Sorbonne Universite

Master 1 - Quantum Information

2021/2022

# Lecture notes
# -
# Quantum Kinematic

# Contents

# 1 Introduction

Physical system which has $d \in \mathbb{N}$ possible distinguishable states. Its physical state $|\psi\rangle \in \mathscr{H}$, the Hilbert space $\mathbb{C}^d$.

$$|\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_d \end{bmatrix} \text{ and } \forall i, \psi_i \in \mathbb{C}. \tag{1}$$

The result of the measurement in the computational basis on $|\psi\rangle$ is $i \in [1, \cdots, d]$ with probability $|\psi_i|^2$.
And $\sum_{i=1}^{d} |\psi_i|^2 = \langle\psi|\psi\rangle = 1$: the state is normalized.

## 1.1 Dirac notation

- Ket:

$$|\psi\rangle = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_d \end{bmatrix} = \psi_1 |1\rangle + \cdots + \psi_d |d\rangle = \sum_{i=1}^{d} \psi_i |i\rangle \tag{2}$$

- Bra:

$$\langle\psi| = |\psi\rangle^\dagger = |\psi^*\rangle^T \tag{3}$$

- Braket:

$$\langle\psi|\phi\rangle = \begin{bmatrix} \psi_1^* \cdots \psi_d^* \end{bmatrix} \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_d \end{bmatrix} = \psi_1^*\phi_1 + \cdots + \psi_d^*\phi_d \tag{4}$$

  $\langle\psi|\phi\rangle$ is the hermitian product of $\psi$ and $\phi$.

## 1.2 Measurement in a basis $B$

$B$ is an orthonormal basis : $B := \{|b_i\rangle\}_{i=1}^{d}$. B has the following properties:

$$\begin{aligned} &\forall i \, \langle b_i|b_i\rangle = \delta_{i,j} \quad \textit{(orthonormality)} \\ &\sum_{i=1}^{d} |b_i\rangle \langle b_j| = I \ \textit{(completeness)} \end{aligned} \tag{5}$$

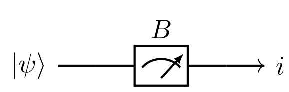


Figure 1: Circuit representation of the measurement of the state $|\psi\rangle$

The probability of the output of a measurement is given by the following formula :

$$\mathbb{P}(out = |b_i\rangle) = |\, \langle b_i|\psi\rangle \,|^2 \tag{6}$$

The physical object is projected into the state $|b_i\rangle$, this is physically called the "wave packet reduction".

### 1.2.1 Qubit

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{7}$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{8}$$

### 1.2.2 Measurement in the basis $\{|\pm\rangle\}$

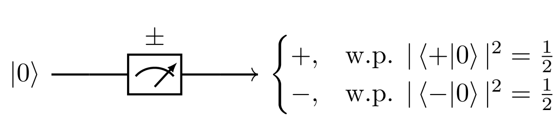


Figure 2: Measure of the state $|0\rangle$ in the basis $|\pm\rangle$

### 1.3 Pauli matrices and properties

There are four extremely useful two by two matrices calles the *Pauli matrices*.

$$\sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma_1 \equiv \sigma_x = X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

**Pauli matrices properties**

- braket decomposition
- commutation relation

$$[X, Y] = 2iZ; \quad [Y, Z] = 2iX; \quad [Z, X] = 2iY \tag{9}$$

### 1.4 Wiesner's Quantum Money

Based on the conjugate coding.

- `bills`:
  - serial number
  - a set of random qubit $E_r \in \{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}^n$
  - `mint` knows {Serial Number + Random}, sends it to the bank.
- `Mint`: makes the bill, and gives it to the forger.
- `Forger`: tries to copy the bill, and spends the two to the bank.
- `Bank`: should accept the true one, reject the fake.

