The self-testing scenario

$\mathscr{L}(\mathscr{H})$ denotes the set of linear operators acting on Hilbert space $\mathscr{H}$.

We know there exist measurement operators $M_{a|x} \in \mathscr{L}(\mathscr{H})$ acting on Alice's Hilbert space and satisfying

$$M_{a|x} \succcurlyeq 0; \forall a, x \sum_a M_{a|x} = Id_A \tag{1}$$

Similarly there exist measurement operators $N_{b|y} \in \mathscr{L}(\mathscr{H})$ acting on Bob's Hilbert space. The measurement operators are therefore projective :

$$\forall a, a' : \quad M_{a|x} M_{a'|x} = \delta_{a,a'} M_{a|x} \tag{2}$$

$$\forall b, b' : \quad N_{b|y} N_{b'|y} = \delta_{b,b'} N_{b|y} \tag{3}$$

Now, from the Born rule, there must exist some quantum state $\rho_{AB} \in \mathscr{L}(\mathscr{H}_\mathrm{A} \otimes \mathscr{H}_\mathrm{B}) \succcurlyeq 0$, and $\mathrm{tr}\rho_{AB} = 1$ such that

$$p(a,b|x,y) = \mathrm{tr}\big[\rho_{AB} M_{a|x} \otimes N_{b|y}\big] \tag{4}$$

In self-testing, one aims to infer the form of the state and the measurement in the trace from knowledge of the correlation $p(a,b|x,y)$ alone, i.e. in device-independent scenario.

Born rule : A key postulate of quantum mechanics which gives the probability that a measurement of a quantum system will yield a given result. More formally, for a state $|\psi\rangle$ and a F_i POVM element (associated with the measurement outcome i), then the probability of obtaining i when measuring $|\psi\rangle$ is given by

$$p(i) = \langle\psi|F_i|\psi\rangle \tag{5}$$

Physical assumptions

1. The experiment admits a quantum description (state and measurement)
2. The laboratories of Alice and Bob are located in separate location in space and there is no communication between the two laboratories.
3. The setting x and y are chosen freely and independently of all other systems in the experiment.
4. Each round of the experiment is independent of all other rounds a physically equivalent to all others (i.e. there exists a single density matrix and measurement operators that are valid in every round).

Impossibility to infer exactly the references

1. *Unitary invariance of the trace* : one can reproduce the statistics of any state $|\psi\rangle$ and measurement $\{M_{a|x}\}, \{N_{b|y}\}$ by instead using the rotated state $U \otimes V |\psi\rangle$ and measurement $\{U M_{a|x} U^\dagger\}, \{V N_{b|y} V^\dagger\}$, where U, V are unitary transformations. Hence, one can never conclude that the state was $|\psi\rangle$ or $U \otimes V |\psi\rangle$.
   On the other hand, considering real reference states ($|\psi\rangle = |\psi\rangle^*$), one can only self-test measurements that are invariant under the complex conjugate $*$, since, assuming a real state $|\psi\rangle$, $p(ab|xy) = \mathrm{tr}\big[\,|\psi\rangle\langle\psi|\, M_{a|x} \otimes N_{b|y}\big] = \mathrm{tr}\big[\,|\psi\rangle\langle\psi|\, M^*_{a|x} \otimes N^*_{b|y}\big]$. Thus any correlation obtained using $\big\{\,|\psi\rangle, M_{a|x}, N_{b|y}\big\}$ can also be obtained using $\big\{\,|\psi\rangle, M^*_{a|x}, N^*_{b|y}\big\}$; but the second is not related to the first one via a local isometry (It's an open problem to list the set of state and measuremend transformation that do not affect the probabilities).

2. *Additional degrees of freedom* : a state $|\psi\rangle \otimes |\xi\rangle$ and measurements $\{M_{a|x} \otimes Id_\xi\}, \{N_{b|y} \otimes Id_\xi\}$ gives the same correlation as $|\psi\rangle$ and $\{M_{a|x}\}, \{N_{b|y}\}$.

Extractability relative to a Bell Inequality

The extractability Ξ is defined as the maximum fidelity of $\Lambda_A \otimes \Lambda_B\left[\rho\right]$ and $|\psi'\rangle$ over all CPTP (Completely positive and trace preserving) maps:

$$\Xi(\rho \rightarrow |\psi'\rangle) = \max_{\Lambda_A,\Lambda_B} F(\Lambda_A \otimes \Lambda_B, |\psi'\rangle) \tag{6}$$

where $\rho \rightarrow |\psi'\rangle$ defines a kind of mapping of the test state ρ to the target state $|\psi'\rangle$. The maximum is taken over all quantum channels (hence the $\Lambda_{A,B}$ are called *quantum channels* ?). This implies that Ξ return the $\Lambda_{A,B}$ such that the fidelity to the reference state is maximal.

In order to test the entanglement characteristics of ρ, $|\psi'\rangle$ is assumed to be a state which achieves the maximal quantum violation. Hence, when the maximal quantum violation is observed in a self-testing scenario, the shared unknown state (ρ) can be mapped to $|\psi'\rangle$, and the resulting extractability is 1.

To get the optimal (robustness-wise) self-testing statement, one can minimize the possible extractability (over all states) when a violation of at least β is observed on a Bell inequality B. This quantity can be captured by the function $\mathcal{Q}$ defined as

$$\mathcal{Q}_{\psi,\mathcal{B}_\mathcal{I}} = \min_{\rho \in S_\mathcal{B}(\beta)} \quad \Xi(\rho \rightarrow |\psi'\rangle) \tag{7}$$

where $S_\mathcal{B}(\beta)$ is the set of states ρ which violate Bell inequality $\mathcal{B}$ with value at least β. One needs to note that the optimal CPTP map generally depends on the observed violation.