TRƯỜNG ĐẠI HỌC BÁCH KHOA HÀ NỘI
VIỆN CÔNG NGHỆ THÔNG TIN VÀ TRUYỀN THÔNG

Data structures and Algorithms

Nguyễn Khánh Phương

Computer Science department
School of Information and Communication technology
E-mail: phuongnk@soict.hust.edu.vn

Course outline

Chapter 1. Fundamentals

Chapter 2. Algorithmic paradigms

Chapter 3. Basic data structures

Chapter 4. Tree

Chapter 5. Sorting

Chapter 6. Searching

Chapter 7. Graph

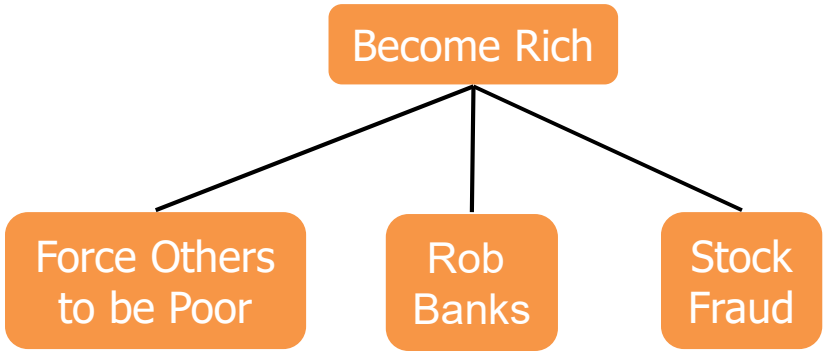
TRƯỜNG ĐẠI HỌC BÁCH KHOA HÀ NỘI
VIỆN CÔNG NGHỆ THÔNG TIN VÀ TRUYỀN THÔNG

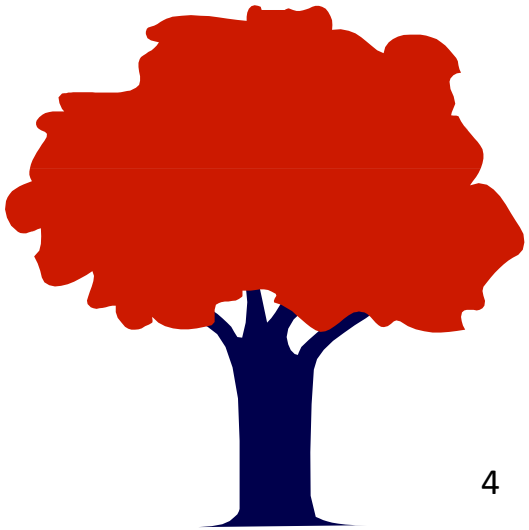
Chapter 4. Tree

Nguyễn Khánh Phương

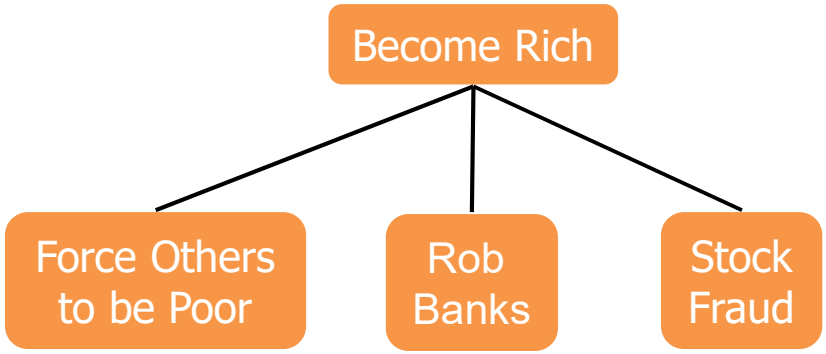
Computer Science department
School of Information and Communication technology
E-mail: phuongnk@soict.hust.edu.vn

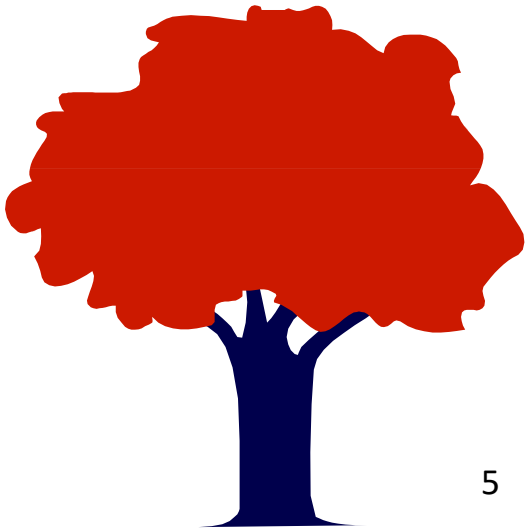
Contents





Contents





4.1. Definitions

4.1.1. Tree definition

4.1.2. Tree terminology

4.1.3. Ordered tree

4.1. Definitions

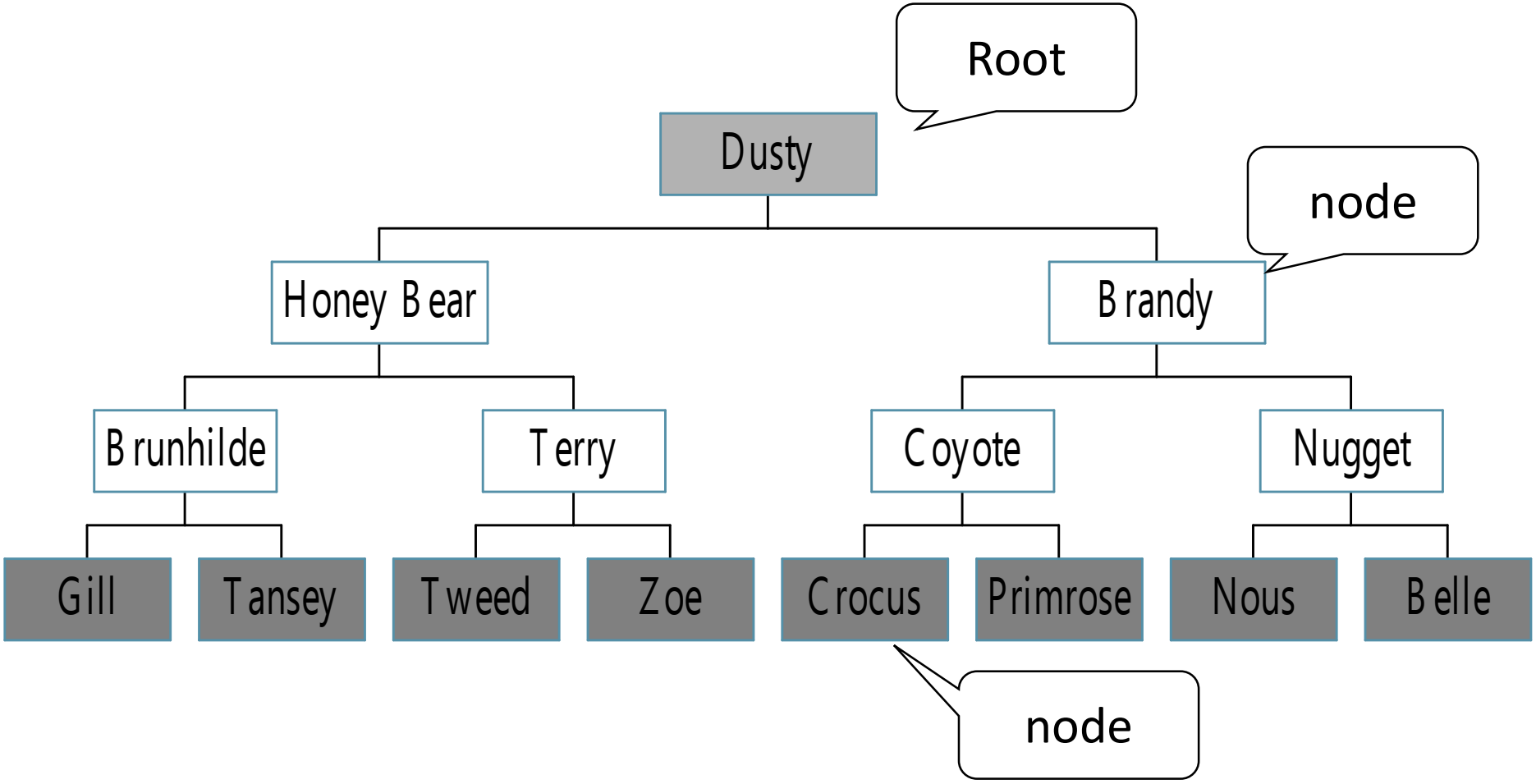
4.1.1. Tree definition

4.1.2. Tree terminology

4.1.3. Ordered tree

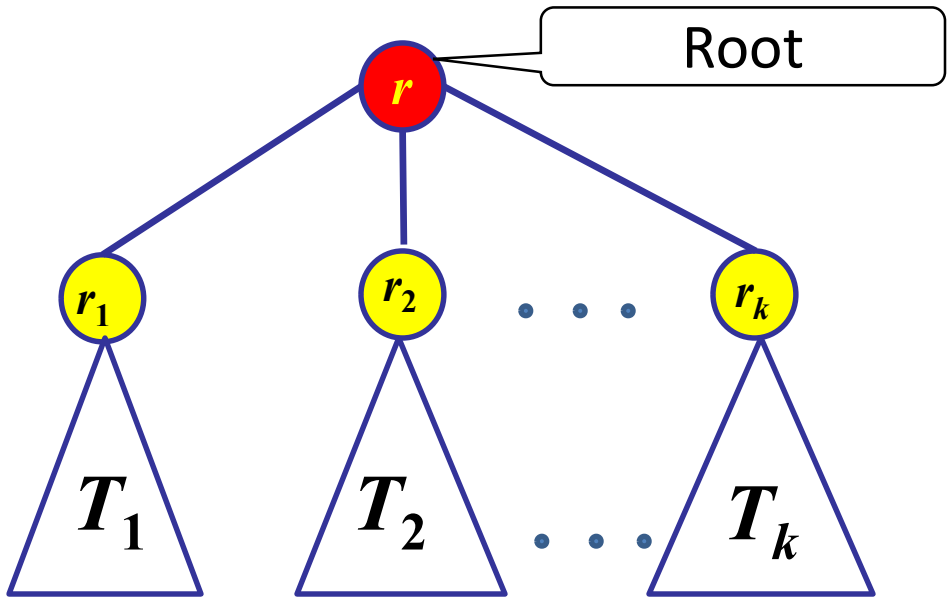
4.1.1. Tree definition

A tree is a collection of elements (nodes) in which there is a special node called the **root**.



Tree: Recursive definition

- Basic Step:
  - The null tree: tree does not have any node
  - The tree has only one node r : node r is called the root of the tree
- Recursive Step:
  - Assume there is a set of trees $T_1,T_2,...,T_k$ with the correspond root $r_1,r_2,...,r_k$
  - We can build a new tree by setting the node r as the parent of nodes $r_1,r_2,....,r_k$:
    - In this new tree: r is the root; T_1, T_2, . . . , T_k are subtrees of the root r; Nodes $r_1,r_2,....,r_k$ are called children of node r.

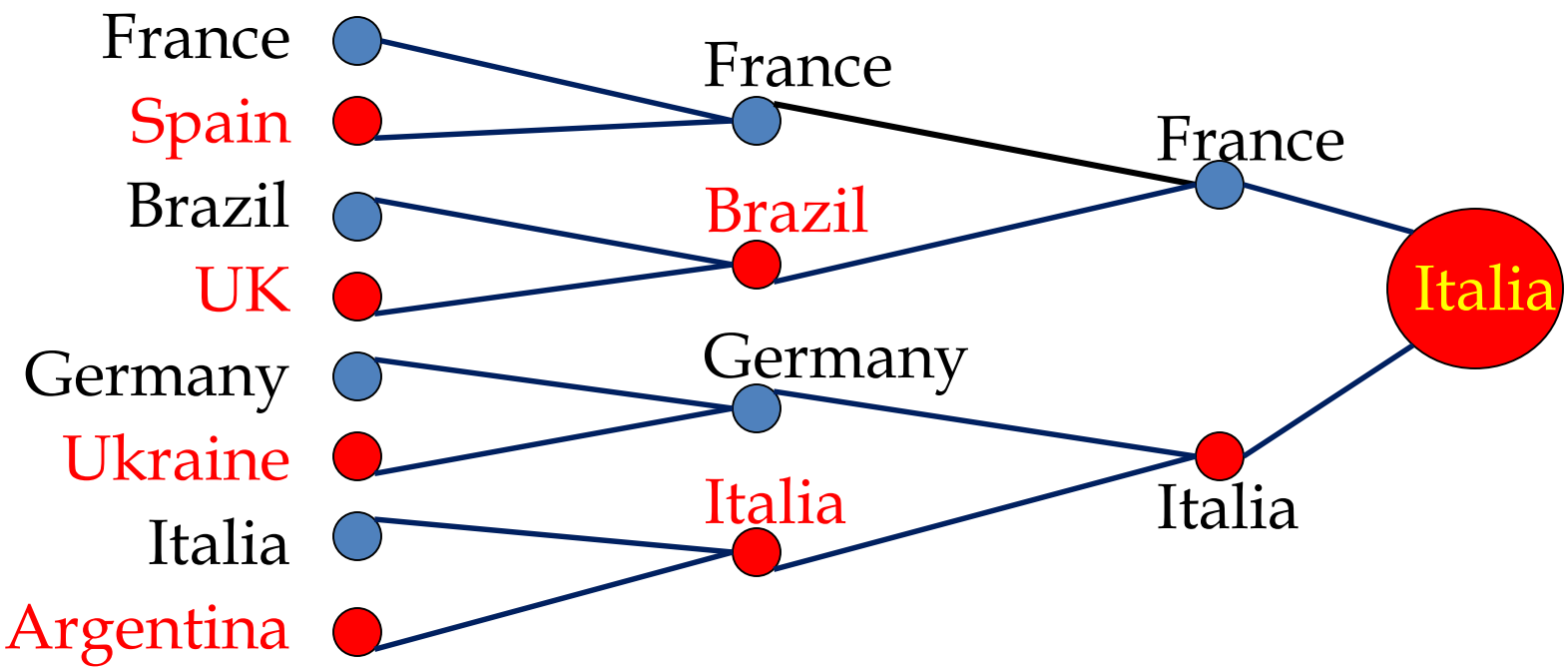


Real applications of tree

- Match Schedule
- Family tree
- Directory tree
- The structure of a book
- Expression tree
- Organization chart
- Collective partition tree
- ….

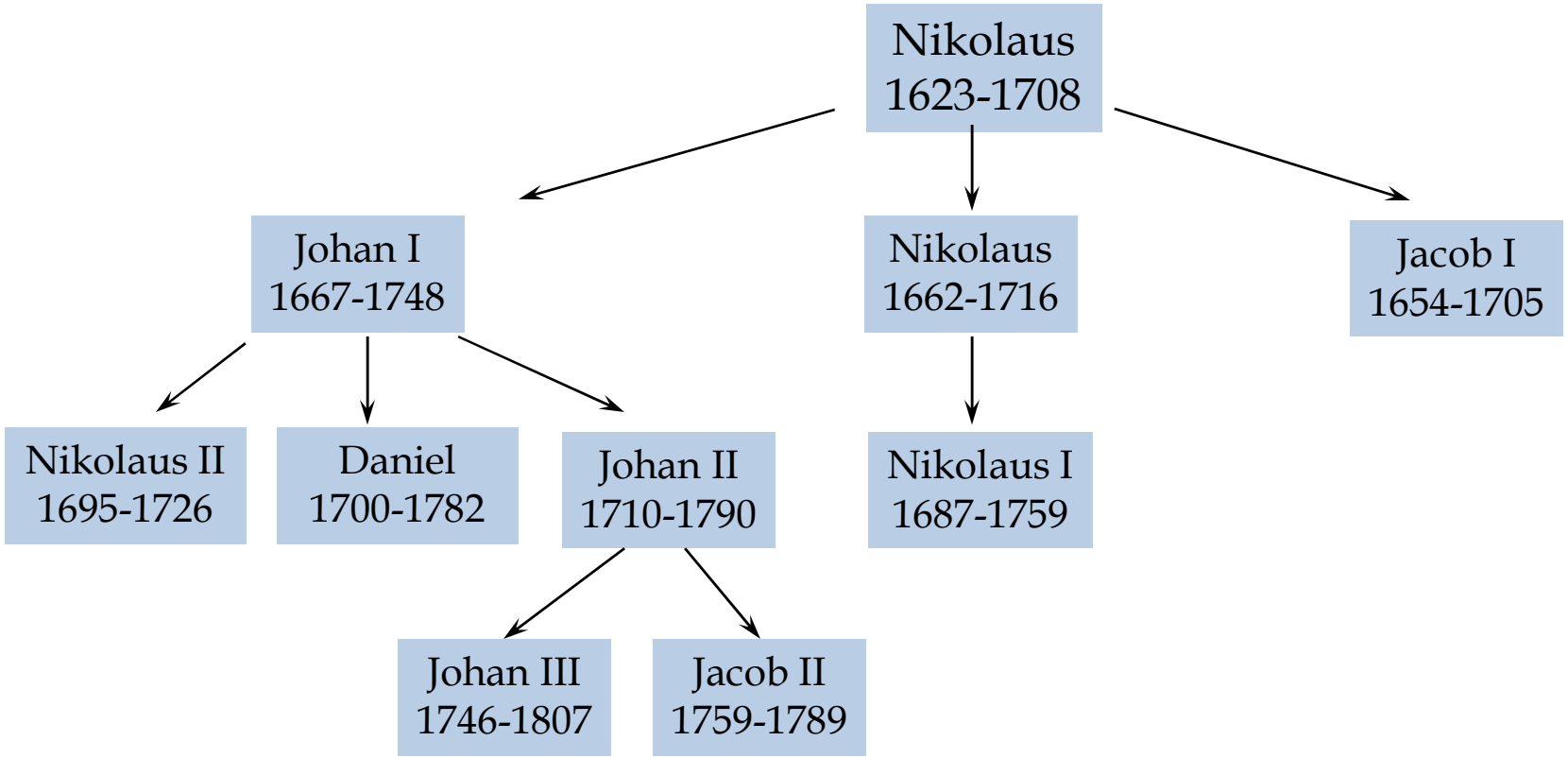
Real applications of tree: Match schedule

- In real life, the tree is used to describe the schedule of sports tournaments in the form of knockout matches



Real applications of tree: family tree

- Family tree of mathematicians of the Bernoulli family

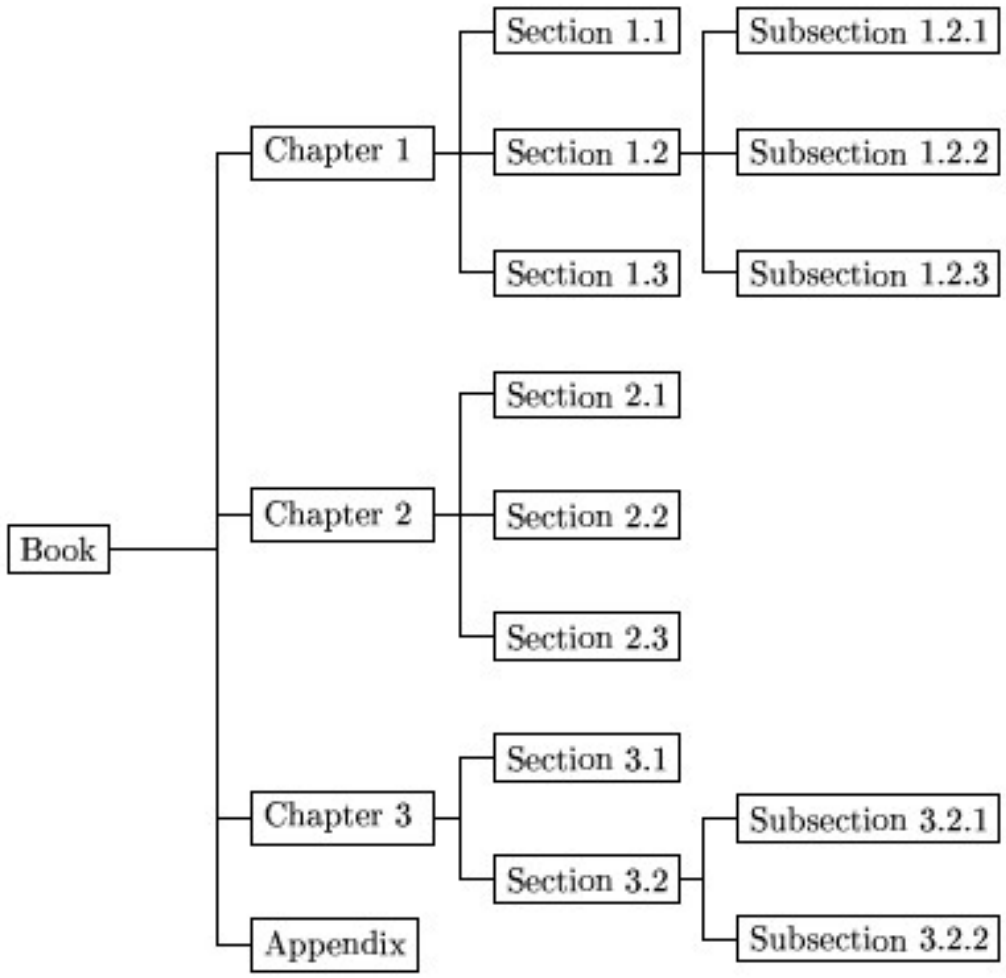


Real applications of tree: directory tree

- Directory tree

Real applications of tree: table of content

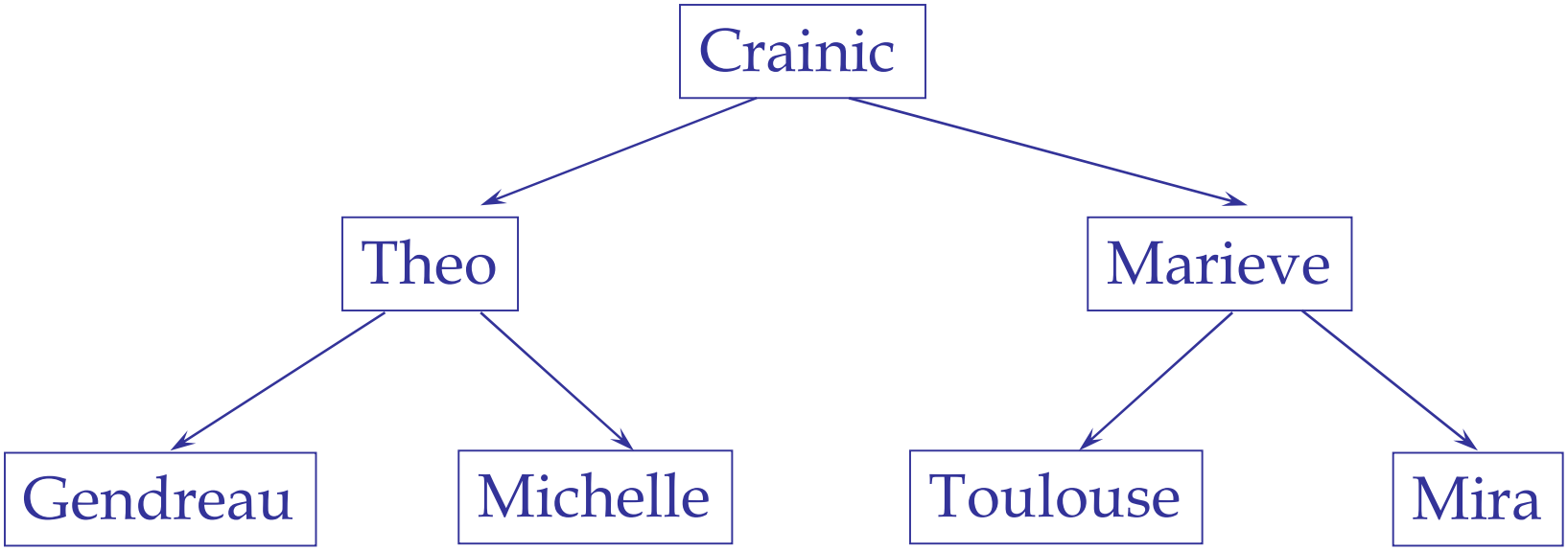
Book
- Chapter 1
  - Section 1.1
  - Section 1.2
    - Subsection 1.2.1
    - Subsection 1.2.2
    - Subsection 1.2.3
  - Section 1.3
- Chapter 2
  - Section 2.1
  - Section 2.2
  - Section 2.3
- Chapter 3
  - Section 3.1
  - Section 3.2
    - Subsection 3.2.1
    - Subsection 3.2.2
- Appendix



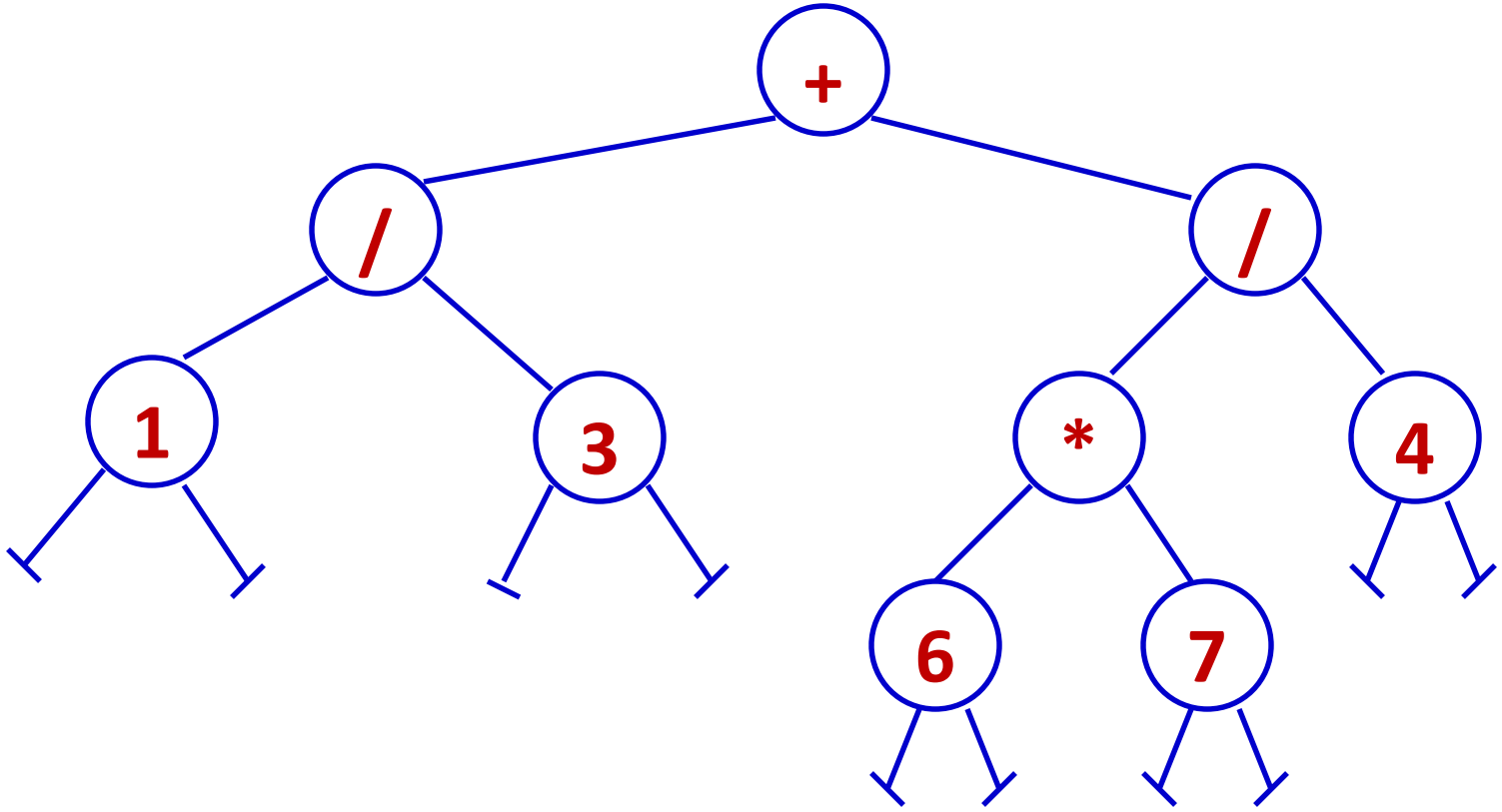
Real applications of tree: Ancestor Tree

- Everyone has parents:

e.g.: Crainic has parents: Theo and Marieve

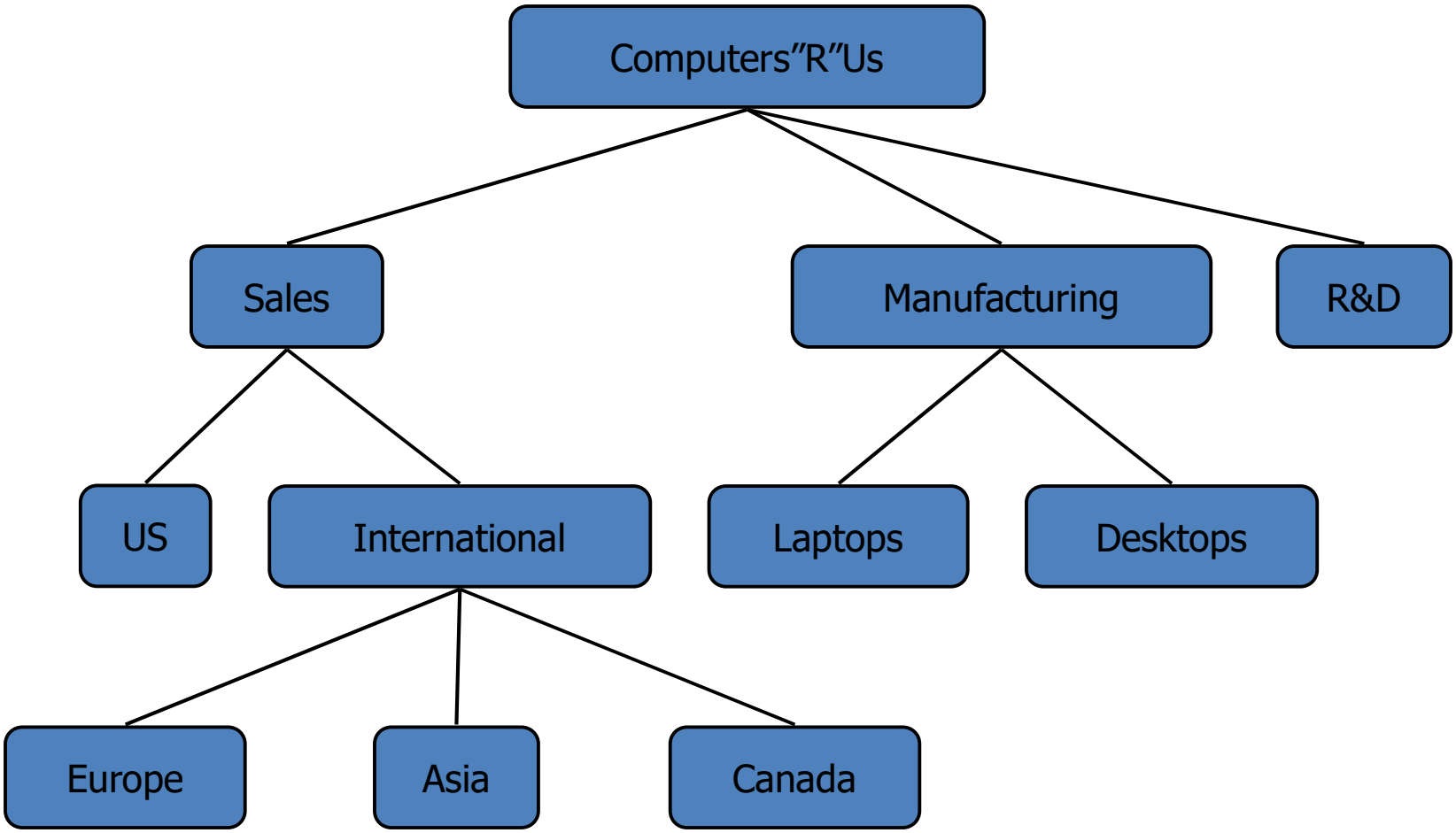


Real applications of tree: Expression Tree



1/3 + 6*7 / 4

Real applications of tree: Organization chart

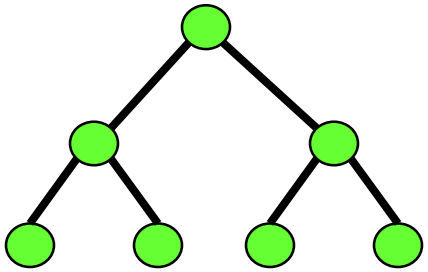


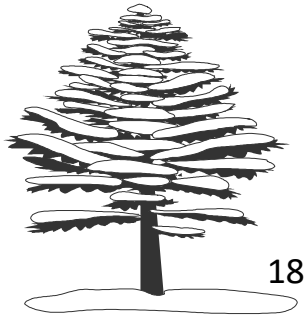
4.1. Definitions

4.1.1. Tree definition

4.1.2. Tree terminology

4.1.3. Ordered tree





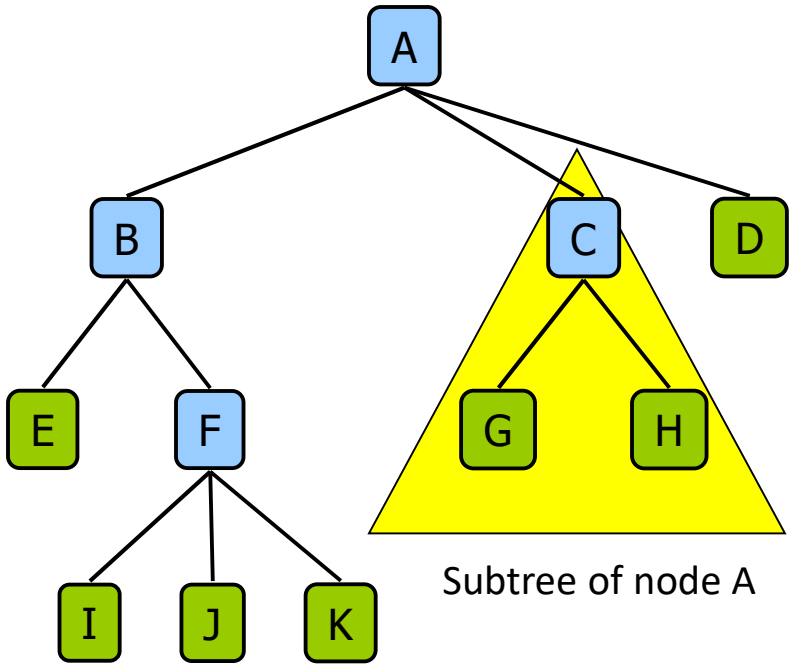
4.1.2. Tree terminology

- Node (nút)
- Root (gốc)
- Leaf (lá)
- Child (con)
- Parent (cha)
- Ancestors (tổ tiên)
- Descendants (hậu duệ)
- Sibling (anh em)
- Internal node (nút trong)
- Height (chiều cao)
- Depth (chiều sâu)

4.1.2. Tree terminology

- **Root**: node without parent (A)
- **Siblings**: nodes share the same parent (B, C, D are siblings; I, J, K are siblings; E and F are siblings; G and H are siblings)
- **Internal node**: node with at least one child (blue nodes: A, B, C, F)
- **External node** (leaf): node without children (green nodes: E, I, J, K, G, H, D)
- **Ancestors** of a node: its' parent, grandparent, grand-grandparent, etc.
- **Descendant** of a node: its' child, grandchild, grand-grandchild, etc.
- **Subtree** of a node: a tree whose root is a child of that node

Example:
Child of B: E, F
Parent of E: B
Ancestor of F: B, A
Descendant of B: E, F, I, J, K

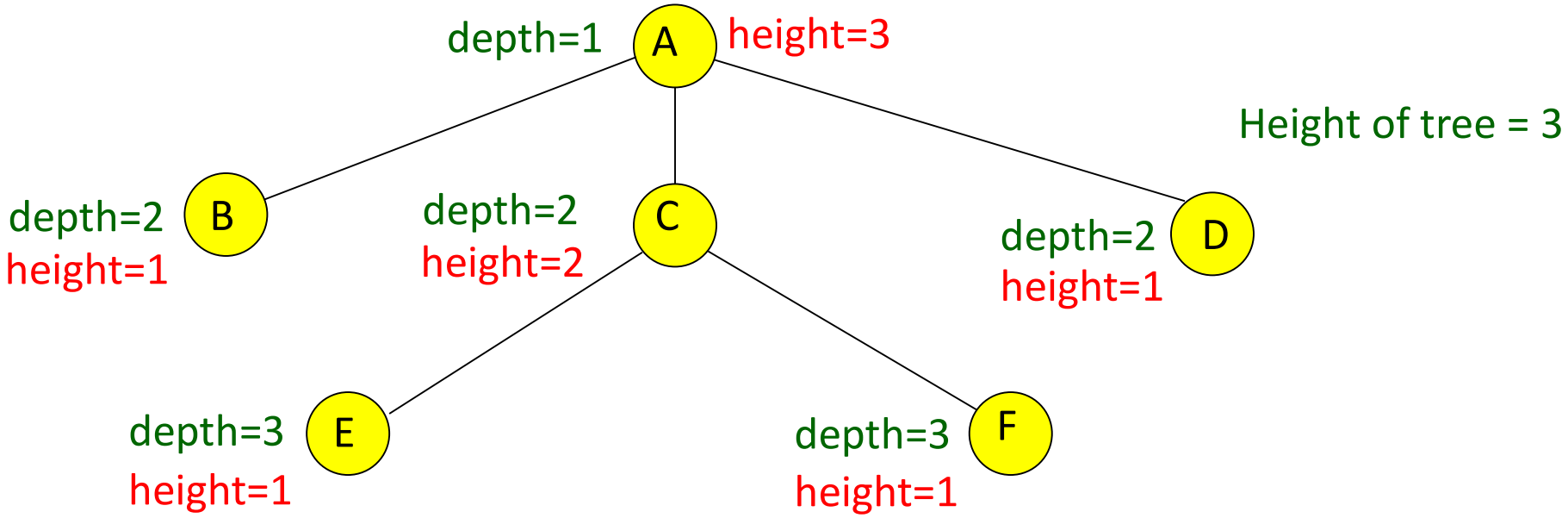


4.1.2. Tree terminology

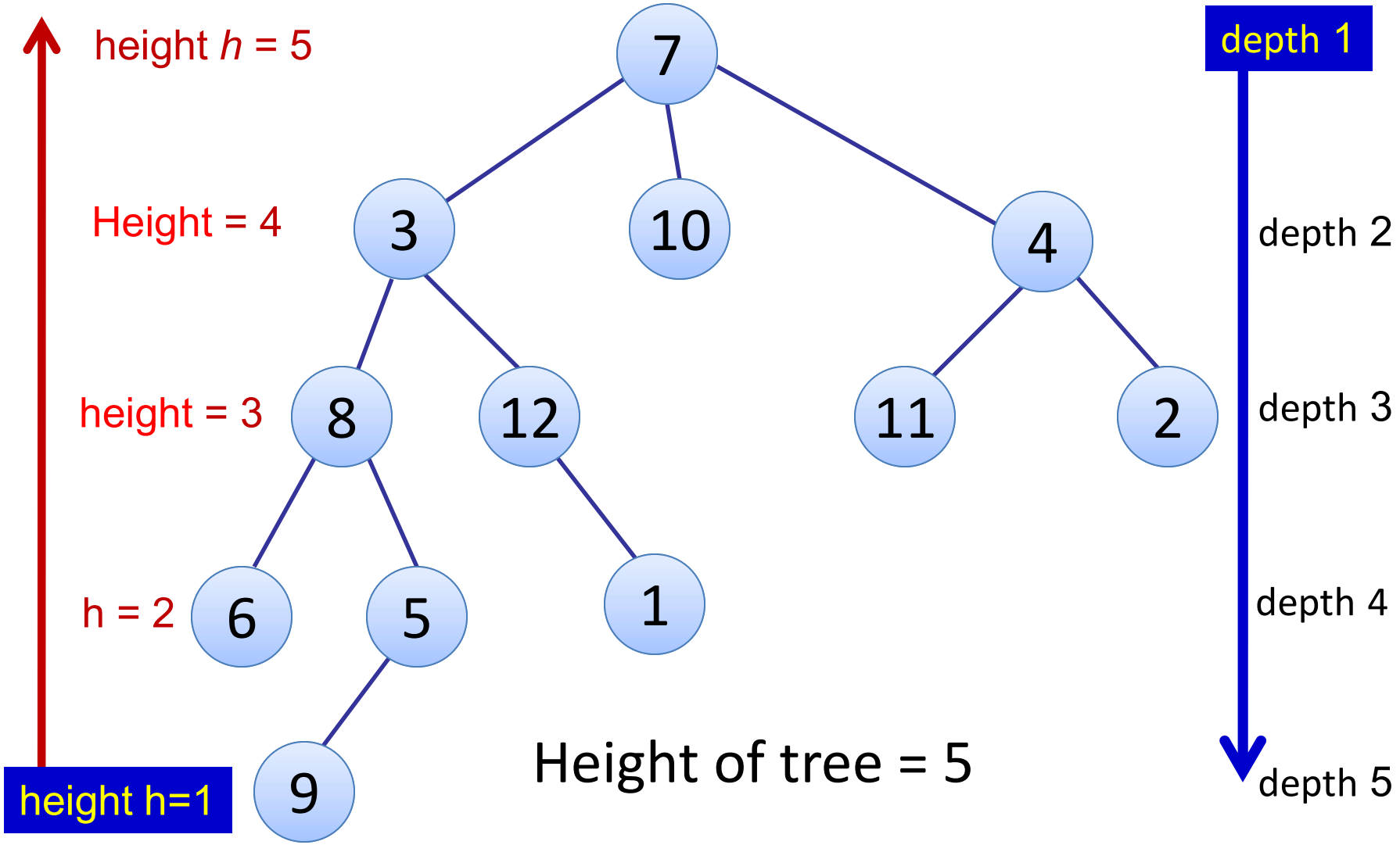
- Path: a sequence of nodes and edges connecting a node with a descendant
- Length of a path = number of edges = number of nodes - 1

(e.g.: Length of path (A ➔ C ➔E) = 2; length of path (C➔F) = 1)

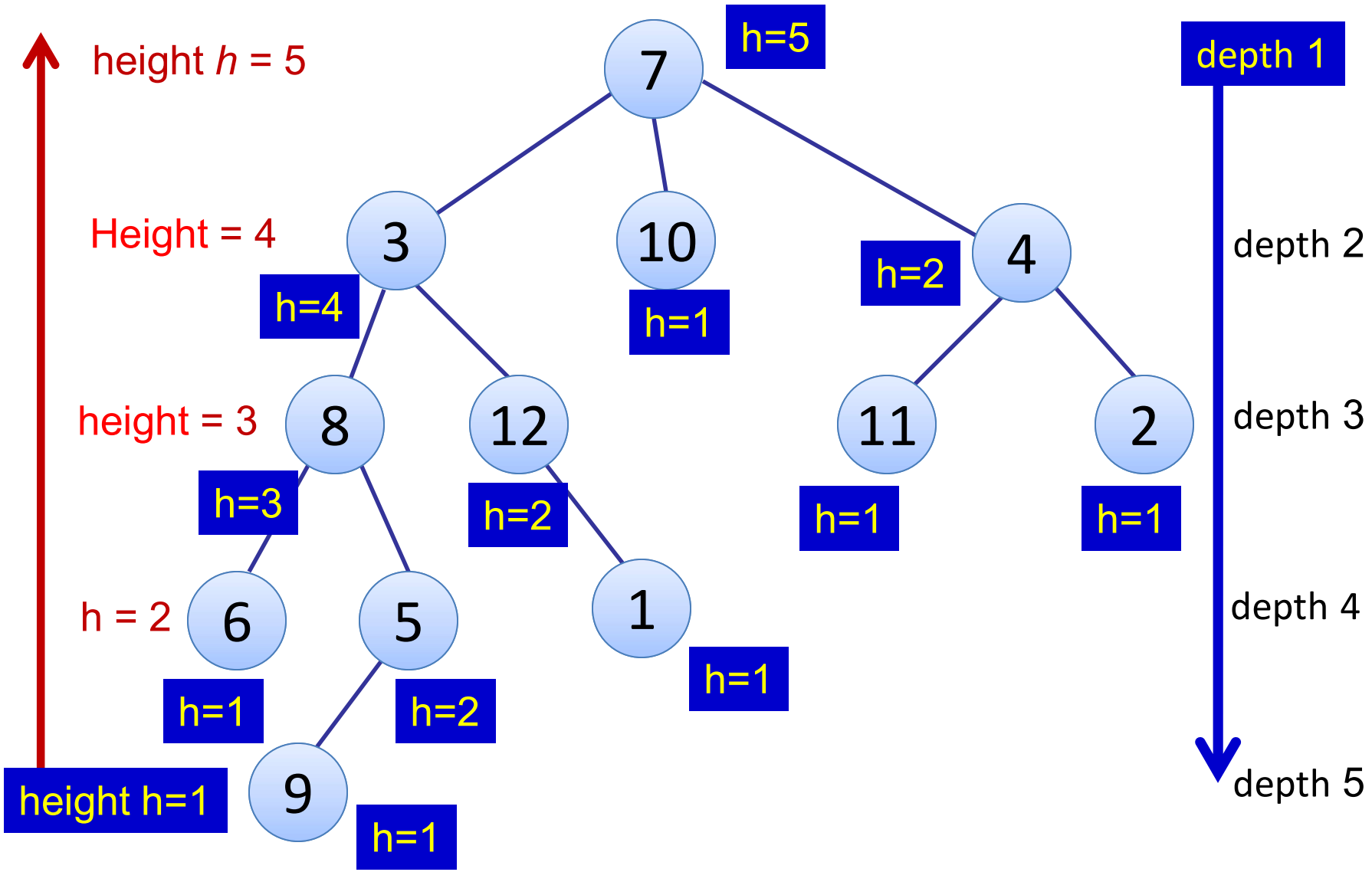
- Depth/level of a node N = 1 + length of path from root to N
- Height of node N = 1 + length of longest path from N to a leaf
- Height (depth) of tree = height of root (= maximum depth of any node on the tree)



Height and depth/level

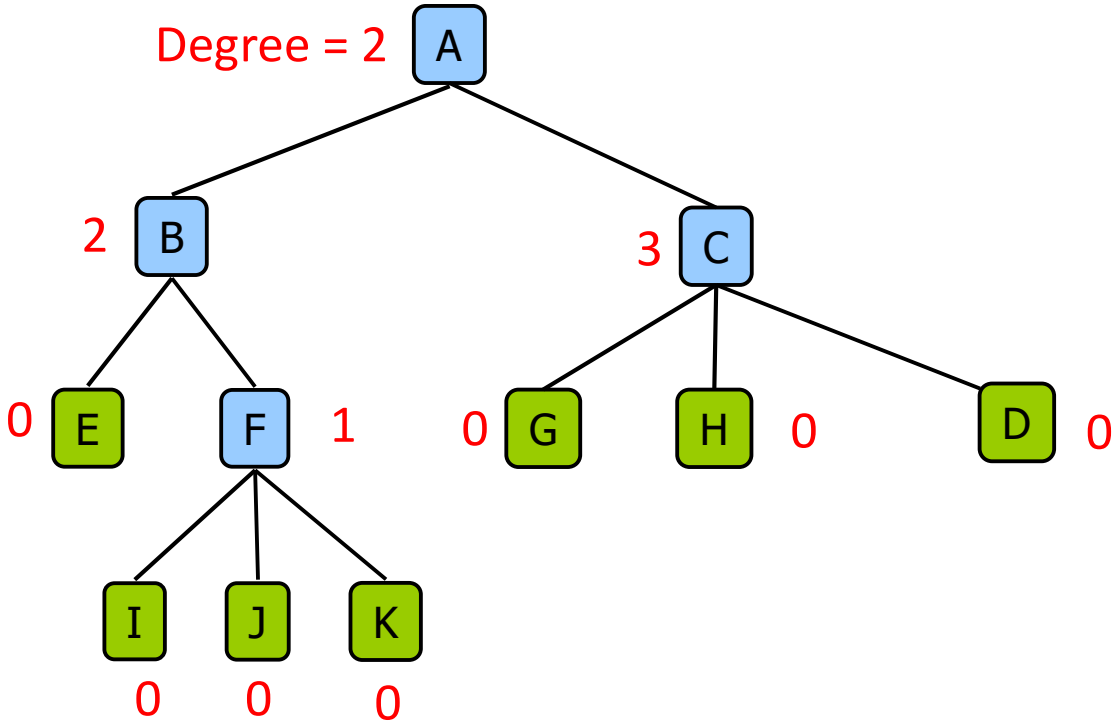


Height of node N = 1 + length of longest path from N to a leaf



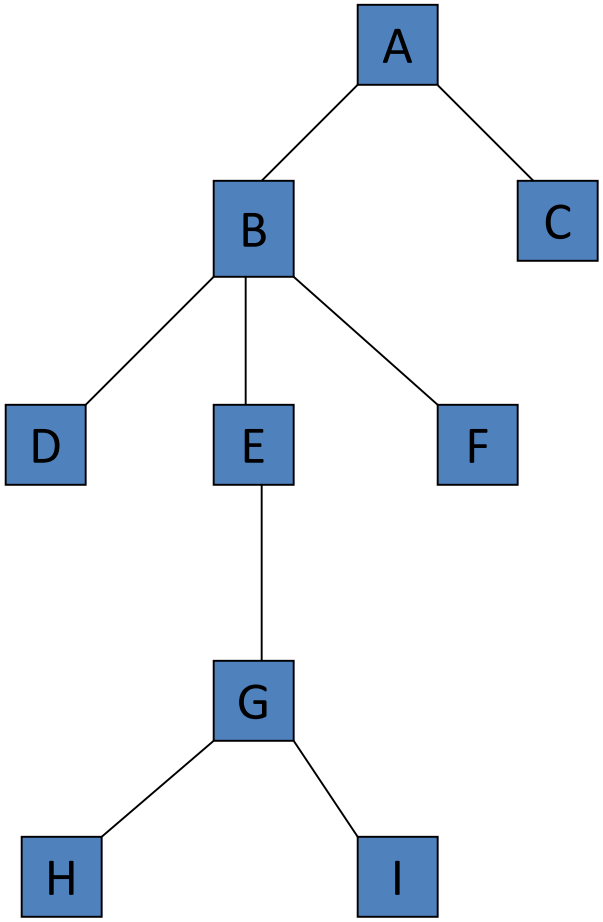
4.1.2. Tree terminology

- **Degree** of a node: the number of its children (the number of its subtree)
- **Degree** of a tree: the maximum degree of any node on the tree



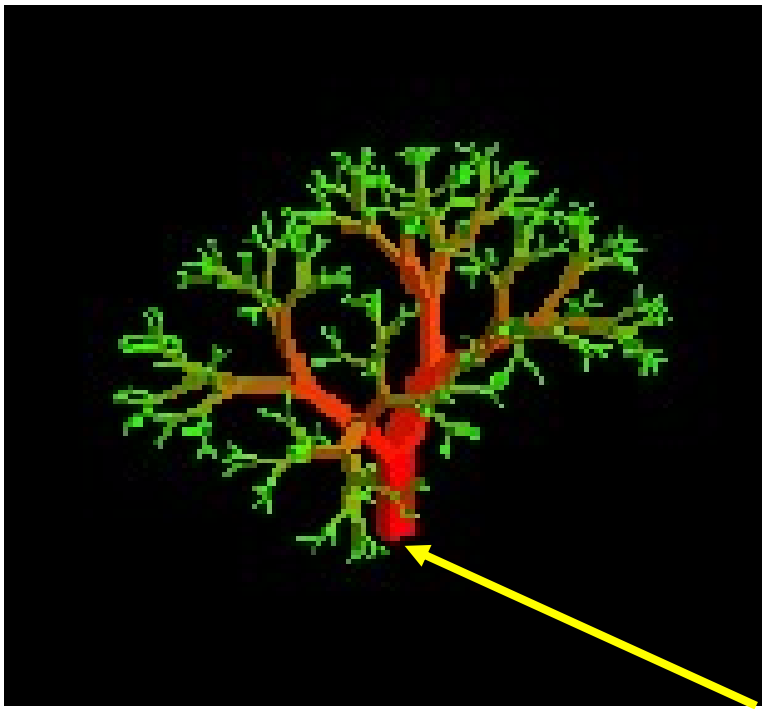
Degree of tree = ?

