Self-test and non-locality detection

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Abstract

1 Introduction

A faire

- local correlations def + can be written as deterministic LHV model (page 8 bell review , local polytope
- Quantum correlations
- self-testing scenario
- Bell example : maximally entangled state = self-tested state
- Mayers Yao

2 Non-locality detection

2.1 Linear Programming formulation

Proposition 1 *A behaviour* $\mathbf{p}$ *is local if and only if it can be written as a convex sum of deterministic behaviours, i.e*

$$\mathbf{p} = \sum_\lambda \mu_\lambda \mathbf{d}_\lambda \ , \ \mu_\lambda \geq 0 \ , \ \sum_\lambda \mu_\lambda = 1 \tag{1}$$

Proposition 2 *The set* $\mathcal{L}$ *of local behaviours is convex, i.e*

$$\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{L} \Rightarrow \forall \alpha \in [0,1] : \alpha\mathcal{P}_1 + (1-\alpha)\mathcal{P}_2 \in \mathcal{L} \tag{2}$$

Hence, it is possible to detect a non-local behaviour using linear programming.
Let $\mathcal{P}$ be the behaviour for which one want to learn whether it is local. Let $\mathbb{1}$ be the behaviour corresponding to the random outcome strategy. It is clear that $\mathbb{1}$ is a local behaviour. It is possible to write the following linear program

$$\min_{\alpha, \vec{\mu}} \alpha$$

$$\begin{cases} (1-\alpha)\mathcal{P} + \alpha\mathbb{1} & = & \sum_\lambda \mu_\lambda \mathbf{d}_\lambda \\ \sum_\lambda \mu_\lambda & = & 1 \\ \alpha & \leq & 1 \\ \forall\lambda, \ \mu_\lambda \geq 0 \ , \ \alpha \geq 0 \end{cases} \tag{3}$$

If the optimal value is $\alpha^* = 0$, then $\mathcal{P}$ is a local behaviour since it can be written as

$$\mathcal{P} = \sum_\lambda \mu^*_\lambda \mathbf{d}_\lambda$$

where μ^* is the coefficients find at the optimum.
On the other hand, if $\alpha^* > 0$, $\mathcal{P}$ is non-local.

The variable α is linked to the visibility $V = 1 - \alpha$.

2.2 CHSH correlations

The behaviour $\mathcal{P}$ obtained by solving the system of equations induced by the quantum correlations (put ref to introduction) is

$\mathcal{P}$	(x,y)			
(a,b)	(0,0)	(0,1)	(1,0)	(1,1)
(-1, -1)	$\cos^2(\pi/8)/2$	$\sin^2(\pi/8)/2$	$\sin^2(\pi/8)/2$	$\cos^2(\pi/8)/2$
(-1, 1)	$\cos^2(\pi/8)/2$	$\sin^2(\pi/8)/2$	$\sin^2(\pi/8)/2$	$\cos^2(\pi/8)/2$
(1, -1)	$\cos^2(\pi/8)/2$	$\sin^2(\pi/8)/2$	$\sin^2(\pi/8)/2$	$\cos^2(\pi/8)/2$
(1, 1)	$\sin^2(\pi/8)/2$	$\cos^2(\pi/8)/2$	$\cos^2(\pi/8)/2$	$\sin^2(\pi/8)/2$

The linear program 2.1 gives an optimal

$$\alpha^*_{CHSH} = 1 - 1/\sqrt{2} \tag{4}$$

, meaning that the visibility is $V^*_{CHSH} = 1/\sqrt{2}$.