| `mint` | `forger` basis | `forger` m. | `bank` m. |
|---|---|---|---|
| $\|0\rangle$ | $\{\|0\rangle, \|1\rangle\}$ | $\|0\rangle$ | $\|0\rangle$ |
| $\|0\rangle$ | $\{\|\pm\rangle\}$ | $\begin{cases} \|+\rangle, & \text{w.p. } \frac{1}{2} \\ \|-\rangle, & \text{w.p. } \frac{1}{2} \end{cases}$ | $\begin{cases} \|0\rangle, & \text{w.p. } \frac{1}{2} \\ \|1\rangle, & \text{w.p. } \frac{1}{2} \end{cases}$ |

We therefore deduce that

$$\mathbb{P}(\text{get caugth}) = 1 - (1 - \frac{1}{4})^n = 1 - (\frac{1}{4})^n \tag{10}$$

### 1.5 Bennett and Brassard Quantum Key Exchange: BB84

Goal: Alice and Bob → share a secret bit string , Eve does not know anything.
Settings: Alice and Bob share a quantum channel and an authenticated classical channel.
Steps:

1. Alice prepares $n$ qubits $E_r \in \{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}^n$, and she sends them to Bob
2. Bob receives . He measures them in the basis $\{B_{0,1}, B_{+,-}\}$
3. They use the public classical channel to compare the basis Bob used. They throw away the *bad basis* qubits.
4. Alice and Bob sample the data and compare the error rate $e$. If $e = 0$, they keep the key; if $e = 25\%$, Eve kowns the key.

What if $0 < e < 25$ ? Eve knows a part of the key.

## 2 Unitary transformation

A transformation is an isolated system, and it is reversible.
Let $T$ to be a transformation.

$$\langle T(|\psi\rangle)|T(|\psi\rangle)\rangle = \langle \psi|\psi\rangle \tag{11}$$

$T$ is linear.

$$T(\alpha |\psi\rangle + \beta |\phi\rangle) = \alpha T(|\psi\rangle) + \beta T(|\phi\rangle) \tag{12}$$

$T$ acts like an unitary operator. $T$ corresponds to a complex matrix $U$: $T(|\phi\rangle) = U|\phi\rangle$, $U \in \mathbb{C}^{n \times n}$, such that $U^\dagger U = Id$.

In the basis $\{|i\rangle\}_{i=0}^n$,$\langle T(|\psi\rangle)|T(|\psi\rangle)\rangle = \langle i|j\rangle = \delta_{i,j}$

We have :

- measurement in computational basis
- a machine making arbitrary unitary $U$

Let's build a measurement in basis $\{|b_i\rangle\}_i$

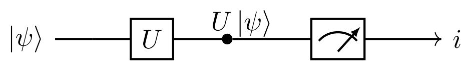


Figure 3: Circuit representation of the measurement unitary exptected behavior

$$\mathbb{P}(i) \overset{\text{def}}{=} |\langle i|U|\psi\rangle|^2 \overset{\text{goal}}{=} |\langle b_i|\psi\rangle|^2 \quad \forall\psi \tag{13}$$

We want $\langle i| U = \langle b_i| \Leftrightarrow U^\dagger |i\rangle = |b_i\rangle \Leftrightarrow U = \sum_i |i\rangle \langle b_i|$
Is $U$ an unitary ?

$$\begin{aligned}
U^\dagger U &= \Big(\sum_i |b_i\rangle \langle i|\Big)\Big(\sum_j |j\rangle \langle b_j|\Big) \\
&= \sum_{i,j} |b_i\rangle \langle i|j\rangle \langle b_j| \\
&= \sum_i |b_i\rangle \langle b_i| \\
&= Id \qquad\qquad\qquad\qquad U \text{ is an unitary.}
\end{aligned} \tag{14}$$