Example: Tree Properties

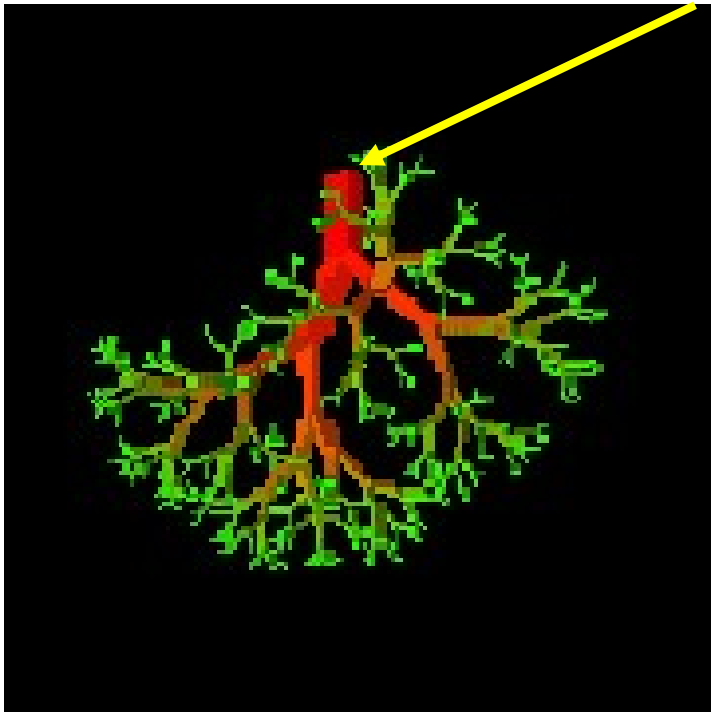


Property	Value
1) Number of nodes	
2) Root Node	
3) Leaves	
4) Internal nodes	
5) Ancestors of H	
6) Descendants of B	
7) Siblings of E	
8) Right subtree of A	
9) Left subtree of A	
10) Height of this tree	
11) Degree of this tree	

How we view a tree



Nature Lovers View



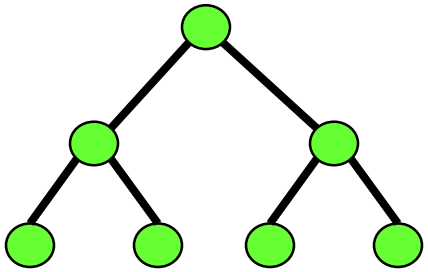
Computer Scientists View

4.1. Definitions

4.1.1. Tree definition

4.1.2. Tree terminology

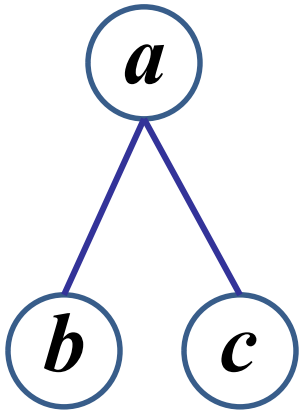
4.1.3. Ordered tree



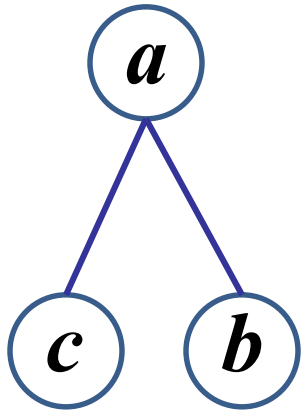
4.1.3. Ordered tree

An ordered tree is an oriented tree in which the children of a node are somehow "ordered."

Example: If T_1 and T_2 are ordered tree then $T_1 \neq T_2$; else $T_1 = T_2$

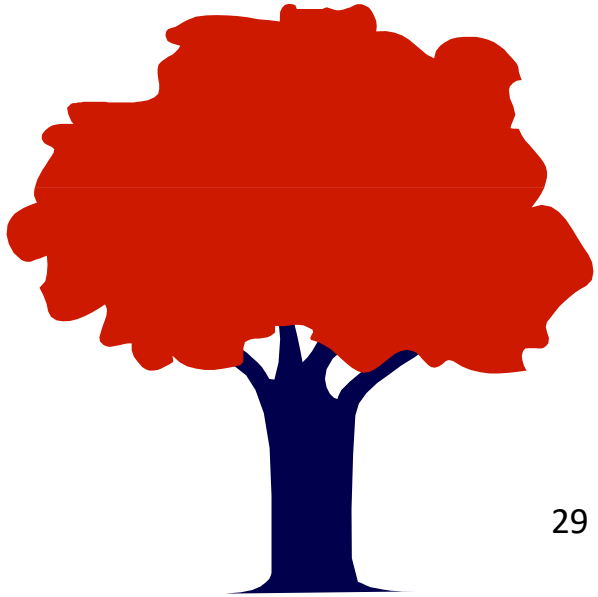


T_1



T_2

Contents

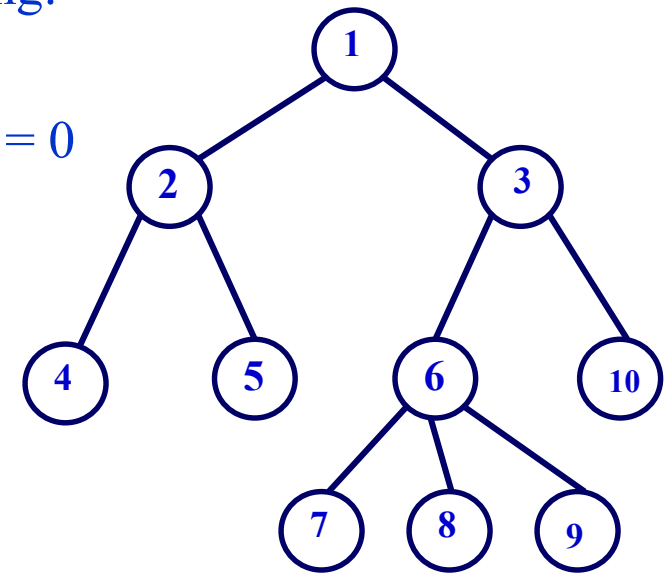


4.2. Tree representation

- There are many methods to implement a tree. Some of them are:
  - Array
  - Lists of Children
  - The Leftmost-Child, Right-Sibling representation

4.2. Tree representation : Array

- Assume ***T*** consists of n nodes, labelled as 1, 2, …, n.
- We use an array ***A*** to represent the tree ***T*** as following:
  - $A[i] = j$ if node j is the parent of node i
  - The root of tree ***T*** does not have parent, so $A[1] = 0$



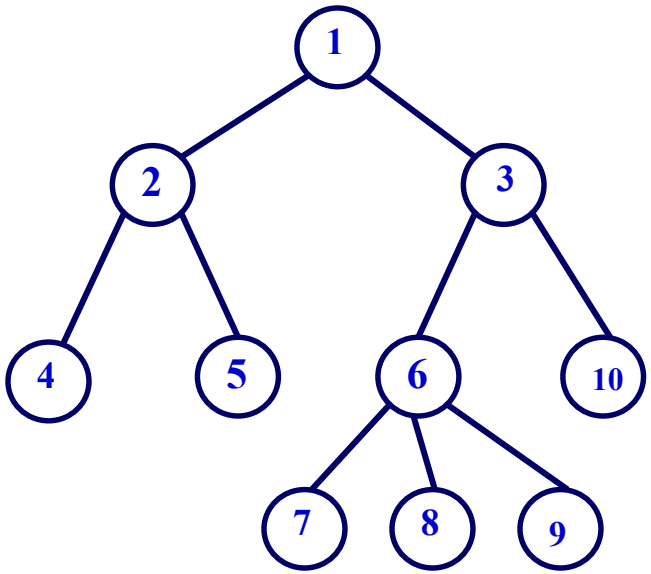
A

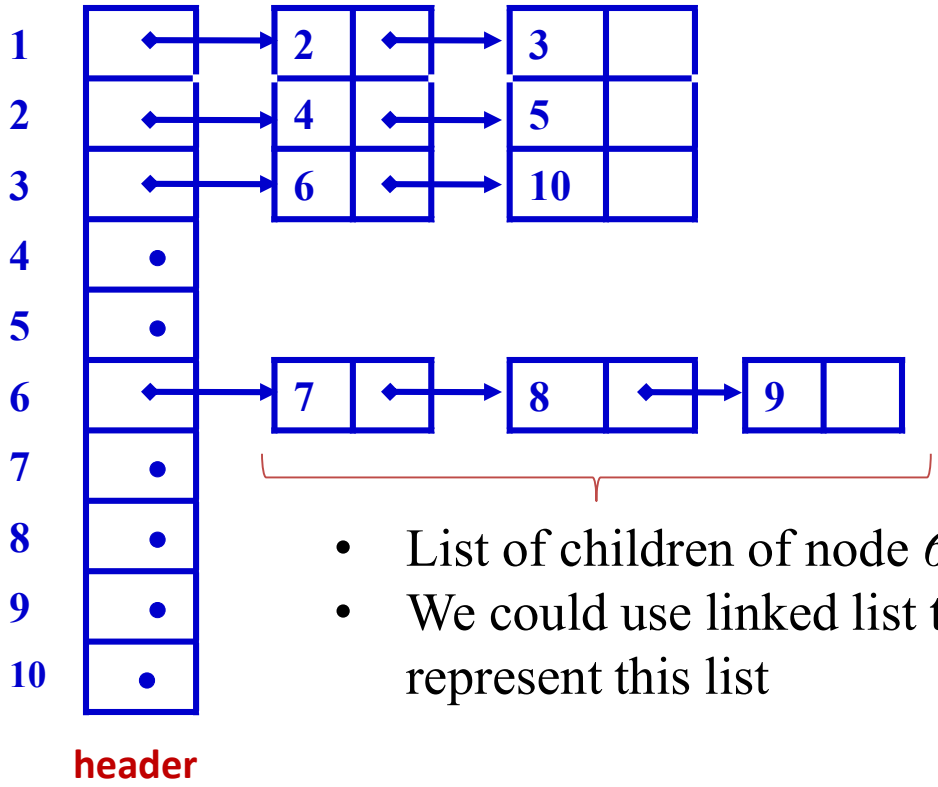
0	1	1	2	2	3	6	6	6	3
A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]	A[10]

4.2. Tree representation

- There are many methods to implement a tree. Some of them are:
  - Array
  - **Lists of Children**
  - The Leftmost-Child, Right-Sibling representation

4.2. Tree representation : List of children





- List of children of node *6*
- We could use linked list to represent this list

`Header` : an array of pointers

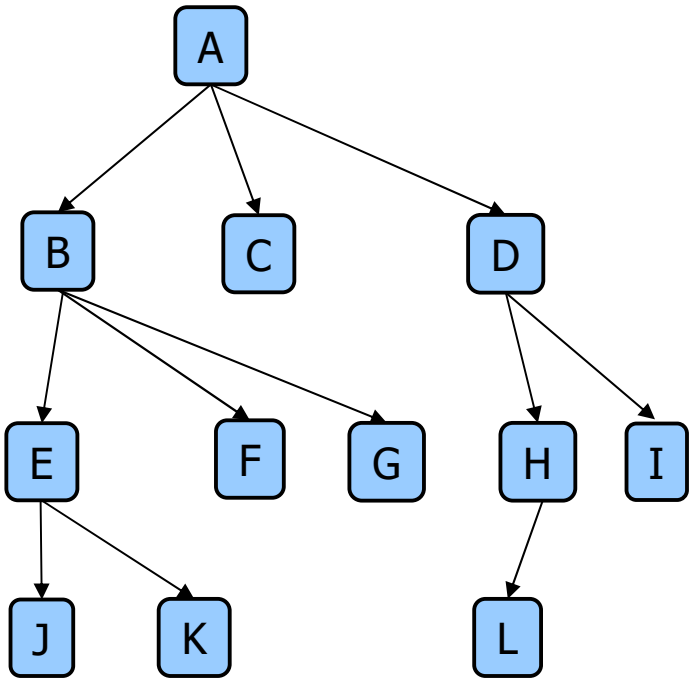
- each pointer `header[i]` points to head of the list of children of the node ***i*** on the tree
- List of children of node ***i***: linked list

4.2. Tree representation

- There are many methods to implement a tree. Some of them are:
  - Array
  - Lists of Children
  - **The Leftmost-Child, Right-Sibling representation**

4.2. Tree representation : Leftmost-Child, Right-Sibling representation

- Each node on the tree could either:
  - Have not any child, or have exactly one leftmost child
  - Have not any right-sibling, or have exactly one right-sibling



Example 1: node B

- Leftmost child of node B: E
- Right-sibling of node B: C

Example 2: node C

- Does not have any child
- Leftmost child of node C:
- Right-sibling of node C:

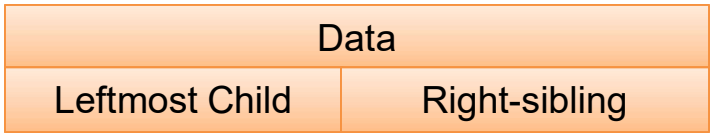
Example 3: node I

- Does not have any child
- Does not have any right-sibling

4.2. Tree representation : Leftmost-Child, Right-Sibling representation

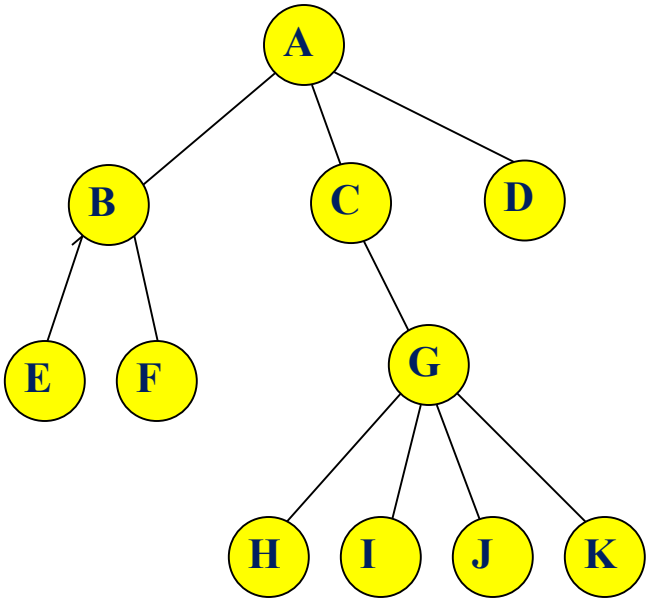
- Each node on the tree could either:
  - Have not any child, or have exactly one leftmost child
  - Have not any right-sibling, or have exactly one right-sibling

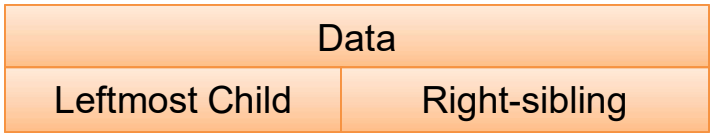
Thus, in order to represent a tree, we could store the information about the leftmost child and right-sibling of each node:



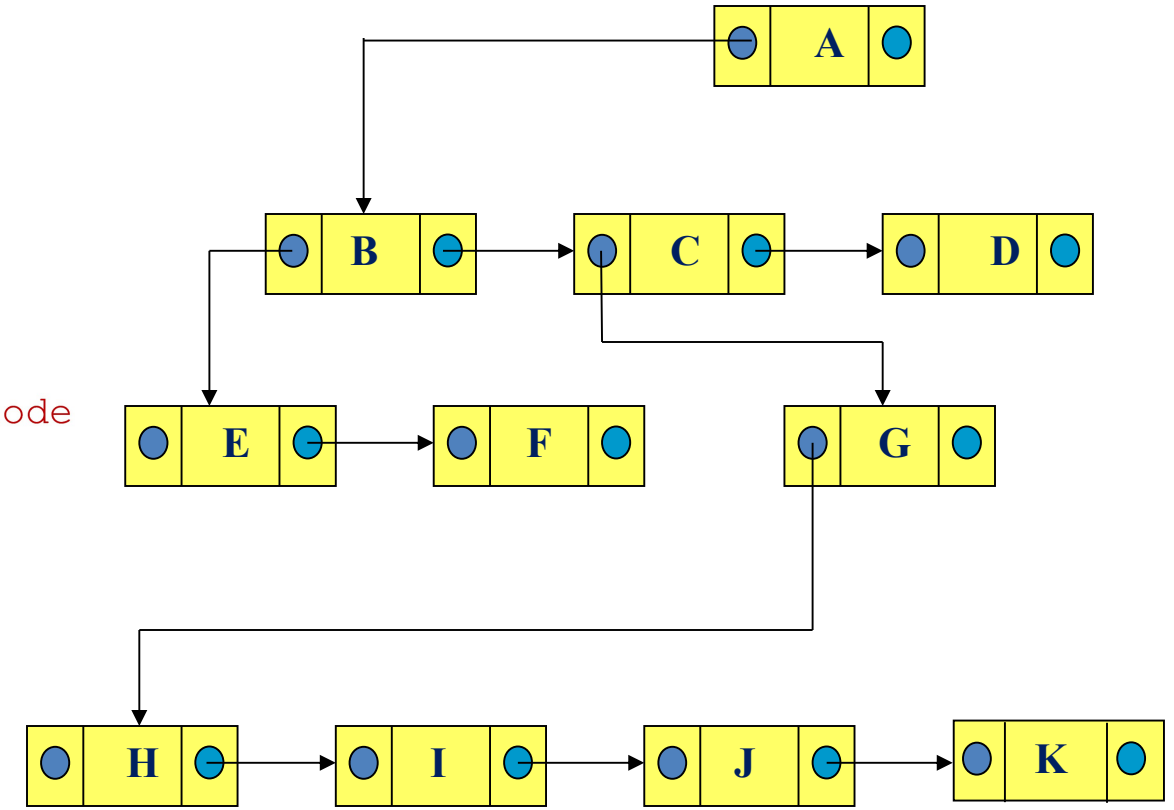

```
struct treeNode
{
   int data;  // data of each node
   treeNode  * leftmost_child;
   treeNode  * right_sibling;
};
treeNode * Root;
```

4.2. Tree representation : Leftmost-Child, Right-Sibling representation

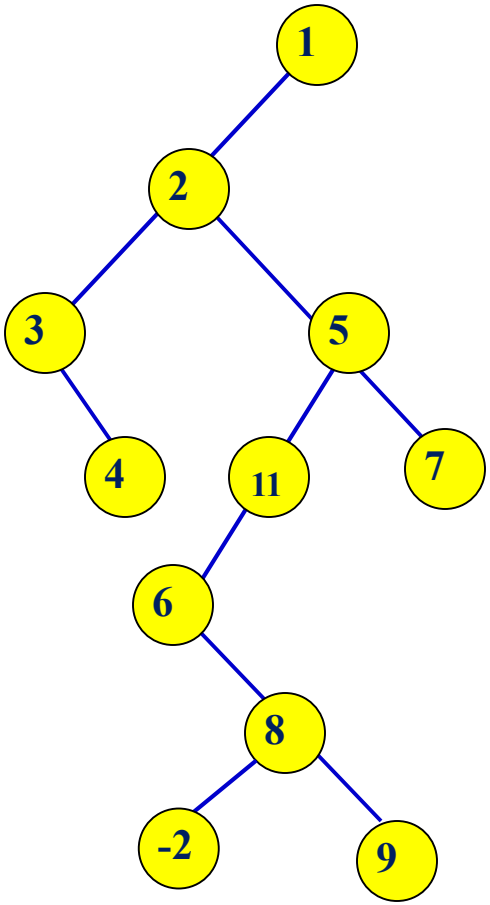


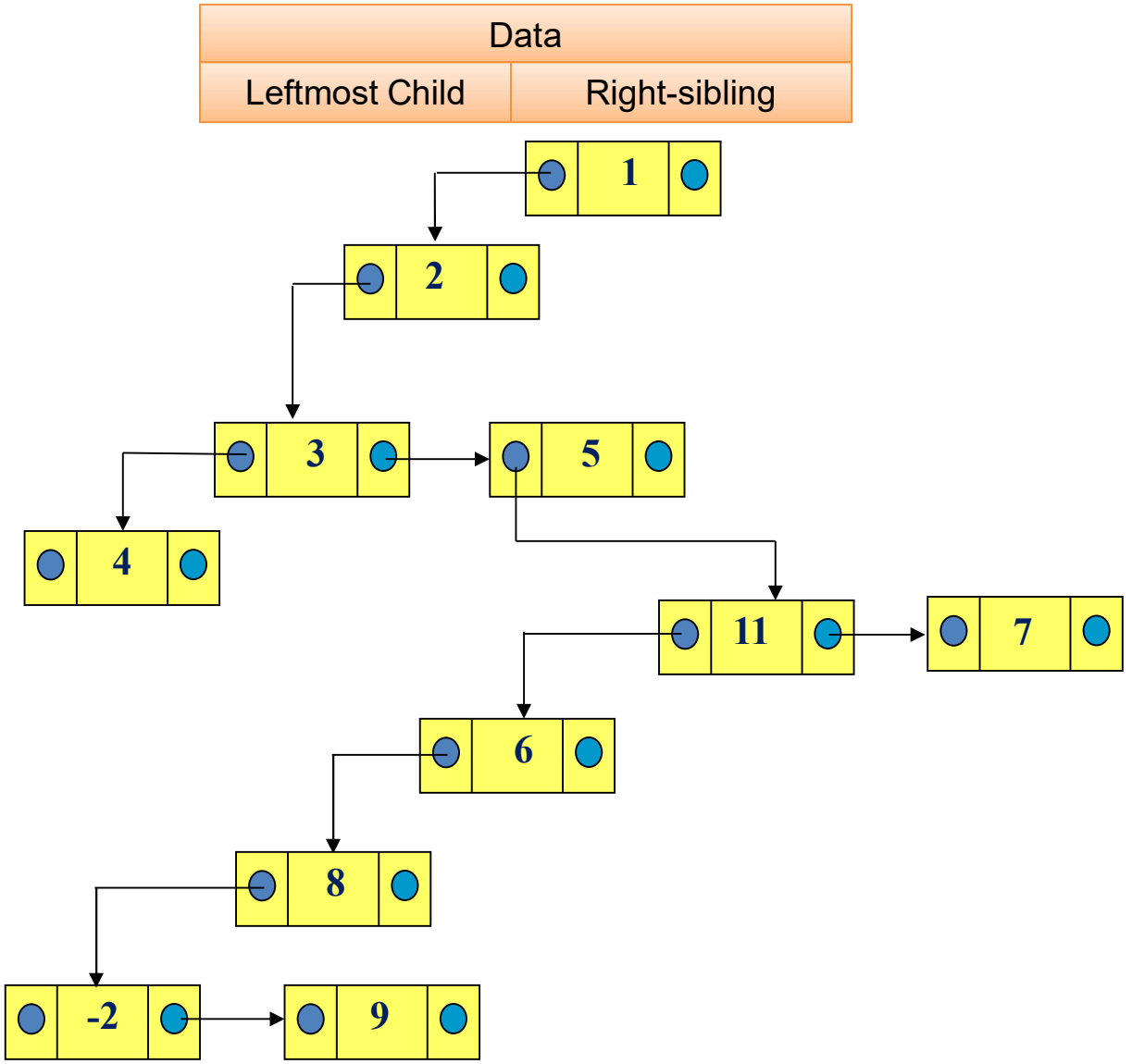



```
struct treeNode
{
   int data;  // data of each node
   treeNode  * leftmost_child;
   treeNode  * right_sibling;
};
treeNode * Root;
```



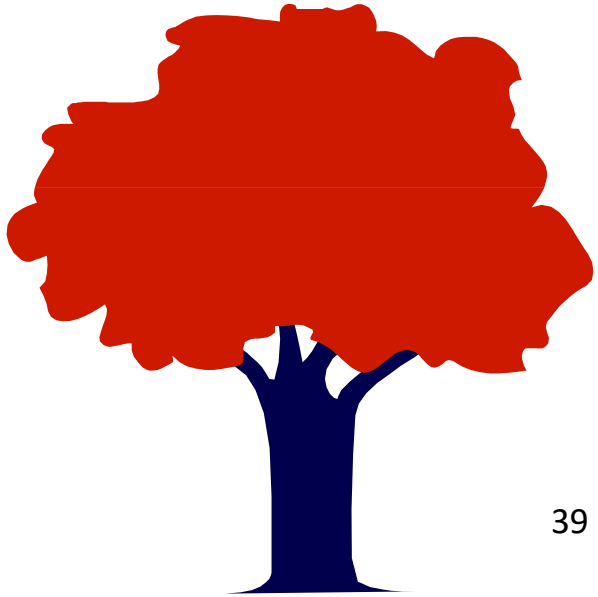
4.2. Tree representation : Leftmost-Child, Right-Sibling representation






```
struct treeNode
{
   int data;  // data of each node
   treeNode  * leftmost_child;
   treeNode  * right_sibling;
};
treeNode * Root;
```

Contents



4.3. Tree Traversal

Three main methods:

- Preorder:
  - visit the root
  - traverse in preorder the children (subtrees)

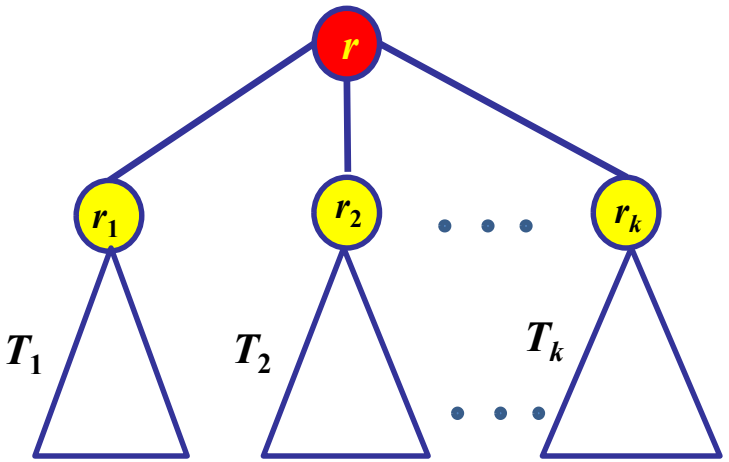
- Postorder
  - traverse in postorder the children (subtrees)
  - visit the root

- Inorder
  - traverse in inorder the left-most children (the leftmost subtree)
  - visit the root
  - traverse in inorder the remaining children that not the leftmost one

Preorder Traversal

- In a preorder traversal, a node is visited before its descendants.

Example: Preorder traversal on tree T:

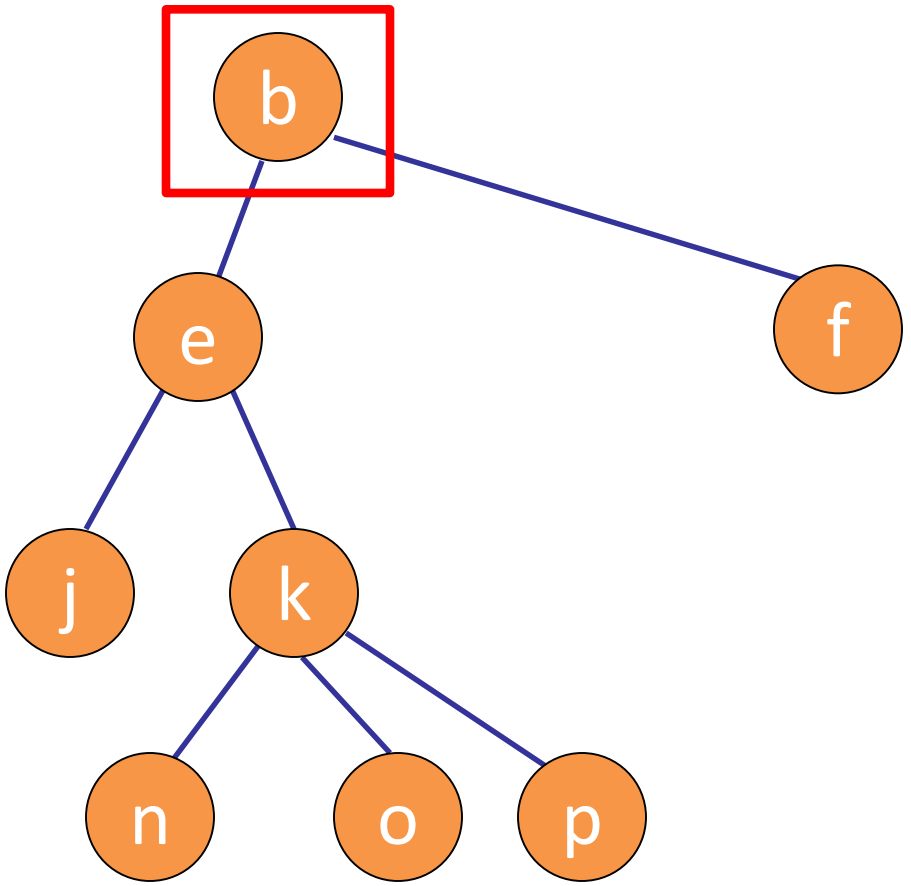



```
procedure preorder(T: ordered rooted tree)
 r := root of T
 visit r                                  Step 1
 for each child c of r from left to right
 begin
    T(c) := subtree with c as its root
    preorder(T(c))                        Step 2, 3…
 end
```

- Step 1: visit root ***r***,
- Step 2: visit T_1 in preorder,
- Step 3: visit T_2 in preorder
- ……..
- Step k+1: visit T_k in preorder

PreOrder

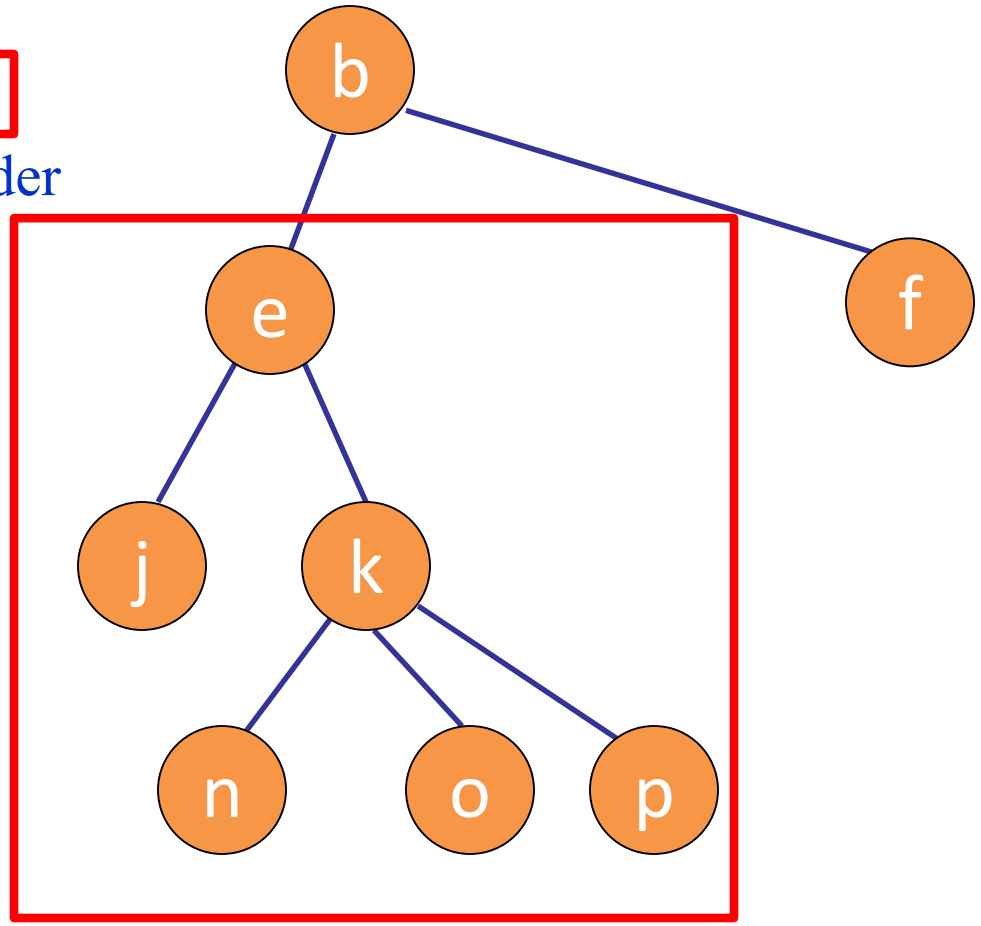
1. Visit root
2. Visit leftmost subtree in preorder
3. Visit remaining subtrees in preorder



b

PreOrder

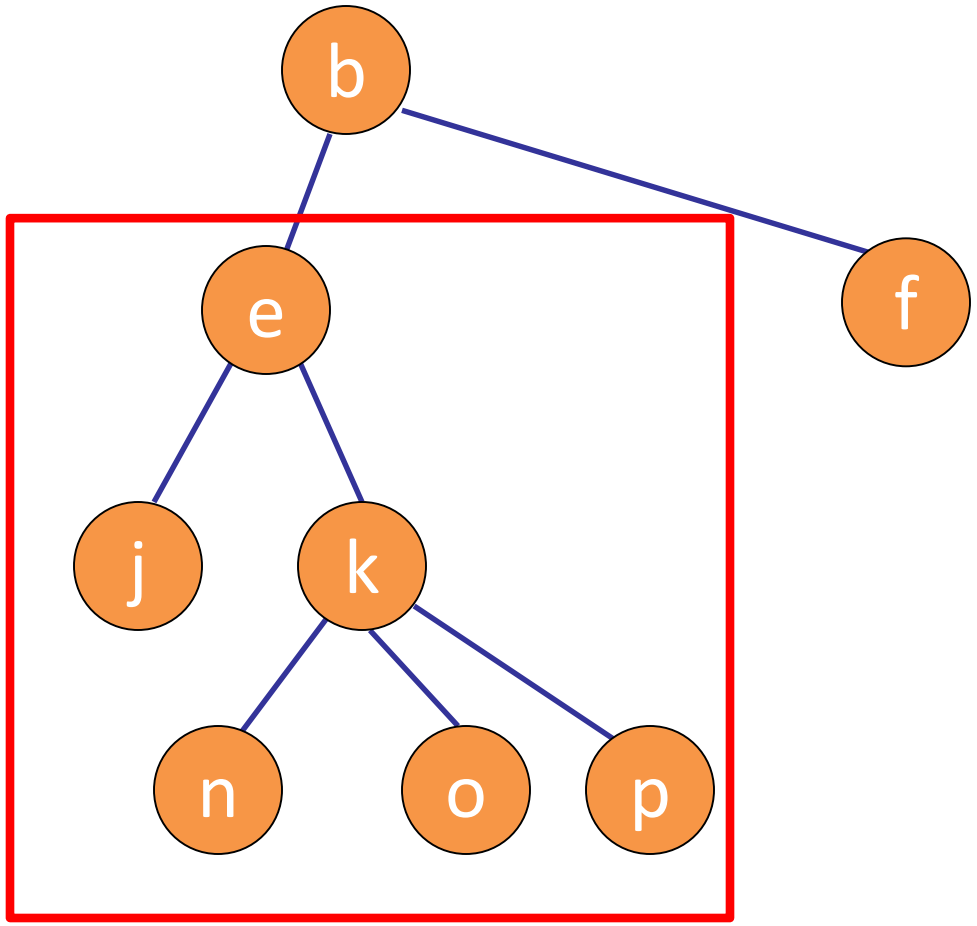
1. Visit root
2. Visit leftmost subtree in preorder
3. Visit remaining subtrees in preorder



b

PreOrder

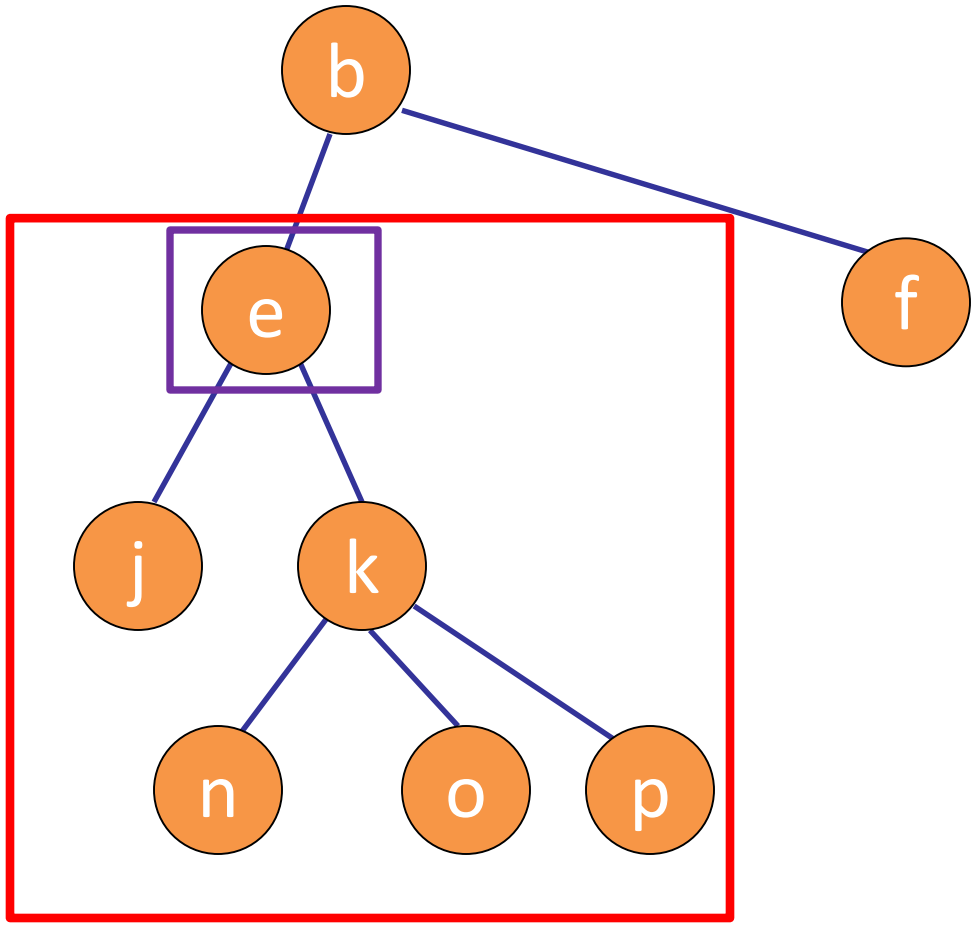
1. Visit root
2. Visit leftmost subtree in preorder
   - 2.1. Visit root
   - 2.2. Visit leftmost subtree in preorder
   - 2.3. Visit remaining subtrees in preorder
3. Visit remaining subtrees in preorder



b

PreOrder

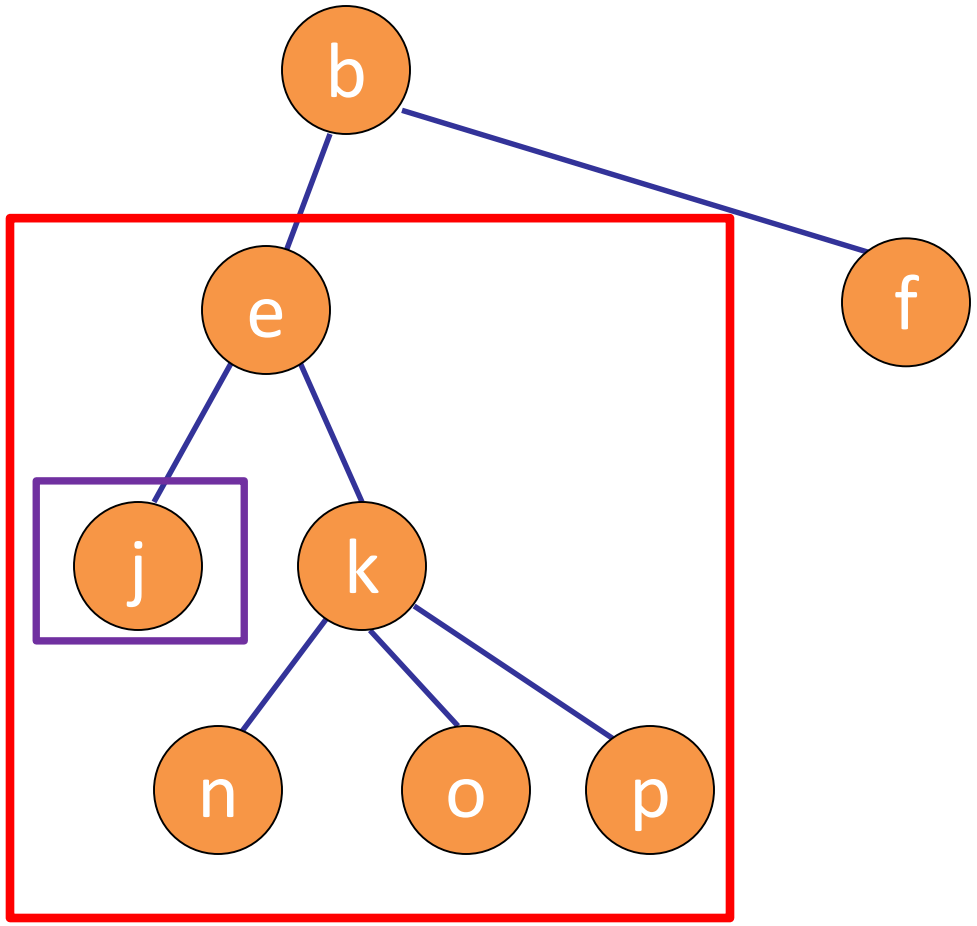
1. Visit root
2. Visit leftmost subtree in preorder
   2.1. Visit root
   2.2. Visit leftmost subtree in preorder
   2.3. Visit remaining subtrees in preorder
3. Visit remaining subtrees in preorder



