Engineering Tripos Part IA - Thermofluids

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1 Fluid Mechanics

This part of the course is concerned with the mechanics of fluids. Our first question is: what exactly is a fluid? One might have seen before that a fluid is defined as a substance that is capable of flowing, i.e. taking the shape of the container it is in, however this is only partially correct.

> **Definition 1** (Fluid). *A fluid is a substance which, when at rest, cannot sustain shear stress.*

1.1 Hydrostatics

In this section of the course, we will first begin by analyzing the statics of fluids, also known as hydrostatics.

> **Theorem 1** (Pascal's law). *Pressure acts equally in all directions.*

Proof. To prove this theorem, we must first imagine an infinitesimally small right triangle on the surface of any object, of length Δx, height Δz, and width b (into the page). Let Δs be the length of the hypotenuse.

By means of equilibrium, in the x direction:

$$p_x b \Delta z - p_n b \Delta s \sin \alpha = 0$$

In the z direction, we deduce that in a similar fashion:

$$p_z b \Delta x - \frac{1}{2} \rho b \Delta x \Delta z g - p_n b \Delta s \cos \alpha = 0$$

The geometry of the system yields us that $\sin \alpha = \frac{\Delta z}{\Delta s}$ and that $\cos \alpha = \frac{\Delta x}{\Delta s}$. By using this in the first equation, we obtain:

$$p_x = p_n$$

In the second equation, this yields us:

$$p_z = p_n + \frac{1}{2} \rho g \Delta z$$

However, this equation is unaffected by us varying Δz, and by our assumption that this is an infinitesimally small triangle, $\Delta z \to dz \to 0$, and thus:

$$p_z = p_n$$

By combining our results, we obtain Pascal's law:

$$p_x = p_z = p_n$$

$\square$

> **Proposition 1** (Hydrostatic pressure). *The pressure in a fluid varies linearly, as given by:*
>
> $$p(z) = p_a + \rho g z$$

> *Where p_a is the atmospheric pressure, and z is the depth below the free surface.*

Proof. By considering a small column of length dz, vertical equilibrium yields:

$$(p(z) + dp)A = (p(z) + \rho g dz)A \iff dp = \rho g dz$$

Integrating this yields:

$$p(z) = p_0 + \rho g z$$

Where p_0 is the pressure at $z = 0$, i.e. at the level of the free surface. Normally, we take this as $p_0 = p_a$, the atmospheric pressure. $\square$

1.1.1 Manometers

The principles of hydrostatics are used to measure pressure using the U-tube manometer. A manometer is a U-tube device that is connected to two pressures, p_A and p_B, with a column of air at both ends separating a column of liquid. We wish to determine the relationship governing the difference between the two aforementioned pressures.

Consider that the height between the levels of liquid on the left and right side is h. By applying hydrostatic pressure:

$$p_2 = p_{1,\text{ left}} + \rho_a g h$$

$$p_2 = p_{1,\text{ right}} + \rho_l g h$$

We know that p_2 must be the same for both columns because of Pascal's law, however p_1 is different because we have two separate fluids (air and liquid). Now, we know that:

$$p_{1,\text{ left}} = p_A + \rho g z$$

$$p_{1,\text{ right}} = p_B + \rho g z$$

Since they are at the same height z. Therefore, we deduce the manometer equation:

$$p_A - p_B = (\rho_l - \rho_a)g h$$

In the case where $\rho_l \gg \rho_a$, this equation further degenerates into:

$$p_A - p_B = \rho_l g h$$

1.1.2 Barometers

A mercury barometer is made from a long tube which is closed at one end and open at the other. It is filled with mercury and then inverted. As the vapour pressure of mercury is so low, we can ignore the pressure in the vacuum of mercury vapour which forms at the top. The height of the column simply indicates atmospheric pressure

At the free surface (where the barometer is put), the pressure is just p_A - this is true for the lower end of the barometer. However, due to the height difference:

$$p_A = p_v + \rho g h$$

Where p_v is the pressure due to the mercury vapour forming on top. However, since $p_v \ll \rho g h$, we can assume that:

$$p_A \approx \rho g h$$

1.1.3 Archimedes' principle

In the first section, we considered the triangular infinitesimally small element of surface from an object. Doing so again, and by considering equilibrium in the z direction, we deduce that:

$$p_z b \Delta x - p_n b \Delta s \cos \alpha - \frac{1}{2} \rho g b \Delta x \Delta z = 0$$

The two pressures give rise to the upthrust, also known as the Archimedic force. By combining these two terms, we obtain:

$$F_A = \rho g V$$

In other terms, the upthrust is equal to the weight of the displaced fluid.

1.1.4 Forces on submerged bodies

Consider that we want to determine the horizontal force acting on a rectangular strip which holds back stationary water. Therefore:

$$F = \int_h^{h+w} \rho g w z \, dz = \rho g \left(h + \frac{w}{2} \right) w^2$$

This is equal to the average force exerted on the wall, multiplied by the surface area. This concept can be further extended to a plane of width given by $b(s)$, inclined at an angle θ, where s is its current length:

$$F = \int_S dF = \int \rho g s \sin \theta b(s) ds = \rho g \sin \theta \int s b(s) ds$$

Furthermore, to obtain the equivalent point of action of the force, we simply need to consider moment equilibrium:

$$F \bar{s} = \int_S s \, dF$$

In general, for hydrostatic pressure distributions (and rectangular laminae), the equivalent point of action of the force is at:

$$\bar{s} = \frac{2}{3} L$$

Where L is the total length of the plane.

1.2 Fluid dynamics

This section introduces the terminology of fluid dynamics required for proceeding with the course.

Definition 2 (Streamline). *A streamline is a curve in space which is always in line (parallel) to the velocity vectors at each point in the flow.*

Note that streamlines cannot cross each other (no mass can cross a boundary), as this would cause the flow two have two different velocity at one given instant.

Definition 3 (Stagnation point). *Any object immersed in a flow has one or more stagnation points - these mark a point on the surface where the flow velocity is null.*

Note that in reality, an interesting effect can be seen - a particle never actually fully touches the boundary, as it would take $t \to \infty$ for that to happen.

Definition 4 (Flow separation). *Streamlines usually follow the surface of an object. However, flows will eventually separate from the surface when they cannot follow its curvature or if the pressure gradients become too large.*

Definition 5 (Steady flow). *We say that a flow is steady if the trajectories of particles passing through each point in space do not change with time. Conversely, we say that a flow is unsteady if it is not steady. Note that streamlines do not make sense in unsteady flow.*

Definition 6 (Compressible flow). *We say that a flow is compressible if the density is constant and uniform throughout the entire region. If, however, density is not constant, we say that the flow is incompressible.*

Definition 7 (Viscosity). *We say that a flow is viscous if it experiences viscous friction. This usually happens when the fluid is very close to the wall. Otherwise, we say that the flow is inviscid.*

Furthermore, the fluid infinitesimally close to the wall does not actually move (because of the interactions between particles) - this is also known as the no-slip condition. Viscous friction is characterised by the Reynolds number:

$$Re = \frac{\rho v L}{\mu}$$

Where ρ is the density of the fluid, v is the velocity of the free stream (very far away), l is a characteristic dimension, and μ is the viscosity.

