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Stochastic Calculus for Finance I

Student's Manual: Solutions to Selected Exercises

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The Binomial No-Arbitrage Pricing Model

1.7 Solutions to Selected Exercises

Exercise 1.2. Suppose in the situation of Example 1.1.1 that the option sells for 1.20 at time zero. Consider an agent who begins with wealth $X_0 = 0$ and at time zero buys Δ_0 shares of stock and Γ_0 options. The numbers Δ_0 and Γ_0 can be either positive or negative or zero. This leaves the agent with a cash position of $-4\Delta_0 - 1.20\Gamma_0$. If this is positive, it is invested in the money market; if it is negative, it represents money borrowed from the money market. At time one, the value of the agent's portfolio of stock, option and money market is

$$X_1 = \Delta_0 S_1 + \Gamma_0 (S_1 - 5)^+ - \frac{5}{4}\left(4\Delta_0 + 1.20\Gamma_0\right).$$

Assume that both H and T have positive probability of occurring. Show that if there is a positive probability that X_1 is positive, then there is a positive probability that X_1 is negative. In other words, one cannot find an arbitrage when the time-zero price of the option is 1.20.

Solution. Considering the cases of a head and of a tail on the first toss, and utilizing the numbers given in Example 1.1.1, we can write:

$$X_1(H) = 8\Delta_0 + 3\Gamma_0 - \frac{5}{4}(4\Delta_0 + 1.20\Gamma_0),$$
$$X_1(T) = 2\Delta_0 + 0 \cdot \Gamma_0 - \frac{5}{4}(4\Delta_0 + 1.20\Gamma_0)$$

Adding these, we get

$$X_1(H) + X_1(T) = 10\Delta_0 + 3\Gamma_0 - 10\Delta_0 - 3\Gamma_0 = 0,$$

or, equivalently,

$$X_1(H) = -X_1(T).$$

In other words, either $X_1(H)$ and $X_1(T)$ are both zero, or they have opposite signs. Taking into account that both $p > 0$ and $q > 0$, we conclude that if there is a positive probability that X_1 is positive, then there is a positive probability that X_1 is negative.

Exercise 1.6 (Hedging a long position - one period.). Consider a bank that has a long position in the European call written on the stock price in Figure 1.1.2. The call expires at time one and has strike price $K = 5$. In Section 1.1, we determined the time-zero price of this call to be $V_0 = 1.20$. At time zero, the bank owns this option, which ties up capital $V_0 = 1.20$. The bank wants to earn the interest rate 25% on this capital until time one, i.e., without investing any more money, and regardless of how the coin tossing turns out, the bank wants to have

$$\frac{5}{4} \cdot 1.20 = 1.50$$

at time one, after collecting the payoff from the option (if any) at time one. Specify how the bank's trader should invest in the stock and money market to accomplish this.

Solution. The trader should use the opposite of the replicating portfolio strategy worked out in Example 1.1.1. In particular, she should short $\frac{1}{2}$ share of stock, which generates \$2 income. She should invest this in the money market. At time one, if the stock goes up in value, the bank has an option worth \$3, has $\$\left(\frac{5}{4} \cdot 2\right) = \$2.50$ in the money market, and must pay \$4 to cover the short position in the stock. This leaves the bank with \$1.50, as desired. On the other hand, if the stock goes down in value, then at time one the bank has an option worth \$0, still has \$2.50 in the money market, and must pay \$1 to cover the short position in stock. Again, the bank has \$1.50, as desired.

Exercise 1.8 (Asian option). Consider the three-period model of Example 1.2.1, with $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$, and take the interest rate $r = \frac{1}{4}$, so that $\tilde{p} = \tilde{q} = \frac{1}{2}$. For $n = 0, 1, 2, 3$, define $Y_n = \sum_{k=0}^{n} S_k$ to be the sum of the stock prices between times zero and n. Consider an *Asian call option* that expires at time three and has strike $K = 4$ (i.e., whose payoff at time three is $\left(\frac{1}{4}Y_3 - 4\right)^+$). This is like a European call, except the payoff of the option is based on the average stock price rather than the final stock price. Let $v_n(s, y)$ denote the price of this option at time n if $S_n = s$ and $Y_n = y$. In particular, $v_3(s, y) = \left(\frac{1}{4}y - 4\right)^+$.

(i) Develop an algorithm for computing v_n recursively. In particular, write a formula for v_n in terms of v_{n+1}.

(ii) Apply the algorithm developed in (i) to compute $v_0(4, 4)$, the price of the Asian option at time zero.

(iii) Provide a formula for $\delta_n(s,y)$, the number of shares of stock which should be held by the replicating portfolio at time n if $S_n = s$ and $Y_n = y$.

Solution.

(i), (iii) Assume that at time n, $S_n = s$ and $Y_n = y$. Then if the $(n+1)$-st toss results in H, we have

$$S_{n+1} = us, \quad Y_{n+1} = Y_n + S_{n+1} = y + us.$$

If the $(n+1)$-st toss results in T, we have instead

$$S_{n+1} = ds, \quad Y_{n+1} = Y_n + S_{n+1} = y + ds.$$

Therefore, formulas (1.2.16) and (1.2.17) take the form

$$\begin{aligned} v_n(s,y) &= \frac{1}{1+r}[\tilde{p}v_{n+1}(us, y+us) + \tilde{q}v_{n+1}(ds, y+ds)], \\ \delta_n(s,y) &= \frac{v_{n+1}(us, y+us) - v_{n+1}(ds, y+ds)}{us - ds}. \end{aligned}$$

(ii) We first list the relevant values of v_3, which are

$$\begin{aligned} v_3(32,60) &= (60/4-4)^+ = 11, \\ v_3(8,36) &= (36/4-4)^+ = 5, \\ v_3(8,24) &= (24/4-4)^+ = 2, \\ v_3(8,18) &= (18/4-4)^+ = 0.50, \\ v_3(2,18) &= (18/4-4)^+ = 0.50, \\ v_3(2,12) &= (12/4-4)^+ = 0, \\ v_3(2,9) &= (9/4-4)^+ = 0, \\ v_3(.50,7.50) &= (7.50/4-4)^+ = 0. \end{aligned}$$

We next use the algorithm from (i) to compute the relevant values of v_2:

$$\begin{aligned} v_2(16,28) &= \frac{4}{5}\left[\frac{1}{2}v_3(32,60) + \frac{1}{2}v_3(8,36)\right] = 6.40, \\ v_2(4,16) &= \frac{4}{5}\left[\frac{1}{2}v_3(8,24) + \frac{1}{2}v_3(2,18)\right] = 1, \\ v_2(4,10) &= \frac{4}{5}\left[\frac{1}{2}v_3(8,18) + \frac{1}{2}v_3(2,12)\right] = 0.20, \\ v_2(1,7) &= \frac{4}{5}\left[\frac{1}{2}v_3(2,9) + \frac{1}{2}v_3(.50,7.50)\right] = 0. \end{aligned}$$

We use the algorithm again to compute the relevant values of v_1:

$$v_1(8,12) = \tfrac{4}{5}\left[\tfrac{1}{2}v_2(16,28) + \tfrac{1}{2}v_2(4,16)\right] = 2.96,$$

$$v_1(2,6) = \tfrac{4}{5}\left[\tfrac{1}{2}v_2(4,10) + \tfrac{1}{2}v_2(1,7)\right] = 0.08.$$

Finally, we may now compute

$$v_0(4,4) = \frac{4}{5}\left[\frac{1}{2}v_1(8,12) + \frac{1}{2}v_1(2,6)\right] = 1.216.$$

Exercise 1.9 (Stochastic volatility, random interest rate). Consider a binomial pricing model, but at each time $n \geq 1$, the "up factor" $u_n(\omega_1\omega_2\ldots\omega_n)$, the "down factor" $d_n(\omega_1\omega_2\ldots\omega_n)$, and the interest rate $r_n(\omega_1\omega_2\ldots\omega_n)$ are allowed to depend on n and on the first n coin tosses $\omega_1\omega_2\ldots\omega_n$. The initial up factor u_0, the initial down factor d_0, and the initial interest rate r_0 are not random. More specifically, the stock price at time one is given by

$$S_1(\omega_1) = \begin{cases} u_0 S_0 \text{ if } \omega_1 = H, \\ d_0 S_0 \text{ if } \omega_1 = T, \end{cases}$$

and, for $n \geq 1$, the stock price at time $n+1$ is given by

$$S_{n+1}(\omega_1\omega_2\ldots\omega_n\omega_{n+1}) = \begin{cases} u_n(\omega_1\omega_2\ldots\omega_n)S_n(\omega_1\omega_2\ldots\omega_n) \text{ if } \omega_{n+1} = H, \\ d_n(\omega_1\omega_2\ldots\omega_n)S_n(\omega_1\omega_2\ldots\omega_n) \text{ if } \omega_{n+1} = T. \end{cases}$$

One dollar invested in or borrowed from the money market at time zero grows to an investment or debt of $1 + r_0$ at time one, and, for $n \geq 1$, one dollar invested in or borrowed from the money market at time n grows to an investment or debt of $1 + r_n(\omega_1\omega_2\ldots\omega_n)$ at time $n+1$. We assume that for each n and for all $\omega_1\omega_2\ldots\omega_n$, the no-arbitrage condition

$$0 < d_n(\omega_1\omega_2\ldots\omega_n) < 1 + r_n(\omega_1\omega_2\ldots\omega_n) < u_n(\omega_1\omega_2\ldots\omega_n)$$

holds. We also assume that $0 < d_0 < 1 + r_0 < u_0$.

(i) Let N be a positive integer. In the model just described, provide an algorithm for determining the price at time zero for a derivative security that at time N pays off a random amount V_N depending on the result of the first N coin tosses.

(ii) Provide a formula for the number of shares of stock that should be held at each time n $(0 \leq n \leq N-1)$ by a portfolio that replicates the derivative security V_N.

(iii) Suppose the initial stock price is $S_0 = 80$, with each head the stock price increases by 10, and with each tail the stock price decreases by 10. In other words, $S_1(H) = 90$, $S_1(T) = 70$, $S_2(HH) = 100$, etc. Assume the interest rate is always zero. Consider a European call with strike price 80, expiring at time five. What is the price of this call at time zero?

Solution.

(i) We adapt Theorem 1.2.2 to this case by defining

$$\tilde{p}_0 = \frac{1 + r_0 - d_0}{u_0 - d_0}, \quad \tilde{q}_0 = \frac{u_0 - 1 - r_0}{u_0 - d_0},$$

and for each and and for all $\omega_1\omega_2\ldots\omega_n$,

$$\tilde{p}_n(\omega_1\omega_2\ldots\omega_n) = \frac{1 + r_n(\omega_1\omega_2\ldots\omega_n) - d_n(\omega_1\omega_2\ldots\omega_n)}{u_n(\omega_1\omega_2\ldots\omega_n) - d_n(\omega_1\omega_2\ldots\omega_n)},$$
$$\tilde{q}_n(\omega_1\omega_2\ldots\omega_n) = \frac{u_n(\omega_1\omega_2\ldots\omega_n) - 1 - r_n(\omega_1\omega_2\ldots\omega_n)}{u_n(\omega_1\omega_2\ldots\omega_n) - d_n(\omega_1\omega_2\ldots\omega_n)}.$$

In place of (1.2.16), we define for $n = N-1, N-2, \ldots, 1$,

$$V_n(\omega_1\omega_2\ldots\omega_n) = \frac{1}{1+r}\big[\tilde{p}_n(\omega_1\omega_2\ldots\omega_n)V_{n+1}(\omega_1\omega_2\ldots\omega_n H)$$
$$+\tilde{q}_n(\omega_1\omega_2\ldots\omega_n)V_{n+1}(\omega_1\omega_2\ldots\omega_n T)\big],$$

and for the the case $n = 0$ we adopt the definition

$$V_0 = \frac{1}{1+r}\big[\tilde{p}_0 V_1(H) + \tilde{q}_0 V_1(T)\big].$$

(ii) The number of shares of stock that should be held at time n is still given by (1.2.17):

$$\Delta_n(\omega_1\ldots\omega_n) = \frac{V_{n+1}(\omega_1\ldots\omega_n H) - V_{n+1}(\omega_1\ldots\omega_n T)}{S_{n+1}(\omega_1\ldots\omega_n H) - S_{n+1}(\omega_1\ldots\omega_n T)}.$$

The proof that this hedge works, i.e., that taking the position Δ_n in the stock at time n and holding it until time $n+1$ results in a portfolio whose value at time $n+1$ is V_{n+1}, is the same as the proof given for Theorem 1.2.2.

(iii) If the stock price at a particular time n is x, then the stock price at the next time is either $x+10$ or $x-10$. That means that the up factor is $u_n = \frac{x+10}{x}$ and the down factor is $d_n = \frac{x-10}{x}$. The corresponding risk-neutral probabilities are

$$\tilde{p}_n = \frac{1 - d_n}{u_n - d_n} = \frac{1 - \frac{x-10}{x}}{\frac{x+10}{x} - \frac{x-10}{x}} = \frac{1}{2},$$

$$\tilde{q}_n = \frac{u_n - 1}{u_n = d_n} = \frac{\frac{x+10}{x} - 1}{\frac{x+10}{x} - \frac{x-10}{x}} = \frac{1}{2}.$$

Because these risk-neutral probabilities do not depend on the time n nor on the coin tosses $\omega_1\ldots\omega_n$, we can easily compute the risk-neutral probability of an arbitrary sequence $\omega_1\omega_2\omega_3\omega_4\omega_5$ to be $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$.

There are three ways for the call with strike 80 to expire in the money at time 5: either the five tosses result in five heads ($S_5 = 130$), result in four heads and one tail ($S_5 = 110$), or result in three heads and two tails ($S_5 = 90$). The risk-neutral probability of five heads is $\frac{1}{32}$. If a tail occurs, it can occur on any toss, and so there are five sequences that have four heads and one tail. Therefore, the risk-neutral probability of four heads and one tail is $\frac{5}{32}$. Finally, if there are two tails in a sequence of five tosses, there 10 ways to choose the two tosses that are tails. Therefore, the risk-neutral probability of three heads and two tails is $\frac{10}{32}$. The time-zero price of the call is

$$V_0 = \frac{1}{32} \cdot (130 - 80) + \frac{5}{32} \cdot (110 - 80) + \frac{10}{32} \cdot (90 - 80) = 9.375.$$

2

Probability Theory on Coin Toss Space

2.9 Solutions to Selected Exercises

Exercise 2.2. Consider the stock price S_3 in Figure 2.3.1.

(i) What is the distribution of S_3 under the risk-neutral probabilities $\tilde{p} = \frac{1}{2}$, $\tilde{q} = \frac{1}{2}$.

(ii) Compute $\widetilde{\mathbb{E}} S_1$, $\widetilde{\mathbb{E}} S_2$, and $\widetilde{\mathbb{E}} S_3$. What is the average rate of growth of the stock price under $\widetilde{\mathbb{P}}$?

(iii) Answer (i) and (ii) again under the actual probabilities $p = \frac{2}{3}$, $q = \frac{1}{3}$.

Solution.

(i) The distribution of S_3 under the risk-neutral probabilities $\tilde{p}$ and $\tilde{q}$ is

32	8	2	.50
$\tilde{p}^3$	$3\tilde{p}^2\tilde{q}$	$3\tilde{p}\tilde{q}^2$	$\tilde{q}^3$

With $\tilde{p} = \frac{1}{2}$, $\tilde{q} = \frac{1}{2}$, this becomes

32	8	2	.50
.125	.375	.375	.125

(ii) By Theorem 2.4.4,

$$\widetilde{\mathbb{E}} \frac{S_3}{(1+r)^3} = \widetilde{\mathbb{E}} \frac{S_2}{(1+r)^2} = \widetilde{\mathbb{E}} \frac{S_1}{(1+r)} = \widetilde{\mathbb{E}} S_0 = S_0 = 4.$$

Therefore,

$$\widetilde{\mathbb{E}}S_1 = (1+r)S_0 = (1.25)(4) = 5,$$
$$\widetilde{\mathbb{E}}S_2 = (1+r)^2 S_0 = (1.25)^2(4) = 6.25,$$
$$\widetilde{\mathbb{E}}S_3 = (1+r)^3 S_0 = (1.25)^3(4) = 7.8125.$$

In particular, we see that

$$\widetilde{\mathbb{E}}S_3 = 1.25 \cdot \widetilde{\mathbb{E}}S_2, \quad \widetilde{\mathbb{E}}S_2 = 1.25 \cdot \widetilde{\mathbb{E}}S_1, \quad \widetilde{\mathbb{E}}S_1 = 1.25 \cdot S_0.$$

Thus, the average rate of growth of the stock price under $\widetilde{\mathbb{P}}$ is the same as the interest rate of the money market.

