50.007 Machine Learning

Hidden Markov Model

(Adapted from Prof. Lu Wei's slides)

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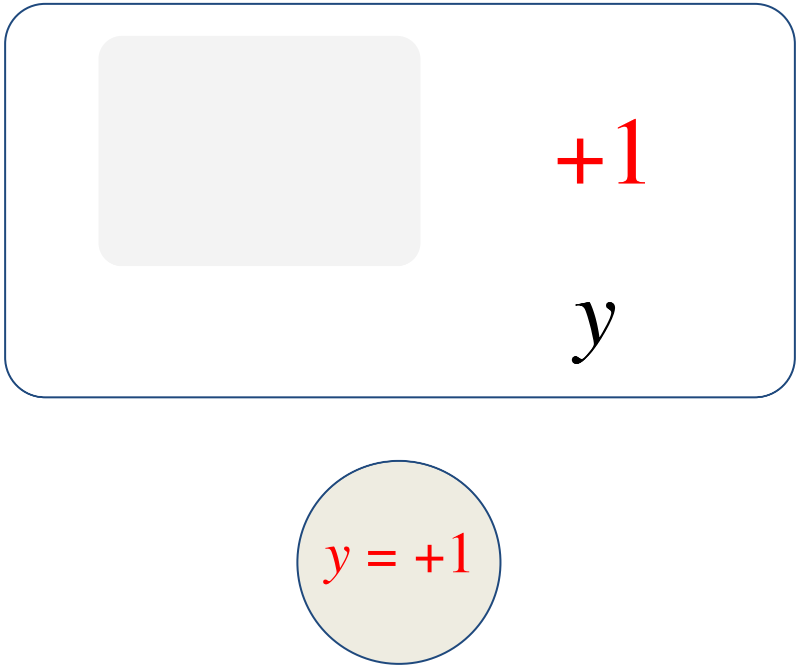
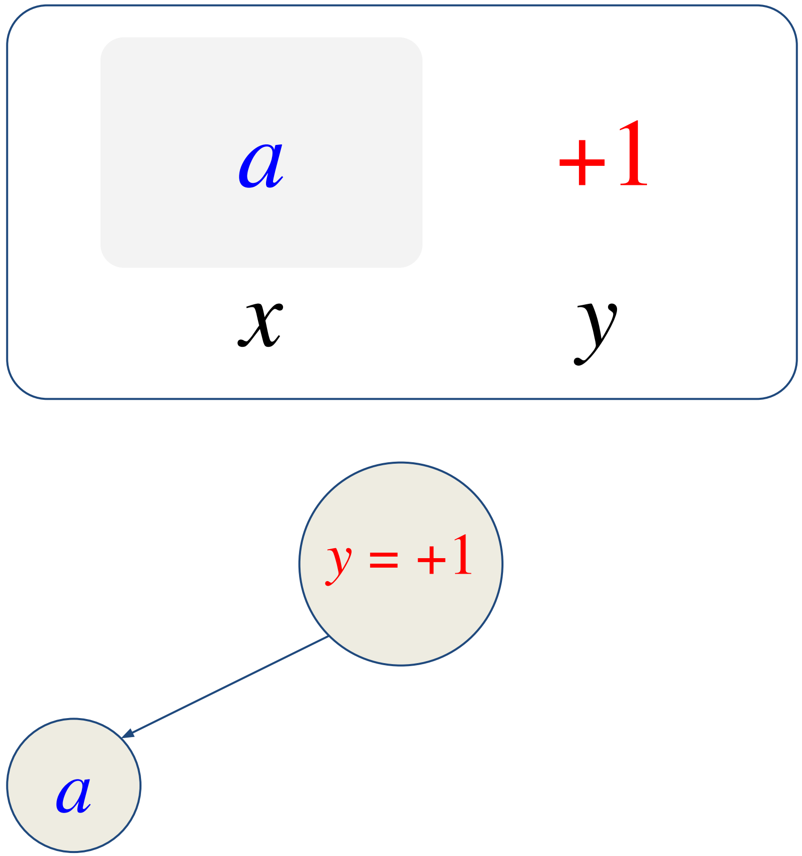
Generative Models