1.2.1 Systems and conservation principles

Definition 8 (Control volume/surface). *A control volume is a region of space. A control surface is the surface of our choice of control volume.*

Proposition 2 (Conservation of mass). *The mass of a system must always remain constant, but the volume may change.*

Suppose we have a control volume which has an inlet and an outlet. Therefore:

$$m_t = \rho_0 V_0 + \rho_{\text{in}} A_{\text{in}} l_{\text{in}}$$

$$m_{t+\Delta t} = \rho_0 V_0 + \rho_{\text{out}} A_{\text{out}} l_{\text{out}}$$

Noting that $l = v\Delta t$ and by dividing both quantities by Δt:

$$\rho_{\text{in}} A_{\text{in}} v_{\text{in}} = \rho_{\text{out}} A_{\text{out}} v_{\text{out}}$$

Equivalently:

$$\dot{m}_{\text{in}} = \dot{m}_{\text{out}}$$

This is known as the continuity equation.

Proposition 3 (Continuity). *In steady flow, the continuity equation states that:*

$$\oiint_{CS} \rho \boldsymbol{v} \cdot \boldsymbol{dA} = 0$$

This is equivalent to saying that the mass flow rate in equals the mass flow rate out. Note that we must always take the velocity component perpendicular to the area (this is why the scalar product is used).

Equivalently, this can be stated as:

$$\sum_i \dot{m}_i = 0$$

Where $\dot{m}_i < 0$ for flow into a control volume (inlet), and $\dot{m}_i > 0$ for flow outside of the control volume (outlet).

Note that the mass flow rate can be calculated using:

$$\dot{m} = \int_A \rho \mathbf{v} \cdot \mathbf{dA}$$

Proposition 4 (Conservation of momentum). *Momentum is conserved in steady flow.*

By Newton's second law, we deduce that:

$$F = \frac{dp}{dt} = \frac{(mv)_{t+\Delta t} - (mv)_t}{\Delta t} = (\dot{m}v)_{\text{out}} - (\dot{m}v)_{\text{in}}$$

Where the quantity $\dot{m}v$ is also known as the momentum flow rate. The force F acting on the fluid is given by all body forces (mechanical, electrostatic, electromagnetic etc.) and pressure forces.

Proposition 5 (Steady-flow momentum equation). *Consider a steady flow problem. Then, we must have that:*

$$\boldsymbol{F} - \oiint_{CS} p\,\boldsymbol{dA} = \oiint_{CS} \rho\boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{dA})$$

Otherwise, this can be stated as:

$$\sum F + \sum pA = \sum(\dot{m}v)_{out} - \sum(\dot{m}v)_{in}$$

Notice that this relationship can be applied in any given direction in our control volume. Furthermore, the momentum flow rate can be calculated from:

$$(\dot{m}v) = \int_A \rho v(\boldsymbol{v} \cdot \boldsymbol{dA})$$

Note that v must be the velocity component in the direction in which we are interested to apply equilibrium.

1.2.2 The equations of motion for fluid flow

In this section, we will further analyze the behaviour of fluid elements under motion. Consider an infinitesimally small fluid particle (cubic) moving in a pressure gradient. Our goal is to find the force due to pressure in each direction.
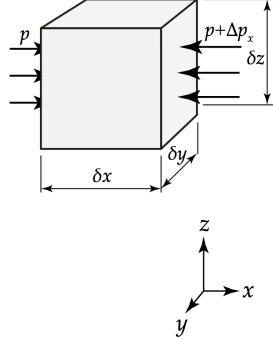

Figure 1: Fluid element

Since we are moving in a pressure gradient ∇p, then the change in any direction is:

$$\Delta p = \nabla p \cdot \mathbf{u}$$

Note that this is just the directional derivative. Let us now consider the x direction. Therefore, the change of pressure is:

$$\Delta p = \frac{\partial p}{\partial x} \delta x$$

By equilibrium (see figure):

$$F = p\delta S - (p + \Delta p)\delta S$$

Because $\delta S = \delta y \delta z$, this can be expressed as:

$$F = -\frac{\partial p}{\partial x} \delta x \delta y \delta z = -\frac{\partial p}{\partial x} \delta V$$

This is true for all directions. Therefore, we can further compress our relationship as:

$$\mathbf{F} = -\nabla p \delta V$$

In other sources, this might appear as:

$$\mathbf{a} = -\frac{1}{\rho} \nabla p$$

Proposition 6 (Bernoulli's equation). *Consider a steady flow problem. If there is negligible viscosity, no mixing/stagnation points, constant density, then along the streamline direction, at any two points:*

$$p_1 + \rho g z_1 + \frac{1}{2}\rho v_1^2 = p_2 + \rho g z_2 + \frac{1}{2}\rho v_2^2$$

Proof. Consider a fluid particle (cubic, as before), and consider an intrinsic coordinate system (s, n) moving along the fluid's path. Then:

$$\ddot{s} = \frac{\partial v}{\partial t} = \frac{\partial s}{\partial t}\frac{\partial v}{\partial s} = v\frac{\partial v}{\partial s}$$

Furthermore, as previously deduced, the force acting in the s direction due to the pressure gradient is:

$$F_p = -\frac{\partial p}{\partial s}\delta V$$

Lastly, the weight of the fluid particle is just:

$$W = -\rho\delta V g\frac{\partial z}{\partial s}$$

By Newton's second law:

$$m\ddot{s} = F_p + W \iff \rho\delta V v\frac{\partial v}{\partial s} = -\frac{\partial p}{\partial s}\delta V - \rho\delta V g\frac{\partial z}{\partial s}$$

By dividing the above through δV and rearranging, we deduce:

$$\frac{\partial p}{\partial s} + \rho g\frac{\partial z}{\partial s} + \rho v\frac{\partial v}{\partial s} = 0$$

Noting that $v\frac{\partial v}{\partial s} = \frac{1}{2}\frac{\partial}{\partial s}(v^2)$, we deduce that:

$$\frac{\partial}{\partial s}(p + \rho g z + \frac{1}{2}\rho v^2) = 0$$

This is equivalent to saying:

$$p + \rho g z + \frac{1}{2}\rho v^2 = \text{const}$$

Thus, our proof is complete. $\qquad\square$

Note that Bernoulli's equation is equivalent to the equation of motion of a fluid applied in the tangential s direction. However, we still need to deduce the equation of motion in the normal direction.

Proposition 7 (Normal equation of motion). *Consider a steady flow problem. Then, the equation of motion for a region of fluid in the normal direction is:*

$$\frac{\partial p}{\partial n} = \frac{\rho v^2}{R}$$

Where R is the radius of curvature of the streamline flow.

Proof. Let us revisit the intrinsic (s, n) coordinate system from mechanics.

