$P_{ij}^n$  -  $n$  - step transition probabilities

Probability that a process in state  $i$  will be in state  $j$  after  $n$  additional transitions. That is,

$$P_{ij}^n = P(X_{n+k} = j | X_k = i), \quad n \geq 0, \quad i, j \geq 0.$$

# Chapman-Kolmogorov Equations

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m \text{ for all } n, m \geq 0, \text{ all } i, j,$$

where,  $P_{ik}^n P_{kj}^m$  represents the probability that starting in  $i$  the process will go to state  $j$  in  $n + m$  with an intermediate stop in state  $k$  after  $n$  steps.

In-class

This can be used to compute  $n$ -step transition probabilities

# In-class

## In-class

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m \text{ for all } n, m \geq 0, \text{ all } i, j.$$

Proof:

## $n$ - step transition matrix

The  $n$ -step transition matrix is

$$\mathbf{P}^{(n)} = \begin{bmatrix} P_{00}^{(n)} & P_{01}^{(n)} & P_{02}^{(n)} & \dots \\ P_{10}^{(n)} & P_{11}^{(n)} & P_{12}^{(n)} & \dots \\ . & . & . & \dots \\ . & . & . & \dots \\ . & . & . & \dots \end{bmatrix}$$

## **$n$ - step transition matrix (cont.)**

The Chapman-Kolmogorov equations imply

$$\mathbf{P}^{(n+m)} = \mathbf{P}^{(n)} \mathbf{P}^{(m)}.$$

In particular,

$$\mathbf{P}^{(2)} = \mathbf{P}^{(1)} \mathbf{P}^{(1)} = \mathbf{P} \mathbf{P} = \mathbf{P}^2.$$

By induction,

$$\mathbf{P}^{(n)} = \mathbf{P}^{(n-1+1)} = \mathbf{P}^{n-1} \mathbf{P} = \mathbf{P}^n.$$

## $n$ - step transition matrix

### Proposition

$$P^{(n)} = P^n = P \times P \times P \times \dots \times P, \quad n \geq 1.$$

That is,  $P^{(n)}$  is equal to  $P$  multiplied by itself  $n$  times.



## Example 1

Let  $X_i = 0$  if it rains on day  $i$ ; otherwise  $X_i = 1$ . Suppose  $P_{00} = 0.7$  and  $P_{10} = 0.4$ . Suppose it rains on Monday. Then, what is the probability that it rains on Friday.

## Example 1 - using R

```
p <- matrix(c(0.7, 0.4, 0.3, 0.6), nrow = 2); p
```

```
      [,1] [,2]  
[1,]  0.7  0.3  
[2,]  0.4  0.6
```

```
p%%p%%p%%p
```

```
      [,1] [,2]  
[1,] 0.5749 0.4251  
[2,] 0.5668 0.4332
```

So that  $P_{00}^{(4)} = 0.5749$

## Example 2

Recall the example from class in which the weather today depends on the weather for the previous two days.

Sate	Yesterday	Today	Tomorrow	Probability
0-RR	1	1	1	0.7
1-SR	0	1	1	0.5
2-RS	1	0	1	0.4
3-SS	0	0	1	0.2

$$\mathbf{P} = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

Now suppose that it was sunny both yesterday and the day before yesterday. What's the probability that it will rain tomorrow?

## Example 2 (cont.)

```
p <- matrix(c(0.7, 0.5, 0, 0, 0, 0, 0.4, 0.2,  
              0.3, 0.5, 0, 0, 0, 0, 0.6, 0.8), ncol=4)
```

```
p%*%p
```

	[,1]	[,2]	[,3]	[,4]
[1,]	0.49	0.12	0.21	0.18
[2,]	0.35	0.20	0.15	0.30
[3,]	0.20	0.12	0.20	0.48
[4,]	0.10	0.16	0.10	0.64

# Unconditional Probabilities

Suppose we know the initial probabilities,

$$\alpha_i = P(X_0 = i), \quad i = 0, 1, 2, \dots$$

and  $\sum_i \alpha_i = 1$ .

According to the Law of total probability

$$\begin{aligned} P(X_n = j) &= \sum_{i=0}^{\infty} P(X_n = j \cap X_0 = i) \\ &= \sum_{i=0}^{\infty} P(X_n = j | X_0 = i) P(X_0 = i) \\ &= \sum_{i=0}^{\infty} P_{ij}^{(n)} \alpha_i \end{aligned}$$

## Example 3 (based on Example 1)

Let  $X_i = 0$  if it rains on day  $i$ ; otherwise  $X_i = 1$ . Suppose  $P_{00} = 0.7$  and  $P_{01} = 0.4$ . Suppose it rains on Monday. Suppose  $P(X_0 = 0) = 0.4$  and  $P(X_0 = 1) = 0.6$ . What is the probability that it will not rain on the 4th day after we start keeping records?