(iii) The distribution of S_3 under the probabilities p and q is

32	8	2	.50
p^3	$3p^2q$	$3pq^2$	q^3

With $p = \frac{2}{3}$, $q = \frac{1}{3}$, this becomes

32	8	2	.50
.2963	.4444	.2222	.0371

To compute the average rate of growth, we reason as follows:

$$\mathbb{E}_n S_{n+1} = \mathbb{E}_n\left(S_n \frac{S_{n+1}}{S_n}\right) = S_n \mathbb{E}_n\left(\frac{S_{n+1}}{S_n}\right) = (pu+qd)S_n.$$

In our case,

$$pu + qd = \frac{2}{3}\cdot 2 + \frac{1}{3}\cdot 12 = 1.5.$$

In other words, the average rate of growth of the stock price under the actual probabilities is 50%. Finally, taking expectations, we have

$$\mathbb{E}S_{n+1} = \mathbb{E}\big(\mathbb{E}_n S_{n+1}\big) = 1.5 \cdot \mathbb{E}S_n,$$

so that

$$\mathbb{E}S_1 = 1.5 \cdot \mathbb{E}S_0 = 6,$$
$$\mathbb{E}S_2 = 1.5 \cdot \mathbb{E}S_1 = 9,$$
$$\mathbb{E}S_3 = 1.5 \cdot \mathbb{E}S_2 = 13.50.$$

Exercise 2.3. Show that a convex function of a martingale is a submartingale. In other words, let $M_0, M_1, \ldots, M_N$ be a martingale and let φ be a convex function. Show that $\varphi(M_0), \varphi(M_1), \ldots, \varphi(M_N)$ is a submartingale.

Solution Let an arbitrary n with $0 \le n \le N-1$ be given. By the martingale property, we have

$$\mathbb{E}_n M_{n+1} = M_n,$$

and hence

$$\varphi(\mathbb{E}_n M_{n+1}) = \varphi(M_n).$$

On the other hand, by the conditional Jensen's inequality, we have

$$\mathbb{E}_n \varphi(M_{n+1}) \geq \varphi(\mathbb{E}_n M_{n+1}).$$

Combining these two, we get

$$\mathbb{E}_n \varphi(M_{n+1}) \geq \varphi(M_n),$$

and since n is arbitrary, this implies that the sequence of random variables $\varphi(M_0)$, $\varphi(M_1), \ldots, \varphi(M_N)$ is a submartingale.

Exercise 2.6 (Discrete-time stochastic integral). Suppose $M_0, M_1, \ldots, M_N$ is a martingale, and let $\Delta_0, \Delta_1, \ldots, \Delta_{N-1}$ be an adapted process. Define the *discrete-time stochastic integral* (sometimes called a *martingale transform*) $I_0, I_1, \ldots, I_N$ by setting $I_0 = 0$ and

$$I_n = \sum_{j=0}^{n-1} \Delta_j (M_{j+1} - M_j), \ n = 1, \ldots, N.$$

Show that $I_0, I_1, \ldots, I_N$ is a martingale.

Solution. Because $I_{n+1} = I_n + \Delta_n(M_{n+1} - M_n)$ and I_n, Δ_n and M_n depend on only the first n coin tosses, we may "take out what is known" to write

$$\mathbb{E}_n[I_{n+1}] = \mathbb{E}_n\big[I_n + \Delta_n(M_{n+1} - M_n)\big] = I_n + \Delta_n\big(\mathbb{E}_n[M_{n+1}] - M_n\big).$$

However, $\mathbb{E}_n[M_{n+1}] = M_n$, and we conclude that $\mathbb{E}_n[I_{n+1}] = I_n$, which is the martingale property.

Exercise 2.8. Consider an N-period binomial model.

(i) Let $M_0, M_1, \ldots, M_N$ and $M'_0, M'_1, \ldots, M'_N$ be martingales under the risk-neutral measure $\widetilde{\mathbb{P}}$. Show that if $M_N = M'_N$ (for every possible outcome of the sequence of coin tosses), then, for each n between 0 and N, we have $M_n = M'_n$ (for every possible outcome of the sequence of coin tosses).

(ii) Let V_N be the payoff at time N of some derivative security. This is a random variable that can depend on all N coin tosses. Define recursively $V_{N-1}, V_{N-2}, \ldots, V_0$ by the algorithm (1.2.16) of Chapter 1. Show that

$$V_0, \frac{V_1}{1+r}, \ldots, \frac{V_{N-1}}{(1+r)^{N-1}}, \frac{V_N}{(1+r)^N}$$

is a martingale under $\widetilde{\mathbb{P}}$.

(iii) Using the risk-neutral pricing formula (2.4.11) of this chapter, define

$$V'_n = \widetilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right],\ n = 0, 1, \ldots, N-1.$$

Show that

$$V'_0, \frac{V'_1}{1+r}, \ldots, \frac{V'_{N-1}}{(1+r)^{N-1}}, \frac{V_N}{(1+r)^N}$$

is a martingale.

(iv) Conclude that $V_n = V'_n$ for every n (i.e., the algorithm (1.2.16) of Theorem 1.2.2 of Chapter 1 gives the same derivative security prices as the risk-neutral pricing formula (2.4.11) of Chapter 2).

Solution.

(i) We are given that $M_n = M'_N$. For n between 0 and $N-1$, this equality and the martingale property imply

$$M_n = \widetilde{\mathbb{E}}_n[M_N] = \widetilde{\mathbb{E}}_n[M'_N] = M_n.$$

(ii) For n between 0 and $N-1$, we compute the following conditional expectation:

$$\begin{aligned}\widetilde{\mathbb{E}}_n&\left[\frac{V_{n+1}}{(1+r)^{n+1}}\right](\omega_1\omega_2\ldots\omega_n)\\ &= \tilde{p}\,\frac{V_{n+1}(\omega_1\omega_2\ldots\omega_n H)}{(1+r)^{n+1}} + \tilde{q}\,\frac{V_{n+1}(\omega_1\omega_2\ldots\omega_n T)}{(1+r)^{n+1}}\\ &= \frac{V_n(\omega_1\omega_2\ldots\omega_n)}{(1+r)^n},\end{aligned}$$

where the second equality follows from (1.2.16). This is the martingal property for $\frac{V_n}{(1+r)^n}$.

(iii) The martingale property for $\frac{V'_n}{(1+r)^n}$ follows from the iterated conditioning property (iii) of Theorem 2.3.2. According to this property, for n between 0 and $n-1$,

$$\begin{aligned}\widetilde{\mathbb{E}}_n\left[\frac{V'_{n+1}}{(1+r)^{n+1}}\right] &= \widetilde{\mathbb{E}}_n\left[\frac{1}{(1+r)^{n+1}}\widetilde{\mathbb{E}}_{n+1}\left[\frac{V_N}{(1+r)^{N-(n+1)}}\right]\right]\\ &= \frac{1}{(1+r)^n}\widetilde{\mathbb{E}}_n\left[\widetilde{\mathbb{E}}_{n+1}\left[\frac{V_N}{(1+r)^{N-n}}\right]\right]\\ &= \frac{1}{(1+r)^n}\widetilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right]\\ &= \frac{V'_n}{(1+r)^n}.\end{aligned}$$

(iv) Since the processes in (ii) and (iii) are martingales under the risk-neutral probability measure and they agree at the final time N, they must agree at all earlier times because of (i).

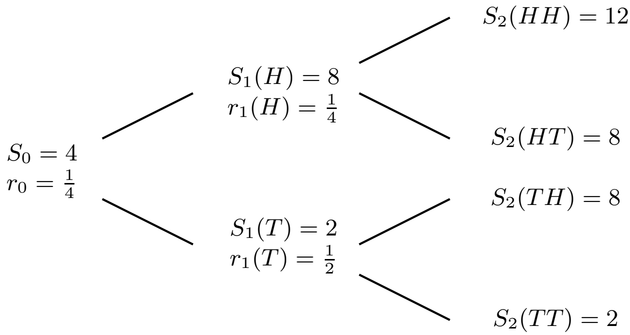

Fig. 2.8.1. A stochastic volatility, random interest rate model.

Exercise 2.9 (Stochastic volatility, random interest rate). Consider a two-period stochastic volatility, random interest rate model of the type described in Exercise 1.9 of Chapter 1. The stock prices and interest rates are shown in Figure 2.8.1.

(i) Determine risk-neutral probabilities

$$\widetilde{\mathbb{P}}(HH),\ \widetilde{\mathbb{P}}(HT),\ \widetilde{\mathbb{P}}(TH),\ \widetilde{\mathbb{P}}(TT),$$

such that the time-zero value of an option that pays off V_2 at time two is given by the risk-neutral pricing formula

$$V_0 = \widetilde{\mathbb{E}}\left[\frac{V_2}{(1+r_0)(1+r_1)}\right].$$

(ii) Let $V_2 = (S_2 - 7)^+$. Compute V_0, $V_1(H)$, and $V_1(T)$.

(iii) Suppose an agent sells the option in (ii) for V_0 at time zero. Compute the position Δ_0 she should take in the stock at time zero so that at time one, regardless of whether the first coin toss results in head or tail, the value of her portfolio is V_1.

(iv) Suppose in (iii) that the first coin toss results in head. What position $\Delta_1(H)$ should the agent now take in the stock to be sure that, regardless of whether the second coin toss results in head or tail, the value of her portfolio at time two will be $(S_2 - 7)^+$?

Solution.

(i) For the first toss, the up factor is $u_0 = 2$ and the down factor is $d_0 = \frac{1}{2}$. Therefore, the risk-neutral probability of a H on the first toss is

$$\tilde{p}_0 = \frac{1 + r_0 - d_0}{u_0 - d_0} = \frac{1 + \frac{1}{4} - \frac{1}{2}}{2 - \frac{1}{2}} = \frac{1}{2},$$

and the risk-neutral probability of T on the first toss is

$$\tilde{q}_0 = \frac{u_0 - 1 - r_0}{u_0 - d_0} = \frac{2 - 1 - \frac{1}{4}}{2 - \frac{1}{2}} = \frac{1}{2}.$$

If the first toss results in H, then the up factor for the second toss is

$$u_1(H) = \frac{S_2(HH)}{S_1(H)} = \frac{12}{8} = \frac{3}{2},$$

and the down factor for the second toss is

$$d_1(H) = \frac{S_2(HT)}{S_1(H)} = \frac{8}{8} = 1.$$

It follows that the risk-neutral probability of getting a H on the second toss, given that the first toss is a H, is

$$\tilde{p}_1(H) = \frac{1 + r_1(H) - d_1(H)}{u_1(H) - d_1(H)} = \frac{1 + \frac{1}{4} - 1}{\frac{3}{2} - 1} = \frac{1}{2},$$

and the risk-neutral probability of T on the second toss, given that the first toss is a H, is

$$\tilde{q}_1(H) = \frac{u_1(H) - 1 - r_1(H)}{u_1(H) - d_1(H)} = \frac{\frac{3}{2} - 1 - \frac{1}{4}}{\frac{3}{2} - 1} = \frac{1}{2},$$

If the first toss results in T, then the up factor for the second toss is

$$u_1(T) = \frac{S_2(TH)}{S_1(T)} = \frac{8}{2} = 4,$$

and the down factor for the second toss is

$$d_1(H) = \frac{S_2(TT)}{S_1(T)} = \frac{2}{2} = 1.$$

It follows that the risk-neutral probability of getting a H on the second toss, given that the first toss is a T, is

$$\tilde{p}_1(T) = \frac{1 + r_1(T) - d_1(T)}{u_1(T) - d_1(T)} = \frac{1 + \frac{1}{2} - 1}{4 - 1} = \frac{1}{6},$$

and the risk-neutral probability of T on the second toss, given that the first toss is a T, is

$$\tilde{q}_1(T) = \frac{u_1(T) - 1 - r_1(T)}{u_1(T) - d_1(T)} = \frac{4 - 1 - \frac{1}{2}}{4 - 1} = \frac{5}{6}.$$

The risk-neutral probabilities are

$$\begin{aligned}
\widetilde{\mathbb{P}}(HH) &= \tilde{p}_0\tilde{p}_1(H) = \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4},\\
\widetilde{\mathbb{P}}(HT) &= \tilde{p}_0\tilde{q}_1(H) = \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4},\\
\widetilde{\mathbb{P}}(TH) &= \tilde{q}_0\tilde{p}_1(T) = \tfrac{1}{2} \cdot \tfrac{1}{6} = \tfrac{1}{12},\\
\widetilde{\mathbb{P}}(TT) &= \tilde{q}_0\tilde{q}_1(T) = \tfrac{1}{2} \cdot \tfrac{5}{6} = \tfrac{5}{12}.
\end{aligned}$$

(ii) We compute

$$\begin{aligned}
V_1(H) &= \frac{1}{1 + r_1(H)}\big[\tilde{p}_1(H)V_2(HH) + \tilde{q}_1(H)V_2(HT)\big]\\
&= \frac{4}{5}\left[\frac{1}{2} \cdot (12 - 7)^+ + \frac{1}{2} \cdot (8 - 7)^+\right]\\
&= 2.40,\\
V_1(T) &= \frac{1}{1 + r_1(T)}\big[\tilde{p}_1(T)V_2(TH) + \tilde{q}_1(T)V_2(TT)\big]\\
&= \frac{2}{3}\left[\frac{1}{6} \cdot (8 - 7)^+ + \frac{5}{6} \cdot (2 - 7)^+\right]\\
&= 0.111111,\\
V_0 &= \frac{1}{1 + r_0}\left[\tilde{p}_0V_1(H) + \tilde{q}_0V_1(T)\right]\\
&= \frac{4}{5}\left[\frac{1}{2} \cdot 2.40 + \frac{1}{2} \cdot 0.1111\right]\\
&= 1.00444.
\end{aligned}$$

We can confirm this price by computing according to the risk-neutral pricing formula in part (i) of the exercise:

$$\begin{aligned} V_0 &= \widetilde{\mathbb{E}}\left[\frac{V_2}{(1+r_0)(1+r_1)}\right] \\ &= \frac{V_2(HH)}{(1+r_0)(1+r_1(H))} \cdot \widetilde{\mathbb{P}}(HH) + \frac{V_2(HT)}{(1+r_0)(1+r_1(H))} \cdot \widetilde{\mathbb{P}}(HT) \\ &\quad + \frac{V_2(TH)}{(1+r_0)(1+r_1(T))} \cdot \widetilde{\mathbb{P}}(TH) + \frac{V_2(TT)}{(1+r_0)(1+r_1(T))} \cdot \widetilde{\mathbb{P}}(TT) \\ &= \frac{(12-7)^+}{(1+\frac{1}{4})(1+\frac{1}{4})} \cdot \frac{1}{4} + \frac{(8-7)^+}{(1+\frac{1}{4})(1+\frac{1}{4})} \cdot \frac{1}{4} \\ &\quad + \frac{(8-7)^+}{(1+\frac{1}{4})(1+\frac{1}{2})} \cdot \frac{1}{12} + \frac{(2-7)^+}{(1+\frac{1}{4})(1+\frac{1}{2})} \cdot \frac{5}{12} \\ &= 0.80 + 0.16 + 0.04444 + 0 \\ &= 1.00444. \end{aligned}$$

(iii) Formula (1.2.17) still applies and yields

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{2.40 - 0.111111}{8-2} = 0.381481.$$

(iv) Again we use formula (1.2.17), this time obtaining

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{(12-7)^+ - (8-7)^+}{12-8} = 1.$$

Exercise 2.11 (Put–call parity). Consider a stock that pays no dividend in an N-period binomial model. A European call has payoff $C_N = (S_N - K)^+$ at time N. The price C_n of this call at earlier times is given by the risk-neutral pricing formula (2.4.11):

$$C_n = \widetilde{\mathbb{E}}_n\left[\frac{C_N}{(1+r)^{N-n}}\right], \quad n = 0, 1, \ldots, N-1.$$

Consider also a put with payoff $P_N = (K - S_N)^+$ at time N, whose price at earlier times is

$$P_n = \widetilde{\mathbb{E}}_n\left[\frac{P_N}{(1+r)^{N-n}}\right], \quad n = 0, 1, \ldots, N-1.$$

Finally, consider a *forward contract* to buy one share of stock at time N for K dollars. The price of this contract at time N is $F_N = S_N - K$, and its price at earlier times is

$$F_n = \widetilde{\mathbb{E}}_n\left[\frac{F_N}{(1+r)^{N-n}}\right], \quad n = 0, 1, \ldots, N-1.$$

(Note that, unlike the call, the forward contract requires that the stock be purchased at time N for K dollars and has a negative payoff if $S_N < K$.)

(i) If at time zero you buy a forward contract and a put, and hold them until expiration, explain why the payoff you receive is the same as the payoff of a call; i.e., explain why $C_N = F_N + P_N$.

(ii) Using the risk-neutral pricing formulas given above for C_n, P_n, and F_n and the linearity of conditional expectations, show that $C_n = F_n + P_n$ for every n.

(iii) Using the fact that the discounted stock price is a martingale under the risk-neutral measure, show that $F_0 = S_0 - \frac{K}{(1+r)^N}$.

(iv) Suppose you begin at time zero with F_0, buy one share of stock, borrowing money as necessary to do that, and make no further trades. Show that at time N you have a portfolio valued at F_N. (This is called a *static replication* of the forward contract. If you sell the forward contract for F_0 at time zero, you can use this static replication to hedge your short position in the forward contract.)

(v) The *forward price* of the stock at time zero is defined to be that value of K that causes the forward contract to have price zero at time zero. The forward price in this model is $(1+r)^N S_0$. Show that, at time zero, the price of a call struck at the forward price is the same as the price of a put struck at the forward price. This fact is called *put–call parity*.

(vi) If we choose $K = (1+r)^N S_0$, we just saw in (v) that $C_0 = P_0$. Do we have $C_n = P_n$ for every n?

Solution

(i) Consider three cases:

Case I: $S_N = K$. Then $C_N = P_N = F_N = 0$;

Case II: $S_N > K$. Then $P_N = 0$ and $C_N = S_N - K = F_N$;

Case III: $S_N < K$. Then $C_N = 0$ and $P_N = K - S_N = -F_N$.

In all three cases, we see that $C_N = F_N + P_N$.

