3-Coloring problem is NP-complete

What is the coloring problem?

A k-coloring of vertices of a given graph $G(V, E)$ is a mapping $c : V(G) \to \{1, 2, ..., k\}$ for which $\{u, v\} \in E(G) \to c(u) \neq c(v)$ for any two vertices $u, v \in G$ and a number $k \in \mathbb{N}$. If the implication holds, we can call the graph proper.

The k-coloring problem asks if a given grap G can be properly colored using at most k colors. If there is a satisfying mapping, we can call the mapping a proper k-coloring.

3-Coloring problem

But for our following proof we explicitly decided to do the proof for the 3-Coloring problem. That's because the 3-coloring problem is the smallest coloring problem which is contained in NP.
The 2-coloring problem is contained in P. We can easily solve this in $O(n)$ using something like Breadth First Search for example.
The 3-coloring problem is contained in NP-complete (proof follows) and if we can show that 3-Coloring is NP-complete, then 4-Coloring, 5-Coloring,..., k-Coloring are automatically NP-complete, since 3-Coloring is just a restriction of them.

Theorem: 3-Coloring is in NP

> **Definition** A language $A \in \{0, 1\}^*$ is in NP if there exist polynomials $p, q : \mathbb{N} \to \mathbb{N}$ and a TM M (verifier) with the following two properties:
>
> - Completeness: if $x \in A$, then there exists a short certificate $y \in \{0, 1\}^*$ with $|y| = p(|x|)$ such that $M(x, y)$ = accept after (at most) $q(|x|)$ steps.
> - Soundness: else if $x \notin A$, then $M(x, y)$ = reject for all possible certificates $y \in \{0, 1\}^*$

Proof.
To show that the problem is in NP, our verifier M takes the Graph $G(V, E)$ and our color mapping c as input and checks in $O(n^2)$ if c is a satisfying mapping.
M does this by accepting if the two connected vertices $u, v \in V$ of every edge $e \in E$ have 2 distinct colors.

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