b e

PreOrder

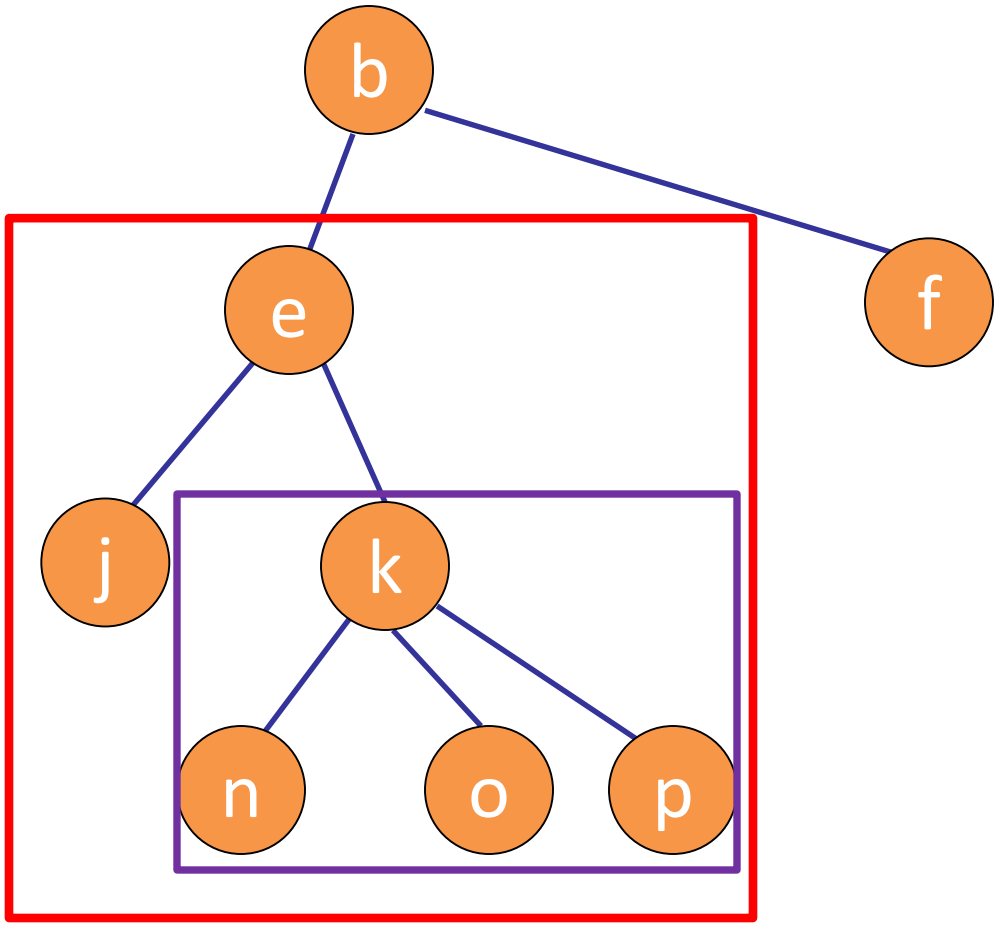
1. Visit root
2. Visit leftmost subtree in preorder
   2.1. Visit root
   2.2. Visit leftmost subtree in preorder
   2.3. Visit remaining subtrees in preorder
3. Visit remaining subtrees in preorder



b e j

PreOrder

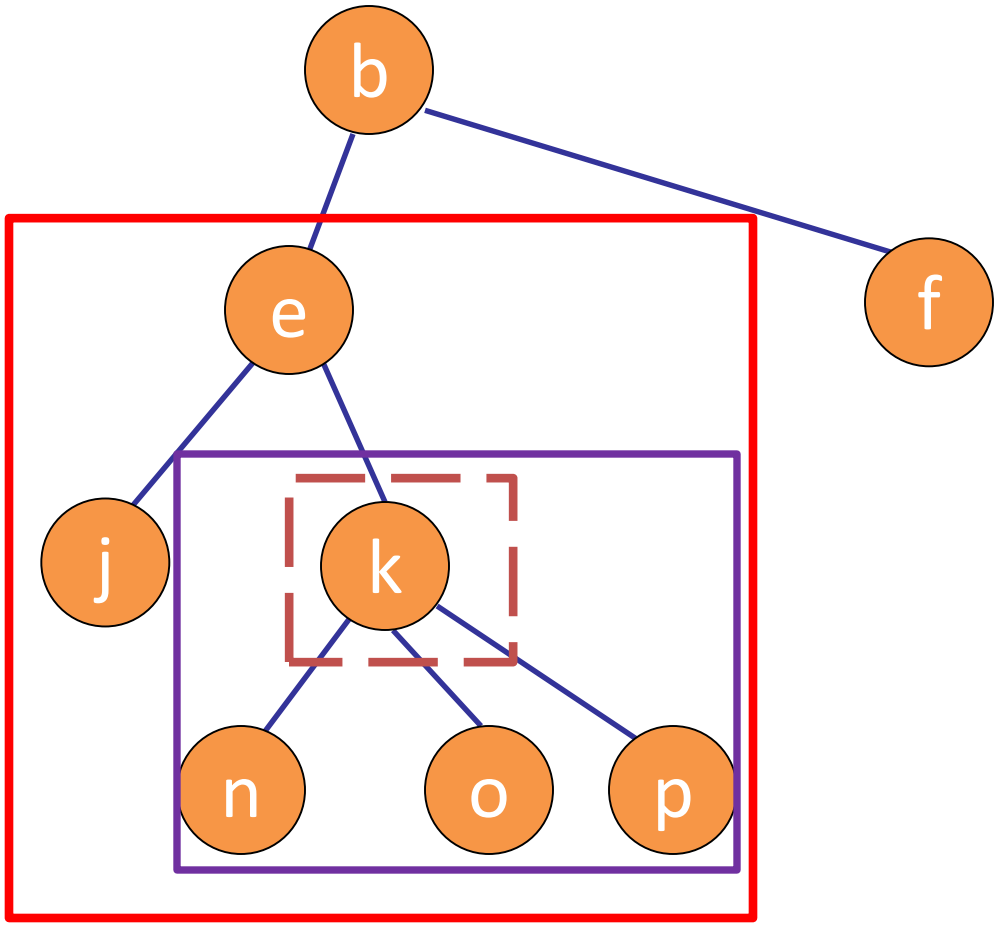
1. Visit root
2. Visit leftmost subtree in preorder
   2.1. Visit root
   2.2. Visit leftmost subtree in preorder
   2.3. Visit remaining subtrees in preorder
3. Visit remaining subtrees in preorder



b e j

PreOrder

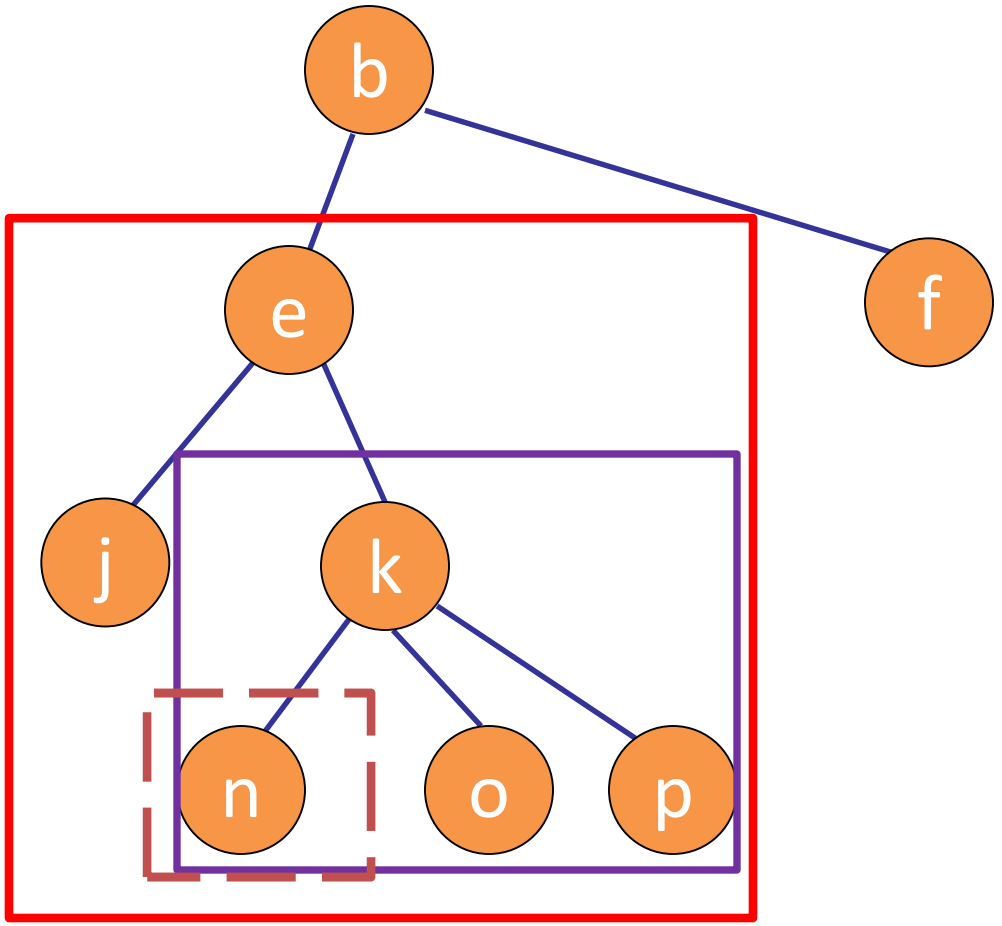
1. Visit root
2. Visit leftmost subtree in preorder
   - 2.1. Visit root
   - 2.2. Visit leftmost subtree in preorder
   - 2.3. Visit remaining subtrees in preorder
     - 2.3.1. Visit root
     - 2.3.2. Visit leftmost subtree in preorder
     - 2.3.3. Visit remaining subtrees in preorder
3. Visit remaining subtrees in preorder



b e j k

PreOrder

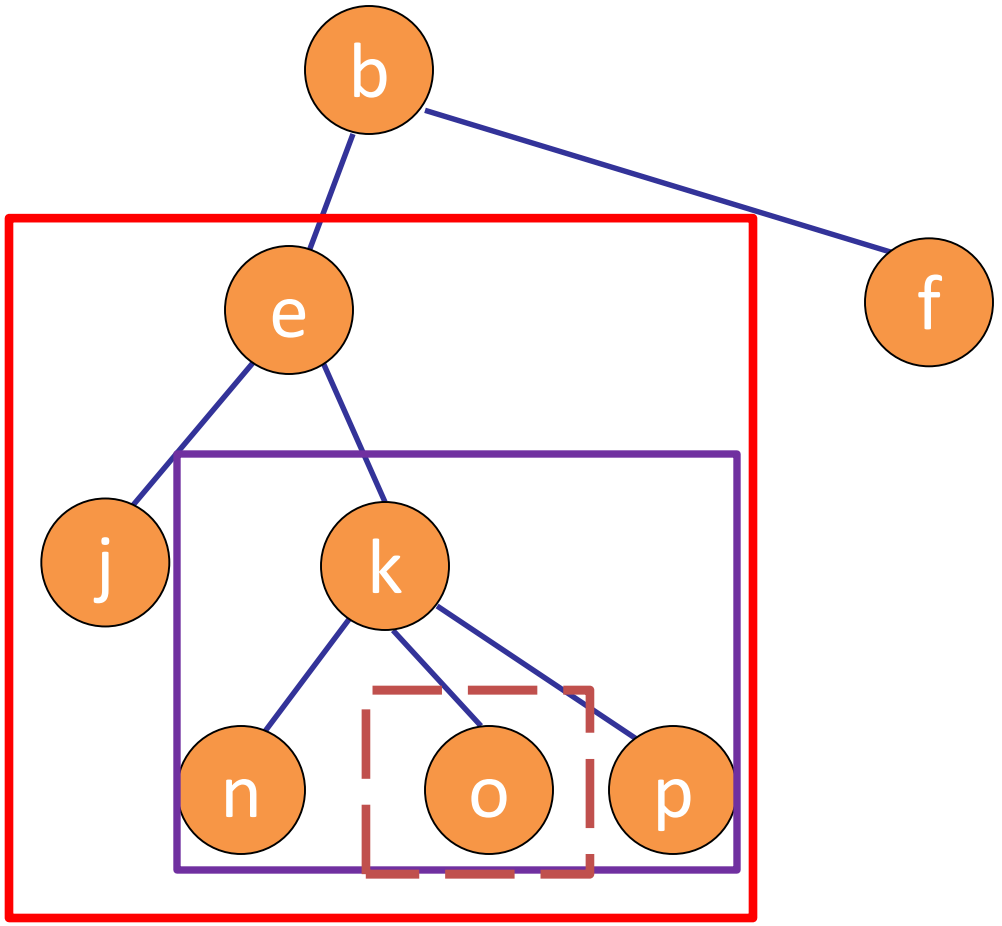
1. Visit root
2. Visit leftmost subtree in preorder
   - 2.1. Visit root
   - 2.2. Visit leftmost subtree in preorder
   - 2.3. Visit remaining subtrees in preorder
     - 2.3.1. Visit root
     - 2.3.2. Visit leftmost subtree in preorder
     - 2.3.3. Visit remaining subtrees in preorder
3. Visit remaining subtrees in preorder



b e j k n

PreOrder

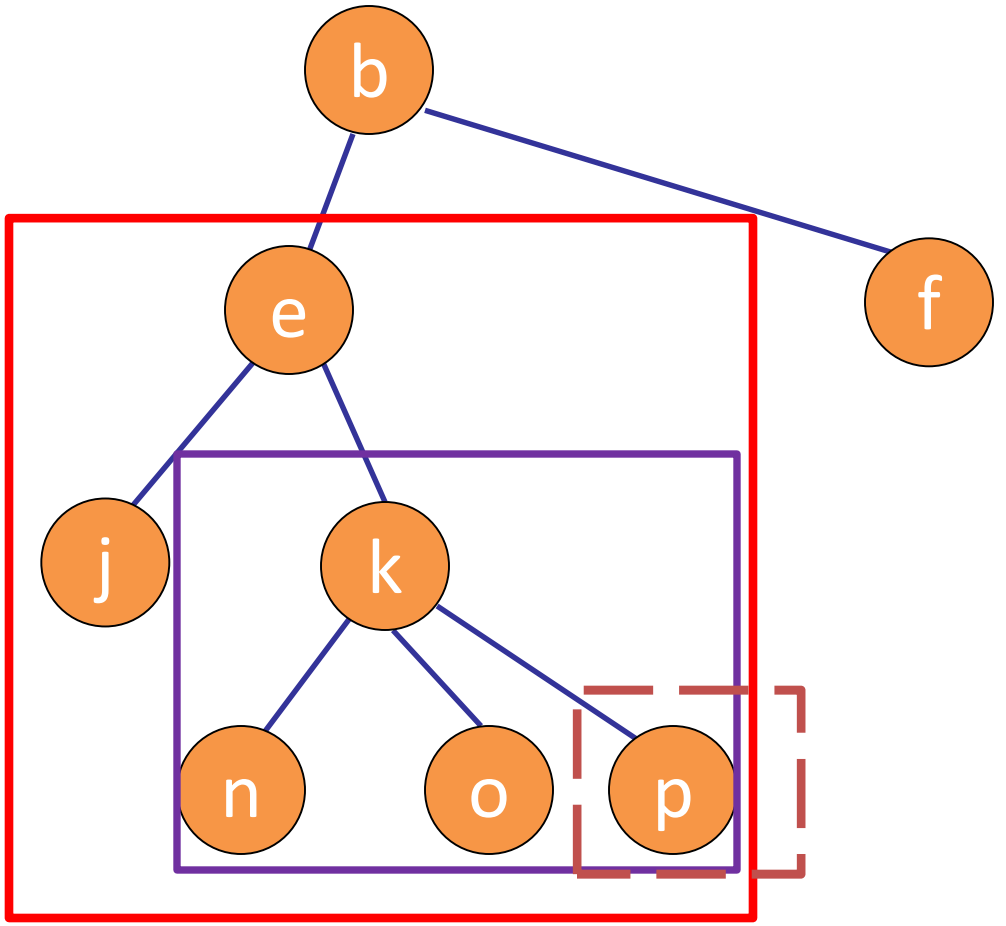
1. Visit root
2. Visit leftmost subtree in preorder
   - 2.1. Visit root
   - 2.2. Visit leftmost subtree in preorder
   - 2.3. Visit remaining subtrees in preorder
     - 2.3.1. Visit root
     - 2.3.2. Visit leftmost subtree in preorder
     - 2.3.3. Visit remaining subtrees in preorder
3. Visit remaining subtrees in preorder



b e j k n o

PreOrder

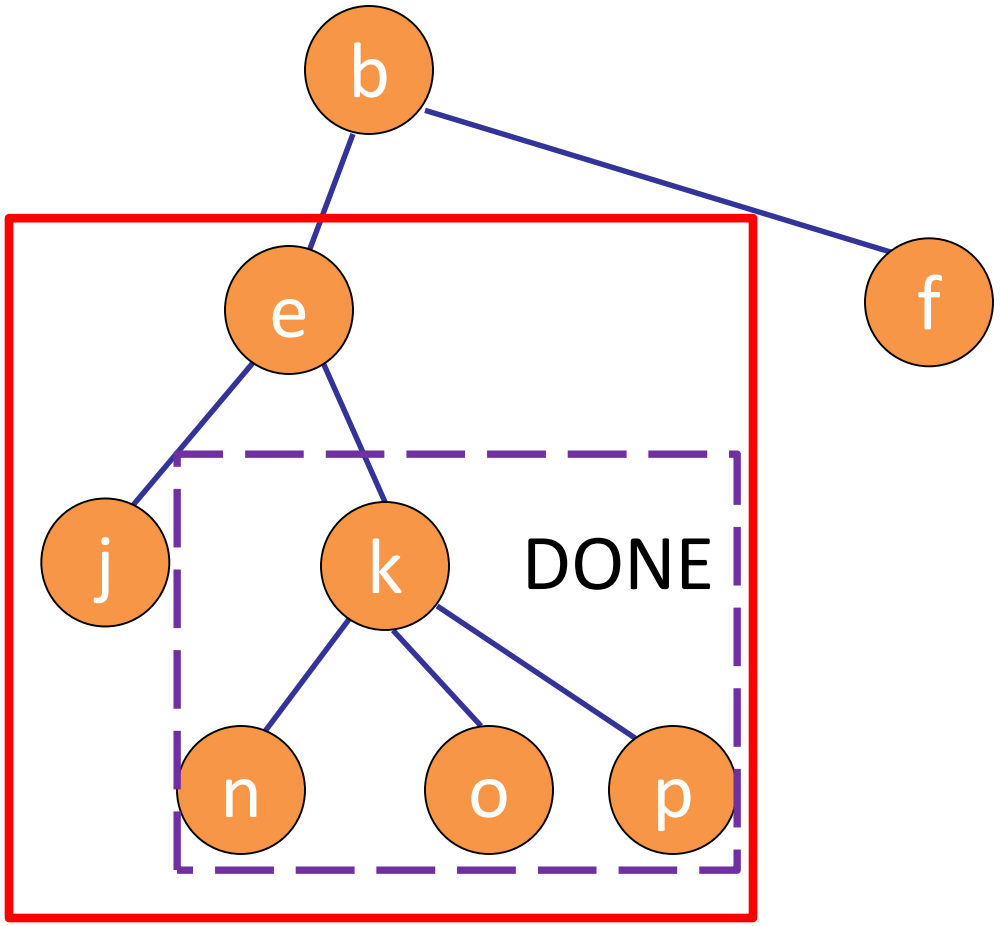
1. Visit root
2. Visit leftmost subtree in preorder
   - 2.1. Visit root
   - 2.2. Visit leftmost subtree in preorder
   - 2.3. Visit remaining subtrees in preorder
     - 2.3.1. Visit root
     - 2.3.2. Visit leftmost subtree in preorder
     - 2.3.3. Visit remaining subtrees in preorder
3. Visit remaining subtrees in preorder



b e j k n o p

PreOrder

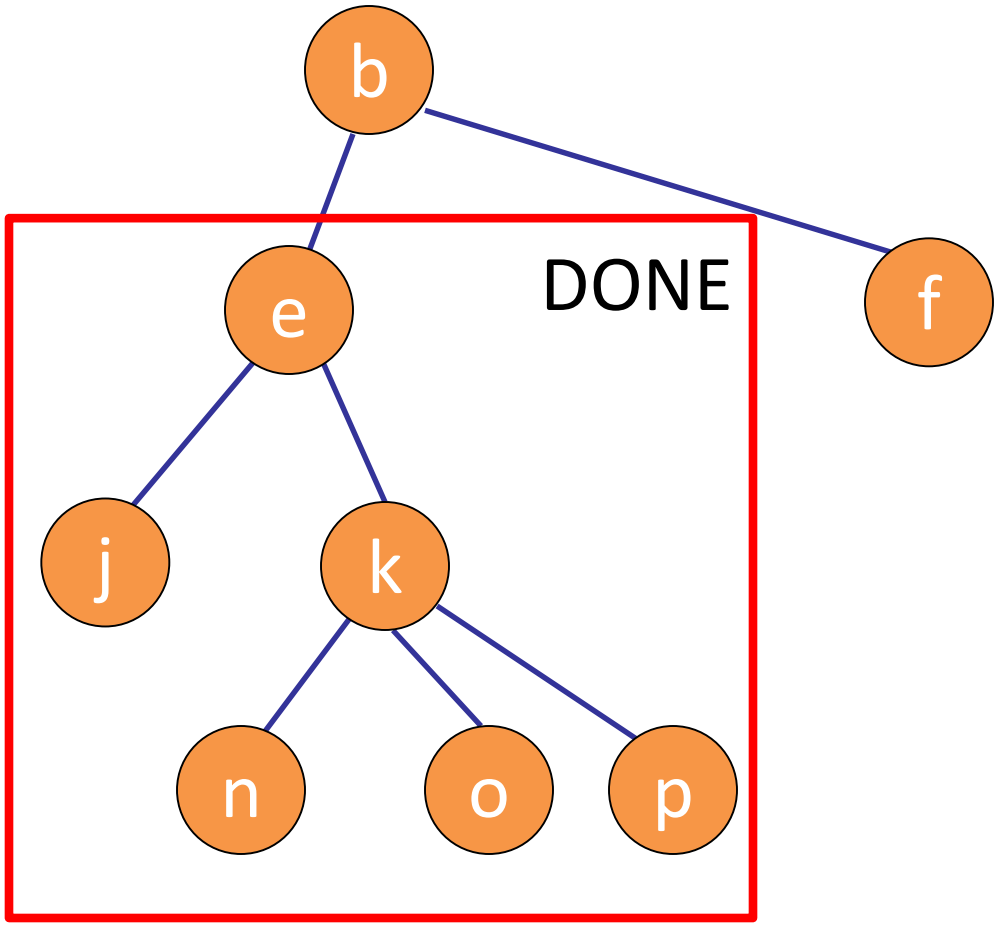
1. Visit root
2. Visit leftmost subtree in preorder
   - 2.1. Visit root
   - 2.2. Visit leftmost subtree in preorder
   - 2.3. Visit remaining subtrees in preorder
     - 2.3.1. Visit root
     - 2.3.2. Visit leftmost subtree in preorder
     - 2.3.3. Visit remaining subtrees in preorder
3. Visit remainingsubtrees in preorder



b e j k n o p

PreOrder

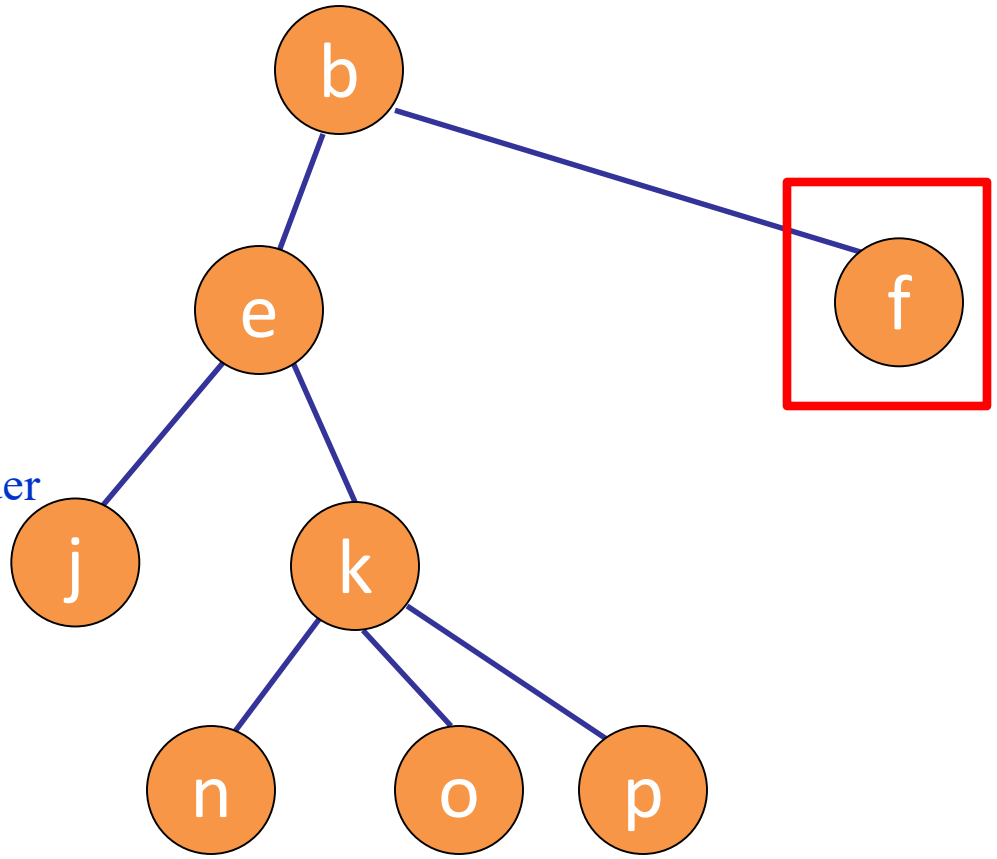
1. Visit root
2. Visit leftmost subtree in preorder
   - 2.1. Visit root
   - 2.2. Visit leftmost subtree in preorder
   - 2.3. Visit remaining subtrees in preorder
     - 2.3.1. Visit root
     - 2.3.2. Visit leftmost subtree in preorder
     - 2.3.3. Visit remaining subtrees in preorder
3. Visit remaining subtrees in preorder



b e j k n o p

PreOrder

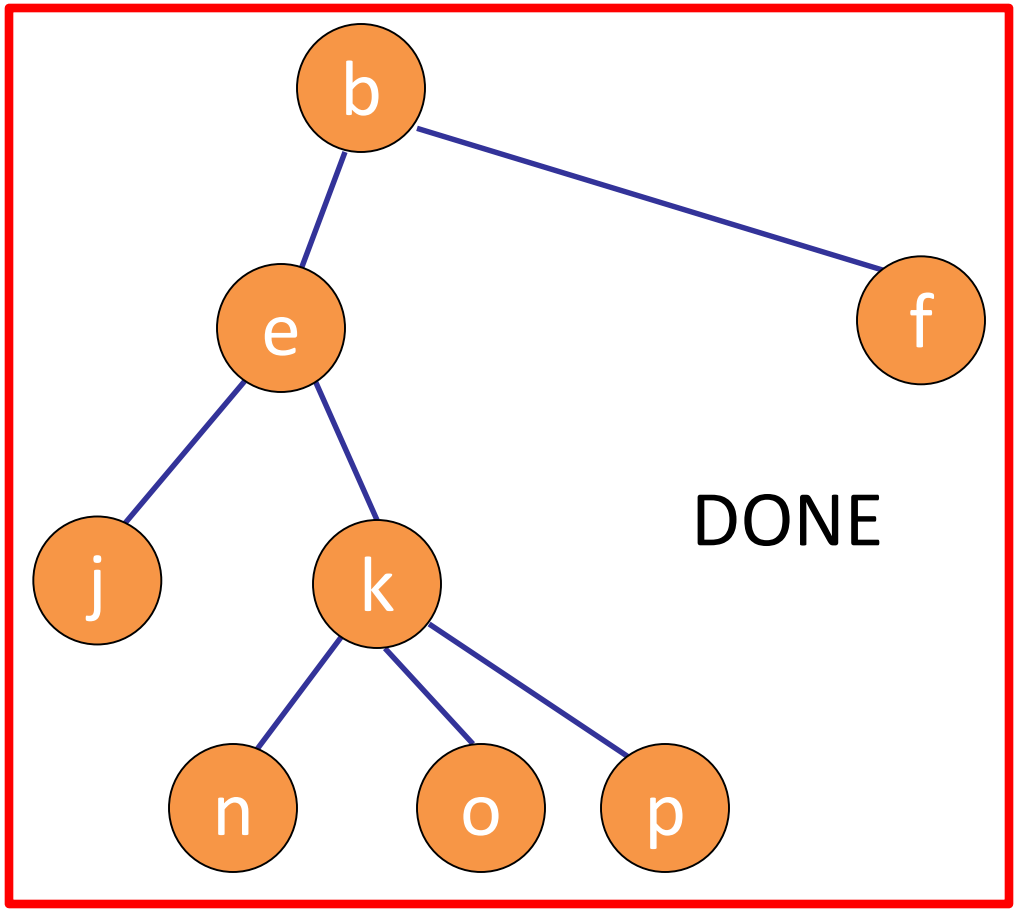
1. Visit root
2. Visit leftmost subtree in preorder
   2.1. Visit root
   2.2. Visit leftmost subtree in preorder
   2.3. Visit remaining subtrees in preorder
      2.3.1. Visit root
      2.3.2. Visit leftmost subtree in preorder
      2.3.3. Visit remaining subtrees in preorder
3. Visit remaining subtrees in preorder



b e j k n o p f

PreOrder

1. Visit root
2. Visit leftmost subtree in preorder
   - 2.1. Visit root
   - 2.2. Visit leftmost subtree in preorder
   - 2.3. Visit remaining subtrees in preorder
     - 2.3.1. Visit root
     - 2.3.2. Visit leftmost subtree in preorder
     - 2.3.3. Visit remaining subtrees in preorder
3. Visit remaining subtrees in preorder

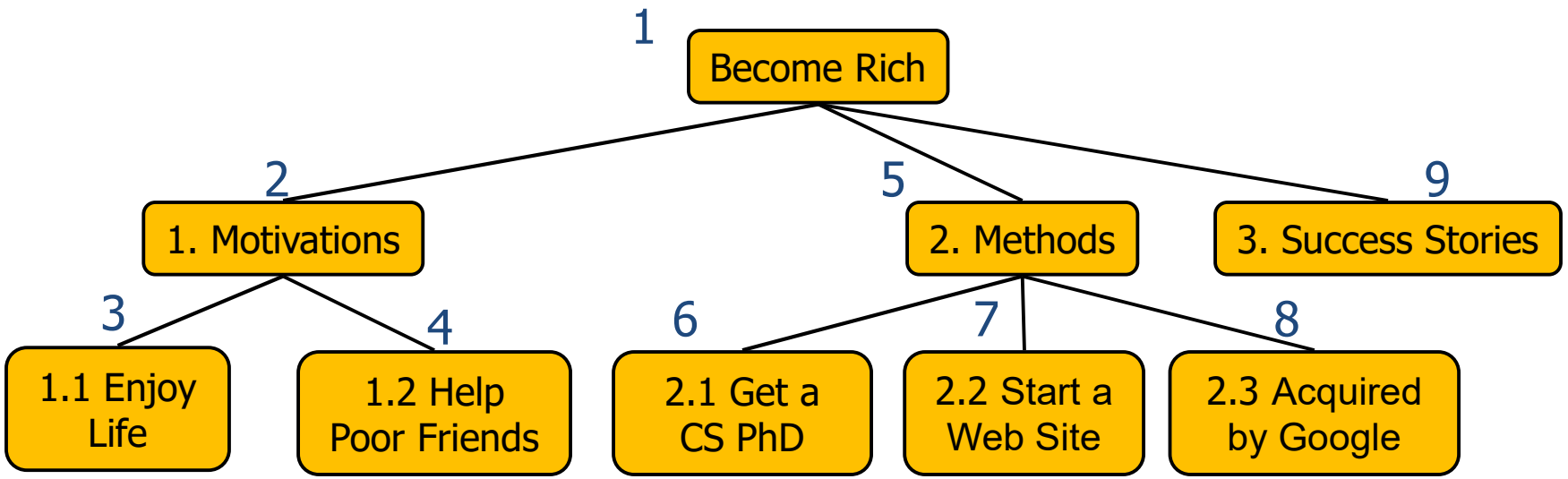


b e j k n o p f

Preorder Traversal

- Application: print a structured document

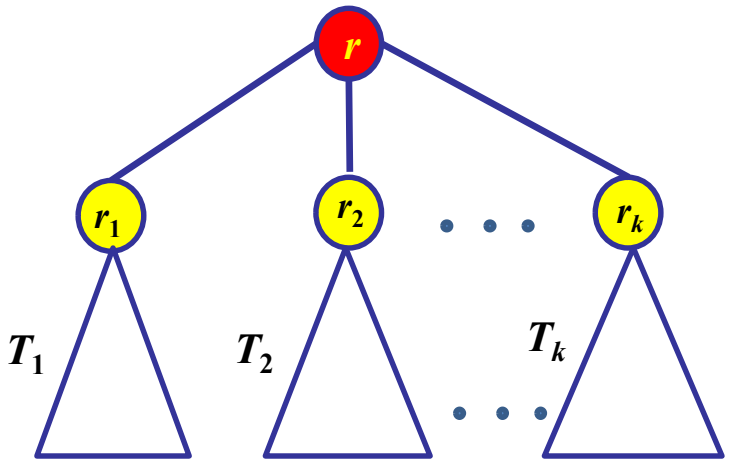
```
procedure preorder(T: ordered rooted tree)
 r := root of T
 visit r
 for each child c of r from left to right
 begin
    T(c) := subtree with c as its root
    preorder(T(c))
 end
```



Postorder Traversal

- In a postorder traversal, a node is visited after its descendants.

Example: Postorder traversal on tree T:

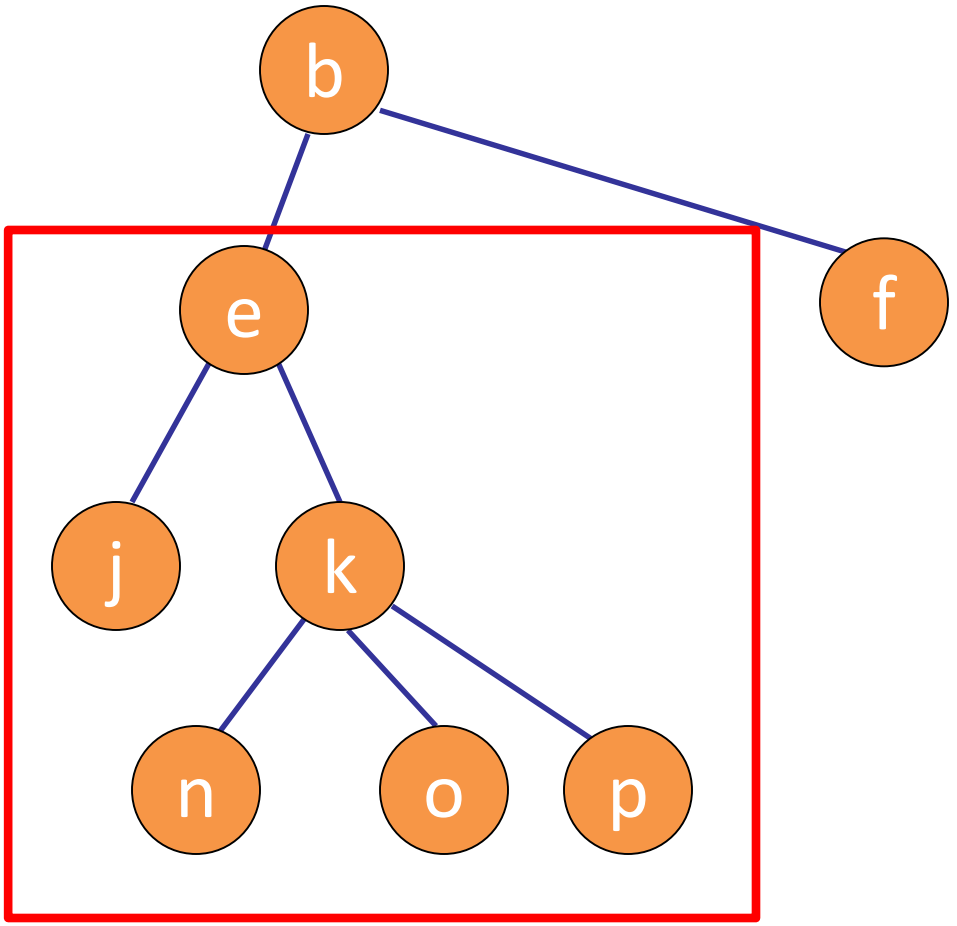



```
procedure postorder(T: ordered rooted tree)
  r := root of T
  for each child c of r from left to right
   begin
      T(c) := subtree with c as its root
      postorder(T(c))
   end
   visit r
```

- Step 1: visit T_1 in postorder,
- Step 2: visit T_2 in postorder,
- ……..
- Step k: visit T_k in postorder,
- Step k+1: visit root r

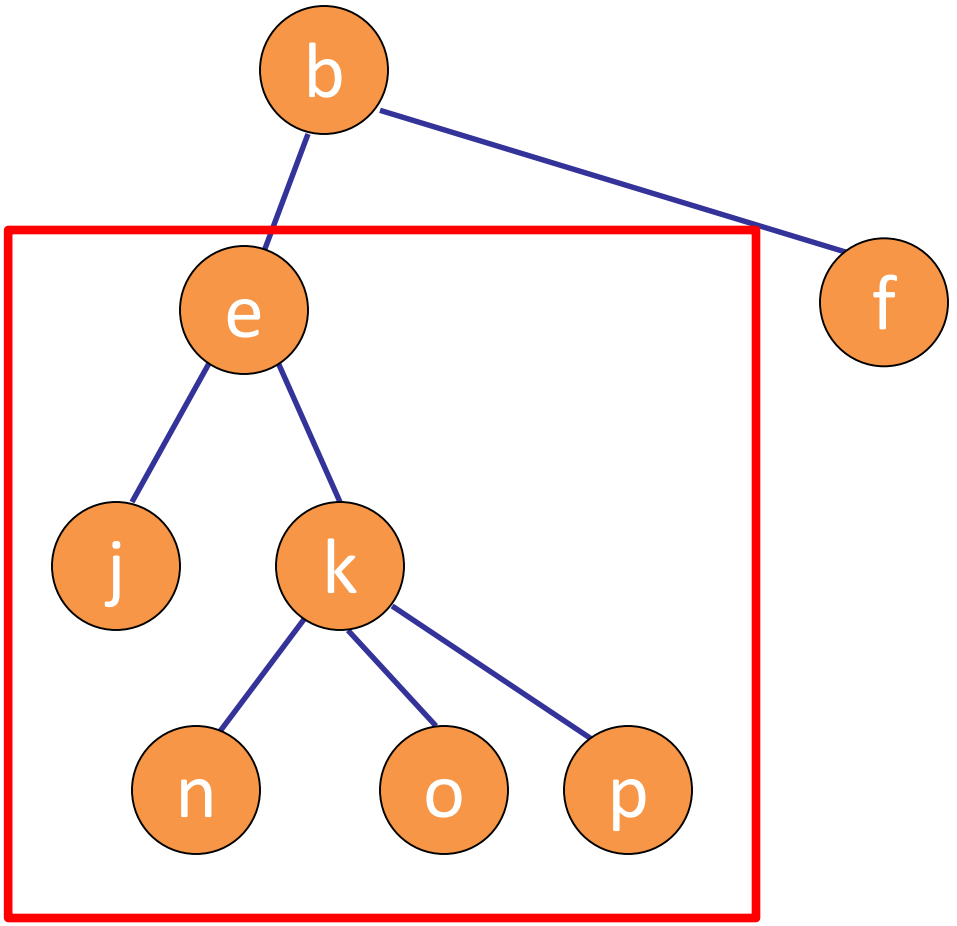
PostOrder

1. Visit leftmost subtree in Postorder
2. Visit remaining subtrees in Postorder
3. Visit root



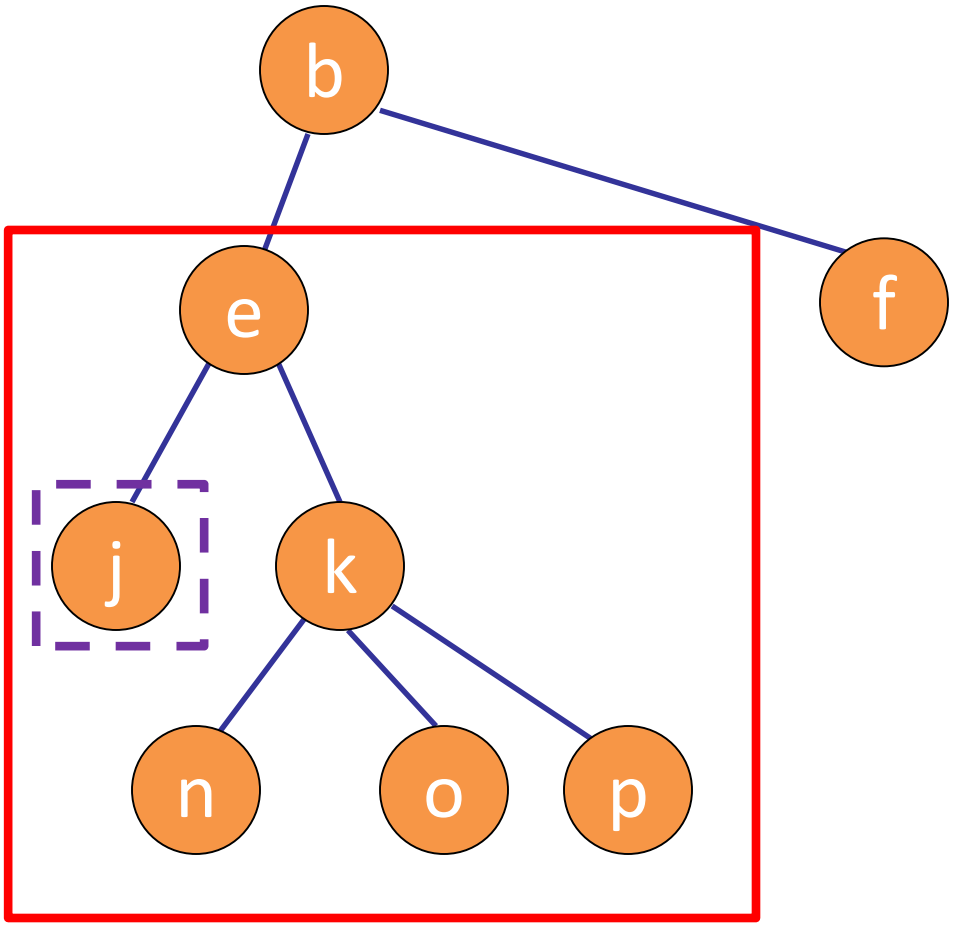
PostOrder

1. Visit leftmost subtree in Postorder
   1.1. Visit leftmost subtree in Postorder
   1.2. Visit remaining subtrees in Postorder
   1.3. Visit root
2. Visit remaining subtrees in Postorder
3. Visit root



