STA 506 2.0 Linear Regression Analysis

Lecture 11-i: Transformations to Correct Model Inadequacies

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Introduction

In this section we are going to learn methods and procedures for building regression models when the assumptions are violated.

Transformations

- Variance-stabilizing transformations
- Transformations to linearize the model

How to get around the problem?

- Transform X variable(s)
- Transform Y
- Transformations on both X variable(s) and Y.

Variance-Stabilizing Transformations

Dataset 1

We will try to model salary as a function of years of experience.

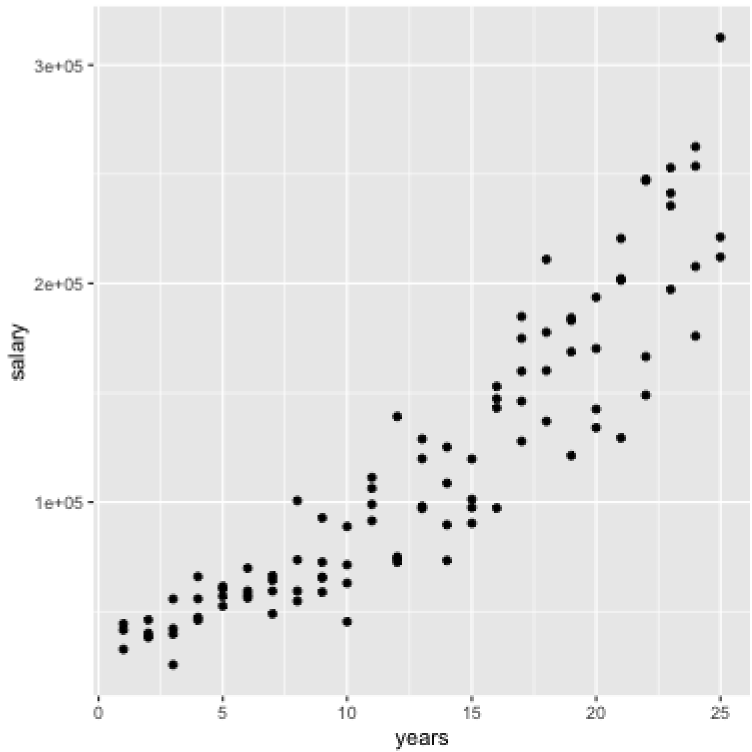
```
library(tidyverse)
salarydata <- read_csv("salarydata.csv")
salarydata
```

```
# A tibble: 100 x 2
   years salary
   <dbl>  <dbl>
 1     1  41504
 2     1  32619
 3     1  44322
 4     2  40038
 5     2  46147
 6     2  38447
 7     2  38163
 8     3  42104
 9     3  25597
10     3  39599
# … with 90 more rows
```

Data are obtained from:
https://daviddalpiaz.github.io/appliedstats/transformations.html

Salary vs Years of Experience

```
ggplot(salarydata, aes(x=years, y=salary)) + geom_point()
```




```
cor(salarydata$years, salarydata$salary)
```

```
[1] 0.9133066
```

Fit a Simple Linear Regression Model

```
salary_fit <- lm(salary ~ years, data = salarydata)
summary(salary_fit)
```

```
Call:
lm(formula = salary ~ years, data = salarydata)

Residuals:
   Min     1Q Median     3Q    Max 
-57225 -18104    241  15589  91332 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)     5302       5750   0.922    0.359    
years           8637        389  22.200   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 27360 on 98 degrees of freedom
Multiple R-squared:  0.8341,	Adjusted R-squared:  0.8324 
F-statistic: 492.8 on 1 and 98 DF,  p-value: < 2.2e-16
```

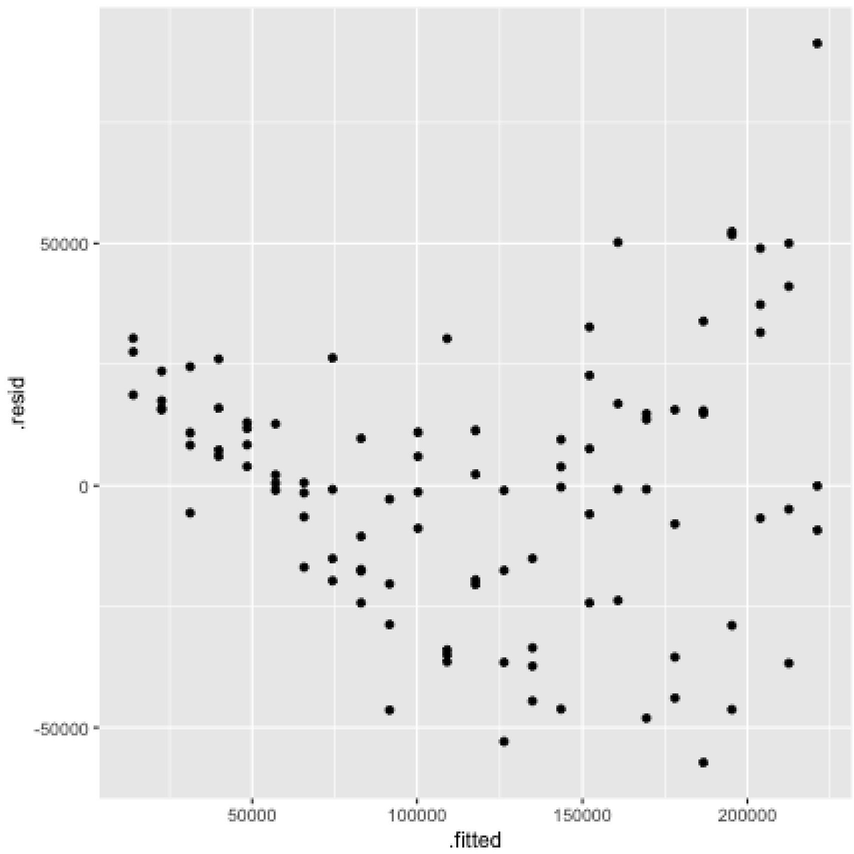
Compute Residuals and Fitted Values

```r
library(broom)
salary_residuals <- augment(salary_fit)
salary_residuals
```

```
# A tibble: 100 x 8
   salary years .fitted .resid .std.resid   .hat .sigma  .cooksd
    <dbl> <dbl>   <dbl>  <dbl>      <dbl>  <dbl>  <dbl>    <dbl>
 1  41504     1  13939. 27565.      1.03  0.0391 27347. 0.0215
 2  32619     1  13939. 18680.      0.697 0.0391 27428. 0.00988
 3  44322     1  13939. 30383.      1.13  0.0391 27315. 0.0261
 4  40038     2  22575. 17463.      0.650 0.0345 27436. 0.00753
 5  46147     2  22575. 23572.      0.877 0.0345 27388. 0.0137
 6  38447     2  22575. 15872.      0.590 0.0345 27447. 0.00622
 7  38163     2  22575. 15588.      0.580 0.0345 27449. 0.00600
 8  42104     3  31212. 10892.      0.404 0.0302 27473. 0.00255
 9  25597     3  31212. -5615.     -0.208 0.0302 27490. 0.000677
10  39599     3  31212.  8387.      0.311 0.0302 27482. 0.00151
# … with 90 more rows
```