(ii)

$$\begin{aligned} C_n &= \widetilde{\mathbb{E}}_n\left[\frac{C_N}{(1+r)^{N-n}}\right] = \widetilde{\mathbb{E}}_n\left[\frac{F_N + P_N}{(1+r)^{N-n}}\right] \\ &= \widetilde{\mathbb{E}}_n\left[\frac{F_N}{(1+r)^{N-n}}\right] + \widetilde{\mathbb{E}}_n\left[\frac{P_N}{(1+r)^{N-n}}\right] = F_n + P_n. \end{aligned}$$

(iii)

$$\begin{aligned} F_0 &= \widetilde{\mathbb{E}}\left[\frac{F_N}{(1+r)^N}\right] = \widetilde{\mathbb{E}}\left[\frac{S_N - K}{(1+r)^N}\right] \\ &= \widetilde{\mathbb{E}}\left[\frac{S_N}{(1+r)^N}\right] - \widetilde{\mathbb{E}}\left[\frac{K}{(1+r)^N}\right] = S_0 - \frac{K}{(1+r)^N}. \end{aligned}$$

(iv) At time zero, your portfolio value is

$$F_0 = S_0 + (F_0 - S_0).$$

At time N, the value of the portfolio is

$$\begin{aligned} S_N + (1+r)^N (F_0 - S_0) &= S_N + (1+r)^N \left(-\frac{K}{(1+r)^N} \right) \\ &= S_N - K = F_N. \end{aligned}$$

(v) First of all, if $K = (1+r)^N S_0$, then, by (iii), $F_0 = 0$. Further, if $F_0 = 0$, then, by (ii), $C_0 = F_0 + P_0 = P_0$.

(vi) No. This would mean, in particular, that $C_N = P_N$, and hence $(S_N - K)^+ = (K - S_N)^+$, which in turn would imply that $S_N(\omega) = K$ for all ω, which is not the case for most values of ω.

Exercise 2.13 (Asian option). Consider an N-period binomial model. An *Asian option* has a payoff based on the average stock price, i.e.,

$$V_N = f\left(\frac{1}{N+1} \sum_{n=0}^{N} S_n \right),$$

where the function f is determined by the contractual details of the option.

(i) Define $Y_n = \sum_{k=0}^{n} S_k$ and use the Independence Lemma 2.5.3 to show that the two-dimensional process (S_n, Y_n), $n = 0, 1, \ldots, N$ is Markov.

(ii) According to Theorem 2.5.8, the price V_n of the Asian option at time n is some function v_n of S_n and Y_n; i.e.,

$$V_n = v_n(S_n, Y_n), \ n = 0, 1, \ldots, N.$$

Give a formula for $v_N(s, y)$, and provide an algorithm for computing $v_n(s, y)$ in terms of v_{n+1}.

Solution

(i) Note first that

$$S_{n+1} = S_n \cdot \frac{S_{n+1}}{S_n}, \quad Y_{n+1} = Y_n + S_n \cdot \frac{S_{n+1}}{S_n},$$

and whereas S_n and Y_n depend only on the first n tosses, $\frac{S_{n+1}}{S_n}$ depends only on toss $n+1$. According to the Independence Lemma 2.5.3, for any function $h_{n+1}(s, y)$ of dummy variables s and y, we have

$$\widetilde{\mathbb{E}}_n\left[h_{n+1}(S_{n+1},Y_{n+1})\right]=\widetilde{\mathbb{E}}_n\left[h_{n+1}\left(S_n\cdot\frac{S_{n+1}}{S_n},Y_n+S_n\cdot\frac{S_{n+1}}{S_n}\right)\right]$$
$$=h_n(S_n,Y_n),$$

where

$$h_n(s,y)=\widetilde{\mathbb{E}}h_{n+1}\left(s\cdot\frac{S_{n+1}}{S_n},y+s\cdot\frac{S_{n+1}}{S_n}\right)$$
$$=\tilde{p}h_{n+1}(su,y+su)+\tilde{q}h_{n+1}(sd,y+sd).$$

Because $\widetilde{\mathbb{E}}_n\left[h_{n+1}(S_{n+1},Y_{n+1})\right]$ can be written as a function of (S_n,Y_n), the two-dimensional process (S_n,Y_n), $n=0,1,\ldots,N$, is a Markov process.

(ii) We have the final condition $V_N(s,y)=f\left(\frac{y}{N+1}\right)$. For $n=N-1,\ldots,1,0$, we have from the risk-neutral pricing formula (2.4.12) and (i) above that

$$V_n=\frac{1}{1+r}\widetilde{\mathbb{E}}_nV_{n+1}=\frac{1}{1+r}\widetilde{\mathbb{E}}_nv_{n+1}(S_{n+1},Y_{n+1})=v_n(S_n,Y_n),$$

where

$$v_n(s,y)=\frac{1}{1+r}\big[\tilde{p}v_{n+1}(su,y+su)+\tilde{q}v_{n+1}(sd,y+sd)\big].$$

3

State Prices

3.7 Solutions to Selected Exercises

Exercise 3.1. Under the conditions of Theorem 3.1.1, show the following analogues of properties (i)–(iii) of that theorem:

(i′) $\widetilde{\mathbb{P}}\left(\frac{1}{Z} > 0\right) = 1$;

(ii′) $\widetilde{\mathbb{E}}\frac{1}{Z} = 1$;

(iii′) for any random variable Y,

$$\mathbb{E}Y = \widetilde{\mathbb{E}}\left[\frac{1}{Z} \cdot Y\right].$$

In other words, $\frac{1}{Z}$ facilitates the switch from $\widetilde{\mathbb{E}}$ to $\mathbb{E}$ in the same way Z facilitates the switch from $\mathbb{E}$ to $\widetilde{\mathbb{E}}$.

Solution

(i′) Because $\mathbb{P}(\omega) > 0$ and $\widetilde{\mathbb{P}}(\omega) > 0$ for every $\omega \in \Omega$, the ratio

$$\frac{1}{Z(\omega)} = \frac{\mathbb{P}(\omega)}{\widetilde{\mathbb{P}}(\omega)}$$

is defined and positive for every $\omega \in \Omega$.

(ii′) We compute

$$\widetilde{\mathbb{E}}\frac{1}{Z} = \sum_{\omega \in \Omega} \frac{1}{Z(\omega)}\widetilde{\mathbb{P}}(\omega) = \sum_{\omega \in \Omega} \frac{\mathbb{P}(\omega)}{\widetilde{\mathbb{P}}(\omega)}\widetilde{\mathbb{P}}(\omega) = \sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1.$$

(iii′) We compute

$$\widetilde{\mathbb{E}}\left[\frac{1}{Z}\cdot Y\right] = \sum_{\omega\in\Omega}\frac{\mathbb{P}(\omega)}{\widetilde{\mathbb{P}}(\omega)}Y(\omega)\widetilde{\mathbb{P}}(\omega) = \sum_{\omega\in\Omega}Y(\omega)\mathbb{P}(\omega) = \mathbb{E}Y.$$

Exercise 3.3. Using the stock price model of Figure 3.1.1 and the actual probabilities $p=\frac{2}{3}$, $q=\frac{1}{3}$, define the estimates of S_3 at various times by

$$M_n = \mathbb{E}_n[S_3],\ n=0,1,2,3.$$

Fill in the values of M_n in a tree like that of Figure 3.1.1. Verify that M_n, $n=0,1,2,3$, is a martingale.

Solution We note that $M_3=S_3$. We compute M_2 from the formula $M_2=\mathbb{E}_2[S_3]$:

$$\begin{aligned}
M_2(HH) &= \tfrac{2}{3}S_3(HHH)+\tfrac{1}{2}S_3(HHT) = \tfrac{2}{3}\cdot 32+\tfrac{1}{3}\cdot 8 = 24,\\
M_2(HT) &= \tfrac{2}{3}S_3(HTH)+\tfrac{1}{3}S_3(HTT) = \tfrac{2}{3}\cdot 8+\tfrac{1}{3}\cdot 2 = 6,\\
M_2(TH) &= \tfrac{2}{3}S_3(THH)+\tfrac{1}{3}S_3(THT) = \tfrac{2}{3}\cdot 8+\tfrac{1}{3}\cdot 2 = 6,\\
M_2(TT) &= \tfrac{2}{3}S_3(THH)+\tfrac{1}{2}S_3(THT) = \tfrac{2}{3}\cdot 2+\tfrac{1}{3}\cdot 0.50 = 1.50.
\end{aligned}$$

We next compute M_1 from the formula $M_1=\mathbb{E}_1[S_3]$:

$$\begin{aligned}
M_1(H) &= \frac{4}{9}S_3(HHH)+\frac{2}{9}S_3(HHT)+\frac{2}{9}S_3(HTH)+\frac{1}{9}S_3(HTT)\\
&= \frac{4}{9}\cdot 32+\frac{2}{9}\cdot 8+\frac{2}{9}\cdot 8+\frac{1}{9}\cdot 2\\
&= 18,\\
M_1(T) &= \frac{4}{9}S_3(THH)+\frac{2}{9}S_3(THT)+\frac{2}{9}S_3(TTH)+\frac{1}{9}S_3(TTT)\\
&= \frac{4}{9}\cdot 8+\frac{2}{9}\cdot 2+\frac{2}{9}\cdot 2+\frac{1}{9}\cdot 0.50\\
&= 4.50.
\end{aligned}$$

Finally, we compute

$$\begin{aligned}
M_0 &= \mathbb{E}[S_3]\\
&= \frac{8}{27}S_3(HHH)+\frac{4}{27}S_3(HHT)+\frac{4}{27}S_3(HTH)+\frac{4}{27}S_3(THH)\\
&\qquad +\frac{2}{27}S_3(HTT)+\frac{2}{27}S_3(THT)+\frac{2}{27}S_3(TTH)+\frac{1}{27}S_3(TTT)\\
&= \frac{8}{27}\cdot 32+\frac{4}{27}\cdot 8+\frac{4}{27}\cdot 8+\frac{4}{27}\cdot 8+\frac{2}{27}\cdot 2+\frac{2}{27}\cdot 2+\frac{2}{27}\cdot 2+\frac{1}{27}\cdot 0.50\\
&= 13.50.
\end{aligned}$$

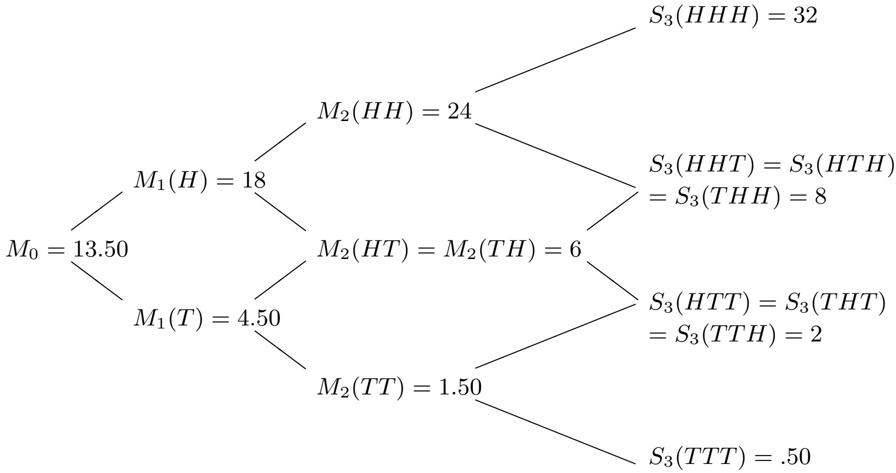

Fig. 3.7.1. An estimation martingale.

We verify the martingale property. We have $M_2 = \mathbb{E}_2[M_3]$ because $M_3 = S_3$ and we used the formula $M_2 = \mathbb{E}_2[S_3]$ to compute M_2. We must check that $M_1 = \mathbb{E}_1[M_2]$ and $M_0 = \mathbb{E}_0[M_1] = \mathbb{E}[M_1]$, which we do below:

$$\begin{aligned}
\mathbb{E}_1[M_2](H) &= \tfrac{2}{3}M_2(HH) + \tfrac{1}{3}M_2(HT) = \tfrac{2}{3}\cdot 24 + \tfrac{1}{3}\cdot 6 = 18 = M_1(H),\\
\mathbb{E}_1[M_2](T) &= \tfrac{1}{2}M_2(TH) + \tfrac{1}{2}M_2(TT) = \tfrac{2}{3}\cdot 6 + \tfrac{1}{3}\cdot 1.50 = 4.50 = M_1(T),\\
M_0 &= \tfrac{2}{3}M_1(H) + \tfrac{1}{3}M_1(T) = \tfrac{2}{3}\cdot 18 + \tfrac{1}{3}\cdot 4.50 = 13.50 = M_0.
\end{aligned}$$

Exercise 3.5 (Stochastic volatility, random interest rate). Consider the model of Exercise 2.9 of Chapter 2. Assume that the actual probability measure is

$$\mathbb{P}(HH) = \frac{4}{9},\ \mathbb{P}(HT) = \frac{2}{9},\ \mathbb{P}(TH) = \frac{2}{9},\ \mathbb{P}(TT) = \frac{1}{9}.$$

The risk-neutral measure was computed in Exercise 2.9 of Chapter 2.

(i) Compute the Radon-Nikodým derivative $Z(HH)$, $Z(HT)$, $Z(TH)$ and $Z(TT)$ of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$

(ii) The Radon-Nikodým derivative process Z_0, Z_1, Z_2 satisfies $Z_2 = Z$. Compute $Z_1(H)$, $Z_1(T)$ and Z_0. Note that $Z_0 = \mathbb{E}Z = 1$.

(iii) The version of the risk-neutral pricing formula (3.2.6) appropriate for this model, which does not use the risk-neutral measure, is

$$
\begin{aligned}
V_1(H) &= \frac{1+r_0}{Z_1(H)} \mathbb{E}_1 \left[\frac{Z_2}{(1+r_0)(1+r_1)} V_2 \right] (H) \\
&= \frac{1}{Z_1(H)(1+r_1(H))} \mathbb{E}_1[Z_2 V_2](H), \\
V_1(T) &= \frac{1+r_0}{Z_1(T)} \mathbb{E}_1 \left[\frac{Z_2}{(1+r_0)(1+r_1)} V_2 \right] (T) \\
&= \frac{1}{Z_1(T)(1+r_1(T))} \mathbb{E}_1[Z_2 V_2](T), \\
V_0 &= \mathbb{E} \left[\frac{Z_2}{(1+r_0)(1+r_1)} V_2 \right].
\end{aligned}
$$

Use this formula to compute $V_1(H)$, $V_1(T)$ and V_0 when $V_2 = (S_2 - 7)^+$. Compare to your answers in Exercise 2.6(ii) of Chapter 2.

Solution

(i) In Exercise 2.9 of Chapter 2, the risk-neutral probabilities are

$$
\widetilde{\mathbb{P}}(HH) = \frac{1}{4}, \ \widetilde{\mathbb{P}}(HT) = \frac{1}{4}, \ \widetilde{\mathbb{P}}(TH) = \frac{1}{12}, \ \widetilde{\mathbb{P}}(TT) = \frac{5}{12}.
$$

Therefore, the Radon-Nikodým derivative is

$$
Z(HH) = \frac{\widetilde{\mathbb{P}}(HH)}{\mathbb{P}(HH)} = \frac{1}{4} \cdot \frac{9}{4} = \frac{9}{16}, \qquad Z(HT) = \frac{\widetilde{\mathbb{P}}(HT)}{\mathbb{P}(HT)} = \frac{1}{4} \cdot \frac{9}{2} = \frac{9}{8},
$$

$$
Z(TH) = \frac{\widetilde{\mathbb{P}}(TH)}{\mathbb{P}(TH)} = \frac{1}{12} \cdot \frac{9}{2} = \frac{3}{8}, \qquad Z(TT) = \frac{\widetilde{\mathbb{P}}(TT)}{\mathbb{P}(TT)} = \frac{5}{12} \cdot \frac{9}{1} = \frac{15}{4},
$$

(ii)

$$
\begin{aligned}
Z_1(H) &= \mathbb{E}_1[Z_2](H) \\
&= Z_2(HH)\mathbb{P}\{\omega_2 = H \text{ given that } \omega_1 = H\} \\
&\qquad + Z_2(HT)\mathbb{P}\{\omega_2 = T \text{ given that } \omega_1 = H\} \\
&= Z_2(HH)\frac{\mathbb{P}(HH)}{\mathbb{P}(HH) + \mathbb{P}(HT)} + Z_2(HT)\frac{\mathbb{P}(HT)}{\mathbb{P}(HH) + \mathbb{P}(HT)} \\
&= \frac{9}{16} \cdot \frac{\frac{4}{9}}{\frac{4}{9} + \frac{2}{9}} + \frac{9}{8} \cdot \frac{\frac{2}{9}}{\frac{4}{9} + \frac{2}{9}} \\
&= \frac{3}{4},
\end{aligned}
$$

$$
\begin{aligned}
Z_1(T) &= \mathbb{E}_1[Z_2](T) \\
&= Z_2(TH)\mathbb{P}\{\omega_2 = H \text{ given that } \omega_1 = T\} \\
&\qquad + Z_2(TT)\mathbb{P}\{\omega_2 = T \text{ given that } \omega_1 = T\} \\
&= Z_2(TH)\frac{\mathbb{P}(TH)}{\mathbb{P}(TH)+\mathbb{P}(TT)} + Z_2(TT)\frac{\mathbb{P}(TT)}{\mathbb{P}(TH)+\mathbb{P}(TT)} \\
&= \frac{3}{8}\cdot\frac{\frac{2}{9}}{\frac{2}{9}+\frac{1}{9}} + \frac{15}{4}\cdot\frac{\frac{1}{9}}{\frac{2}{9}+\frac{1}{9}} \\
&= \frac{3}{2}, \\
Z_0 &= \mathbb{E}_0[Z_1] \\
&= \mathbb{E}[Z_1] \\
&= Z_1(H)\big(\mathbb{P}(HH)+\mathbb{P}(HT)\big) + Z_1(T)\big(\mathbb{P}(TH)+\mathbb{P}(TT)\big) \\
&= \frac{3}{4}\cdot\left(\frac{4}{9}+\frac{2}{9}\right) + \frac{3}{2}\cdot\left(\frac{2}{9}+\frac{1}{9}\right) \\
&= 1.
\end{aligned}
$$