PostOrder

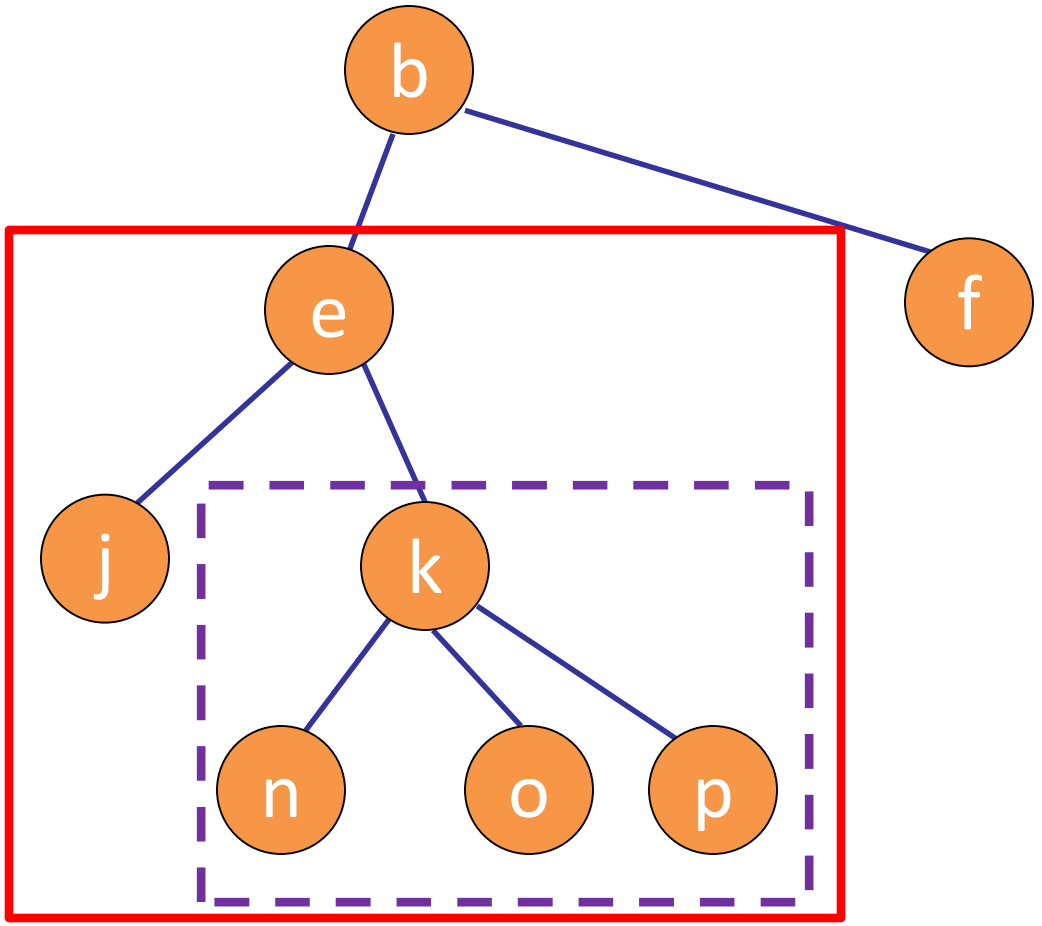
1. Visit leftmost subtree in Postorder
   1.1. Visit leftmost subtree in Postorder
   1.2. Visit remaining subtrees in Postorder
   1.3. Visit root
2. Visit remaining subtrees in Postorder
3. Visit root



j

PostOrder

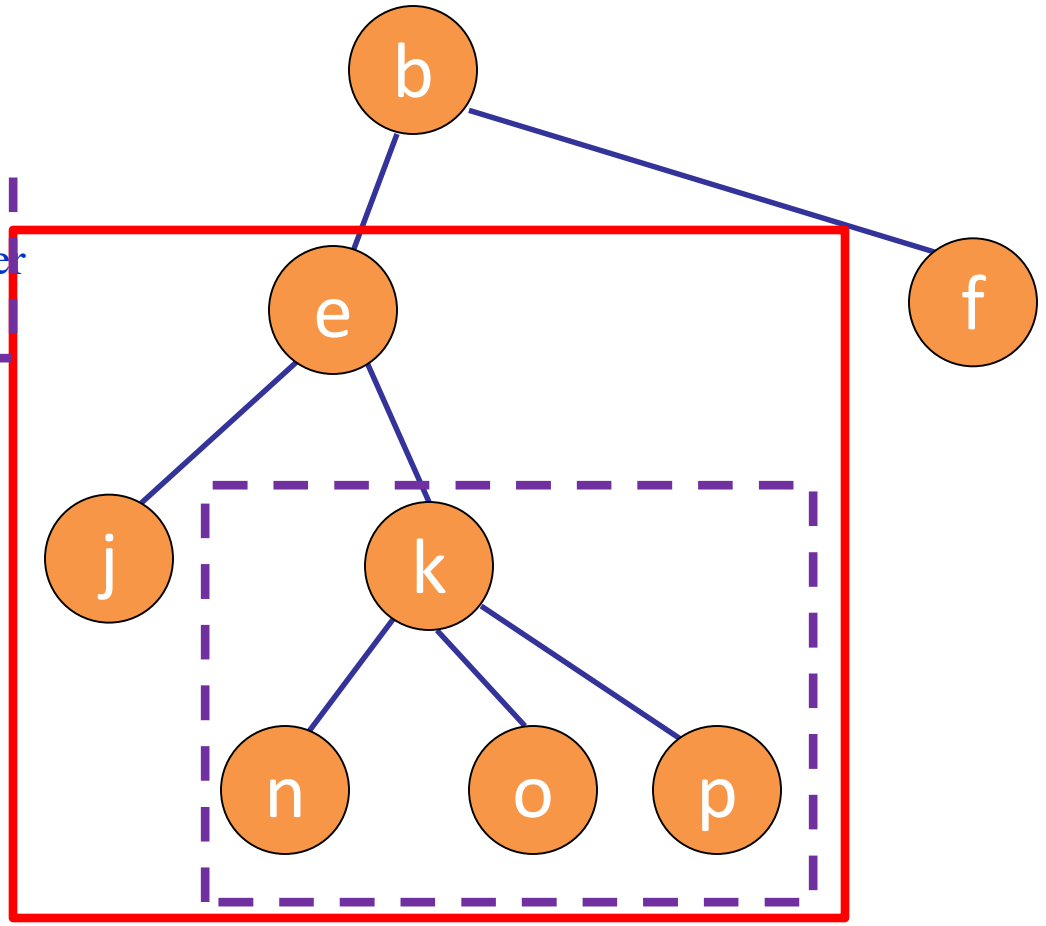
1. Visit leftmost subtree in Postorder
   1.1. Visit leftmost subtree in Postorder
   1.2. Visit remaining subtrees in Postorder
   1.3. Visit root
2. Visit remaining subtrees in Postorder
3. Visit root



j

PostOrder

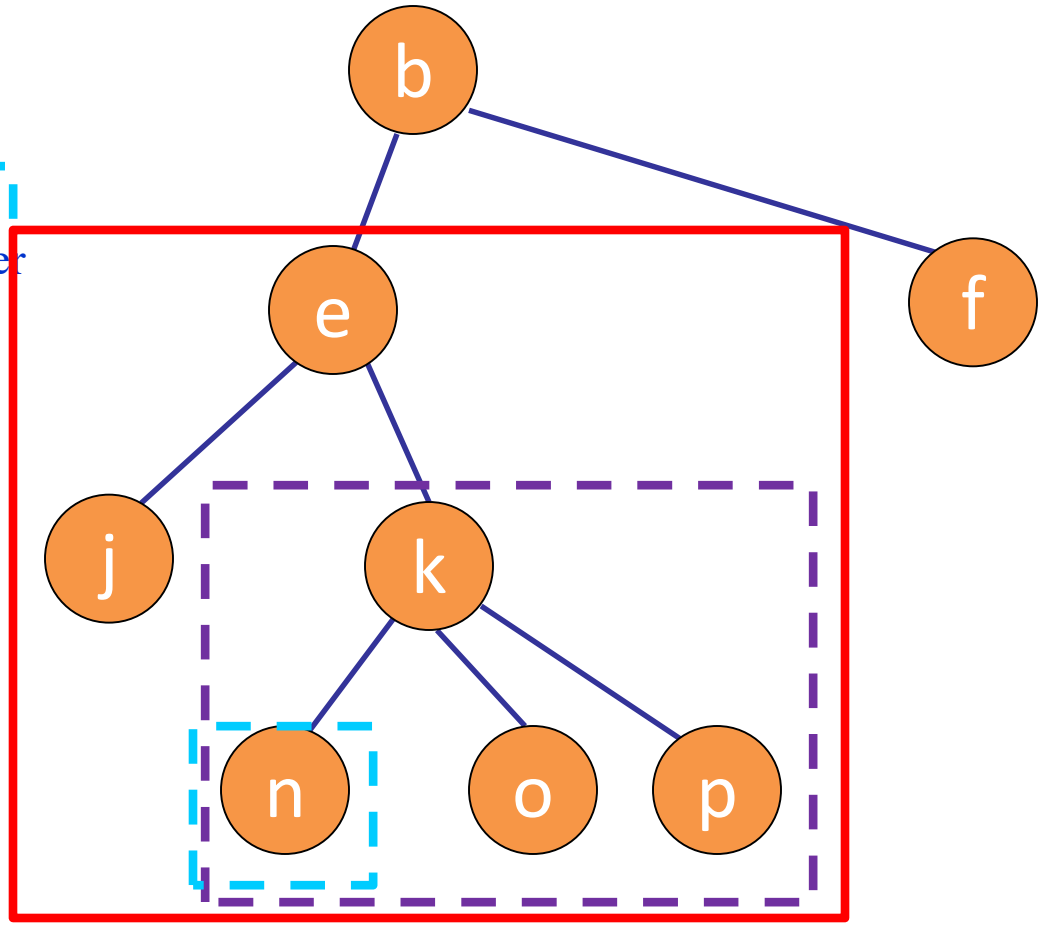
1. Visit leftmost subtree in Postorder
   1.1. Visit leftmost subtree in Postorder
   1.2. Visit remaining subtrees in Postorder
      1.2.1. Visit leftmost subtree in Postorder
      1.2.2. Visit remaining subtrees in Postorder
      1.2.3. Visit root
   1.3. Visit root
2. Visit remaining subtrees in Postorder
3. Visit root



j

PostOrder

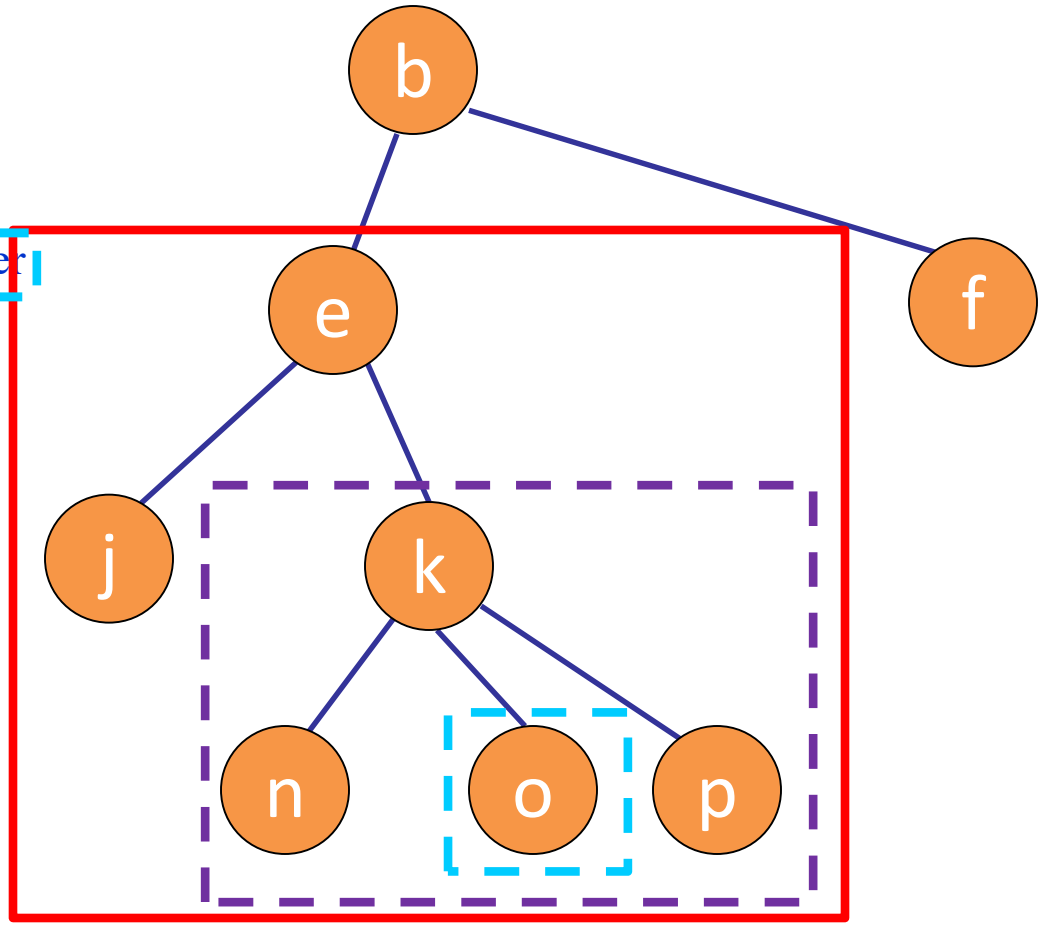
1. Visit leftmost subtree in Postorder
   1.1. Visit leftmost subtree in Postorder
   1.2. Visit remaining subtrees in Postorder
      1.2.1. Visit leftmost subtree in Postorder
      1.2.2. Visit remaining subtrees in Postorder
      1.2.3. Visit root
   1.3. Visit root
2. Visit remaining subtrees in Postorder
3. Visit root



j n

PostOrder

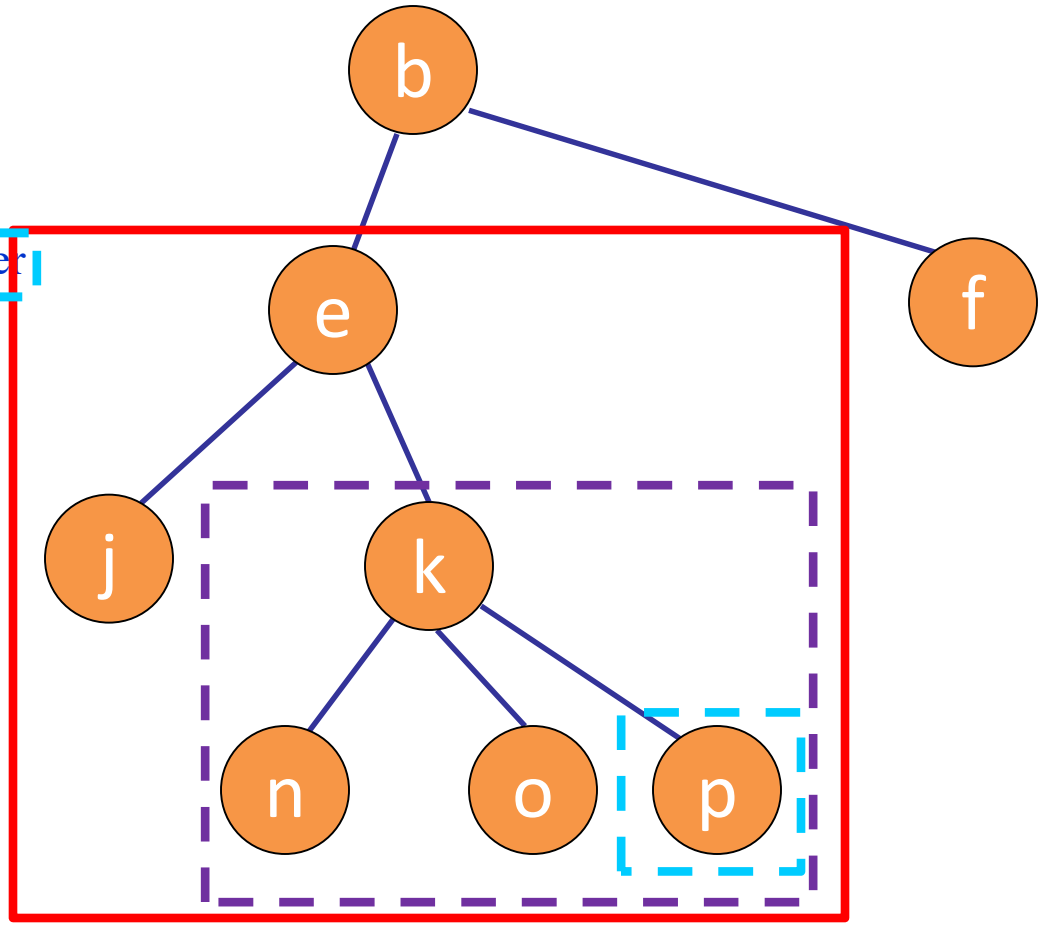
1. Visit leftmost subtree in Postorder
   1.1. Visit leftmost subtree in Postorder
   1.2. Visit remaining subtrees in Postorder
      1.2.1. Visit leftmost subtree in Postorder
      1.2.2. Visit remaining subtrees in Postorder
      1.2.3. Visit root
   1.3. Visit root
2. Visit remaining subtrees in Postorder
3. Visit root



j n o

PostOrder

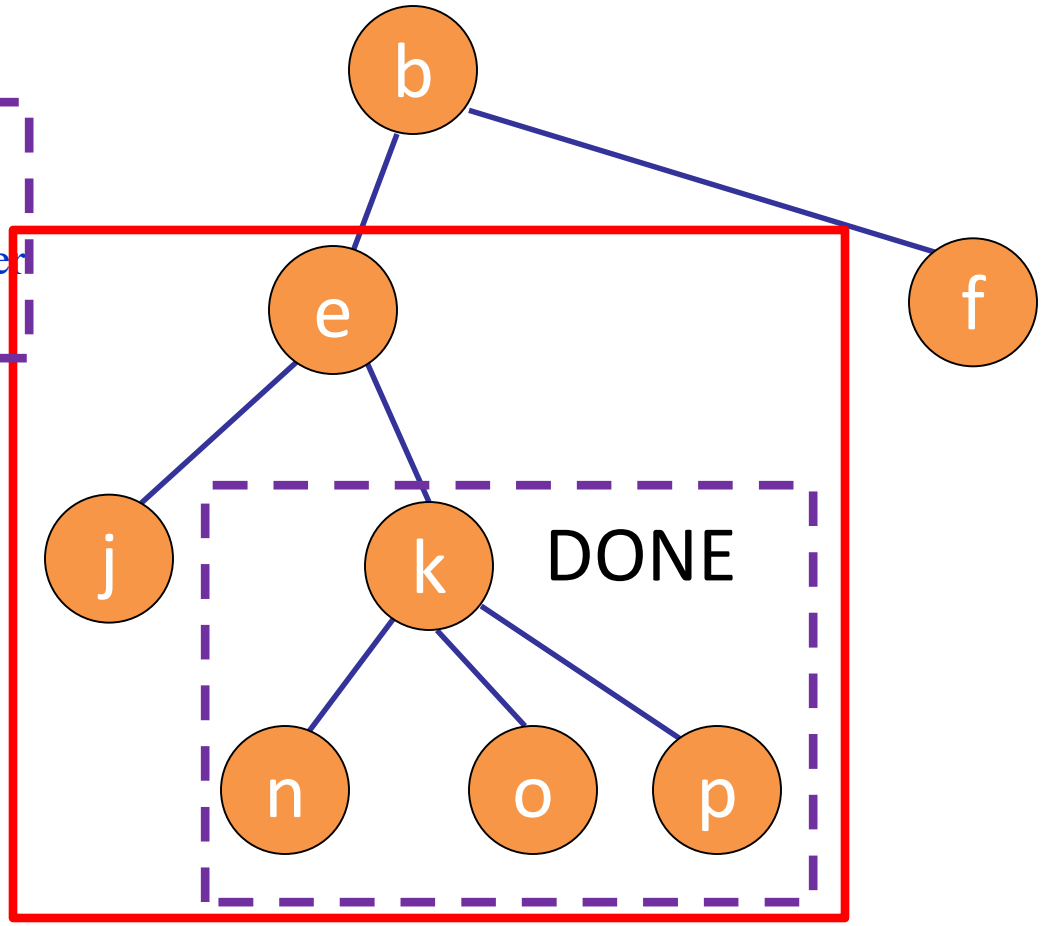
1. Visit leftmost subtree in Postorder
   1.1. Visit leftmost subtree in Postorder
   1.2. Visit remaining subtrees in Postorder
      1.2.1. Visit leftmost subtree in Postorder
      1.2.2. Visit remaining subtrees in Postorder
      1.2.3. Visit root
   1.3. Visit root
2. Visit remaining subtrees in Postorder
3. Visit root



j n o p

PostOrder

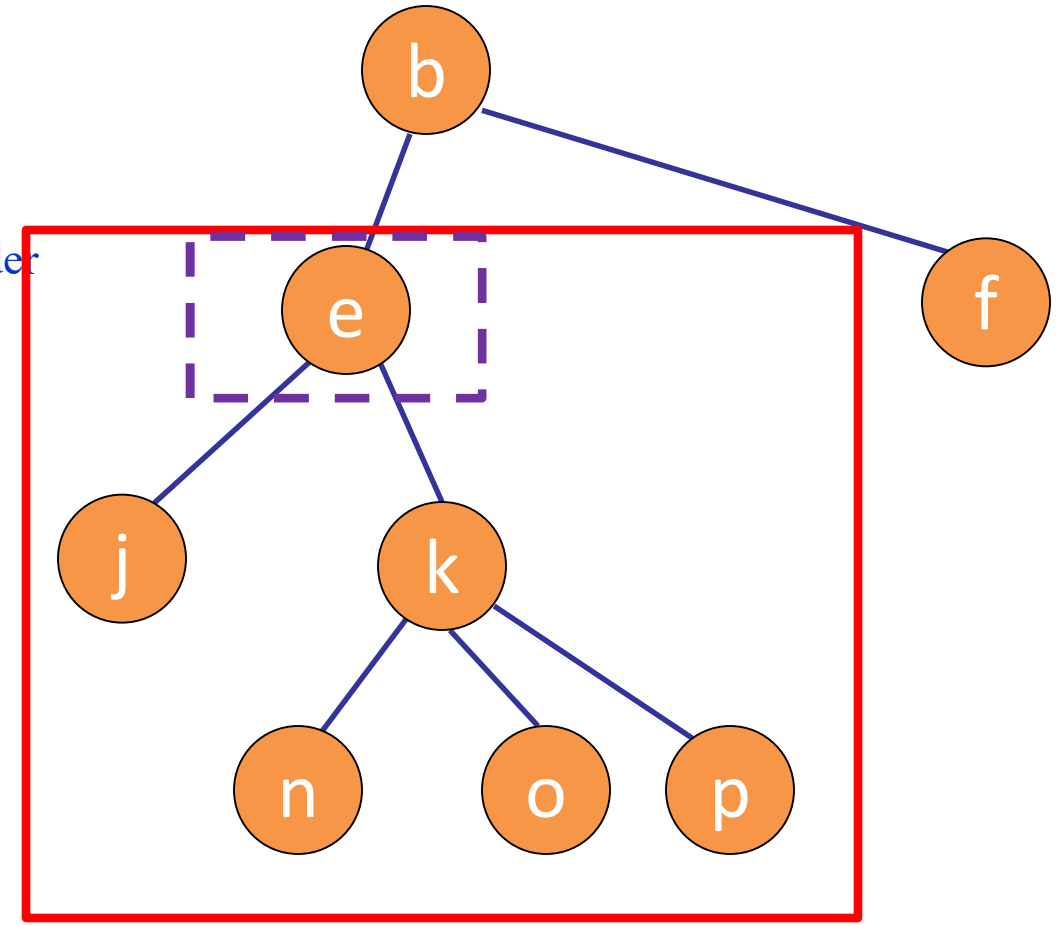
1. Visit leftmost subtree in Postorder
   1.1. Visit leftmost subtree in Postorder
   1.2. Visit remaining subtrees in Postorder
      1.2.1. Visit leftmost subtree in Postorder
      1.2.2. Visit remaining subtrees in Postorder
      1.2.3. Visit root
   1.3. Visit root
2. Visit remaining subtrees in Postorder
3. Visit root



j n o p k

PostOrder

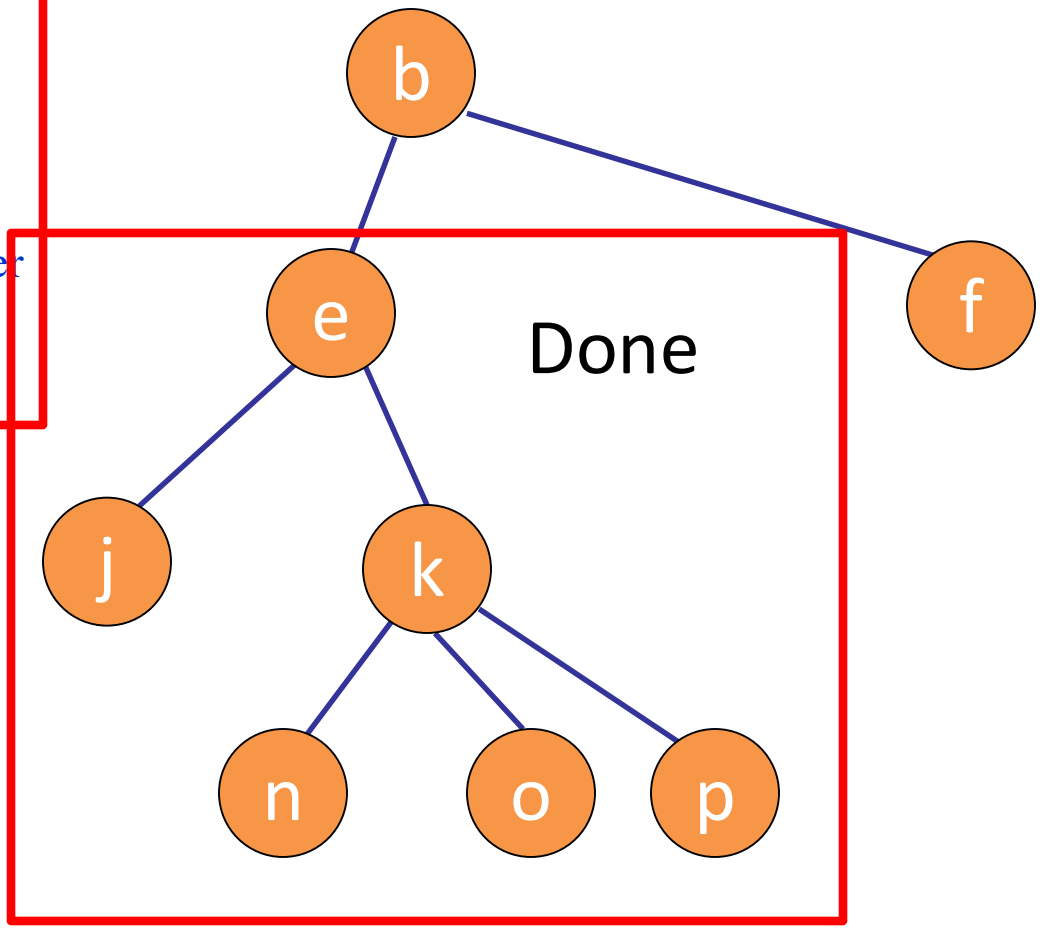
1. Visit leftmost subtree in Postorder
   1.1. Visit leftmost subtree in Postorder
   1.2. Visit remaining subtrees in Postorder
      1.2.1. Visit leftmost subtree in Postorder
      1.2.2. Visit remaining subtrees in Postorder
      1.2.3. Visit root
   1.3. Visit root
2. Visit remaining subtrees in Postorder
3. Visit root



j n o p k e

PostOrder

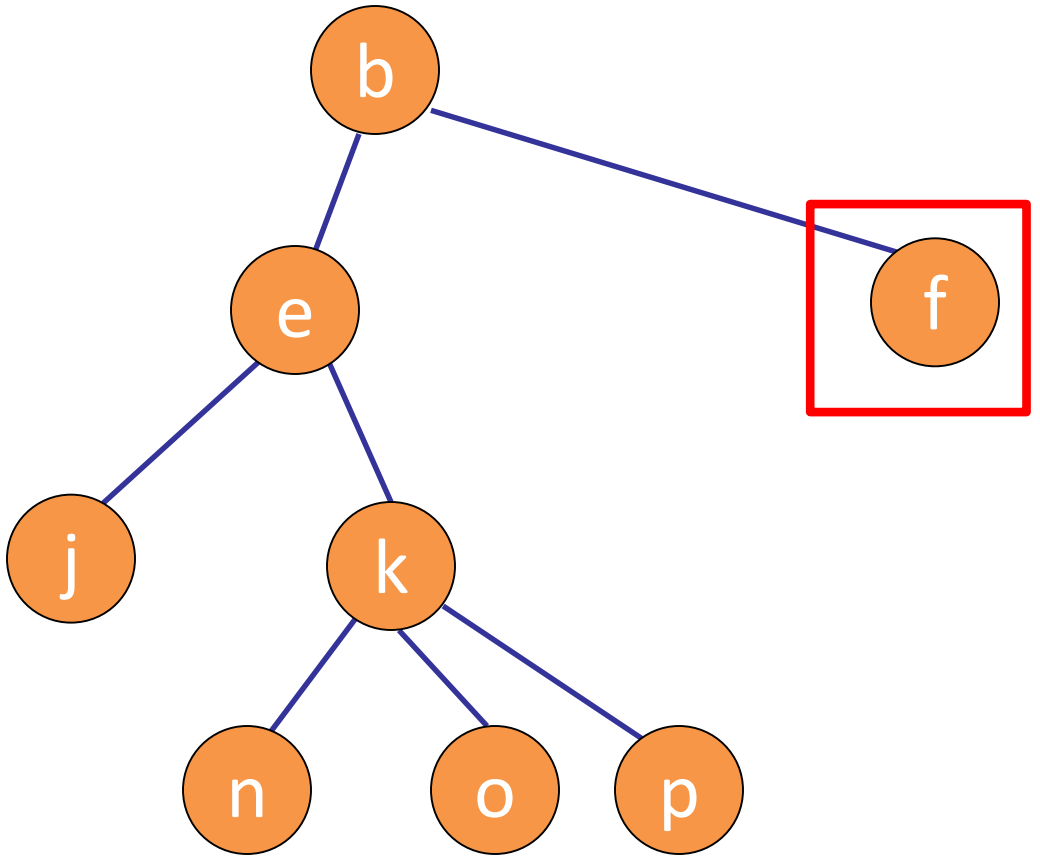
1. Visit leftmost subtree in Postorder
   1.1. Visit leftmost subtree in Postorder
   1.2. Visit remaining subtrees in Postorder
      1.2.1. Visit leftmost subtree in Postorder
      1.2.2. Visit remaining subtrees in Postorder
      1.2.3. Visit root
   1.3. Visit root
2. Visit remaining subtrees in Postorder
3. Visit root



j n o p k e

PostOrder

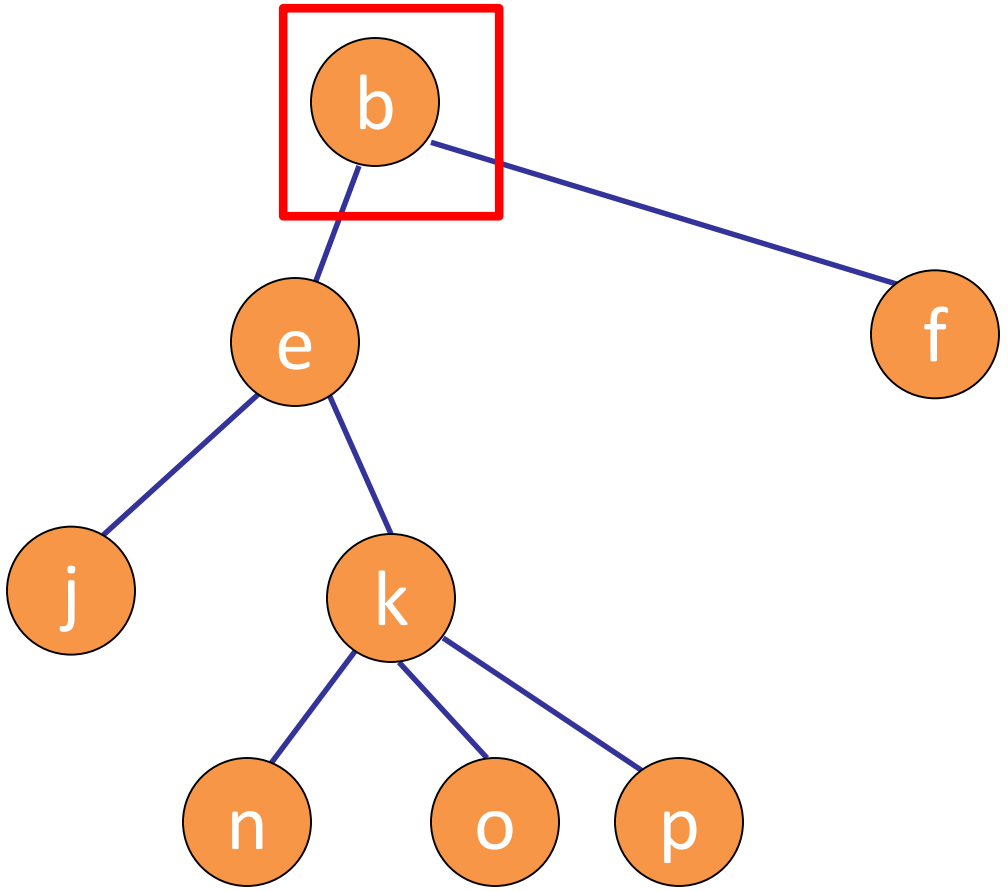
1. Visit leftmost subtree in Postorder
   1.1. Visit leftmost subtree in Postorder
   1.2. Visit remaining subtrees in Postorder
      1.2.1. Visit leftmost subtree in Postorder
      1.2.2. Visit remaining subtrees in Postorder
      1.2.3. Visit root
   1.3. Visit root
2. Visit remaining subtrees in Postorder
3. Visit root



j n o p k e f

PostOrder

1. Visit leftmost subtree in Postorder
   1.1. Visit leftmost subtree in Postorder
   1.2. Visit remaining subtrees in Postorder
      1.2.1. Visit leftmost subtree in Postorder
      1.2.2. Visit remaining subtrees in Postorder
      1.2.3. Visit root
   1.3. Visit root
2. Visit remaining subtrees in Postorder
3. Visit root

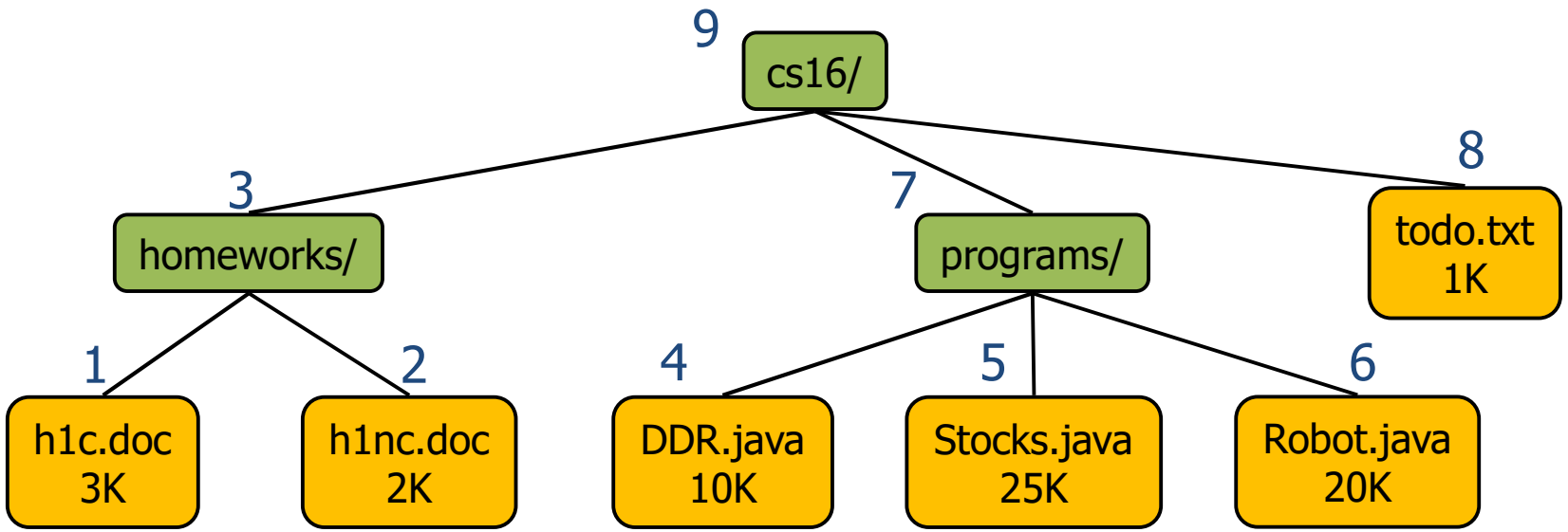


j n o p k e f b

Postorder Traversal

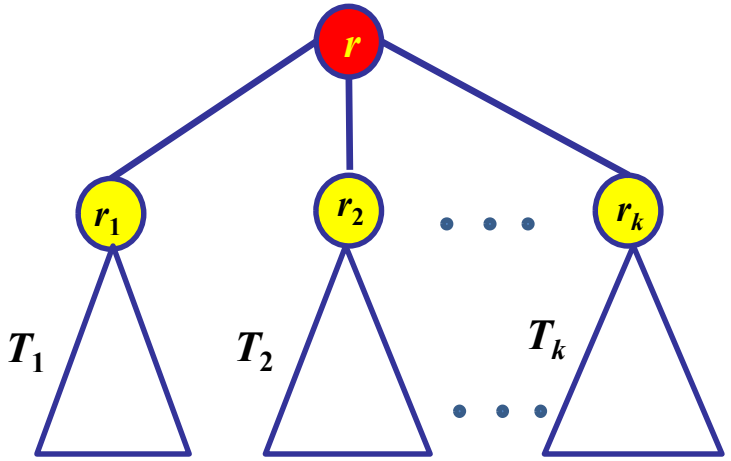
- Application: compute space used by files in a directory and its subdirectories

```
procedure postorder(T: ordered rooted tree)
  r := root of T
  for each child c of r from left to right
  begin
      T(c) := subtree with c as its root
      postorder(T(c))
  end
  visit r
```



Inorder Traversal

- Inorder traversal on tree T:

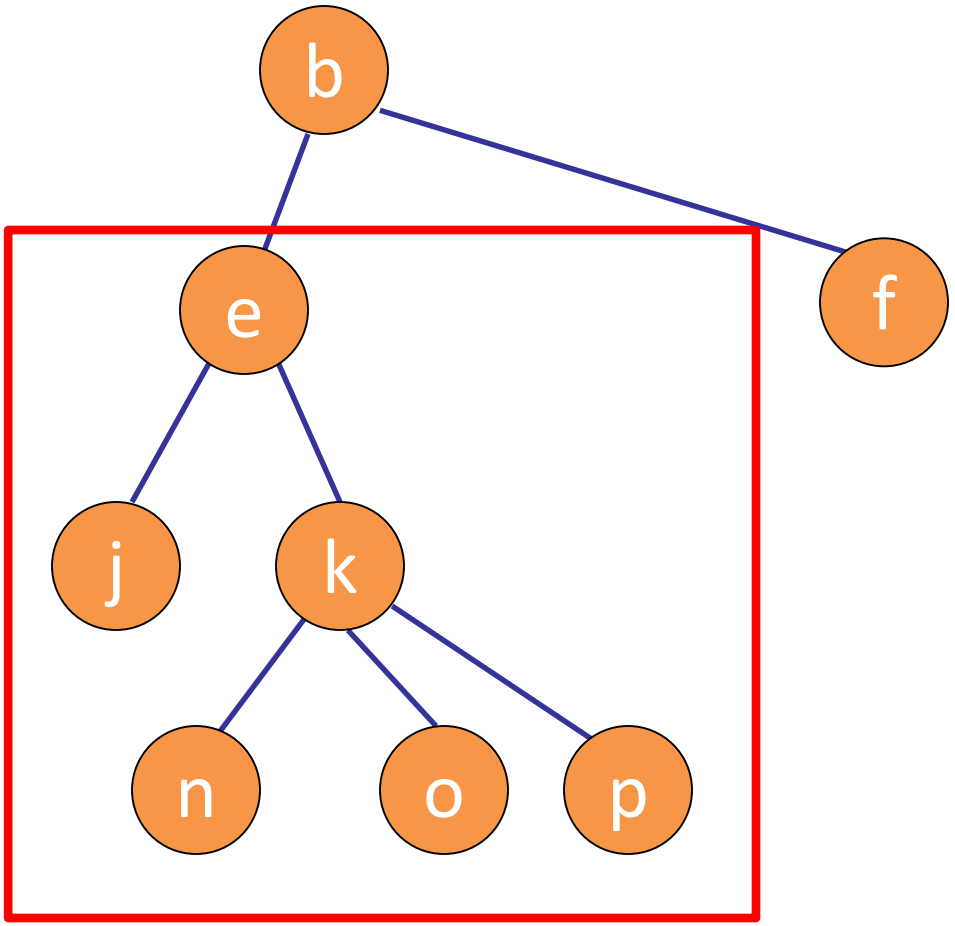


- Step 1: visit T_1 in inorder,
- Step 2: visit root ***r***,
- Step 3: visit T_2 in inorder,
- ……..
- Step k+1: visit T_k in inorder

```
procedure inorder(T: ordered rooted tree)
  r := root of T
  if r is a leaf then visit r
  else
  begin
     l := first child of r from left to right
     T(l) := subtree with l as its root
     inorder(T(l))        Step 1

     visit r              Step 2
     for each child c of r except for l left to right
        T(c) := subtree with c as its root
        inorder(T(c))     Step 3, 4..
  end
```

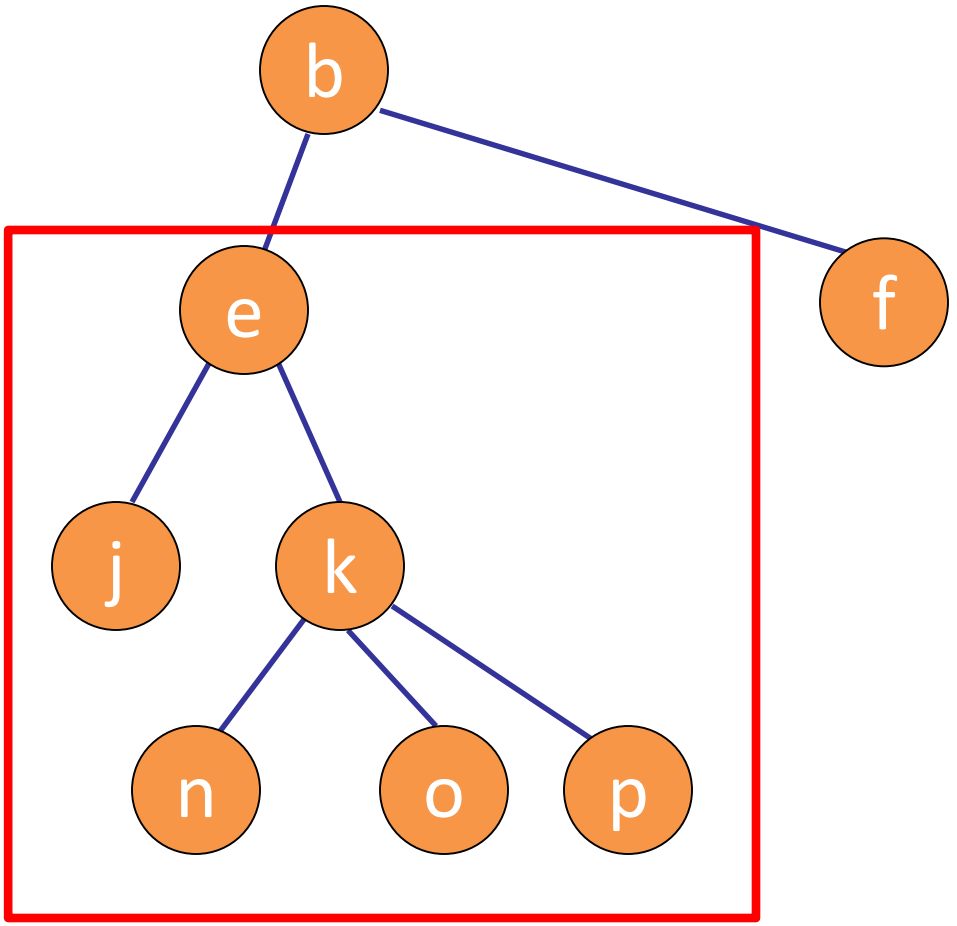
InOrder

1. Visit leftmost subtree in Inorder
2. Visit root
3. Visit remaining subtrees in Inorder



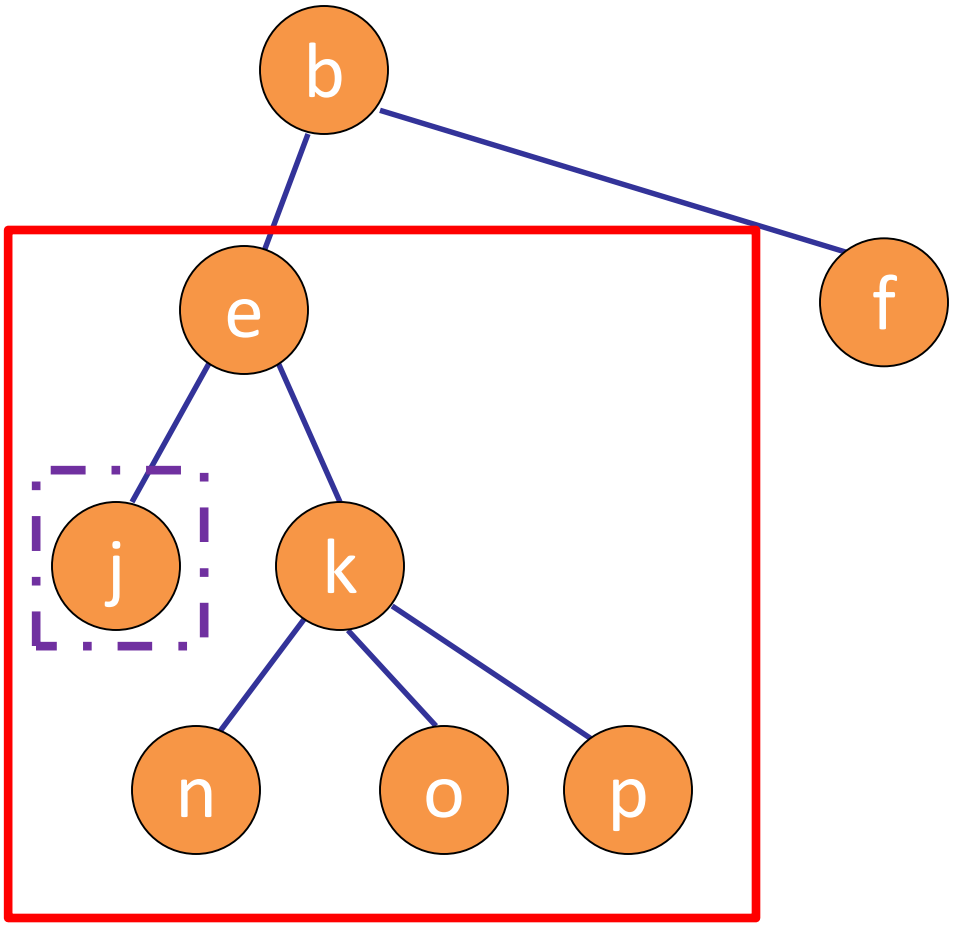
InOrder

1. Visit leftmost subtree in Inorder
   1.1. Visit leftmost subtree in Inorder
   1.2. Visit root
   1.3. Visit remaining subtrees in Inorder
2. Visit root
3. Visit remaining subtrees in Inorder