x	y
$a\ b\ c$	$+1$

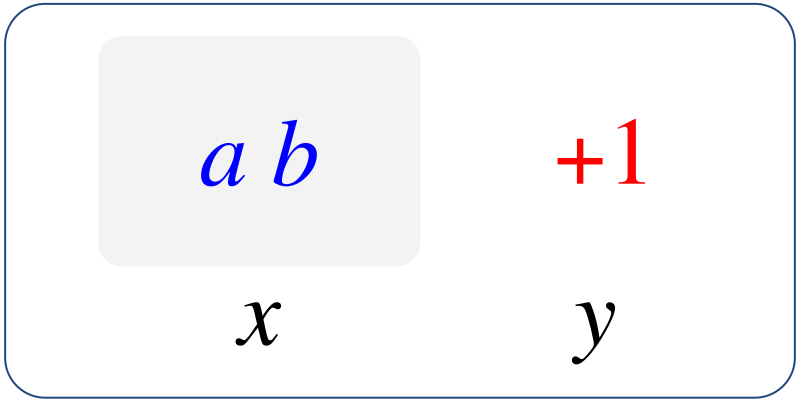
Generative Models

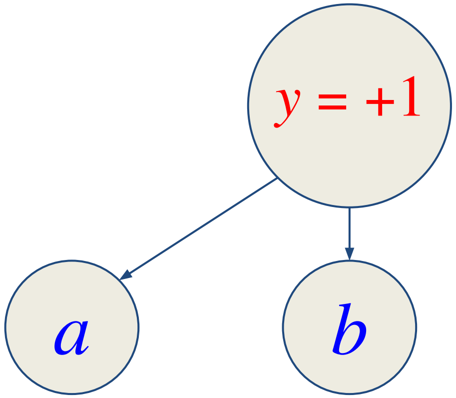


Generative Models

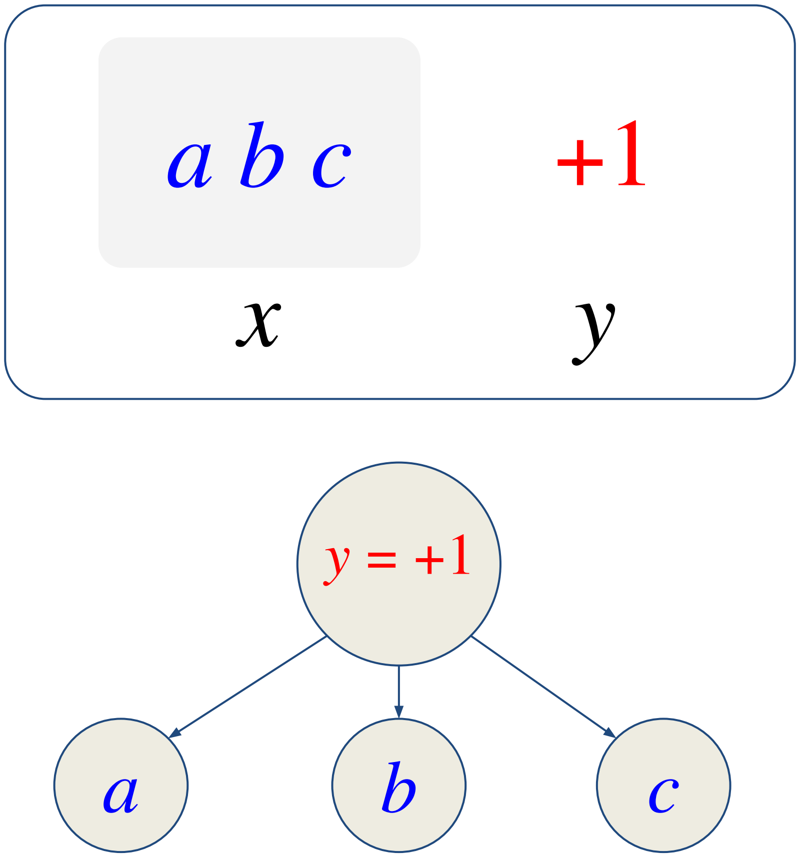


Generative Models

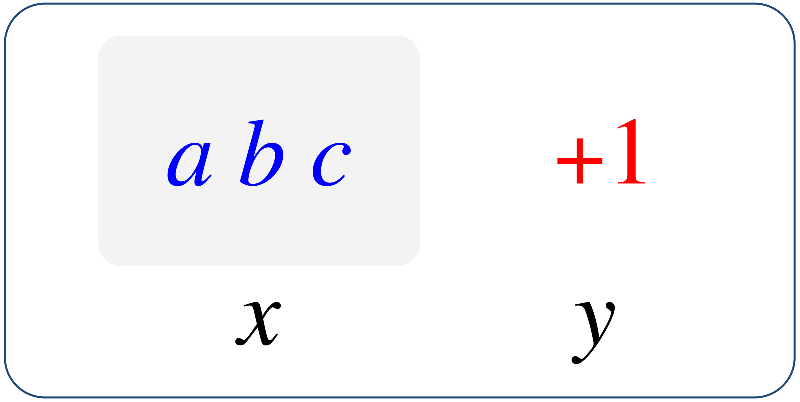


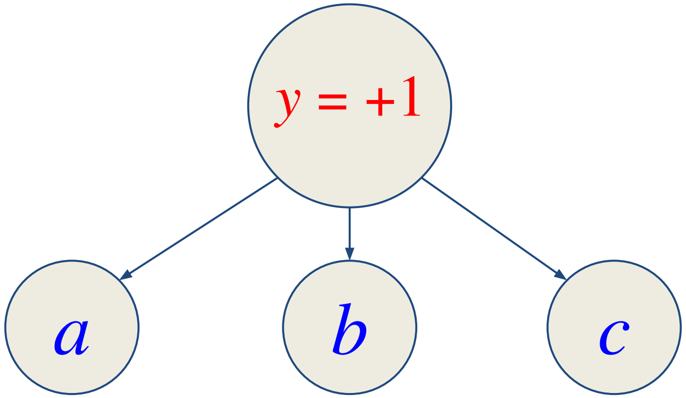


Generative Models



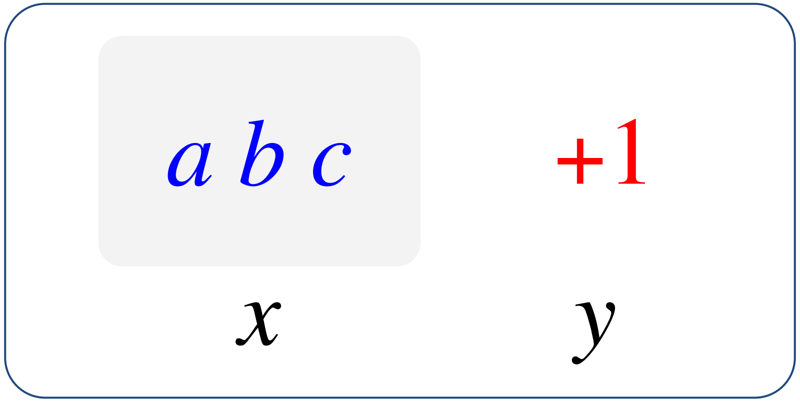
Generative Models

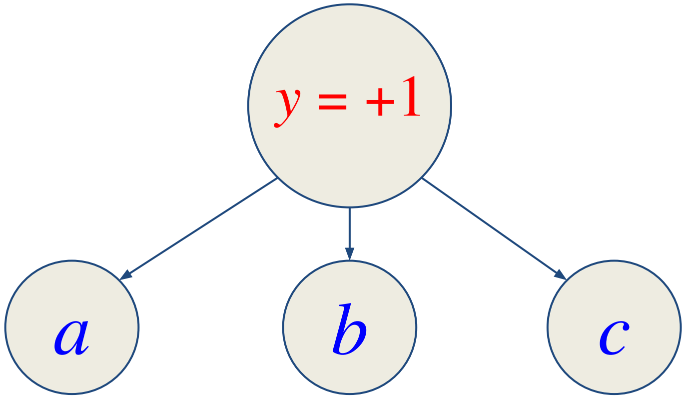




$p(x\text{=“}a, b, c\text{”}, y = +1)$

Generative Models





$$p(x=\text{“}a, b, c\text{”}, y = +1) = p(y = +1)\, p(a \mid y = +1)\, p(b \mid y = +1)\, p(c \mid y = +1)$$

Sequence Labeling

Faith is a fine invention

Sequence Labeling

Noun	Verb	Determiner	Adjective	Noun
N	**V**	**D**	**A**	**N**
Faith	is	a	fine	invention

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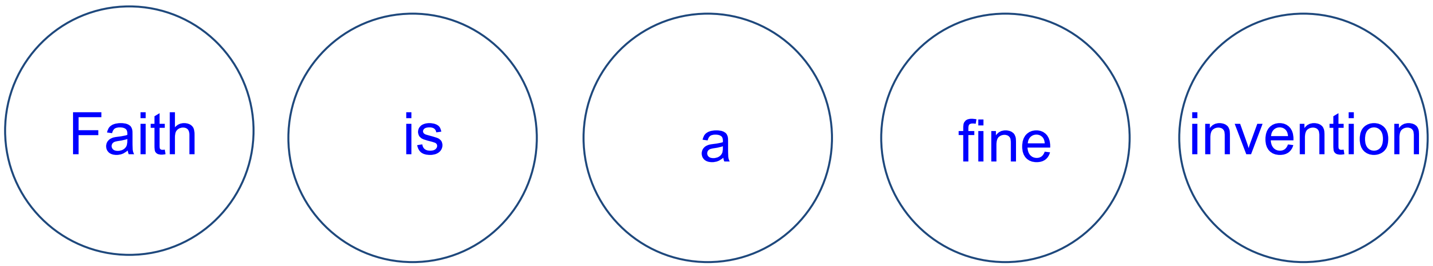
Sequence Labeling

Sequence Labeling

Sequence Labeling



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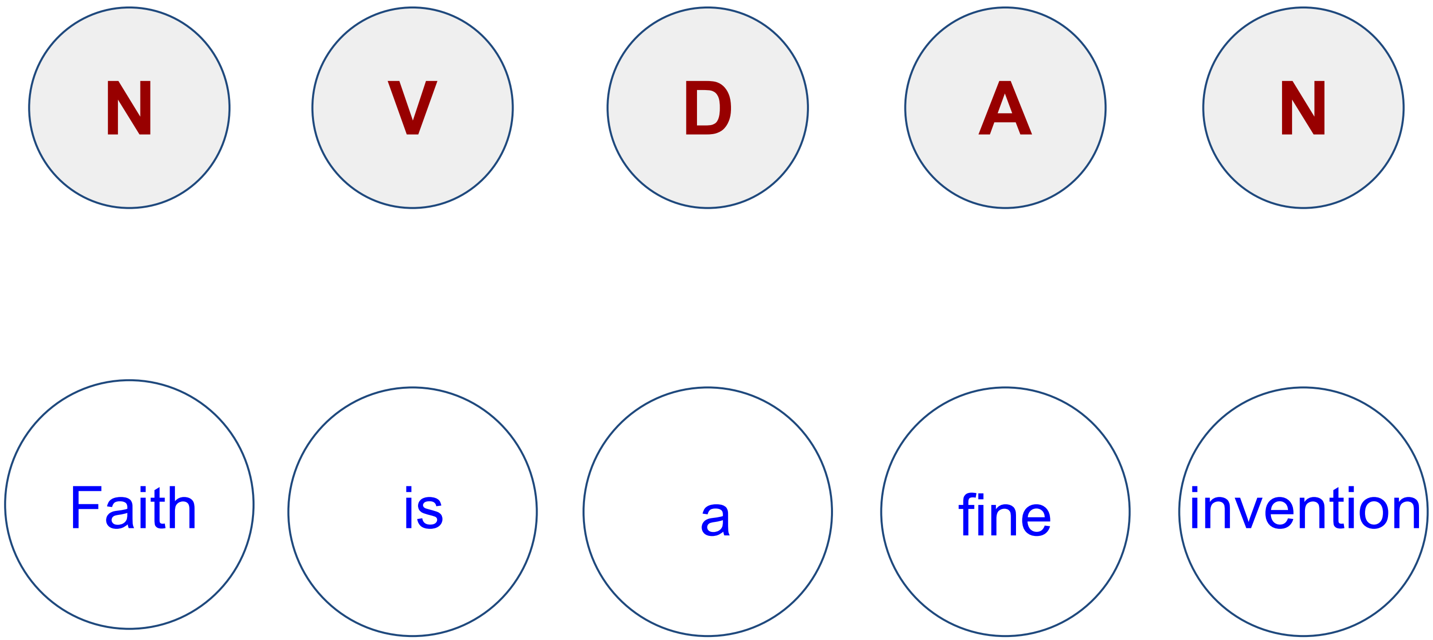

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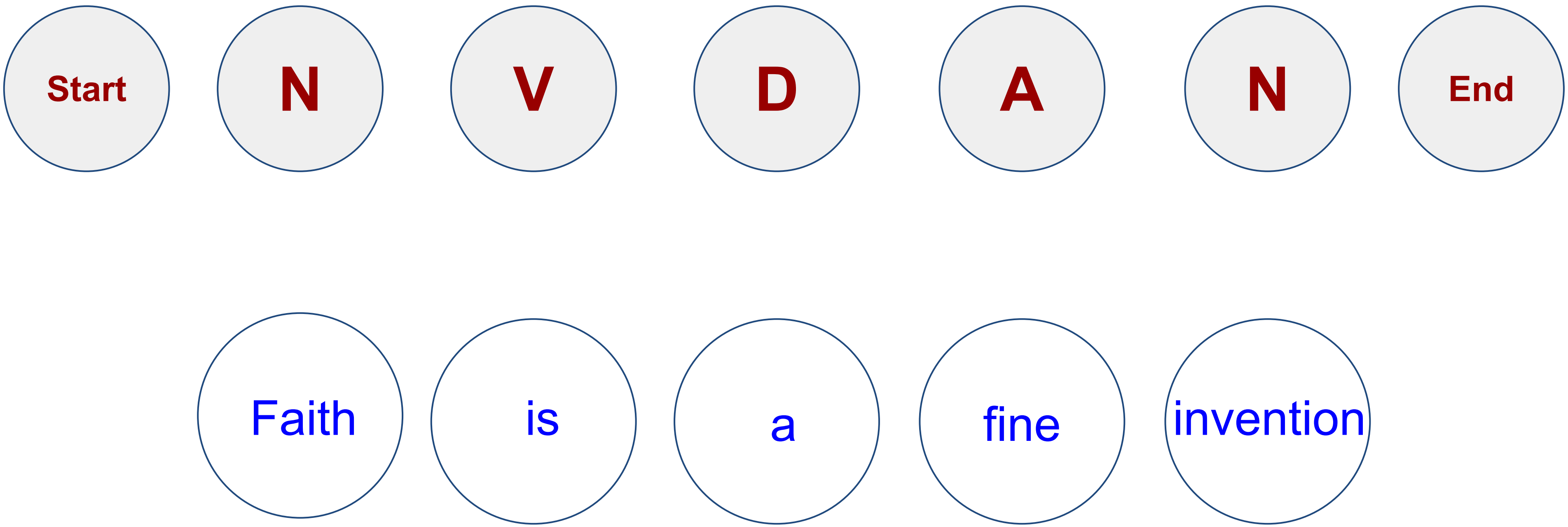

Sequence Labeling

Start

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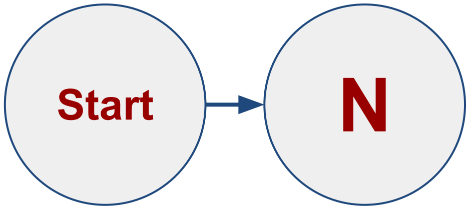
Sequence Labeling



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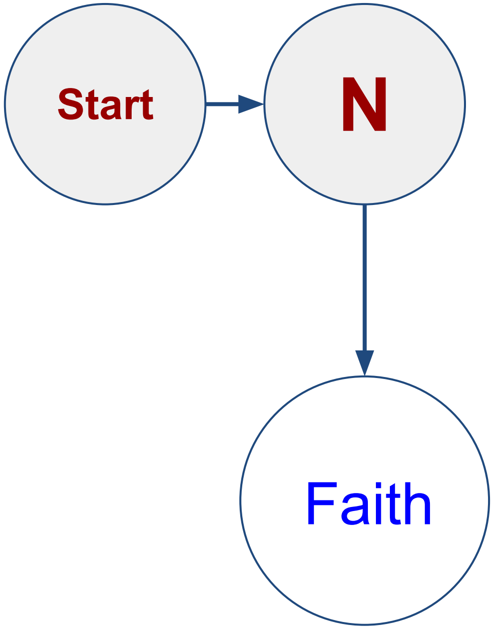

