Wave equation

$$\frac{1}{c^2} \cdot \frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} = 0 \tag{1}$$

The general form of the solution is $f(x,t) = g(x - ct) + h(x + ct)$

Progressive plane wave equation

$$f(x,t) = A\cos(\kappa x - \omega t) = A\ \mathrm{Re}(e^{i(\kappa x - \omega t)}) \tag{2}$$

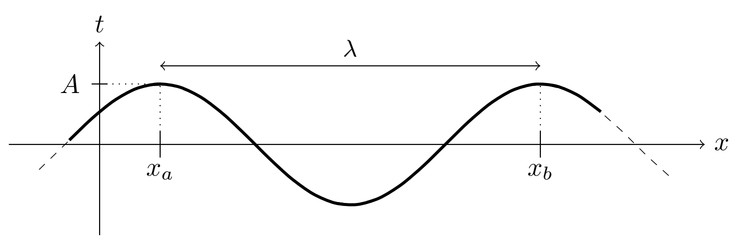

Figure 1: Wave : spacial representation

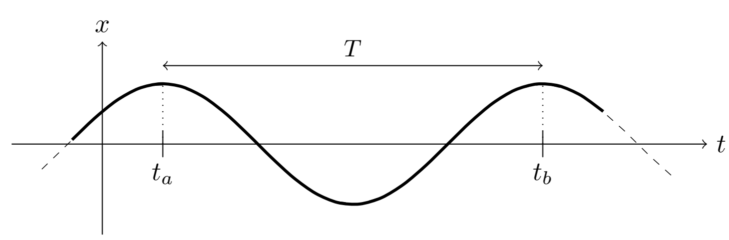

Figure 2: Wave : time representation

Dispersion relation

Proof p. 2

By plotting (2) in exponential form into (1), with found that f if a wave if and only if:

$$\kappa^2 = \frac{\omega^2}{c^2} \tag{3}$$

Wave length relation

Proof p. 2

$$\lambda = \frac{2\pi}{\kappa} \tag{4}$$

Where κ is the wave number.

Period relation

Proof p. 2

$$T \stackrel{\text{def}}{=} \frac{1}{\nu} = \frac{\lambda}{c} \tag{5}$$

Proofs

Dispersion relation (3)

Referring to Fig. 2.

The wave is given by the equation $f(x,t) = A\ \mathrm{Re}(e^{i(\kappa x - \omega t)})$.

$$\frac{\partial f}{\partial t} = A(-i\omega)e^{i(\kappa x-\omega t)} \qquad \frac{\partial^2 f}{\partial t^2} = -A\omega^2 e^{i(\kappa x-\omega t)}$$
$$\frac{\partial f}{\partial x} = A(i\kappa)e^{i(\kappa x-\omega t)} \qquad \frac{\partial f}{\partial x^2} = -A\kappa^2 e^{i(\kappa x-\omega t)}$$

Using the equation (1), we obtain the relation

$$-\frac{\omega^2}{c^2} + \kappa^2 = 0$$

Hence, f is a wave if and only if

$$\kappa^2 = \frac{\omega^2}{c^2}$$

Wave length (4)

Referring to Fig. 1.

$$\begin{aligned} f(x_b,t) &= f(x_a,t) \\ A\cos(\kappa x_b - \omega t) &= A\cos(\kappa x_a - \omega t) \\ \kappa x_b - \omega t &= \kappa x_a - \omega t + 2\pi \\ \kappa(x_b - x_a) &= 2\pi \\ x_b - x_a &= \frac{2\pi}{\kappa} \\ \lambda &= \frac{2\pi}{\kappa} \end{aligned}$$

Period (5)

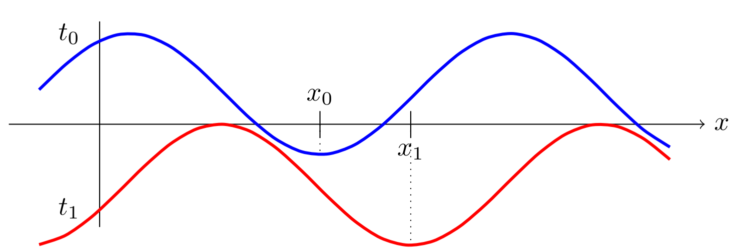

Figure 3: Delayed waves

Considering Fig. 3, the red wave is delayed from the blue one by a time $t_1 - t_0$. And we have the following equality.

$$\begin{aligned} e^{i(\kappa x_0 - \omega t_0)} &= e^{i(\kappa x_1 - \omega t_1)} \\ \kappa x_0 - \omega t_0 &= \kappa x_1 - \omega t_1 \\ \omega(t_1 - t_0) &= \kappa(x_1 - x_0) \\ x_1 - x_0 &= \frac{\omega}{\kappa}(t_0 - t_1) \end{aligned}$$

And $c = \frac{\omega}{\kappa}$, $\lambda = x_1 - x_0$, $T = t_0 - t_1$. Hence $T = \frac{\lambda}{c}$.