InOrder

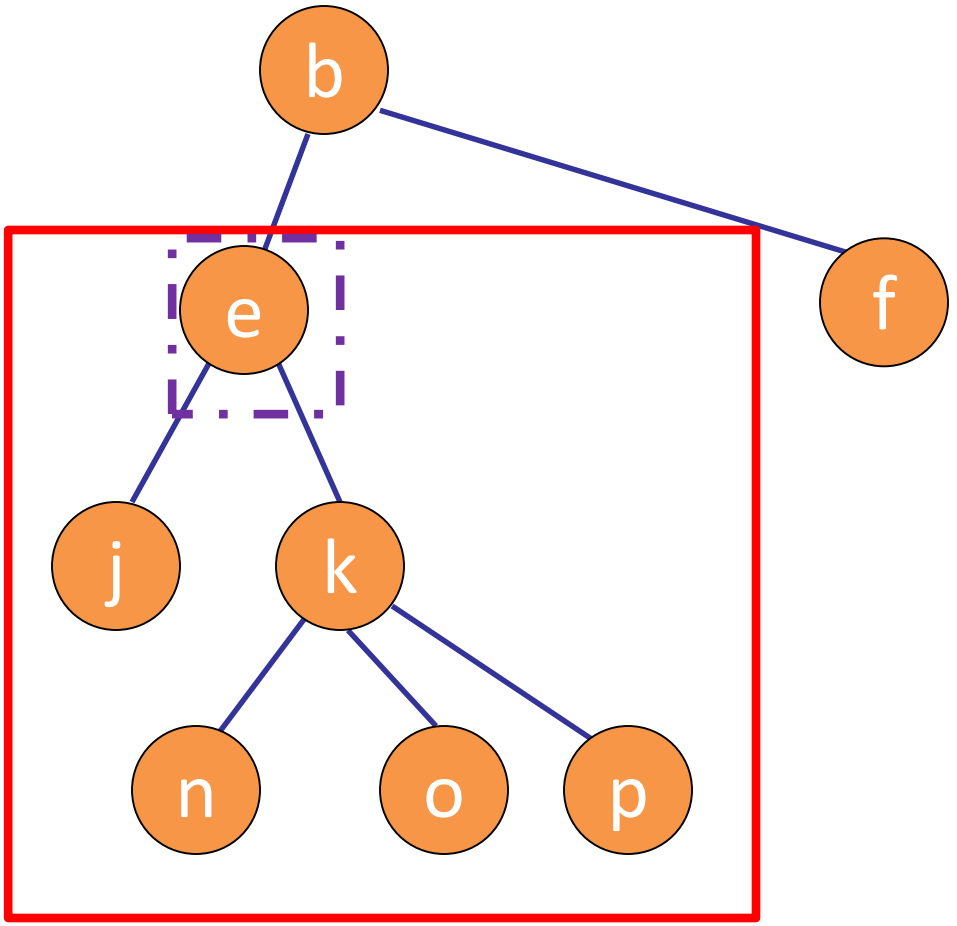
1. Visit leftmost subtree in Inorder
   1.1. Visit leftmost subtree in Inorder
   1.2. Visit root
   1.3. Visit remaining subtrees in Inorder
2. Visit root
3. Visit remaining subtrees in Inorder



j

InOrder

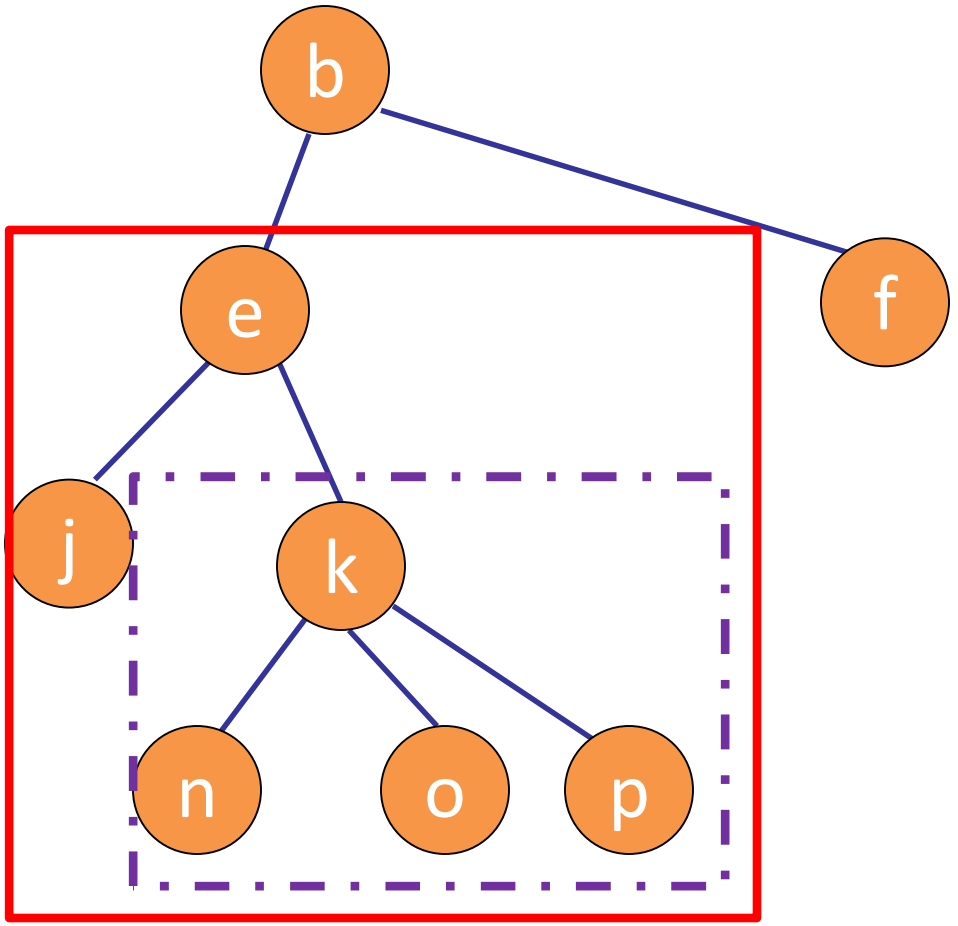
1. Visit leftmost subtree in Inorder
   1.1. Visit leftmost subtree in Inorder
   1.2. Visit root
   1.3. Visit remaining subtrees in Inorder
2. Visit root
3. Visit remaining subtrees in Inorder



j e

InOrder

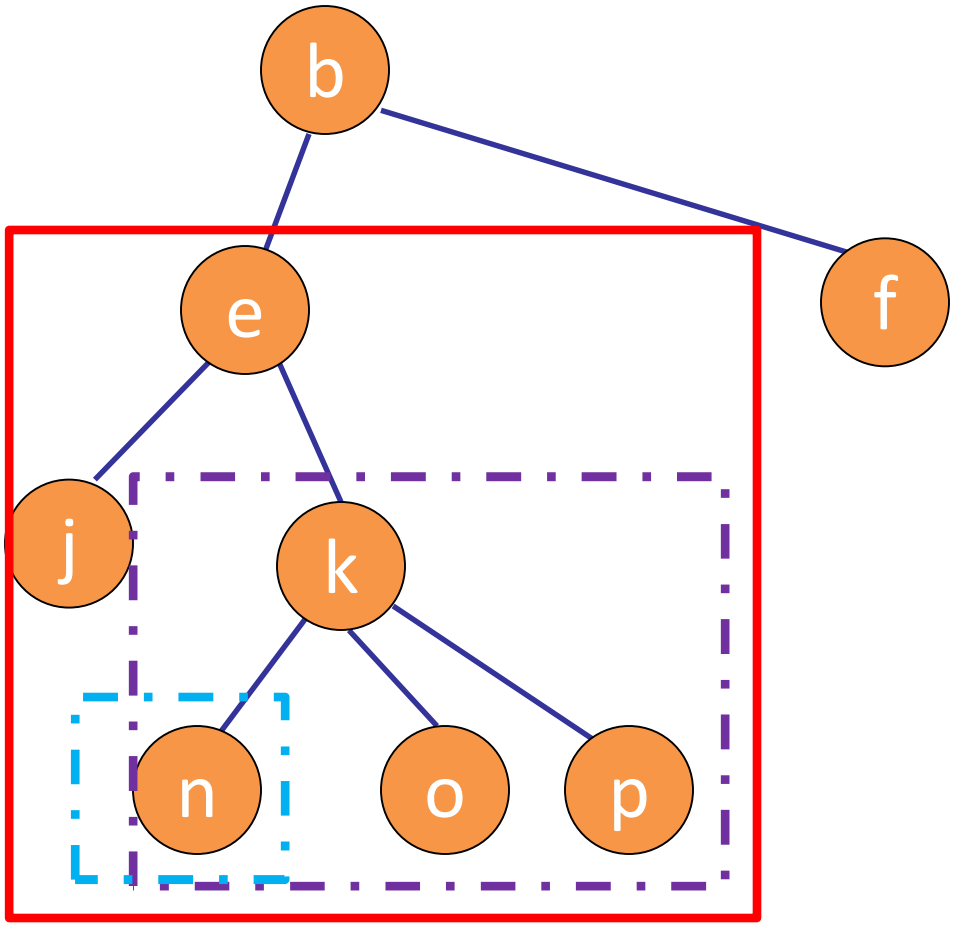
1. Visit leftmost subtree in Inorder
   1.1. Visit leftmost subtree in Inorder
   1.2. Visit root
   1.3. Visit remaining subtrees in Inorder
2. Visit root
3. Visit remaining subtrees in Inorder



j e

InOrder

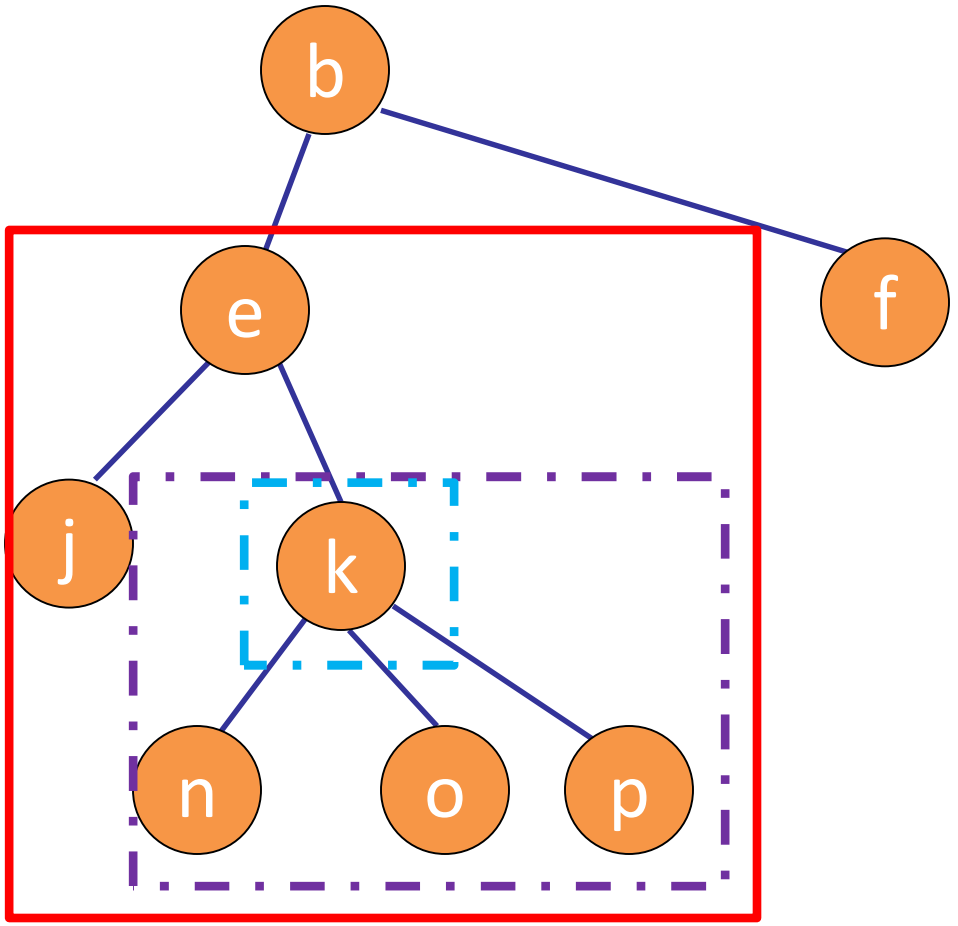
1. Visit leftmost subtree in Inorder
   1.1. Visit leftmost subtree in Inorder
   1.2. Visit root
   1.3. Visit remaining subtrees in Inorder
      1.3.1. Visit leftmost subtree in Inorder
      1.3.2. Visit root
      1.3.3. Visit remaining subtrees in Inorder
2. Visit root
3. Visit remaining subtrees in Inorder



j e n

InOrder

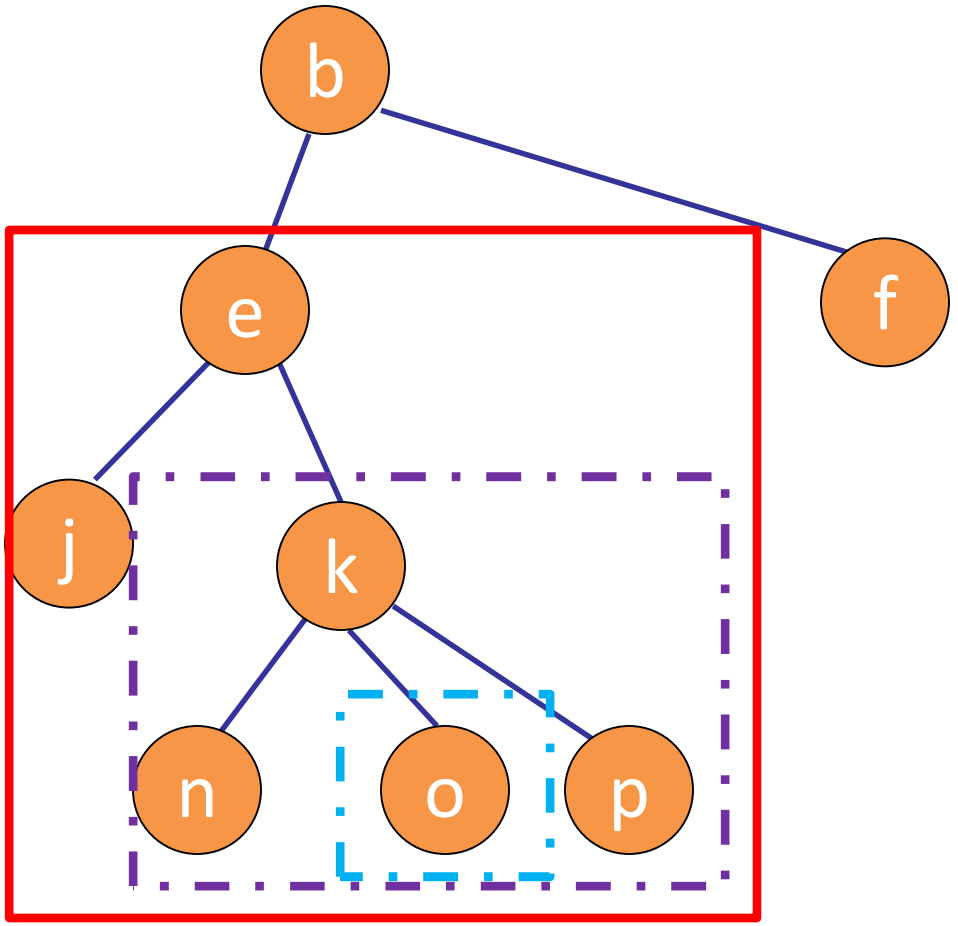
1. Visit leftmost subtree in Inorder
   1.1. Visit leftmost subtree in Inorder
   1.2. Visit root
   1.3. Visit remaining subtrees in Inorder
      1.3.1. Visit leftmost subtree in Inorder
      1.3.2. Visit root
      1.3.3. Visit remaining subtrees in Inorder
2. Visit root
3. Visit remaining subtrees in Inorder



j e n k

InOrder

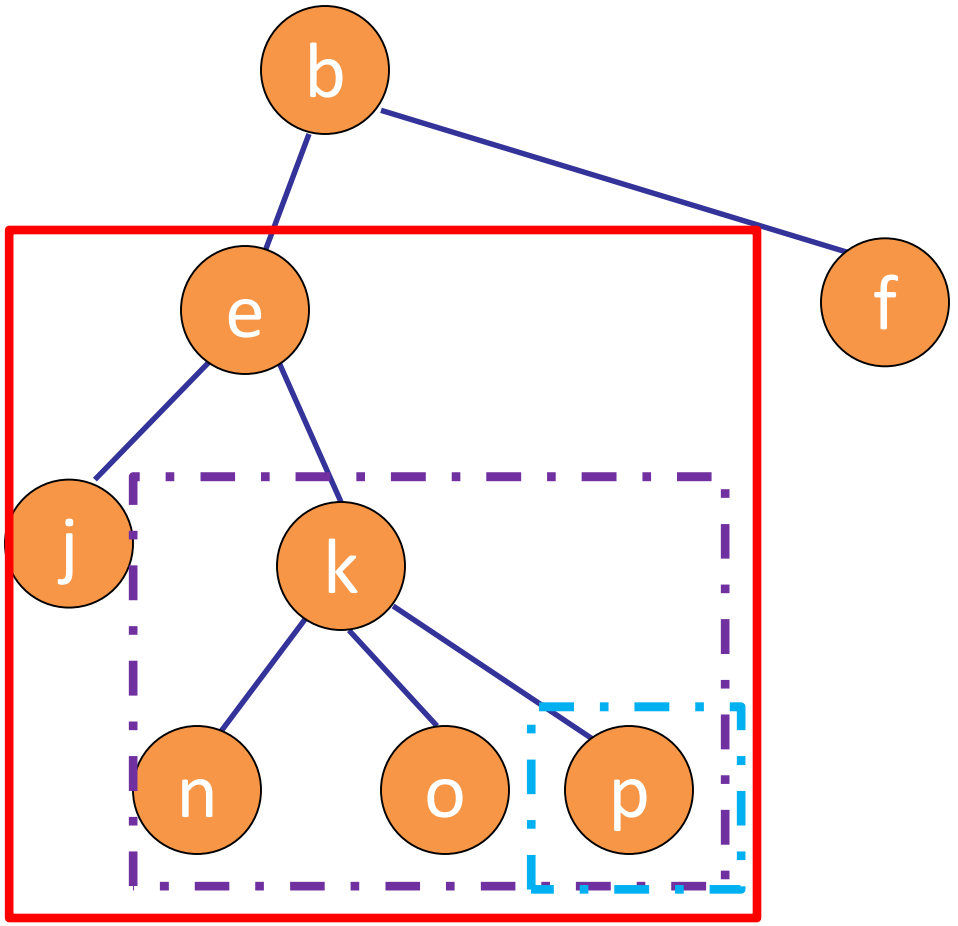
1. Visit leftmost subtree in Inorder
   1.1. Visit leftmost subtree in Inorder
   1.2. Visit root
   1.3. Visit remaining subtrees in Inorder
      1.3.1. Visit leftmost subtree in Inorder
      1.3.2. Visit root
      1.3.3. Visit remaining subtrees in Inorder
2. Visit root
3. Visit remaining subtrees in Inorder



j e n k o

InOrder

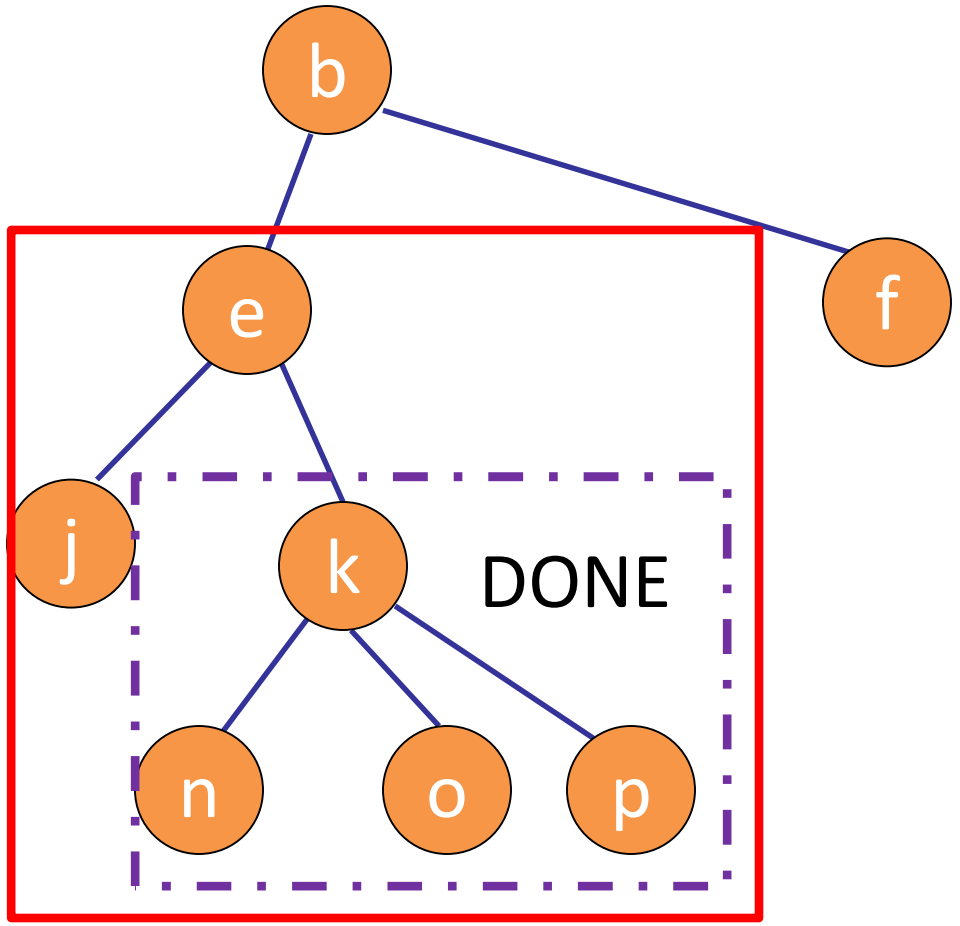
1. Visit leftmost subtree in Inorder
   1.1. Visit leftmost subtree in Inorder
   1.2. Visit root
   1.3. Visit remaining subtrees in Inorder
      1.3.1. Visit leftmost subtree in Inorder
      1.3.2. Visit root
      1.3.3. Visit remaining subtrees in Inorder
2. Visit root
3. Visit remaining subtrees in Inorder



j e n k o p

InOrder

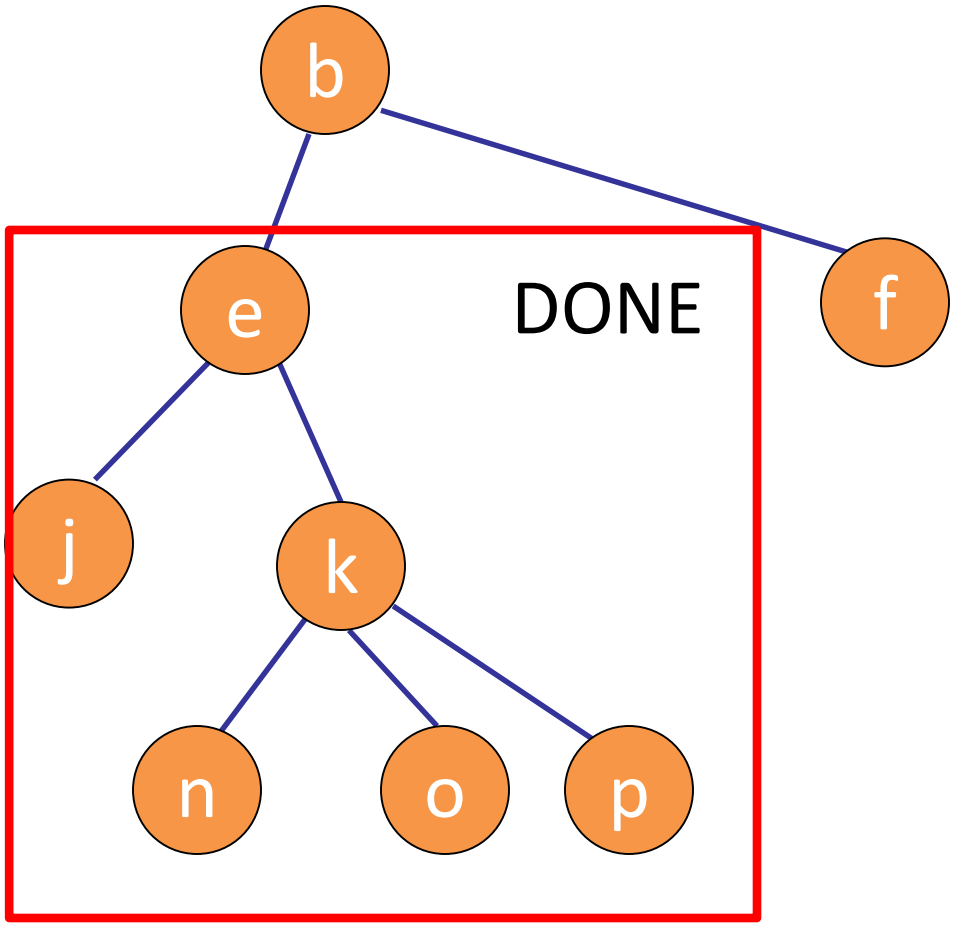
1. Visit leftmost subtree in Inorder
   1.1. Visit leftmost subtree in Inorder
   1.2. Visit root
   1.3. Visit remaining subtrees in Inorder
      1.3.1. Visit leftmost subtree in Inorder
      1.3.2. Visit root
      1.3.3. Visit remaining subtrees in Inorder
2. Visit root
3. Visit remaining subtrees in Inorder



j e n k o p

InOrder

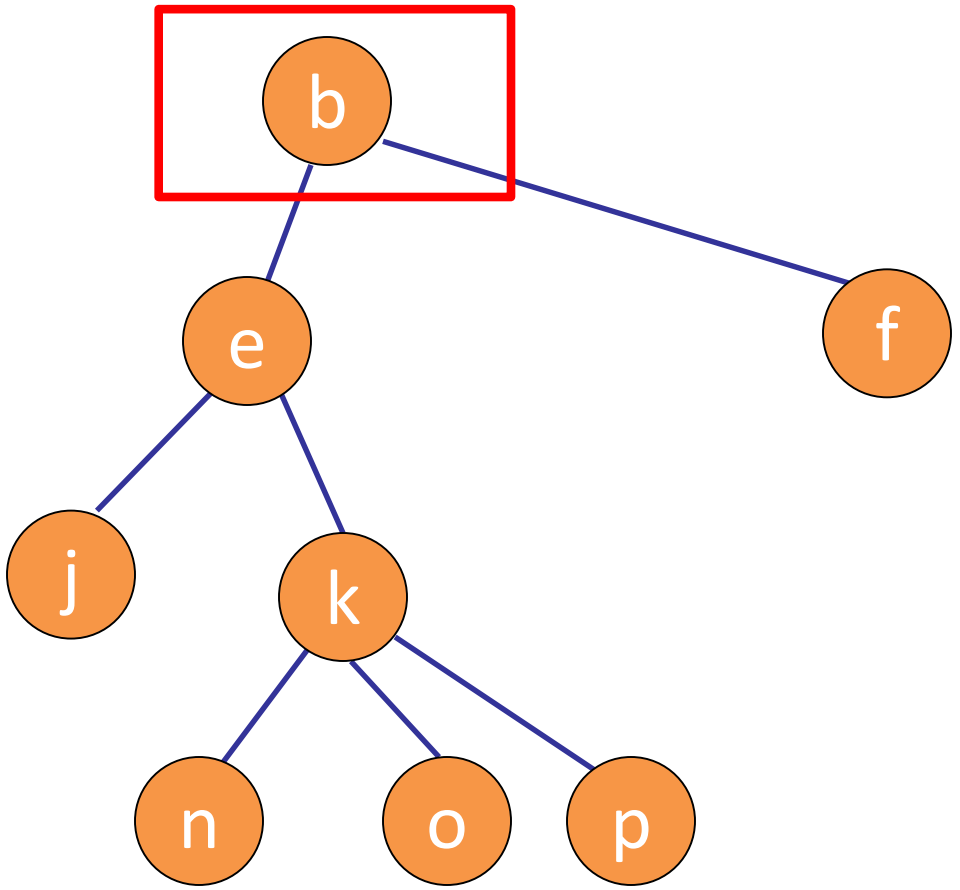
1. Visit leftmost subtree in Inorder
   1.1. Visit leftmost subtree in Inorder
   1.2. Visit root
   1.3. Visit remaining subtrees in Inorder
      1.3.1. Visit leftmost subtree in Inorder
      1.3.2. Visit root
      1.3.3. Visit remaining subtrees in Inorder
2. Visit root
3. Visit remaining subtrees in Inorder



j e n k o p

InOrder

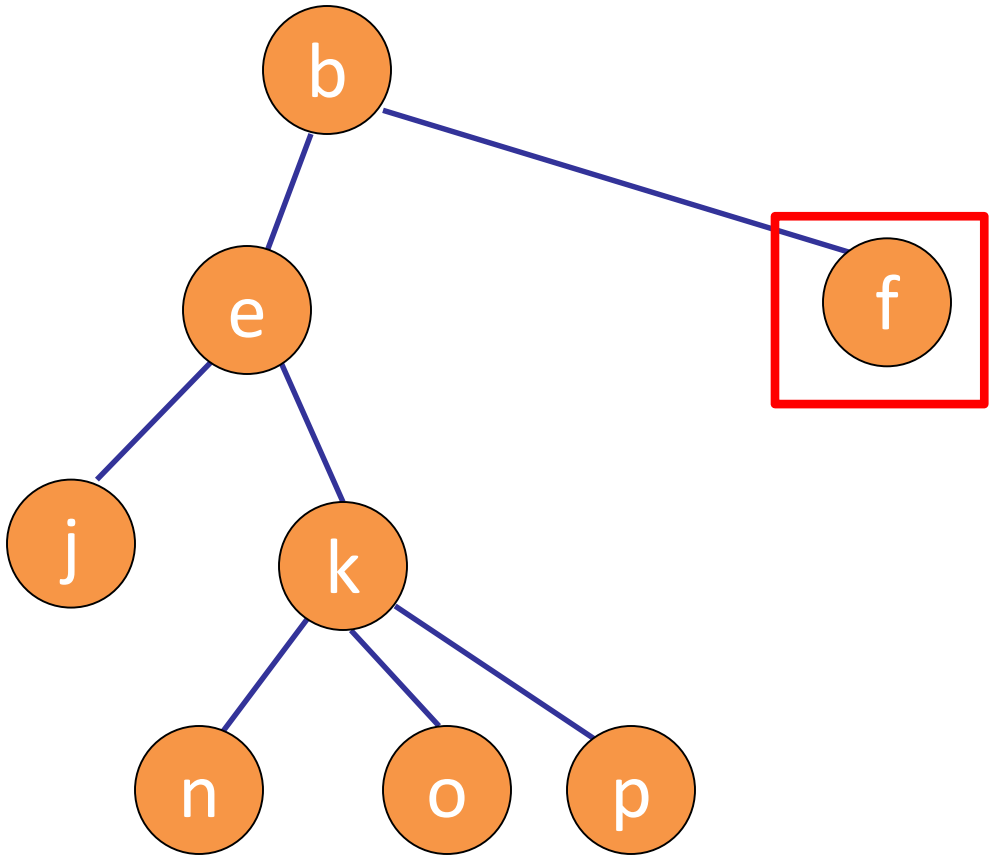
1. Visit leftmost subtree in Inorder
   1.1. Visit leftmost subtree in Inorder
   1.2. Visit root
   1.3. Visit remaining subtrees in Inorder
      1.3.1. Visit leftmost subtree in Inorder
      1.3.2. Visit root
      1.3.3. Visit remaining subtrees in Inorder
2. Visit root
3. Visit remaining subtrees in Inorder



j e n k o p b

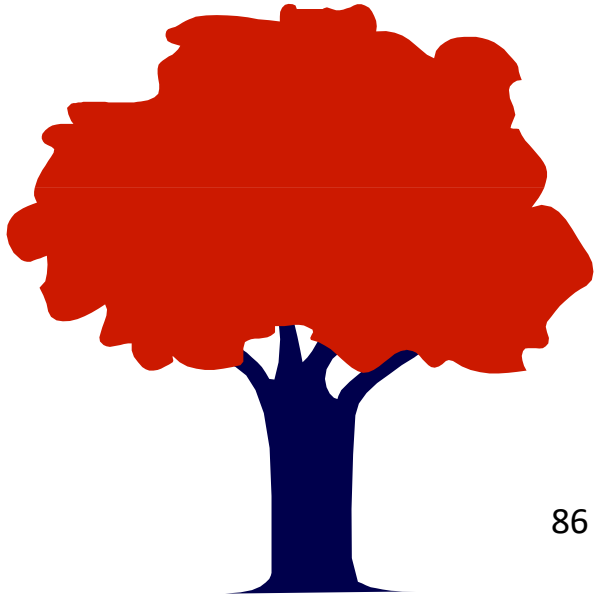
InOrder

1. Visit leftmost subtree in Inorder
   1.1. Visit leftmost subtree in Inorder
   1.2. Visit root
   1.3. Visit remaining subtrees in Inorder
      1.3.1. Visit leftmost subtree in Inorder
      1.3.2. Visit root
      1.3.3. Visit remaining subtrees in Inorder
2. Visit root
3. Visit remaining subtrees in Inorder



j e n k o p b f

Contents



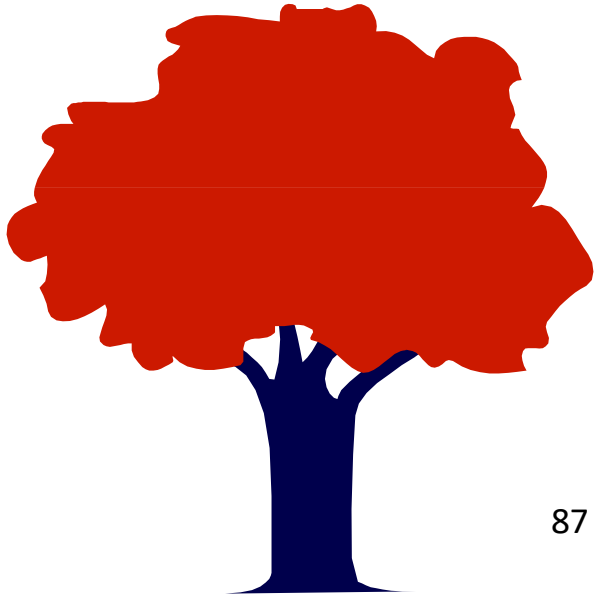
4.4. Binary tree

4.4.1. Definitions

4.4.2. Binary tree representation

4.4.3. Binary tree traversal

4.4.4. Some applications



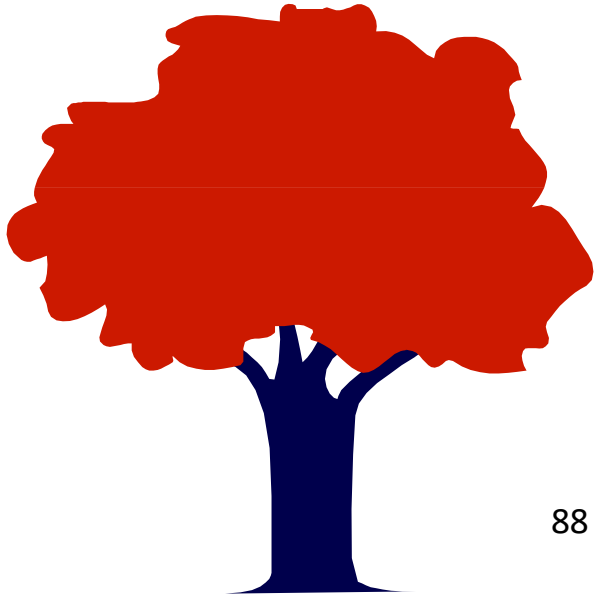
4.4. Binary tree

4.4.1. Definitions

4.4.2. Binary tree representation

4.4.3. Binary tree traversal

4.4.4. Some applications

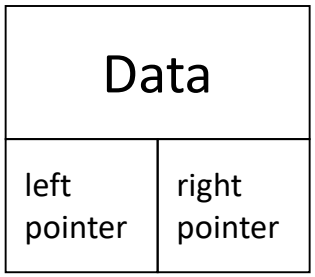


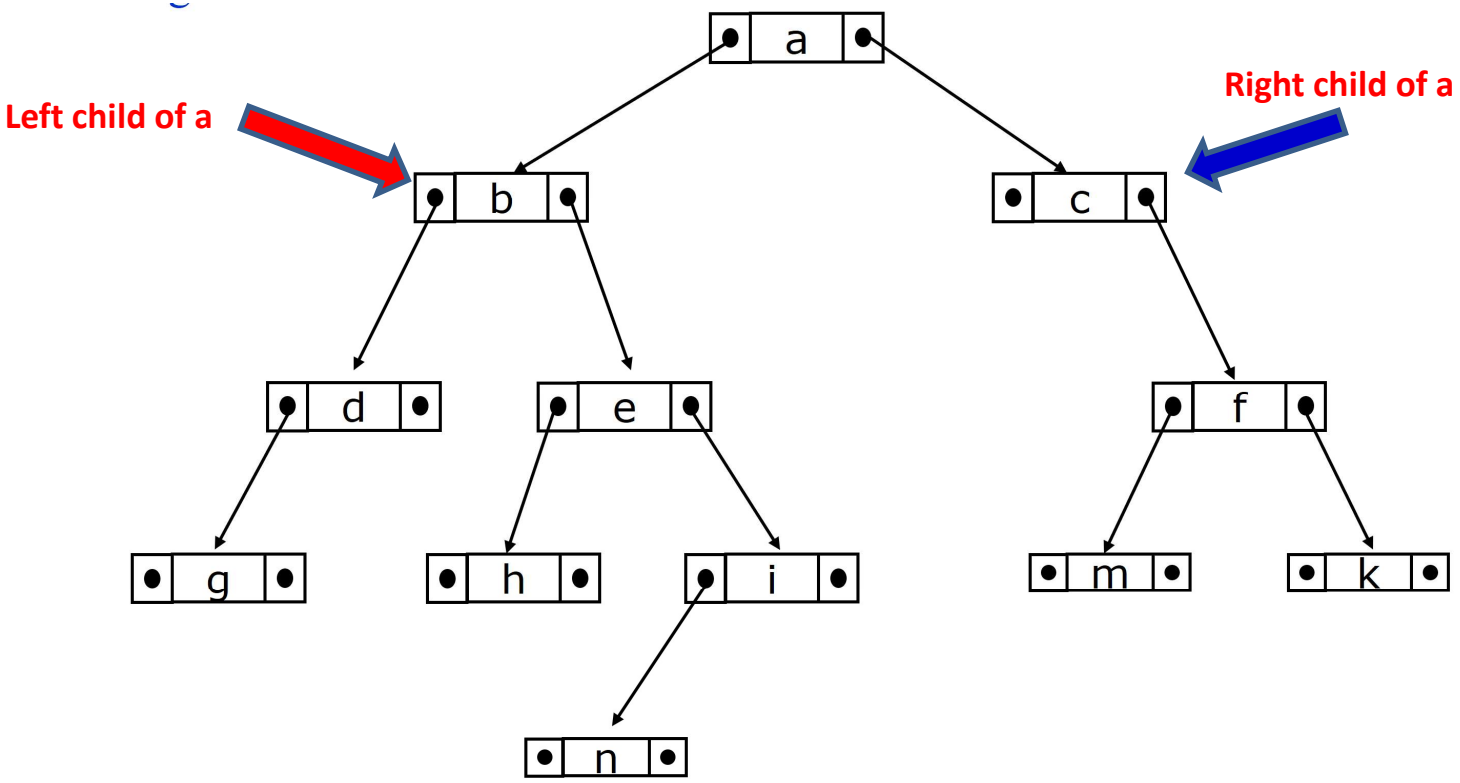
4.4.1. Definitions: Binary tree

- A binary tree is a tree in which no node can have more than two children.

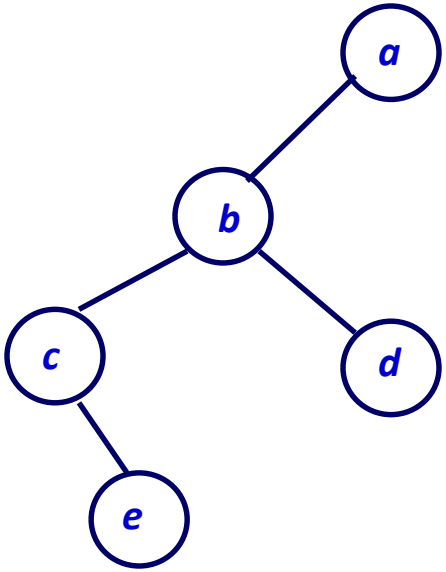
➔ Each node could either: (1) has no child, (2) has only left child, (3) has only right child, (4) has both left child and right child

- Each node has the data, a reference to a left child and a reference to a right child.

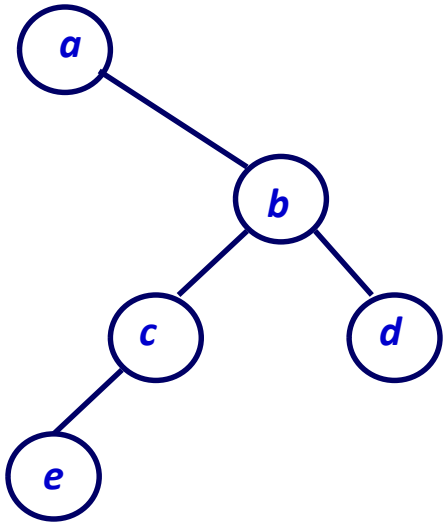




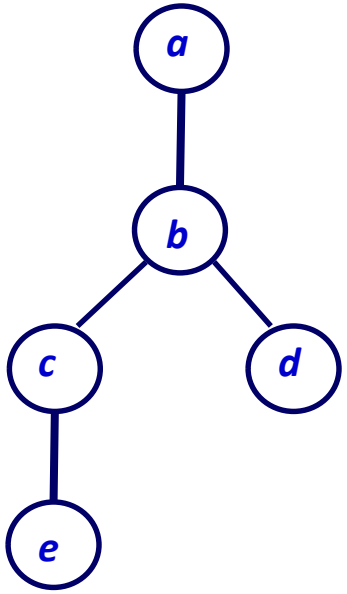
Note



Binary tree T_1



Binary tree T_2



General tree

$$T_1 \neq T_2$$

Given a binary tree of *n* nodes

height(tree) = depth(tree)

= MAX {depth(leaf)} = height(root)

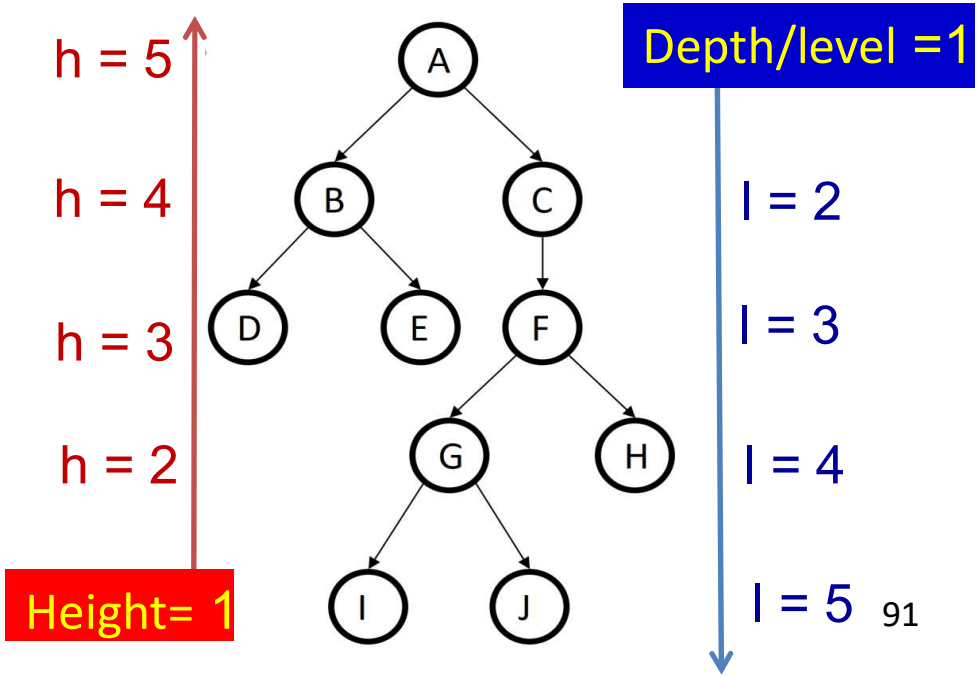
Properties:

1) max number of nodes on level $i = 2^{i-1}$ $(i \geq 1)$

   max number of leaves on tree = $2^{depth(tree)-1}$

2) max number of nodes on tree= $2^{depth(tree)} - 1$

3) Binary tree with *n* nodes: depth(tree) $\geq \lceil \log_2(n) \rceil$



Given a binary tree of *n* nodes

Properties:

1) max number of nodes on level $i = 2^{i-1}$ ($i \geq 1$) : On level i of a binary tree, there are maximum 2^{i-1} nodes

Proof: Induction.

- ***Base case:*** Root is the only node on level i=1. So the maximum number of nodes on level i=1 is $2^0 = 2^{i-1}$.
- ***Inductive step:*** Assume the property is true with every value of j, $1 \leq j < i$: it means that the maximum number of nodes on level j is 2^{j-1}.
  - As the maximum number of nodes on level i-1 is 2^{i-2}, and on the other hand, by definition of binary tree, each node has <=2 child

➔The maximum number of nodes on level $i \leq 2 *$ max nodes on level i-1

$2 * 2^{i-2} = 2^{i-1}$

Given a binary tree of *n* nodes

Properties:

2) max number of nodes on tree = $2^{depth(tree)} - 1$: on binary tree with the height k, there are maximum number of node = 2^k-1, $k \geq 1$.

Proof:

Use 1) we have: Level i of binary tree has max number of nodes = 2^{i-1} ($i \geq 1$)

Binary tree with the height k ➔ there are k levels

➔ The maximum number of nodes on this tree is

$$\sum_{i=1}^{k} 2^{i-1} = 2^k - 1.$$

Given a binary tree of *n* nodes

Properties:

3) The binary tree with n nodes: depth(tree) $\geq \lceil \log_2(n+1) \rceil$

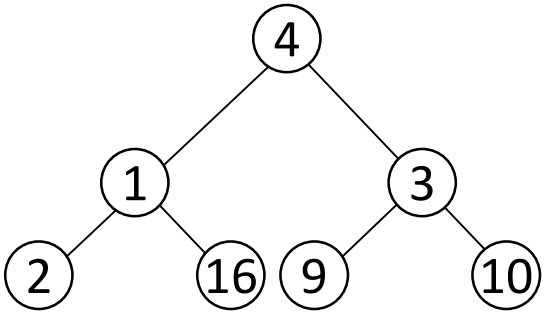
Proof:

- Binary tree with n nodes has the minimum height k only when the number of nodes on each level $i = 1, 2, ..., k$ is as max as possible. Therefore, we have:

$$n = \sum_{i=1}^{k} 2^{i-1} = 2^k - 1, \text{ so we have } 2^k = n+1, \text{ it means } k = \lceil \log_2(n+1) \rceil.$$

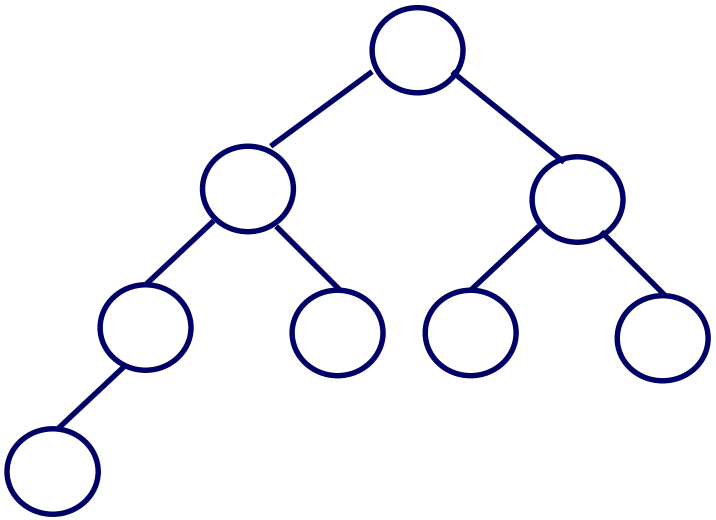
Full binary tree vs complete binary tree

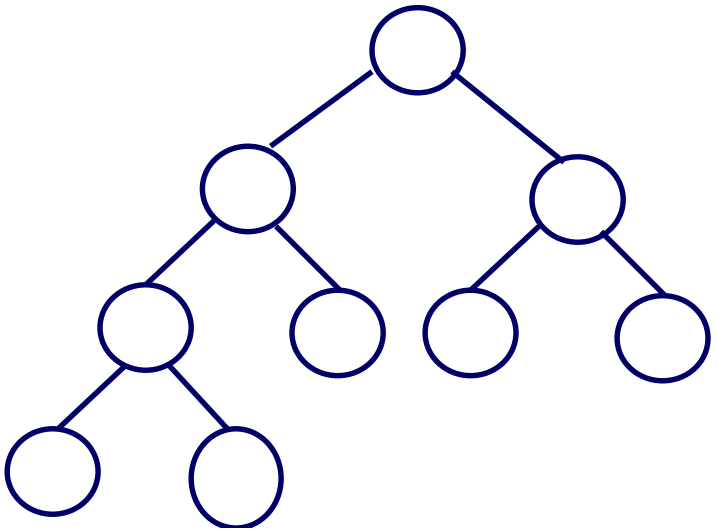
- ***Full*** binary tree: a binary tree in which
  - every parent has 2 children,
  - every leaf has equal depth



Full binary tree

- ***Complete*** binary tree: a binary tree in which
  - every level is full except possibly the deepest level
  - if the deepest level isn't full, leaf nodes are as far to the left as possible

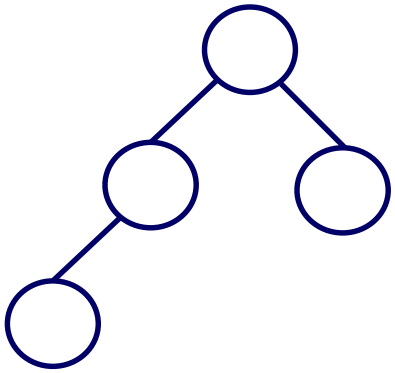




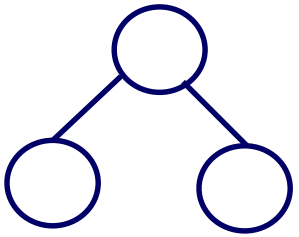
Which tree is full binary tree / complete binary tree?

