MA 591 Fall 2024 HW1

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1. Prove the Cauchy-Schwarz inequality over a general inner product space. That is, for all $\boldsymbol{x}$, $\boldsymbol{y}$ in an inner product space, it is always true that $|\langle \boldsymbol{x}, \boldsymbol{y}\rangle| \leq \|\boldsymbol{x}\|\|\boldsymbol{y}\|$, and that equality holds if and only if either $\boldsymbol{x} = \alpha\boldsymbol{y}$ or $\boldsymbol{y} = \boldsymbol{0}$.

Proof.
Let $\boldsymbol{x}, \boldsymbol{y}$ be two general vectors in a general inner product space.

- If $\boldsymbol{x} = \boldsymbol{0}$ or $\boldsymbol{y} = \boldsymbol{0}$, then $|\langle \boldsymbol{x}, \boldsymbol{y}\rangle| = 0$, and $\|\boldsymbol{x}\|\|\boldsymbol{y}\| = 0$. Thus, $|\langle \boldsymbol{x}, \boldsymbol{y}\rangle| = 0 = \|\boldsymbol{x}\|\|\boldsymbol{y}\|$.
- If $\boldsymbol{x} \neq \boldsymbol{0}$ and $\boldsymbol{y} \neq \boldsymbol{0}$, then for all $c \in \mathbb{C}$, $\langle \boldsymbol{x} + c\boldsymbol{y}, \boldsymbol{x} + c\boldsymbol{y}\rangle = \|\boldsymbol{x} + c\boldsymbol{y}\|^2 \geq 0$.

  Moreover, $$\begin{aligned} 0 \leq \langle \boldsymbol{x} + c\boldsymbol{y}, \boldsymbol{x} + c\boldsymbol{y}\rangle &= \langle \boldsymbol{x}, \boldsymbol{x} + c\boldsymbol{y}\rangle + \langle c\boldsymbol{y}, \boldsymbol{x} + c\boldsymbol{y}\rangle \\ &= \langle \boldsymbol{x}, \boldsymbol{x}\rangle + \langle \boldsymbol{x}, c\boldsymbol{y}\rangle + \langle c\boldsymbol{y}, \boldsymbol{x}\rangle + \langle c\boldsymbol{y}, c\boldsymbol{y}\rangle \\ &= \langle \boldsymbol{x}, \boldsymbol{x}\rangle + \langle \boldsymbol{x}, c\boldsymbol{y}\rangle + \overline{\langle \boldsymbol{x}, c\boldsymbol{y}\rangle} + \overline{c}c\langle \boldsymbol{y}, \boldsymbol{y}\rangle \\ &= \|\boldsymbol{x}\|^2 + \overline{c}\langle \boldsymbol{x}, \boldsymbol{y}\rangle + c\,\overline{\langle \boldsymbol{x}, \boldsymbol{y}\rangle} + |c|^2\|\boldsymbol{y}\|^2 \text{ for all } c \in \mathbb{C}. \end{aligned}$$

  Since $\boldsymbol{y} \neq 0$, $\|\boldsymbol{y}\|^2 \neq 0$, choose $c = -\frac{|\langle \boldsymbol{x}, \boldsymbol{y}\rangle|}{\|\boldsymbol{y}\|^2}$ specifically,

  $$\begin{aligned} 0 &\leq \|\boldsymbol{x}\|^2 - \frac{|\langle \boldsymbol{x}, \boldsymbol{y}\rangle|}{\|\boldsymbol{y}\|^2}\langle \boldsymbol{x}, \boldsymbol{y}\rangle - \frac{|\langle \boldsymbol{x}, \boldsymbol{y}\rangle|}{\|\boldsymbol{y}\|^2}\,\overline{\langle \boldsymbol{x}, \boldsymbol{y}\rangle} + \left| -\frac{|\langle \boldsymbol{x}, \boldsymbol{y}\rangle|}{\|\boldsymbol{y}\|^2}\right|^2 \|\boldsymbol{y}\|^2 \\ &= \|\boldsymbol{x}\|^2 - 2\frac{|\langle \boldsymbol{x}, \boldsymbol{y}\rangle|^2}{\|\boldsymbol{y}\|^2} + \frac{|\langle \boldsymbol{x}, \boldsymbol{y}\rangle|^2}{\|\boldsymbol{y}\|^2} \\ &= \|\boldsymbol{x}\|^2 - \frac{|\langle \boldsymbol{x}, \boldsymbol{y}\rangle|^2}{\|\boldsymbol{y}\|^2}. \end{aligned}$$