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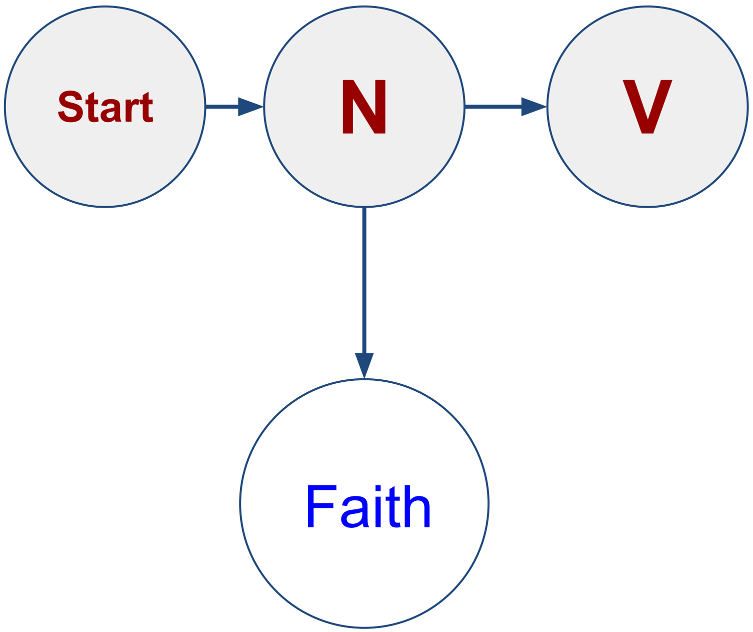

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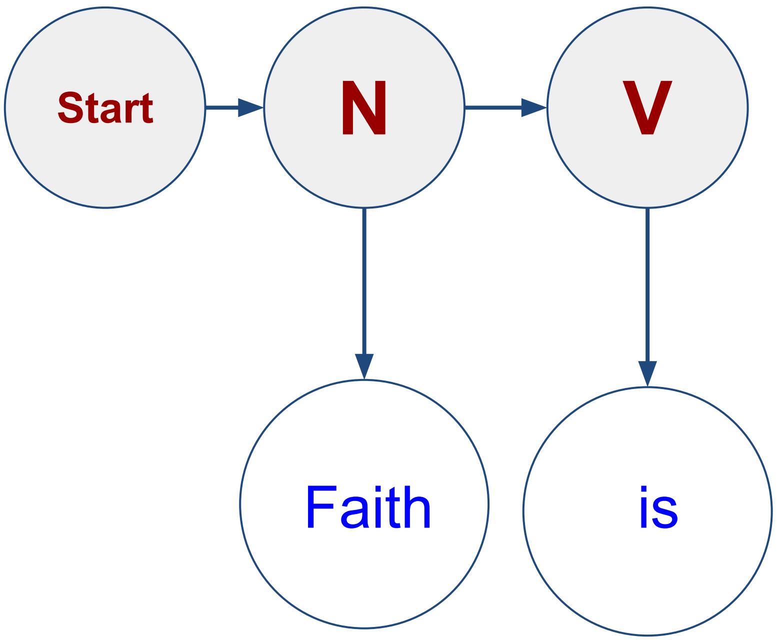

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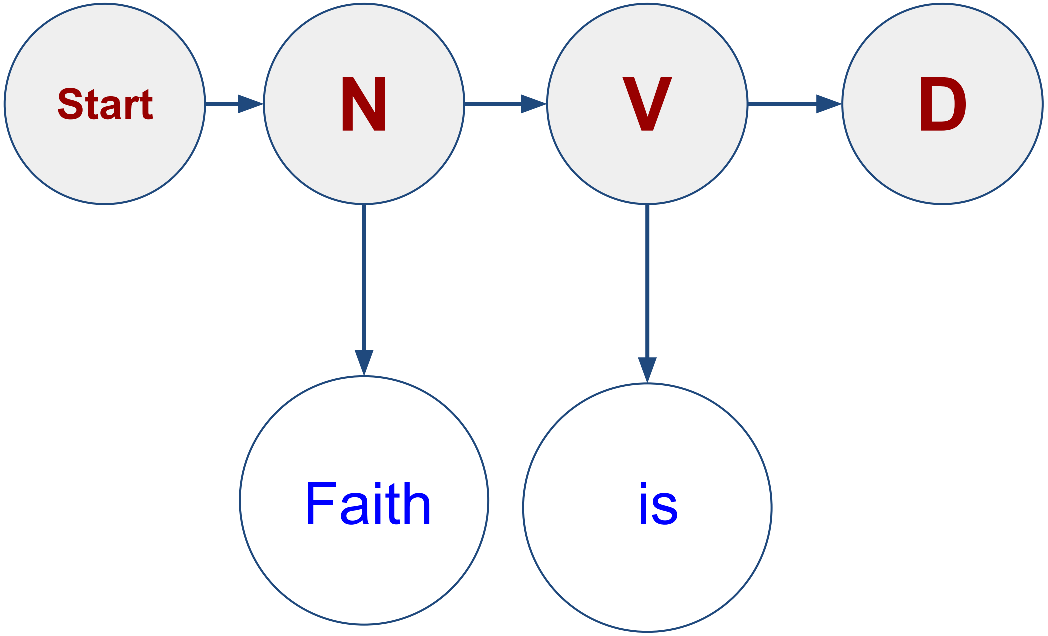

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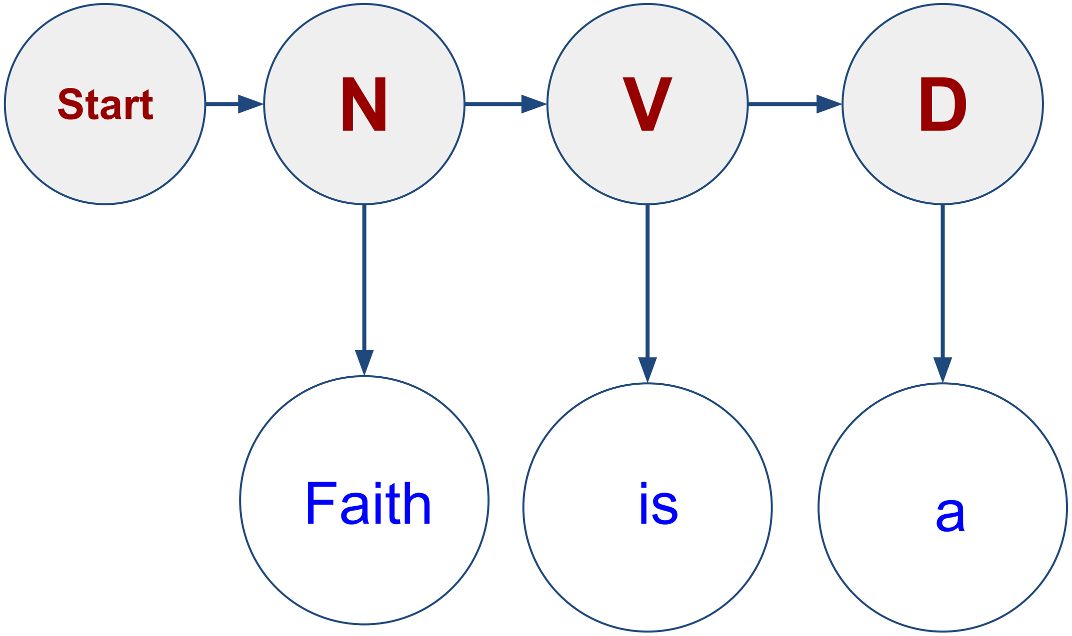

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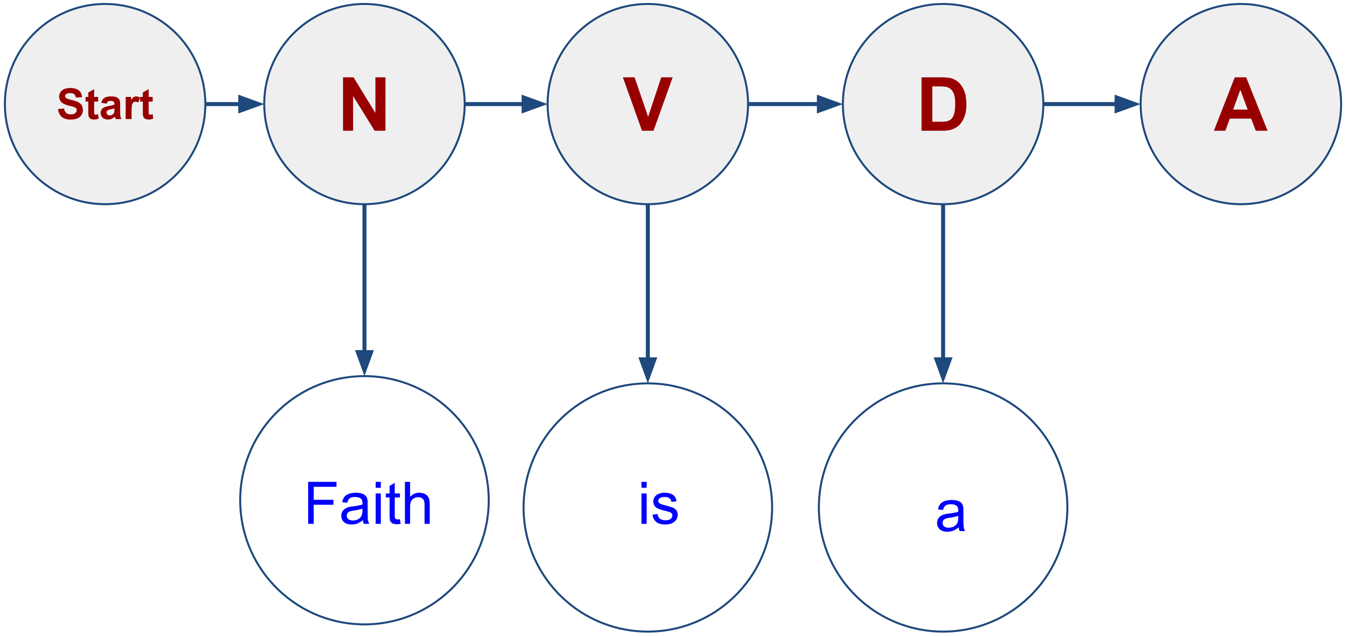