Because we are, again, considering a streamline, we must remember from the definition of a streamline that its velocity is only in the s direction. Furthermore, form mechanics, the normal acceleration is:

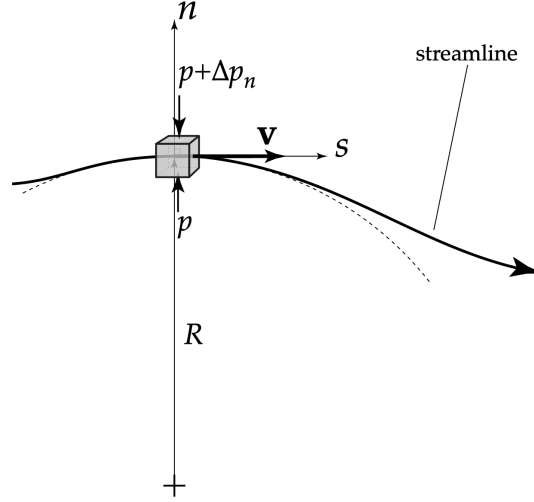$$\ddot{n} = \dot{s}\dot{\psi} = \frac{\dot{s}^2}{R}$$

Figure 2: Fluid particle

Where R is the radius of curvature. Note that in mechanics, R is pointing towards the centre of curvature, whereas in fluid mechanics, we always choose R to be pointing outwards. Hence, by the pressure gradient formula:

$$-\frac{\partial p}{\partial n}\delta V = -\rho\delta V\frac{v^2}{R}$$

By dividing both sides by δV (note that $m = \rho\delta V$), we obtain:

$$\frac{\partial p}{\partial n} = \rho\frac{v^2}{R}$$

Thus, our proof is concluded. Another important thing to note is that $\frac{\partial p}{\partial n} > 0$, since the right hand side of the equation is always positive. This means that if we go in the positive n direction, pressure is always increasing, and if we go in the negative n direction, pressure is decreasing. This is a fundamental result that allows us to explain many events we usually observe in nature, and other important effects, such as the Coandă effect.

<div align="right">□</div>

This result also allows us to explain why parallel flow has no pressure gradient. If we are in parallel flow, $R \to \infty$, and so, therefore:

$$\frac{\partial p}{\partial n} \to 0$$

Furthermore, it is important to note that in diverging flow, $\frac{\partial p}{\partial s} > 0$, and in converging flow, $\frac{\partial p}{\partial s} < 0$.

Definition 9 (Coefficient of pressure). *We define the pressure coefficient as:*

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho v_\infty^2}$$

Note that the Bernoulli term $\frac{1}{2}\rho v_\infty^2$ is oftentimes called the dynamic pressure or dynamic head.

Proposition 8 (The Pitot-Static Tube). *The Pitot-Static tube measures the difference between the pressure and the Pitot pressure. This can then be used to measure the velocity of the freestream flow. Consider a fluid path going from infinity (freestream), until the stagnation point. This yields:*

$$p_\infty + \frac{1}{2}\rho v_\infty^2 = p_1$$

Since $v_1 = 0$, due to it being a stagnation point. Now, consider the second entrance:

$$p_\infty + \frac{1}{2}\rho v_\infty^2 = p_2 + \frac{1}{2}\rho v_\infty^2$$

Because $v_2 = v_\infty$ (undisturbed flow). Hence, by combining the two equations:

$$p_1 = p_2 + \frac{1}{2}\rho v_\infty^2$$

Yielding us:

$$v_\infty = \sqrt{\frac{2(p_1 - p_2)}{\rho}}$$

Proposition 9 (The Coandă effect). *Any jet of fluid sticks to a curved object's surface even after its curvature begins to decrease. This is explained by the streamline curvature equation:*

$$\frac{\partial p}{\partial n} = \frac{\rho v^2}{R}$$

Since this is always positive, we observe that because we traverse in the negative n direction, the pressure must fall the closer we get to the surface. Therefore, p_∞ is higher than the pressure at the surface of the object, leading to the fluid being pushed onto it, and sticking for longer than it should.

Proposition 10 (The Magnus effect). *If an object is rotated in a freestream flow, then the streamlines curve in a way which is similar to an aerofoil. Therefore, because of the streamline curvature equation, we expect lift to occur.*

2 Thermodynamics

Definition 10 (Thermodynamic system). *A thermodynamic system is an arbitrary geometrical portion of the universe with fixed or movable boundaries which may contain matter, energy, or both.*

We say that a thermodynamic system is closed if it represents a fixed quantity of matter, around which a boundary can be fixed. Everything inside the boundary is the system and everything outside is the surroundings (or the environment).

Note that since no matter crosses the boundary, mass conservation is automatically satisfied for a closed system. Also note that energy can cross the boundary in the form of heat and work (which shall be defined later).

Definition 11 (Isolated system). *We say that a (closed) system is isolated if there is no heat or work exchange between itself and its surroundings.*

Definition 12 (Thermodynamic properties and state). *A system possesses a number of thermodynamic properties, such as pressure, volume and temperature, which together define its thermodynamic state. Thermodynamic properties depend only on the state of the system, and not how the system arrived at that state.*

Proposition 11 (Kelvin-Celsius relationship). *Note that in Thermodynamics, we only work with absolute temperatures. Therefore:*

$$T(K) = T(°C) + 273.15$$

Definition 13 (Extensive property). *We say that a property of a thermodynamic system is extensive if it depends on the size (extent) of the system. An example of this is the volume.*

Definition 14 (Intensive property). *We say that a property is intensive if it is not extensive. Typical examples include pressure and temperature.*

Definition 15 (Specific property). *We say that a property is specific if it is defined per unit mass. These are a subset of intensive properties.*

Proposition 12 (The two property rule). *For simple compressible systems at rest, two independent intensive properties and the mass are sufficient to fully define the state of the system, when it is in equilibrium. Thus:*

$$p = p(T, V) \iff T = T(p, V) \iff V = V(p, T)$$

Definition 16 (Thermodynamic equilibrium). *A thermodynamic system is in equilibrium when none of its thermodynamic properties are changing in time at a measurable rate. Note that Thermodynamics can only furnish relationships connecting equilibrium states.*

However, if any process is carried out slowly, departures from equilibrium may be kept very small and the process effectively passes through a series of equilibrium states. This is referred to as a **quasi-equilibrium** *or* **quasi-static** *process.*

2.1 The first law of Thermodynamics

Theorem 2 (The first law of Thermodynamics for closed systems). *Consider a closed system - it does not exchange matter with the surroundings (no mass crosses the boundary), however, it may exchange energy in the form of either **heat** or **work**. Hence:*

$$\Delta E = Q - W$$

Where ΔE encapsulates kinetic, potential, and internal energy. If we neglect potential and kinetic energy terms, then:

$$\Delta U = Q - W$$

In differential form, the statement of this law is:

$$dU = \delta Q - \delta W$$

Note that heat and work are modes of energy transfer only, they do not reside in a system. Thus, they are not properties of any specific state.

Proposition 13 (Resisted work). *One type of work that we will frequently encounter is that due to a fully resisted expansion of a gas. Consider a system comprised of a gas held by a piston at a pressure p on both sides. Then:*

$$W = \int p \, dV$$

Proof. The force acting on both sides of the piston is therefore $F = pS$. A small displacement dx causes work to be done:

$$\delta W = F dx$$

Therefore:

$$W = \int F dx = \int pS dx = \int p \, dV$$

Note that this is only true in the case of a fully resisted transformation. $\qquad\square$

Definition 17 (Cyclic process). *A cyclic process is one for which the system is returned to its original state. Because of this, in a cyclic process it is always true that:*

$$\Delta U = 0$$

Furthermore, $\Delta U = Q - W \iff Q = W$ for the entire cycle. Hence:

$$Q = W = \oint p \, dV$$

Proposition 14 (The ideal gas equation of state). *For many gases, experiments show that they obey the following relationship:*

$$pV = n\overline{R}T$$

This can be written as:

$$pV = mRT \ \text{or} \ pv = RT$$

Where R is the specific gas constant, $R = \frac{\overline{R}}{\mu}$, where μ is the molar mass of the gas.