# 3 Composition of systems

Let $A \in \mathscr{H}_A = \mathbb{C}^{d_A}$ and $B \in \mathscr{H}_B = \mathbb{C}^{d_B}$ to be two systems in their respective vector spaces. Then we can construct the space

$$\mathscr{H}_{AB} = \mathscr{H}_A \otimes \mathscr{H}_B \tag{15}$$

Its orthonormal basis is $\{|ij\rangle_{AB}\}_{i,j}$, and

$$\dim\mathscr{H}_{AB} = \dim\mathscr{H}_A \cdot \dim\mathscr{H}_B \tag{16}$$

If $|\alpha\rangle = \sum_i \alpha_i |i\rangle_A$ and $|\beta\rangle = \sum_i \beta_i |i\rangle_B$, then

$$|\phi\rangle_{AB} = |\alpha\rangle \otimes |\beta\rangle = \sum_{i,j} \alpha_i\beta_j |i\rangle_A |j\rangle_B \tag{17}$$

and $|\phi\rangle_{AB} \in \mathscr{H}_{AB}$. $|\phi\rangle_{AB}$ is a joint state of systems $A$ and $B$.
The inner product between two basis states can be defined as

$$\langle i,j|k,l\rangle = \langle i|k\rangle_A \langle j|l\rangle_B = \delta_{ik}\delta_{jl} \tag{18}$$

The most general state in the space $\mathscr{H}_{AB}$ can be written

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \tag{19}$$

with the usual condition for $|\psi\rangle$ to be normalized:

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1 \tag{20}$$

Not all states of $\mathscr{H}_{AB}$ are separable into one state of $\mathscr{H}_A$ and one state of $\mathscr{H}_B$
For example : $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathscr{H}_{AB}$, but $\nexists |\alpha\rangle \in \mathscr{H}_A, |\beta\rangle \in \mathscr{H}_B$, such that $|\alpha\rangle \otimes |\beta\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
Necessary condition on the coefficients $(\alpha, \beta, \gamma, \delta)$ of a state to be separable:

$$\alpha\delta = \beta\gamma \tag{21}$$

## 3.1 No cloning theorem

**The no cloning theorem**

$$\text{There is no } U \text{ such that } \forall |\psi\rangle \in \mathscr{H}, U|\psi\rangle = |\psi\rangle \otimes |\psi\rangle \in \mathscr{H} \otimes \mathscr{H}. \tag{22}$$

**Proof**

Suppose there exists a such unitary U, then U is a cloning operator.

$$\begin{aligned} U\left|0\right\rangle &\stackrel{\text{def}}{=} \left|0\right\rangle\left|0\right\rangle \\ U\left|1\right\rangle &\stackrel{\text{def}}{=} \left|1\right\rangle\left|1\right\rangle \end{aligned} \tag{23}$$

By computing the application of $U$ on the state $\left|+\right\rangle$, we get on the one hand, by linearity of unitaries

$$U(\frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}}) = \frac{\left|00\right\rangle + \left|11\right\rangle}{\sqrt{2}} \tag{24}$$

and on the other hand, by definition of the operator behavior

$$U(\frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}}) = \frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}} \otimes \frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}} = \frac{1}{2}(\left|00\right\rangle + \left|01\right\rangle + \left|10\right\rangle + \left|11\right\rangle) \tag{25}$$

which is a contradiction. Then such a $U$ operator can not exist.

## 3.2 Superdense coding

*Superdense coding* involves two parties, `Alice` and `Bob`. The protocol allows `Alice` and `Bob` to share two bits of information by exchanging just one qubit. Basically, `Alice` is in possession of two classical bits of information, which she wishes to send to `Bob`.

Suppose `Alice` and `Bob` initially share a pair of qubits in the entangled state

$$\left|\psi\right\rangle = \frac{\left|00\right\rangle + \left|11\right\rangle}{\sqrt{2}} \tag{26}$$

Here is the procedure.

| The state `Alice` wants to send | The gate she applies | The states after |
|---|---|---|
| 00 | $I$ | $\frac{\left|00\right\rangle+\left|11\right\rangle}{\sqrt{2}} = \left|\phi^+\right\rangle$ |
| 01 | $Z$ | $\frac{\left|00\right\rangle-\left|11\right\rangle}{\sqrt{2}} = \left|\phi^-\right\rangle$ |
| 10 | $X$ | $\frac{\left|10\right\rangle+\left|01\right\rangle}{\sqrt{2}} = \left|\psi^+\right\rangle$ |
| 11 | $Y$ | $\frac{\left|01\right\rangle-\left|10\right\rangle}{\sqrt{2}} = \left|\psi^-\right\rangle$ |

`Alice` send her qubit to `Bob` and he measures the resulting pair in the base
$\{\left|\phi^+\right\rangle, \left|\phi^-\right\rangle, \left|\psi^+\right\rangle, \left|\psi^-\right\rangle\}$. This is indeed a basis, and its name is the *Bell basis*, and the states are called the *Bell states*.