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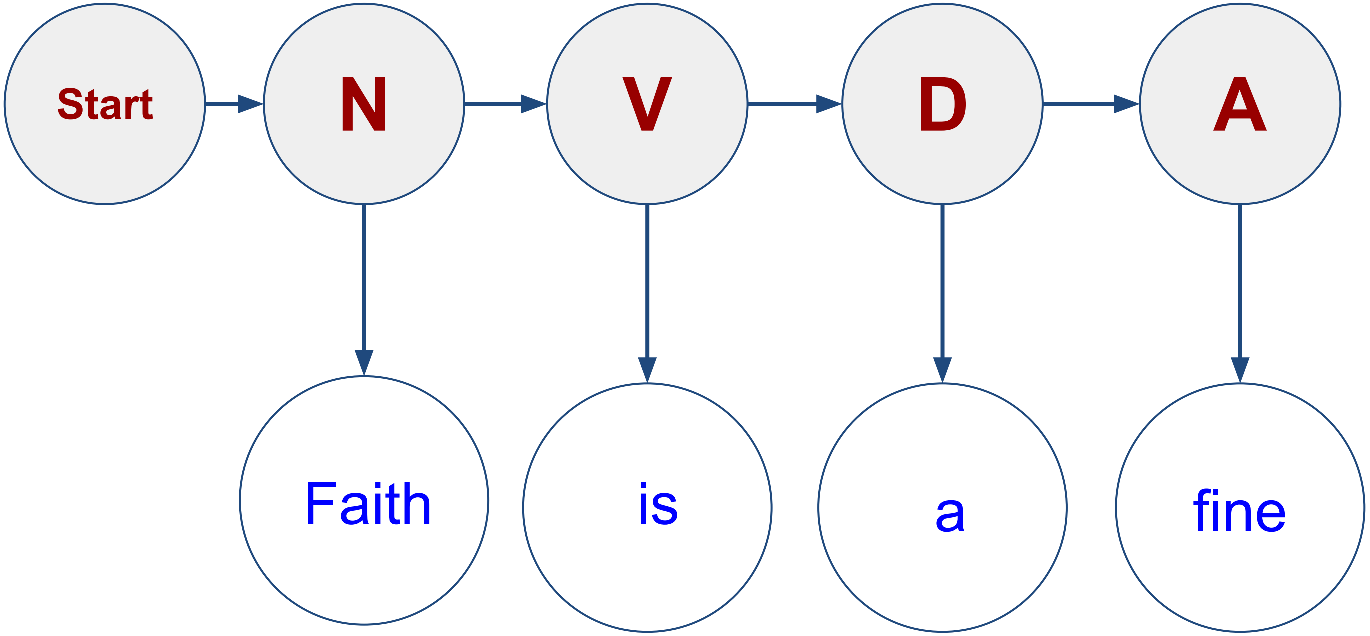

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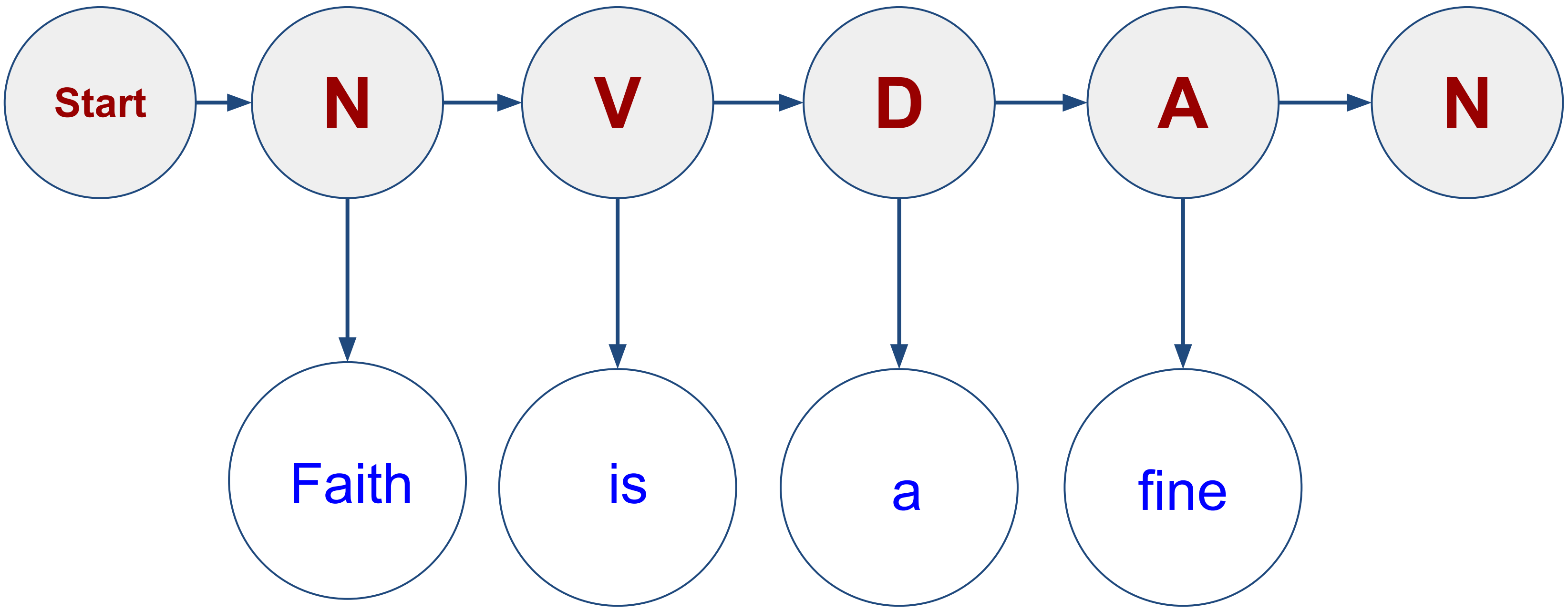
Sequence Labeling



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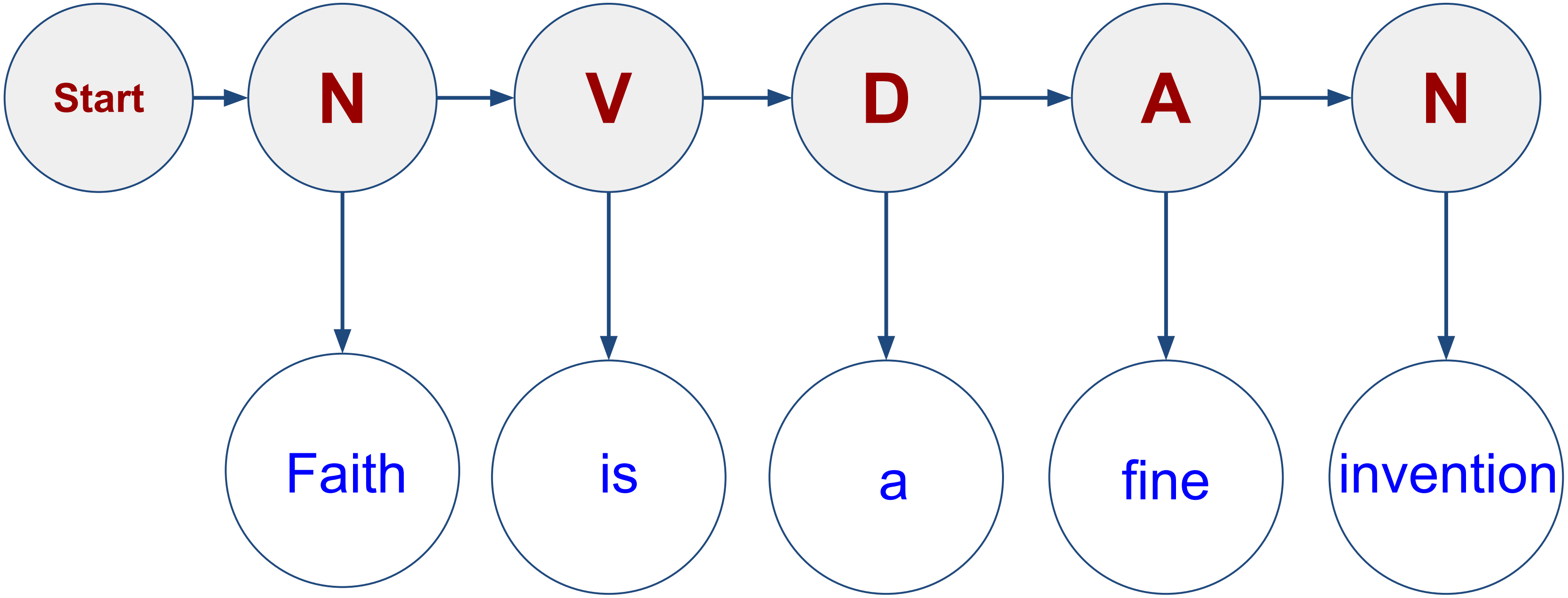