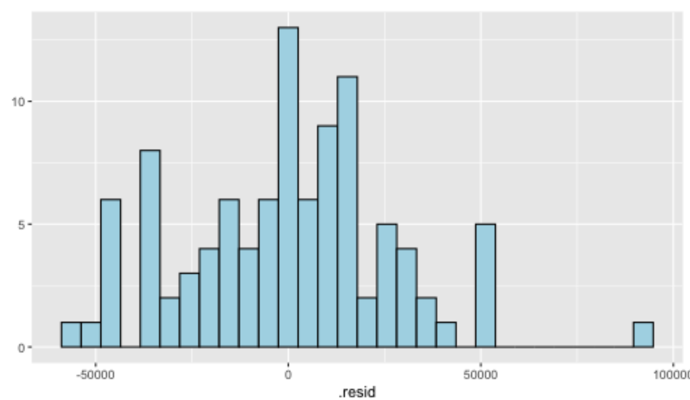
Residuals vs Fitted Values

```
ggplot(salary_residuals, aes(x=.fitted, y=.resid)) + geom_point()
```



Normality assumption

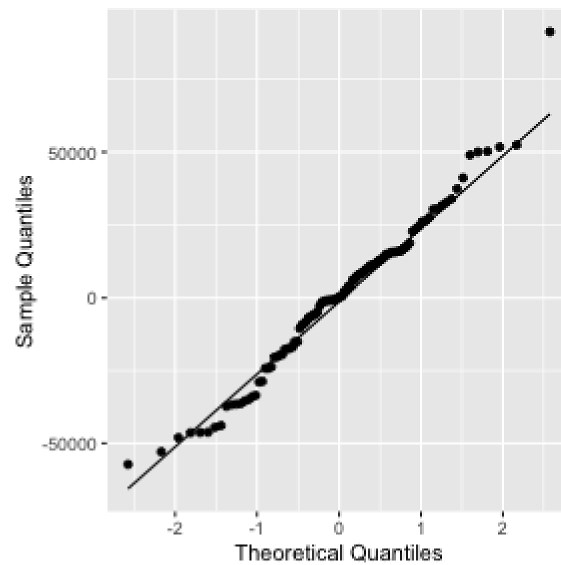
```
qplot(data=salary_residuals,
      x=.resid,)+
  geom_histogram(color="black",
                 fill="lightblue
```



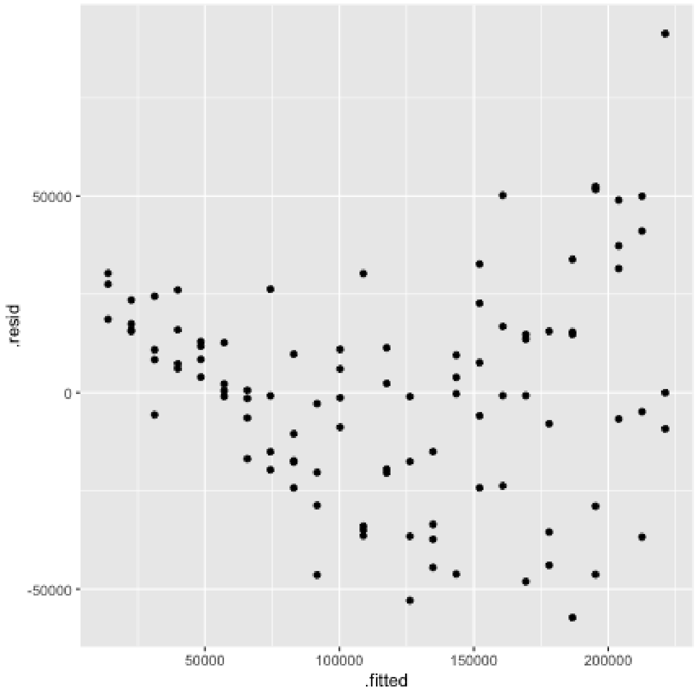

```
shapiro.test(salary_residuals$.r
```

```
    Shapiro-Wilk normality test

data:  salary_residuals$.resid
W = 0.98258, p-value = 0.2101
```

```
ggplot(salary_residuals,
       aes(sample=.resid))+
  stat_qq() +
  stat_qq_line() +
  labs(x="Theoretical Quantiles"
       y="Sample Quantiles")
```



Variance - Stabilizing Transformations



- Useful variance-stabilizing transformations
  - square root: $\sqrt{y}$
  - log transformation: $log(y)$
  - reciprocal: y^{-1}
  - reciprocal square root: $y^{-1/2}$

Apply log transformation

$$log(Y) = \beta_0 + \beta_1x_ + \epsilon$$

```
salarydata$log.salary <- log(salarydata$salary)
salarydata
```

```
# A tibble: 100 x 3
   years salary log.salary
   <dbl>  <dbl>      <dbl>
 1     1  41504       10.6
 2     1  32619       10.4
 3     1  44322       10.7
 4     2  40038       10.6
 5     2  46147       10.7
 6     2  38447       10.6
 7     2  38163       10.5
 8     3  42104       10.6
 9     3  25597       10.2
10     3  39599       10.6
# … with 90 more rows
```

Fit a regression model with log transformation

```
salary_fit_log <- lm(log.salary ~ years, data = salarydata)
salary_fit_log
```

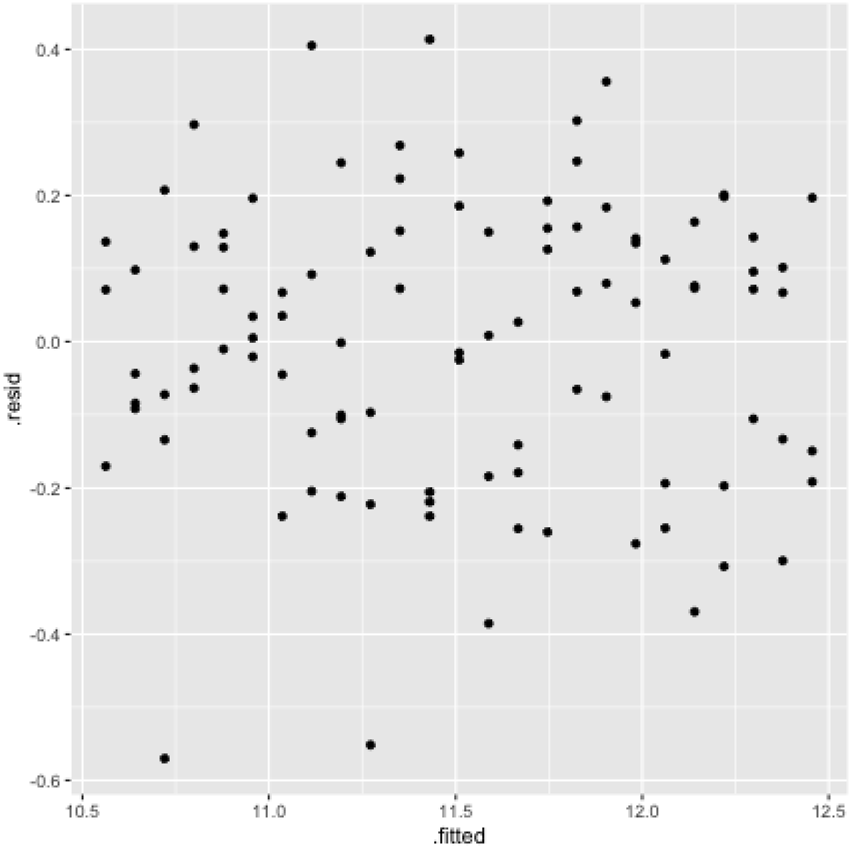
```

Call:
lm(formula = log.salary ~ years, data = salarydata)

Coefficients:
(Intercept)        years
   10.48381      0.07888
```