Note that for a property X, we label x as its specific equivalent, i.e.:

$$x = \frac{X}{m}$$

This allows us to fully remove mass from the equation and only focus on the more important aspects of the problem. Also note that the specific volume $v = \frac{V}{m}$ is just the inverse density:

$$v = \frac{1}{\rho}$$

Definition 18 (Enthalpy). *Consider a constant pressure process which is also fully resisted. Then:*

$$W = \int pdV = p\Delta V$$

The first law gives:

$$\Delta U = Q - W \iff Q = \Delta U + W \iff Q = \Delta U + p\Delta V$$

This can then be written as:

$$Q = \Delta(U + pV)$$

By defining $H := U + pV$ as the enthalpy of the system, then:

$$Q = \Delta H$$

The units of enthalpy are the same as for energy, but note that enthalpy is **not** energy. It is merely just a short-hand for $U + pV$, and it is introduced because this combination appears very frequently.

2.1.1 Specific heat capacities

As we have previously seen, for a constant volume process, the first law gives:

$$Q = \Delta U$$

And as previously, for a constant pressure process:

$$Q = \Delta H$$

Accordingly, there must be two specific heat capacities, defined as follows:

(a) The constant volume (isochoric) specific heat capacity:

$$c_v = \left(\frac{\partial u}{\partial T} \right)_v$$

(b) The constant pressure (isobaric) specific heat capacity:

$$c_p = \left(\frac{\partial h}{\partial T} \right)_p$$

Therefore, we can introduce the following definitions.

Definition 19 (Ideal gas). *A gas is said to be ideal if its internal energy $U = U(T)$ and enthalpy $H = H(T)$ are functions of temperature only.*

Definition 20 (Semi-perfect gas). *A gas is said to be semi-perfect if $c_v = c_v(T)$ and $c_p = c_p(T)$ are both functions of temperature.*

Definition 21 (Perfect gas). *A gas is said to be perfect if c_v and c_p are constants.*

Definition 22 (Adiabatic exponent). *We define the adiabatic exponent γ of any gas as:*

$$\gamma = \frac{c_p}{c_v}$$

Proposition 15 (The Robert-Mayer relationship). *For any ideal gas, it is true that:*

$$c_p = c_v + R$$

Proof. We know from our definition that:

$$c_p = \left(\frac{\partial h}{\partial T} \right)_p = \left(\frac{\partial (u + pv)}{\partial T} \right)_p = \left(\frac{\partial u}{\partial T} \right)_p + \left(\frac{\partial (pv)}{\partial T} \right)_p$$

Now, $pv = RT$, and therefore:

$$\left(\frac{\partial (pv)}{\partial T} \right)_p = R$$

Because we have an ideal gas, the internal energy only depends on temperature, and then:

$$\left(\frac{\partial u}{\partial T}\right)_p = \left(\frac{\partial u}{\partial T}\right)_v = c_v$$

Finally:

$$c_p = c_v + R$$

$\square$

Proposition 16. *For a perfect gas, by the above definitions, we have that:*

$$du = c_v dT \ \text{ and } \ dh = c_p dT$$

2.1.2   Thermodynamic processes

Proposition 17 (The isobaric process). *The isobaric process is a thermodynamic process which happens at constant pressure. Hence:*

$$W = \int p dV = p\Delta V$$

By applying the first law, $\Delta U = Q - W$, we obtain that:

$$Q = \Delta U + W = mc_v\Delta T + p\Delta V = mc_v\Delta T + mR\Delta T = mc_p\Delta T = \Delta H$$

Proposition 18 (The isochoric process). *The isochoric process is a thermodynamic process which happens at constant volume. Hence:*

$$W = \int p dV = 0$$

Since $\Delta U = Q - W$, we obtain that the heat transfer is equal to the change in internal energy, i.e.:

$$Q = \Delta U = mc_v\Delta T$$

Proposition 19 (The isothermal process). *The isothermal process is a thermodynamic process which happens at constant temperature. Hence:*

$$W = \int_A^B p dV = \int_A^B \frac{mRT}{V}dV = mRT\int_A^B \frac{1}{V}dV = mRT\ln\frac{V_B}{V_A}$$

Because the temperature is constant, $\Delta U = mc_v\Delta T = 0$, and hence:

$$0 = Q - W \iff Q = W$$

Proposition 20 (The adiabatic process)**.** *The adiabatic process is a thermodynamic process in which there is no heat exchange. Let us start from the differential form of the first law:*

$$dU = \delta Q - \delta W$$

Since there is no heat exchange:

$$dU = -\delta W \iff mc_v dT = -pdV \iff mc_v dT + pdV = 0$$

Since $pV = mRT \iff p = \frac{mRT}{V}$, and so:

$$mc_v dT + \frac{mRT}{V}dV = 0 \iff c_v \frac{dT}{T} + R\frac{dV}{V} = 0$$

Noting that $\frac{R}{c_v} = \gamma - 1$, we obtain:

$$\frac{dT}{T} + (\gamma - 1)\frac{dV}{V} = 0 \iff \ln\frac{T_2}{T_1} + (\gamma - 1)\ln\frac{V_2}{V_1} = 0$$

Therefore:

$$\ln\frac{T_2}{T_1} = (\gamma - 1)\ln\frac{V_1}{V_2}$$

And therefore, by exponentiating and multiplying:

$$T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1} \iff TV^{\gamma-1} = const.$$

This equation can be manipulated into the two following forms:

$$pV^\gamma = const. \; or \; \frac{p}{T^{\frac{\gamma}{\gamma-1}}} = const.$$

Beware of the assumptions involved in this process - it has to be adiabatic ($Q = 0$), the gas has to be perfect, and we have to be in quasi-equilibrium. This is also referred as the isentropic process.

Proposition 21 (The polytropic process)**.** *The polytropic process is a thermodynamic process in which the pressure and volume are related by:*

$$pV^n = const.$$

Note that:

(a) $n = \gamma$ represents an isentropic process

(b) $n = 1$ represents an isothermal process

(c) $n = 0$ is an isobaric process

(d) $n \to \infty$ is an isochoric process

Furthermore, for $n \neq 1$, the work done is equal to:

$$W = \frac{p_1 V_1 - p_2 V_2}{n - 1}$$

The first law can then be used to calculate the total heat transfer.

2.2 The second law of Thermodynamics

The first law is the law of energy conservation - it makes no distinction between heat and work and only deals with their combined effect. However, the second law is the law of efficiency of energy conversion - it makes a great distinction between heat and work. We are now concerned with either irreversible processes (which take place in only one direction) and reversible processes (which can take place in either direction).

Definition 23 (Irreversible process)**.** *In general, after a system undergoes an irreversible process, it can be restored to its initial state through suitable flows of heat and work from the surroundings. However, in doing so, the surroundings are changed permanently.*

The following types of processes are irreversible:

(a) *Processes involving fluid friction of friction between solid surfaces*

(b) *Heat transfer across a finite temperature difference*

(c) *The unresisted or partially resisted expansion of a gas*

(d) *A rapid chemical reaction, such as an explosion*

(e) *The mixing of two different fluids*

Definition 24 (Reversible process)**.** *A process is said to be reversible if the system and its surroundings can be returned to ther initial state.*

Now, we will look at statements of the second law of thermodynamics. Both are statement of what cannot be achieved within a process.