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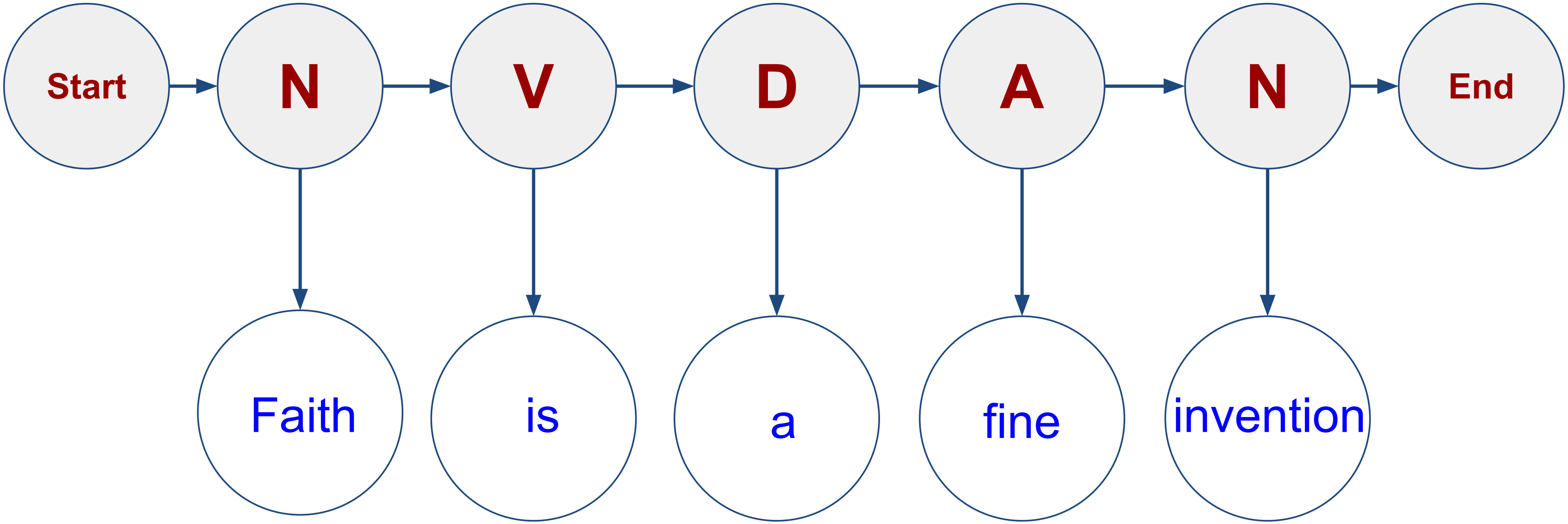

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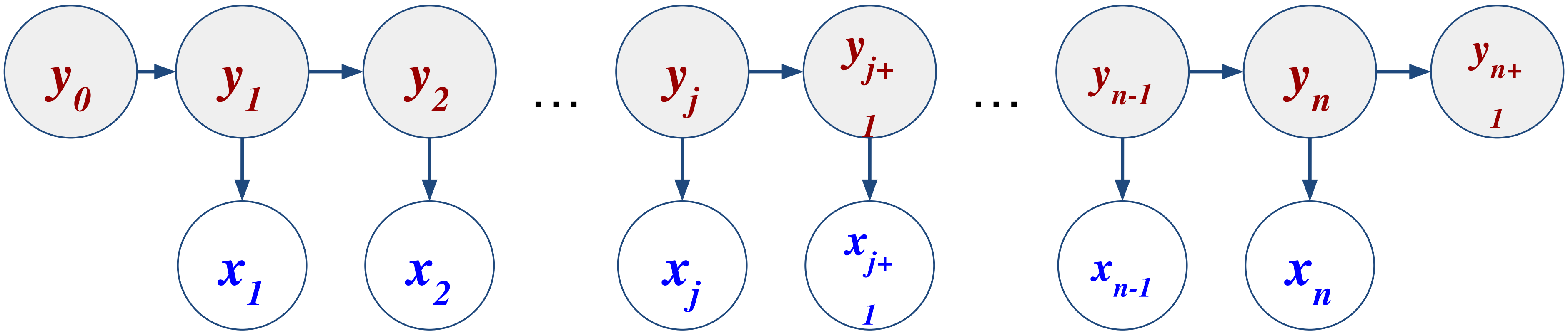
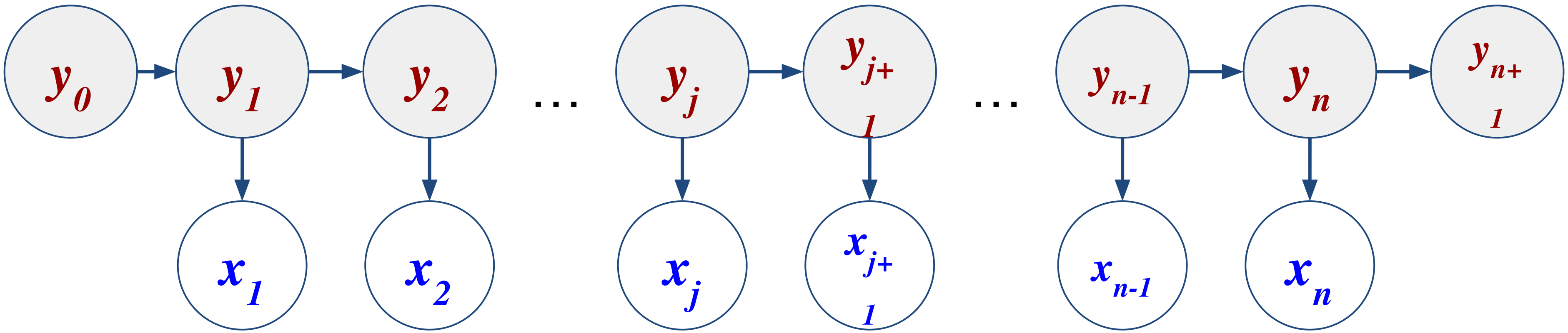


$$p(x_1, x_2, \ldots, x_{n-1}, x_n, y_0, y_1, y_2, \ldots, y_{n-1}, y_n, y_{n+1})$$

START

END

Sequence Labeling



$$p(x_1, x_2, \ldots, x_{n-1}, x_n, y_0, y_1, y_2, \ldots, y_{n-1}, y_n, y_{n+1})$$

$$\prod_{j=0}^{n} p(y_{j+1} | y_j) \times \prod_{j=1}^{n} p(x_j | y_j)$$

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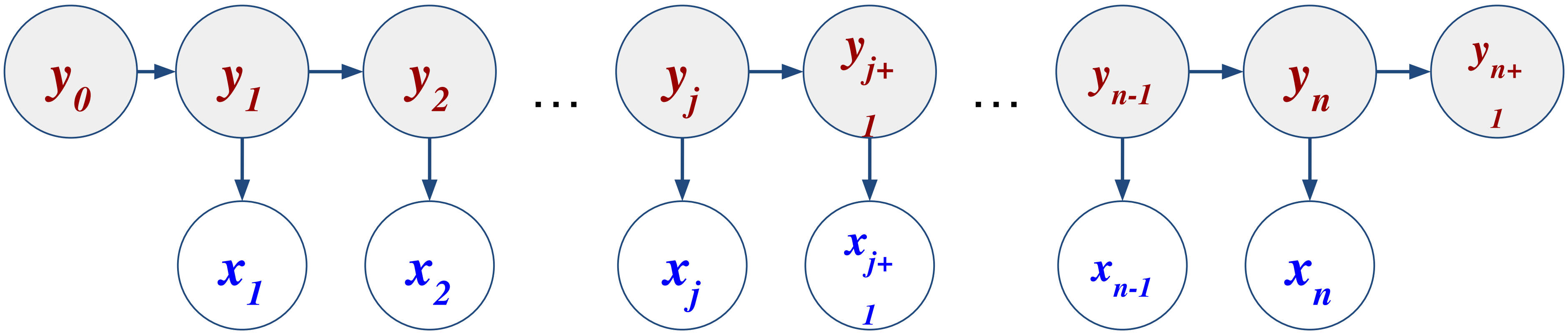
Sequence Labeling



$$p(x_1, x_2, \ldots, x_{n-1}, x_n, y_0, y_1, y_2, \ldots, y_{n-1}, y_n, y_{n+1})$$

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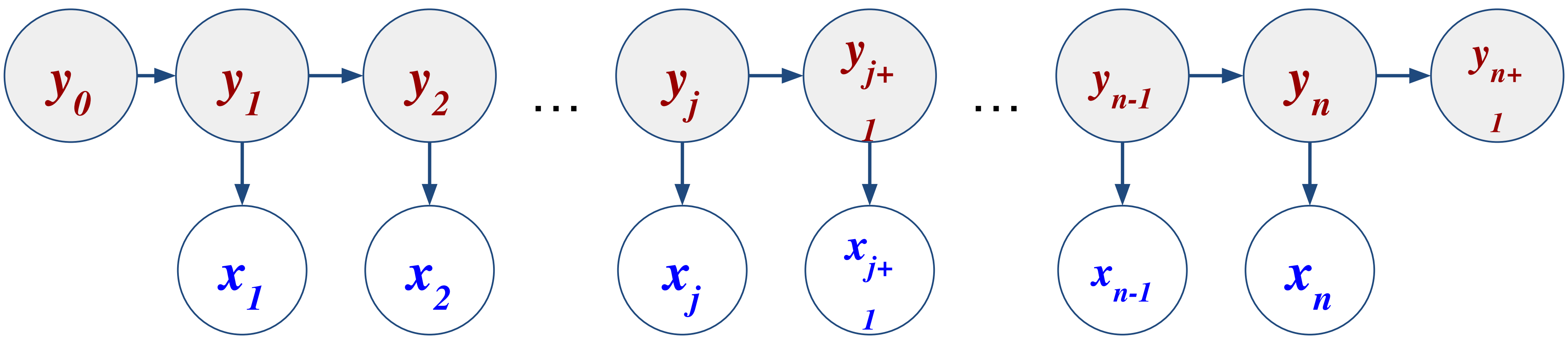

$$p(x_1, x_2, \ldots, x_{n-1}, x_n, y_0, y_1, y_2, \ldots, y_{n-1}, y_n, y_{n+1})$$

$$\underbrace{\prod_{j=0}^{n} a_{y_j, y_{j+1}}}_{\textbf{Transition Probabilities}} \times \underbrace{\prod_{j=1}^{n} b_{y_j}(x_j)}_{\textbf{Emission Probabilities}}$$

Hidden Markov Model

- An HMM is defined by a tuple $\langle \mathcal{T}, \mathcal{O}, \theta \rangle$, where
  - $\mathcal{T}$: a set of states including START and END states
  - $\mathcal{O}$: a set of observation symbols
  - θ : Transition and emission parameters $a_{u,v}$, and $b_u(o)$

Hidden Markov Model (Example)

$$\mathcal{T} = \{\texttt{START}, A, B, \texttt{STOP}\}$$

$$\mathcal{O} = \{\text{"the"}, \text{"dog"}\}$$

$u \backslash v$	A	B	$\texttt{STOP}$
$\texttt{START}$	1.0	0.0	0.0
A	0.5	0.5	0.0
B	0.0	0.8	0.2

$$a_{u,v}$$

$u \backslash o$	"the"	"dog"
A	0.9	0.1
B	0.1	0.9

$$b_u(o)$$

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Hidden Markov Model (Example)

$$a_{u,v}$$

$u \backslash v$	A	B	STOP
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$$(\mathbf{x}, \mathbf{y}) = \text{the}/A, \text{dog}/B, \text{the}/A$$

What is $p(\mathbf{x}, \mathbf{y})$?