2.3 Mayer-Yao's correlations

The behaviour $\mathcal{P}$ obtained by solving the system of equations induced by the quantum correlations (put ref to introduction) is

$\mathcal{P}$	(a,b)			
(x,y)	(-1,-1)	(-1,1)	(1,-1)	(1,1)
$(0,0)$	1/2	0	0	1/2
$(0,1)$	1/4	1/4	1/4	1/4
$(0,2)$	$\cos^2(\pi/8)/2$	$\sin^2(\pi/8)/2$	$\sin^2(\pi/8)/2$	$\cos^2(\pi/8)/2$
$(1,0)$	1/4	1/4	1/4	1/4
$(1,1)$	1/2	0	0	1/2
$(1,2)$	$\cos^2(\pi/8)/2$	$\sin^2(\pi/8)/2$	$\sin^2(\pi/8)/2$	$\cos^2(\pi/8)/2$
$(2,0)$	$\cos^2(\pi/8)/2$	$\sin^2(\pi/8)/2$	$\sin^2(\pi/8)/2$	$\cos^2(\pi/8)/2$
$(2,1)$	$\cos^2(\pi/8)/2$	$\sin^2(\pi/8)/2$	$\sin^2(\pi/8)/2$	$\cos^2(\pi/8)/2$
$(2,2)$	1/2	0	0	1/2

The linear program 2.1 gives an optimal

$$\alpha^*_{MY} = 0.1715 \tag{5}$$

, meaning that the visibility is $V^*_{CHSH} = 0.8284$.

3 Inequalities from duality

The linear program for non-locality detection also comes in a useful dual form

$$\max_{\gamma, \mathbf{y}} -\mathcal{P} \cdot \mathbf{y} + \gamma - \omega$$

$$\begin{cases} (\mathbb{1} - \mathcal{P}) \cdot \mathbf{y} - \omega & \leq & 1 \\ \gamma + \mathbf{d}_\lambda \cdot \mathbf{y} & \leq & 0, \quad \forall\lambda \\ \mathbf{y} \in \mathbb{R}^n, \ \gamma \in \mathbb{R}, \omega \geq 0 \end{cases}$$

(6)

The results from CHSH and Mayers-Yao show that (FIND WHY) one always has at the optimum

$$\mathcal{P}\cdot\mathbf{y}^* = -\alpha^* \quad (7)$$
$$\mathbb{1}\cdot\mathbf{y}^* = 1-\alpha^* \quad (8)$$

and thus, multiplying the first constraint of the primal by $\mathbf{y}^*$, one finds that

$$\sum_\lambda \mu_\lambda^* \mathbf{d}_\lambda \cdot \mathbf{y}^* = 0 \quad (9)$$

which means that the solution $\mathbf{y}^*$ of the dual problem is perpendicular to the optimal convex sum of deterministic behaviours. Therefore, the solution $\mathbf{y}^*$ corresponds to the non-zero coefficients which maximally violate the Bell inequality surement très faux

$$\mathbf{y}^*\cdot\mathcal{P} \geq S_l \quad (10)$$

but this dual form only inform that

$$\mathbf{y}^*\cdot\mathcal{P} \geq \alpha$$

Therefore, it does not give any information on the local bound mais peut etre que si.

3.1 CHSH correlations

$\mathbf{y}^*$	(a,b)			
(x,y)	$(-1,-1)$	$(-1,1)$	$(1,-1)$	$(1,1)$
(0,0)	$-\sqrt{2}$	0	0	0
$(0,1)$	0	$\sqrt{2}$	0	0
(1,0)	0	0	$\sqrt{2}$	0
$(1,1)$	$\sqrt{2}$	0	0	0

3.2 Mayer-Yao's correlation

3.3 Another Dual find better name

One can notice that another primal is possible for detecting non-locality

$$\min_{\alpha,\mu} \alpha$$
$$\begin{cases} (1-\alpha)\mathcal{P}+\alpha\mathbb{1} \leq \sum_\lambda \mu_\lambda \mathbf{d}_\lambda \\ \sum_\lambda \mu_\lambda = 1 \\ \alpha \leq 1 \\ \forall\lambda,\ \mu_\lambda \geq 0\ ,\ \alpha \geq 0 \end{cases} \quad (11)$$

and the results are exactly the same at the optimum.