## 3.3 Quantum teleportation

# 4 Measurements

## 4.1 Projective measurement

A projective measurement is described by an observable, a Hermitian operator. They are defined by a set of projectors $\{\Pi_j\}_{j=1}^k, k \leq d$.
Projectors properties:

$$\forall j, \Pi_j^2 = \Pi_j \qquad \Pi_j \Pi_i = \delta_{i,j}\Pi_j \tag{27}$$

A projector is defined as follows:

$$\Pi_j = \sum_{l=1}^{d_j} \left|l_l^j\right\rangle\left\langle l_l^j\right| \tag{28}$$

Upon measuring the state $\left|\psi\right\rangle$, the probability of getting result $j$ is given by

$$\left\langle\psi\middle|\Pi_j\middle|\psi\right\rangle = \|\Pi_j\left|\psi\right\rangle\|^2 \tag{29}$$

Given that outcome $j$ occured, the state of the quantum system immediately after the measurement is

$$\frac{\Pi_j\left|\psi\right\rangle}{\|\Pi_j\left|\psi\right\rangle\|^2} \tag{30}$$

## 4.2 Observables

Observables correspond to physical quantities, with values in $\mathbb{R}$. They are well defined in a basis $\{\left|b_i\right\rangle\}_i$ (i.e $\forall\left|b_i\right\rangle, \exists a_i \in \mathbb{R}$)

**Note :** $\alpha |b_1\rangle + \beta |b_2\rangle$ has **not always** a well defined value.

An observable is defined as follow:

$$O \stackrel{\text{def}}{=} \sum_i o_i \underbrace{|b_i\rangle \langle b_i|}_{\text{projector on } |b_i\rangle} = \sum_j o_j \Pi_j \tag{31}$$

$O$ is diagonalizable by definition and $O^\dagger = O$: $O$ is hermitian.

$$\text{Shape of } O \ : \begin{pmatrix} o_1 & 0 & \cdots & 0 \\ 0 & o_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_d \end{pmatrix}$$

The probability of getting the result $i$ by measuring $O$ on a state $|\psi\rangle$ is $\langle\psi|\Pi_i|\psi\rangle$

### 4.2.1 Expectation value and standard deviation

The expectation value of $O$, written $\langle O\rangle$, is given by

$$\begin{aligned} \langle O\rangle &= \sum_i o_i \mathbb{P}(i\,|\psi\rangle) \\ &= \sum_i o_i \|\Pi_i\,|\psi\rangle\,\|^2 \\ &= \sum_i o_i \,\langle\psi|\Pi_i|\psi\rangle \\ &= \langle\psi| \sum_i o_i \Pi_i\,|\psi\rangle \\ &= \langle\psi|O|\psi\rangle \end{aligned} \tag{32}$$

From this formula for the expectation value follows a formula for the standard deviation associated to the observation of $O$

$$\Delta^2 O = \langle (O - \langle O\rangle)^2\rangle = \langle O^2\rangle - \langle O\rangle^2 \tag{33}$$

**Note:** If $|\psi\rangle$ is an eigenstate of $O$, then $O\,|\psi\rangle = \lambda\,|\psi\rangle$.
Hence:

$$\begin{aligned} \langle O\rangle &= \langle\psi|O|\psi\rangle \\ &= \langle\psi|\lambda|\psi\rangle \\ &= \lambda\,\langle\psi|\psi\rangle \\ &= \lambda \end{aligned} \tag{34}$$

And:

$$\begin{aligned} O\,|\psi\rangle = \langle O\rangle\,|\psi\rangle &\Rightarrow \Delta^2 O = (\lambda^2 - \lambda^2) = 0 \\ &\Rightarrow \Delta O = 0 \end{aligned} \tag{35}$$