We may also check directly that $\mathbb{E}Z = 1$, as follows:

$$
\begin{aligned}
\mathbb{E}Z &= Z(HH)\mathbb{P}(HH) + Z(HT)\mathbb{P}(HT) + Z(TH)\mathbb{P}(TH) + Z(TT)\mathbb{P}(TT) \\
&= \frac{9}{16}\cdot\frac{4}{9} + \frac{9}{8}\cdot\frac{2}{9} + \frac{3}{8}\cdot\frac{2}{9} + \frac{15}{4}\cdot\frac{1}{9} \\
&= \frac{1}{4} + \frac{1}{4} + \frac{1}{12} + \frac{5}{12} \;=\; 1.
\end{aligned}
$$

(iii) We recall that

$$V_2(HH) = 5,\ V_2(HT) = 1,\ V_2(TH) = 1,\ V_2(TT) = 0.$$

We computed in part (ii) that

$$
\begin{aligned}
\mathbb{P}\{\omega_2 = H \text{ given that } \omega_1 = H\} &= \frac{\frac{4}{9}}{\frac{4}{9}+\frac{2}{9}} = \frac{2}{3}, \\
\mathbb{P}\{\omega_2 = T \text{ given that } \omega_1 = H\} &= \frac{\frac{2}{9}}{\frac{4}{9}+\frac{2}{9}} = \frac{1}{3}, \\
\mathbb{P}\{\omega_2 = H \text{ given that } \omega_1 = T\} &= \frac{\frac{2}{9}}{\frac{2}{9}+\frac{1}{9}} = \frac{2}{3}, \\
\widetilde{\mathbb{P}}\{\omega_2 = T \text{ given that } \omega_1 = T\} &= \frac{\frac{1}{9}}{\frac{2}{9}+\frac{1}{9}} = \frac{1}{3}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
V_1(H) &= \frac{1}{Z_1(H)\big(1+r_1(H)\big)}\mathbb{E}_1[Z_2V_2](H) \\
&= \left(\frac{3}{4}\cdot\frac{5}{4}\right)^{-1}\big[Z_2(HH)V_2(HH)\mathbb{P}\{\omega_2 = H \text{ given that } \omega_1 = H\} \\
&\qquad\qquad + Z_2(HT)V_2(HT)\mathbb{P}\{\omega_2 = T \text{ given that } \omega_1 = H\}\big] \\
&= \frac{16}{15}\left[\frac{9}{16}\cdot 5\cdot\frac{2}{3} + \frac{9}{8}\cdot 1\cdot\frac{1}{3}\right] \\
&= 2.40, \\
V_1(T) &= \frac{1}{Z_1(T)\big(1+r_1(T)\big)}\mathbb{E}_1[Z_2V_2](T) \\
&= \left(\frac{3}{2}\cdot\frac{3}{2}\right)^{-1}\big[Z_2(TH)V_2(TH)\mathbb{P}\{\omega_2 = H \text{ given that } \omega_1 = T\} \\
&\qquad\qquad + Z_2(TT)V_2(TT)\mathbb{P}\{\omega_2 = T \text{ given that } \omega_1 = T\}\big] \\
&= \frac{4}{9}\left[\frac{3}{8}\cdot 1\cdot\frac{2}{3} + \frac{15}{4}\cdot 0\cdot\frac{1}{3}\right] \\
&= 0.111111,
\end{aligned}
$$

and

$$
\begin{aligned}
V_0 &= \mathbb{E}\left[\frac{Z_2V_2}{(1+r_0)(1+r_1)}\right] \\
&= \frac{Z_2(HH)V_2(HH)}{(1+r_0)(1+r_1(H))}\mathbb{P}(HH) + \frac{Z_2(HT)V_2(HT)}{(1+r_0)(1+r_1(H))}\mathbb{P}(HT) \\
&\quad + \frac{Z_2(TH)V_2(TH)}{(1+r_0)(1+r_1(T))}\mathbb{P}(TH) + \frac{Z_2(TT)V_2(TT)}{(1+r_0)(1+r_1(T))}\mathbb{P}(TT) \\
&= \left(\frac{5}{4}\cdot\frac{5}{4}\right)^{-1}\frac{9}{16}\cdot 5\cdot\frac{4}{9} + \left(\frac{5}{4}\cdot\frac{5}{4}\right)^{-1}\frac{9}{8}\cdot 1\cdot\frac{2}{9} + \left(\frac{5}{4}\cdot\frac{3}{2}\right)^{-1}\frac{3}{8}\cdot 1\cdot\frac{2}{9} \\
&\quad + \left(\frac{5}{4}\cdot\frac{3}{2}\right)^{-1}\frac{15}{4}\cdot 0\cdot\frac{1}{9} \\
&= \frac{16}{25}\cdot\frac{5}{4} + \frac{16}{25}\cdot\frac{1}{4} + \frac{8}{15}\cdot\frac{1}{12} \\
&= 1.00444.
\end{aligned}
$$

Exercise 3.6. Consider Problem 3.3.1 in an N-period binomial model with the utility function $U(x) = \ln x$. Show that the optimal wealth process corresponding to the optimal portfolio process is given by $X_n = \frac{X_0}{\zeta_n}$, $n = 0, 1, \ldots, N$, where ζ_n is the state price density process defined in (3.2.7).

Solution From (3.3.25) we have

$$
X_N = I\left(\frac{\lambda Z}{(1+r)^N}\right) = I(\lambda\zeta_N).
$$

When $U(x) = \ln x$, $U'(x) = \frac{1}{x}$ and the inverse function of U' is $I(y) = \frac{1}{y}$. Therefore,

$$X_N = \frac{1}{\lambda \zeta_N}$$

We must choose λ to satisfy (3.3.26), which in this case takes the form

$$X_0 = \mathbb{E}\big[\zeta_N I(\lambda \zeta_N)\big] = \frac{1}{\lambda}.$$

Substituting this into the previous equation, we obtain

$$X_N = \frac{X_0}{\zeta_N}.$$

Because $\frac{X_n}{(1+r)^n}$ is a martingale under the risk-neutral measure $\widetilde{\mathbb{P}}$, we have

$$\frac{X_n}{(1+r)^n} = \widetilde{\mathbb{E}}_n\left[\frac{X_N}{(1+r)^N}\right] = \frac{1}{Z_n}\mathbb{E}_n\left[\frac{Z_N X_N}{(1+r)^N}\right] = \frac{1}{Z_n}\mathbb{E}_n\left[\zeta_N X_N\right] = \frac{X_0}{Z_n}.$$

Therefore,

$$X_n = \frac{(1+r)^n X_0}{Z_n} = \frac{X_0}{\zeta_n}.$$

Exercise 3.8. The Lagrange Multiplier Theorem used in the solution of Problem 3.3.5 has hypotheses that we did not verify in the solution of that problem. In particular, the theorem states that if the gradient of the constraint function, which in this case is the vector $(p_1\zeta_1, \dots, p_m\zeta_m)$, is not the zero vector, then the optimal solution must satisfy the Lagrange multiplier equations (3.3.22). This gradient is not the zero vector, so this hypothesis is satisfied. However, even when this hypothesis is satisfied, the theorem does not guarantee that there is an optimal solution; the solution to the Lagrange multiplier equations may in fact minimize the expected utility. The solution could also be neither a maximizer nor a minimizer. Therefore, in this exercise, we outline a different method for verifying that the random variable X_N given by (3.3.25) maximizes the expected utility.

We begin by changing the notation, calling the random variable given by (3.3.25) X_N^* rather than X_N. In other words,

$$X_N^* = I\left(\frac{\lambda}{(1+r)^N}Z\right), \tag{3.6.1}$$

where λ is the solution of equation (3.3.26). This permits us to use the notation X_N for an arbitrary (not necessarily optimal) random variable satisfying (3.3.19). We must show that

$$\mathbb{E}U(X_N) \le \mathbb{E}U(X_N^*). \tag{3.6.2}$$

(i) Fix $y > 0$, and show that the function of x given by $U(x) - yx$ is maximized by $y = I(x)$. Conclude that

$$U(x) - yx \leq U(I(y)) - yI(y) \text{ for every } x. \tag{3.6.3}$$

(ii) In (3.6.3), replace the dummy variable x by the random variable X_N and replace the dummy variable y by the random variable $\frac{\lambda Z}{(1+r)^N}$. Take expectations of both sides and use (3.3.19) and (3.3.26) to conclude that (3.6.2) holds.

Solution

(i) Because $U(x)$ is concave, and for each fixed $y > 0$, yx is a linear function of x, the difference $U(x) - yx$ is a concave function of x. The derivative of this function is $U'(x) - y$, and this is zero if and only if $U'(x) = y$, which is equivalent to $x = I(y)$. A concave function has its maximum at the point where its derivative is zero. The inequality (3.6.2) is just this statement.

(ii) Making the suggested replacements, we obtain

$$U(X_N) - \frac{\lambda Z X_N}{(1+r)^N} \leq U\left(I\left(\frac{\lambda Z}{(1+r)^N}\right)\right) - \frac{\lambda Z}{(1+r)^N} I\left(\frac{\lambda Z}{(1+r)^N}\right).$$

Taking expectations under $\mathbb{P}$ and using the fact that Z is the Radon-Nikodým derivative of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$, we obtain

$$\begin{aligned}\mathbb{E}U(X_N) &- \lambda\widetilde{\mathbb{E}}\frac{X_N}{(1+r)^N}\\ &\leq \mathbb{E}U\left(I\left(\frac{\lambda Z}{(1+r)^N}\right)\right) - \lambda\mathbb{E}\left[\frac{Z}{(1+r)^N} I\left(\frac{\lambda Z}{(1+r)^N}\right)\right].\end{aligned}$$

From (3.3.19) and (3.3.26), we have

$$\widetilde{\mathbb{E}}\frac{X_N}{(1+r)^N} = X_0 = \mathbb{E}\left[\frac{Z}{(1+r)^N} I\left(\frac{\lambda Z}{(1+r)^N}\right)\right].$$

Cancelling these terms on the left- and right-hand sides of the above equation, we obtain (3.6.2).

4

American Derivative Securities

4.9 Solutions to Selected Exercises

Exercise 4.1. In the three-period model of Figure 1.2.2 of Chapter 1, let the interest rate be $r = \frac{1}{4}$ so the risk-neutral probabilities are $\tilde{p} = \tilde{q} = \frac{1}{2}$.

(i) Determine the price at time zero, denoted V_0^P, of the American put that expires at time three and has intrinsic value $g_P(s) = (4 - s)^+$.

(ii) Determine the price at time zero, denoted V_0^C, of the American call that expires at time three and has intrinsic value $g_C(s) = (s - 4)^+$.

(iii) Determine the price at time zero, denoted V_0^S, of the American straddle that expires at time three and has intrinsic value $g_S(s) = g_P(s) + g_C(s)$.

(iv) Explain why $V_0^S < V_0^P + V_0^C$.

Solution

(i) The payoff of the put at expiration time three is

$$\begin{aligned}
V_3^P(HHH) &= (4-32)^+ = 0,\\
V_3^P(HHT) &= V_3^P(HTH) = V_3^P(THH) = (4-8)^+ = 0,\\
V_3^P(HTT) &= V_3^P(THT) = V_3^P(TTH) = (4-2)^+ = 2,\\
V_3^P(TTT) &= (4-0.50)^+ = 3.50.
\end{aligned}$$

Because $\frac{1}{1+r}\tilde{p} = \frac{1}{1+r}\tilde{q} = \frac{2}{5}$, the value of the put at time two is

$$
\begin{aligned}
V_2^P(HH) &= \max\left\{\big(4 - S_2(HH)\big)^+, \frac{2}{5}V_3^P(HHH) + \frac{2}{5}V_3^P(HHT)\right\} \\
&= \max\left\{(4-16)^+, \frac{2}{5}\cdot 0 + \frac{2}{5}\cdot 0\right\} \\
&= \max\{0, 0\} \\
&= 0, \\
V_2^P(HT) &= \max\left\{\big(4 - S_2(HT)\big)^+, \frac{2}{5}V_3^P(HTH) + \frac{2}{5}V_3^P(HTT)\right\} \\
&= \max\left\{(4-4)^+, \frac{2}{5}\cdot 0 + \frac{2}{5}\cdot 2\right\} \\
&= \max\{0, 0.80\} \\
&= 0.80, \\
V_2^P(TH) &= \max\left\{\big(4 - S_2(TH)\big)^+, \frac{2}{5}V_3^P(THH) + \frac{2}{5}V_3^P(THT)\right\} \\
&= \max\left\{(4-4)^+, \frac{2}{5}\cdot 0 + \frac{2}{5}\cdot 2\right\} \\
&= \max\{0, 0.80\} \\
&= 0.80, \\
V_2^P(TT) &= \max\left\{\big(4 - S_2(TT)\big)^+, \frac{2}{5}V_3^P(TTH) + \frac{2}{5}V_3^P(TTT)\right\} \\
&= \max\left\{(4-1)^+, \frac{2}{5}\cdot 2 + \frac{2}{5}\cdot 3.50\right\} \\
&= \max\{3, 2.20\} \\
&= 3.
\end{aligned}
$$

At time one the value of the put is

$$
\begin{aligned}
V_1^P(H) &= \max\left\{\big(4 - S_1(H)\big)^+, \frac{2}{5}V_2^P(HH) + \frac{2}{5}V_2^P(HT)\right\} \\
&= \max\left\{(4-8)^+, \frac{2}{5}\cdot 0 + \frac{2}{5}\cdot 0.80\right\} \\
&= \max\{0, 0.32\} \\
&= 0.32, \\
V_1^P(T) &= \max\left\{\big(4 - S_1(T)\big)^+, \frac{2}{5}V_2^P(TH) + \frac{2}{5}V_2^P(TT)\right\} \\
&= \max\left\{(4-2)^+, \frac{2}{5}\cdot 0.80 + \frac{2}{5}\cdot 3\right\} \\
&= \max\{2, 1.52\} \\
&= 2.
\end{aligned}
$$

The value of the put at time zero is

$$\begin{aligned}
V_0^P &= \max\left\{(4-S_0)^+, \frac{2}{5}V_1^P(H) + \frac{2}{5}V_1^P(T)\right\}\\
&= \max\left\{(4-4)^+, \frac{2}{5}\cdot 0.32 + \frac{2}{5}\cdot 2\right\}\\
&= \max\{0, 0.928\}\\
&= 0.928.
\end{aligned}$$

(ii) The payoff of the call at expiration time three is

$$\begin{aligned}
V_3^C(HHH) &= (32-4)^+ = 28,\\
V_3^C(HHT) &= V_3^C(HTH) = V_3^C(THH) = (8-4)^+ = 4,\\
V_3^C(HTT) &= V_3^C(THT) = V_3^C(TTH) = (2-4)^+ = 0,\\
V_3^C(TTT) &= (0.50-4)^+ = 0.
\end{aligned}$$

Because $\frac{1}{1+r}\tilde{p} = \frac{1}{1+r}\tilde{q} = \frac{2}{5}$, the value of the call at time two is

$$\begin{aligned}
V_2^C(HH) &= \max\left\{\big(S_2(HH)-4\big)^+, \frac{2}{5}V_3^C(HHH) + \frac{2}{5}V_3^C(HHT)\right\}\\
&= \max\left\{(16-4)^+, \frac{2}{5}\cdot 28 + \frac{2}{5}\cdot 4\right\}\\
&= \max\{12, 12.8\}\\
&= 12.8,\\
V_2^C(HT) &= \max\left\{\big(S_2(HT)-4\big)^+, \frac{2}{5}V_3^C(HTH) + \frac{2}{5}V_3^C(HTT)\right\}\\
&= \max\left\{\big(4-4\big)^+, \frac{2}{5}\cdot 4 + \frac{2}{5}\cdot 0\right\}\\
&= \max\{0, 1.60\}\\
&= 1.60,\\
V_2^C(TH) &= \max\left\{\big(S_2(TH)-4\big)^+, \frac{2}{5}V_3^C(THH) + \frac{2}{5}V_3^C(THT)\right\}\\
&= \max\left\{\big(4-4\big)^+, \frac{2}{5}\cdot 4 + \frac{2}{5}\cdot 0\right\}\\
&= \max\{0, 1.60\}\\
&= 1.60,\\
V_2^C(TT) &= \max\left\{\big(S_2(TT)-4\big)^+, \frac{2}{5}V_3^C(TTH) + \frac{2}{5}V_3^C(TTT)\right\}\\
&= \max\left\{(1-4)^+, \frac{2}{5}\cdot 0 + \frac{2}{5}\cdot 0\right\}\\
&= \max\{0, 0\}\\
&= 0.
\end{aligned}$$