  Thus, $\dfrac{|\langle \boldsymbol{x}, \boldsymbol{y}\rangle|^2}{\|\boldsymbol{y}\|^2} \leq \|\boldsymbol{x}\|^2$, which implies that $|\langle \boldsymbol{x}, \boldsymbol{y}\rangle|^2 \leq \|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2$.

  Taking the square root both sides, $|\langle \boldsymbol{x}, \boldsymbol{y}\rangle| \leq \|\boldsymbol{x}\|\|\boldsymbol{y}\|$.

Moreover, if $\boldsymbol{x} = \alpha\boldsymbol{y}$, then $|\langle \boldsymbol{x}, \boldsymbol{y}\rangle| = |\langle \alpha\boldsymbol{y}, \boldsymbol{y}\rangle| = |\overline{\alpha}|\|\boldsymbol{y}\|^2$ and $\|\boldsymbol{x}\|\|\boldsymbol{y}\| = \|\alpha\boldsymbol{y}\|\|\boldsymbol{y}\| = |\alpha|\|\boldsymbol{y}\|^2$.
Since $|\overline{\alpha}|\|\boldsymbol{y}\|^2 = |\alpha|\|\boldsymbol{y}\|^2$, $|\langle \boldsymbol{x}, \boldsymbol{y}\rangle| = \|\boldsymbol{x}\|\|\boldsymbol{y}\|$.
Therefore, if $\boldsymbol{x} = \alpha\boldsymbol{y}$ or $\boldsymbol{y} = \boldsymbol{0}$, then $|\langle \boldsymbol{x}, \boldsymbol{y}\rangle| = \|\boldsymbol{x}\|\|\boldsymbol{y}\|$.
On the other hand, assume that $|\langle \boldsymbol{x}, \boldsymbol{y}\rangle| = \|\boldsymbol{x}\|\|\boldsymbol{y}\|$ and $\boldsymbol{y} \neq \boldsymbol{0}$.

Thus, $\|\boldsymbol{x}\|= \frac{|\langle \boldsymbol{x},\boldsymbol{y}\rangle|}{\|\boldsymbol{y}\|}$ and $\|\boldsymbol{x}\|^2= \frac{|\langle \boldsymbol{x},\boldsymbol{y}\rangle|^2}{\|\boldsymbol{y}\|^2}$.

Moreover, $\boldsymbol{x} = \dfrac{|\langle \boldsymbol{x},\boldsymbol{y}\rangle|}{\|\boldsymbol{y}\|^2}\boldsymbol{y} + \left(\boldsymbol{x} - \dfrac{|\langle \boldsymbol{x},\boldsymbol{y}\rangle|}{\|\boldsymbol{y}\|^2}\boldsymbol{y}\right)$

$$\begin{aligned}
\|\boldsymbol{x}\|^2 &= \left\| \frac{|\langle \boldsymbol{x},\boldsymbol{y}\rangle|}{\|\boldsymbol{y}\|^2}\boldsymbol{y} + (\boldsymbol{x} - \frac{|\langle \boldsymbol{x},\boldsymbol{y}\rangle|}{\|\boldsymbol{y}\|^2}\boldsymbol{y}) \right\|^2 \\
&\leq \left\| \frac{|\langle \boldsymbol{x},\boldsymbol{y}\rangle|}{\|\boldsymbol{y}\|^2}\boldsymbol{y} \right\|^2 + \left\| \boldsymbol{x} - \frac{|\langle \boldsymbol{x},\boldsymbol{y}\rangle|}{\|\boldsymbol{y}\|^2}\boldsymbol{y} \right\|^2 \\
&= \frac{|\langle \boldsymbol{x},\boldsymbol{y}\rangle|^2}{\|\boldsymbol{y}\|^2} + \left\| \boldsymbol{x} - \frac{|\langle \boldsymbol{x},\boldsymbol{y}\rangle|}{\|\boldsymbol{y}\|^2}\boldsymbol{y} \right\|^2
\end{aligned}$$

