Quantum information project - reading notes
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Abstract

This documents gathers reading notes and solutions found during the project.

Self-testing of quantum systems: a review [ŠB20]

The self-testing scenario

$\mathscr{L}(\mathscr{H})$ denotes the set of linear operators acting on Hilbert space $\mathscr{H}$. We know there exist measurement operators $M_{a|x} \in \mathscr{L}(\mathscr{H})$ acting on Alice's Hilbert space and satisfying

$$M_{a|x} \succcurlyeq 0; \forall a, x \sum_a M_{a|x} = \mathbb{1}_A \tag{1}$$

Similarly, there exist measurement operators $N_{b|y} \in \mathscr{L}(\mathscr{H})$ acting on Bob's Hilbert space. The measurement operators are therefore projective :

$$\begin{aligned} \forall a, a' : \quad & M_{a|x} M_{a'|x} = \delta_{a,a'} M_{a|x} \\ \forall b, b' : \quad & N_{b|y} N_{b'|y} = \delta_{b,b'} N_{b|y} \end{aligned} \tag{2}$$

Now, from the Born rule, there must exist some quantum state $\rho_{AB} \in \mathscr{L}(\mathscr{H}_\mathrm{A} \otimes \mathscr{H}_\mathrm{B}) \succcurlyeq 0$, and $\mathrm{tr}\rho_{AB} = 1$ such that

$$p(a, b|x, y) = \mathrm{tr}\big[\rho_{AB} M_{a|x} \otimes N_{b|y}\big] \tag{3}$$

In self-testing, one aims to infer the form of the state and the measurement in the trace from knowledge of the correlation $p(a, b|x, y)$ alone, i.e. in device-independent scenario.
Born rule : A key postulate of quantum mechanics which gives the probability that a measurement of a quantum system will yield a given result. More formally, for a state $|\psi\rangle$ and an F_i POVM element (associated with the measurement outcome i), then the probability of obtaining i when measuring $|\psi\rangle$ is given by

$$p(i) = \langle\psi|F_i|\psi\rangle \tag{4}$$

Physical assumptions

1. The experiment admits a quantum description (state and measurement)

2. The laboratories of Alice and Bob are located in separate location in space and there is no communication between the two laboratories.

3. The setting x and y are chosen freely and independently of all other systems in the experiment.

4. Each round of the experiment is independent of all other rounds a physically equivalent to all others (i.e. there exists a single density matrix and measurement operators that are valid in every round).

Impossibility to infer exactly the references

1. *Unitary invariance of the trace* : one can reproduce the statistics of any state $|\psi\rangle$ and measurement $\{M_{a|x}\}, \{N_{b|y}\}$ by instead using the rotated state $U \otimes V |\psi\rangle$ and measurement $\{U M_{a|x} U^\dagger\}, \{V N_{b|y} V^\dagger\}$, where U, V are unitary transformations. Hence, one can never conclude that the state was $|\psi\rangle$ or $U \otimes V |\psi\rangle$. On the other hand, considering real reference states ($|\psi\rangle = |\psi\rangle^*$), one can only self-test measurements that are invariant under the complex conjugate $*$, since, assuming a real state $|\psi\rangle$, $p(ab|xy) = \mathrm{tr}\big[\, |\psi\rangle\langle\psi| M_{a|x} \otimes N_{b|y}\big] = \mathrm{tr}\big[\, |\psi\rangle\langle\psi| M^*_{a|x} \otimes N^*_{b|y}\big]$. Thus, any correlation obtained using $\big\{\, |\psi\rangle, M_{a|x}, N_{b|y}\big\}$ can also be obtained using $\big\{\, |\psi\rangle, M^*_{a|x}, N^*_{b|y}\big\}$; but the second is not related to the first one via a local isometry (It's an open problem to list the set of state and measurement transformation that do not affect the probabilities).

2. *Additional degrees of freedom* : a state $|\psi\rangle \otimes |\xi\rangle$ and measurements $\{M_{a|x} \otimes \mathbb{1}_\xi\}, \{N_{b|y} \otimes \mathbb{1}_\xi\}$ gives the same correlation as $|\psi\rangle$ and $\{M_{a|x}\}, \{N_{b|y}\}$.