At time one the value of the call is

$$\begin{aligned}
V_1^C(H) &= \max\left\{\left(S_1(H)-4\right)^+, \frac{2}{5}V_2^C(HH)+\frac{2}{5}V_2^C(HT)\right\}\\
&= \max\left\{(8-4)^+, \frac{2}{5}\cdot 12.8+\frac{2}{5}\cdot 1.60\right\}\\
&= \max\{4, 5.76\} \;=\; 5.76,\\
V_1^C(T) &= \max\left\{\left(S_1(T)-4\right)^+, \frac{2}{5}V_2^C(TH)+\frac{2}{5}V_2^C(TT)\right\}\\
&= \max\left\{(2-4)^+, \frac{2}{5}\cdot 1.60+\frac{2}{5}\cdot 0\right\}\\
&= \max\{0, 0.64\} \;=\; 0.64.
\end{aligned}$$

The value of the call at time zero is

$$\begin{aligned}
V_0^C &= \max\left\{(S_0-4)^+, \frac{2}{5}V_1^C(H)+\frac{2}{5}V_1^C(T)\right\}\\
&= \max\left\{(4-4)^+, \frac{2}{5}\cdot 5.76+\frac{2}{5}\cdot 0.64\right\}\\
&= \max\{0, 2.56\}\\
&= 2.56.
\end{aligned}$$

(iii) Note that $g_S(s)=|s-4|$. The payoff of the straddle at expiration time three is

$$\begin{aligned}
V_3^S(HHH) &= \quad |32-4| \quad = 28,\\
V_3^S(HHT) &= V_3^S(HTH) = V_3^S(THH) = \quad |8-4| \quad = 4,\\
V_3^S(HTT) &= V_3^S(THT) = V_3^S(TTH) = |2-4| = 2,\\
V_3^S(TTT) &= \quad |0.50-4| \quad = 3.50.
\end{aligned}$$

We see that the payoff of the straddle is the payoff of the put given in the solution to (i) plus the payoff of the call given in the solution to (ii). Because $\frac{1}{1+r}\tilde{p}=\frac{1}{1+r}\tilde{q}=\frac{2}{5}$, the value of the straddle at time two is

$$\begin{aligned}
V_2^S(HH) &= \max\left\{\left|S_2(HH)-4\right|, \frac{2}{5}V_3^S(HHH)+\frac{2}{5}V_3^S(HHT)\right\}\\
&= \max\left\{|16-4|, \frac{2}{5}\cdot 28+\frac{2}{5}\cdot 4\right\}\\
&= \max\{12, 12.8\}\\
&= 12.8,\\
V_2^S(HT) &= \max\left\{\left|S_2(HT)-4\right|^+, \frac{2}{5}V_3^S(HTH)+\frac{2}{5}V_3^S(HTT)\right\}\\
&= \max\left\{(4-4)^+, \frac{2}{5}\cdot 4+\frac{2}{5}\cdot 2\right\}\\
&= \max\{0, 2.40\}\\
&= 2.40,
\end{aligned}$$

$$
\begin{aligned}
V_2^S(TH) &= \max\left\{\left(S_2(TH)-4\right)^+, \frac{2}{5}V_3^S(THH)+\frac{2}{5}V_3^S(THT)\right\}\\
&= \max\left\{(4-4)^+, \frac{2}{5}\cdot 4+\frac{2}{5}\cdot 2\right\}\\
&= \max\{0, 2.40\}\\
&= 2.40,\\
V_2^S(TT) &= \max\left\{\left|S_2(TT)-4\right|, \frac{2}{5}V_3^S(TTH)+\frac{2}{5}V_3^S(TTT)\right\}\\
&= \max\left\{|1-4|, \frac{2}{5}\cdot 2+\frac{2}{5}\cdot 3.50\right\}\\
&= \max\{3, 2.20\}\\
&= 3.
\end{aligned}
$$

One can verify in every case that $V_2^S = V_2^P + V_2^C$. At time one the value of the straddle is

$$
\begin{aligned}
V_1^S(H) &= \max\left\{\left|S_1(H)-4\right|, \frac{2}{5}V_2^S(HH)+\frac{2}{5}V_2^S(HT)\right\}\\
&= \max\left\{|8-4|, \frac{2}{5}\cdot 12.8+\frac{2}{5}\cdot 2.40\right\}\\
&= \max\{4, 6.08\}\\
&= 6.08,\\
V_1^S(T) &= \max\left\{\left|S_1(T)-4\right|, \frac{2}{5}V_2^S(TH)+\frac{2}{5}V_2^S(TT)\right\}\\
&= \max\left\{|2-4|, \frac{2}{5}\cdot 2.40+\frac{2}{5}\cdot 3\right\}\\
&= \max\{2, 2.16\}\\
&= 2.16.
\end{aligned}
$$

We have $V_1^S(H) = 6.08 = 0.32 + 5.76 = V_1^P(H) + V_1^C(H)$, but $V_1^S(T) = 2.16 < 2 + 0.64 = V_1^P(T) + V_1^C(T)$.

The value of the straddle at time zero is

$$
\begin{aligned}
V_0^S &= \max\left\{|S_0-4|, \frac{2}{5}V_1^S(H)+\frac{2}{5}V_1^S(T)\right\}\\
&= \max\left\{|4-4|, \frac{2}{5}\cdot 6.08+\frac{2}{5}\cdot 2.16\right\}\\
&= \max\{0, 3.296\}\\
&= 3.296.
\end{aligned}
$$

We have $V_0^S = 3.296 < 0.928 + 2.56 = V_0^P + V_0^C$.

(iv) For the put, if there is a tail on the first toss, it is optimal to exercise at time one. This can be seen from the equation

$$\begin{aligned} V_1^P(T) &= \max\left\{\big(4 - S_1(T)\big)^+, \frac{2}{5}V_2^P(TH) + \frac{2}{5}V_2^P(TT)\right\} \\ &= \max\left\{(4-2)^+, \frac{2}{5}\cdot 0.80 + \frac{2}{5}\cdot 3\right\} \\ &= \max\{2, 1.52\} \\ &= 2, \end{aligned}$$

which shows that the intrinsic value at time one if the first toss results in T is greater than the value of continuing. On the other hand, for the call the intrinsic value at time one if there is a tail on the first toss is $(S_1(T) - 4)^+ = (2-4)^+ = 0$, whereas the value of continuing is 0.64. Therefore, the call should not be exercised at time one if there is a tail on the first toss.

The straddle has the intrinsic value of a put plus a call. When it is exercised, both parts of the payoff are received. In other words, it is not an American put plus an American call, because these can be exercised at different times whereas the exercise of a straddle requires both the put payoff and the call payoff to be received. In the computation of the straddle price

$$\begin{aligned} V_1^S(T) &= \max\left\{\big|S_1(T) - 4\big|, \frac{2}{5}V_2^S(TH) + \frac{2}{5}V_2^S(TT)\right\} \\ &= \max\left\{|2-4|, \frac{2}{5}\cdot 2.40 + \frac{2}{5}\cdot 3\right\} \\ &= \max\{2, 2.16\} \\ &= 2.16, \end{aligned}$$

we see that it is not optimal to exercise the straddle at time one if the first toss results in T. It would be optimal to exercise the put part, but not the call part, and the straddle cannot exercise one part without exercising the other. Greater value is achieved by not exercising both parts than would be achieved by exercising both. However, this value is less than would be achieved if one could exercise the put part and let the call part continue, and thus $V_1^S(T) < V_1^P(T) + V_1^C(T)$. This loss of value at time one results in a similar loss of value at the earlier time zero: $V_0^S < V_0^P + V_0^C$.

Exercise 4.3. In the three-period model of Figure 1.2.2 of Chapter 1, let the interest rate be $r = \frac{1}{4}$ so the risk-neutral probabilities are $\tilde{p} = \tilde{q} = \frac{1}{2}$. Find the time-zero price and optimal exercise policy (optimal stopping time) for the path-dependent American derivative security whose intrinsic value at each time n, $n = 0, 1, 2, 3$, is $\left(4 - \frac{1}{n+1}\sum_{j=0}^{n} S_j\right)^+$. This intrinsic value is a put on the average stock price between time zero and time n.

Solution The intrinsic value process for this option is

$$\begin{aligned}
G_0 &= (4-S_0)^+ &&= (4-4)^+ &&= 0,\\
G_1(H) &= \left(4-\tfrac{S_0+S_1(H)}{2}\right)^+ &&= \left(4-\tfrac{4+8}{2}\right)^+ &&= 0,\\
G_1(T) &= \left(4-\tfrac{S_0+S_1(T)}{2}\right)^+ &&= \left(4-\tfrac{4+2}{2}\right)^+ &&= 1,\\
G_2(HH) &= \left(4-\tfrac{S_0+S_1(H)+S_2(HH)}{3}\right)^+ &&= \left(4-\tfrac{4+8+16}{3}\right)^+ &&= 0,\\
G_2(HT) &= \left(4-\tfrac{S_0+S_1(H)+S_2(HT)}{3}\right)^+ &&= \left(4-\tfrac{4+8+4}{3}\right) &&= 0,\\
G_2(TH) &= \left(4-\tfrac{S_0+S_1(T)+S_2(TH)}{3}\right)^+ &&= \left(4-\tfrac{4+2+4}{3}\right)^+ &&= 0.6667,\\
G_2(TT) &= \left(4-\tfrac{S_0+S_1(T)+S_2(TT)}{3}\right)^+ &&= \left(r-\tfrac{4+2+1}{3}\right) &&= 1.6667.
\end{aligned}$$

At time three, the intrinsic value G_3 agrees with the option value V_3. In other words,

$$\begin{aligned}
V_3(HHH) &= G_3(HHH)\\
&= \left(4-\frac{S_0+S_1(H)+S_2(HH)+S_3(HHH)}{4}\right)^+\\
&= \left(4-\frac{4+8+16+32}{4}\right)^+\\
&= 0,\\
V_3(HHT) &= G_3(HHT)\\
&= \left(4-\frac{S_0+S_1(H)+S_2(HH)+S_3(HHT)}{4}\right)^+\\
&= \left(4-\frac{4+8+16+8}{4}\right)^+\\
&= 0,\\
V_3(HTH) &= G_3(HTH)\\
&= \left(4-\frac{S_0+S_1(H)+S_2(HT)+S_3(HTH)}{4}\right)^+\\
&= \left(4-\frac{4+8+4+8}{4}\right)^+\\
&= 0,\\
V_3(HTT) &= G_3(HTT)\\
&= \left(4-\frac{S_0+S_1(H)+S_2(HT)+S_3(HTT)}{4}\right)^+\\
&= \left(4-\frac{4+8+4+2}{4}\right)^+\\
&= 0,
\end{aligned}$$

$$
\begin{aligned}
V_3(THH) &= G_3(THH) \\
&= \left(4 - \frac{S_0 + S_1(T) + S_2(TH) + S_3(THH)}{4}\right)^+ \\
&= \left(4 - \frac{4+2+4+8}{4}\right)^+ \\
&= 0, \\
V_3(THT) &= G_3(THT) \\
&= \left(4 - \frac{S_0 + S_1(T) + S_2(TH) + S_3(THT)}{4}\right)^+ \\
&= \left(4 - \frac{4+2+4+2}{4}\right)^+ \\
&= 1, \\
V_3(TTH) &= G_3(TTH) \\
&= \left(4 - \frac{S_0 + S_1(T) + S_2(TT) + S_3(TTH)}{4}\right)^+ \\
&= \left(4 - \frac{4+2+1+2}{4}\right)^+ \\
&= 1.75, \\
V_3(TTT) &= G_3(TTT) \\
&= \left(4 - \frac{S_0 + S_1(T) + S_2(TT) + S_3(TTT)}{4}\right)^+ \\
&= \left(4 - \frac{4+2+1+0.50}{4}\right)^+ \\
&= 2.125.
\end{aligned}
$$

We use the algorithm of Theorem 4.4.3, noting that $\frac{\tilde{p}}{1+r} = \frac{\tilde{q}}{1+r} = \frac{2}{5}$, to obtain

$$
\begin{aligned}
V_2(HH) &= \max\left\{G_2(HH), \frac{2}{5}V_3(HHH) + \frac{2}{5}V_3(HHT)\right\} \\
&= \max\left\{0, \frac{2}{5}\cdot 0 + \frac{2}{5}\cdot 0\right\} \\
&= 0, \\
V_2(HT) &= \max\left\{G_2(HT), \frac{2}{5}V_3(HTH) + \frac{2}{5}V_3(HTT)\right\} \\
&= \max\left\{0, \frac{2}{5}\cdot 0 + \frac{2}{5}\cdot 0\right\} \\
&= 0,
\end{aligned}
$$

$$\begin{aligned}
V_2(TH) &= \max\left\{G_2(TH), \frac{2}{5}V_3(THH) + \frac{2}{5}V_3(THT)\right\}\\
&= \max\left\{0.6667, \frac{2}{5}\cdot 0 + \frac{2}{5}\cdot 1\right\}\\
&= \max\{0.6667, 0.40\}\\
&= 0.6667,\\
V_2(TT) &= \max\left\{G_2(TT), \frac{2}{5}V_3(TTH) + \frac{2}{5}V_3(TTT)\right\}\\
&= \max\left\{1.6667, \frac{2}{5}\cdot 1.75 + \frac{2}{5}\cdot 2.125\right\}\\
&= \max\{1.6667, 1.55\}\\
&= 1.6667.
\end{aligned}$$

Continuing, we have

$$\begin{aligned}
V_1(H) &= \max\left\{G_1(H), \frac{2}{5}V_2(HH) + \frac{2}{5}V_2(HT)\right\}\\
&= \max\left\{0, \frac{2}{5}\cdot 0 + \frac{2}{5}\cdot 0\right\}\\
&= 0,\\
V_1(T) &= \max\left\{G_1(T), \frac{2}{5}V_2(TH) + \frac{2}{5}V_2(TT)\right\}\\
&= \max\left\{1, \frac{2}{5}\cdot 0.6667 + \frac{2}{5}\cdot 1.6667\right\}\\
&= \max\{1, 0.9334\}\\
&= 1,\\
V_0 &= \max\left\{G_0, \frac{2}{5}V_1(H) + \frac{2}{5}V_1(T)\right\}\\
&= \max\left\{0, \frac{2}{5}\cdot 0 + \frac{2}{5}\cdot 1\right\}\\
&= \max\{0, 0.40\}\\
&= 0.40.
\end{aligned}$$

To find the optimal exercise time, we work forward. Since $V_0 > G_0$, one should not exercise at time zero. However, $V_1(T) = G_1(T)$, so it is optimal to exercise at time one if there is a T on the first toss. If the first toss results in H, the option is destined always be out of the money. With the intrinsic value $G_n = \left(4 - \frac{1}{n+1}\sum_{j=1}^{n} S_j\right)^+$ defined in the exercise, it does not matter what exercise rule we choose in this case. If the payoff were $\left(4 - \frac{1}{n+1}\sum_{j=1}^{n} S_j\right)$, so that exercising out of the money is costly (as one would expect in practice), then one should allow the option to expire unexercised.

Exercise 4.5. In equation (4.4.5), the maximum is computed over all stopping times in $\mathcal{S}_0$. List all the stopping times in $\mathcal{S}_0$ (there are 26), and from among those, list the stopping times that never exercise when the option is out of the money (there are 11). For each stopping time τ in the latter set, compute $\mathbb{E}\left[\mathbb{I}_{\{\tau\le 2\}}\left(\frac{4}{5}\right)^{\tau} G_{\tau}\right]$. Verify that the largest value for this quantity is given by the stopping time of (4.4.6), the one which makes this quantity equal to the 1.36 computed in (4.4.7).

Solution A stopping time is a random variable, and we can specify a stopping time by listing its values $\tau(HH)$, $\tau(HT)$, $\tau(TH)$, and $\tau(TT)$. The stopping time property requires that $\tau(HH) = 0$ if and only if $\tau(HT) = \tau(TH) = \tau(TT) = 0$. Similarly, $\tau(HH) = 1$ if and only if $\tau(HT) = 1$ and $\tau(TH) = 1$ if and only if $\tau(TT) = 1$. The 26 stopping times in the two-period binomial model are tabulated below.