Full binary tree
- every parent has 2 children,
- every leaf has equal depth

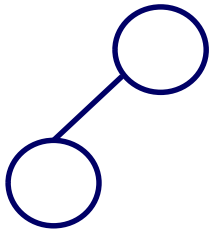
Complete binary tree
- every level is full except possibly the deepest level
- if the deepest level isn't full, leaf nodes are as far to the left as possible



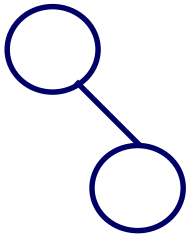
Complete binary tree



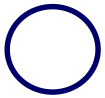
Full and complete



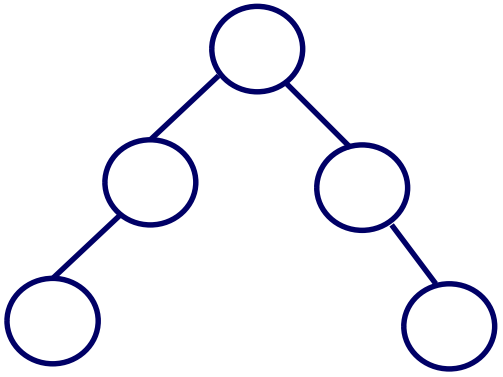
Complete



Neither full nor complete



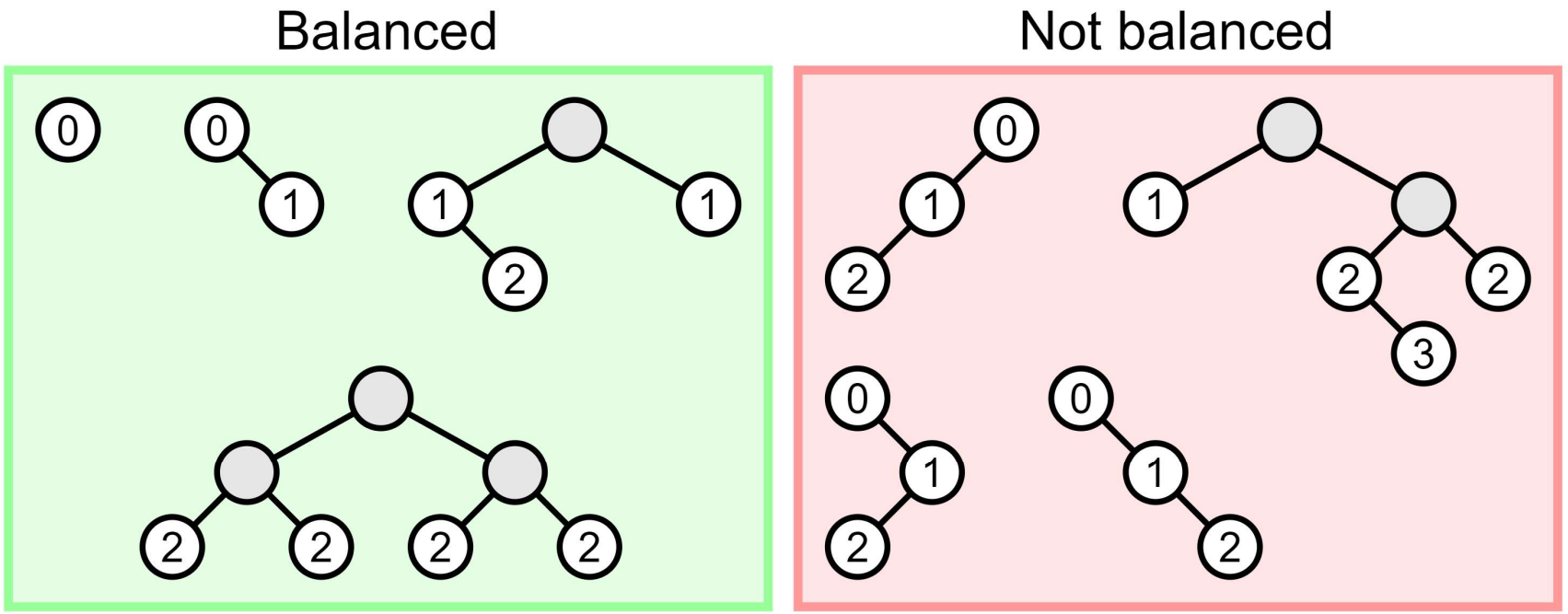
Full and complete



Neither full nor complete

Balanced binary tree (Cây nhị phân cân bằng)

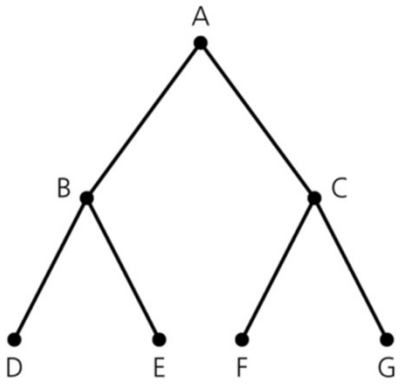
- **Definition.** A balanced binary tree is a binary tree in which the height of the left and right subtree of any node differ by not more than 1.



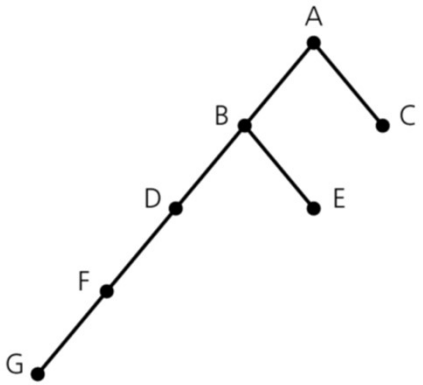
Balanced binary tree (Cây nhị phân cân bằng)

- **Definition.** A balanced binary tree is a binary tree in which the height of the left and right subtree of any node differ by not more than 1.
- Comment:
  - If a binary tree is full binary tree then it is also a complete binary tree.
  - If a binary tree is complete binary tree, then it is also a balanced binary tree.

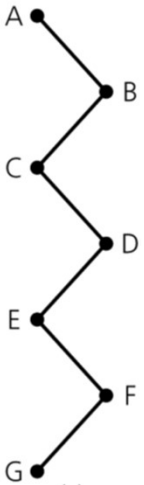
Example:



(a)



(b)



(c)

1. Which one is full binary tree?
2. Which one is complete binary tree?
3. Which one is balanced binary tree?

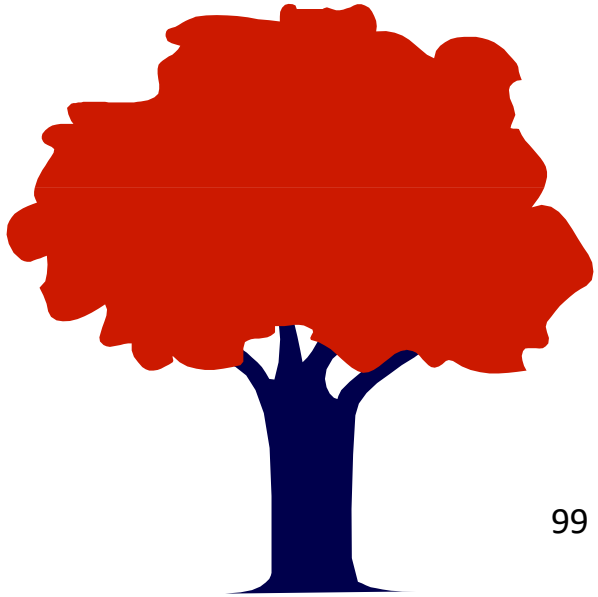
4.4. Binary tree

4.4.1. Definitions

4.4.2. Binary tree representation

4.4.3. Binary tree traversal

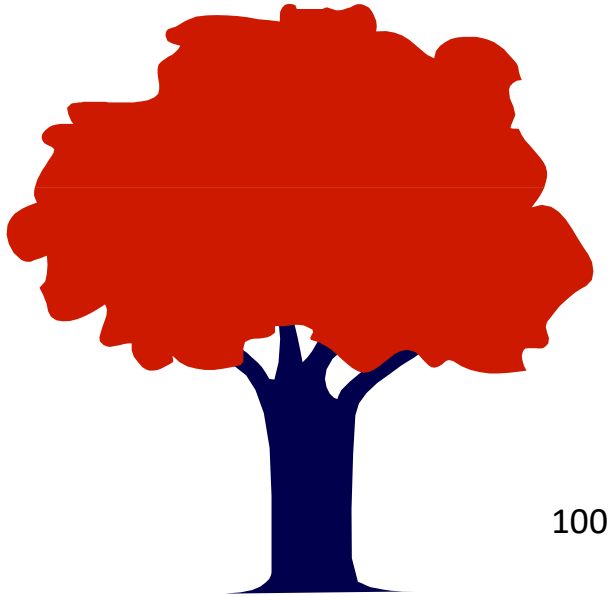
4.4.4. Some applications



4.4.2. Binary tree representation

4.4.2.1. Array

4.4.2.2. Pointer

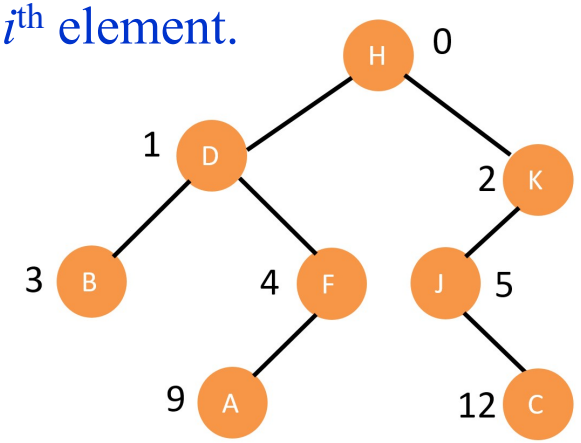


1D array representation of a binary tree

Nodes are numbered / indexed as following:

- giving 0 to root,
- then all the nodes are numbered from left to right level by level from top to bottom. Empty nodes are also numbered.

Then each node having an index i is put into the array as its i^{th} element.



➔ An 1D array ***A*** can be used to represent a binary tree:

- Root is $A[0]$
- for any node at $A[i]$:
  - its parent node is $A[(i\text{-}1)/2]$
  - its left child $A[2i + 1]$ and its right child is $A[2i + 2]$
  - If any node does not have any of its child, then null value is stored at the corresponding index of the array.

	H	D	K	B	F	J	NULL	NULL	NULL	A	NULL	NULL	C	NULL
A	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	.....	25	26	.....	30

Right child of K

Left child of B

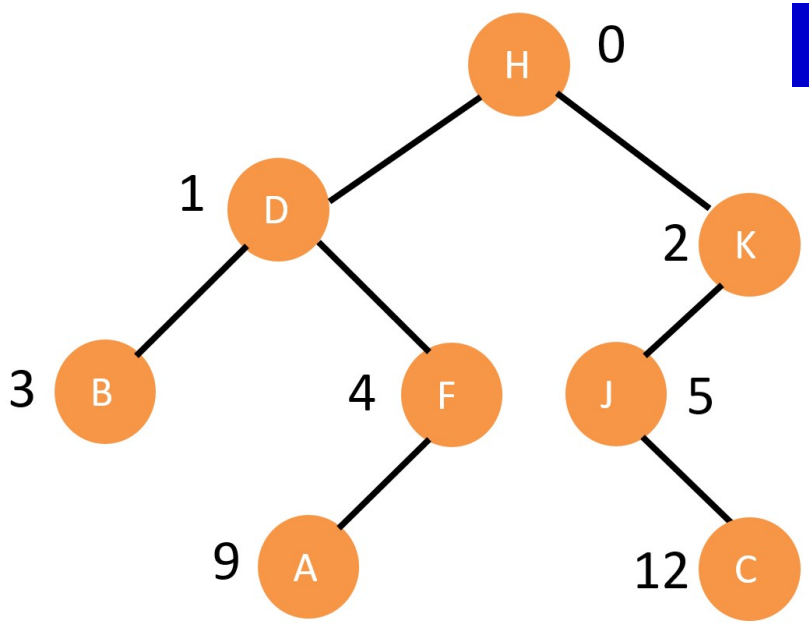
Right child of B

left child of J

Left child of C

Right child of C

1D array representation of a binary tree



Depth/level =1 Max #nodes = 2^0=1

l = 2 Max #nodes = 2^1=2

l = 3 Max #nodes = 2^2=4

l = 4 Max #nodes = 2^3=8

Max #nodes = 2^4=16

$\sum max\#nodes = 31$

for any node at $A[i]$:

- its parent node is $A[(i\text{-}1)/2]$
- its left child $A[2i + 1]$ and its right child is $A[2i + 2]$

	H	D	K	B	F	J	NULL	NULL	NULL	A	NULL	NULL	C	NULL
A	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	.....	25	26	.....	30

Right child of K

Left child of B

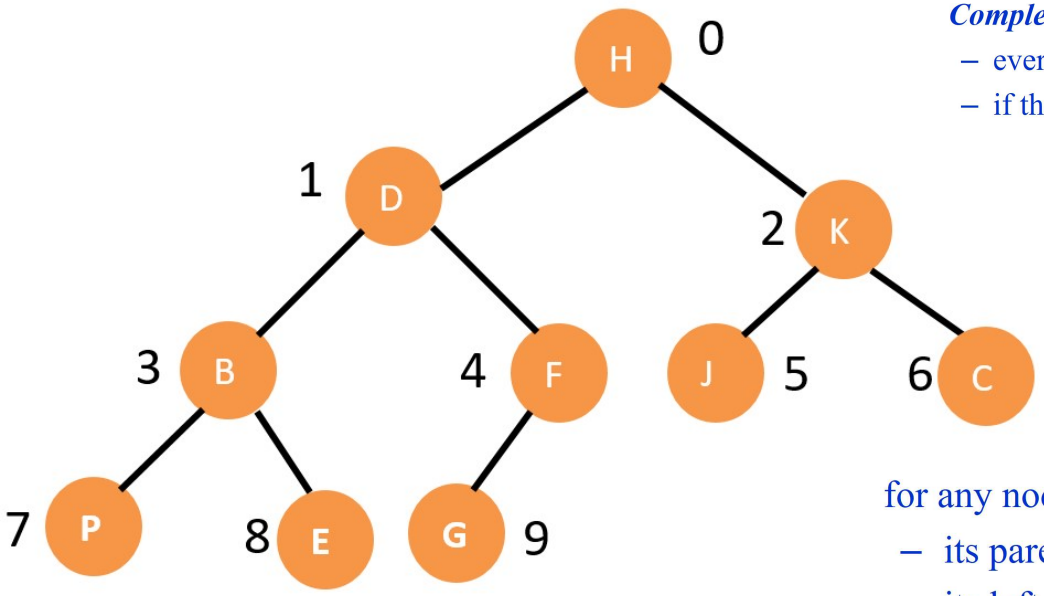
Right child of B

left child of J

Left child of C

Right child of C

1D array representation of a complete binary tree



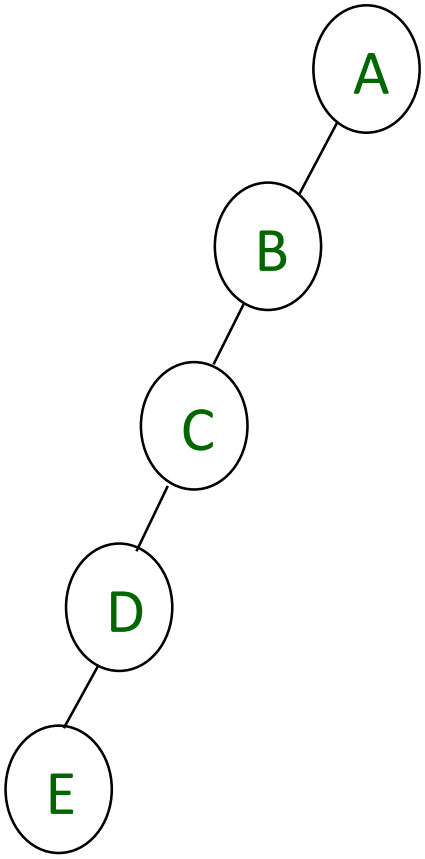
Complete binary tree
- every level is full except possibly the deepest level
- if the deepest level isn't full, leaf nodes are as far to the left as possible

for any node at $A[i]$:
- its parent node is $A[(i\text{-}1)/2]$
- its left child $A[2i + 1]$ and its right child is $A[2i + 2]$

H	D	K	B	F	J	C	P	E	G
A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]

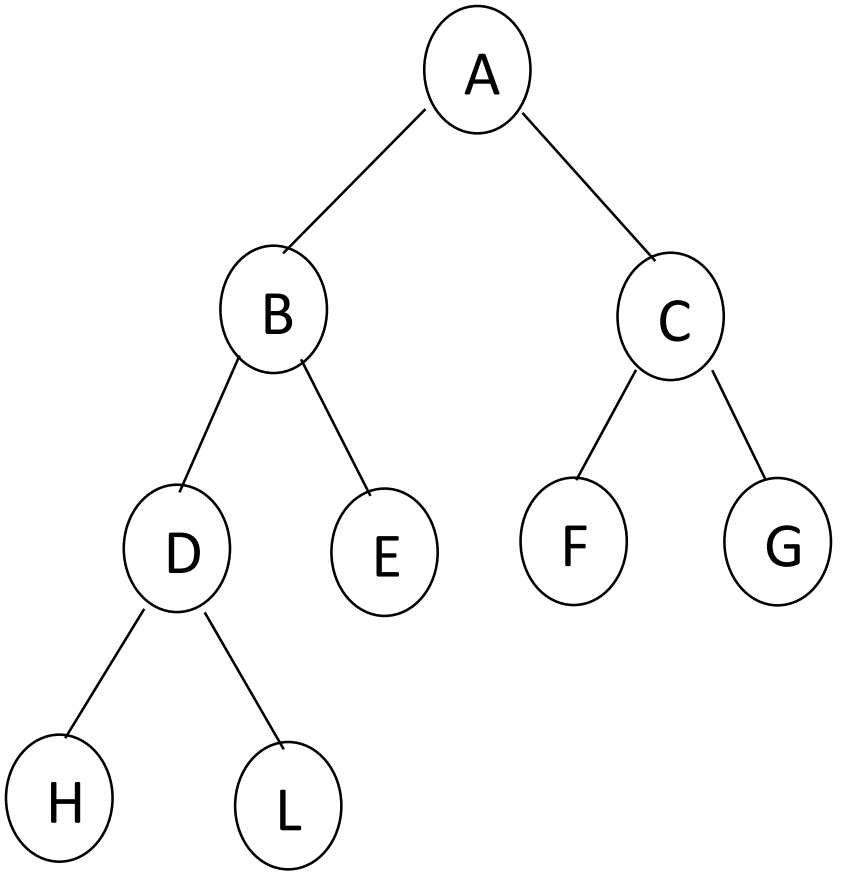
If using 1D array to represent a complete binary tree of n nodes:
- We only need the array of n elements
- To check a node $A[i]$ is leaf or not ?

1D array representation of a binary tree



[0]	A
[1]	B
[2]	NULL
[3]	C
[4]	NULL
[5]	NULL
[6]	NULL
[7]	D
	NULL
[15]	E

(1) waste space
(2) insertion/deletion problem

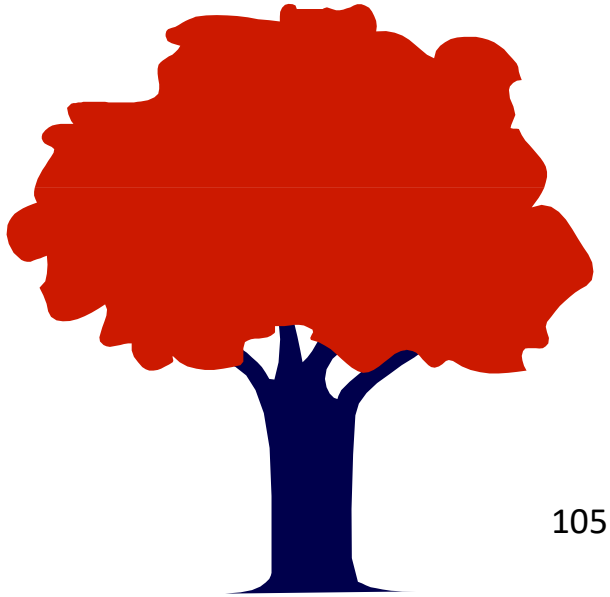


[0]	A
[1]	B
[2]	C
[3]	D
[4]	E
[5]	F
[6]	G
[7]	H
[8]	L

4.4.2. Binary tree representation

4.4.2.1. Array

4.4.2.2. Pointer

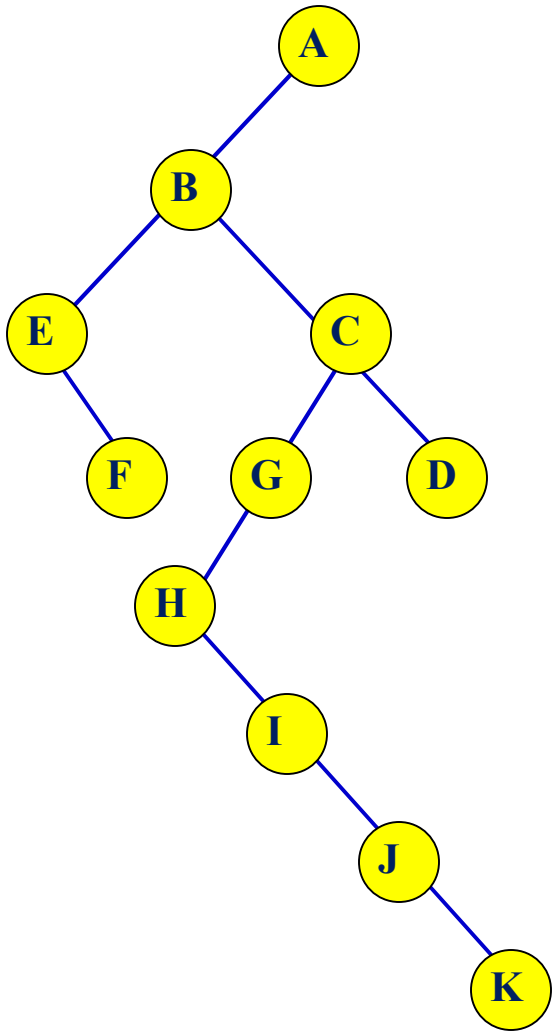


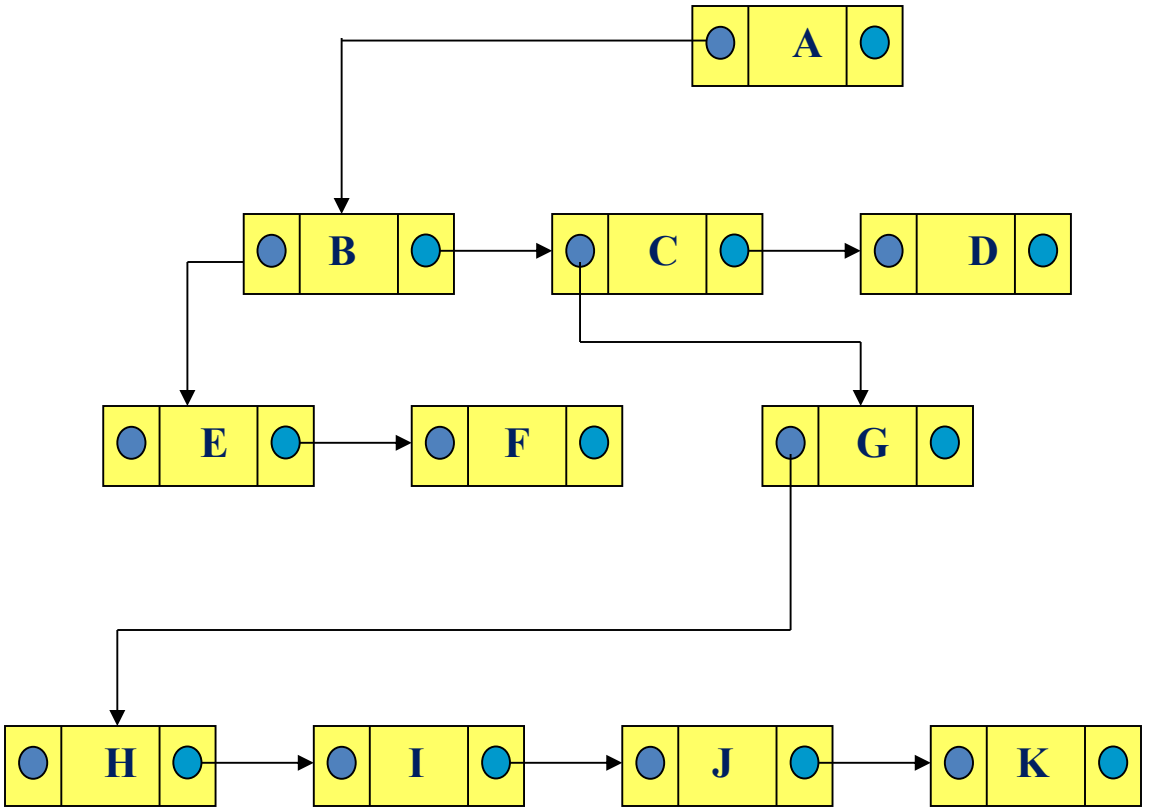
Pointer representation of a binary tree

- Each node contains the address of the left child and the right child.
- If any node has its left or right child empty then it will have in its respective link field, a null value.
- A leaf node has null value in both of its links.
- The structure defining a node of binary tree in C is as follows:

```
struct node
{
   DataType data; /*data of node; DataType: int, char, double..*/
   node *left ;   /* points to the left child */
   node *right;   /* points to the right child */
};
```

Pointer representation of a binary tree: Example





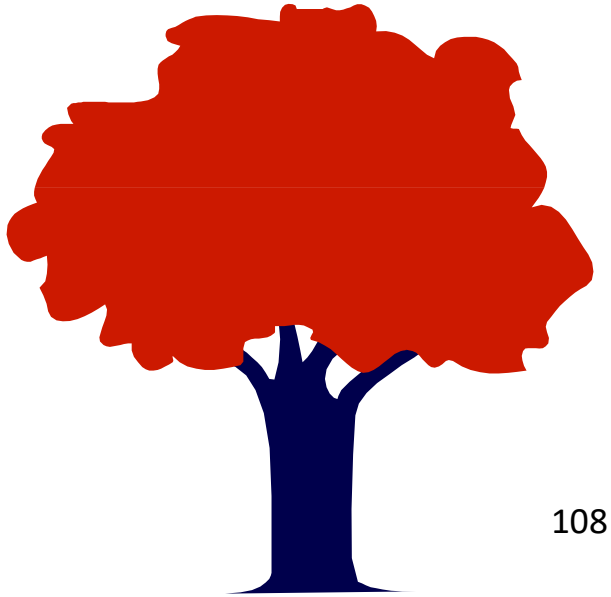
4.4. Binary tree

4.4.1. Definitions

4.4.2. Binary tree representation

4.4.3. Binary tree traversals

4.4.4. Some applications

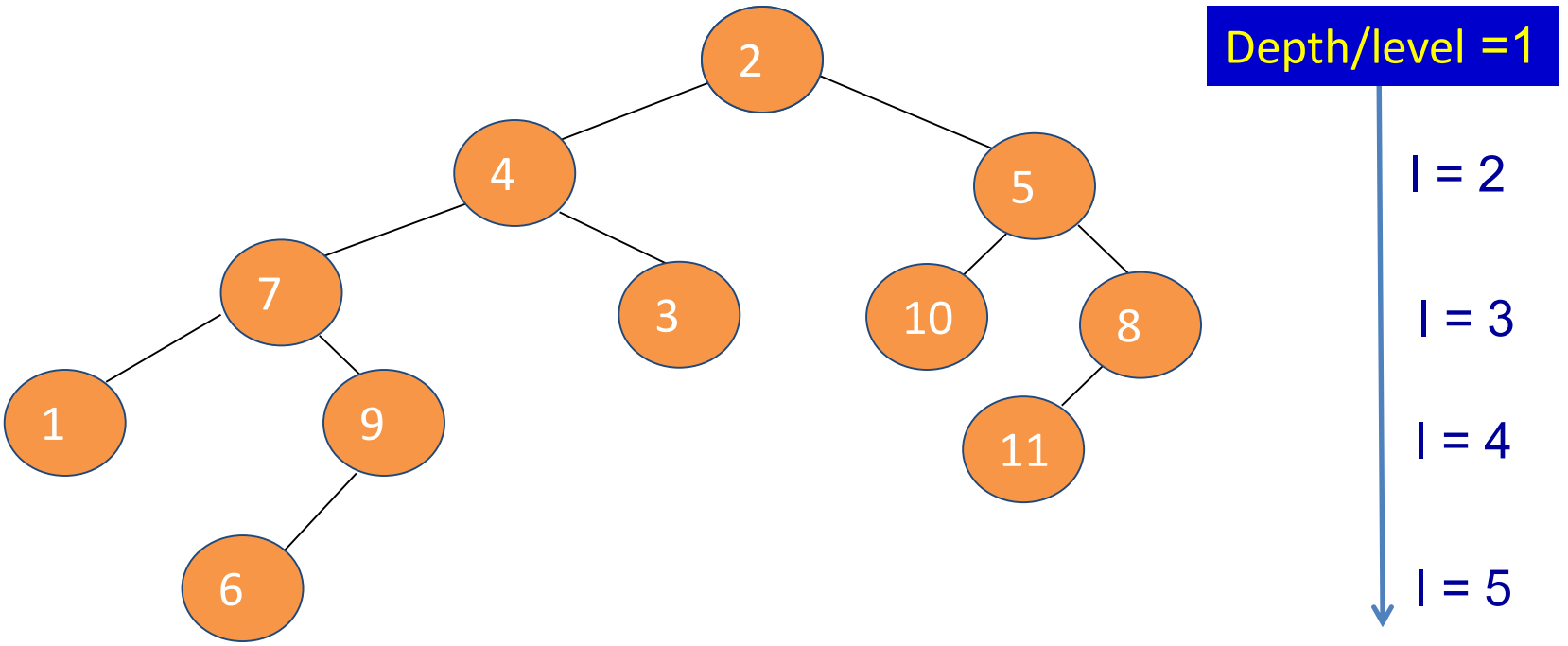


4.4.3. Binary Tree Traversals

- A traversal is where each node in a tree is visited and visited once
- There are two very common traversals
  - Breadth First
  - Depth First

4.4.3. Binary Tree Traversals: Breadth First

- In a breadth first traversal all of the nodes on a given level are visited and then all of the nodes on the next level are visited.
- Usually in a left to right fashion



- Breadth_First_Search(2): 2, 4, 5, 7, 3, 10, 8, 1, 9, 11, 6

4.4.3. Binary Tree Traversals: Depth First

- In a depth first traversal all the nodes on a branch are visited before any others are visited
- There are three common depth first traversals
  - Inorder
  - Preorder
  - Postorder
- Each type has its use and specific application

4.4.3. Binary Tree Traversals: Depth First

- ***Preorder* NLR**
  - Visit a **n**ode,
  - Visit **l**eft subtree in preorder,
  - Visit **r**ight subtree in preorder
- ***Inorder* LNR**
  - Visit **l**eft subtree in inorder,
  - Visit a **n**ode,
  - Visit **r**ight subtree in inorder
- ***Postorder* LRN**
  - Visit **l**eft subtree in postorder,
  - Visit **r**ight subtree in postorder,
  - Visit a **n**ode

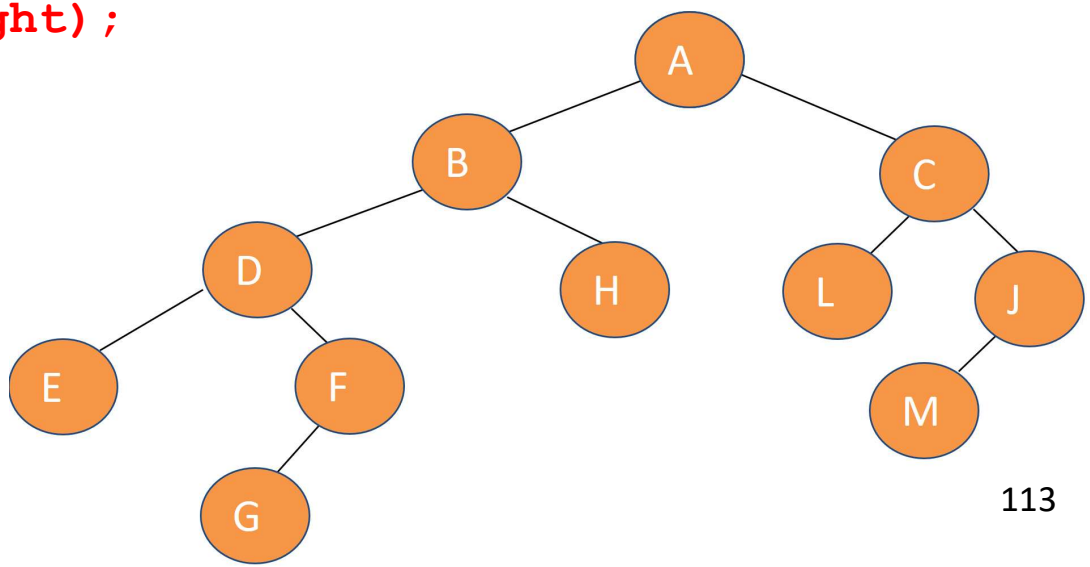
Preorder traversal – NLR

- Visit the node.
- Traverse the left subtree.
- Traverse the right subtree.

```
void printPreorder(node *root)
{    if( root != NULL )
     {
       cout<<root->word;
       printPreorder(root->left);
       printPreorder(root->right);
     }
}
Call: printPreorder(A);
```

A B D E F G H C L J M

Left subtree Right subtree



Inorder traversal – LNR

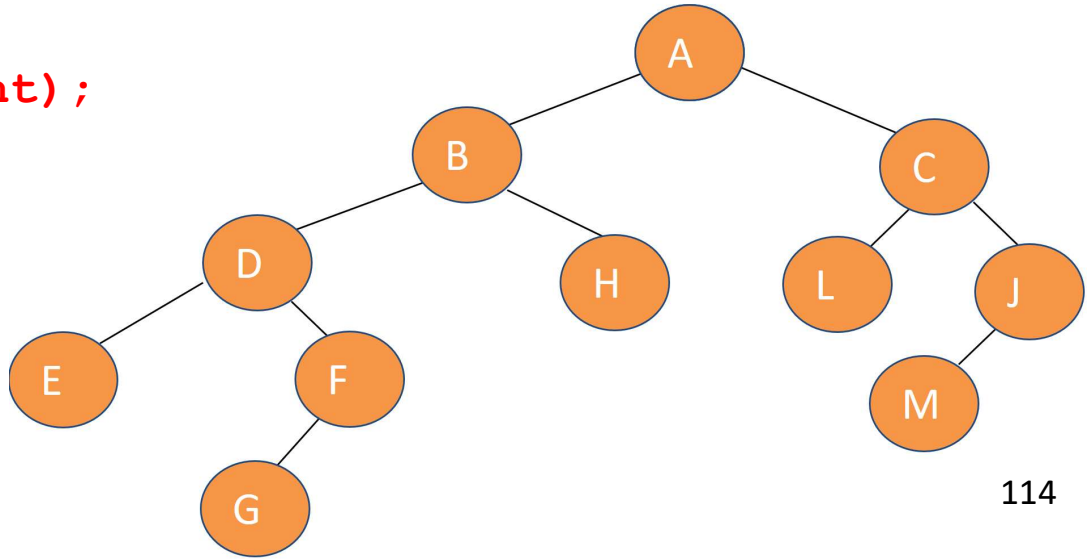
- Traverse the left subtree
- Visit the node
- Traverse the right subtree

```
void printInorder(node *tree)
{
   if( tree != NULL )
   {
       printInorder(tree->left);
       cout<<tree->word;
       printInorder(tree->right);
   }
}
```

Gọi: printInorder(A);

E D G F B H A L C M J

Cây con trái Cây con phải



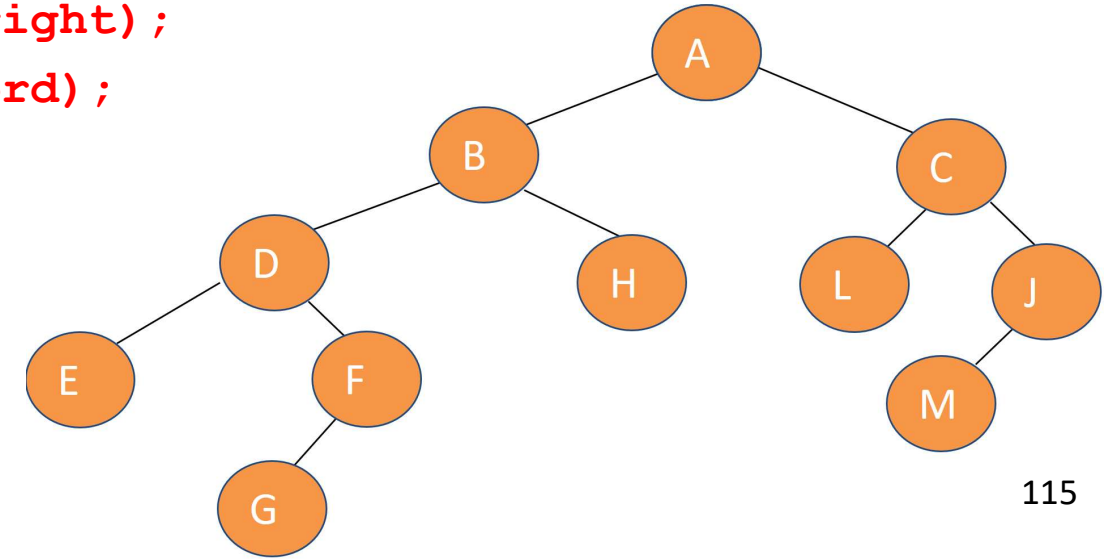
Postorder traversal – LRN

- Traverse the left subtree
- Traverse the right subtree
- Visit the node

```
void printPostorder(node *tree)
{
   if( tree != NULL )
     {
       printPostorder(tree->left);
       printPostorder(tree->right);
       printf("%s ", tree->word);
   }
}
```

```
Call: printPostorder(A);
E G F D H B L M J C A
```

Left subtree Right subtree



Basic operations on binary tree

```
struct node
{
    char word[20];  // Data of node
    node * left;
    node *right;
};

node* makeTreeNode(char *word);
node *insertNode(node* root, char *word, bool LEFT);
int countNodes(node *root);
int depth(node *root);
void freeTree(node *root);
void printPreorder(node *root);
void printPostorder(node *root);
void printInorder(node *root);
```

Basic operations on binary tree: MakeNode

```
node* makeTreeNode(char *word);
```

- Input parameter: data of the inserted node
- Output: Return the pointer to (address of) the new node

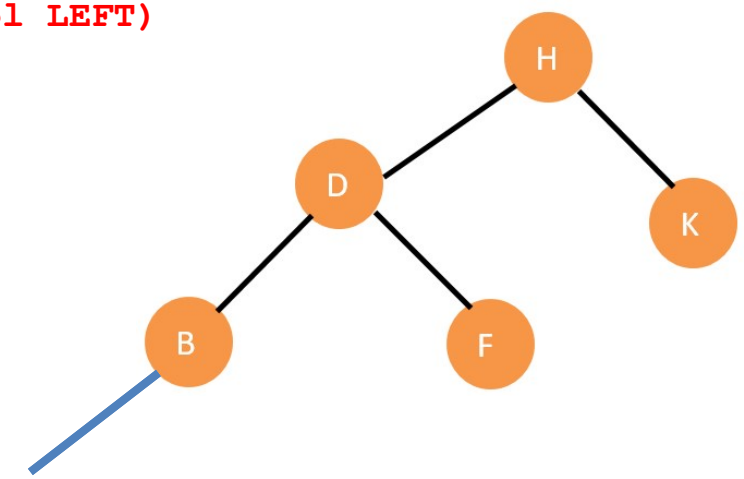
Steps:

- Allocate memory for new node. Check whether the allocation is successful?
  - If Yes: assign data of new node = word
    pointer of left child = NULL
    pointer to right child = NULL
- Return the pointer to (address of) the new node

```
node *makeTreeNode(char *word)
{
    node *newode = NULL;
    newNode = new node;
    if (newNode == NULL) { //Memory allocation is unsuccessful
         cout<<"Out of memory"<<endl;
         exit(1);
    }
    else {
         strcpy(newNode->word, word);
         newNode->left =NULL;
         newNode->right =NULL;
    }
    return newNode;
}
```

Basic operations on binary tree: insert new node into the tree

```
node *insertNode(node *tree, char *word, bool LEFT)
{
    node *newNode, *p;
    newNode = makeTreeNode(word);
    if ( tree == NULL ) return newNode;
    if (LEFT) //insert node on the leftmost of the tree
    {
        p = tree;
        while (p->left !=NULL) p = p->left;
        p->left = newNode;
        cout<<"Node "<<word<<" is left child of "<<(*p).word<<endl;
    }
    else  { //insert node on the rightmost of the tree
        p = tree;
        while (p->right !=NULL) p = p->right;
        p->right = newNode;
        cout<<"Node "<<word<<" is right child of "<<(*p).word<<endl;
    }
    return tree;
}
```

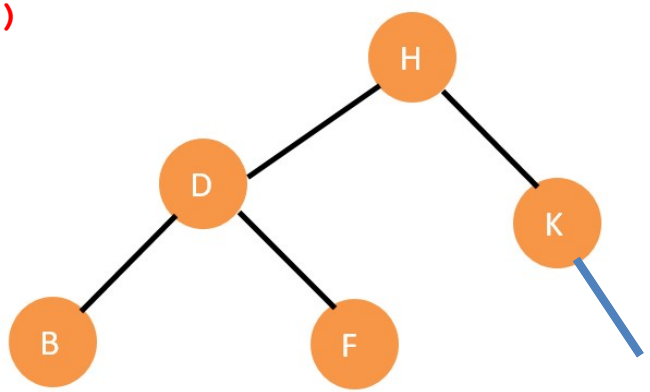


C

Call: `RandomInsert(H, "C", 1);`

Basic operations on binary tree: insert new node into the tree

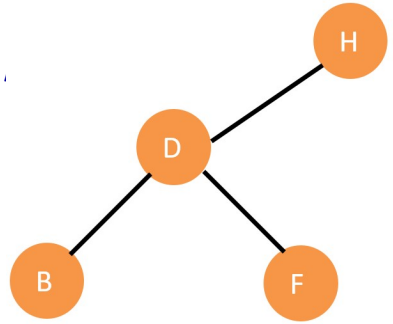
```
node *insertNode(node *tree, char *word, bool LEFT)
{
    node *newNode, *p;
    newNode = makeTreeNode(word);
    if ( tree == NULL ) return newNode;
    if (LEFT) //insert node on the leftmost of the tree
    {
        p = tree;
        while (p->left !=NULL) p = p->left;
        p->left = newNode;
        printf("Node %s  is left child of  %s \n",word,(*p).word);
    }
    else  { //insert node on the rightmost of the tree
        p = tree;
        while (p->right !=NULL) p = p->right;
        p->right = newNode;
        printf("Node %s  is right child of  %s \n", word,(*p).word);
    }
    return tree;
}
```



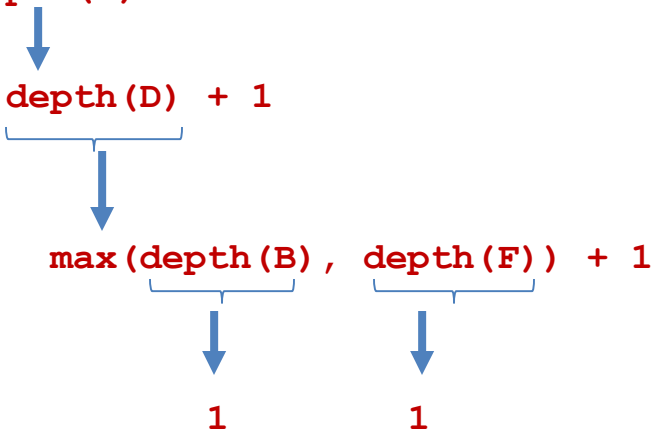
Call: `RandomInsert(H, "C", 0);`

Basic operations on binary tree: calculate depth of tree

```
int depth(node *root)  {  /* the function computes the depth of a tree */
    if( root == NULL ) return 0;
     // Base case : Leaf Node. This accounts for height = 1.
     if (root->left == NULL && root->right == NULL) return 1;

     // If left subtree is NULL, recur for right subtree
     if (root->left==NULL) return depth(root->right) + 1;

     // If right subtree is NULL, recur for left subtree
     if (root->right==NULL) return depth(root->left) + 1;

     return max(depth(root->left), depth(root->right)) + 1;
}
```

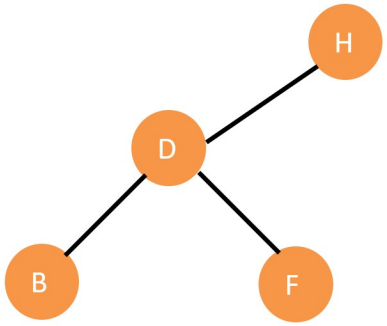


Call depth(H):



Basic operations on binary tree: calculate depth of tree

```
int depth(node *root)  {  /* the function computes the depth of a tree */
    if (root == NULL ) return 0;
    int lDepth = depth(root->left);
    int rDepth = depth(root->right);
    return max(lDepth, rDepth) + 1;
}
```