However, the following dual will give different results

$$max_{\gamma,\mathbf{y}}\,\mathcal{P}\cdot\mathbf{y}+\gamma-\omega$$
$$\begin{cases} (\mathcal{P}-\mathbb{1})\cdot\mathbf{y}-\omega \leq 1 \\ \gamma+\mathbf{d}_\lambda\cdot\mathbf{y} \leq 0 \\ \mathbf{y}\in\mathbb{R}_+^n,\ \gamma\in\mathbb{R}, \omega\geq 0 \end{cases}$$
(12)

From this dual, we have that

$$\sum_\lambda \mu_\lambda \mathbf{d}_\lambda \leq -\gamma \text{ and } \mathbb{1}\cdot\mathbf{y} \leq -\gamma \quad (13)$$

Hence we can write

$$\begin{aligned} (1-\alpha)\mathcal{P}\cdot\mathbf{y}+\alpha\mathbb{1}\cdot\mathbf{y} &\leq \sum_\lambda \mu_\lambda \mathbf{d}_\lambda\cdot\mathbf{y} \\ \Leftrightarrow (1-\alpha)\mathcal{P}\cdot\mathbf{y}+\alpha\mathbb{1}\cdot\mathbf{y} &\leq \sum_\lambda \mu_\lambda(-\gamma) \\ \Leftrightarrow (1-\alpha)\mathcal{P}\cdot\mathbf{y}+\alpha\mathbb{1}\cdot\mathbf{y} &\leq -\gamma \\ \Leftrightarrow \mathcal{P}\cdot\mathbf{y} &\leq \frac{-\gamma-\alpha\mathbb{1}\cdot\mathbf{y}}{1-\alpha} \\ \Leftrightarrow \mathcal{P}\cdot\mathbf{y} &\leq \frac{-\gamma+\alpha\gamma}{1-\alpha} \end{aligned}$$

and conclude on a maximum violation of the Bell inequality

4 Robustness against white noise

The aim of this section is to study robustness against the addition of a white noise.

Consider a Werner state

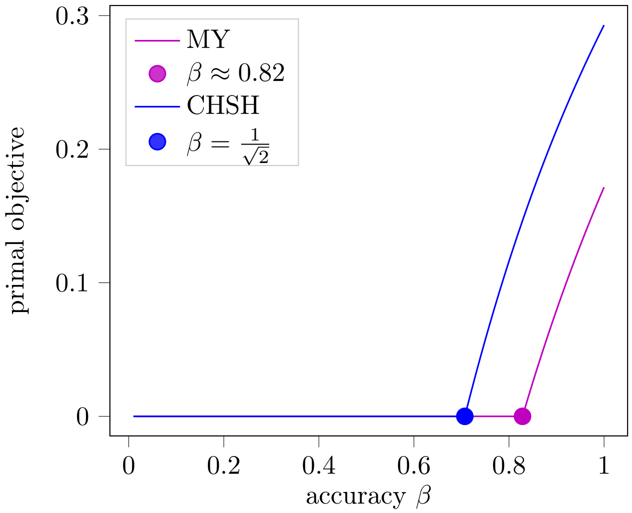
$$\rho = \beta\,|\Phi\rangle\langle\Phi| + (1-\beta)\mathbb{1} \quad (14)$$

where $|\Phi\rangle$ is a maximally entangled state and $\mathbb{1}$ a fully randomized behaviour.

Procedure 1 *1. Start with $\beta = 1$ and a given precision δ*

2. While the result of the primal is non local for the state ρ : $\beta = \beta - \delta$

Procedure 1 is applied by a dichotomic algorithm in order to have a good precision with the fewer iterations possible.



For the CHSH game, we obtained $\beta^*_{CHSH} = 1/\sqrt{2}$ and for Mayer-Yao's correlations, $\beta^*_{MY} =\approx 0.82$. In each case, one can noticed that $\beta^* = 1-\alpha^*$ where α^* was the optimal objective obtained in Section .. The study of the robustness was already given by the primal detecting non-local behaviour since it was written using a convex combination of the behaviour and a fully randomized behaviour which is actually a white noise. However, if one wants to study another type of noise, this study would be necessary.

A conclusion we can draw is that the maximally entangled state is more robust for CHSH correlations than for Mayer-Yao's correlations.

References