50.007 Machine Learning

Lecture 3

Hinge Loss

Recap

Training data

$$\mathcal{S}_n = \{ (x^{(i)}, y^{(i)}) \mid i = 1, \dots, n \}$$

- Features/Inputs $x^{(i)} = \left(x_1^{(i)}, \dots, x_d^{(i)}\right)^\top \in \mathbb{R}^d$
- Labels/Output $y^{(i)} \in \{-1, +1\}$

Model

Set of *linear* classifiers $h: \mathbb{R}^d \rightarrow \{-1, +1\}$

$$h(x; \theta, \theta_0) = \mathrm{sign}(\theta_d x_d + \cdots + \theta_1 x_1 + \theta_0)$$

$$= \mathrm{sign}(\theta^\top x + \theta_0) = \begin{cases} +1 & \text{if } \theta^\top x + \theta_0 \geq 0, \\ -1 & \text{if } \theta^\top x + \theta_0 < 0. \end{cases}$$

Model Parameters

$$\theta \in \mathbb{R}^d, \theta_0 \in \mathbb{R}$$

offset

$[\![\cdot]\!]$ is the *indicator* function that returns a 1 if its argument is true, and 0 otherwise.

Training Error

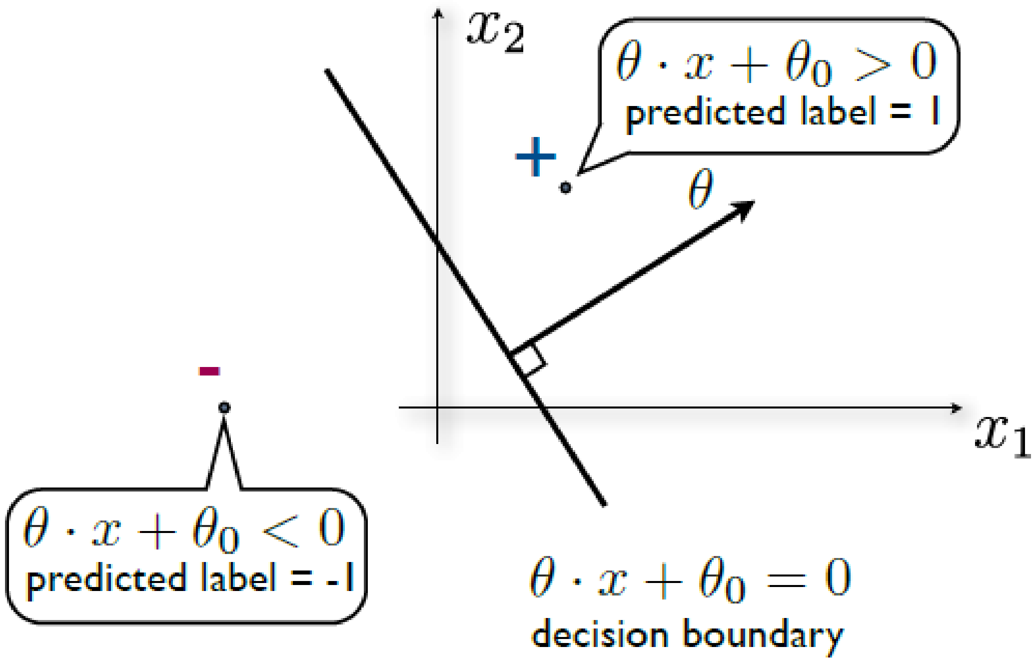
$$Loss(\theta, \theta_0; x, y) = [\![y \neq h(x; \theta, \theta_0)]\!]$$

$$\mathcal{R}(\theta, \theta_0; \mathcal{S}_*) = \frac{1}{n} \sum_{(x,y)\in\mathcal{S}_*} Loss(\theta, \theta_0; x, y)$$

Decision Boundaries

Vector θ is orthogonal to the decision boundary.

Vector θ points in direction of region labelled $+1$.



Linear Classifier

- **Linear classifier (with offset):**

$$h(x;\theta,\theta_0) = \mathrm{sign}(\theta^\top x + \theta_0) = \begin{cases} +1 & \text{if } \theta^\top x + \theta_0 \geq 0, \\ -1 & \text{if } \theta^\top x + \theta_0 < 0. \end{cases}$$

- **Training error:**

$$\mathcal{E}_n(\hat{\theta},\hat{\theta}_0) = \frac{1}{n}\sum_{t=1}^{n} [\![y^{(t)}(\hat{\theta}\cdot x^{(t)} + \hat{\theta}_0) \leq 0]\!]$$

$[\![\cdot]\!]$ is the *indicator* function that returns a 1 if its argument is true, and 0 otherwise.