Extractability relative to a Bell Inequality

The extractability Ξ is defined as the maximum fidelity of $\Lambda_A \otimes \Lambda_B[\rho]$ and $|\psi'\rangle$ over all CPTP (Completely positive and trace preserving) maps:

$$\Xi(\rho \to |\psi'\rangle) = \max_{\Lambda_A,\Lambda_B} F(\Lambda_A \otimes \Lambda_B, |\psi'\rangle) \quad (5)$$

where $\rho \to |\psi'\rangle$ defines a kind of mapping of the test state ρ to the target state $|\psi'\rangle$. The maximum is taken over all quantum channels (why are the $\Lambda_{A,B}$ called *quantum channels* ?). This implies that Ξ return the $\Lambda_{A,B}$ such that the fidelity to the reference state is maximal.
In order to test the entanglement characteristics of ρ, $|\psi'\rangle$ is assumed to be a state which achieves the maximal quantum violation. Hence, when the maximal quantum violation is observed in a self-testing scenario, the shared unknown state (ρ) can be mapped to $|\psi'\rangle$, and the resulting extractability is 1.
To get the optimal (robustness-wise) self-testing statement, one can minimize the possible extractability (over all states) when a violation of at least β is observed on a Bell inequality B. This quantity can be captured by the function $\mathcal{Q}$ defined as

$$\mathcal{Q}_{\psi,\mathcal{B}_\mathcal{I}} = \min_{\rho\in S_\mathcal{B}(\beta)} \quad \Xi(\rho \to |\psi'\rangle) \quad (6)$$

where $S_\mathcal{B}(\beta)$ is the set of states ρ which violate Bell inequality $\mathcal{B}$ as defined in [ZCP+18] with value at least β. One needs to note that the optimal CPTP map generally depends on the observed violation.

The device-independent outlook on quantum physics [Sca15]

Formal characterization of local variables

The local variable paradigm is related to the idea of pre-established agreement between the two parties, i.e. the output of each run is fully determined by a variable λ:

$$\mathbb{P}(a,b|x,y,\lambda) = \mathbb{P}(a|x,\lambda)\mathbb{P}(b|y,\lambda) \quad (7)$$

In other words, one can write the probability of having (a, b) given the two inputs (x, y) as:

$$\mathbb{P}(a,b|x,y) = \int d\lambda \mathbb{P}(a|x,\lambda)\mathbb{P}(b|y,\lambda) \quad (8)$$

This pre-established agreement is called *shared randomness.*

No-signalling condition : The fact that the statistics of a (b) should nod depend on $y(x)$, that is, the following conditions hold for all a,b,x,y,λ:

$$\mathbb{P}(a|x,y,\lambda) = \mathbb{P}(a|x,\lambda) \qquad \mathbb{P}(b|x,y,\lambda) = \mathbb{P}(b|y,\lambda) \quad (9)$$

From the above conditions one can deduce the *no-signalling constraints* :

$$\begin{aligned} \forall x,x' \in \mathcal{Y} : \sum_b \mathbb{P}(a,b|x,y) &= \sum_b \mathbb{P}(a,b|x,y') = \mathbb{P}(a|x) \\ \forall x,x' \in \mathcal{X} : \sum_a \mathbb{P}(a,b|x,y) &= \sum_a \mathbb{P}(a,b|x',y) = \mathbb{P}(b|y) \end{aligned} \quad (10)$$

Bell inequalities as a polytope

The set $\mathcal{L}$ of all families of probability distributions that can be obtained with LV is convex. In other words, if $\mathcal{P}_1 \in \mathcal{L}$ and $\mathcal{P}_2 \in \mathcal{L}$, then $\forall\alpha \in [0,1] : \alpha\mathcal{P}_1 + (1-\alpha)\mathcal{P}_2 \in \mathcal{L}$. Furthermore, each deterministic local point is an extremal point of $\mathcal{L}$.
A polytope $\mathcal{L}$ in $\mathbb{R}^d$ is delimited by finitely many $(d-1)$-dimensional hyperplanes called facets (of the polytope).