Compute Residuals and Fitted values

```
salary_log_residuals <- augment(salary_fit_log)
salary_log_residuals
```

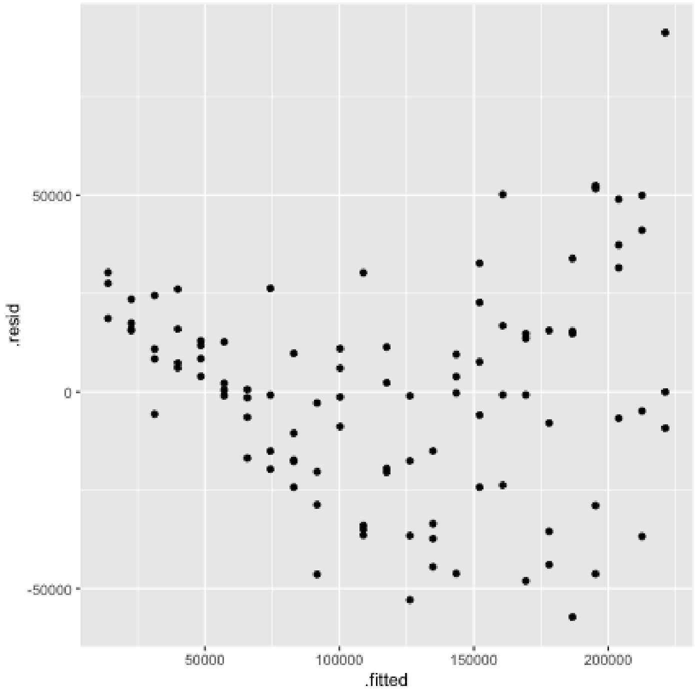
```
# A tibble: 100 x 8
   log.salary years .fitted  .resid .std.resid   .hat .sigma  .cooksd
        <dbl> <dbl>   <dbl>   <dbl>      <dbl>  <dbl>  <dbl>    <dbl>
 1       10.6     1    10.6  0.0709      0.370 0.0391  0.196 0.00278
 2       10.4     1    10.6 -0.170      -0.887 0.0391  0.196 0.0160
 3       10.7     1    10.6  0.137       0.713 0.0391  0.196 0.0103
 4       10.6     2    10.6 -0.0440     -0.229 0.0345  0.196 0.000936
 5       10.7     2    10.6  0.0980      0.510 0.0345  0.196 0.00465
 6       10.6     2    10.6 -0.0845     -0.440 0.0345  0.196 0.00346
 7       10.5     2    10.6 -0.0919     -0.479 0.0345  0.196 0.00409
 8       10.6     3    10.7 -0.0725     -0.377 0.0302  0.196 0.00221
 9       10.2     3    10.7 -0.570      -2.96  0.0302  0.187 0.137
10       10.6     3    10.7 -0.134      -0.696 0.0302  0.196 0.00754
# … with 90 more rows
```

Residuals vs Fitted values

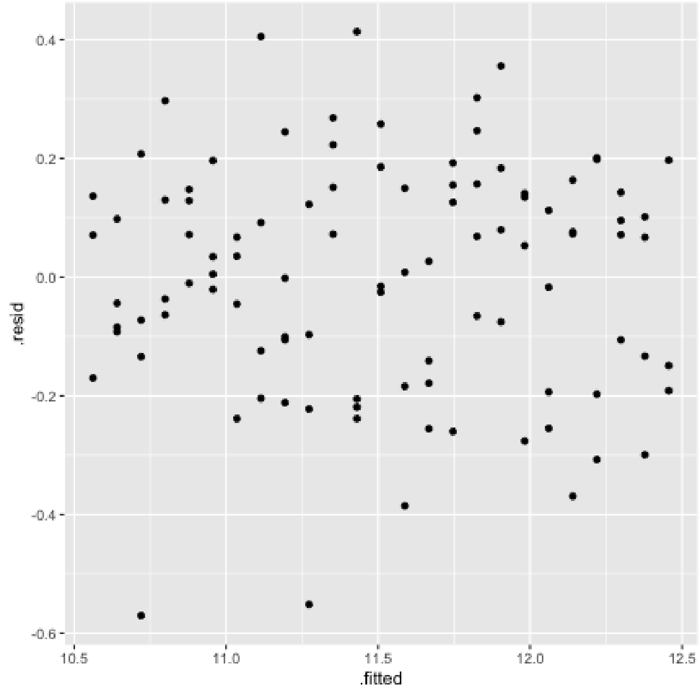


Residuals vs Fitted

$$Y = \beta_0 + \beta_1x + \epsilon$$



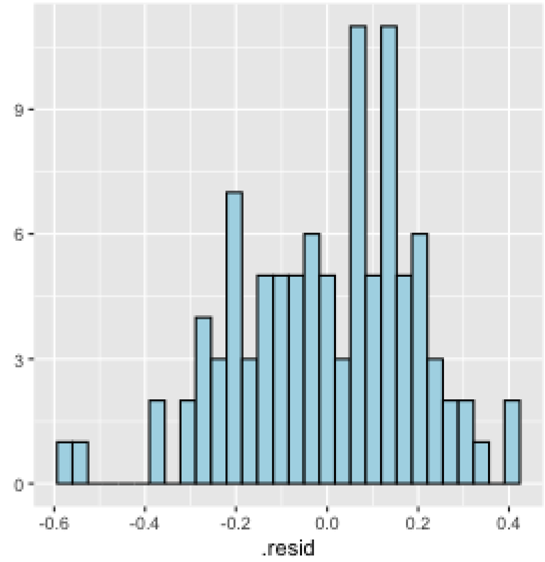
$$log(Y) = \beta_0 + \beta_1x + \epsilon$$



Normality assumption

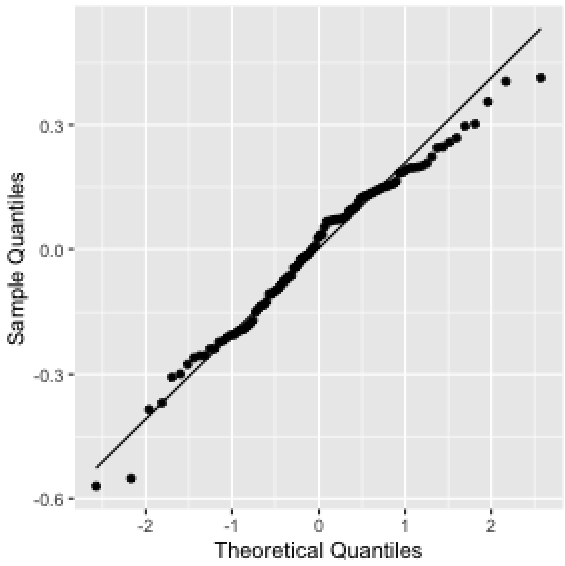
\(log(Y) = \beta_0 + \beta_1x + \epsilon\)

```
qplot(data=salary_log_residuals, x=.resid,)+
  geom_histogram(color="black", fill="lightblue")
```