### 4.2.2 Commutators

A key property of quantum physics is the existence of incompatible measurements: for any physical property $A$, there exists another physical property $B$ which is incompatible with $A$. The incompatible means it is physically impossible to prepare a state $|\psi\rangle$ which gives perfectly predictible outputs for both measurements $A$ and $B$. Let us first assume $A$ and $B$ to be observables. A key property of this pair of observable is their commutator

$$[A, B] := AB - BA \tag{36}$$

If $A$ and $B$ commute (i.e $[A, B] = 0 \Leftrightarrow AB = BA$), there exists a basis such that the result of a measurement of $A$ and a measurement of $B$ are perfectly defined.

Conversely, if such a basis exists, then $[A, B] = 0$

Therefore, if $A$ and $B$ do not commute, they correspond to incompatible measurements. (The proofs are in the $4^{th}$ tutorial.)

### 4.2.3 The Robertson-Heisenberg uncertainty relation

This relation evaluates the sharpness of two observables we will call $A$ and $B$ through the standard deviations $\Delta A$ and $\Delta B$, and the states that, for any state $|\psi\rangle$ and any observable $A$ and $B$

$$\Delta A \Delta B \geq \frac{1}{2}|\,\langle\psi|[A, B]|\psi\rangle \tag{37}$$

**Example**

Using the Pauli matrix $\sigma_x = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |+\rangle\langle+| - |-\rangle\langle-|$.
Known results : $X|+\rangle = |+\rangle$ and $X|-\rangle = -|-\rangle$.
We define $|\theta\rangle := \cos\theta|0\rangle + \sin\theta|1\rangle$
Then

$$\begin{aligned} \langle X\rangle_{|\theta\rangle} &= \langle\theta|X|\theta\rangle \\ &= [\cos\theta \sin\theta] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta \\ sin\theta \end{bmatrix} \\ &= 2\sin\theta\cos\theta \\ &= sin2\theta \end{aligned} \tag{38}$$

## 4.3 Generalized measurements

A generalized measurement is defined by

$$\{K_i\}_i \text{ such that } \sum_i K_i^\dagger K_i = Id \tag{39}$$

where the $K_i$ are called Kraus Operators. The probability of getting the result $i$ from a general measurement operator is given by $\mathbb{P}(i) = \|K_i|\psi\rangle\|^2$ , and the state of the system just after the measurement is $K_i|\psi\rangle = \frac{K_i|\psi\rangle}{\|K_i|\psi\rangle\|^2}$

**Generalized measurement → Operator**

If $i \in \{1\}$ then $K_1^\dagger K_1 = Id \Rightarrow K_1$ is unitary.

**Generalized measurement → Set of projectors**

If $K_i := \Pi_i$ then $\sum_i K_i^\dagger K_i = \sum_i \Pi_i^\dagger \Pi_i = \sum_i \Pi_i = Id$

**Example**

With prob. $P_j$, I measure $\{\Pi_{ij}\}_i$ ($\sum_i \Pi_{ij} = Id$) and I measure $U_j$ on the output state. Probability of getting $ij$ :

$$\begin{aligned} \mathbb{P}(ij) &= P_j\langle\psi|\Pi_{ij}U^\dagger U\Pi_{ij}|\psi\rangle \\ &= P_j\langle\psi|\Pi_{ij}|\psi\rangle \end{aligned} \tag{40}$$

And the resulting state is $\dfrac{U\Pi_{ij}|\psi\rangle}{\|\Pi_{ij}|\psi\rangle\|^2}$
Let $\{K_{ij} = \sqrt{P_j}U\Pi_{ij}\}_{ij}$, then

$$\begin{aligned} \sum_{ij} K_{ij}^\dagger K_{ij} &= \sum_{ij} P_j\Pi_{ij}U^\dagger U\Pi_{ij} \\ &= \sum_j P_j \sum_i \Pi_{ij} \\ &= \sum_j P_j \\ &= Id \end{aligned} \tag{41}$$

Can we associate each set $\{K_i\}_i$ with a $U$ and a $\{\Pi_i\}_i$ ?