Link between Bell inequalities and linear programming

It is possible to state to which set ($\mathcal{L},\mathcal{Q},\mathcal{NS}$) a probability distribution P belongs to. Consider the linear program given below:

$$\begin{aligned} &\min \quad q \\ &s.t. \quad \begin{cases} (1-q)P + q\mathbb{1} & = \sum_\lambda \mu_\lambda D_\lambda \\ \sum_\lambda \mu_\lambda & = 1 \\ q & \leq 1 \\ \forall\lambda, \mu_\lambda \geq 0, q \geq 0 \end{cases} \end{aligned} \quad (11)$$

where $\{D_\lambda\}_\lambda$ is the set of all deterministic behaviors as described in [BCP+14] and $\mathbb{1}$ is the vector $\big(0.5 \cdots 0.5\big)^T$ that represents a fully random noise. If the result is $q = 0$, then the probability vector P describes a local model, otherwise, if $q > 0$, the probability vector describes a quantum behavior.

The dual program of the program (11) is the following:

$$
\begin{aligned}
&\max \quad -\Gamma \cdot P + \phi - \omega \\
&s.t. \quad \begin{cases} -\Gamma \cdot D_\lambda + \phi & \leq 0 \quad , \forall \lambda \in 1, \cdots, \Delta^{2m} \\ \Gamma \cdot P - \omega & \leq 1 \\ \phi, \forall \lambda \ \mu_\lambda \in \mathbb{R}, \omega \geq 0 \end{cases}
\end{aligned} \tag{12}
$$

We need to discuss the above program.

Non-linear programming to determine a probability vector

Considering the linear program 11, one can transform the vector P into a vector of variables. Adding the constraints for P to be a probability vector, one can solve the associated non-linear program. It yields the following linear program

$$
\begin{aligned}
&\max \quad CHSH \\
&s.t. \quad \begin{cases} (1-q)P + q\mathbb{1} & = \sum_\lambda \mu_\lambda D_\lambda \\ \sum_\lambda \mu_\lambda & = 1 \\ q & \leq 1 \\ \forall \lambda, \mu_\lambda \geq 0, q \geq 0 \end{cases}
\end{aligned} \tag{13}
$$

where $CHSH$ is the CHSH inequality:

$$
\langle A_0 \otimes B_0 \rangle + \langle A_1 \otimes B_0 \rangle + \langle A_0 \otimes B_1 \rangle - \langle A_1 \otimes B_1 \rangle \tag{14}
$$

and expectation value can be decomposed in terms of measurement operators such that:

$$
\langle A_x \otimes B_y \rangle = \langle (M_{1|x} - M_{-1|x}) \otimes (M_{1|y} - M_{-1|y}) \rangle \tag{15}
$$

and considering

$$
p(ab|xy) = \langle M_{a|x} \otimes M_{b|y} \rangle \tag{16}
$$

This can be resolved by Gurobi Solver [Gur22] even though the program found is non-convex.

Note: It should be possible to add constraint to ensure the resulting probability vector P stays in $\mathcal{Q}$ to get the maximal violation of the CHSH inequality. Otherwise, the maximal value is 4 meaning that the probability vector is in $\mathcal{NS}$. Considering the additional constraint $CHSH \leq 2\sqrt{2}$ (that ensure that P is *at most* in $\mathcal{Q}$), we obtain in P the probabilities that maximally violates the CHSH inequality, and $q = 0.91$ which is consistent with what we could expect considering the program 11.

References

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[Gur22] Gurobi Optimization, LLC, *Gurobi Optimizer Reference Manual*, 2022.

[Sca15] Valerio Scarani, *The device-independent outlook on quantum physics (lecture notes on the power of bell's theorem)*, 2015.

[ZCP+18] Wen-Hao Zhang, Geng Chen, Xing-Xiang Peng, Xiang-Jun Ye, Peng Yin, Ya Xiao, Zhi-Bo Hou, Ze-Di Cheng, Yu-Chun Wu, Jin-Shi Xu, Chuan-Feng Li, and Guang-Can Guo, *Experimentally robust self-testing for bipartite and tripartite entangled states*, Physical Review Letters **121** (2018Dec), no. 24.