Engineering Tripos Part IA - Mechanical Vibrations

Morărescu Mihnea-Theodor

April 7, 2024

Contents

1 Introduction to linear systems

In this course, we are concerned with linear systems that are described by ordinary differential equations.

Proposition 1 (Governing differential equation). *The governing differential equation of such a system is:*

$$a_n \frac{d^n y}{dt^n} + \ldots + a_1 \frac{dy}{dt} + a_0 y = x$$

1.1 System elements

Proposition 2 (Mechanical systems for translation). *The mechanical components for translational motion are the mass, the damper, and the spring.*

1. *The mass exerts a force of the type:* $f = m\ddot{y}$
2. *The damper (or the dashpot) exerts a force of the type:* $f = \lambda\dot{y}$
3. *The spring exerts a force of the type:* $f = ky$

Proposition 3 (Mechanical systems for rotation). *The mechanical components for rotational motion are the intertia, the torsional damper, and the torsional spring.*

1. *The intertia exerts a torque of the type:* $\tau = J\ddot{\theta}$
2. *The torsional damper exerts a torque of the type:* $\tau = \lambda\dot{\theta}$
3. *The torsional spring exerts a torque of the type:* $\tau = k\theta$

Proposition 4 (Electrical systems). *The electrical components for electrical system as the inductance, the resistance and the capacitor.*

1. *The inductor has a voltage:* $v = L\frac{di}{dt} = L\ddot{q}$
2. *The resistor has a voltage:* $v = iR = R\dot{q}$

3. The capacitor has a voltage: $v = \frac{1}{C}q$

Note that in mechanical systems, for translation the coordinate y is the degree of freedome, whereas for rotation the angle θ is the degree of freedom. In electrical systems, the charge q is usually the degree of freedom.

2 Response of first order systems

Consider a spring and a damper in series, acted upon by a force f. Therefore:

$$0 = f - k(x-y) \iff f = k(x-y)$$
$$0 = -k(y-x) - \lambda\dot{y} \iff \lambda\dot{y} = k(x-y)$$

The second equation yields us:

$$\lambda\dot{y} + ky = kx \iff \frac{\lambda}{k}\dot{y} + y = x$$

By defining the time constant of the system $T := \frac{\lambda}{k}$, we obtain the governing differential equation for first-order systems:

$$T\frac{dy}{dt} + y = x$$

2.1 Integration of the equation

The equation $T\dot{y} + y = x$ can now be integrated to obtain its solution. For the time being, let us assume that we do not know anything about x. The complementary function is the solution to the equation:

$$T\dot{y} + y = 0 \iff \dot{y} + \frac{1}{T}y = 0$$

By standard procedure as per differential equations, the complementary function is:

$$y_{CF} = Ae^{-\frac{t}{T}}$$

2.1.1 Step response

Definition 1 (Step response)**.** *The step response is the response of the system to the Heaviside step function, i.e.:*

$$H(t) = \begin{cases} 0, t < 0 \\ 1, t \geq 0 \end{cases}$$

We will further consider an input of the type $x_0 H(t), x_0 \in \mathbb{R}$. Therefore, a particular solution to the differential equation is:

$$y_0(t) = x_0, t \geq 0$$

Hence, the output is:

$$y(t) = Ae^{-\frac{t}{T}} + x_0$$

By means of the initial condition $y(0) = 0$, we obtain the step response of a first-order system:

$$y(t) = x_0 \left(1 - e^{-\frac{t}{T}}\right)$$

2.1.2 Impulse response

Definition 2 (Impulse response)**.** *The impulse response of a system is its response to the Dirac-δ function, defined as $\delta(t) = 0, \forall t \neq 0$, and:*

$$\int_a^b \delta(t)dt = 1, \forall a < 0 < b$$

From the Mathematics course, we know that the impulse response is the derivative of the step response. Therefore, if we have an impulse of amplitude x_0, the output of the system will be:

$$y(t) = \frac{x_0}{T} e^{-\frac{t}{T}}$$

2.1.3 Ramp response

Definition 3 (Ramp response)**.** *The ramp response of a system is its response to an input of the type $x(t) = x_0 t$.*