Call depth(H):

max (depth(D), depth(NULL)) + 1

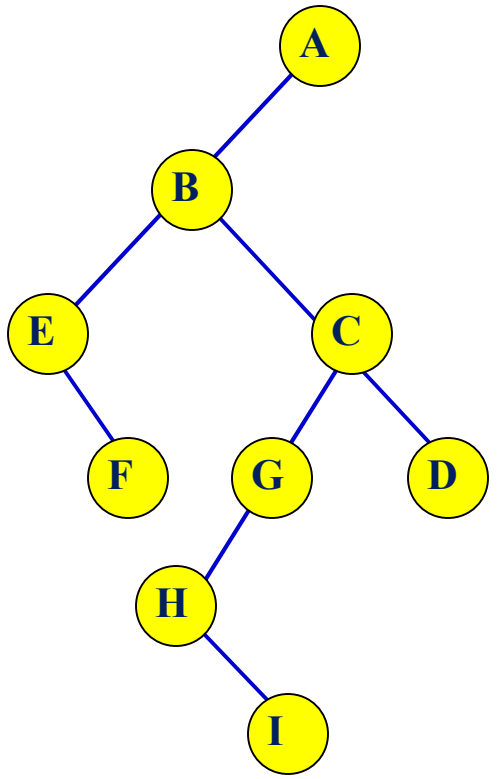
max(depth(B), depth(F))+1

max(depth(NULL), depth(NULL)) + 1 = max(0, 0) + 1 = 1

max(depth(NULL), depth(NULL)) + 1 = max(0, 0) + 1 = 1

Basic operations on binary tree: count number of nodes on the tree

```
int countNodes(node *root) {
/* the function counts the number of nodes of a tree*/
   if (root == NULL ) return 0;
   else {
      int ld = countNodes(root->left);
      int rd = countNodes(root->right);
      return 1+ld+rd;
   }
}
```



#nodes on the tree = 1 + #nodes on left subtree + #nodes on right subtree

Call countNodes(A);

Basic operations on binary tree: free the tree

```
void freeTree(node *root)
{
     if( root == NULL ) return;
      freeTree(root->left);
      freeTree(root->right);
      delete root;
}
```

Program in C++

```
/* The program for testing binary tree traversal */
#include <bits/stdc++.h>
using namespace std;
struct node {
    char word[20];
    node * left;
    node *right;
};
node *makeTreeNode(char *word);
node *insertNode(node* root,char *word);
void freeTree(node *tree);
void printPreorder(node *tree);
void printPostorder(node *tree);
void printInorder(node *tree);
int countNodes(node *tree);
int depth(node *tree);
```

Program in C++

```
int main() {
    node *randomTree=NULL;
    char word[20] = "a";
     while( strcmp(word,"~")!=0)
    {    cout<<"\nEnter item (~ to finish): ";
         fflush(stdin);cin>>word;
         if (strcmp(word,"~")==0) cout<<"Finish to input data for node... \n";
         else randomTree  = insertNode(randomTree,word, rand()%2);
    }
    cout<<"The tree in preorder:\n"; printPreorder(randomTree);
    cout<<"The tree in postorder:\n"; printPostorder(randomTree);
    cout<<"The tree in inorder:\n"; printInorder(randomTree);
    cout<<"The number of nodes is: "<<countNodes(randomTree)<<endl;
   cout<<"The depth of the tree is: "<<depth(randomTree)<<endl;
    freeTree(randomTree);
}
```

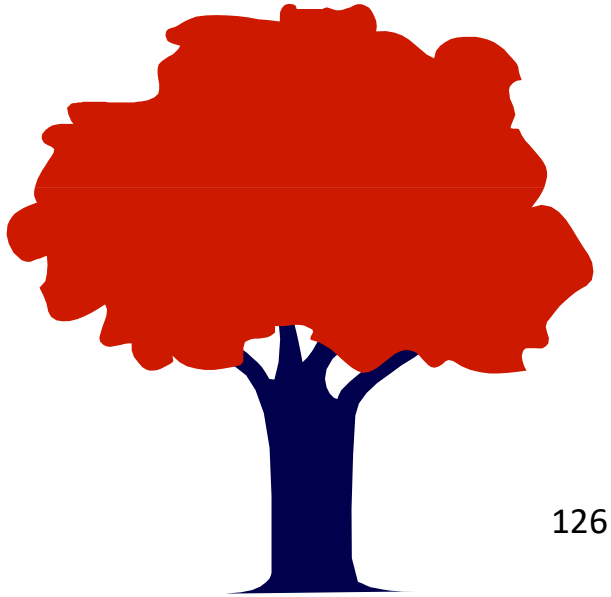
4.4. Binary tree

4.4.1. Definitions

4.4.2. Binary tree representation

4.4.3. Binary tree traversals

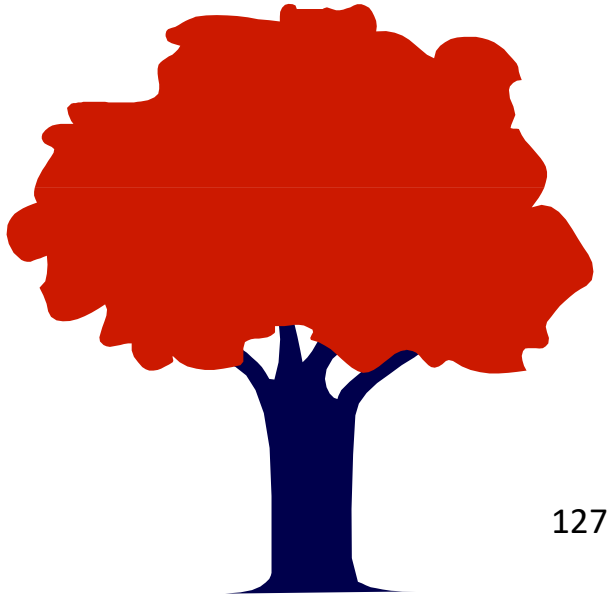
4.4.4. Some applications



4.4.4. Some applications

4.4.4.1. Arithmetic expression

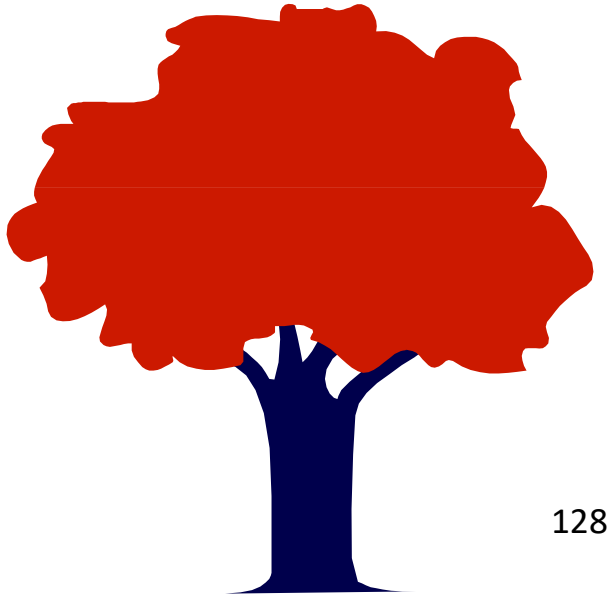
4.4.4.2. Huffman code



4.4.4. Some applications

4.4.4.1. Arithmetic expression

4.4.4.2. Huffman code

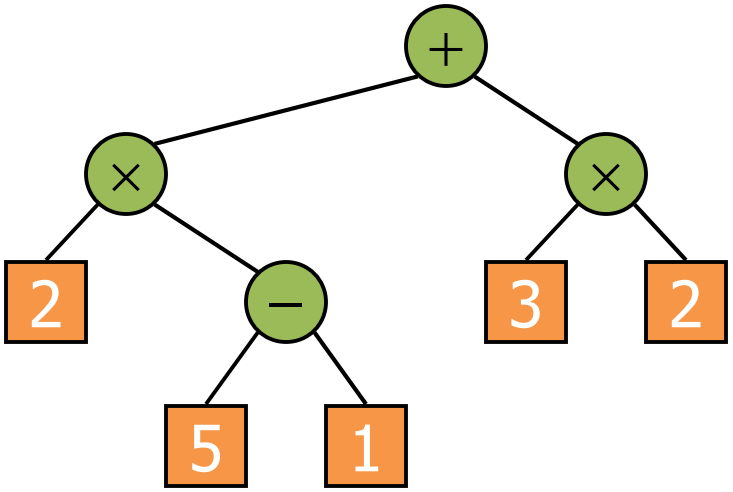


4.4.4.1. Arithmetic expression

- Binary tree are used to represent for an arithmetic expression:
  - internal nodes: operators
  - leaves: operands

Example 1: arithmetic expression tree for the expression

$$((2 \times (5 - 1)) + (3 \times 2))$$



Traversing expression tree:

1. preorder traversal

+ x 2 – 5 1 x 3 2

give us: prefix expression

2. inorder traversal

2 x 5 – 1 + 3 x 2

give us: infix expression

3. postorder traversal

2 5 1 – x 3 2 x +

give us: postfix expression

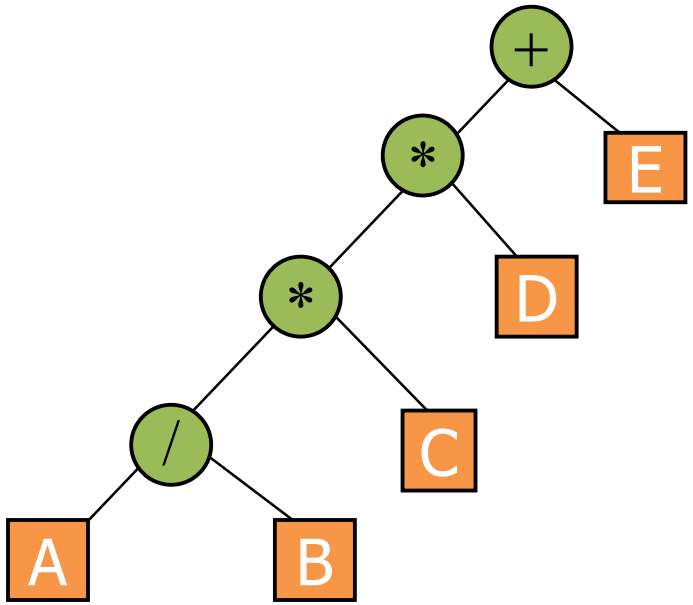
4. level order traversal (breath first traveral)

+ x x 2 – 3 2 5 1

4.4.4.1. Arithmetic expression

- Binary tree are used to represent for an arithmetic expression:
  - internal nodes: operators
  - leaves: operands

Example 2:



Traversing expression tree:

1. preorder traversal

+ * * / A B C D E

give us: prefix expression

2. inorder traversal

A / B * C * D + E

give us: infix expression

3. postorder traversal

A B / C * D * E +

give us: postfix expression

4. level order traversal (breath first traveral)

+ * E * D / C A B

Infix/Postfix/Prefix Expressions

- **INFIX**: the expressions in which operands surround the operator

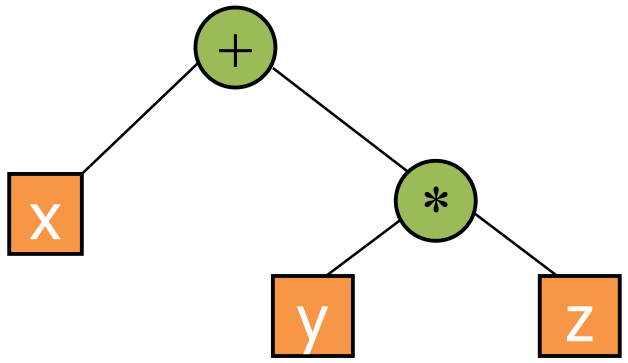
Example: x+y, x+y*z

- **POSTFIX** (also known as Reverse Polish Notation): operator comes after the operands

Example: xy+ , xyz*+

- **PREFIX** (also Known as Polish notation): operator comes before the operands

Example: +xy, +x*yz



Infix Expressions

- **INFIX**: the expressions in which operands surround the operator

How do you evaluate :

a * b + c / d

1) This is done by assigning operator priorities:

priority(*) = priority(/) > priority(+) = priority(-)

2) When an operand lies between two operators, the operand associates with the operator that has higher priority.

➔ Operand b lies between * and + ==> b associates with *

➔ Operand c lies between + and / ==> c associates with /

Therefore: a* b + c / d = (a*b) + (c/d)

Infix/Postfix/Prefix Expressions (Kí pháp trung/hậu/tiền tố)

- Khi lập trình, việc để cho máy tính tính giá trị một biểu thức toán học là điều quá đỗi bình thường, nhưng để trình bày làm sao cho máy tính có thể đọc và hiểu được quy trình tính toán đó không phải là điều đơn giản.
  - Ví dụ: "giải quyết" các biểu thức có dấu ngoặc như (a+b)*c + (d+e)*f , thì các phương pháp tách chuỗi đơn giản đều không khả thi. Trong tình huống này, ta phải dùng đến Ký Pháp Nghịch Đảo Ba Lan (Reserve Polish Notation – RPN), một thuật toán "kinh điển" trong lĩnh vực trình biên dịch.
  - Ví dụ: 5 + ((1 + 2) * 4) + 3 ➔ 5 1 2 + 4 * + 3 +

infix postfix / Reverse Polish notation

Ký pháp nghịch đảo Ba Lan được phát minh vào khoảng giữa thập kỷ 1950 bởi Charles Hamblin – một triết học gia và khoa học gia máy tính người Úc – dựa theo công trình về ký pháp Ba Lan của nhà Toán học người Ba Lan Jan Łukasiewicz. Hamblin trình bày nghiên cứu của mình tại một hội nghị khoa học vào tháng 6 năm 1957 và chính thức công bố vào năm 1962.

Evaluate postfix expression

6 3 6 + 5 * 9 / -

The process of evaluating the value of postfix expression is quite natural for the computer. The idea is as follows: scan the expression from left to right:

- If the current character is operator (denote `OPT`), then take the two operands immediately preceding it in the expression (denote A and B, respectively) …. A B `OPT`…

And do the calculation C = A `OPT` B, then push C back into the expression

When the process is finished, the last number remaining in the expression is the value of that expression.

Evaluate postfix expression

6 3 6 + 5 * 9 / -

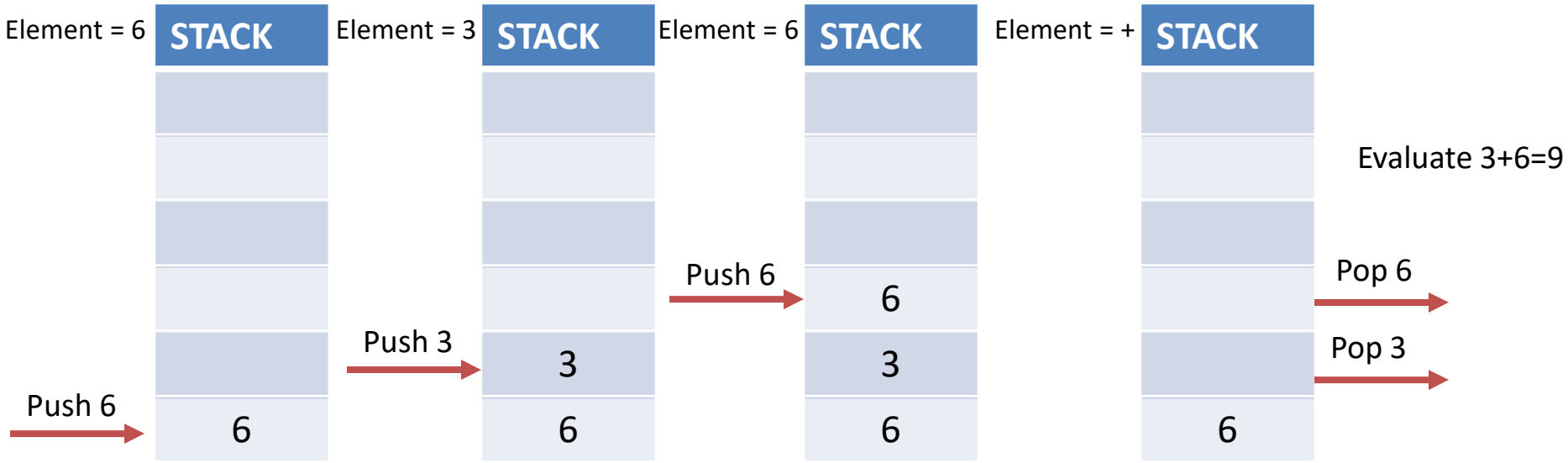
The process of evaluating the value of postfix expression is quite natural for the computer. The idea is as follows: scan the expression from left to right:

- If the current character is operand (number or variable), then push this operand into stack;
- If the current character is operator (denote by `OPT`), then pop 2 operands out of stack (denote A and B, respectively), and do the calculation C = A `OPT` B, then push C into stack

When the process is finished, the last number remaining in the stack is the value of that expression.

Example 1: Evaluate postfix expressions: using stack

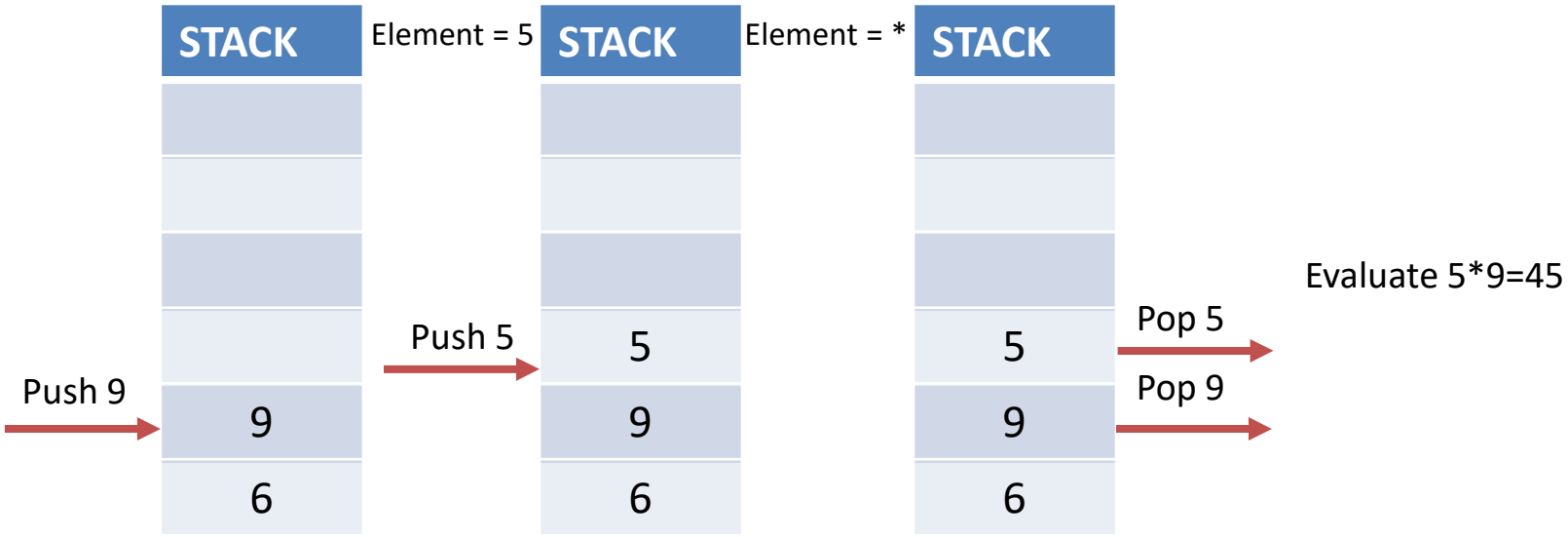
- Evaluate the following postfix expression using stacks: **6 3 6 + 5 * 9 / -**
- The expression is evaluated by using stack as follows:



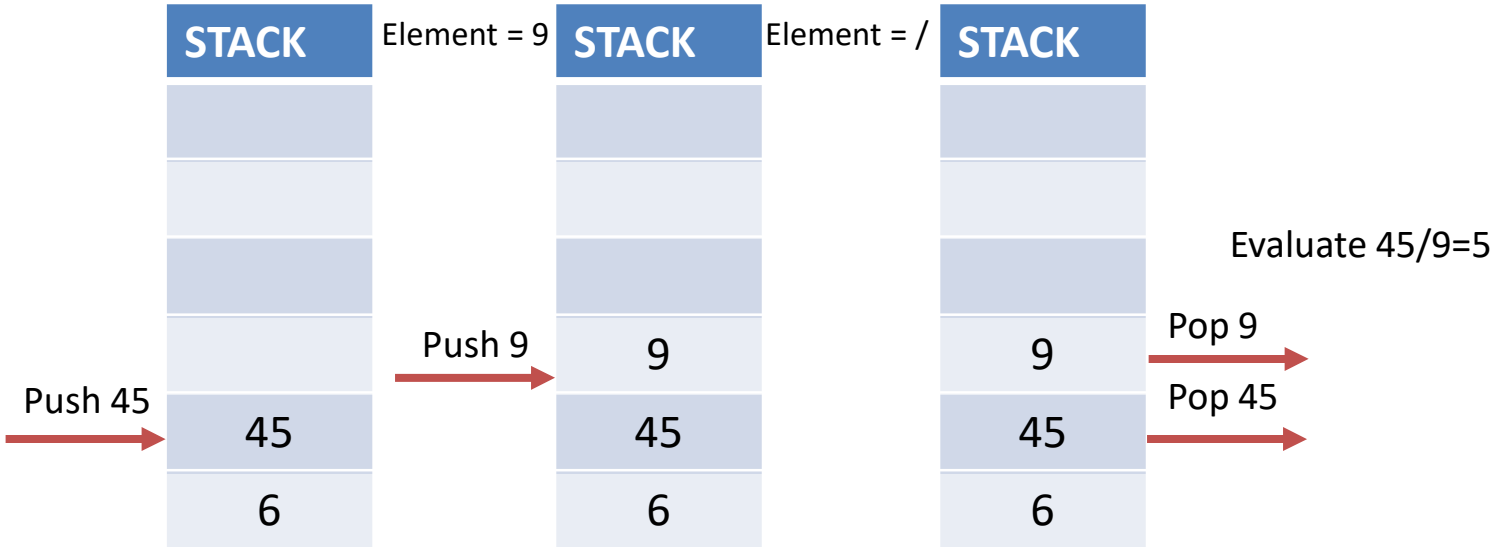
Push 9

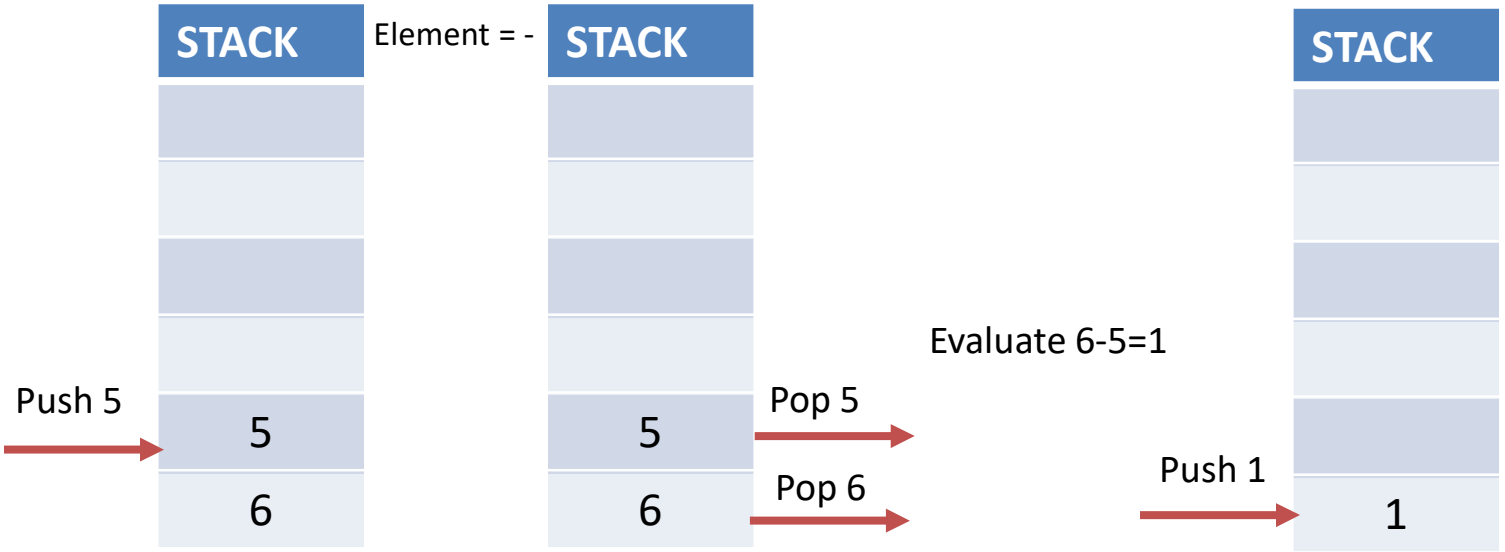
Example 1: Evaluate postfix expressions: using stack

- Evaluate the following postfix expression using stacks: **6 3 6 + 5 * 9 / -**
- The expression is evaluated by using stack as follows:



Evaluate postfix expressions using stack: 6 3 6 + 5 * 9 / -





Final state of stack with result

Evaluate postfix expressions: using stack

```
evaluationPostfix(stack S, postfixExpression E)
{ //Expression is stored in array E
  //Read expression from left to right:
   i = 0;
   while ( i < number of characters_in_postfix_expression)
   {
      if (E[i] is an operand)
             push E[i] onto stack
      else if (E[i] is an operator)
      {
        1.  Pop the top element from stack and store it in operand2
        2.  Pop the next top element from stack and store it in operand2
        3.  Evaluate: operand2 op operand1 and push the result onto stack
      }
      i = i + 1;
   }//end while
    Pop the top element, store it in Result;
    return Result; //which is value of expression
}
```

Example 1:Evaluate postfix expression:

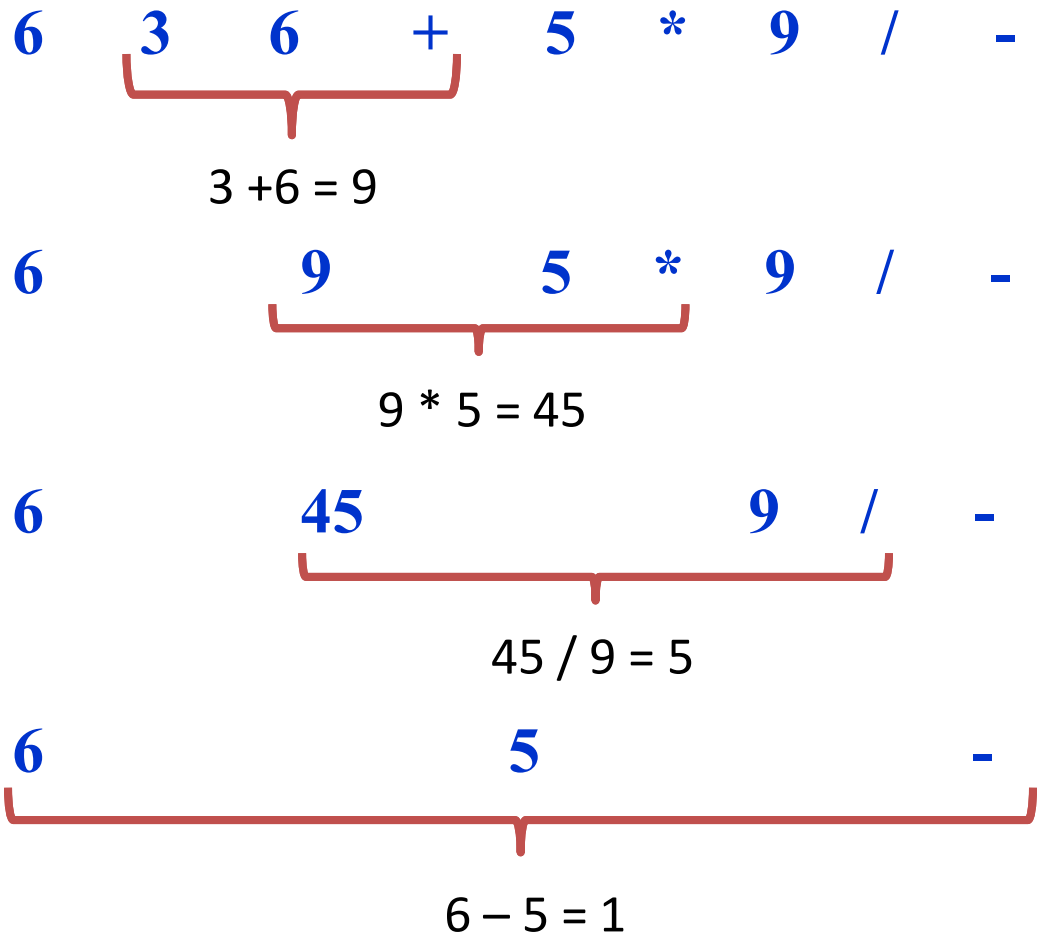
636+5*9/-

Value of expression = 1

2nd way to display the steps of algorithm

	Element of expression	Action performed	Stack status
1	6	Push 6 to stack	6
2	3	Push 3 to stack	6 3
3	6	Push 6 to stack	6 3 6
4	+	Pop 6	6 3
5		Pop 3	6
6		Evaluate 3 + 6 = 9	6
7		Push 9 to stack	6 9
8	5	Push 5 to stack	6 9 5
9	*	Pop 5	6 9
10		Pop 9	6
11		Evaluate 9*5=45	6
12		Push 45 to stack	6 45
13	9	Push 9 to stack	6 45 9
14	/	Pop 9	6 45
15		Pop 45	6
16		Evaluate 45/9=5	6
17		Push 5 to stack	6 5
18	-	Pop 5	6
19		Pop 6	Empty
20		Evaluate 6-5=1	Empty
21		Push 1 to stack	1
22		Pop value = 1	Empty

Example 1: Evaluate postfix expressions: using stack

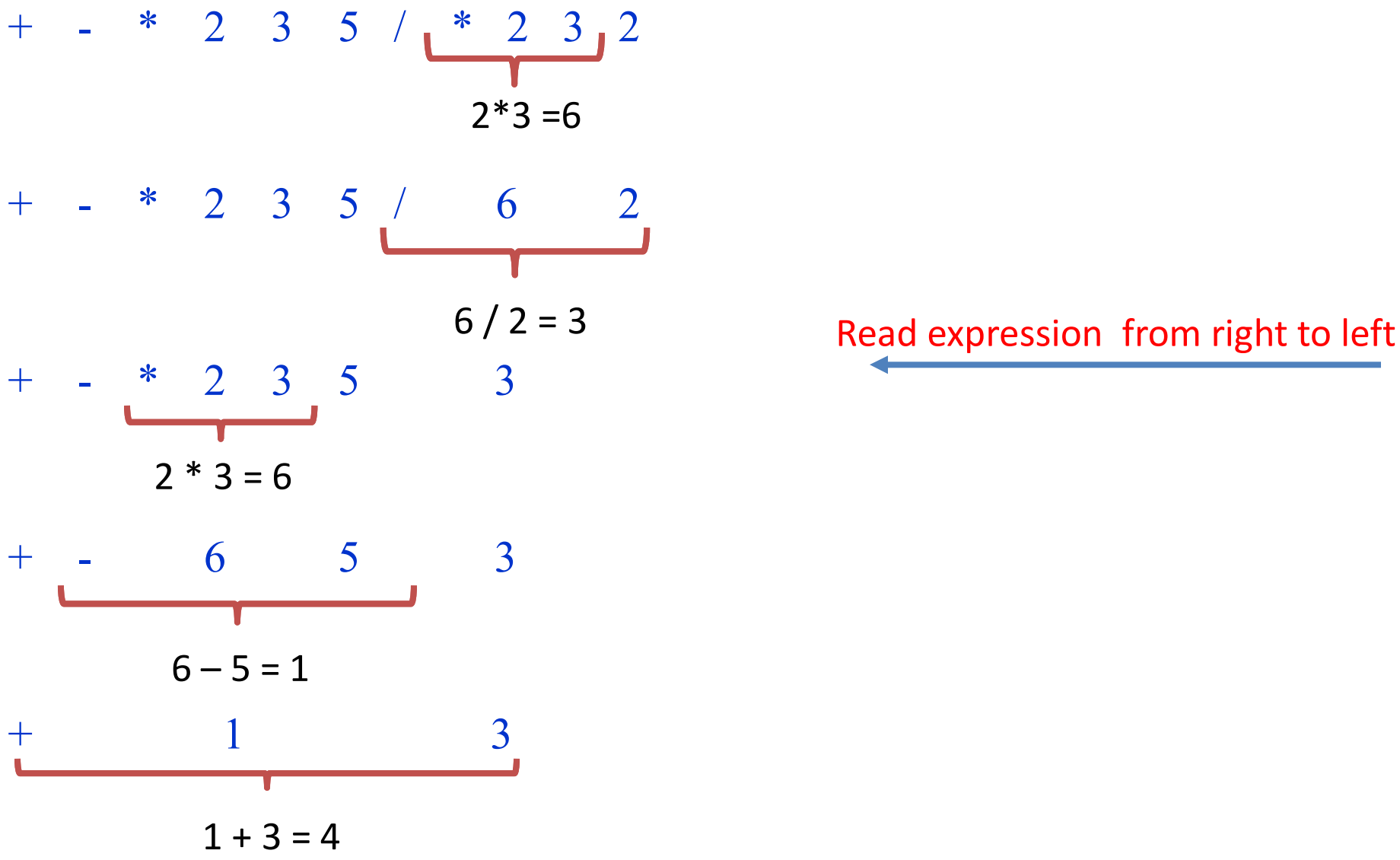


Read expression from left to right

Value of expression = 1

3rd way to display the steps of algorithm

Example 2: Evaluate prefix expressions: using stack



Read expression from right to left

Value of expression = 4

Example 2: Evaluate prefix expression:

+-*235/*232

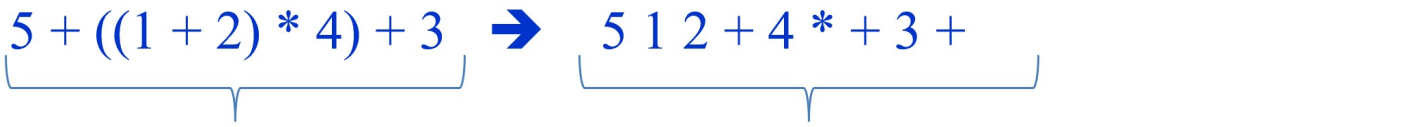
	Element of expression	Action performed	Stack status
1	2	Push 2 to stack	2
2	3	Push 3 to stack	2 3
3	2	Push 2 to stack	2 3 2
4	*	Pop 2	2 3
5		Pop 3	2
6		Evaluate 2 * 3 = 6	2
7		Push 6 to stack	2 6
8	/	Pop 6 to stack	2
9		Pop 2	Empty
10		Evaluate: 6/2=3	Empty
11		Push 3 to stack	3
12	5	Push 5 to stack	3 5
13	3	Push 3 to stack	3 5 3
14	2	Push 2 to stack	3 5 3 2

	Element of expression	Action performed	Stack status
15	*	Pop 2	3 5 3
16		Pop 3	3 5
17		Evaluate 2*3=6	3 5
18		Push 6 to stack	3 5 6
19	-	Pop 6	3 5
20		Pop 5	3
21		Evaluate 6-5 = 1	3
22		Push 1 to stack	3 1
23	+	Pop 1	3
24		Pop 3	Empty
25		Evaluate 1 + 3 = 4	Empty
26		Push 4 to stack	4
27		Pop value = 4	Empty

Value of expression = 4

Convert from infix to postfix

- Already know how to evaluate the postfix : using stack
- How to convert from infix to postfix ?



infix postfix / Reverse Polish notation

Convert from infix to postfix: application of Stack

EH is the postfix need to build. Initialize EH = ∅

Step 1: Scan the infix expression from left to right: current character is x

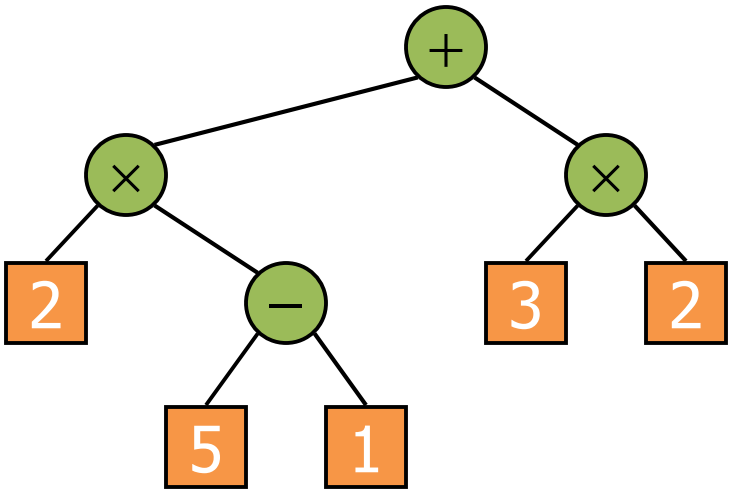
- If x is operand then insert it on the right of EH.
- If x is opening bracket " (", then push it into Stack
- If x is operator, then push x into Stack as following way:
  a. If Stack is empty or the top of stack is opening bracket "(" then push x into stack.
  b. If top of the stack is operator and its priority is lower than that of x, then push x into stack.
  c. If top of stack is operator and its priority is higher or equal to than that of x, then pop y out of stack, insert y on the right of EH. Back to step (a).
- If x is closing bracket ")" then:
  a. If top of the Stack is operator y, then pop y out of Stack and insert it on the right of EH. Repeat the process until Stack is empty or top of Stack is opening bracket. When top of Stack is opening bracket, the pop it out of stack.
  b. If top of the Stack is opening bracket, then pop it out of Stack.

Step 2: When we finish scanning all characters in EH, then we pop elements out of stack in turn, and insert on the right of EH until Stack is empty.

Print Arithmetic Expressions

Given a binary tree having root pointer pointed to the root of the tree, and this tree represents an arithmetic expression. Write a function to print this expression.

- Inorder traversal:
  - print "(" before traversing left subtree
  - print operand or operator when visiting node
  - print ")" after traversing right subtree



Call `printTree(+);`

The expression is:

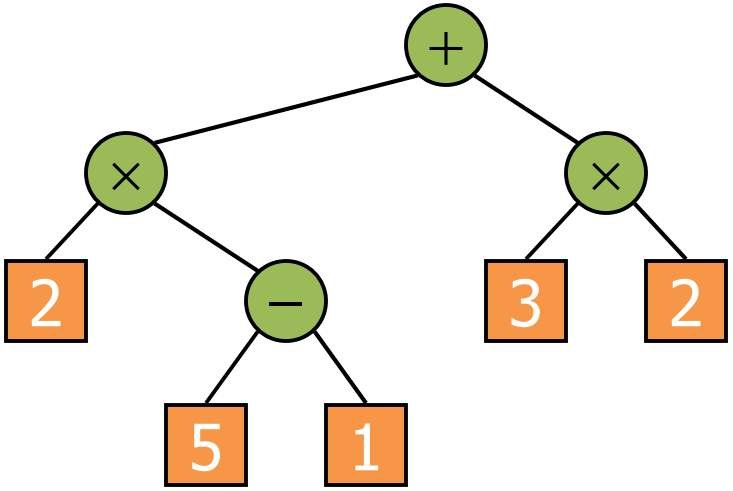
((2 × (5 - 1)) + (3 × 2))

```
void printTree(node *root)
    if (root.left != NULL)
    {
        print("(");
        printTree (root.left);
    }
    print(root.data);
    if (root.right != NULL)
    {
        printTree (root.right);
        print (")");
    }
```

Evaluate Arithmetic Expressions

Given a binary tree having root pointer pointed to the root of the tree, and this tree represents an arithmetic expression. Write a function to evaluate this expression.

- Postorder traversal:
  - Recursively evaluate subtrees
  - Apply the operator after subtrees are evaluated



Call `A = evaluate(+);`

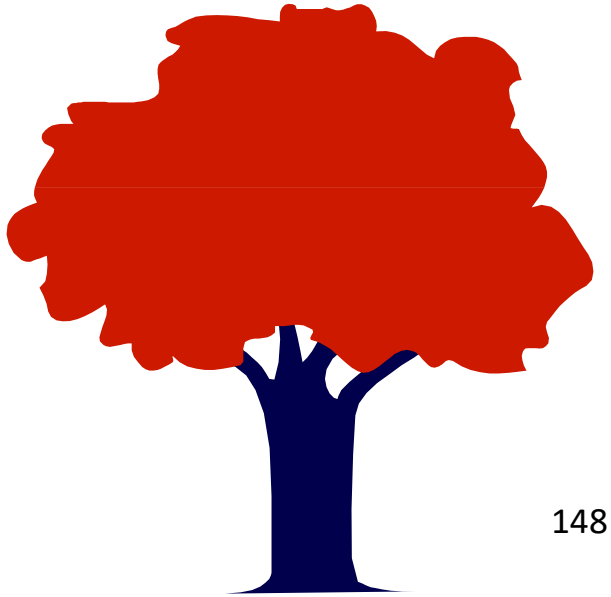
The value of A is:

```
int evaluate (node *root)
  if (root.left == NULL)//external node
    return root.data;
   else  //internal node
     x = evaluate (root.left);
     y = evaluate (root.right);
     let o be the operator root.data
     z = apply o to x and y
     return z;
```

4.4.4. Some applications

4.4.4.1. Arithmetic expression

4.4.4.2. Huffman code



Exercise

- Suppose that we have a 100,000 character data file that we wish to store. The file contains only 6 characters, appearing with the following frequencies:

	a	b	c	d	e	f
Frequency	45000	13000	12000	16000	9000	5000

- We encode each character as a binary string (called **codeword**). Thus, we need to find a binary code that encodes the file using as few bits as possible, i.e., compresses it as much as possible.
- There are two possible ways to encode:
  - A fixed-length code: each codeword has the same length.
  - A variable-length code: codewords may have different lengths.
- Example: if using fixed-length code for our problem, we need at least 3 bits per codeword as there are 6 characters ($2^2 = 4 < 6$, $2^3 = 8$)

	a	b	c	d	e	f
Frequency	45000	13000	12000	16000	9000	5000
A fixed-length	000	001	010	011	100	101
A variable-length	0	101	100	111	1101	1100

Exercise

- Example: if using fixed-length code for our problem, we need at least 3 bits per codeword as there are 6 characters ($2^2 = 4 < 6$, $2^3 = 8$)

	a	b	c	d	e	f
Frequency	45000	13000	12000	16000	9000	5000
A fixed-length	000	001	010	011	100	101
A variable-length	0	101	100	111	1101	1100

- The fixed length-code requires:

$$\underbrace{3}_{\text{3 bits for each character}} * \underbrace{100000}_{\text{Number of characters}} = 300000 \text{ bits to store the file}$$

- The variable-length code uses only:

$$\underbrace{1 * 45000}_{\text{\#bits used to encode character "a"}} + 3*13000 + 3*12000 + 3 * 16000 + 4 * 9000 + \underbrace{4 * 5000}_{\text{\#bits used to encode character "f"}} = 224000 \text{ bits}$$

➔ saving a lot of space: 25%

Exercise

- There are two possible ways to encode:
  - A fixed-length code: each codeword has the same length

    ➔ easy to encode and decode, but requires larger memory.
  - A variable-length code: codewords may have different lengths
    - Assign the longest codes to the most infrequent events.
    - Assign the shortest codes to the most frequent events.
    - Each code word must be uniquely identifiable regardless of length➔ no codeword is a prefix of another one (example: {a = 1, b = 110, c = 10, d = 111}, then when encoding, what is "1101111"?)

    ➔ Prefix code: a code is called a prefix code if no codeword is a prefix of another one (Example: {a = 0, b = 110, c = 10, d = 111} is a prefix code)

	a	b	c	d	e	f
Frequency	45000	13000	12000	16000	9000	5000
A fixed-length	000	001	010	011	100	101
A variable-length	0	101	100	111	1101	1100

Optimum source coding problem

- The variable-length code uses only 224000 bits rather than 300000 bits of fixed-length code ➔ save 25%. Question: Could we do better ? Or is this already the optimum (lowest cost) prefix code ?

	a	b	c	d	e	f
Frequency	45000	13000	12000	16000	9000	5000
A fixed-length	000	001	010	011	100	101
A variable-length	0	101	100	111	1101	1100

- The optimum source coding problem: Given an alphabet $A = \{a_1, \ldots, a_n\}$ with frequency distribution $f(a_i)$, find a binary prefix code C for A that minimizes the number of bits

$$B(C) = \sum_{a=1}^{n} f(a_i)L(c(a_i))$$

needed to encode a message of $\sum_{a=1}^{n} f(a)$ characters, where:

  - $c(a_i)$ is the codeword for encoding a_i ,
  - $L(c(a_i))$ is the length of the codeword $c(a_i)$.

Optimum source coding problem

- The optimum source coding problem: Given an alphabet $A = \{a_1, \ldots, a_n\}$ with frequency distribution $f(a_i)$, find a binary prefix code C for A that **minimizes** the number of bits

$$B(C) = \sum_{a=1}^{n} f(a_i) L(c(a_i))$$

needed to encode a message of $\sum_{a=1}^{n} f(a)$ characters, where:

  - $c(a_i)$ is the codeword for encoding a_i ,
  - $L(c(a_i))$ is the length of the codeword $c(a_i)$.

Remark: Huffman developed a nice greedy algorithm for solving this problem and producing a minimum cost (optimum) prefix code. The code that it produces is called a **Huffman code**.

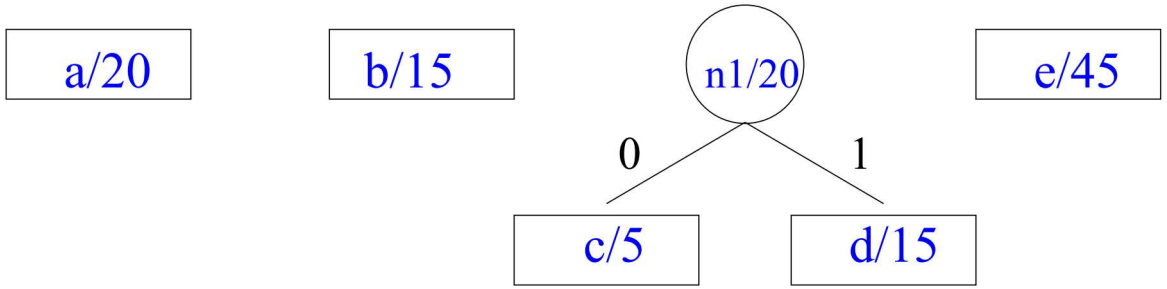
4.4.4.2 Huffman code

- Step 1: Pick two letters x, y from alphabet A with the smallest frequencies and create a subtree that has these two characters as leaves. (greedy idea) Label the root of this subtree as z.
- Step 2: Set frequency f(z) = f(x) + f(y). Remove x, y and add z creating new alphabet: $A' = A \cup \{z\} - \{x, y\}$, note that $|A'| = |A| - 1$
- Repeat this procedure, called **merge**, with new alphabet A' until an alphabet with only one symbol is left.