Stopping Time	HH	HT	TH	TT
τ_1	0	0	0	0
τ_2	1	1	1	1
τ_3	1	1	2	2
τ_4	1	1	2	∞
τ_5	1	1	∞	2
τ_6	1	1	∞	∞
τ_7	2	2	1	1
τ_8	2	2	2	2
τ_9	2	2	2	∞
τ_{10}	2	2	∞	2
τ_{11}	2	2	∞	∞
τ_{12}	2	∞	1	1
τ_{13}	2	∞	2	2
τ_{14}	2	∞	2	∞
τ_{15}	2	∞	∞	2
τ_{16}	2	∞	∞	∞
τ_{17}	∞	2	1	1
τ_{18}	∞	2	2	2
τ_{19}	∞	2	2	∞
τ_{20}	∞	2	∞	2
τ_{21}	∞	2	∞	∞
τ_{22}	∞	∞	1	1
τ_{23}	∞	∞	2	2
τ_{24}	∞	∞	2	∞
τ_{25}	∞	∞	∞	2
τ_{26}	∞	∞	∞	∞

The intrinsic value process for this option is given by

$$G_0 = 1,\ G_1(H) = -3,\ G_1(T) = 3,$$
$$G_2(HH) = -11,\ G_2(HT) = G_2(TH) = 1,\ G_2(TT) = 4.$$

The stopping times that take the value 1 when there is an H on the first toss are mandating an exercise out of the money ($G_1(H) = -3$). This rules out τ_2–τ_6. Also, the stopping times that take the value 2 when there is an HH on the first two tosses are mandating an exercise out of the money ($G_2(HH) = -11$). This rules out τ_7–τ_{16}. For all other exercise situations, G is positive, so the option is in the money. This leaves us with τ_1 and the ten stopping times τ_{17}–τ_{26}. We evaluate the risk-neutral expected payoff of these eleven stopping times.

$$\begin{aligned}
\mathbb{E}\left[\mathbb{I}_{\{\tau_1 \le 2\}} \left(\frac{4}{5}\right)^{\tau} G_{\tau_1}\right] &= G_0 \ = \ 1,\\
\mathbb{E}\left[\mathbb{I}_{\{\tau_{17} \le 2\}} \left(\frac{4}{5}\right)^{\tau} G_{\tau_{17}}\right] &= \frac{1}{4}\cdot\frac{16}{26}G_2(HT) + \frac{1}{2}\cdot\frac{4}{5}G_1(T)\\
&= \frac{4}{25}\cdot 1 + \frac{2}{5}\cdot 3 \ = \ 1.36,\\
\mathbb{E}\left[\mathbb{I}_{\{\tau_{18} \le 2\}} \left(\frac{4}{5}\right)^{\tau} G_{\tau_{18}}\right] &= \frac{1}{4}\cdot\frac{16}{26}G_2(HT) + \frac{1}{4}\cdot\frac{16}{25}G_2(TH) + \frac{1}{4}\cdot\frac{16}{25}G_2(TT)\\
&= \frac{4}{25}\cdot 1 + \frac{4}{25}\cdot 1 + \frac{4}{25}\cdot 4 \ = \ 0.96,\\
\mathbb{E}\left[\mathbb{I}_{\{\tau_{19} \le 2\}} \left(\frac{4}{5}\right)^{\tau} G_{\tau_{19}}\right] &= \frac{1}{4}\cdot\frac{16}{26}G_2(HT) + \frac{1}{4}\cdot\frac{16}{25}G_2(TH)\\
&= \frac{4}{25}\cdot 1 + \frac{4}{25}\cdot 1 \ = \ 0.32,\\
\mathbb{E}\left[\mathbb{I}_{\{\tau_{20} \le 2\}} \left(\frac{4}{5}\right)^{\tau} G_{\tau_{20}}\right] &= \frac{1}{4}\cdot\frac{16}{26}G_2(HT) + \frac{1}{4}\cdot\frac{16}{25}G_2(TT)\\
&= \frac{4}{25}\cdot 1 + \frac{4}{25}\cdot 4 \ = \ 0.80,\\
\mathbb{E}\left[\mathbb{I}_{\{\tau_{21} \le 2\}} \left(\frac{4}{5}\right)^{\tau} G_{\tau_{21}}\right] &= \frac{1}{4}\cdot\frac{16}{26}G_2(HT)\\
&= \frac{4}{25}\cdot 1 \ = \ 0.16,\\
\mathbb{E}\left[\mathbb{I}_{\{\tau_{22} \le 2\}} \left(\frac{4}{5}\right)^{\tau} G_{\tau_{22}}\right] &= \frac{1}{2}\cdot\frac{4}{5}G_1(T)\\
&= \frac{2}{5}\cdot 3 \ = \ 1.20,\\
\mathbb{E}\left[\mathbb{I}_{\{\tau_{23} \le 2\}} \left(\frac{4}{5}\right)^{\tau} G_{\tau_{23}}\right] &= +\frac{1}{4}\cdot\frac{16}{25}G_2(TH) + \frac{1}{4}\cdot\frac{16}{25}G_2(TT)\\
&= \frac{4}{25}\cdot 1 + \frac{4}{25}\cdot 4 \ = \ 0.80,
\end{aligned}$$

$$\mathbb{E}\left[\mathbb{I}_{\{\tau_{24}\le 2\}}\left(\frac{4}{5}\right)^{\tau} G_{\tau_{24}}\right] = +\frac{1}{4}\cdot\frac{16}{25}G_2(TH)$$
$$= \frac{4}{25}\cdot 1 = 0.16,$$
$$\mathbb{E}\left[\mathbb{I}_{\{\tau_{25}\le 2\}}\left(\frac{4}{5}\right)^{\tau} G_{\tau_{25}}\right] = +\frac{1}{4}\cdot\frac{16}{25}G_2(TT)$$
$$= +\frac{4}{25}\cdot 4 = 0.64,$$
$$\mathbb{E}\left[\mathbb{I}_{\{\tau_{26}\le 2\}}\left(\frac{4}{5}\right)^{\tau} G_{\tau_{26}}\right] = 0.$$

The largest value, 1.36, is obtained by the stopping time τ_{17}.

Exercise 4.7. For the class of derivative securities described in Exercise 4.6 whose time-zero price is given by (4.8.3), let $G_n = S_n - K$. This derivative security permits its owner to buy one share of stock in exchange for a payment of K at any time up to the expiration time N. If the purchase has not been made at time N, it must be made then. Determine the time-zero value and optimal exercise policy for this derivative security. (Assume $r \ge 0$.)

Solution Set $Y_n = \frac{1}{(1+r)^n}(S_n - K)$, $n = 0, 1, \ldots, N$. We assume $r \ge 0$. Because the discounted stock price is a martingale under the risk-neutral measure and $\frac{K}{(1+r)^{n+1}}$ is not random, we have

$$\widetilde{\mathbb{E}}_n[Y_{n+1}] = \widetilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] - \widetilde{\mathbb{E}}_n\left[\frac{K}{(1+r)^{n+1}}\right]$$
$$= \frac{S_n}{(1+r)^n} - \frac{K}{(1+r)^{n+1}}$$
$$\ge \frac{S_n}{(1+r)^n} - \frac{K}{(1+r)^n}.$$

This shows that Y_n, $n = 0, 1, \ldots, N$, is a submartingale. According to Theorem 4.3.3 (Optional Sampling—Part II), $\widetilde{\mathbb{E}}Y_{N\wedge\tau} \le \widetilde{\mathbb{E}}Y_N$ whenever τ is a stopping time. If τ is a stopping time satisfying $\tau(\omega) \le N$ for every sequence of coin tosses ω, this becomes $\widetilde{\mathbb{E}}Y_\tau \le \widetilde{\mathbb{E}}Y_N$, or equivalently,

$$\widetilde{\mathbb{E}}\left[\frac{1}{(1+r)^\tau}G_\tau\right] \le \widetilde{\mathbb{E}}\left[\frac{1}{(1+r)^N}G_N\right].$$

Therefore,

$$V_0 = \max_{\tau\in\mathcal{S}_0,\tau\le N}\widetilde{\mathbb{E}}\left[\frac{1}{(1+r)^\tau}G_\tau\right] \le \widetilde{\mathbb{E}}\left[\frac{1}{(1+r)^N}G_N\right].$$

On the other hand, because the stopping time that is equal to N regardless of the outcome of the coin tossing is in the set of stopping times over which the above maximum is taken, we must in fact have equality:

$$V_0 = \max_{\tau \in \mathcal{S}_0, \tau \leq N} \widetilde{\mathbb{E}} \left[\frac{1}{(1+r)^\tau} G_\tau \right] = \widetilde{\mathbb{E}} \left[\frac{1}{(1+r)^N} G_N \right].$$

Hence, it is optimal to exercise at the final time N regardless of the outcome of the coin tossing.

5

Random Walk

5.8 Solutions to Selected Exercises

Exercise 5.2. (First passage time for random walk with upward drift) Consider the asymmetric random walk with probability p for an up step and probability $q = 1 - p$ for a down step, where $\frac{1}{2} < p < 1$ so that $0 < q < \frac{1}{2}$. In the notation of (5.2.1), let τ_1 be the first time the random walk starting from level 0 reaches the level 1. If the random walk never reaches this level, then $\tau_1 = \infty$.

(i) Define $f(\sigma) = pe^{\sigma} + qe^{-\sigma}$. Show that $f(\sigma) > 1$ for all $\sigma > 0$.

(ii) Show that when $\sigma > 0$, the process

$$S_n = e^{\sigma M_n} \left(\frac{1}{f(\sigma)} \right)^n$$

is a martingale.

(iii) Show that for $\sigma > 0$,

$$e^{-\sigma} = \mathbb{E} \left[\mathbb{I}_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right].$$

Conclude that $\mathbb{P}\{\tau_1 < \infty\} = 1$.

(iv) Compute $\mathbb{E}\alpha^{\tau_1}$ for $\alpha \in (0, 1)$.

(v) Compute $\mathbb{E}\tau_1$.

Solution.

(i) The function $f(\sigma)$ satisfies $f(0) = 1$ and $f'(\sigma) = pe^{\sigma} - qe^{-\sigma}$, $f''(\sigma) = f(\sigma)$. Since f is always positive, f'' is always positive and f is convex. Under the assumption $p > 1/2 > q$, we have $f'(0) = p - q > 0$, and the convexity of f implies that $f(\sigma) > 1$ for all $\sigma > 0$.

(ii) We compute

$$\begin{aligned}
\widetilde{\mathbb{E}}_n[S_{n+1}] &= \left(\frac{1}{f(\sigma)}\right)^{n+1} \widetilde{\mathbb{E}}_n\left[e^{\sigma(M_n+X_{n+1})}\right] \\
&= \left(\frac{1}{f(\sigma)}\right)^{n+1} e^{\sigma M_n}\mathbb{E}\left[e^{\sigma X_{n+1}}\right] \\
&= \left(\frac{1}{f(\sigma)}\right)^{n+1} e^{\sigma M_n}\left(pe^{\sigma}+qe^{-\sigma}\right) \\
&= e^{\sigma M_n}\left(\frac{1}{f(\sigma)}\right)^{n} = S_n.
\end{aligned}$$

(iii) Because $S_{n\wedge\tau_1}$ is a martingale starting at 1, we have

$$1 = \mathbb{E}S_{n\wedge\tau_1} = \mathbb{E}e^{\sigma M_{n\wedge\tau_1}}\left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1}. \tag{5.8.1}$$

For $\sigma > 0$, the positive random variable $e^{\sigma M_{n\wedge\tau_1}}$ is bounded above by e^{σ}, and $0 < 1/f(\sigma) < 1$. Therefore,

$$\lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}}\left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1} = \mathbb{I}_{\{\tau_1<\infty\}}e^{\sigma}\left(\frac{1}{f(\sigma)}\right)^{\tau}.$$

Letting $n\to\infty$ in (5.8.1), we obtain

$$1 = \mathbb{E}\left[\mathbb{I}_{\{\tau_1<\infty\}}e^{\sigma}\left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right],$$

or equivalently,

$$e^{-\sigma} = \mathbb{E}\left[\mathbb{I}_{\{\tau_1<\infty\}}\left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right]. \tag{5.8.2}$$

This equation holds for all positive σ. We let $\sigma\downarrow 0$ to obtain the formula

$$1 = \mathbb{E}\mathbb{I}_{\{\tau_1<\infty\}} = \mathbb{P}\{\tau_1<\infty\}.$$

(iv) We now introduce $\alpha\in(0,1)$ and solve the equation $\alpha = \frac{1}{f(\sigma)}$ for $e^{-\sigma}$. This equation can be written as

$$\alpha pe^{\sigma} + \alpha qe^{-\sigma} = 1,$$

which may be rewritten as

$$\alpha q\left(e^{-\sigma}\right)^2 - e^{-\sigma} + \alpha p = 0,$$

and the quadratic formula gives

$$e^{-\sigma} = \frac{1 \pm \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}. \tag{5.8.3}$$

We want σ to be positive, so we need $e^{-\sigma}$ to be less than 1. Hence we take the negative sign, obtaining

$$e^{-\sigma} = \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}.$$

Substituting this into (5.8.2), we obtain the formula

$$\mathbb{E}\alpha^{\tau_1} = \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}. \tag{5.8.4}$$

(v) Differentiating (5.8.4) with respect to α leads to

$$\mathbb{E}\tau\alpha^{\tau-1} = \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha^2 q\sqrt{1 - 4\alpha^2 pq}},$$

and letting $\alpha \uparrow 1$, we obtain

$$\mathbb{E}\tau_1 = \frac{1 - \sqrt{1 - 4pq}}{2q\sqrt{1 - 4pq}} = \frac{1 - \sqrt{(1-2q)^2}}{2q\sqrt{(1-2q)^2}} = \frac{1 - (1-2q)}{2q(1-2q)} = \frac{1}{1-2q} = \frac{1}{p-q}.$$

Exercise 5.4 (Distribution of τ_2). Consider the symmetric random walk, and let τ_2 be the first time the random walk, starting from level 0, reaches the level 2. According to Theorem 5.2.3,

$$\mathbb{E}\alpha^{\tau_2} = \left(\frac{1 - \sqrt{1 - \alpha^2}}{\alpha}\right)^2 \quad \text{for all } \alpha \in (0, 1).$$

Using the power series (5.2.21), we may write the right-hand side as

$$\begin{aligned}
\left(\frac{1 - \sqrt{1 - \alpha^2}}{\alpha}\right)^2 &= \frac{2}{\alpha} \cdot \frac{1 - \sqrt{1 - \alpha^2}}{\alpha} - 1 \\
&= -1 + \sum_{j=1}^{\infty} \left(\frac{\alpha}{2}\right)^{2j-2} \frac{(2j-2)!}{j!(j-1)!} \\
&= \sum_{j=2}^{\infty} \left(\frac{\alpha}{2}\right)^{2j-2} \frac{(2j-2)!}{j!(j-1)!} \\
&= \sum_{k=1}^{\infty} \left(\frac{\alpha}{2}\right)^{2k} \frac{(2k)!}{(k+1)!k!}.
\end{aligned}$$

(i) Use the above power series to determine $\mathbb{P}\{\tau_2 = 2k\}$, $k = 1, 2, \ldots$.

(ii) Use the reflection principle to determine $\mathbb{P}\{\tau_2 = 2k\}$, $k = 1, 2, \ldots$.

Solution

(i) Since the random walk can reach the level 2 only on even-numbered steps, $\mathbb{P}\{\tau_2 = 2k-1\} = 0$ for $k = 1, 2, \ldots$. Therefore,

$$\mathbb{E}\alpha^{\tau_2} = \sum_{k=1}^{\infty} \alpha^{2k}\mathbb{P}\{\tau_2 = 2k\} = \sum_{k=1}^{\infty} \left(\frac{\alpha}{2}\right)^{2k} \frac{(2k)!}{(k+1)!k!}.$$

Equating coefficients, we conclude that

$$\mathbb{P}\{\tau = 2k\} = \frac{(2k)!}{(k+1)!k!}\left(\frac{1}{2}\right)^{2k}, \quad k = 1, 2, \ldots.$$

(ii) For $k = 1, 2, \ldots$, the number of paths that reach or exceed level 2 by time $2k$ is equal to the number of paths that are at level 2 at time $2k$ plus the number of paths that exceed level 2 at time $2k$ plus the number of paths that reach the level 2 before time $2k$ but are below level 2 at time $2k$. A path is of the last type if and only if its reflected path exceeds level 2 at time $2k$. Thus, the number of paths that reach or exeed level 2 by time $2k$ is equal to the number of paths that are at level 2 at time $2k$ plus twice the number of paths that exeed level 2 at time $2k$. For the symmetric random walk, every path has the same probability, and hence

$$\begin{aligned}
\mathbb{P}\{\tau_2 \le 2k\} &= \mathbb{P}\{M_{2k} = 2\} + 2\mathbb{P}\{M_{2k} \ge 4\} \\
&= \mathbb{P}\{M_{2k} = 2\} + \mathbb{P}\{M_{2k} \ge 4\} + \mathbb{P}\{M_{2k} \le -4\} \\
&= 1 - \mathbb{P}\{M_{2k} = 0\} - \mathbb{P}\{M_{2k} = -2\} \\
&= 1 - \frac{(2k)!}{k!k!}\left(\frac{1}{2}\right)^{2k} - \frac{(2k)!}{(k+1)!(k-1)!}\left(\frac{1}{2}\right)^{2k}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbb{P}\{\tau_2 = 2k\} \\
&= \mathbb{P}\{\tau_2 \le 2k\} - \mathbb{P}\{\tau_2 \le 2k-2\} \\
&= \left(\frac{1}{2}\right)^{2k-2}\left[\frac{(2k-2)!}{(k-1)!(k-1)!} + \frac{(2k-2)!}{k!(k-2)!}\right] \\
&\qquad - \left(\frac{1}{2}\right)^{2k}\left[\frac{(2k)!}{k!k!} + \frac{(2k)!}{(k+1)!(k-1)!}\right] \\
&= \left(\frac{1}{2}\right)^{2k}\left[\frac{4(2k-2)!}{(k-1)!(k-1)!} - \frac{(2k)!}{k!k!}\right] \\
&\qquad + \left(\frac{1}{2}\right)^{2k}\left[\frac{4(2k-2)!}{k!(k-2)!} - \frac{(2k)!}{(k+1)!(k-1)!}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{2}\right)^{2k} \left[\frac{2k \cdot 2k(2k-2)! - 2k(2k-1)(2k-2)!}{k!k!}\right] \\
&\qquad + \left(\frac{1}{2}\right)^{2k} \left[\frac{(2k+2)(2k-2)(2k-2)! \;\; - 2k(2k-1)(2k-2)!}{(k+1)!(k-1)!}\right] \\
&= \left(\frac{1}{2}\right)^{2k} \left[\frac{4k^2(2k-2)! - (4k^2-2k)(2k-2)!}{k!k!}\right] \\
&\qquad + \left(\frac{1}{2}\right)^{2k} \left[\frac{(4k^2-4)(2k-2)! \;\; - (4k^2-2k))(2k-2)!}{(k+1)!(k-1)!}\right] \\
&= \left(\frac{1}{2}\right)^{2k} \left[\frac{2k(2k-2)!}{k!k!} + \frac{2(k-2)(2k-2)!}{(k+1)!(k-1)!}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{2}\right)^{2k} \frac{(2k-2)!}{(k+1)!k!} \left[(2k(k+1) + 2(k-2)k\right] \\
&= \left(\frac{1}{2}\right)^{2k} \frac{(2k-2)!}{(k+1)!k!} 2k(2k-1) \\
&= \left(\frac{1}{2}\right)^{2k} \frac{(2k)!}{(k+1)!k!}.
\end{aligned}
$$

Exercise 5.7. (Hedging a short position in the perpetual American put). Suppose you have sold the perpetual American put of Section 5.4 and are hedging the short position in this put. Suppose at the current time the stock price is s and the value of your hedging portfolio is $v(s)$. Your hedge is to first consume the amount

$$
c(s) = v(s) - \frac{4}{5}\left[\frac{1}{2}v(2s) + \frac{1}{2}v\left(\frac{s}{2}\right)\right] \tag{5.7.3}
$$

and then take a position

$$
\delta(s) = \frac{v(2s) - v\left(\frac{s}{2}\right)}{2s - \frac{s}{2}} \tag{5.7.4}
$$

in the stock. (See Theorem 4.2.2 of Chapter 4. The processes C_n and Δ_n in that theorem are obtained by replacing the dummy variable s by the stock price S_n in (5.7.3) and (5.7.4); i.e., $C_n = c(S_n)$ and $\Delta_n = \delta(S_n)$.) If you hedge this way, then regardless of whether the stock goes up or down on the next step, the value of your hedging portfolio should agree with the value of the perpetual American put.

