$$\min_{r,w} \quad \frac{1}{2}\|r\|_2^2 + \lambda\|w\|_1$$
$$\text{st} \quad r = Xw - y$$

Lagrangian is $\mathcal{L}(w,r,\mu) = \frac{1}{2}r^T r + \mu^T r + \lambda\|w\|_1 - \mu^T X w + \mu^T y$

Thus, recall that, $g(\mu) = \inf_{w,r} \mathcal{L}(w,r,\mu)$

$$= \inf_r h(r) - \lambda \sup_w \left(\frac{1}{\lambda}(X^T\mu)^T w - \|w\|_1 \right) + \mu^T y$$

We've shown: $r^* = -\mu^*$ for the 1^{st} part

For the 2^{nd} part: $\sup_w \frac{1}{\lambda}(X^T\mu)^T w - \|w\|_1 = \left\|\frac{X^T\mu}{\lambda}\right\|_1^*$

When considering $\|\cdot\|_1^*$:

- for any $y \in \mathbb{R}^m$, $\forall x \in \mathbb{R}^m$: $y^T x - \|x\|_1 = \sum_{i=1}^m y_i x_i - \sum_{i=1}^m |x_i|$

$$= \sum_{i=1}^m y_i \cdot \mathrm{sgn}(x_i) \cdot |x_i| - \sum_{i=1}^m |x_i|$$
$$= \sum_{i=1}^m |x_i| \left(y_i\, \mathrm{sgn}(x_i) - 1 \right)$$

This quantity explodes whenever $+y_i > 1$ or $-y_i < 1$
(if $\mathrm{sgn}(x_i) = +1$) (if $\mathrm{sgn}(x_i) = -1$)

So y needs to be s.t $\|y\|_\infty \leq 1$

- Then $y^T x - \|x\|_1 \leq \sum_{i=1}^m |x_i| (|y_i| - 1)$ because $y_i\,\mathrm{sgn}(x_i) \leq |y_i|$

Equality holds when $\forall i \in [1; m]$, $|x_i|(|y_i| - 1) = 0$

ie $x_i = 0$ OR $|y_i| - 1 = 0$

- In our case, μ^*, w^* satisfies $w_i^* \left(\left|\frac{X^T\mu^*}{\lambda}\right|_i - 1 \right) = 0 \quad \forall i \in [1; d]$

ie $\boxed{w_i^* \left(|X^T\mu^*|_i - \lambda \right) = 0}$ ie either $w_i^* = 0$ OR $|X^T\mu^*|_i - \lambda = 0$