Normality assumption (cont.)

```
ggplot(salary_log_residuals,
       aes(sample=.resid))+
  stat_qq() +
  stat_qq_line() +
  labs(x="Theoretical Quantiles",
       y="Sample Quantiles")
```



Normality assumption (cont.)

```
shapiro.test(salary_log_residuals$.resid)
```

```

	Shapiro-Wilk normality test

data:  salary_log_residuals$.resid
W = 0.98033, p-value = 0.141
```

Model Statistics

```
summary(salary_fit_log)
```

```
Call:
lm(formula = log.salary ~ years, data = salarydata)

Residuals:
     Min       1Q   Median       3Q      Max 
-0.57022 -0.13560  0.03048  0.14157  0.41366 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 10.48381    0.04108  255.18   <2e-16 ***
years        0.07888    0.00278   28.38   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1955 on 98 degrees of freedom
Multiple R-squared:  0.8915,	Adjusted R-squared:  0.8904 
F-statistic: 805.2 on 1 and 98 DF,  p-value: < 2.2e-16
```

Hypothesis testing

Intercept

$H_0: \beta_0 = 0$ vs $H_1: \beta_0 \neq 0$

Decision:

p-value < 0.05. We reject H_0 under 0.05 level of significance.

Conclusion: We can conclude that population regression line intercept is significantly different from 0.

Slope

$H_0: \beta_1 = 0$ vs $H_1: \beta_1 \neq 0$

Decision:

p-value < 0.05. We reject H_0 under 0.05 level of significance.

Conclusion: The variable `years` contributes significantly to the model.

Re-scale

log scale to the original scale of the data

Preliminary Maths

$$Y=10$$ $$log(10) = 2.302585$$ $$e^{2.302585} = 10$$ $$e^{a+b} = e^ae^b$$

New fitted regression model

$$\hat{log(Y)} = 10.48 + 0.079X$$

Convert to original scale

$$e^{\hat{log(Y)}} = e^{10.48 + 0.079X}$$

$$Y = e^{10.48}e^{0.079X}$$

Interpretation of slope

When $X=x_1$

$$Y' = e^{10.48}e^{0.079x_1}$$

When $X={x_1 + 1}$

$$Y'' = e^{10.48}e^{0.079(x_1 + 1)} = e^{10.48}e^{0.079(x_1)}e^{0.079}$$

$$e^{0.079} = 1.0822$$

We see that for every one additional year of experience, average (median) salary increases 1.0822 times. We are now multiplying, not adding.

Interpretation of slope

$$\hat{log(Y)} = \hat{\beta_0} + \hat{\beta_1}x$$

Interpretation of $\hat{\beta_0}$

When $x=0$, the median of Y is expected to be $e^{\hat{\beta}_0}$.

Interpretation of $\hat{\beta_1}$

For every one unit increase in x, the median of Y is expected to multiply by a factor of $e^{\hat{\beta}_1}$.

Why median not mean?: Read here
https://www2.stat.duke.edu/courses/Spring20/sta210.001/slides/lec-slides/09-transformations.html#29

Confidence interval for β_j

The confidence interval for the coefficient of x describing its relationship with Y is:

- Confidence intervals β_0

$$[\hat{\beta_0} - t_{\alpha/2, n-2}se(\hat{\beta_0}), \hat{\beta_0} + t_{\alpha/2, n-2}se(\hat{\beta_0})]$$

- Confidence intervals β_1

$$[\hat{\beta_1} - t_{\alpha/2, n-2}se(\hat{\beta_1}), \hat{\beta_1} + t_{\alpha/2, n-2}se(\hat{\beta_1})]$$

```r
confint(salary_fit_log, level=0.95)
```

```
                 2.5 %      97.5 %
(Intercept) 10.40227614 10.56533628
years        0.07336341  0.08439611
```

Confidence interval for β_j - backtransform

The confidence interval for the coefficient of x describing its relationship with $log(Y)$ is:

- Confidence intervals β_0

$$[e^{\hat{\beta_0} - t_{\alpha/2, n-2}se(\hat{\beta_0})}, e^{\hat{\beta_0} + t_{\alpha/2, n-2}se(\hat{\beta_0})}]$$

- Confidence intervals β_1

$$[e^{\hat{\beta_1} - t_{\alpha/2, n-2}se(\hat{\beta_1})}, e^{\hat{\beta_1} + t_{\alpha/2, n-2}se(\hat{\beta_1})}]$$

```
exp(confint(salary_fit_log, level=0.95))
```

```
                    2.5 %      97.5 %
(Intercept) 32934.503906 38767.45083
years           1.076122     1.08806
```

Transformations to Linearize the Model

source: https://srilankamirror.com/biz/20026-coconuts-to-be-measured-by-weight-instead-of-circumference

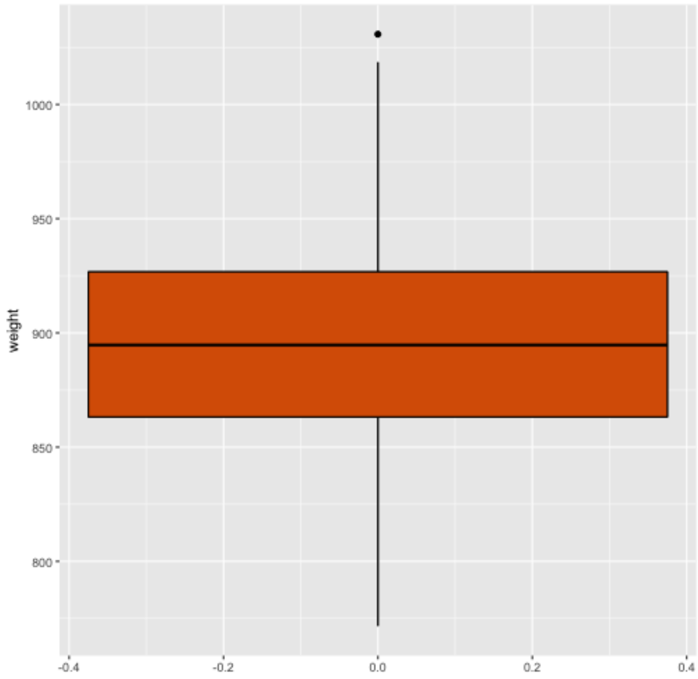
Data

```r
coconut <- read_csv("coconut.csv") # Ignore the warning message
coconut
```