The resulting tree is the Huffman code

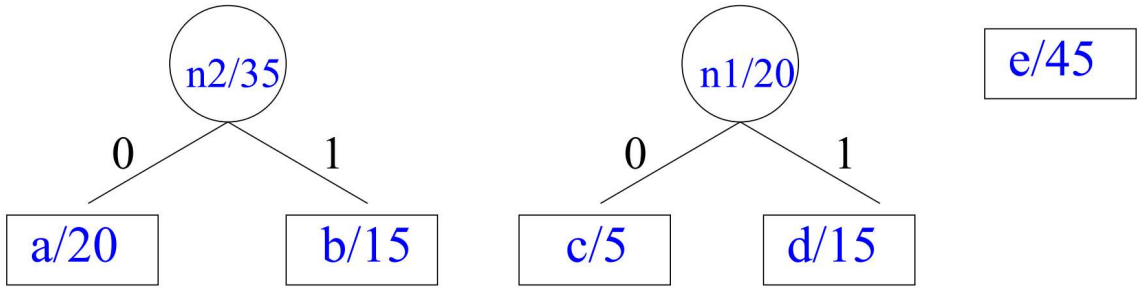
4.4.4.2 Huffman code

- Let A = {**a** / 20, **b** / 15, **c** / 5, **d** /15, **e** / 45} be the alphabet and its frequency distribution.
- In the first step Huffman coding merges c and d



Alphabet is now A_1 = {**a** / 20, **b** / 15, **n1** / 20, **e** / 45}

- Algorithm merges a and b (could also b and n1)

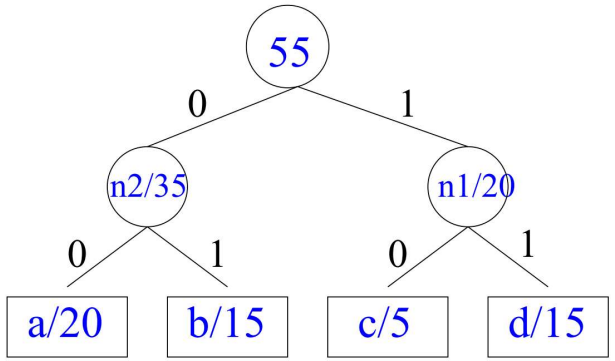


Alphabet is now A_2 = {**n2** / 35, **n1**: / 20, **e** / 45}

4.4.4.2 Huffman code

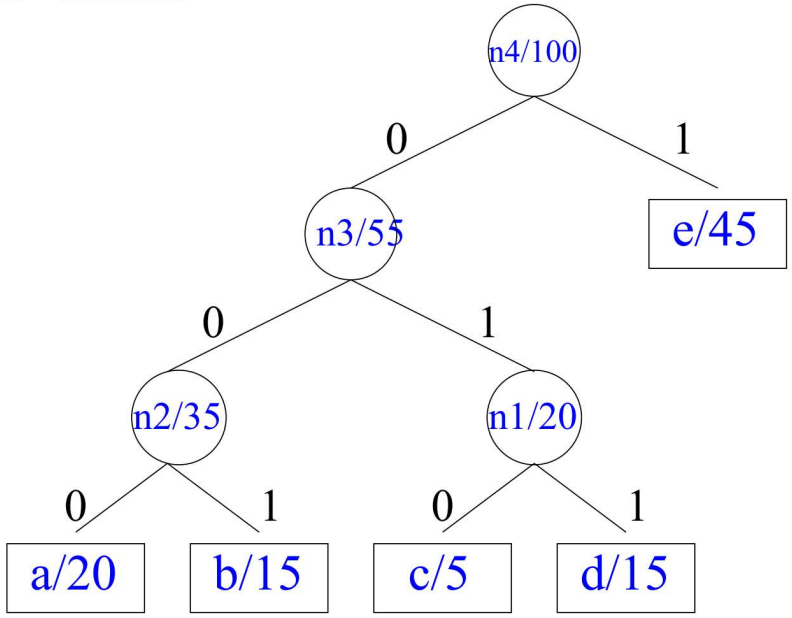
Alphabet is now A_2 = {**n2** / 35, **n1** / 20, **e** / 45}

Algorithm merges n1 and n2:



Alphabet is now A_2 = {**n3** / 55, **e** / 45}

Algorithm merges e and n3 and finishes



Huffman code is:

a = 000, b = 001, c = 010, d = 011, e = 1

This is the optimum (minimum-cost) prefix code for this distribution.

4.4.4.2 Huffman code: Creating Tree

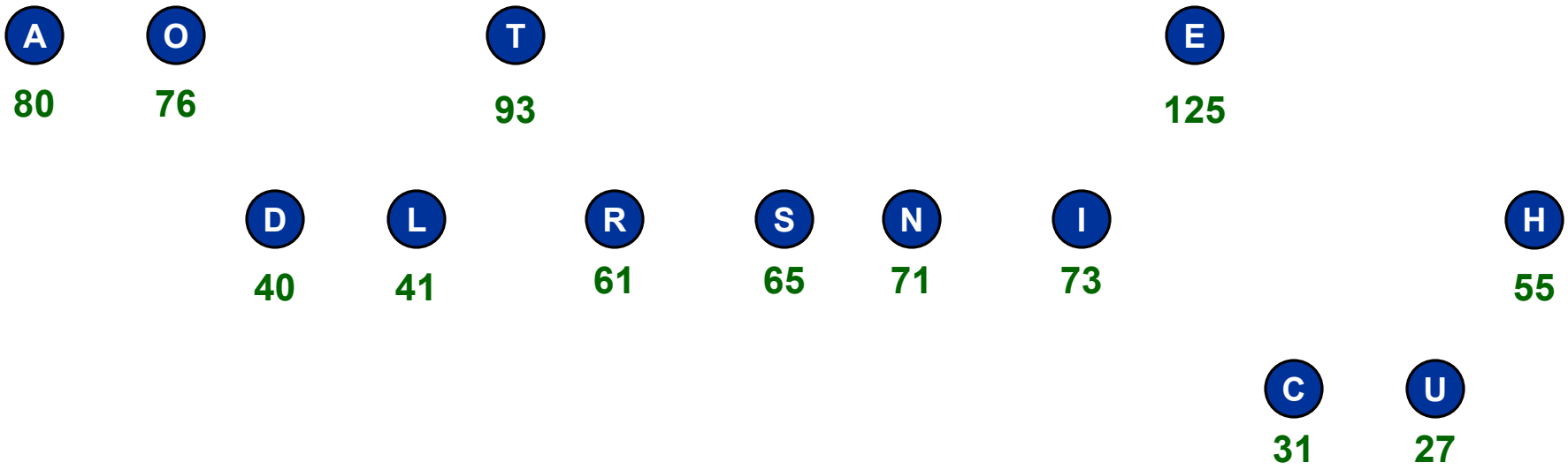
- Algorithm:
  - Place each symbol in leaf
    - Weight of leaf = symbol frequency
  - Select two trees L and R (initially leafs)
    - Such that L, R have lowest frequencies in tree
  - Create new (internal) node
    - Left child $\Rightarrow$ L
    - Right child $\Rightarrow$ R
    - New frequency $\Rightarrow$ frequency(L) + frequency(R)
  - Repeat until all nodes merged into one tree

4.4.4.2 Huffman code

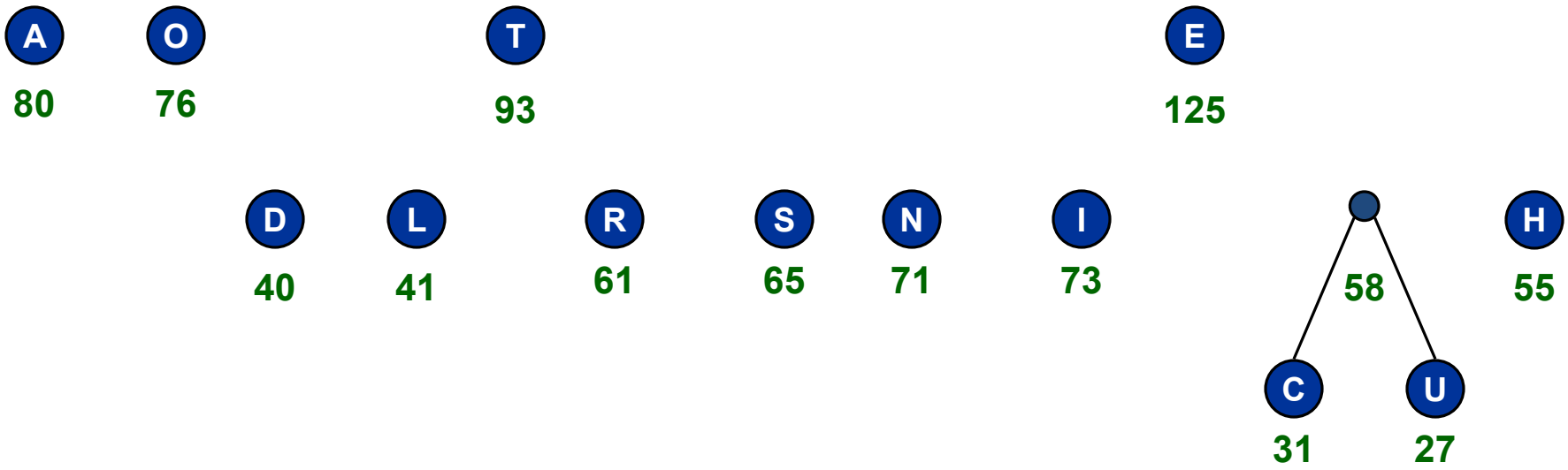
- Frequency of characters used in a text file:

Char	Freq
E	125
T	93
A	80
O	76
I	72
N	71
S	65
R	61
H	55
L	41
D	40
C	31
U	27

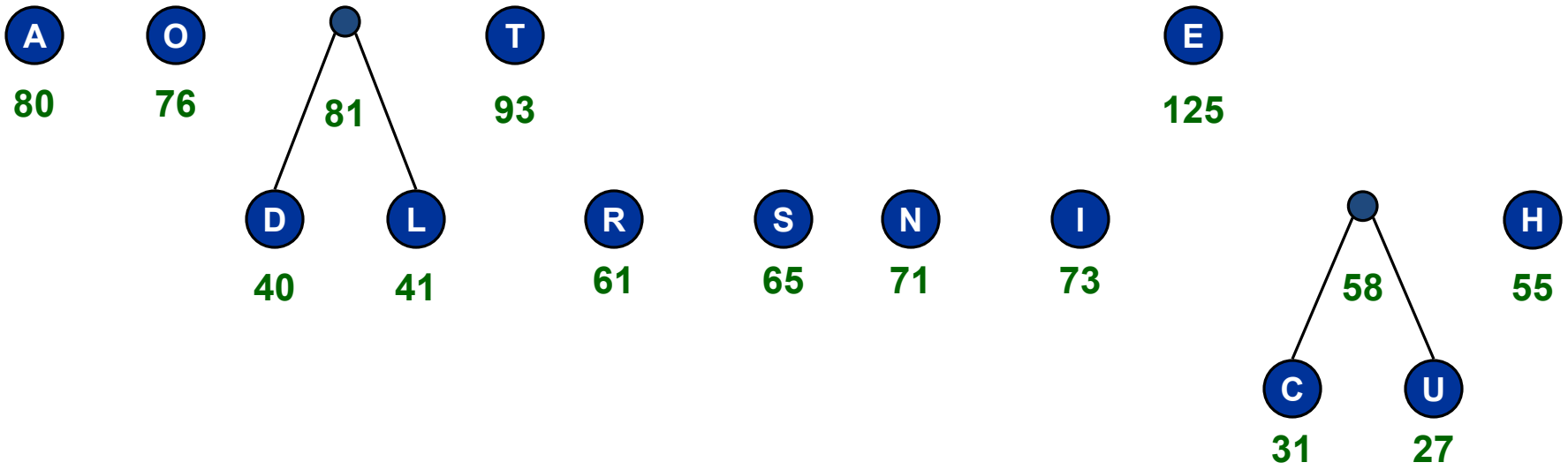
4.4.4.2 Huffman code



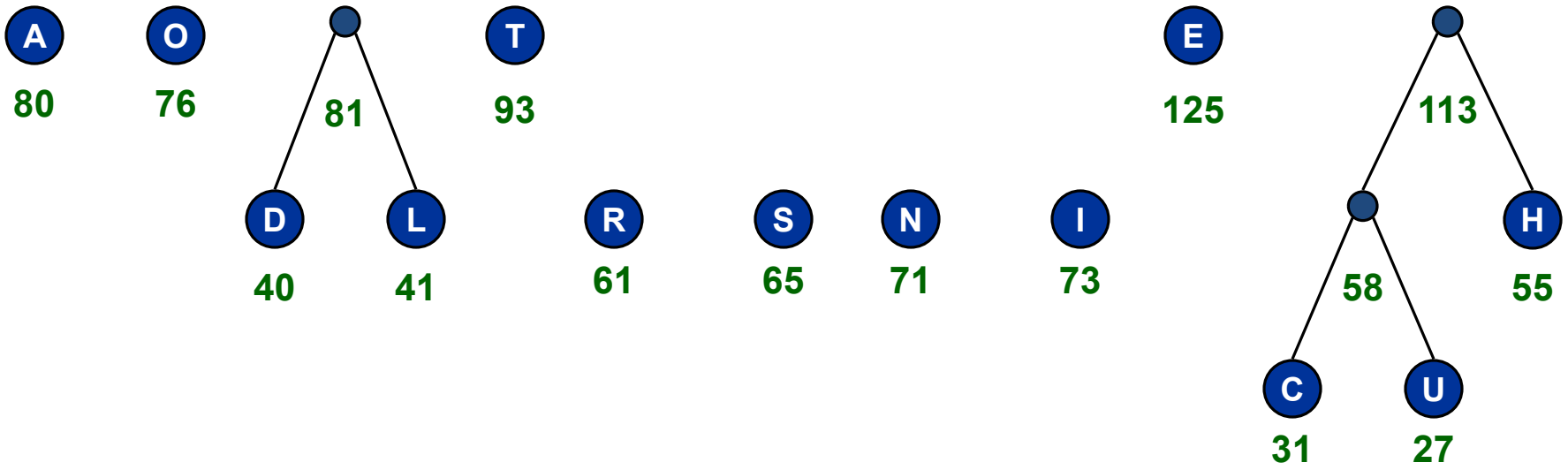
4.4.4.2 Huffman code



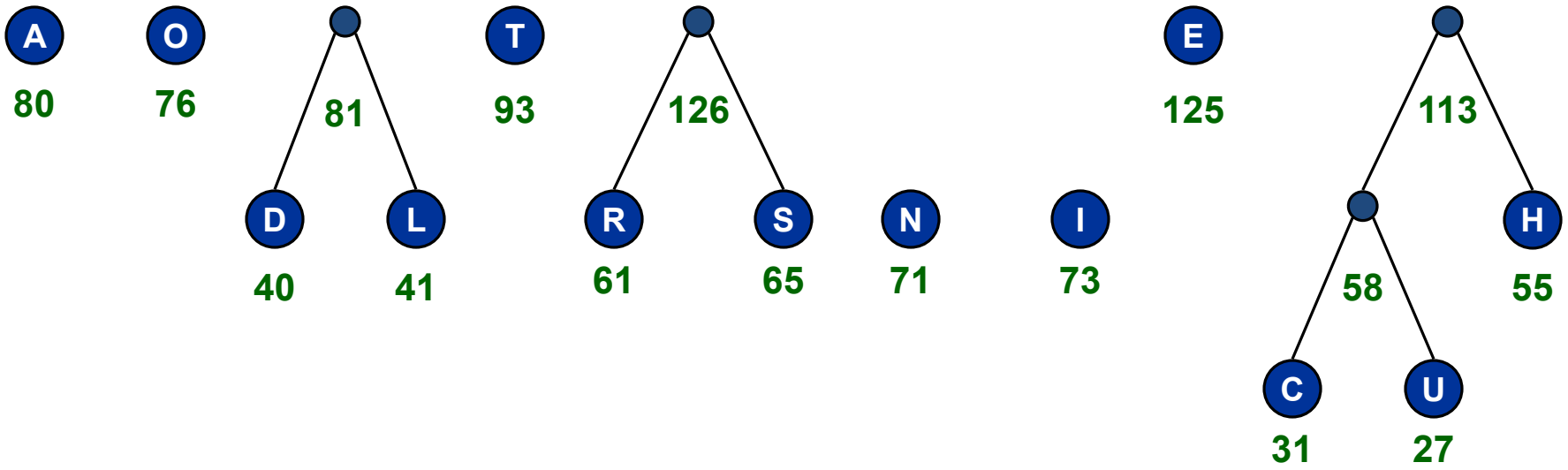
4.4.4.2 Huffman code



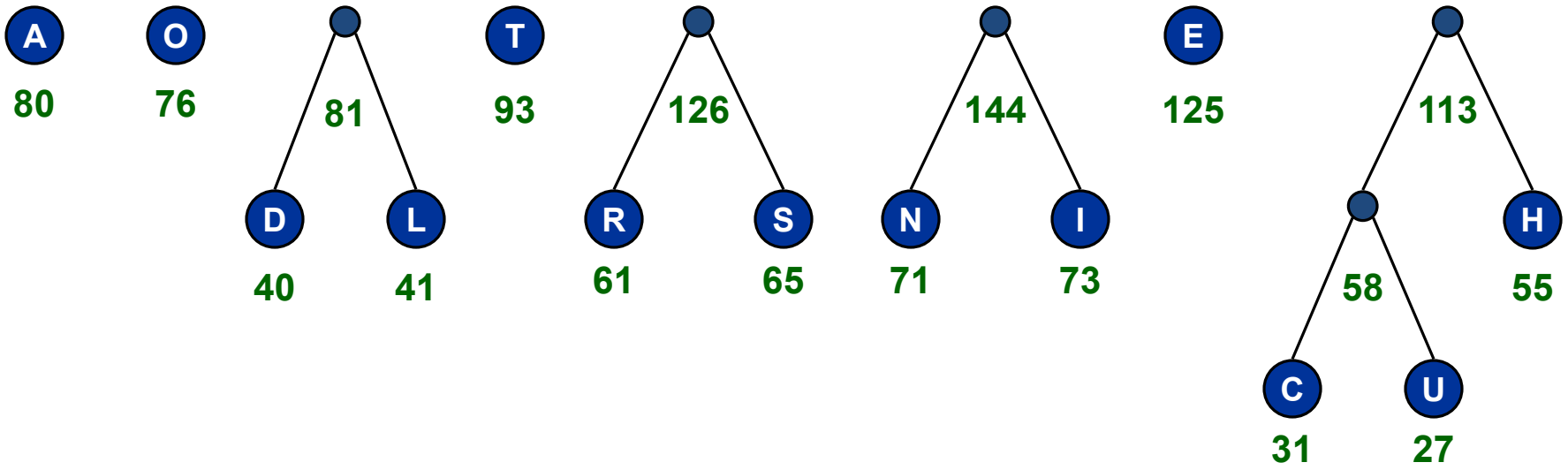
4.4.4.2 Huffman code



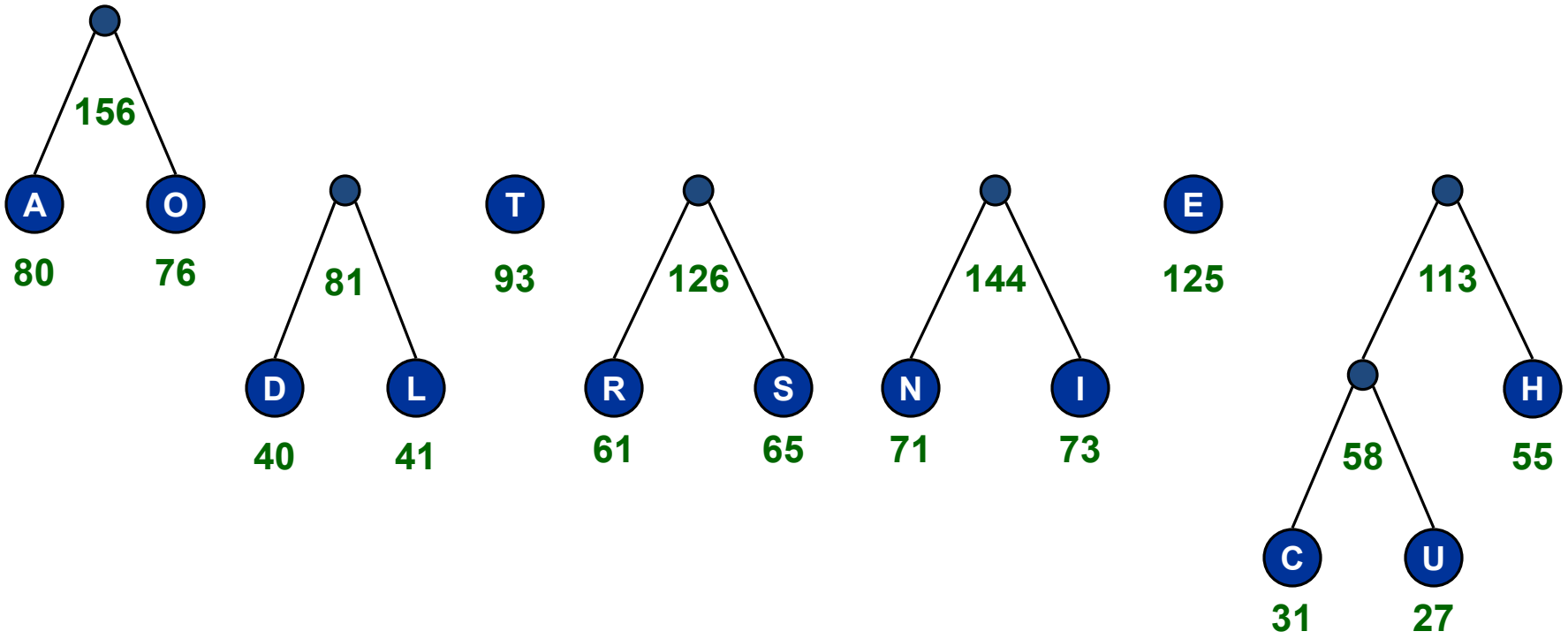
4.4.4.2 Huffman code



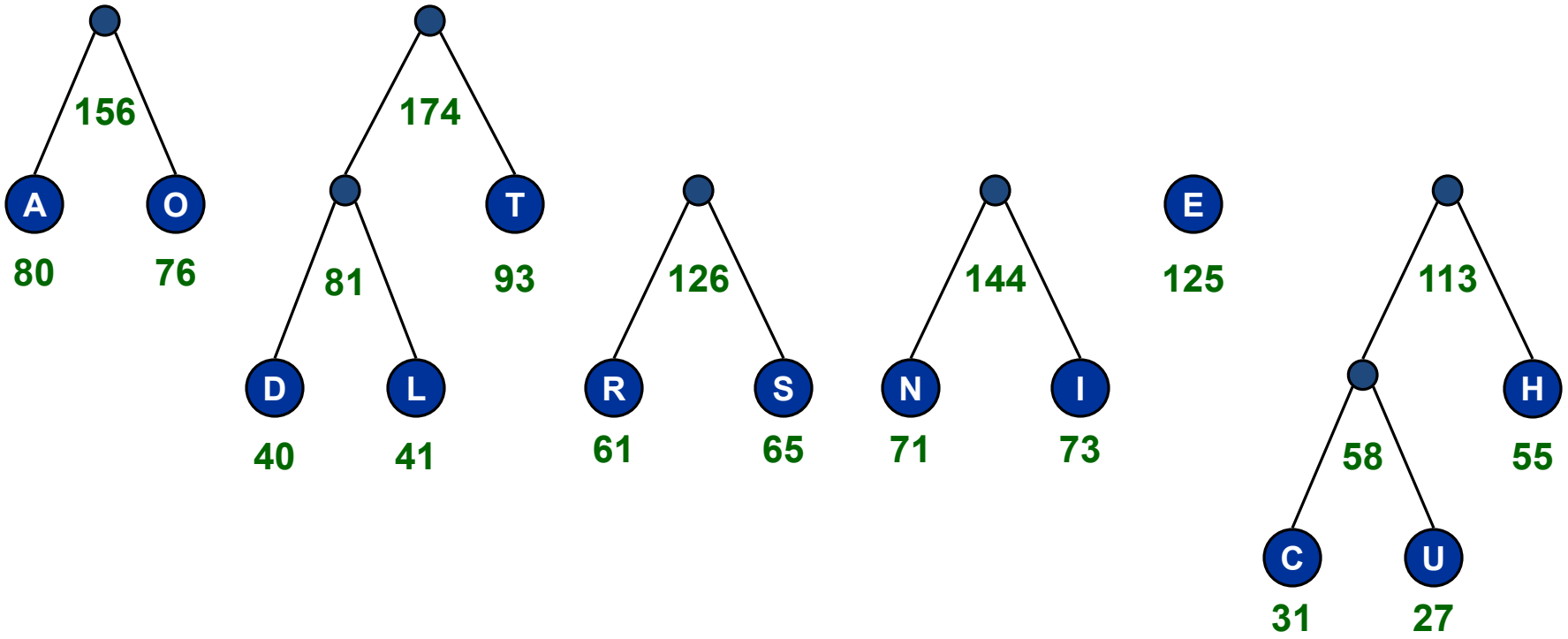
4.4.4.2 Huffman code



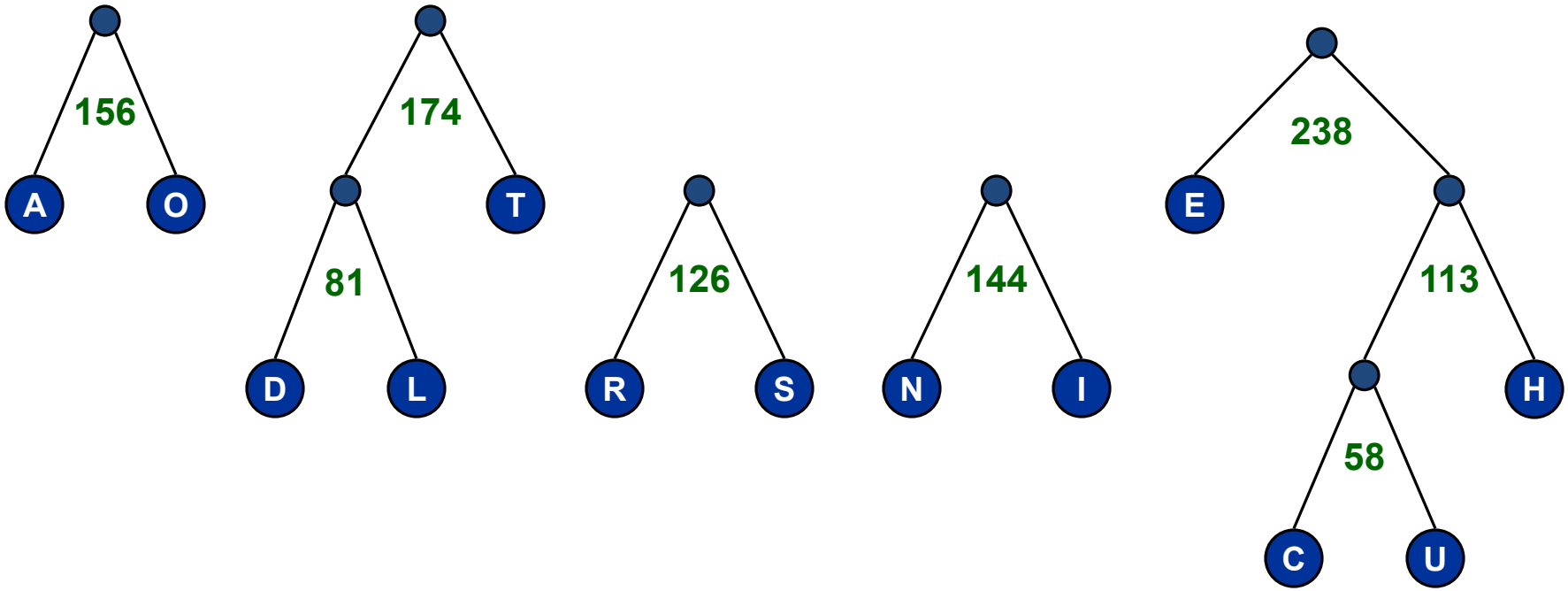
4.4.4.2 Huffman code



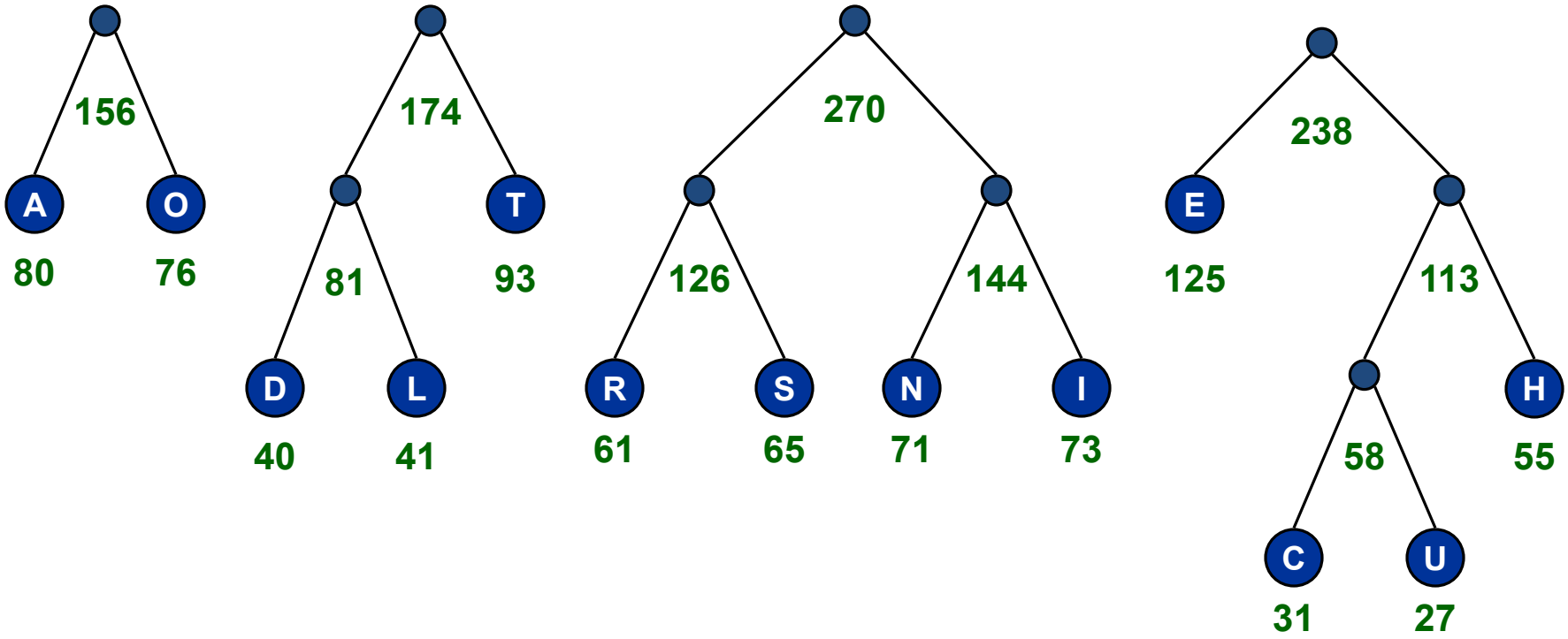
4.4.4.2 Huffman code



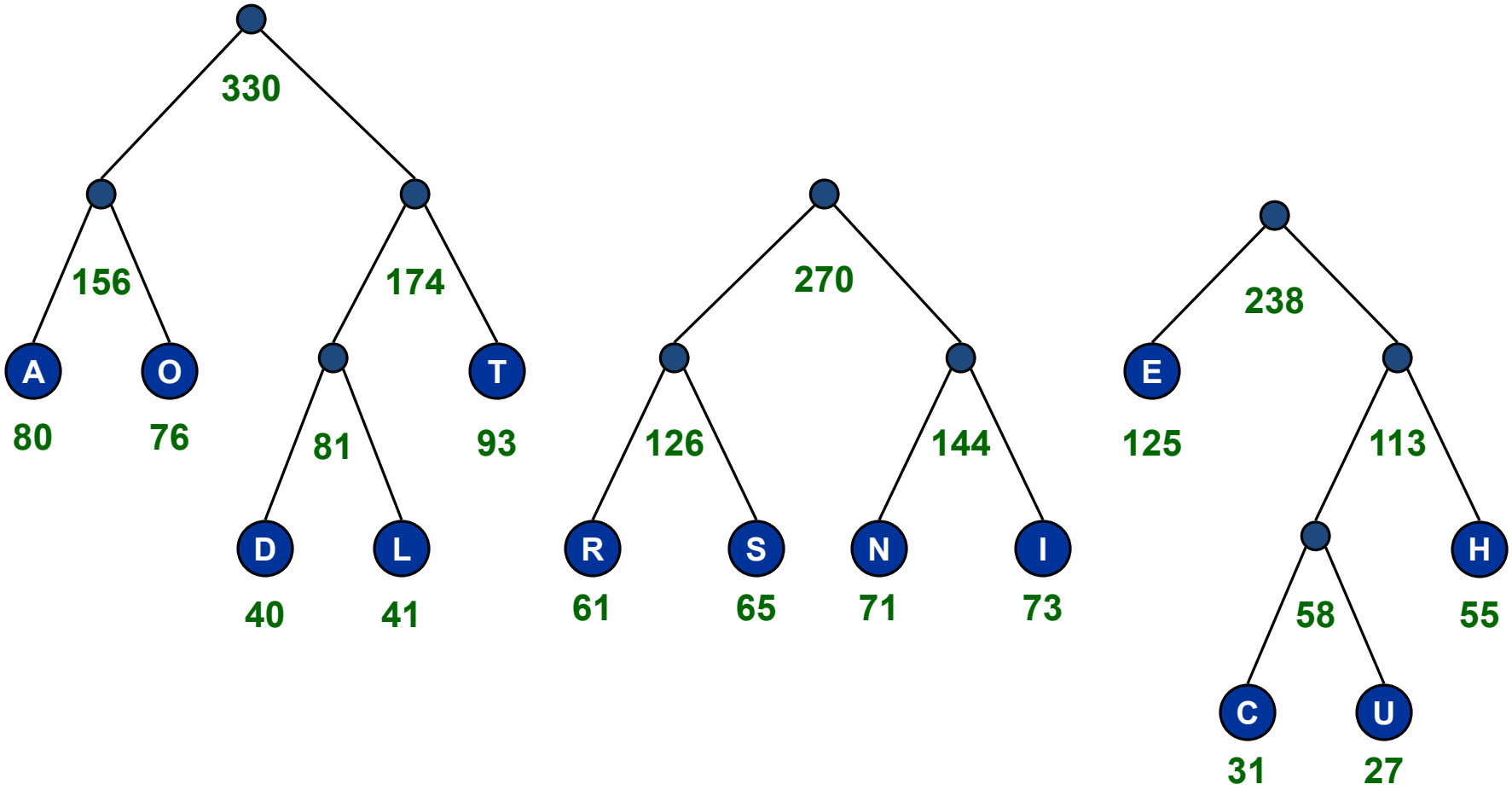
4.4.4.2 Huffman code



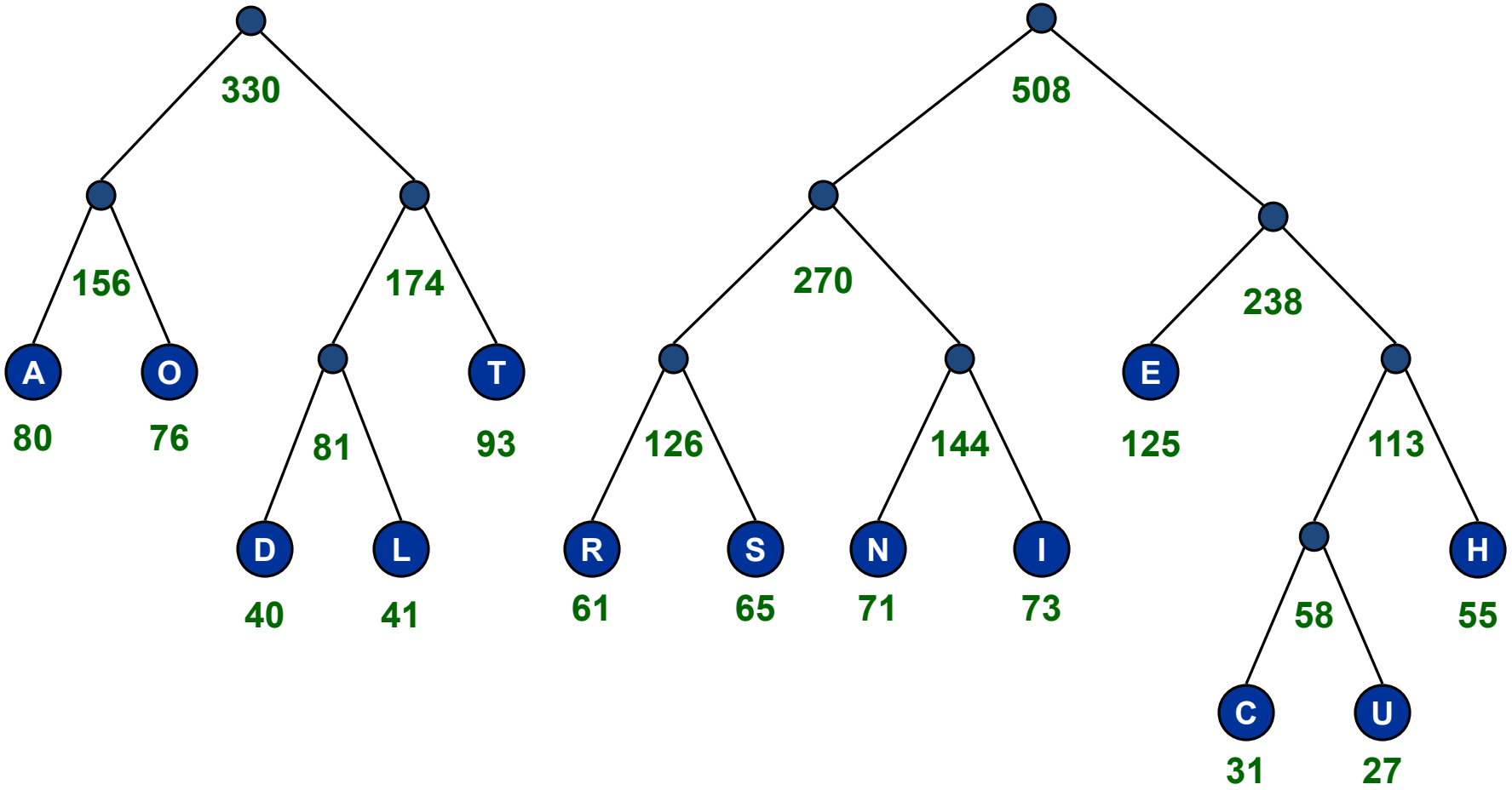
4.4.4.2 Huffman code



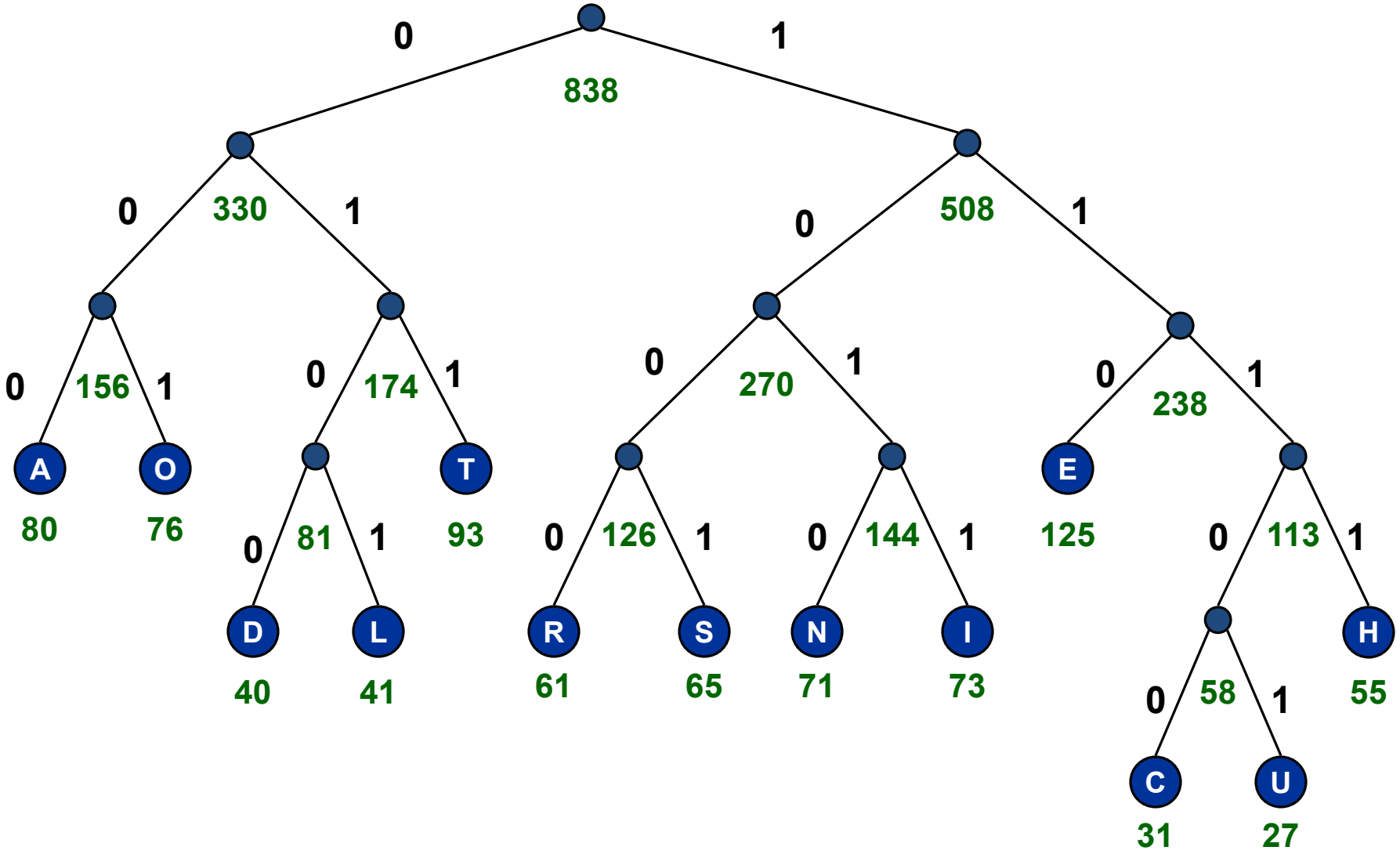
4.4.4.2 Huffman code



4.4.4.2 Huffman code



4.4.4.2 Huffman code



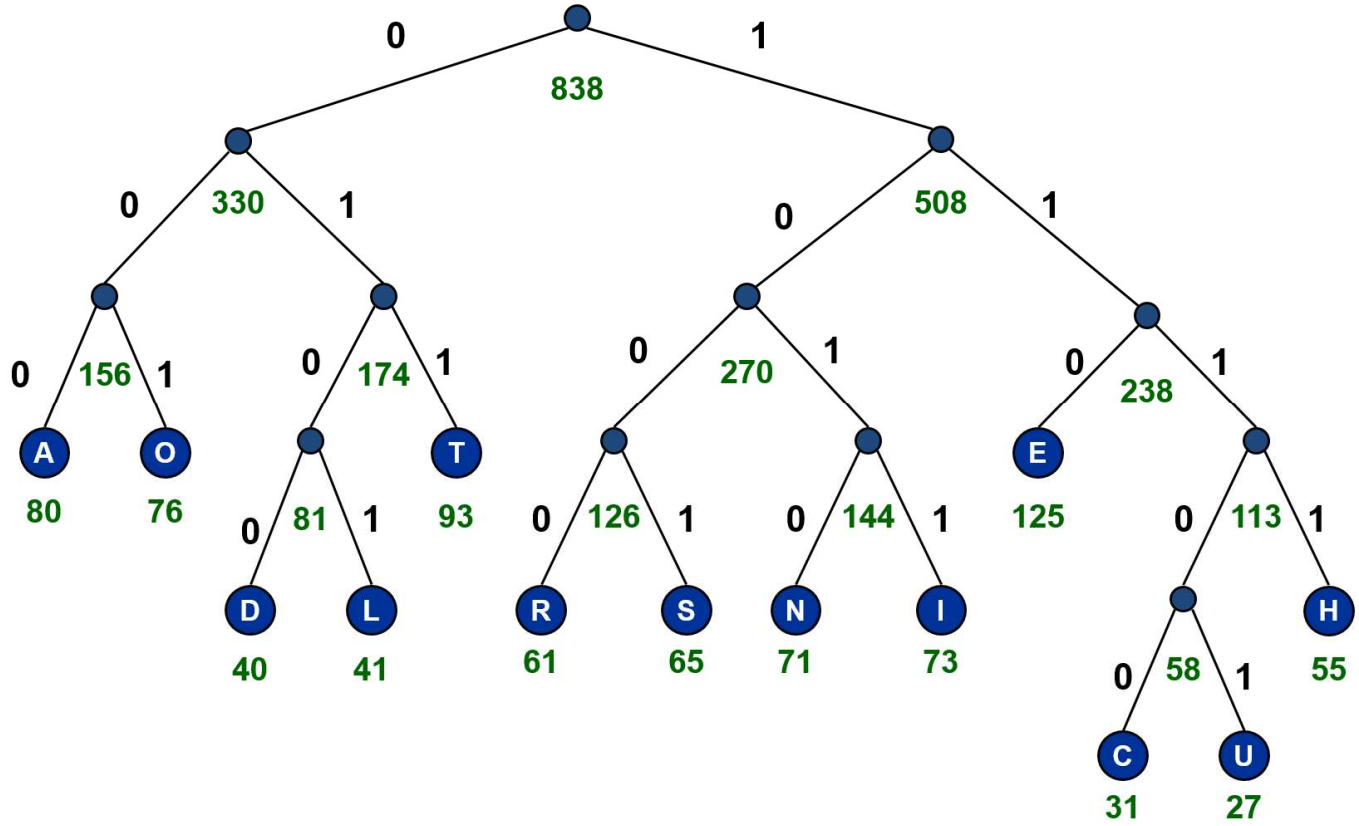
4.4.4.2 Huffman code

Char	Frequency	Fixed-length code	Huffman code
E	125	0000	110
T	93	0001	011
A	80	0010	000
O	76	0011	001
I	73	0100	1011
N	71	0101	1010
S	65	0110	1001
R	61	0111	1000
H	55	1000	1111
L	41	1001	0101
D	40	1010	0100
C	31	1011	11100
U	27	1100	11101
Sum	838	3352	3036

- Use Huffman code: save 10%
- Huffman encoding is a simple example of **data compression**: representing data in fewer bits than it would otherwise need

4.4.4.2 Huffman code: Decoding

- Algorithm decoding:
  - Read compressed file & binary tree
  - Use binary tree to decode file
    - Follow path from root to leaf
- Example: Decode “11100000011”



Data compression

- Huffman encoding is a simple example of data compression: representing data in fewer bits than it would otherwise need
- A more sophisticated method is GIF (Graphics Interchange Format) compression, for .gif files
- Another is JPEG (Joint Photographic Experts Group), for .jpg files
  - Unlike the others, JPEG is lossy—it loses information
  - Generally OK for photographs (if you don't compress them *too* much), because decompression adds "fake" data very similar to the original