Theorem 3 (The Kelvin-Planck statement)**.** *It is impossible to construct a cyclic device whose sole effect is to produce positive work whilst receiving heat from a single thermal reservoir.*

Theorem 4 (The Clausius statement)**.** *It is impossible to construct a cyclic device whose sole effect is the transfer of heat from a cooler to a hotter body.*

2.2.1 Heat engines

Definition 25 (Thermal efficiency)**.** *The thermal efficiency of a cycle is defined as the net work done over the total heat input, i.e.:*

$$\eta = \frac{W}{Q_H} = \frac{Q_H - Q_C}{Q_H} = 1 - \frac{Q_C}{Q_H}$$

Therefore, using a Carnot cycle (two isothermal processes and two adiabatic processes), one can construct an engine with:

$$\eta = 1 - \frac{T_C}{T_H}$$

Theorem 5. *The maximum efficiency of a cyclic heat engine operating between two thermal reservoirs is attained when the cycle is reversible.*

Theorem 6. *All reversible heat engines operating between the same thermal reservoirs are equally efficient.*

By the second theorem, because the Carnot cycle is reversible, the maximum efficiency we can obtain when oberating between two temperatures is the Carnot efficiency. Note that this also implies that:

$$\frac{Q_H}{T_H} = \frac{Q_C}{T_C}$$

2.2.2   Refrigerators and heat pumps

Refrigerators and heat pumps are devices for extracting heat from a cold thermal reservoir and delivering (a greater quantity of) heat to a hotter thermal reservoir. Such devices do not violate the Clausius statement of the second law because they require a work input.

Definition 26. *For both of these, we define the coefficient of performance as:*

(a) For a refrigerator, $COP_R = \frac{Q_C}{W}$

(b) For a heat pump, $COP_P = \frac{Q_H}{W}$

2.3 Thermal equilibrium and the zeroth law of Thermodynamics

The following statement might be both trivial and obvious, but it is impossible to prove, and thus we need to take it as an axiom. Moreover, without it temperature cannot be defined.

> **Theorem 7** (The zeroth law of Thermodynamics). *If systems B and C are each separately in thermal equilibrium with system A, then they would be in thermal equilibrium with each other if brought into thermal contact.*

Lord Kelvin created a temperature scale based on the second law, which is completely independent of any thermometric substance. The starting point is the fact that all reversible heat engines operating between the same two fixed temperatures have the same efficiency.

Consequently, for a heat engine:

$$\frac{Q_1}{Q_2} = f(\theta_1, \theta_2)$$

On dimensional grounds:

$$\frac{Q_1}{Q_2} = f\left(\frac{\theta_1}{\theta_2}\right)$$

The choice of this function is arbitrary, and the simplest form is used to define the thermodynamic temperature scale as:

$$\frac{Q_1}{Q_2} = \frac{\theta_1}{\theta_2} = \frac{T_1}{T_2}$$

The equality between θ and T follows from the previous section regarding heat engines. A single fixed point is required to define the size of the temperature unit (the Kelvin), and this is taken to be the triple point of pure water. This is the point at which ice, water and steam all coexist in equilibrium, and it is given the exact value of $0.01°\text{C} = 273.16\text{K}$ on the thermodynamic scale.

> **Theorem 8** (The Clausius inequality). *The Clausius inequality states that:*
>
> $$\oint \frac{dQ}{T} \leq 0$$
>
> *This is the most concise mathematical statement of the second law.*

This is a useful starting point for the next chapter of this course, where we will discuss entropy. Note that the convention is that $Q > 0$ if heat is transfered to the system, and $Q < 0$ when it is rejected. Also note that the inverse is true for work, i.e. $W < 0$ for work input and $W > 0$ fork work output.

2.4   Entropy

Entropy is a property associated with the second law, just as energy is associated with the first law. In a previous chapter, we introduced the concept of irreversibility. It is therefore useful to have some means of quantifying the irreversibility that occurs during any process. This is achieved through entropy.

> **Definition 27** (Entropy). *The entropy of a system is an extensive thermodynamic property, defined as such:*
>
> $$dS = \left(\frac{dQ}{T}\right)_{rev}$$

Note that if we wish to integrate the following and deduce the change in entropy, the integration must always be carried over a reversible path. Also note that if a process is reversible, then:

$$dS = \frac{dQ}{T}$$

This shows that if a process is reversible and adiabatic, then it is also isentropic, i.e.:

$$dS = 0$$

> **Proposition 22** (Equality of areas). *Let us consider a cyclic reversible process. Then, the area under the (p, V) diagram is equal to the area under the (T, S) diagram.*

Proof. Note that because of the cycle being reversible, then $dS = \frac{dQ}{T} \iff dQ = TdS$. Then, by using the fact that we are within a cycle and the change in internal energy over the entire cycle is null:

$$\oint dQ = \oint dW \iff \oint TdS = \oint pdV$$

$\square$

> **Proposition 23** (Entropy changes for an irreversible process). *For an irreversible process, we have that:*
>
> $$dS \geq \frac{dQ}{T} \iff dS = \frac{dQ}{T} + dS_{irrev}$$

2.4.1   Applications of entropy

The entropy of a system cannot be measured directly, and so it must be inferred from the other properties, such as pressure, volume and temperature.

> **Proposition 24** (The entropy equations). *Consider a simple compressible system undergoing an infinitesimal change of state by heat and work interactions with the surroundings. Therefore:*
>
> $$dU = \delta Q - \delta W$$

If the process is reversible, then $dS = \frac{dQ}{T} \iff \delta Q = TdS$. Also, $\delta W = pdV$. Therefore:

$$dU = TdS - pdV \iff TdS = dU + pdV$$

Noting that $dU = mc_v dT$ and that $pdV = \frac{mRT}{V}dV$, we obtain:

$$TdS = mc_v dT + \frac{mRT}{V}dV \iff dS = \frac{mc_v dT}{T} + \frac{mR}{V}dV$$

Taking the specific entropy of the system:

$$ds = c_v \frac{dT}{T} + R\frac{dv}{v}$$

This can then be integrated to obtain the change of entropy in terms of temperature and volume. Furthermore:

$$H = U + pV \iff dH = dU + pdV + Vdp$$

Noting that $TdS = dU + pdV$ from above:

$$dH = TdS + Vdp \iff TdS = dH - Vdp$$

Therefore:

$$dS = mc_p \frac{dT}{T} - \frac{mR}{p}dp$$

By taking the specific entropy:

$$ds = c_p \frac{dT}{T} - R\frac{dp}{p}$$

This can then be integrated to obtain the change of entropy in terms of pressure and temperature.

Also note that any two of the words "adiabatic", "reversible" and "isentropic" imply the third. Also note that entropy is quantifiable through the Boltzmann relationship:

$$S = k_B \ln \Omega$$

Where k_B is the Boltzmann constant, and Ω is the number of microstates of the isolated system (what cannot be measured). However, we will only stick to the classical definition of entropy for the sake of practicality.

2.5 Reciprocating engines

In this section, the theory developed so far is applied to analyse reciprocating engines, including petrol and diesel engines.