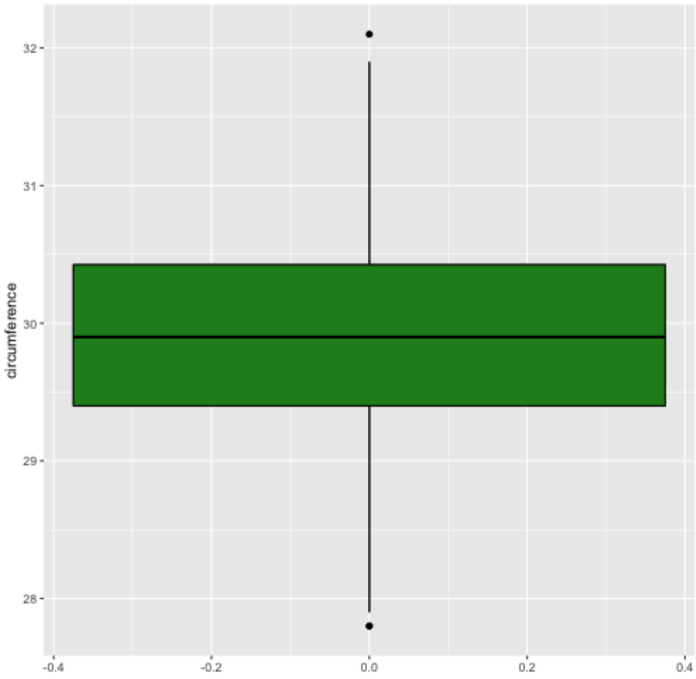
```
# A tibble: 100 x 3
      X1 weight circumference
   <dbl>  <dbl>         <dbl>
 1     1   773.          27.8
 2     2   772.          27.8
 3     3   780.          27.9
 4     4   790.          28.1
 5     5   806.          28.4
 6     6   813.          28.5
 7     7   817.          28.6
 8     8   820.          28.6
 9     9   818.          28.6
10    10   830.          28.8
# … with 90 more rows
```

EDA

```
qplot(data=coconut, y=weight, ge
  geom_boxplot(color="black", fi
```

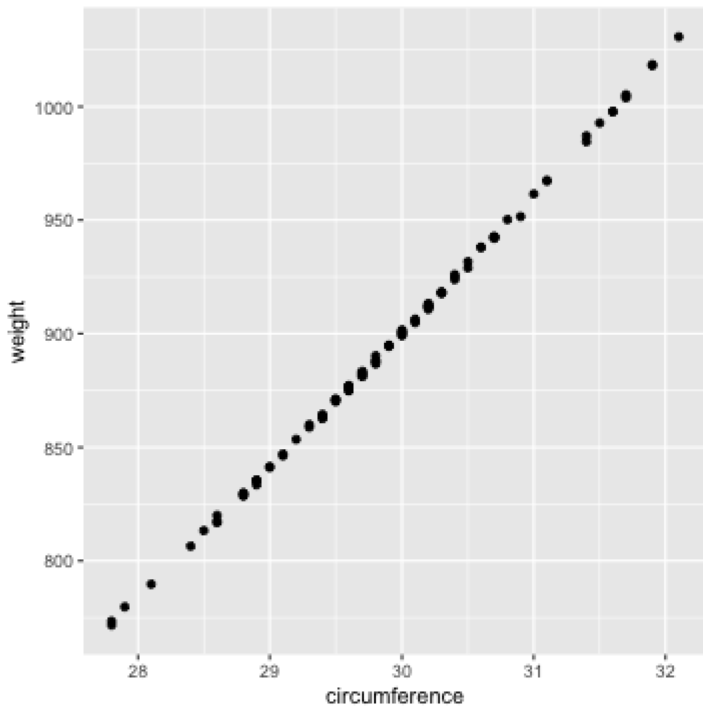



```
qplot(data=coconut, y=circumfere
  geom_boxplot(color="black", fi
```



Weight vs Circumference

```
ggplot(coconut, aes(x=circumference, y=weight)) + geom_point()
```




```
cor(coconut$circumference, coconut$weight)
```

```
[1] 0.9996482
```

Fit a regression model

```
coconut.lm <- lm(weight ~ circumference, data=coconut)
coconut.lm
```

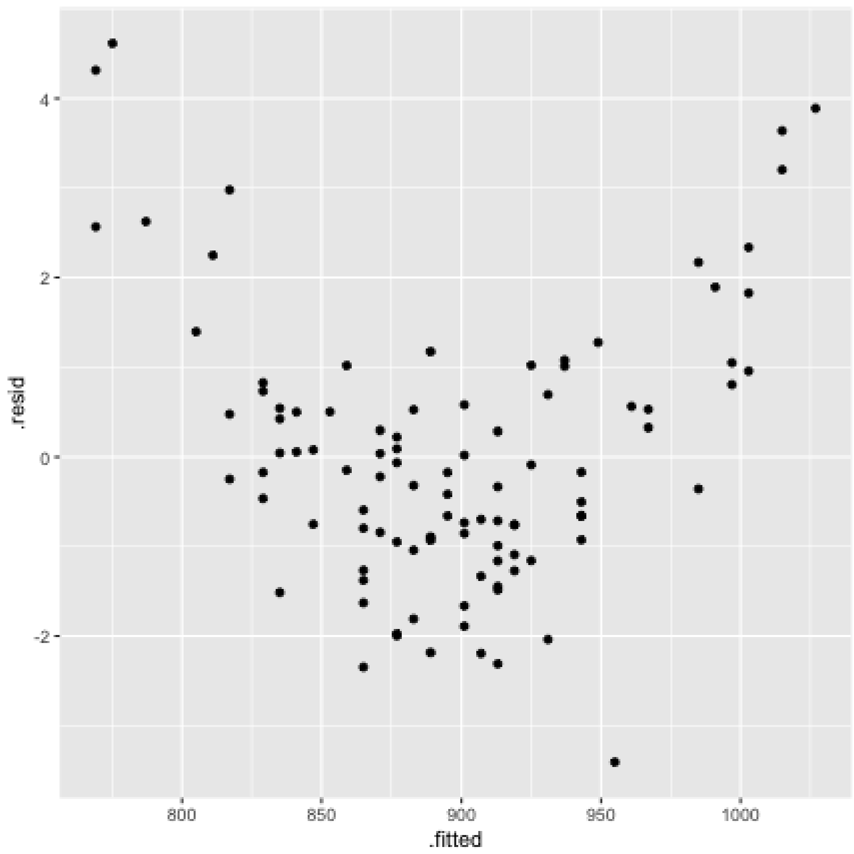
```

Call:
lm(formula = weight ~ circumference, data = coconut)

Coefficients:
  (Intercept)  circumference  
      -897.25          59.94  
```

Compute Residuals and Fitted Values

```
coconut.lm.result <- broom::augment(coconut.lm)
coconut.lm.result
```

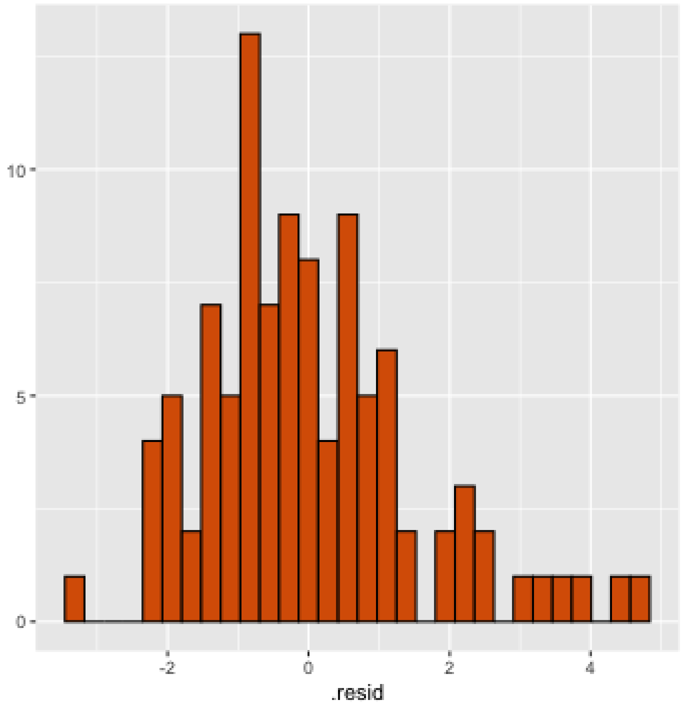
```
# A tibble: 100 x 8
   weight circumference .fitted .resid .std.resid   .hat .sigma   .cooksd
    <dbl>         <dbl>   <dbl>  <dbl>      <dbl>  <dbl>  <dbl>     <dbl>
 1   773.          27.8    769.  4.32       2.93  0.0602   1.46 0.275
 2   772.          27.8    769.  2.57       1.74  0.0602   1.50 0.0971
 3   780.          27.9    775.  4.62       3.12  0.0556   1.45 0.287
 4   790.          28.1    787.  2.63       1.77  0.0470   1.50 0.0772
 5   806.          28.4    805.  1.39       0.934 0.0359   1.52 0.0162
 6   813.          28.5    811.  2.25       1.50  0.0326   1.51 0.0380
 7   817.          28.6    817. -0.247     -0.165 0.0295   1.53 0.000413
 8   820.          28.6    817.  2.98       1.99  0.0295   1.50 0.0601
 9   818.          28.6    817.  0.473      0.316 0.0295   1.53 0.00152
10   830.          28.8    829.  0.825      0.549 0.0240   1.53 0.00371
# … with 90 more rows
```

