FOURIER-SERIES-BASED VIRTUAL FIELDS METHOD FOR THE IDENTIFICATION OF 3-D STIFFNESS DISTRIBUTIONS

T. T. Nguyen[1], J. M. Huntley[1], I. A. Ashcroft[2], P. D. Ruiz[1] and F. Pierron[3]

[1] *Wolfson School of Mechanical and Manufacturing Engineering, Loughborough University, UK*
[2] *Faculty of Engineering, University of Nottingham, UK*
[3] *Faculty of Engineering and the Environment, University of Southampton, UK*

1. INTRODUCTION

The virtual fields method (VFM) is a powerful technique for determining stiffness distributions of a sample, based on measured full-field strain data. The advantage of the VFM over many other methods is its ability to solve inverse problems of this type without any iteration [1]. A key step in any application of the VFM is the selection of the virtual fields. Several techniques are based on the use of polynomials of spatial variables (either on the whole domain or in a piecewise form), and the material properties are considered as having single values (homogeneous) within the domain. The first attempt to parameterise the material properties as a function of spatial variables was proposed in [2] for reconstruction of the stiffness map of a plate with impact damage.

In this paper, we retain the basic concepts underlying the VFM but approach the parameterisation of the material properties in the spatial frequency, rather than spatial, domain by performing a 3-D Fourier series expansion of the stiffness distribution over the region of interest. Furthermore, the virtual fields are not selected as polynomials of spatial variables as in the previous VFM literature, but from a set of simple cosine or sine functions of different spatial frequencies. The abbreviation F-VFM will be used to denote the VFM in which both a Fourier series is used for the material property parameterization, and cosine/sine functions for the virtual fields. The F-VFM was developed originally for 2-D geometries [3]; here it is extended to volumetric datasets resulting, for example, from measurements with Digital Volume Correlation or Phase Contrast Magnetic Resonance Imaging.

2. THE FOURIER VIRTUAL FIELDS METHOD

The fundamental equation for the VFM, written for a deformable body subject to quasi-static loading, can be simplified by neglecting the integrals involving the body force and acceleration terms as follows [1]:

$$-\int_V \boldsymbol{\epsilon}^* \mathbf{Q} \boldsymbol{\epsilon} \, dV + \int_{S_f} \mathbf{T} \mathbf{u}^* dS_f = 0 \tag{1}$$

where $\boldsymbol{\epsilon}$ is the measured strain field within volume V, $\mathbf{T}$ is the traction distribution on the surface S_f of V, $\boldsymbol{\epsilon}^*$ and $\mathbf{u}^*$ are the virtual strain and displacement fields, respectively, and $\mathbf{Q}$ is the stiffness matrix. For elastic isotropic materials, $\mathbf{Q}$ is only dependent on two elastic parameters, Q_{xx} and νQ_{xx}, where ν is Poisson's ratio. Although we allow for spatial variation in Q_{xx} we assume that ν is a known constant. The basic idea of the F-VFM is to expand Q_{xx} as a 3-D Fourier series in the spatial variables (x,y,z). The unknown coefficients of this expansion are then determined by successive application of Eqn. (1), each time with a different virtual field. In the F-VFM, these virtual fields are chosen to be cosine and sine waves of different spatial frequencies. Provided the number of virtual fields is at least equal to the number of unknown Fourier coefficients, the coefficients are determined uniquely by a simple inversion of a matrix representation of these simultaneous equations, without any iteration.

2.1. Fast algorithm

The choice of cosine and sine waves for the virtual strain fields has an important benefit in that a fast algorithm based on the fast Fourier transform can be implemented [3]. Computational effort can in practice be reduced by some 4-5 orders of magnitude compared to a non-fast implementation.

2.2. Unknown boundary conditions

In many cases the traction distribution $\mathbf{T}$ over the surface of the region of interest will be unknown. In such cases, the virtual fields can be modified through the use of a window function that tapers the $\mathbf{u}^*$ field to zero on S_f. The surface integral in Eqn. (1) will thus be equal to zero regardless of the precise surface traction distribution. Only non-dimensional stiffness distributions, normalised with respect to the dc term in the stiffness expansion, can be obtained in such situations.

2.3 Example application

The F-VFM has been applied to a number of simulated datasets created by finite element analysis of a sample with specified modulus distribution, and its performance in the presence of noise has also been tested, as described in [3]. An example of its application to an experimental dataset is shown in Figure 1. The data came from an experiment first presented in [4] in which a tissue-mimicking phantom of rectangular cuboid shape of size 80×64×154 mm^3, incorporating a spherical inclusion of 25 mm diameter and stiffness approximately 4× that of the background, was loaded cyclically in compression. All three displacement components were measured using phase contrast Magnetic Resonance Imaging. For incompressible materials such as this, application of the VFM leads to reconstruction of the shear modulus distribution [3,5]. Figure 1(a) shows the computed distribution for the case where 7219 unknown coefficients were determined. Total computation time using the fast algorithm was approximately 3 minutes on an Intel Core™ i7 CPU 2.79 GHz computer with 8GB of memory. The average shear modulus computed from the region known to contain the inclusion was 4.26, which is close to the expected value of 4.