Hidden Markov Model (Example)

$a_{u,v}$

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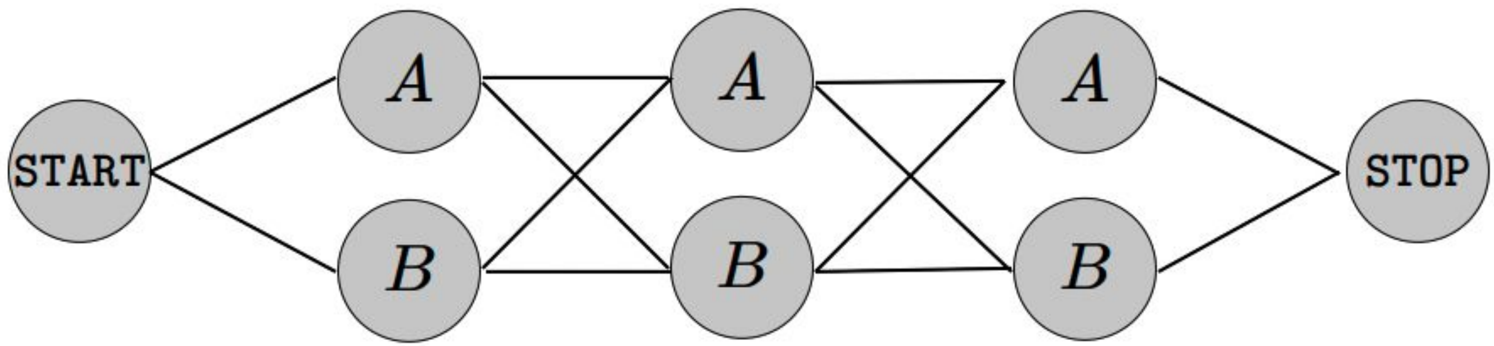

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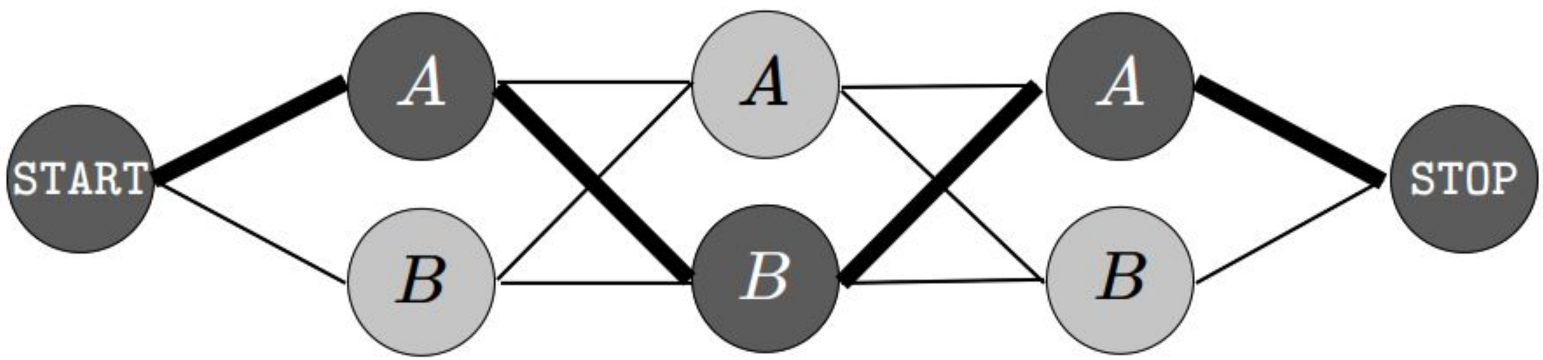

$a_{\texttt{START},A}$

Hidden Markov Model (Example)

$a_{u,v}$

$u \backslash v$	A	B	STOP
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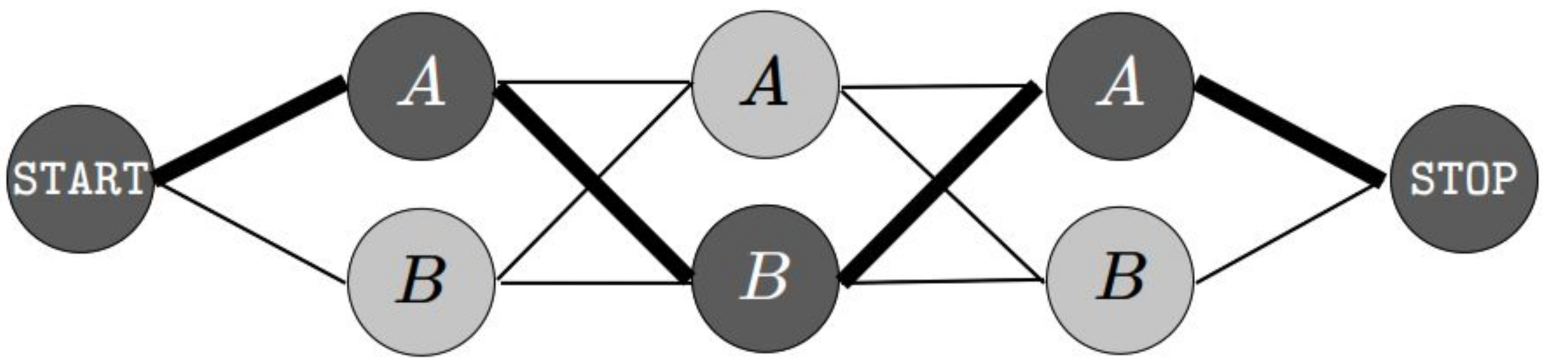

$a_{\texttt{START},A} \times b_A(\text{"the"})$

Hidden Markov Model (Example)

$a_{u,v}$

$u \backslash v$	A	B	STOP
START	1.0	0.0	0.0
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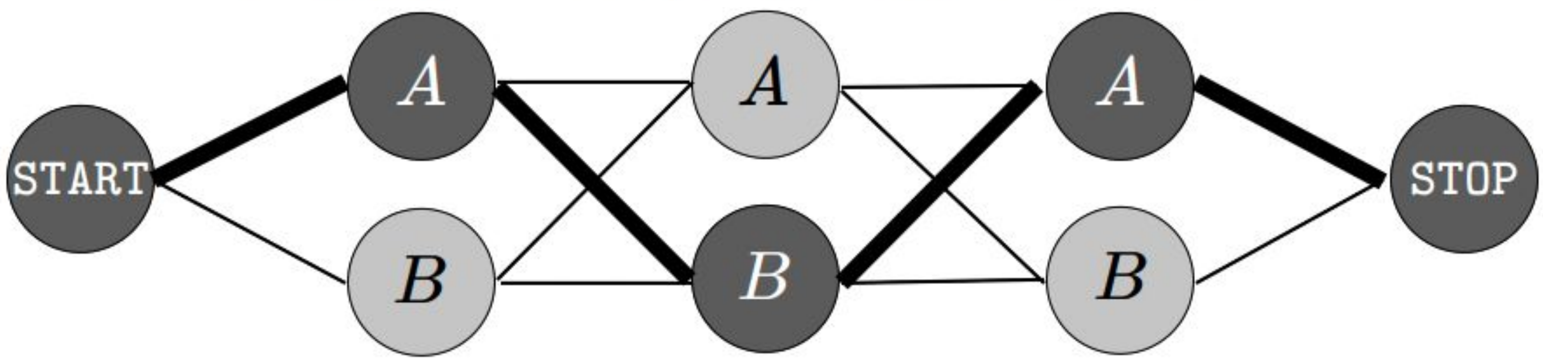

$a_{\texttt{START},A} \times b_A(\text{"the"}) \times a_{A,B}$

Hidden Markov Model (Example)

$$a_{u,v}$$

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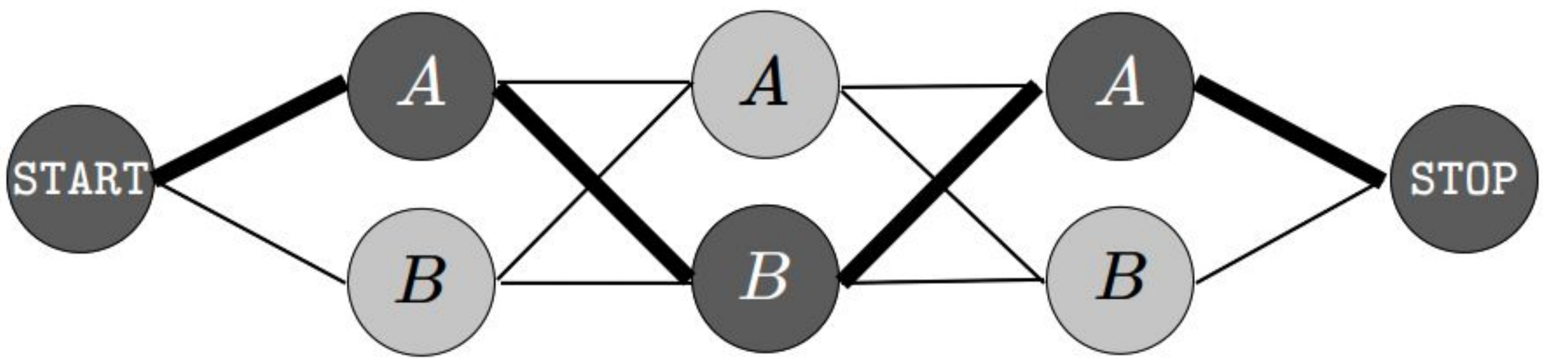

$$a_{\texttt{START},A} \times b_A(\text{"the"}) \times a_{A,B} \times b_B(\text{"dog"})$$

Hidden Markov Model (Example)