Residuals vs Fitted values



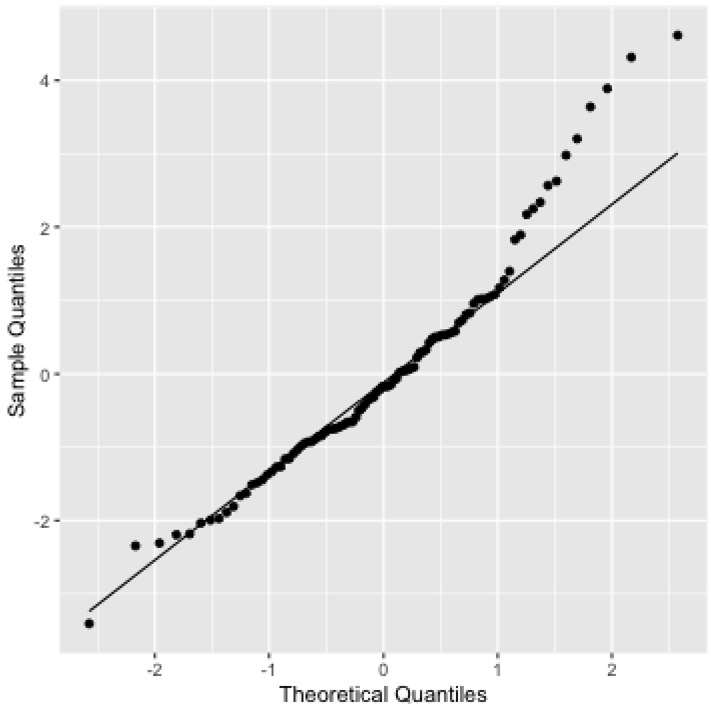
Normality assumption

```
qplot(data=coconut.lm.result, x=.resid, geom=c("histogram"))+
  geom_histogram(color="black", fill="#d95f02")
```



Normality assumption

```
ggplot(coconut.lm.result, aes(sample=.resid))+
  stat_qq() + stat_qq_line() +
  labs(x="Theoretical Quantiles",  y="Sample Quantiles")
```



Normality assumption (cont.)

```
shapiro.test(coconut.lm.result$.resid)
```

```

	Shapiro-Wilk normality test

data:  coconut.lm.result$.resid
W = 0.95463, p-value = 0.001697
```

Transform Y

$$\sqrt(Y) = \beta_0 + \beta_1 x + \epsilon$$

```
coconut$sqrt.weight <- sqrt(coconut$weight)
coconut
```

```
# A tibble: 100 x 4
      X1 weight circumference sqrt.weight
   <dbl>  <dbl>         <dbl>       <dbl>
 1     1   773.          27.8        27.8
 2     2   772.          27.8        27.8
 3     3   780.          27.9        27.9
 4     4   790.          28.1        28.1
 5     5   806.          28.4        28.4
 6     6   813.          28.5        28.5
 7     7   817.          28.6        28.6
 8     8   820.          28.6        28.6
 9     9   818.          28.6        28.6
10    10   830.          28.8        28.8
# … with 90 more rows
```

Estimate parameters of \(\sqrt(Y) = \beta_0 + \beta_1 x + \epsilon\)

```
coconut.lm2 <- lm(sqrt.weight ~ circumference, data=coconut)
coconut.lm2
```

```

Call:
lm(formula = sqrt.weight ~ circumference, data = coconut)

Coefficients:
  (Intercept)  circumference  
      0.01631        0.99951  
```

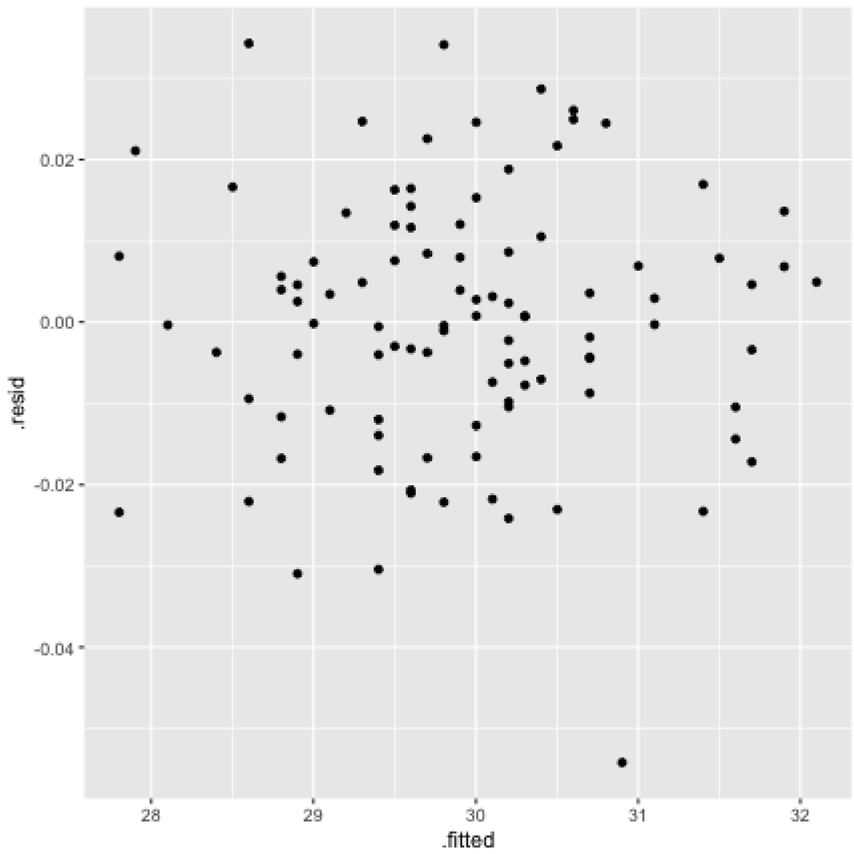
Compute Residuals and Fitted Values

```
coconut.lm.result2 <- broom::augment(coconut.lm2)
coconut.lm.result2
```

```
# A tibble: 100 x 8
   sqrt.weight circumference .fitted    .resid .std.resid   .hat .sigma  .coo
         <dbl>         <dbl>   <dbl>     <dbl>      <dbl>  <dbl>  <dbl>    <d
 1        27.8          27.8    27.8  0.00810      0.536  0.0602 0.0157  9.20
 2        27.8          27.8    27.8 -0.0234      -1.55   0.0602 0.0155  7.66
 3        27.9          27.9    27.9  0.0211       1.39   0.0556 0.0155  5.69
 4        28.1          28.1    28.1 -0.000372    -0.0244 0.0470 0.0157  1.47
 5        28.4          28.4    28.4 -0.00373     -0.244  0.0359 0.0157  1.10
 6        28.5          28.5    28.5  0.0166       1.08   0.0326 0.0156  1.98
 7        28.6          28.6    28.6 -0.0221      -1.43   0.0295 0.0155  3.13
 8        28.6          28.6    28.6  0.0343       2.23   0.0295 0.0153  7.58
 9        28.6          28.6    28.6 -0.00946     -0.615  0.0295 0.0157  5.75
10        28.8          28.8    28.8  0.00562      0.365  0.0240 0.0157  1.64
# … with 90 more rows
```