Consider the differential equation:

$$T\frac{dy}{dt} + y = x_0 t \iff \frac{dy}{dt} + \frac{1}{T}y = \frac{x_0}{T}t$$

The integrating factor is $I = e^{\int p(x)dx} = e^{\int \frac{1}{T}dt} = e^{\frac{t}{T}}$. Multiplying:

$$\left(\frac{dy}{dt} + \frac{1}{T}y\right) e^{\frac{t}{T}} = \frac{x_0}{T} t e^{\frac{t}{T}} \iff \left(ye^{\frac{t}{T}}\right)' = \frac{x_0}{T} t e^{\frac{t}{T}}$$

By integrating this relationship, we obtain the ramp response:

$$y(t) = x_0 T \left(e^{-\frac{t}{T}} - 1\right) + x_0 t$$

2.1.4 Harmonic response

Definition 4 (Harmonic response)**.** *The harmonic response of a system is its response to an input of the type $x(t) = X \cos \omega t$.*

The trick to easily solving harmonic response problems is to instead work using sines and cosines, utilize complex numbers. The form of the input is equivalent to:

$$x(t) = \mathfrak{Re}(Xe^{i\omega t})$$

By means of causality, the output has to be of the form $y(t) = \mathfrak{Re}(Ye^{i\omega t})$. Note that $X, Y \in \mathbb{C}$. The variable X can also be real, but allowing it to be complex allows us to also encode an initial phase into our problem. By dropping the real specification and substituting into our initial equation, we obtain:

$$Ti\omega Ye^{i\omega t} + Ye^{i\omega t} = Xe^{i\omega t}$$

This is equivalent to:

$$(1 + i\omega T)Y = X \iff Y = \frac{X}{1 + i\omega T}$$

Now, by taking the real part, we can obtain the particular integral:

$$y(t) = \mathfrak{Re}\left(Ye^{i\omega t}\right) = \mathfrak{Re}\left(\frac{X}{1 + i\omega T}e^{i\omega T}\right)$$

By writing $X = |X|e^{i\phi_x}$ and amplifying the denominator, we have:

$$y(t) = \mathfrak{Re}\left(\frac{1 - i\omega T}{1 + \omega^2 T^2}|X|e^{i(\omega t + \phi_x)}\right)$$

By using Euler's identity:

$$y(t) = \mathfrak{Re}\left(\frac{|X|(1 - i\omega T)}{1 + \omega^2 T^2}(\cos(\omega t + \phi_x) + i \sin(\omega t + \phi_x))\right)$$

Therefore:

$$y(t) = \frac{|X|}{\sqrt{1 + \omega^2 T^2}} \cos(\omega t + \phi)$$

Where $\phi = \phi_x - \arctan \omega T$. Note that this is only the particular integral. By adding in the complimentary function, we obtain the full output of this system when excited harmonically:

$$y(t) = Ae^{-\frac{t}{T}} + \frac{|X|}{\sqrt{1 + \omega^2 T^2}} \cos(\omega t + \phi)$$

With the initial condition of $y(0) = 0$.

3 Response of second order systems

Proposition 5 (Standard form). *The standard form of a second order system is given by the differential equation:*

$$\frac{1}{\omega_n^2}\ddot{y} + \frac{2\zeta}{\omega_n}\dot{y} + y = x$$

Proof. Let us consider a mass with a dashpot and a spring in parallel, having an applied force f. The governing differential equation is:

$$m\ddot{y} = f - ky - \lambda\dot{y} \iff m\ddot{y} + \lambda\dot{y} + ky = f$$

By dividing this equation through k, we deduce:

$$\frac{m}{k}\ddot{y} + \frac{\lambda}{k}\dot{y} + y = \frac{f}{k}$$

Now, we can define the natural frequency of this system as $\frac{1}{\omega_n^2} = \frac{m}{k} \iff \omega_n = \sqrt{\frac{k}{m}}$. Likewise, by identifying $\frac{2\zeta}{\omega_n} = \frac{\lambda}{k}$, we can deduce that the damping factor ζ is given by:

$$\zeta = \frac{\lambda}{2\sqrt{mk}}$$

□

3.1 Free vibration

Definition 5 (Free vibration). *Free vibration occurs when the input x is equal to $x(t) = 0$.*