## Example 3 (cont.)

Let  $X_i = 0$  if it rains on day  $i$ ; otherwise  $X_i = 1$ . Suppose  $P_{00} = 0.7$  and  $P_{01} = 0.4$ . Suppose it rains on Monday. Suppose  $P(X_0 = 0) = 0.4$  and  $P(X_0 = 1) = 0.6$ . What is the probability that it will not rain on the 4th day after we start keeping records?

```
p <- matrix(c(0.7, 0.4, 0.3, 0.6), nrow = 2)
p%%p%%p%%p
```

	[,1]	[,2]
[1,]	0.5749	0.4251
[2,]	0.5668	0.4332

## Example 4

Suppose that a taxi driver operates between Wijerama and Nugegoda. If the driver is in Wijerama the probability that he gets a trip to Nugegoda from one passenger or a group of travelling together is 0.2 and that for him to get a trip nearby Wijerama is 0.8. If the driver is in Nugegoda he has equal chance of getting a trip to Wijerama or nearby Nugegoda. The behaviour of the driver evolves over time in a probabilistic manner.

0 - Wijerama, 1 - Nugegoda

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix}$$



## Example 4 (cont.)

- i) If the driver is currently at Wijerama, what is the probability that he will be back at Wijerama after three trips?

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```
p <- matrix(c(0.8, 0.5, 0.2, 0.5), ncol=2)  
p%%p%%p
```

```
      [,1] [,2]  
[1,] 0.722 0.278  
[2,] 0.695 0.305
```

## Example 4 (cont.)

- ii) If the driver is at Nugegoda, how many trips on the average will be in Nugegoda before he next goes to Wijerama?

## Example 4 (cont.): In-class

## Example 4 (cont.): In-class

Suppose  $P^{(0)} = (0.5, 0.5)$ , equal chance for driver be in either Wijerama or Nugegoda. What is the probability he will be in Wijerama after the first trip.

*In-class: Method 1*

## Probability after n-th step

$$\mathbf{P}^{(n)} = \mathbf{P}^{(0)} \mathbf{P}^n$$

## In-class: Method 2

# Types of States

*Definition:* If  $P_{ij}^{(n)} > 0$  for some  $n \geq 0$ , state  $j$  is **accessible** from  $i$ .

Notation:  $i \rightarrow j$ .

*Definition:* If  $i \rightarrow j$  and  $j \rightarrow i$ , then  $i$  and  $j$  **communicate**.

Notation:  $i \leftrightarrow j$ .



# Theorem:

Communication is an equivalence relation:

- (i)  $i \leftrightarrow i$  for all  $i$  (reflexive).
- (ii)  $i \leftrightarrow j$  implies  $j \leftrightarrow i$  (symmetric).
- (iii)  $i \leftrightarrow j$  and  $j \leftrightarrow k$  imply  $i \leftrightarrow k$  (transitive).

## In-class: Proof

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## In-class: Proof

(ii)  $i \leftrightarrow j$  implies  $j \leftrightarrow i$  (symmetric).

## In-class: Proof

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# In-class: Proof

## Note:

- Two states that communicate are said to be in the same **class**.
- The concept of communication divides the state space up into a number of separate classes.

In-class: demonstration

## Theorem (cont.)

**Definition:** An equivalence class consists of all states that communicate with each other.

Remark: Easy to see that two equivalence classes are disjoint.

Example: The following  $P$  has equivalence classes  $\{0, 1\}$  and  $\{2, 3\}$

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.75 & 0.25 \\ 0 & 0 & 0.25 & 0.75 \end{bmatrix}$$

## Equivalence class (cont.)

What about this?

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.3 & 0.2 & 0 \\ 0 & 0 & 0.75 & 0.25 \\ 0 & 0 & 0.25 & 0.75 \end{bmatrix}$$



# Irreducible

*Definition:* A MC is irreducible if there is only one equivalence class (i.e., if all states communicate with each other).

What about these?

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.75 & 0.25 \\ 0 & 0 & 0.25 & 0.75 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.3 & 0.2 & 0 \\ 0 & 0 & 0.75 & 0.25 \\ 0 & 0 & 0.25 & 0.75 \end{bmatrix}$$

## Irreducible (cont.)

What about these?

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 0.25 & 0 & 0.75 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

# Identify the equivalence classes

Consider a Markov chain with a state space  $S = \{0, 1, 2, 3, 4\}$  and having the following one-step transition probability matrix.

$$\mathbf{P} = \begin{bmatrix} 0.4 & 0.2 & 0 & 0.4 & 0 \\ 0.2 & 0.4 & 0.1 & 0.3 & 0 \\ 0.1 & 0.2 & 0.5 & 0.1 & 0.1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

# Problems <sup>1</sup>

Example 4.10

Example 4.11

Example 4.12

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<sup>1</sup>Introduction to Probability Models, Sheldon M. Ross

# Classification of States - next week

Reading Section 4.3: Classification of States<sup>2</sup>

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<sup>2</sup>Introduction to Probability Models, Sheldon M. Ross