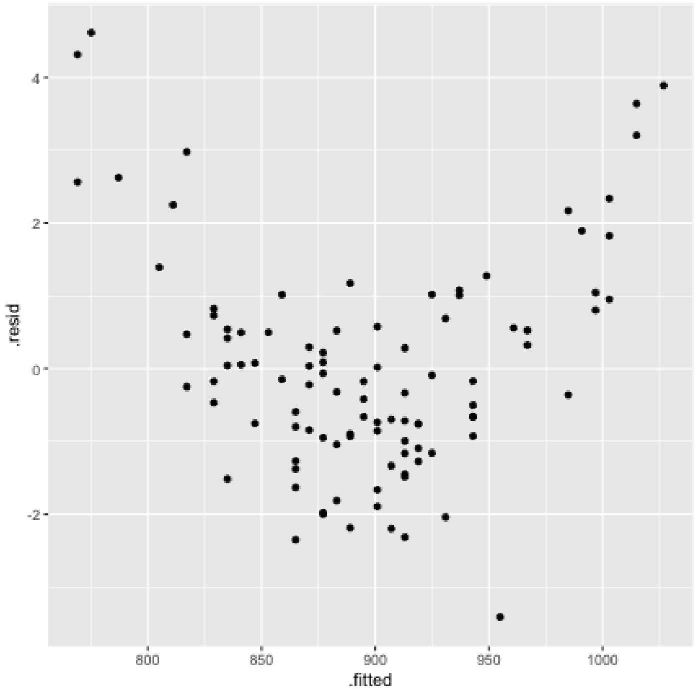
Residuals vs Fitted values

```
ggplot(coconut.lm.result2, aes(x=.fitted, y=.resid)) + geom_point()
```

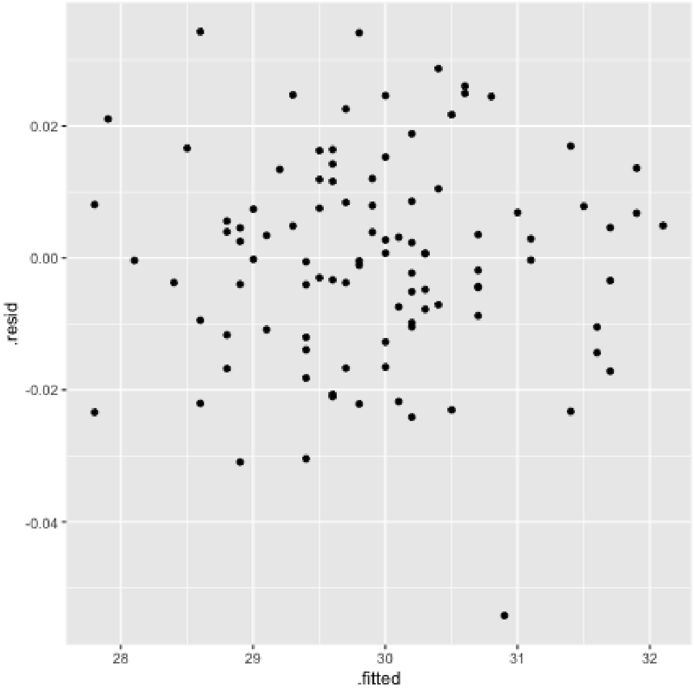


Residuals vs Fitted values

$Y = \beta_0 + \beta_1 x + \epsilon$

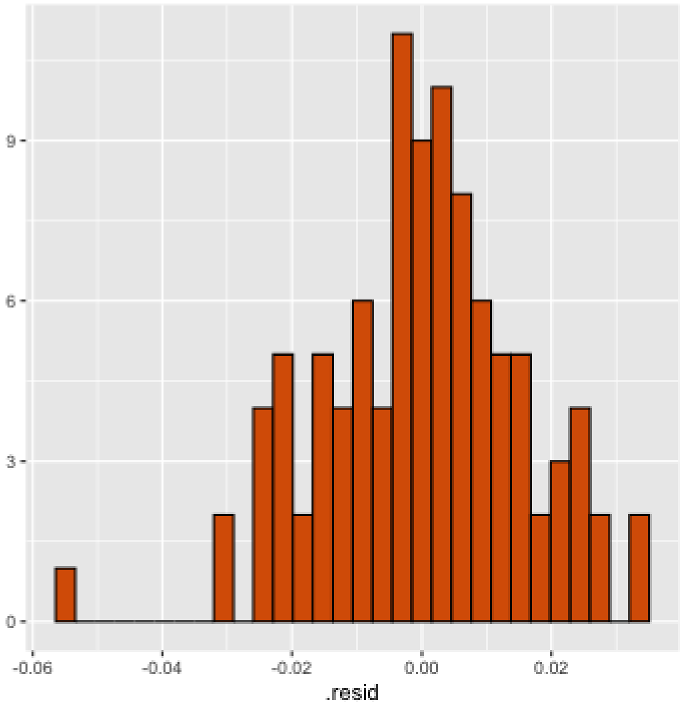


$\sqrt(Y) = \beta_0 + \beta_1 x + \epsilon$



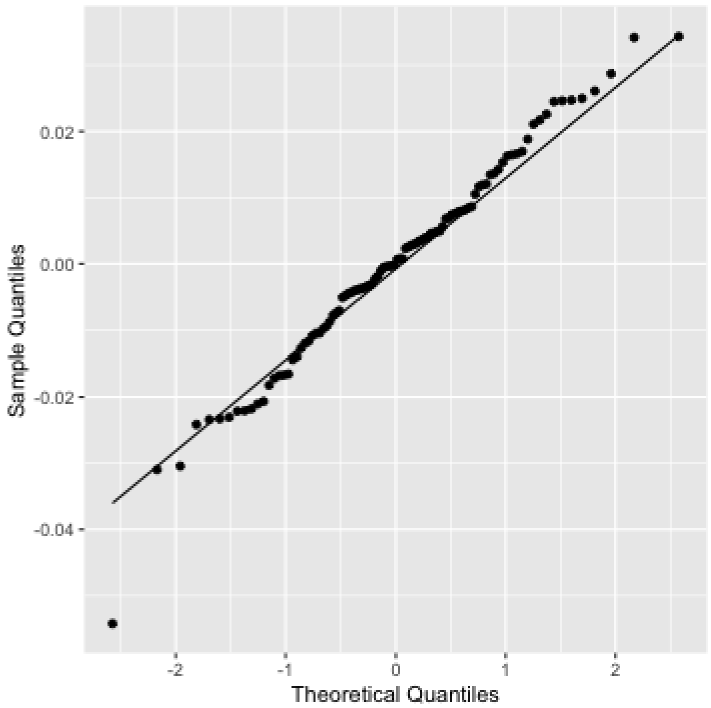
Normality assumption

```
qplot(data=coconut.lm.result2, x=.resid, geom=c("histogram"))+
  geom_histogram(color="black", fill="#d95f02")
```