Thus, $\|\boldsymbol{x}\|^2= \frac{|\langle \boldsymbol{x},\boldsymbol{y}\rangle|^2}{\|\boldsymbol{y}\|^2}$ implies $\left\| \boldsymbol{x} - \frac{|\langle \boldsymbol{x},\boldsymbol{y}\rangle|}{\|\boldsymbol{y}\|^2}\boldsymbol{y} \right\|^2 = 0$, which further implies $\boldsymbol{x} - \frac{|\langle \boldsymbol{x},\boldsymbol{y}\rangle|}{\|\boldsymbol{y}\|^2}\boldsymbol{y} = \boldsymbol{0}$.
In other words, $\boldsymbol{x} = \alpha\boldsymbol{y}$, where $\alpha = \frac{|\langle \boldsymbol{x},\boldsymbol{y}\rangle|}{\|\boldsymbol{y}\|^2}$.
Therefore $|\langle \boldsymbol{x},\boldsymbol{y}\rangle|\leq \|\boldsymbol{x}\|\|\boldsymbol{y}\|$, and that equality holds if and only if either $\boldsymbol{x} = \alpha\boldsymbol{y}$ or $\boldsymbol{y} = \boldsymbol{0}$. ☐

2. Show that if S is any orthonormal subset (not necessarily maximal, nor countable) of a Hilbert space $\mathcal{H}$ and $\boldsymbol{x} \in \mathcal{H}$, then the inner product $\langle \boldsymbol{z},\boldsymbol{x}\rangle$ is nonzero for at most a countable number of $\boldsymbol{z} \in S$. Therefore, the Fourier series of any $\boldsymbol{x} \in \mathcal{H}$ defined by

$$\mathcal{F}(\boldsymbol{x}) := \sum_{\boldsymbol{z}\in S} \langle \boldsymbol{z},\boldsymbol{x}\rangle \boldsymbol{z} \tag{1}$$

always makes sense in any Hilbert space.

Proof.
Given $\boldsymbol{x} \in \mathcal{H}$, show that $\{\boldsymbol{z} \in S \mid |\langle \boldsymbol{z},\boldsymbol{x}\rangle|\neq 0\}$ is countable (in other words, the inner product $\langle \boldsymbol{z},\boldsymbol{x}\rangle$ is nonzero for at most a countable number of $\boldsymbol{z} \in S$).
For each $\epsilon > 0$, choose $n \in \mathbb{Z}$ such that $n > \frac{\|\boldsymbol{x}\|^2}{\epsilon^2}$. In other words, $n\epsilon^2 > \|\boldsymbol{x}\|^2$.
Define $H_{x,\epsilon} = \{z \in S \mid |\langle \boldsymbol{z},\boldsymbol{x}\rangle|> \epsilon\}$.
Then, $H_{x,\epsilon}$ is finite. Otherwise, if $H_{x,\epsilon}$ is infinite, choose $\{z_1,\ldots,z_n\} \subset H_{x,\epsilon}$.
Then, $n\epsilon^2 < \sum_{i=1}^{n} |\langle \boldsymbol{z}_i,\boldsymbol{x}\rangle|^2\leq \|\boldsymbol{x}\|^2$. This is a contradiction to the assumption $n\epsilon^2 > \|\boldsymbol{x}\|^2$.
Thus, $H_{x,\epsilon}$ is finite.
Then, define $\mathcal{H}_x = \bigcup_{\epsilon>0} H_{x,\epsilon}$. Specifically, if $\epsilon = m \in \mathbb{N}$, then $\mathcal{H}_x = \bigcup_{m} H_{x,\frac{1}{m}}$, which is countable. ☐

3. The Riesz representation theorem claims that $\Lambda : \mathcal{H} \to \mathbb{C}$ is a bounded linear functional if and only if there exists $\boldsymbol{x} \in \mathcal{H}$ such that $\Lambda(\boldsymbol{y}) = \langle \boldsymbol{x},\boldsymbol{y}\rangle$ for every $\boldsymbol{y} \in \mathcal{H}$. Show that this representation $\boldsymbol{x}$ is unique.