- Linear classifier that achieves zero training error is called **realizable**.

Zero-One Loss

$$\mathcal{E}_n(\hat{\theta}, \hat{\theta}_0) = \frac{1}{n}\sum_{t=1}^{n} [\![y^{(t)}(\hat{\theta} \cdot x^{(t)} + \hat{\theta}_0) \leq 0]\!]$$

z

Let $[\![y^{(t)}(\hat{\theta} \cdot x^{(t)} + \hat{\theta}_0) \leq 0]\!] = 1$ (0 otherwise) if

- $y \neq h(x; \theta)$, or [misclassified]
- (x, y) is on decision boundary [boundary]

Note that $y(\theta^\top x) \leq 0$ if

- $\theta^\top x$ and y differ in sign, or [misclassified]
- $\theta^\top x$ is zero [boundary]

Perceptron Algorithm

- Initialize the **weight** ($\theta = 0$).
- For each training example 't' in S_n, classify the instance
  - if correct, continue
  - else, $\theta^{(k+1)} = \theta^{(k)} + y^{(t)} x^{(t)}$

    $\theta_0^{(k+1)} = \theta_0^{(k)} + y^{(t)}$
- Terminate if the **training error** is zero (realizable) or a pre-determined number of iterations are completed (non-realizable).

Perceptron update rule

Linearly separable case

Theorem 2.1 *The perceptron update rule converges after a finite number of mistakes when the training examples are linearly separable through origin.*

- The above theorem implies, **zero training error** can be achieved using **perceptron update rule** for **linearly separable** training examples.

$$\mathcal{E}_n(\hat{\theta}, \hat{\theta}_0) = \frac{1}{n} \sum_{t=1}^{n} [\![y^{(t)}(\hat{\theta} \cdot x^{(t)} + \hat{\theta}_0) \leq 0]\!] = 0$$

http://www.cs.columbia.edu/~mcollins/courses/6998-2012/notes/perc.converge.pdf

Perceptron Algorithm

1. Training Set (Linearly Separable)

$$\left(x^{(1)}, y^{(1)}\right), \left(x^{(2)}, y^{(2)}\right), \dots, \left(x^{(n)}, y^{(n)}\right)$$

2. Model (Set of Perceptrons)

$$h(x; \theta) = \operatorname{sign}(\theta_1 x_1 + \cdots + \theta_d x_d)$$

3. Training Loss (Fraction of Misclassified/Boundary Points)

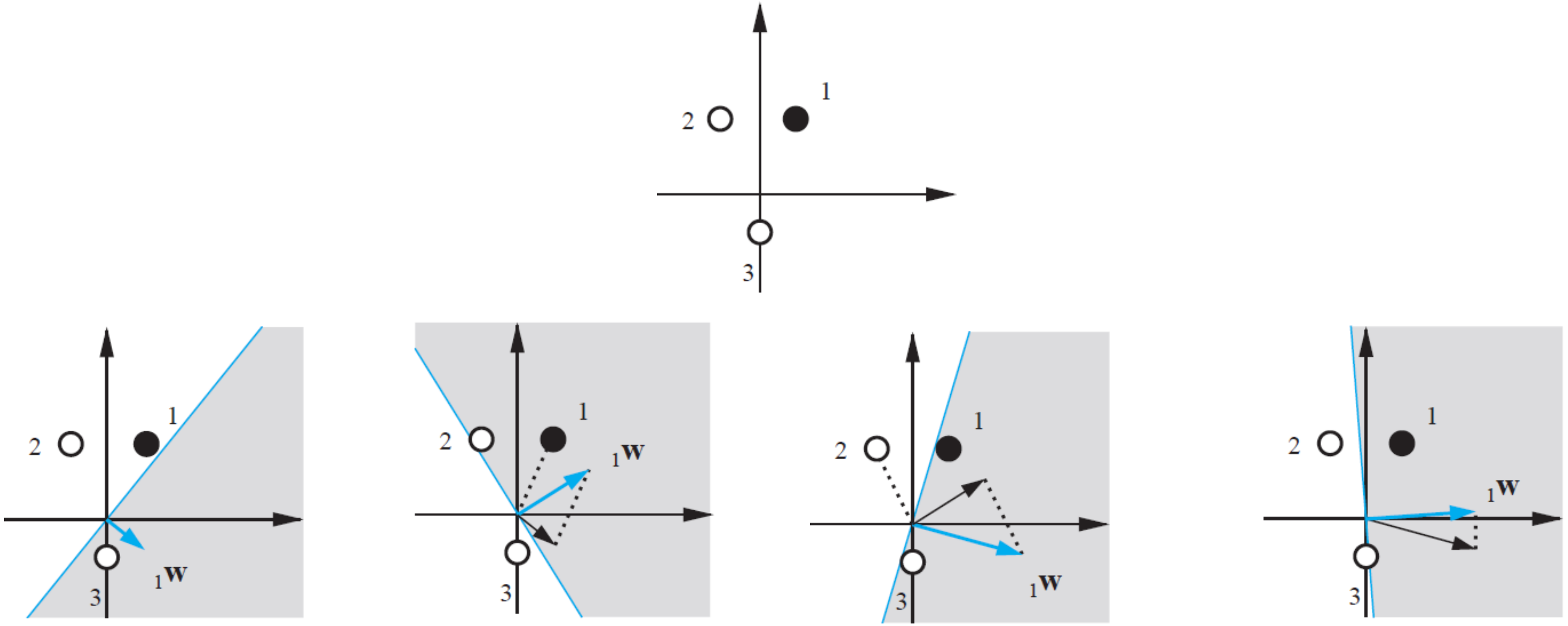
$$\boldsymbol{\varepsilon}_n(\theta) = \frac{1}{n} \sigma_{(x,y)\in\mathcal{S}_n} [\![\, y(\theta^\top x) \leq 0 \,]\!]$$