$a_{u,v}$

$u\backslash v$	A	B	STOP
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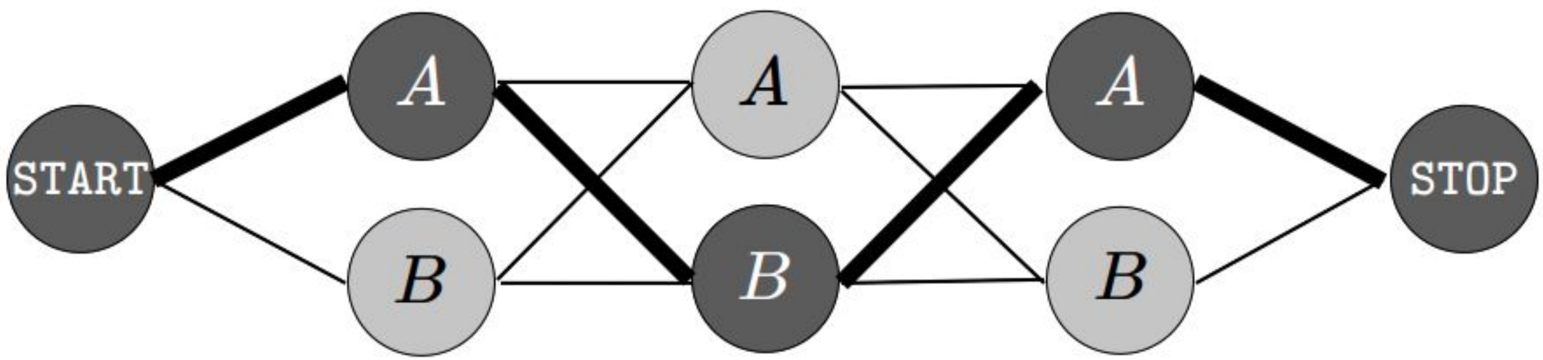

$$a_{\texttt{START},A} \times b_A(\text{"the"}) \times a_{A,B} \times b_B(\text{"dog"}) \times a_{B,A}$$

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Hidden Markov Model (Example)

$a_{u,v}$

$u \backslash v$	A	B	STOP
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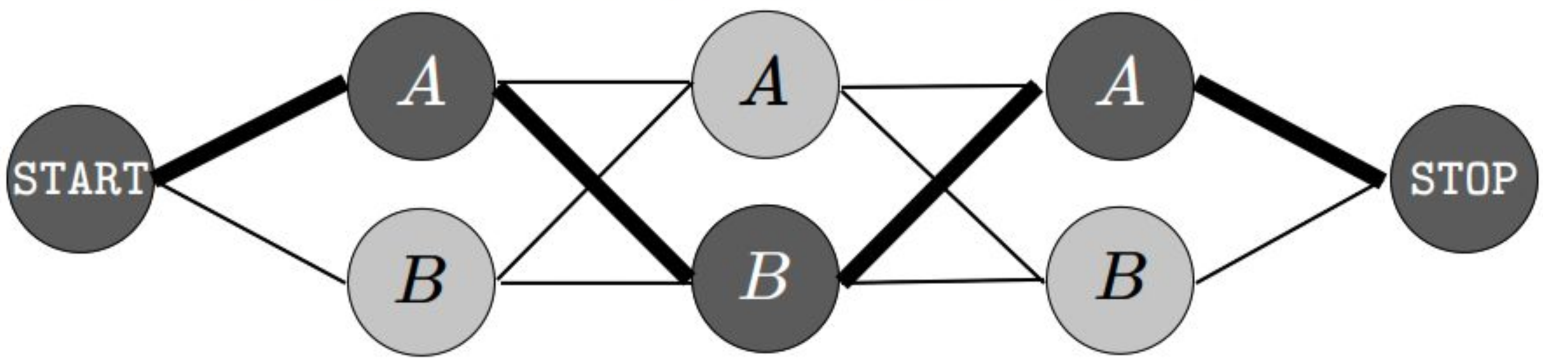

$$a_{\texttt{START},A} \times b_A(\text{"the"}) \times a_{A,B} \times b_B(\text{"dog"}) \times a_{B,A} \times b_A(\text{"the"})$$

Hidden Markov Model (Example)

$$a_{u,v}$$

$u \backslash v$	A	B	STOP
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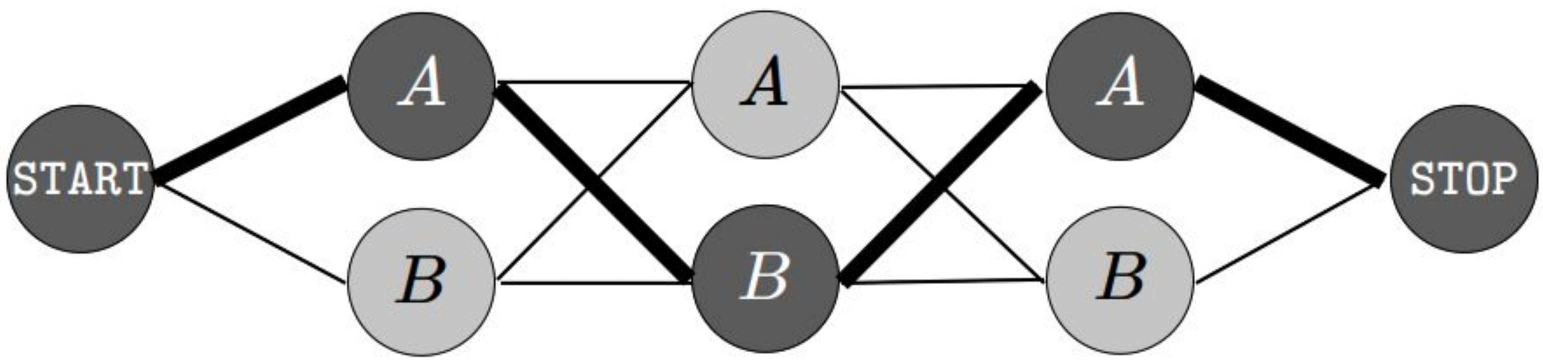

$$a_{\texttt{START},A} \times b_A(\text{``the''}) \times a_{A,B} \times b_B(\text{``dog''}) \times a_{B,A} \times b_A(\text{``the''}) \times a_{A,\texttt{STOP}}$$

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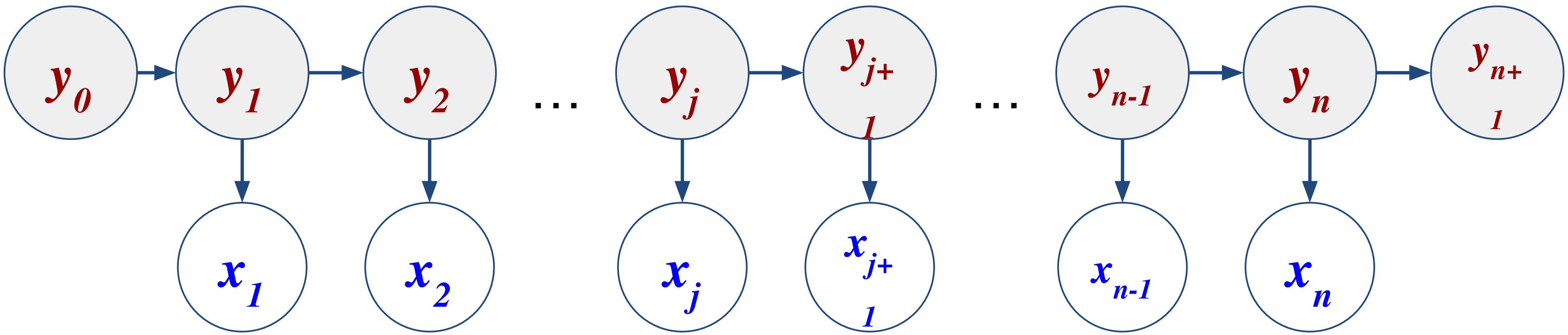
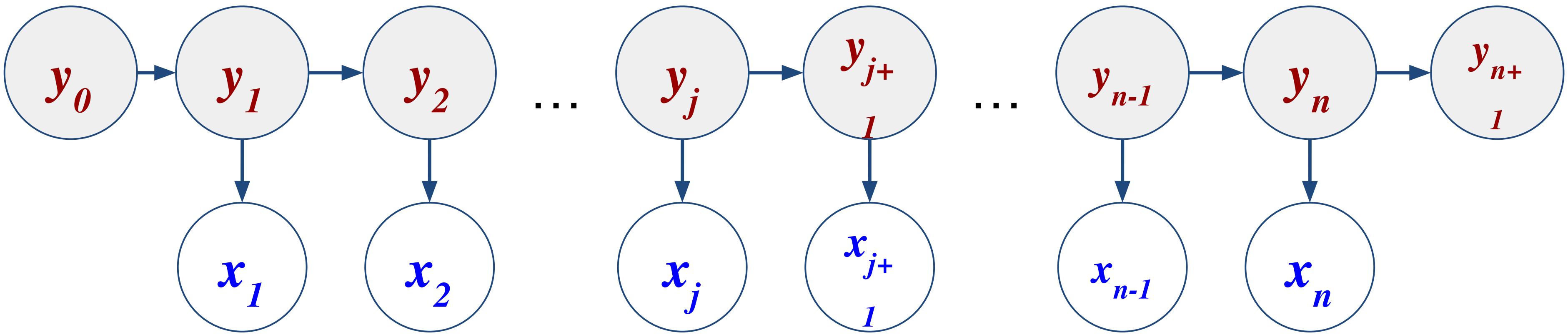
Hidden Markov Model



$$p(\mathbf{x},\mathbf{y}) = \underbrace{\prod_{j=0}^{n} a_{y_j,y_{j+1}}}_{\text{Transition probabilities}} \times \underbrace{\prod_{j=1}^{n} b_{y_j}(x_j)}_{\text{Emission probabilities}}$$

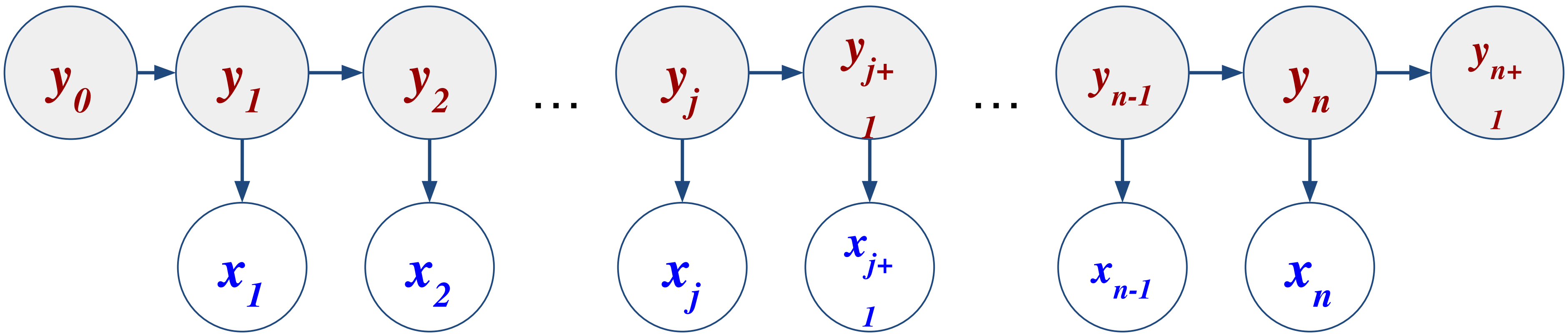
Now that we know what are the model parameters, how do we estimate them? In other words, how to do learning?