Proof.

Assume that the representation of $\boldsymbol{x}$ is not unique.

In other words, there exist $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{H}$ such that $\Lambda(\boldsymbol{y}) = \langle \boldsymbol{x}_1, \boldsymbol{y} \rangle = \langle \boldsymbol{x}_2, \boldsymbol{y} \rangle$ for every $\boldsymbol{y} \in \mathcal{H}$.

Thus, $\langle \boldsymbol{x}_1, \boldsymbol{y} \rangle - \langle \boldsymbol{x}_2, \boldsymbol{y} \rangle = 0$, and since $\langle \boldsymbol{x}_1, \boldsymbol{y} \rangle - \langle \boldsymbol{x}_2, \boldsymbol{y} \rangle = \langle \boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{y} \rangle$,

$$\langle \boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{y} \rangle = 0 \text{ for every } \boldsymbol{y} \in \mathcal{H}$$

Specifically, chose $\boldsymbol{y} = \boldsymbol{x}_1 - \boldsymbol{x}_2$, which implies $\langle \boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{x}_1 - \boldsymbol{x}_2 \rangle = 0$.

Since $\langle \boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{x}_1 - \boldsymbol{x}_2 \rangle = \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^2$,

$$\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^2 = 0$$

Thus, $\boldsymbol{x}_1 - \boldsymbol{x}_2 = \boldsymbol{0}$, which implies that $\boldsymbol{x}_1 = \boldsymbol{x}_2$.

Therefore, the representation of $\boldsymbol{x}$ is unique. $\square$

4. Let

$$\ell^2 := \{\{a_n\}_{n=0}^{\infty} \subset \mathbb{C} | \sum_{n=0}^{\infty} |a_n|^2 < \infty\} \tag{2}$$

denote the space of all square-summable sequences over the complex field $\mathbb{C}$. Define the inner product of any two sequences $\{a_n\}, \{b_n\} \in \ell^2$ via

$$\langle \{a_n\}, \{b_n\} \rangle := \sum_{n=0}^{\infty} \overline{a_n} b_n \tag{3}$$

Define the function $\Lambda : \ell^2 \to \mathbb{C}$ by

$$\Lambda(\{a_n\}) := a_0 + 10a_{10} + 20a_{20} + 30a_{30} \tag{4}$$

which obviously is bounded and linear. What is the Riesz representation of Λ in ℓ^2?

Proof.

Define the sequence $\{b_n\}$ with each term equal to $\begin{cases} 1 & \text{if } n = 0 \\ 10 & \text{if } n = 10 \\ 20 & \text{if } n = 20 \\ 30 & \text{if } n = 30 \\ 0 & \text{otherwise.} \end{cases}$

The sequence $\{b_n\} \in \ell^2$, since $\{b_n\}_{n=0}^{\infty} \subset \mathbb{C}$,

$$\text{and } \sum_{n=0}^{\infty} |b_n|^2 = |b_0|^2 + |b_{10}|^2 + |b_{20}|^2 + |b_{30}|^2 = 1^2 + 10^2 + 20^2 + 30^2 < \infty$$

$$\begin{aligned}
\text{The inner product } \langle \{b_n\}, \{a_n\} \rangle \ &= \sum_{n=0}^{\infty} \overline{b_n} a_n \\
&= 1 \cdot a_0 + 10 \cdot a_{10} + 20 \cdot a_{20} + 30 \cdot a_{30} \\
&= a_0 + 10a_{10} + 20a_{20} + 30a_{30} \\
&= \Lambda(\{a_n\}) \text{ for every } \{a_n\} \ \text{ in } \ell^2.
\end{aligned}$$

Therefore, there exists a sequence $\{b_n\} \in \ell^2$ such that $\Lambda(\{a_n\}) = \langle \{b_n\}, \{a_n\} \rangle$ for every $\{a_n\} \in \ell^2$, where

$\{b_n\}$ has each term equal to $\begin{cases} 1 & \text{if } n = 0 \\ 10 & \text{if } n = 10 \\ 20 & \text{if } n = 20 \\ 30 & \text{if } n = 30 \\ 0 & \text{otherwise.} \end{cases}$ $\square$

References

Lecture's Note