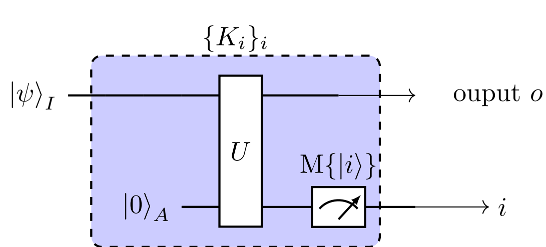


Figure 4: Circuit representation of such $U$ and $\{\Pi_i\}_i$.
Note : $\mathscr{H}_A \otimes \mathscr{H}_I = \mathscr{H}_O \otimes \mathscr{H}_M$

$\forall i$, the output state of the system is

$$(I_o \otimes |i\rangle_M\langle i|)U|\psi\rangle \otimes |0\rangle_A = |i\rangle_M\langle i|U|0\rangle_A|\psi\rangle_I \tag{42}$$

Assume $K_i =_M \langle i|U|0\rangle_A$. With (42), we deduce that the output state is $K_i |\psi\rangle$, w.p. $\langle\psi|K_i^\dagger K_i|\psi\rangle$. Is $\{K_i\}_i$ a valid set of operators ?

$$\begin{aligned}
\sum_i K_i^\dagger K_i &= \sum_i ({}_A\langle 0| \otimes I_I) U^\dagger (|i\rangle_M \otimes I_O)(I_O \otimes_M \langle i|) U (I_I \otimes |0\rangle_A) \\
&= ({}_A\langle 0| \otimes I_I)\, U^\dagger \underbrace{(\underbrace{\underbrace{\sum_i |i\rangle_M}_{=I_M} \otimes I_O)}_{=I_{OM}}(I_O \otimes_M \langle i|)}_{=I_{OA}} (I_I \otimes |0\rangle_A) \\
&= ({}_A\langle 0| \otimes I_I) I_{OA} (I_I \otimes |0\rangle_A) \\
&= I_O \qquad \{K_i\}_i \text{ is a valid set.}
\end{aligned} \tag{43}$$

**$\{K_i\}_i \to$ Unitary**

Let $U := \sum_i K_i \otimes |i\rangle_{MA} \langle 0| + \cdots$. The $\cdots$ represents extra terms used to make $U$ a unitary, but can be neglected in the computation. By tensoring with $|0\rangle_A$, we obtain

$$U |\psi\rangle \otimes |0\rangle_A = \sum_i K_i |\psi\rangle \otimes |i\rangle \tag{44}$$

And then

$$\begin{aligned}
{}_A\langle 0|U^\dagger U|0\rangle_A &=_A \langle 0| (\sum_i |0\rangle_{AM} \langle i| K_i^\dagger \cdot \sum_j K_j |j\rangle_{AM} \langle 0|) |0\rangle_A \\
&= \underbrace{{}_A\langle 0|0\rangle_A}_{=1} \cdot \sum_{ij} ({}_M\langle i| \otimes K_i^\dagger)(|j\rangle_M \otimes K_j) \underbrace{{}_A\langle 0|0\rangle_A}_{=1} \\
&= \sum_{ij} \underbrace{\langle i|j\rangle}_{\delta_{ij}} K_i^\dagger K_j \\
&= \sum_i K_i^\dagger K_i \\
&= Id
\end{aligned} \tag{45}$$

## 4.4 POVMs

POVMs means Projective Operator Valued Measure : we "only care about the output $i$, not the state $K_i |\psi\rangle$". The probability of getting $i$ is $\langle\psi|K_i^\dagger K_i|\psi\rangle$.
Let $E_i = K_i^\dagger K_i$. POVMs are then defined by the set $\{E_i\}_i$, such that $\sum_i E_i = Id, E_i \geq 0$.

$E_i$ is semi-definite positive: $\forall\psi, \langle\psi|E_i|\psi\rangle \geq 0$. This implies that $E_i$ is hermitian, and all its eigenvalues are $\geq 0$.

**POVM $\to$ Kraus operators**

Let $K_i = \sqrt{E_i}$. $\{K_i\}_i$ is a well defined set of operators.