2.5.1 The Stirling engine

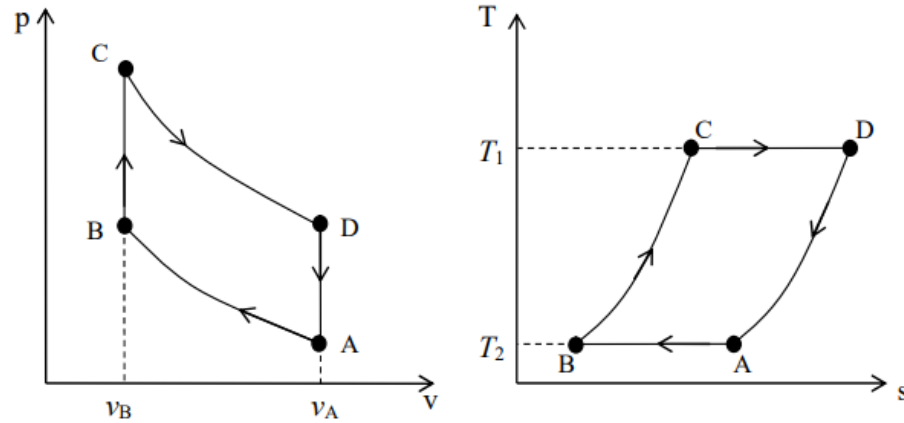Figure 3: The Sirling engine cycle

A Stirling engine is an example of a reciprocating external combusting engine - this is because the heat is transferred from an external source rather than by burning fuel internally, as is done within spark/compression ignition engines. A Stirling engine is comrpised of the following cycle:

(a) A $\to$ B: Isothermal compression at T_2

(b) B $\to$ C: Constant volume head addition

(c) C $\to$ D: Isothermal expansion at T_1

(d) D $\to$ A: Constant volume heat rejection

Note that it can also be shown that the efficiency of a Stirling cycle is:

$$\eta = 1 - \frac{T_2}{T_1}$$

Where $T_2 < T_1$ are the operating temperatures of the cycle.

2.5.2 Spark & compression ignition engines

Definition 28 (Spark ignition). *A spark ignition engine is an engine (usually petrol engines) in which a mixture of air and vaporised fuel is introduced into the cylinder and ignited by a spark.*

> **Definition 29** (Compression ignition). *A compression ignition engine is an engine (usually diesel engines) in which air is compressed and reaches a high enough temperature that the fuel ignites spontaneously. The fuel is usually injected directly into the cylinder in the form of a liquid droplet spray.*

In general, a spark ignition engine functions in the following way:

(a) The induction stroke (A → B) - the inlet valve is opened and the piston moves downward, drawing the fuel-air mixture into the cylinder. The pressure falls below atmospheric pressure due to frictional losses in the intake system.

(b) The compression stroke (B → C) - both valves are closed and the piston moves upward compressing the mixture. The spark occurs before the top dead centre, and after a delay initiates rapid combustion at almost constant volume, causing a rapid increase in temperature and pressure.

(c) The power stroke (C → E) - the valves remain closed as the piston returns to the bottom dead centre. The pressure initially continues to rise as combustion is completed (D), and then falls as the hot gases expand. Work is done on the piston and transferred via the connecting rod to the crankshaft.

(d) The exhaust stroke (E → A) - the exhaust valve opens and the products of combustion are expelled as the piston returns to the top dead centre.
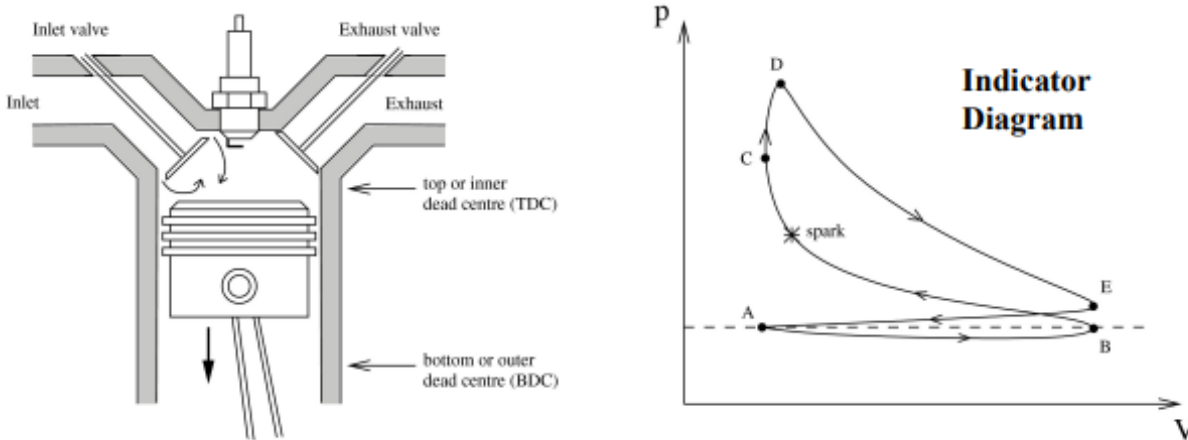

Figure 4: Spark ignition engine

The compression ignition engine is very similar, except that instead of the spark, fuel is injected, and at a later stage. Combustion occurs more slowly, resulting in a flatter top to the indicator diagram.

2.5.3 The air-standard cycles

Air-standard cycles provide a very simplified model of the processes described above. The main approximations involved are:

(a) The working fluid is assumed to be air throughout, with constant specific heat capacities..

(b) The air is treated as a closed system and the combustion of the fuel is modelled as a heat addition from an external source.

(c) The compression and expansion processes are assumed to be adiabatic and reversible, thus isentropic. Note that heat losses may be significant in reality, especially during the power stroke.

Proposition 25 (The air-standard Otto cycle). *The Otto cycle provides an approximate model of spark ignition engines and comprises the following four processes:*

(a) $A \rightarrow B$: Isentropic compression

(b) $B \rightarrow C$: Heat addition at constant volume

(c) $C \rightarrow D$: Isentropic expansion

(d) $D \rightarrow A$: Heat rejection at constant volume
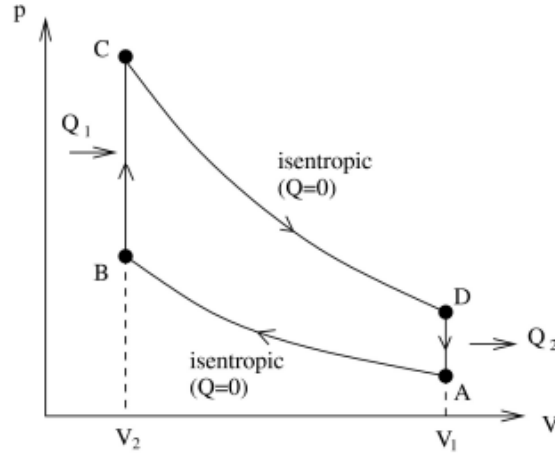

Figure 5: The Otto cycle

Note that in a real engine, the final process replaces expulsion of the exhaust gas and renewal with fresh charge in the real engine. Also, the ratio between the maximum and minimum volume is called the compression ratio, $r_v = \frac{V_1}{V_2}$.