Since the governing differential equation is:

$$\frac{1}{\omega_n^2}\ddot{y} + \frac{2\zeta}{\omega_n}\dot{y} + y = 0$$

By solving the characteristic equation, we obtain the solutions:

$$\alpha_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

We now have to distinguish between three separate cases:

1. $\zeta > 1$, i.e. both roots are real and we refer to this system as "over-damped"

2. $\zeta = 0$, i.e. both roots are repeated and we refer to the system as "critically damped"

3. $\zeta < 1$, i.e. both roots are cmplex conjugate and we refer to the system as "under damped"

3.1.1 Over-damped systems

This is equivalent to the system being in the first case, which is equivalent to $\zeta > 1$. Therefore, the complementary function is:

$$y_{CF} = Ae^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t} + Be^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t}$$

3.1.2 Critically damped systems

This is equivalent to the system being in the second case, which is equivalent to $\zeta = 1$. Therefore, the complementary function is of the type:

$$y_{CF} = (A + Bt)e^{-\omega_n t}$$

3.1.3 Under-damped systems

This is equivalent to the system being in the third case, which is equivalent to $\zeta < 1$. Therefore, before proceeding, let us define:

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

This is also referred to as the damped natural frequency. Hence, the complementary function is of the type:

$$y_{CF} = e^{-\zeta\omega_n t}\left(A\sin\omega_d t + B\cos\omega_d t\right)$$

However, we want this to be in an even closer form. Let us consider the equation:

$$A\cos\left(x+\phi\right) = B\sin x + C\cos x \iff A\cos x\cos\phi - A\sin x\sin\phi = B\sin x + C\cos x$$

By identifying terms, we deduce that $C = A\cos\phi$ and $B = -A\sin\phi$. Therefore, the two constants A and ϕ are:

$$\phi = -\arctan\frac{B}{C} \iff A = \sqrt{B^2 + C^2}$$

Therefore, we can write the response of under-damped system as:

$$y_{CF} = De^{-\zeta\omega_n t}\cos\left(\omega_d t - \psi\right)$$

Where D and ψ are constants which are yet to be determined.

3.2 Step response

It is trivial to observe that if $f(t) = x_0 H(t)$, $x_0 \in \mathbb{R}$, then the particular integral is simply $y_{PI} = x_0, \forall t > 0$. Let us now only consider an under-damped system. Hence:

$$y(t) = De^{-\zeta\omega_n t}\cos(\omega_d t - \psi) + x_0$$

By setting the boundary conditions $y(0) = 0$ and $\dot{y}(0) = 0$ due to causality, we deduce that:

$$D = -\frac{x_0}{\cos\psi} \iff \tan\psi = \zeta\frac{\omega_n}{\omega_d}$$

Also, note that $\sin\psi = \zeta$ and $\cos\psi = \sqrt{1-\zeta^2}$. Therefore, the output (response) of the system is:

$$y(t) = x_0\left(1 - \frac{e^{-\zeta\omega_n t}\cos(\omega_d t - \psi)}{\cos\psi}\right)$$

If $\zeta << 1$, then this expression can be approximated as:

$$y(t) \approx x_0\left(1 - e^{-\zeta\omega_n t}\cos(\omega_n t)\right)$$

The graph for the step response can be found on page 6 of the Mechanics databook.

3.3 Impulse response

As mentioned previously, the impulse response is the response of the system to $\delta(t)$. Hence, it is also the derivative of the step response. By simply differentiating the expression above:

$$y(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t$$

The graph for the impulse response can be found on page 7 of the Mechanics databook.

3.4 Logarithmic decrement

Definition 6 (Logarithmic decrement)**.** *The logarithmic decrement is a method of measuring the damping ratio from the transient response of a system. It is defined as:*

$$\Delta = \ln\frac{y_1}{y_2} = \ln e^{\zeta\omega_n T} = \zeta\omega_n T = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

Where $T = \frac{2\pi}{\omega_d}$ is the period of the damped oscillations.