(i) Compute $c(s)$ when $s = 2^j$ for the three cases $j \le 0$, $j = 1$ and $j \ge 2$.

(ii) Compute $\delta(s)$ when $s = 2^j$ for the three cases $j \le 0$, $j = 1$ and $j \ge 2$.

(iii) Verify in each of the three cases $s = 2^j$ for $j \le 0$, $j = 1$ and $j \ge 2$ that the hedge works (i.e., regardless of whether the stock goes up or down, the value of your hedging portfolio at the next time is equal to the value of the perpetual American put at that time).

Solution

(i) Throughout the following computations, we use (5.4.6):

$$v(2^j) = \begin{cases} 4 - 2^j, & \text{if } j \le 1, \\ \dfrac{4}{2^j}, & \text{if } j \ge 1. \end{cases}$$

For $j \le 0$, we have

$$\begin{aligned} c(2^j) &= v(2^j) - \frac{4}{5}\left[\frac{1}{2}v(2^{j+1}) + \frac{1}{2}v(2^{j-1})\right] \\ &= 4 - 2^j - \frac{2}{5}\left[4 - 2^{j+1} + 4 - 2^{j-1}\right] \\ &= 4 - 2^j - \frac{2}{5}\left[8 - 5 \cdot 2^{j-1}\right] \\ &= \frac{4}{5}. \end{aligned}$$

For $j = 1$, we have

$$\begin{aligned} c(2) &= v(2) - \frac{4}{5}\left[\frac{1}{2}v(4) + \frac{1}{2}v(1)\right] \\ &= 2 - \frac{2}{5}[1 + 3] \\ &= \frac{2}{5}. \end{aligned}$$

Finally, for $j \ge 2$,

$$\begin{aligned} c(2^j) &= v(2^j) - \frac{4}{5}\left[\frac{1}{2}v(2^{j+1}) + \frac{1}{2}v(2^{j-1})\right] \\ &= \frac{4}{2^j} - \frac{2}{5}\left[\frac{4}{2^{j+1}} + \frac{4}{2^{j-1}}\right] \\ &= \frac{4}{2^j} - \frac{2}{5}\left[\frac{4}{2^{j+1}} + \frac{16}{2^{j+1}}\right] \\ &= 0. \end{aligned}$$

(ii) For $j \le 0$, we have

$$\delta(2^j) = \frac{v(2^{j+1}) - v(2^{j-1})}{2^{j+1} - 2^{j-1}} = \frac{\left(4 - 2^{j+1}\right) - \left(4 - 2^{j-1}\right)}{2^{j+1} - 2^{j-1}} = -1.$$

For $j = 1$, we have

$$\delta(2) = \frac{v(4) - v(1)}{4 - 1} = \frac{1 - 3}{4 - 1} = -\frac{2}{3}.$$

Finally, for $j \geq 2$,

$$\begin{aligned}\delta(2^j) &= \frac{v(2^{j+1}) - v(2^{j-1})}{2^{j+1} - 2^{j-1}} \\ &= \frac{\frac{4}{2^{j+1}} - \frac{4}{2^{j-1}}}{2^{j+1} - 2^{j-1}} \\ &= \frac{4 - 16}{2^{j+1}} \cdot \frac{1}{(4-1)2^{j-1}} \\ &= -\frac{4}{2^{2j}}.\end{aligned}$$

(iii) If the stock price is 2^j, where $j \leq 0$, then we have a portfolio valued at $v(2^j) = 4 - 2^j$. We consume $c(2^j) = \frac{4}{5}$ and short one share of stock ($\delta(2^j) = -1$). This creates a cash position of

$$4 - 2^j - \frac{4}{5} + 2^j = \frac{16}{5},$$

which is invested in the money market. When we go to the next period, the money market investment grows to $\frac{5}{4} \cdot \frac{16}{5} = 4$. If the stock goes up to 2^{j+1}, our short stock position has value -2^{j+1}, so our portfolio is valued at $4 - 2^{j+1}$, which is the option value in this case. If the stock goes down to 2^{j-1}, our portfolio is valued at $4 - 2^{j-1}$, which is the option value also in this case.

When the stock price is 2^j with $j \leq 0$, the owner of the option should exercise, and we are prepared for that by being short one share of stock. When she exercises, she will deliver the share of stock, which will cover our short position. But when she exercises, we must pay her 4, so we are maintaining a cash position of 4. At the beginning of any period in which she fails to exercise, we get to consume the present value of the interest that will accrue to this cash position over the next period.

If the stock price is 2, then we have a portfolio valued at $v(2) = 2$. We consume $c(2) = \frac{2}{5}$ and short $\frac{2}{3}$ of a share of stock ($\delta(2) = -\frac{2}{3}$). This creates a cash position of

$$2 - \frac{2}{5} + \frac{2}{3} \cdot 2 = \frac{44}{15},$$

which is invested in the money market. When we go to the next period, the money market investment grows to $\frac{5}{4} \cdot \frac{44}{15} = \frac{11}{3}$. If the stock goes up to 4, the short stock position has value $-\frac{2}{3} \cdot 4 = -\frac{8}{3}$, so the portfolio is valued at $\frac{11}{3} - \frac{8}{3} = 1$, which is the option value in this case. If the stock goes down to 1, the short stock position has value $-\frac{2}{3} \cdot 1 = -\frac{2}{3}$, so the portfolio is valued at $\frac{11}{3} - \frac{2}{3} = 3$, which is the option value also in this case.

Finally, if the stock price is 2^j, where $j \geq 2$, then we have a portfolio valued at $\frac{4}{2^j}$. We consume $c(2^j) = 0$ and short $\frac{4}{2^{2j}}$ shares of stock ($\delta(2^j) = -\frac{4}{2^{2j}}$). This creates a cash position of

$$\frac{4}{2^j} + \frac{4}{2^{2j}} \cdot 2^j = \frac{8}{2^j},$$

which is invested in the money market. When we go to the next period, the money market investment grows to

$$\frac{5}{4} \cdot \frac{8}{2^j} = \frac{10}{2^j}.$$

If the stock goes up to 2^{j+1}, the short stock position has value

$$-\frac{4}{2^{2j}} \cdot 2^{j+1} = -\frac{8}{2^j},$$

so the portfolio is valued at

$$\frac{10}{2^j} - \frac{8}{2^j} = \frac{2}{2^j} = \frac{4}{2^{j+1}},$$

which is the option value in this case. If the stock goes down to 2^{j-1}, the short stock position has value

$$-\frac{4}{2^{2j}} \cdot 2^{j-1} = -\frac{2}{2^j},$$

so the portfolio is valued at

$$\frac{10}{2^j} - \frac{2}{2^j} = \frac{8}{2^j} = \frac{4}{2^{j-1}},$$

which is the option value also in this case.

Exercise 5.9. (Provided by Irene Villegas.) Here is a method for solving equation (5.4.13) for the value of the perpetual American put in Section 5.4.

(i) We first determine $v(s)$ for large values of s. When s is large, it is not optimal to exercise the put, so the maximum in (5.4.13) will be given by the second term,

$$\frac{4}{5}\left[\frac{1}{2}v(2s) + \frac{1}{2}v\left(\frac{s}{2}\right)\right] = \frac{2}{5}v(2s) + \frac{2}{5}v\left(\frac{s}{2}\right).$$

We thus seek solutions to the equation

$$v(s) = \frac{2}{5}v(2s) + \frac{2}{5}v\left(\frac{s}{2}\right). \tag{5.7.5}$$

All such solutions are of the form s^p for some constant p, or linear combinations of functions of this form. Substitute s^p into (5.7.5), obtain a quadratic equation for 2^p, and solve to obtain $2^p = 2$ or $2^p = \frac{1}{2}$. This leads to the values $p = 1$ and $p = -1$, i.e., $v_1(s) = s$ and $v_2(s) = \frac{1}{s}$ are solutions to (5.7.5).

(ii) The general solution to (5.7.5) is a linear combination of $v_1(s)$ and $v_2(s)$, i.e.,

$$v(s) = As + \frac{B}{s}. \tag{5.7.6}$$

For large values of s, the value of the perpetual American put must be given by (5.7.6). It remains to evaluate A and B. Using the second boundary condition in (5.4.15), show that A must be zero.

(iii) We have thus established that for large values of s, $v(s) = \frac{B}{s}$ for some constant B still to be determined. For small values of s, the value of the put is its intrinsic value $4 - s$. We must choose B so these two functions coincide at some point, i.e., we must find a value for B so that, for some $s > 0$,

$$f_B(s) = \frac{B}{s} - (4 - s)$$

equals zero. Show that, when $B > 4$, this function does not take the value 0 for any $s > 0$, but, when $B \leq 4$, the equation $f_B(s) = 0$ has a solution.

(iv) Let B be less than or equal to 4 and let s_B be a solution of the equation $f_B(s) = 0$. Suppose s_B is a stock price which can be attained in the model (i.e., $s_B = 2^j$ for some integer j). Suppose further that the owner of the perpetual American put exercises the first time the stock price is s_B or smaller. Then the discounted risk-neutral expected payoff of the put is $v_B(S_0)$, where $v_B(s)$ is given by the formula

$$v_B(s) = \begin{cases} 4 - s, & \text{if } s \leq s_B, \\ \frac{B}{s}, & \text{if } s \geq s_B. \end{cases} \tag{5.7.7}$$

Which values of B and s_B give the owner the largest option value?

(v) For $s < s_B$, the derivative of $v_B(s)$ is $v'_B(s) = -1$. For $s > s_B$, this derivative is $v'_B(s) = -\frac{B}{s^2}$. Show that the best value of B for the option owner makes the derivative of $v_B(s)$ continuous at $s = s_B$ (i.e., the two formulas for $v'_B(s)$ give the same answer at $s = s_B$).

Solution

(i) With $v(s) = s^p$, (5.7.5) becomes

$$s^p = \frac{2}{5} \cdot 2^p s^p + \frac{2}{5} \cdot \frac{1}{2^p} s^p,$$

Mulitplication by $\frac{2^p}{s^p}$ leads to

$$2^p = \frac{2}{5} (2^p)^2 + \frac{2}{5},$$

and we may rewrite this as

$$(2^p)^2 - \frac{5}{2} \cdot 2^p + 1 = 0,$$

a quadratic equation in 2^p. The solution to this equation is

$$2^p = \frac{1}{2} \left(\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4} \right) = \frac{1}{2} \left(\frac{5}{2} \pm \frac{3}{2} \right).$$

We thus have either $2^p = 2$ or $2^p = \frac{1}{2}$, and hence either $p = 1$ or $p = -1$.

(ii) With $v(s) = As + \frac{B}{s}$, we have $\lim_{s\to\infty} v(s) = \pm\infty$ unless $A = 0$. The boundary condition $\lim_{s\to\infty} v(s) = 0$ implies therefore that A is zero.

(iii) Since $v(s)$ is positive, we must have $B > 0$. We note that $\lim_{s\downarrow 0} f_B(s) = \lim_{s\to\infty} f_B(s) = \infty$. Therefore, $f_B(s)$ takes the value zero for some $s \in (0,\infty)$ if and only if its minimium over $(0,\infty)$ is less than or equal to zero. To find the minimizing value of s, we set the derivative of $f_B(s)$ equal to zero:

$$-\frac{B}{s^2} + 1 = 0.$$

This results in the critical point $s_c = \sqrt{B}$. We note that the second derivative, $\frac{2B}{s^3}$, is positive on $(0,\infty)$, so the function is convex, and hence f_B attains a minimum at s_c. The minimal value of f_B on $(0,\infty)$ is

$$f_B(s_c) = 2\sqrt{B} - 4.$$

This is positive if $B > 4$, in which case $f_B(s) = 0$ has no solution in $(0,\infty)$. If $B = 4$, then $f_B(s_c) = 0$ and s_c is the only solution to the equation $f_B(s) = 0$ in $(0,\infty)$. If $0 < B < 4$, then $f_B(s_c) < 0$ and the equation $f_B(s) = 0$ has two solutions in $(0,\infty)$.

(iv) Since $v_B(s) = \frac{B}{s}$ for all large values of s, we maximize this by choosing B as large as possible, i.e., $B = 4$. For values of $B < 4$, the curve $\frac{B}{s}$ lies below the curve $\frac{4}{s}$ (see Figure 5.8.1), and values of $B > 4$ are not possible because of part (iii).

(iv) We see from the tangency of the curve $y = \frac{4}{s}$ with the intrinsic value $y = 4 - s$ at the point (2,2) in Figure 5.8.1 that $y = \frac{4}{s}$ and $y = 4 - s$ have the same derivative at $s = 2$. Indeed,

$$\frac{d}{ds}\frac{4}{s}\bigg|_{s=2} = -\frac{4}{s^2}\bigg|_{s=2} = -1,$$

and as noted in the statement of the exercise,

$$\frac{d}{ds}(4 - s) = -1.$$

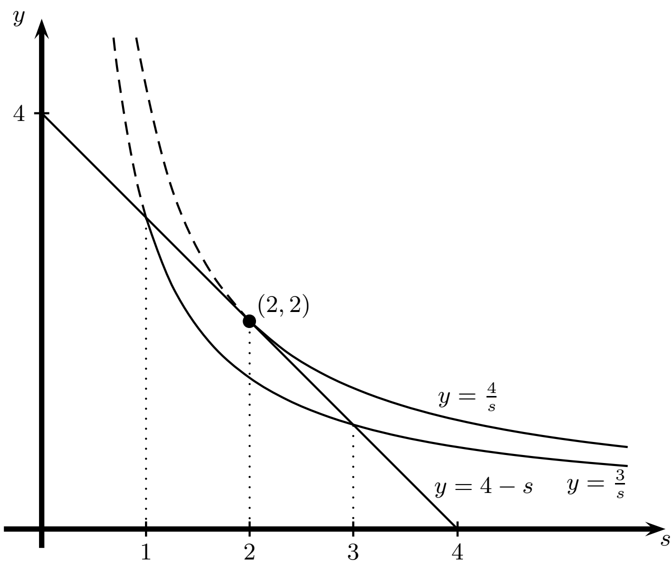

Fig. 5.8.1. The curve $y = \frac{B}{s}$ for $B = 3$ and $B = 4$.

6

Interest-Rate-Dependent Assets

6.9 Solutions to Selected Exercises

Exercise 6.1. Prove parts (i), (ii), (iii) and (v) of Theorem 2.3.2 when conditional expectation is defined by Definition 6.2.2. (Part (iv) is not true in the form stated in Theorem 2.3.2 when the coin tosses are not independent.)

Solution We take each of parts (i), (ii), (iii) and (v) of Theorem 2.3.2 in turn.