This gives us the following efficiency for the Otto cycle:

$$\eta = 1 - \frac{1}{r_v^{\gamma-1}}$$

Proposition 26 (The air-standard Diesel cycle). *The Diesel cycle provides an approximate model of compression ignition engines, and comprises the following four processes:*

(a) $A \rightarrow B$: Isentropic compression

(b) $B \rightarrow C$: Heat addition at constant pressure

(c) $C \rightarrow D$: Isentropic expansion

(d) $D \rightarrow A$: Heat rejection at constant volume

Note the similarities between the two engines - only one process differs. Also, the constant pressure heat addition is intended to model the slower combustion process. As before, we define the same compression ratio, and also the cut-off ratio:
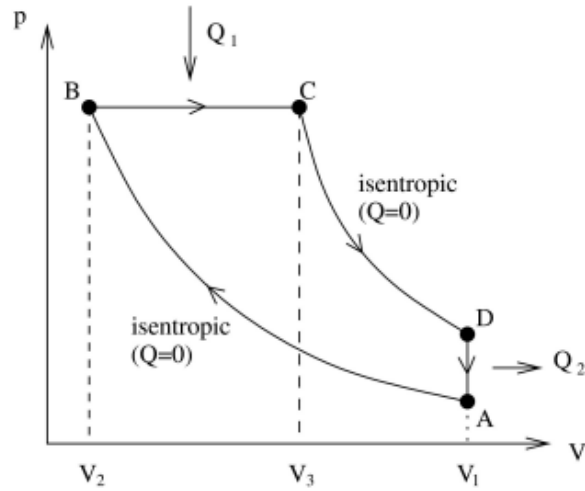
$$\alpha = \frac{V_3}{V_2}$$

Figure 6: The Diesel cycle

This then gives us the following efficiency:

$$\eta = 1 - \frac{1}{r_v^{\gamma-1}} \left(\frac{\alpha^\gamma - 1}{\gamma(\alpha - 1)} \right)$$

Both of these efficiencies are exercises that can be found in the Thermofluids example sheets.

2.6 Control volume analysis

Note that in the following section, we will borrow some concepts from fluid mechanics. We define the mass flow rate as previously:

$$\dot{m} = \int_A \rho \mathbf{v} \cdot \mathbf{dA}$$

Theorem 9 (Mass conservation)**.** *Consider a control volume. Therefore, under non-steady conditions:*

$$\sum \dot{m_{in}} - \sum \dot{m_{out}} = \frac{dm_{cv}}{dt}$$

Proof. At time t, the mass in the system is $m_{\text{cv}} + \delta m_i$. At time $t + \delta t$, the mass in the system will be $m_{\text{cv}} + \delta m_{\text{cv}} + \delta m_e$. Since no mass crosses the boundary:

$$m_{\text{cv}} + \delta m_i = m_{\text{cv}} + \delta m_{\text{cv}} + \delta m_e$$

This is equivalent to:

$$\delta m_i - \delta m_e = \delta m_{\text{cv}}$$

Dividing the above by δt yields:

$$\dot{m}_i - \dot{m}_e = \dot{m}_{\text{cv}}$$

And by converting the above into a summation, we obtain the result. $\square$

Proposition 27 (Steady flow mass conservation)**.** *Under steady flow conditions, the theorem above degenerates into:*

$$\sum \dot{m_{in}} = \sum \dot{m_{out}}$$

This is trivial, because $\dot{m}_{cv}$ vanishes.

Theorem 10 (Energy conservation)**.** *Consider a control volume. Then, under non-steady conditions:*

$$\dot{Q} - \dot{W}_x = \frac{dE_{cv}}{dt} + \sum \dot{m}_e (h_e + \frac{1}{2} V_e^2 + g z_e) - \sum \dot{m}_i (h_i + \frac{1}{2} V_i^2 + g z_i)$$

Where the e subscript denotes an outflow, while the i subscript denotes an inflow.

Proof. We will denote e as the specific energy of the system. The energy of the system at time t is $E_{\text{cv}} + \delta m_i e_i$. At time $t + \delta t$, the energy becomes $E_{\text{cv}} + \delta E_{\text{cv}} + \delta m_e e_e$. The work done by the sistem is:

$$\delta W = \delta W_x + p_e v_e \delta m_e - p_i v_i \delta m_i$$

By applying the first law in the form $\delta E = Q - W$, we obtain:

$$\delta Q - (\delta W_x + p_e v_e \delta m_e - p_i v_i \delta m_i) = (E_{\text{cv}} + \delta E_{\text{cv}} + \delta m_e e_e) - (E_{\text{cv}} + \delta m_i e_i)$$

Therefore:

$$\delta Q - \delta W_x = \delta E_{\text{cv}} + \delta m_e (e_e + p_e v_e) - \delta m_i (e_i + p_i v_i)$$

Note that:

$$e + pv = u + \frac{1}{2} V^2 + gz + pv = h + \frac{1}{2} V^2 + gz$$

By substituting this in, generalizing to multiple inflows and outflows, and dividing by δt, we obtain the above relationship. □

Note that under steady flow conditions this degenerates into:

$$\dot{Q} - \dot{W}_x = \sum \dot{m}_e (h_e + \frac{1}{2} V_e^2 + gz_e) - \sum \dot{m}_i (h_i + \frac{1}{2} V_i^2 + gz_i)$$

Furthermore, if we have one inflow and one outflow, this further degenerates into:

$$\dot{Q} - \dot{W}_x = \dot{m}(h_e + \frac{1}{2} V_e^2 + gz_e) - \dot{m}(h_i + \frac{1}{2} V_i^2 + gz_i)$$

Dividing through the mass flow rate yields:

$$q - w_x = (h_e + \frac{1}{2} V_e^2 + gz_e) - (h_i + \frac{1}{2} V_i^2 + gz_i)$$

In a lot of problems, the potential and kinetic energy terms can be neglected. This means that we can use the following approximation:

$$q - w_x = h_e - h_i$$

If the process is adiabatic, we obtain, in differential form:

$$-dw_x = dh$$

Remember that $dh = du + pdv + vdp$ and that $Tds = du + pdv$, so then $dh = Tds + vdp$. Furthermore, if the process is reversible, then it is isentropic, and we obtain that for an adiabatic and reversible process (if we neglect kinetic and potential energy terms).

$$dw_x = -vdp$$

Note that the above relationship can be used to prove Bernoulli's equation, if we do not neglect energy terms and simply integrate it and assume that the net shaft work is null.

Theorem 11 (Entropy changes). *Consider a control volume. Then, under non-steady conditions:*

$$\frac{dS_{cv}}{dt} + \sum \dot{m}_e s_e - \sum \dot{m}_i s_i = \sum \frac{\dot{Q}_k}{T_k} + \dot{S}_{irrev}$$

Proof. We proceed with the proof as previously shown. At time t, the entropy of the system is $S_\text{cv} + \delta m_i s_i$. At time $t + \delta t$, the entropy is $S_\text{cv} + \delta S_\text{cv} + \delta m_e s_e$. Therefore, the entropy increase is:

$$\delta S_\text{sys} = (S_\text{cv} + \delta S_\text{cv} + \delta m_e s_e) - (S_\text{cv} + \delta m_i s_i) = \delta S_\text{cv} + \delta m_e s_e - \delta m_i s_i$$

The system's entropy increase can be written as:

$$\delta S_\text{sys} = \sum \frac{\delta Q_k}{T_k} + \delta S_\text{irrev}$$

By allowing for a summation and dividing by δt, we can then obtain the theorem above. $\qquad\square$

Note that in the case where the flow is steady, then $\frac{dS_\text{cv}}{dt} = 0$, and hence, the theorem above degenerates in:

$$\sum \dot{m}_e s_e - \sum \dot{m}_i s_i = \sum \frac{\dot{Q}_k}{T_k} + \dot{S}_\text{irrev}$$

If we have only one inflow and outflow, then:

$$s_e - s_i = \sum \frac{q_k}{T_k} + \Delta s_\text{irrev}$$

Furthermore, if the flow is reversible, then the irreverisble term disappears, and the equation further degenerates into:

$$s_e - s_i = \sum \frac{q_k}{T_k}$$

In differential form, this is equivalent to $ds = \frac{dq}{T}$, which is simply the definition of entropy for reversible steady flow.

