3-Coloring problem is NP-complete

Introduction

What is the coloring problem?

Given a graph $G(V, E)$, the graph coloring problem assigns a color-mapping for each vertex to its corresponding color $c : V(G) \to \{1, 2, ..., k\}$ for which $\forall (u, v) \in E(G) : c(u) \neq c(v)$ for any two vertices $u, v \in G$ and a number $k \in \mathbb{N}$. If this statement holds, we can call the graph **proper**.

The k-coloring problem asks if a given graph G can be properly colored using at most k distinct colors. If such a satisfying mapping exists, we may call the mapping proper k-coloring.

3-Coloring problem

For our following proof we explicitly decided to do the proof for the **3-coloring-problem**. That is because the 3-coloring problem is the smallest coloring problem which is contained in **NP**.
The 2-coloring problem is contained in **P**. We can easily solve this in $O(n)$ using something like Breadth-first search for example.

If we can prove that 3-Coloring is **NP-complete**, then 4-Coloring, 5-Coloring, ..., k-Coloring are NP-complete as well, since 3-Coloring is just a restriction of them.

Justification of problem choice

The graph coloring problem caught our attention due to us being already introduced to this problem in the lecture *Logic* before.

The fact that the nature of this problem is easily recognisable, and may be even understood by a child, but requires deeper knowledge and understanding when trying to get a hold of efficient solving and ultimately proving its membership in **NP-completeness**, was appealing to us.

We believe, that the choice of this problem will lead us to a deeper understanding of the concept of *Karp reduction* and the wide-ranging impact it has on theoretical computer science.

Theorem: 3-Coloring is in NP

Definition A language $A \in \{0, 1\}^*$ is in NP if there exist polynomials $p, q : \mathbb{N} \to \mathbb{N}$ and a TM M (verifier) with the following two properties:

- Completeness: if $x \in A$, then there exists a short certificate $y \in \{0, 1\}^*$ with $|y| = p(|x|)$ such that $M(x, y)$ = accept after (at most) $q(|x|)$ steps.
- Soundness: else if $x \notin A$, then $M(x, y)$ = reject for all possible certificates $y \in \{0, 1\}^*$

Proof.
To show that the problem is in NP, our verifier M takes the Graph $G(V, E)$ and our color mapping c as input and checks in $O(n^2)$ if c is a satisfying mapping.
M does this by accepting if the two connected vertices $u, v \in V$ of every edge $e \in E$ have 2 distinct colors.

□

Theorem: 3-Coloring is NP-hard