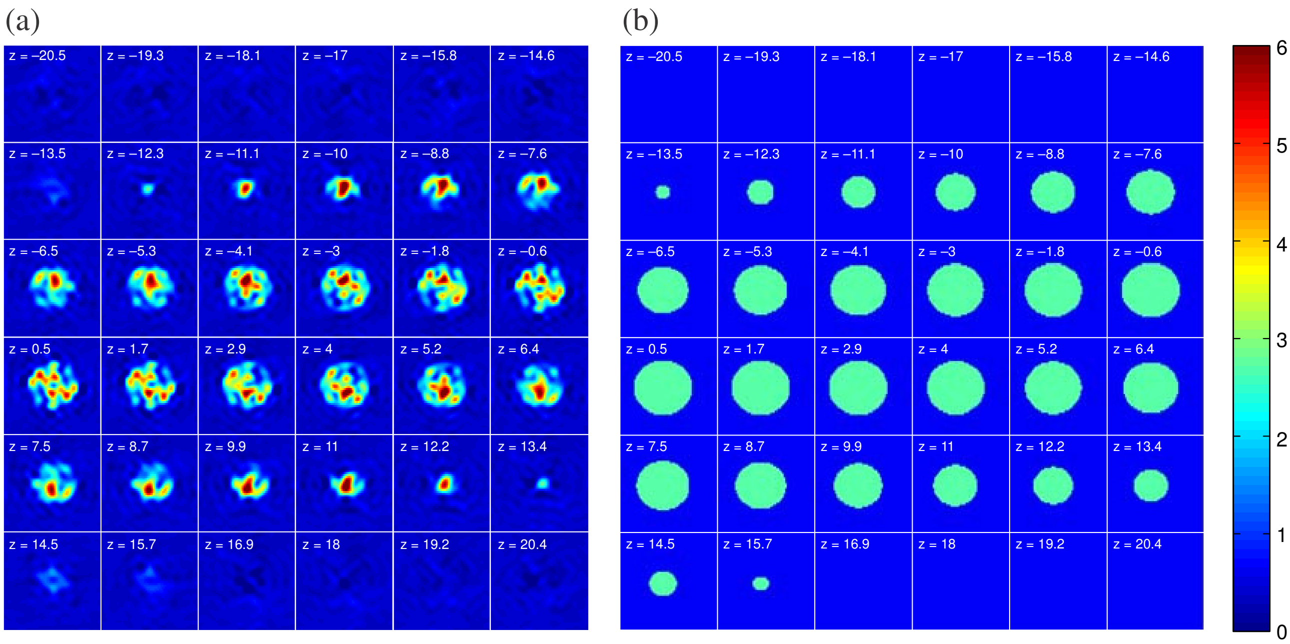

Figure 1. (a) Slices through the reconstructed 3-D shear modulus distribution of a phantom from its MRI data by the F-VFM compared with (b) the reference distribution determined from the magnitude of the MRI signals

3. CONCLUSIONS

The paper presents a development of the virtual fields method to reconstruct 3-D stiffness distributions from measured full-field data. Like the previously-published VFM algorithms, stiffness distributions are reconstructed after a single computation step without any iteration. By using a 3-D Fourier series expansion of the unknown stiffness distribution, and cosine/sine waves for the virtual fields, an efficient numerical algorithm based on the fast Fourier Transform allows volume identification problems with ~10^4 degrees of freedom to be solved in just a few minutes. Adaptation to the case of unknown boundary conditions is straightforward, although the technique then only provides stiffness distributions normalized with respect to the average stiffness value.

References

[1] F. Pierron, M. Grédiac, *The Virtual Fields Method*, Springer New-York, 2012. ISBN 978-1-4614-1823-8.
[2] J.H. Kim, F. Pierron, M.R. Wisnom, K. Syed-Muhamad, Identification of the local stiffness reduction of a damaged composite plate using the virtual fields method, Composites Part A-Applied Science and Manufacturing 38, pp. 2065-2075, 2007.
[3] T.T. Nguyen, *The Fourier virtual fields method for the identification of material property distributions*, PhD thesis, Loughborough University, 2013. https://dspace.lboro.ac.uk/2134/12658
[4] D.D. Steele, T.L. Chenevert, A.R. Skovoroda, S.Y. Emelianov, Three-dimensional static displacement, stimulated echo NMR elasticity imaging, Physics in Medicine and Biology 45, pp. 1633-1648, 2000.
[5] S. Avril, J.M. Huntley, F. Pierron, and D. Steele, 3D heterogeneous stiffness reconstruction using MRI and the virtual fields method, Experimental Mechanics 48, pp. 479-494, 2008.