Normality assumption

```
ggplot(coconut.lm.result2, aes(sample=.resid))+
  stat_qq() + stat_qq_line() +
  labs(x="Theoretical Quantiles",  y="Sample Quantiles")
```



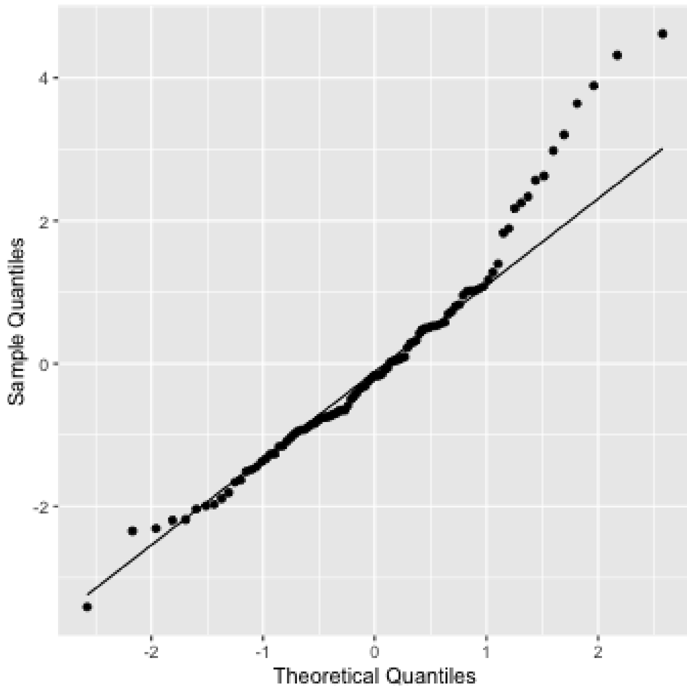
Normality assumption (cont.)

```
shapiro.test(coconut.lm.result2$.resid)
```

```
	Shapiro-Wilk normality test

data:  coconut.lm.result2$.resid
W = 0.98624, p-value = 0.3885
```

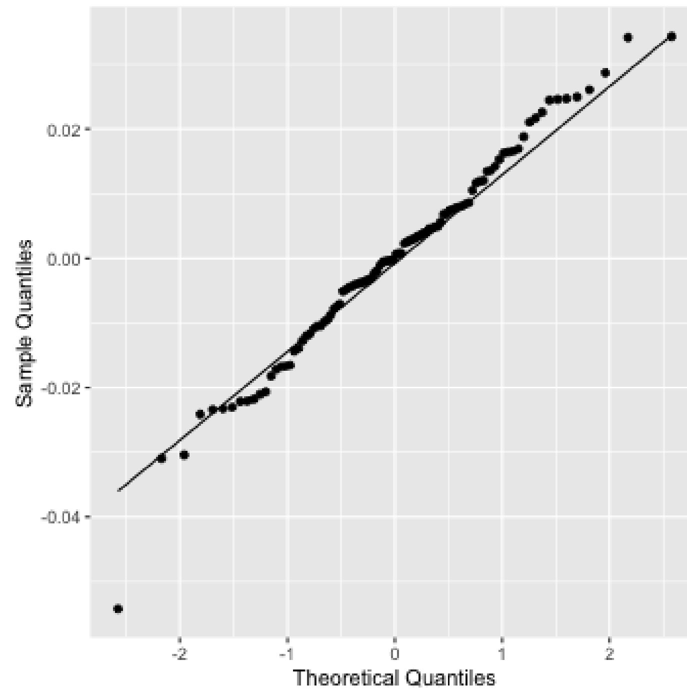
Normality test

\(Y = \beta_0 + \beta_1 x + \epsilon\)




```
    Shapiro-Wilk normality test

data:  coconut.lm.result$.resid
W = 0.95463, p-value = 0.001697
```

\(\sqrt(Y) = \beta_0 + \beta_1 x + \epsilon\)




```
    Shapiro-Wilk normality test

data:  coconut.lm.result2$.resid
```

Your turn: Tests on individual regression coefficients

```
summary(coconut.lm2)
```

```
Call:
lm(formula = sqrt.weight ~ circumference, data = coconut)

Residuals:
      Min        1Q    Median        3Q       Max 
-0.054212 -0.009988  0.000234  0.008474  0.034324 

Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
(Intercept)   0.016311   0.049346   0.331    0.742    
circumference 0.999507   0.001648 606.382   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.0156 on 98 degrees of freedom
Multiple R-squared:  0.9997,	Adjusted R-squared:  0.9997 
F-statistic: 3.677e+05 on 1 and 98 DF,  p-value: < 2.2e-16
```

Back transformation: sqrt

- The back transformation is to square the number. If you have negative numbers, you can't take the square root; you should add a constant to each number to make them all positive.

- Square-root transformation when the variable is a count of something, or the variables that can take positive values.

Back transformation: sqrt

$$\sqrt(Y) = 0.016 + 0.99x$$

$$Y = (0.016 + 0.99x)^2$$

$$Y = 0.016^2 + 2(0.016 \times0.99)x + (0.99x)^2$$

> "Although the popular square root transformation can be useful for simplifying relationships with quadratic effects, and also for stabilizing variances (Baguley, 2012), this transformation does not aid in interpretation."
>
> Data Transformations for Inference with Linear Regression: Clarifications and Recommendations. [J. Pek, O. Wong, A. C. Wong]

Link to the paper: https://scholarworks.umass.edu/cgi/viewcontent.cgi?article=1360&context=pare

Note

- When the model is presented to the professional community or to the general public/ when making preidctions, transformations done to the dependent variable Y should be transformed back to the **original units**

Model with transformation on X

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1log(x)$$

Interpretation of intercept:

When $log(x)=0$, that is $x=1$, mean of Y is expected to be $\hat{\beta_0}$

Interpretation of slope:

When X is multiplied by a factor of K, the mean of Y changes by $\hat{\beta}_1log(K)$.

Example: when $K=2$ (some constant value)

When X is multiplied by a factor of 2, the mean of Y changes by $\hat{\beta}_1log(2)$.

Help but Not RULES

Transformations on X

- Suppose the assumption of normally and independently distributed responses with constant variance are at least approximately satisfied, however the relationship between Y and one or more of the regressor variables is nonlinear.

Transformations on Y

- To correct nonnormality assumption and/or nonconstant variance assumption of the error term.

Acknowledgement

Introduction to Linear Regression Analysis, Douglas C. Montgomery, Elizabeth A. Peck, G. Geoffrey Vining