The last simplification is in the case where the flow is isentropic, giving that:

$$s_e - s_i = 0 \iff s_e = s_i$$

Proposition 28 (Turbines and compressors). *Flow through compressors and turbines is sufficiently fast for heat transfer to be negligible. As an approximation, we will assume that the flow is also reversible. Therefore, when operating under these assumptions in steady state, the flow is thus isentropic (adiabatic, reversible, steady).*
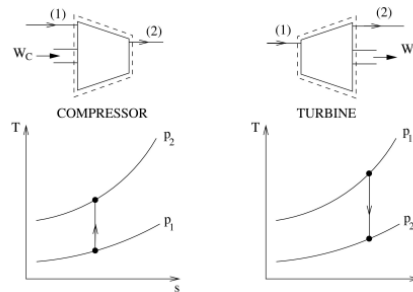

Figure 7: Compressor and turbine

2.7   Gas turbines and jet engines

In this final section, we will apply steady flow control volume analysis to study gas turbines and the simplest form of a jet engine. As with reciprocating internal combustion engines, a simplified air-standard cycle analysis will be adopted.

Definition 30 (Gas turbine). *A gas turbine is a device whose function is to produce shaft work in order to drive a generator, or a pump, or some other piece of rotating machinery.*

Definition 31 (Jet engine). *A jet engine is a device that produces no net shaft work - its function is to generate thrust by means of a high velocity exhaust jet.*

2.7.1   The air-standard Joule cycle

The main components of a typical industrial gas turbine are shown below.
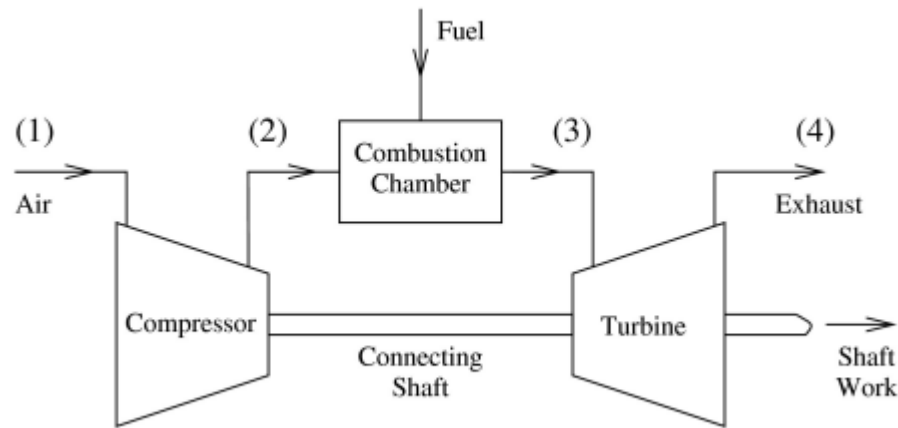

Figure 8: Gas turbine

Air is taken in at (1) and compressed at (2), at which its pressure is typically 12 - 40 times less than at (1). Fuel is then added and combustion occurs at more or less constant pressure. Hot, high pressure combustion gases then enter the turbine at (3) and expand down to the exhaust pressure at (4), which is usually the same as the ambient pressure p.

The gas turbine is clearly a flow device: there are two inflows (air and fuel) and one exit flow (the exhaust). It would be quite possible to draw a control surface around the whole gas turbine and then apply the steady-flow energy equation to compute the output shaft work. However, this requires accounting for the enthalpy changes during combustion, a topic which has not yet been covered. Instead, we consider the working fluid as air throughout and we model the combustion process as a constant pressure heat input.

Proposition 29 (The air-standard Joule cycle). *The air-standard Joule cycle (sometimes called the Brayton cycle) is an approximate model of the gas turbine that replaces the open-circuit "cycle" with a closed-circuit one, as shown below. Note that each device (compressor, turbine, etc.) operates as a*

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steady flow device so must be analysed by the steady-flow energy equation (and not the First Law for a system).
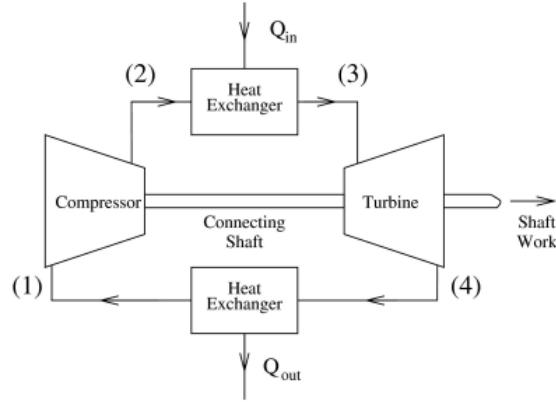

Figure 9: The Joule cycle

The processes are as follows:

(a) $(1) \rightarrow (2)$: Isentropic compression

(b) $(2) \rightarrow (3)$: Constant pressure heat addition

(c) $(3) \rightarrow (4)$: Isentropic expansion

(d) $(4) \rightarrow (1)$: Constant pressure heat rejection

2.7.2 The jet engine

The major components of the simplest form of the jet engine (a turbojet) are shown below.
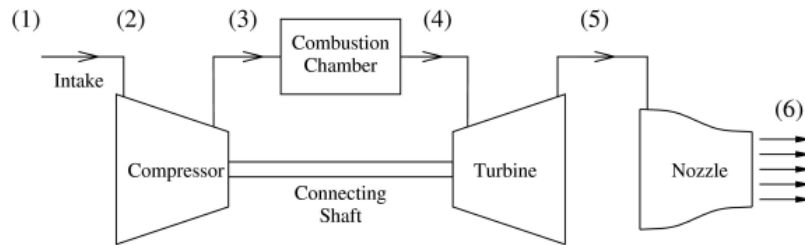

Figure 10: The turbojet engine

The major difference between the jet engine and the gas turbine is that, rather than producing net shaft work, the turbine work output for a jet engine balances the compressor work input.

The processes for a jet engine are as follows:

(a) $(1) \rightarrow (2)$: Isentropic stagnation (intake)

(b) (2) → (3): Isentropic compression (compressor)

(c) (3) → (4): Constant pressure heat input (combustor)

(d) (4) → (5): Isentropic expansion (turbine)
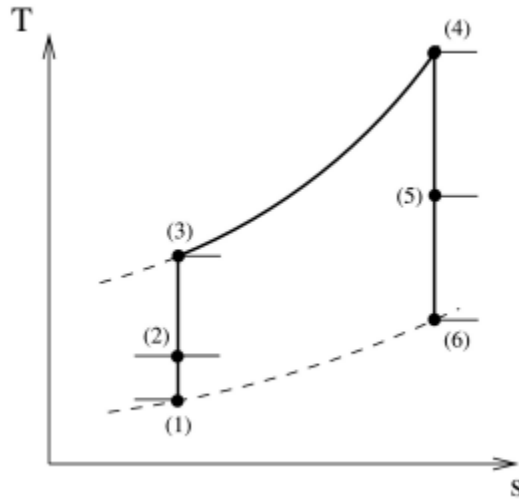
(e) (5) → (6): Isentropic acceleration (nozzle)

Figure 11: The $T - S$ diagram for the turbojet

It is not usually necessary to close the cycle with a fictitious process from (6) → (1), because we are normally interested in calculating the engine thrust, rather than efficiency. To do this, we simply apply the steady flow momentum equation for the whole engine.