(i) **Linearity of conditional expectations.** According to Definition 6.2.2,

$$
\begin{aligned}
&\widetilde{\mathbb{E}}_n[c_1X + c_2Y](\overline{\omega}_1 \ldots \overline{\omega}_n)\\
&= \sum_{\overline{\omega}_{n+1},\ldots,\overline{\omega}_N} \big(c_1X(\overline{\omega}_1 \ldots \overline{\omega}_n\overline{\omega}_{n+1} \ldots \overline{\omega}_N) + c_2Y(\overline{\omega}_1 \ldots \overline{\omega}_n\overline{\omega}_{n+1} \ldots \overline{\omega}_N)\big)\\
&\qquad\qquad \times\widetilde{\mathbb{P}}\{\omega_{n+1} = \overline{\omega}_{n+1}, \ldots, \omega_N = \overline{\omega}_N | \omega_1 = \overline{\omega}_1, \ldots, \omega_n = \overline{\omega}_n\}\\
&= c_1 \sum_{\overline{\omega}_{n+1},\ldots,\overline{\omega}_N} X(\overline{\omega}_1 \ldots \overline{\omega}_n\overline{\omega}_{n+1} \ldots \overline{\omega}_N)\\
&\qquad\qquad \times\widetilde{\mathbb{P}}\{\omega_{n+1} = \overline{\omega}_{n+1}, \ldots, \omega_N = \overline{\omega}_N | \omega_1 = \overline{\omega}_1, \ldots, \omega_n = \overline{\omega}_n\}\\
&\quad +c_2 \sum_{\overline{\omega}_{n+1},\ldots,\overline{\omega}_N} Y(\overline{\omega}_1 \ldots \overline{\omega}_n\overline{\omega}_{n+1} \ldots \overline{\omega}_N)\\
&\qquad\qquad \times\widetilde{\mathbb{P}}\{\omega_{n+1} = \overline{\omega}_{n+1}, \ldots, \omega_N = \overline{\omega}_N | \omega_1 = \overline{\omega}_1, \ldots, \omega_n = \overline{\omega}_n\}\\
&= c_1\widetilde{\mathbb{E}}_n[X](\overline{\omega}_1 \ldots \overline{\omega}_n) + c_2\widetilde{\mathbb{E}}_n[Y](\overline{\omega}_1 \ldots \overline{\omega}_n).
\end{aligned}
$$

(ii) **Taking out what is known.** If X depends only on the first n coin tosses, then

$$\begin{aligned}
&\widetilde{\mathbb{E}}_n[XY](\overline{\omega}_1 \dots \overline{\omega}_n) \\
&= \sum_{\overline{\omega}_{n+1},\dots,\overline{\omega}_N} X(\overline{\omega}_1 \dots \overline{\omega}_n) Y(\overline{\omega}_1 \dots \overline{\omega}_n \overline{\omega}_{n+1} \dots \overline{\omega}_N) \\
&\qquad\qquad \times \widetilde{\mathbb{P}}\{\omega_{n+1} = \overline{\omega}_{n+1}, \dots, \omega_N = \overline{\omega}_N | \omega_1 = \overline{\omega}_1, \dots, \omega_n = \overline{\omega}_n\} \\
&= X(\overline{\omega}_1 \dots \overline{\omega}_n) \sum_{\overline{\omega}_{n+1},\dots,\overline{\omega}_N} Y(\overline{\omega}_1 \dots \overline{\omega}_n \overline{\omega}_{n+1} \dots \overline{\omega}_N) \\
&\qquad\qquad \times \widetilde{\mathbb{P}}\{\omega_{n+1} = \overline{\omega}_{n+1}, \dots, \omega_N = \overline{\omega}_N | \omega_1 = \overline{\omega}_1, \dots, \omega_n = \overline{\omega}_n\} \\
&= X(\overline{\omega}_1 \dots \overline{\omega}_n) \widetilde{\mathbb{E}}_n[Y](\overline{\omega}_1 \dots \overline{\omega}_n)
\end{aligned}$$

(iii) **Iterated conditioning.** If $0 \le n \le m \le N$, then because $\widetilde{\mathbb{E}}_m[X]$ depends only on the first m coin tosses and

$$\begin{aligned}
&\sum_{\overline{\omega}_{m+1},\dots,\overline{\omega}_N} \widetilde{\mathbb{P}}\{\omega_{n+1} = \overline{\omega}_{n+1}, \dots, \omega_m = \overline{\omega}_m, \omega_{m+1} = \overline{\omega}_{m+1}, \dots, \omega_N = \overline{\omega}_N \\
&\qquad\qquad |\omega_1 = \overline{\omega}_1, \dots, \omega_n = \overline{\omega}_n\} \\
&\qquad\qquad\quad = \widetilde{\mathbb{P}}\{\omega_{n+1} = \overline{\omega}_{n+1}, \dots, \omega_m = \overline{\omega}_m | \omega_1 = \overline{\omega}_1, \dots, \omega_n = \overline{\omega}_n\},
\end{aligned}$$

we have

$$\begin{aligned}
&\widetilde{\mathbb{E}}_n\big[\widetilde{\mathbb{E}}_m[X]\big](\overline{\omega}_1 \dots \overline{\omega}_n) \\
&= \sum_{\overline{\omega}_{n+1},\dots,\overline{\omega}_N} \widetilde{\mathbb{E}}_m[X](\overline{\omega}_1 \dots \overline{\omega}_m) \\
&\qquad \times \widetilde{\mathbb{P}}\{\omega_{n+1} = \overline{\omega}_{n+1}, \dots, \omega_m = \overline{\omega}_m, \omega_{m+1} = \overline{\omega}_{m+1}, \dots, \omega_N = \overline{\omega}_N \\
&\qquad\qquad |\omega_1 = \overline{\omega}_1, \dots, \omega_n = \overline{\omega}_n\} \\
&= \sum_{\overline{\omega}_{n+1},\dots,\overline{\omega}_m} \widetilde{\mathbb{E}}_m[X](\overline{\omega}_1 \dots \overline{\omega}_m) \sum_{\overline{\omega}_{m+1},\dots,\overline{\omega}_N} \\
&\qquad \times \widetilde{\mathbb{P}}\{\omega_{n+1} = \overline{\omega}_{n+1}, \dots, \omega_m = \overline{\omega}_m, \omega_{m+1} = \overline{\omega}_{m+1}, \dots, \omega_N = \overline{\omega}_N \\
&\qquad\qquad |\omega_1 = \overline{\omega}_1, \dots, \omega_n = \overline{\omega}_n\} \\
&= \sum_{\overline{\omega}_{n+1},\dots\overline{\omega}_m} \widetilde{\mathbb{E}}_m[X](\overline{\omega}_1 \dots \overline{\omega}_m) \\
&\qquad \times \widetilde{\mathbb{P}}\{\omega_{n+1} = \overline{\omega}_{n+1}, \dots, \omega_m = \overline{\omega}_m | \omega_1 = \overline{\omega}_1, \dots, \omega_n = \overline{\omega}_n\} \\
&= \sum_{\overline{\omega}_{n+1},\dots,\overline{\omega}_m} \sum_{\overline{\omega}_{m+1},\dots,\overline{\omega}_N} X(\overline{\omega}_1 \dots \overline{\omega}_m \overline{\omega}_{m+1} \dots \overline{\omega}_N) \\
&\qquad \times \widetilde{\mathbb{P}}\{\omega_{m+1} = \overline{\omega}_{m+1}, \dots, \omega_N = \overline{\omega}_N | \omega_1 = \overline{\omega}_1, \dots, \omega_m = \overline{\omega}_m\} \\
&\qquad \times \widetilde{\mathbb{P}}\{\omega_{n+1} = \overline{\omega}_{n+1}, \dots, \omega_m = \overline{\omega}_m | \omega_1 = \overline{\omega}_1, \dots, \omega_n = \overline{\omega}_n\},
\end{aligned}$$

where we have used the definition

$$
\begin{aligned}
&\widetilde{\mathbb{E}}_m[X](\overline{\omega}_1 \dots \overline{\omega}_m) \\
&= \sum_{\overline{\omega}_{m+1},\dots,\overline{\omega}_N} X(\overline{\omega}_1 \dots \overline{\omega}_m \overline{\omega}_{m+1} \dots \overline{\omega}_N) \\
&\qquad\qquad \times \widetilde{\mathbb{P}}\{\omega_{m+1} = \overline{\omega}_{m+1}, \dots, \omega_N = \overline{\omega}_N | \omega_1 = \overline{\omega}_1, \dots, \omega_m = \overline{\omega}_m\}
\end{aligned}
$$

in the last step. Using the fact that

$$
\begin{aligned}
&\widetilde{\mathbb{P}}\{\omega_{m+1} = \overline{\omega}_{m+1}, \dots, \omega_N = \overline{\omega}_N | \omega_1 = \overline{\omega}_1, \dots, \omega_m = \overline{\omega}_m\} \\
&\times \widetilde{\mathbb{P}}\{\omega_{n+1} = \overline{\omega}_{n+1}, \dots, \omega_m = \overline{\omega}_m | \omega_1 = \overline{\omega}_1, \dots, \omega_n = \overline{\omega}_n\} \\
&\quad = \widetilde{\mathbb{P}}\{\omega_{n+1} = \overline{\omega}_{n+1}, \dots, \omega_m = \overline{\omega}_m, \omega_{m+1} = \overline{\omega}_{m+1}, \dots, \omega_N = \overline{\omega}_N \\
&\qquad\qquad | \omega_1 = \overline{\omega}_1, \dots, \omega_n = \overline{\omega}_n\} \\
&\quad = \widetilde{\mathbb{P}}\{\omega_{n+1} = \overline{\omega}_{n+1}, \dots, \omega_N = \overline{\omega}_N | \omega_1 = \overline{\omega}_1, \dots, \omega_n = \overline{\omega}_n\},
\end{aligned}
$$

we may write the last term in the above formula for $\widetilde{\mathbb{E}}_n\big[\widetilde{\mathbb{E}}_m[X]\big](\overline{\omega}_1 \dots \overline{\omega}_n)$ as

$$
\begin{aligned}
&\widetilde{\mathbb{E}}_n\big[\widetilde{\mathbb{E}}_m[X]\big](\overline{\omega}_1 \dots \overline{\omega}_n) \\
&= \sum_{\overline{\omega}_{n+1},\dots,\overline{\omega}_m} \sum_{\overline{\omega}_{m+1},\dots,\overline{\omega}_N} X(\overline{\omega}_1 \dots \overline{\omega}_m \overline{\omega}_{m+1} \dots \overline{\omega}_N) \\
&\qquad\qquad \times \widetilde{\mathbb{P}}\{\omega_{n+1} = \overline{\omega}_{n+1}, \dots, \omega_N = \overline{\omega}_N | \omega_1 = \overline{\omega}_1, \dots, \omega_n = \overline{\omega}_n\} \\
&= \sum_{\overline{\omega}_{n+1},\dots,\overline{\omega}_N} X(\overline{\omega}_1 \dots \overline{\omega}_N) \\
&\qquad\qquad \times \widetilde{\mathbb{P}}\{\omega_{n+1} = \overline{\omega}_{n+1}, \dots, \omega_N = \overline{\omega}_N | \omega_1 = \overline{\omega}_1, \dots, \omega_n = \overline{\omega}_n\} \\
&= \widetilde{\mathbb{E}}_n[X](\overline{\omega}_1 \dots \overline{\omega}_n).
\end{aligned}
$$

(iv) **Conditional Jensen's inequality.** This follows from part (i) just like the proof of part (v) in Appendix A.

Exercise 6.2. Verify that the discounted value of the static hedging portfolio constructed in the proof of Theorem 6.3.2 is a martingale under $\widetilde{\mathbb{P}}$.

Solution The static hedging portfolio in Theorem 6.3.2 is, at time n, to short $\frac{S_n}{B_{n,m}}$ zero coupon bonds maturing at time m and to hold one share of the asset with price S_n. The value of this portfolio at time k, where $n \le k \le m$, is

$$X_k = S_k - \frac{S_n}{B_{n,m}} B_{k,m}, \quad k = n, n+1, \dots, m.$$

For $n \le k \le m-1$, we have

$$\widetilde{\mathbb{E}}_k\left[D_{k+1} X_{k+1}\right] = \widetilde{\mathbb{E}}_k[D_{k+1} S_{k+1}] - \frac{S_n}{B_{n,m}} \widetilde{\mathbb{E}}_k\left[D_{k+1} B_{k+1,m}\right].$$

Using the fact that the discounted asset price is a martingale under the risk-neutral measure and also using (6.2.5) first in the form $D_{k+1}B_{k+1,m} = \widetilde{\mathbb{E}}_{k+1}[D_m]$ and then in the form $D_kB_{k,m} = \widetilde{\mathbb{E}}_k[D_m]$, we may rewrite this as

$$\begin{aligned}
\widetilde{\mathbb{E}}_k[D_{k+1}X_{k+1}] &= D_kS_k - \frac{S_n}{B_{n,m}}\widetilde{\mathbb{E}}_k\left[\widetilde{\mathbb{E}}_{k+1}[D_m]\right]\\
&= D_kS_k - \frac{S_n}{B_{n,m}}\widetilde{\mathbb{E}}_k[D_m]\\
&= D_kS_k - \frac{S_n}{B_{n,m}}D_kB_{k,m}\\
&= D_kX_k.
\end{aligned}$$

This is the martingale property.

Exercise 6.4. Using the data in Example 6.3.9, this exercise constructs a hedge for a short position in the caplet paying $(R_2 - \frac{1}{3})^+$ at time three. We observe from the second table in Example 6.3.9 that the payoff at time three of this caplet is

$$V_3(HH) = \frac{2}{3}, \quad V_3(HT) = V_3(TH) = V_3(TT) = 0.$$

Since this payoff depends on only the first two coin tosses, the price of the caplet at time two can be determined by discounting:

$$V_2(HH) = \frac{1}{1+R_2(HH)}V_3(HH) = \frac{1}{3}, \quad V_2(HT) = V_2(TH) = V_2(TT) = 0.$$

Indeed, if one is hedging a short position in the caplet and has a portfolio valued at $\frac{1}{3}$ at time two in the event $\omega_1 = H, \omega_2 = H$, then one can simply invest this $\frac{1}{3}$ in the money market in order to have the $\frac{2}{3}$ required to pay off the caplet at time three.

In Example 6.3.9, the time-zero price of the caplet is determined to be $\frac{2}{21}$ (see (6.3.10)).

(i) Determine $V_1(H)$ and $V_1(T)$, the price at time one of the caplet in the events $\omega_1 = H$ and $\omega_1 = T$, respectively.

(ii) Show how to begin with $\frac{2}{21}$ at time zero and invest in the money market and the maturity two bond in order to have a portfolio value X_1 at time one that agrees with V_1, regardless of the outcome of the first coin toss. Why do we invest in the maturity two bond rather than the maturity three bond to do this?

(iii) Show how to take the portfolio value X_1 at time one and invest in the money market and the maturity three bond in order to have a portfolio value X_2 at time two that agrees with V_2, regardless of the outcome of the first two coin tosses. Why do we invest in the maturity three bond rather than the maturity two bond to do this?

Solution.

(i) We determine V_1 by the risk-neutral pricing formula. In particular,

$$\begin{aligned}
V_1(H) &= \frac{1}{D_1(H)}\widetilde{\mathbb{E}}_1[D_2V_2](H)\\
&= \widetilde{\mathbb{P}}\{\omega_2 = H|\omega_1 = H\}D_2(HH)V_2(HH)\\
&\qquad +\widetilde{\mathbb{P}}\{\omega_2 = T|\omega_1 = H\}D_2(HT)V_2(HT)\\
&= \frac{2}{3}\cdot\frac{6}{7}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{6}{7}\cdot 0 \;=\; \frac{4}{21},\\
V_1(T) &= \frac{1}{D_1(T)}\widetilde{\mathbb{E}}_1[D_2V_2](T) \;=\; 0.
\end{aligned}$$

(ii) We compute the number of shares of the time-two maturity bond by the usual formula:

$$\Delta_0 = \frac{V_1(H)-V_1(T)}{B_{1,2}(H)-B_{1,2}(T)} = \frac{\frac{4}{21}-0}{\frac{6}{7}-\frac{5}{7}} = \frac{4}{3}.$$

It is straight-forward to verify that this works. Set $X_0 = V_0 = \frac{2}{21}$ and compute

$$\begin{aligned}
X_1(H) &= \Delta_0 B_{1,2}(H) + (1+R_0)(X_0-\Delta_0 B_{0,2})\\
&= \frac{4}{3}\cdot\frac{6}{7}+\frac{2}{21}-\frac{4}{3}\cdot\frac{11}{14} \;=\; \frac{4}{21} \;=\; V_1(H),\\
X_1(T) &= \Delta_0 B_{1,2}(T) + (1+R_0)(X_0-\Delta_0 B_{0,2})\\
&= \frac{4}{3}\cdot\frac{5}{7}+\frac{2}{21}-\frac{4}{3}\cdot\frac{11}{14} \;=\; 0 \;=\; V_1(T).
\end{aligned}$$

We do not use the maturity three bond because $B_{1,3}(H) = B_{1,3}(T)$, and this bond therefore provides no hedge against the first coin toss.

(iii) In the event of a T on the first coin toss, the caplet price is zero, the hedging portfolio has zero value, and no further hedging is required. In the event of a H on the first toss, we hedge by taking in the maturity three bond the position

$$\Delta_1(H) = \frac{V_2(HH)-V_2(HT)}{B_{2,3}(HH)-B_{2,3}(HT)} = \frac{\frac{1}{3}-0}{\frac{1}{2}-1} = -\frac{2}{3}.$$

It is straight-forward to verify that this works. We compute

$$\begin{aligned}
X_2(HH) &= \Delta_1(H)B_{2,3}(HH) + (1+R_1(H))(X_1(H)-\Delta_1(H)B_{1,3}(H))\\
&= -\frac{2}{3}\cdot\frac{1}{2}+\frac{7}{6}\left(\frac{4}{21}+\frac{2}{3}\cdot\frac{4}{7}\right) \;=\; \frac{1}{3} \;=\; V_2(HH),\\
X_2(HT) &= \Delta_1(H)B_{2,3}(HT) + (1+R_1(H))(X_1(H)-\Delta_1(H)B_{1,3}(H))\\
&= -\frac{2}{3}\cdot 1+\frac{7}{6}\left(\frac{4}{21}+\frac{2}{3}\cdot\frac{4}{7}\right) \;=\; 0 \;=\; V_2(HT).
\end{aligned}$$

We do not use the maturity two bond because $B_{2,2}(HH) = B_{2,2}(HT)$, and this bond therefore provides no hedge against the second coin toss.