Hidden Markov Model



$$a_{u,v} = \frac{\text{count}(u,v)}{\text{count}(u)} \qquad b_u(o) = \frac{\text{count}(u \to o)}{\text{count}(u)}$$

Hidden Markov Model



Number of times we see a transition from u to v

$$a_{u,v} = \frac{\boxed{\text{count}(u,v)}}{\boxed{\text{count}(u)}}$$

Number of times we see the state u in the training set

Number of times we see observation o generated from u

$$b_u(o) = \frac{\boxed{\text{count}(u \rightarrow o)}}{\boxed{\text{count}(u)}}$$

Number of times we see the state u in the training set

Hidden Markov Model

$$\begin{array}{ccc} y^{(1)} & = & (A, B) \\ x^{(1)} & & (e, g) \end{array} \qquad \begin{array}{ccc} y^{(2)} & = & (A, B) \\ x^{(2)} & & (e, h) \end{array} \qquad \begin{array}{ccc} y^{(3)} & = & (A, B) \\ x^{(3)} & & (f, h) \end{array}$$

$$a_{\mathtt{START},A} \times b_A(e) \times a_{A,B} \times b_B(g) \times a_{B,\mathtt{STOP}}$$

$$\times a_{\mathtt{START},A} \times b_A(e) \times a_{A,B} \times b_B(h) \times a_{B,\mathtt{STOP}}$$

$$\times a_{\mathtt{START},A} \times b_A(f) \times a_{A,B} \times b_B(h) \times a_{B,\mathtt{STOP}}$$

(1) Write down the likelihood

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Hidden Markov Model

$$\begin{array}{rcl} y^{(1)} & = & (A, B) \\ x^{(1)} & & (e, g) \end{array} \qquad \begin{array}{rcl} y^{(2)} & = & (A, B) \\ x^{(2)} & & (e, h) \end{array} \qquad \begin{array}{rcl} y^{(3)} & = & (A, B) \\ x^{(3)} & & (f, h) \end{array}$$

$$\boxed{a_{\mathtt{START},A}} \times \boxed{b_A(e)} \times \boxed{a_{A,B}} \times \boxed{b_B(g)} \times \boxed{a_{B,\mathtt{STOP}}}$$

$$\times \boxed{a_{\mathtt{START},A}} \times \boxed{b_A(e)} \times \boxed{a_{A,B}} \times \boxed{b_B(h)} \times \boxed{a_{B,\mathtt{STOP}}}$$

$$\times \boxed{a_{\mathtt{START},A}} \times \boxed{b_A(f)} \times \boxed{a_{A,B}} \times \boxed{b_B(h)} \times \boxed{a_{B,\mathtt{STOP}}}$$

Hidden Markov Model

$$\begin{array}{ccc} y^{(1)} & = & (A, B) \\ x^{(1)} & & (e, g) \end{array} \qquad \begin{array}{ccc} y^{(2)} & = & (A, B) \\ x^{(2)} & & (e, h) \end{array} \qquad \begin{array}{ccc} y^{(3)} & = & (A, B) \\ x^{(3)} & & (f, h) \end{array}$$

$$\left(\boxed{a_{\texttt{START},A}}\right)^3 \times \left(\boxed{a_{A,B}}\right)^3 \times \left(\boxed{a_{B,\texttt{STOP}}}\right)^3$$
$$\times \left(\boxed{b_A(f)}\right)^1 \times \left(\boxed{b_A(e)}\right)^2$$
$$\times \left(\boxed{b_B(h)}\right)^2 \times \left(\boxed{b_B(g)}\right)^1$$

Hidden Markov Model

$$\begin{array}{ccc} y^{(1)} & & (A, B) \\ x^{(1)} & = & (e, g) \end{array} \qquad \begin{array}{ccc} y^{(2)} & & (A, B) \\ x^{(2)} & = & (e, h) \end{array} \qquad \begin{array}{ccc} y^{(3)} & & (A, B) \\ x^{(3)} & = & (f, h) \end{array}$$

$$(a_{\texttt{START},A})^3 \times (a_{A,B})^3 \times (a_{B,\texttt{STOP}})^3$$

$$\times (b_A(f))^1 \times (b_A(e))^2$$

$$\times (b_B(h))^2 \times (b_B(g))^1$$

Hidden Markov Model

$$\begin{array}{rcl} y^{(1)} & \multirow{2}{*}{=} & (A, B) \\ x^{(1)} & & (e, g) \end{array} \qquad \begin{array}{rcl} y^{(2)} & \multirow{2}{*}{=} & (A, B) \\ x^{(2)} & & (e, h) \end{array} \qquad \begin{array}{rcl} y^{(3)} & \multirow{2}{*}{=} & (A, B) \\ x^{(3)} & & (f, h) \end{array}$$

$$\begin{aligned} & \left(a_{\mathtt{START},A}\right)^{\mathrm{Count}(\mathtt{START},A)} \\ \times & \left(a_{A,B}\right)^{\mathrm{Count}(A,B)} \\ \times & \left(a_{B,\mathtt{STOP}}\right)^{\mathrm{Count}(B,\mathtt{STOP})} \end{aligned}$$

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Hidden Markov Model

$$\begin{array}{ccc} y^{(1)} & & (A, B) \\ x^{(1)} & = & (e, g) \end{array} \qquad \begin{array}{ccc} y^{(2)} & & (A, B) \\ x^{(2)} & = & (e, h) \end{array} \qquad \begin{array}{ccc} y^{(3)} & & (A, B) \\ x^{(3)} & = & (f, h) \end{array}$$

$$\prod_{u,v} (a_{u,v})^{\mathrm{Count}(u,v)}$$

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Hidden Markov Model

$$\begin{matrix} y^{(1)} \\ x^{(1)} \end{matrix} = \begin{matrix} (A, B) \\ (e, g) \end{matrix} \qquad \begin{matrix} y^{(2)} \\ x^{(2)} \end{matrix} = \begin{matrix} (A, B) \\ (e, h) \end{matrix} \qquad \begin{matrix} y^{(3)} \\ x^{(3)} \end{matrix} = \begin{matrix} (A, B) \\ (f, h) \end{matrix}$$

$$\prod_{u,v} (a_{u,v})^{\mathrm{Count}(u,v)}$$

$$\times \prod_{u,o} (b_u(o))^{\mathrm{Count}(u \to o)}$$

(2) This is the likelihood. We can consider the log-likelihood.

Hidden Markov Model

$$\begin{array}{ccc} y^{(1)} \\ x^{(1)} \end{array} = \begin{array}{c} (A, B) \\ (e, g) \end{array} \qquad \begin{array}{c} y^{(2)} \\ x^{(2)} \end{array} = \begin{array}{c} (A, B) \\ (e, h) \end{array} \qquad \begin{array}{c} y^{(3)} \\ x^{(3)} \end{array} = \begin{array}{c} (A, B) \\ (f, h) \end{array}$$

$$\sum_{u,v} \mathrm{Count}(u, v) \log(a_{u,v})$$

$$+ \sum_{u,o} \mathrm{Count}(u \rightarrow o) \log(b_u(o))$$

(3) Next, take the partial derivative with respect to each individual parameter, and set it to zero. Solve the equation.

Hidden Markov Model

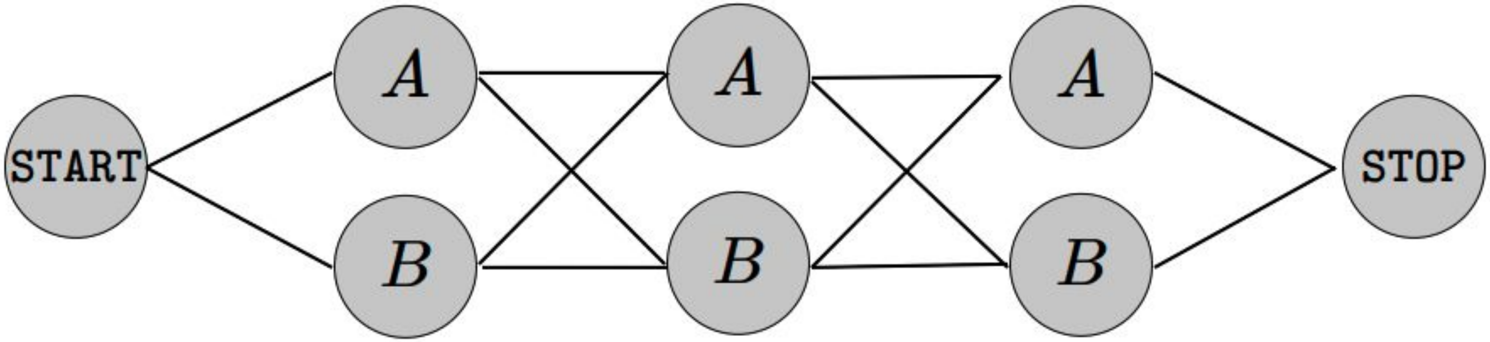
$a_{u,v}$

$u\backslash v$	A	B	STOP
START	1.0	0.0	0.0
A	0.5	0.5	0.0
B	0.0	0.8	0.2

$b_u(o)$

$u\backslash o$	"the"	"dog"
A	0.9	0.1
B	0.1	0.9

$$(\mathbf{x}, \mathbf{y}) = \text{the}/A, \text{dog}/B, \text{the}/A$$



Which label sequence y is the most probable given the word sequence x?