4. Algorithm (Mistake-Driven Algorithm)

$$\theta^{(k+1)} = \theta^{(k)} + y^{(t)} x^{(t)}$$

Example (Linearly Separable)

- Perceptron algorithm oscillates and terminates with zero error in linearly separable case.



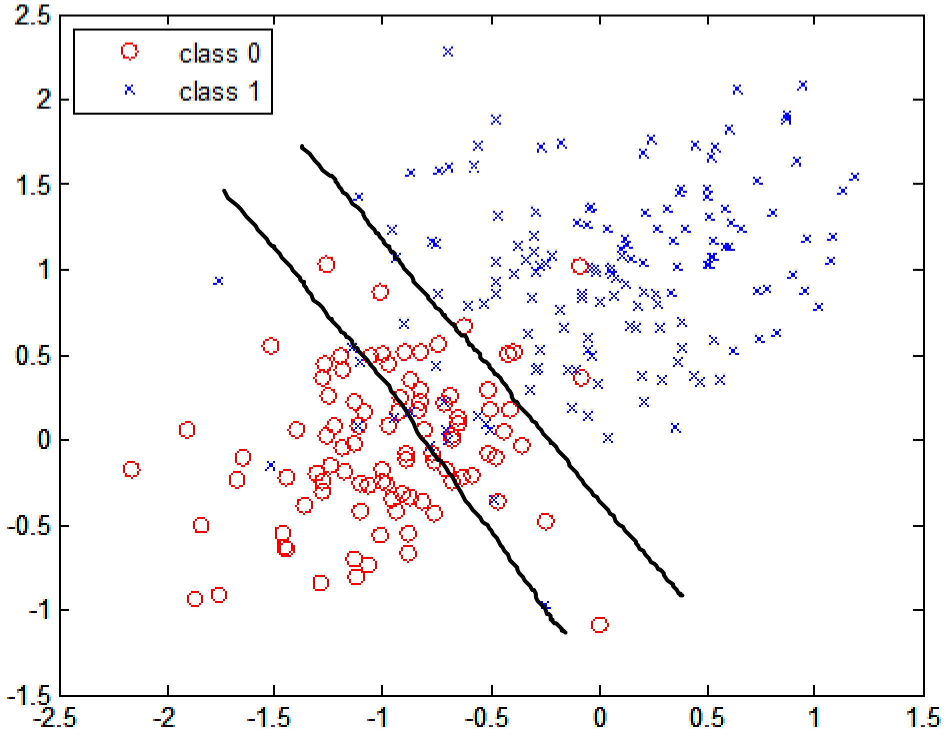
Linear Classifier

Non-Separable Case

Non-Separable case

- **Perceptron algorithm** will **not converge nor** will it find the classifier with the **smallest error**, if training examples are **linearly not separable**.

- **Goal:** Find a **classifier that minimizes the training error** in the **non-realizable case.**



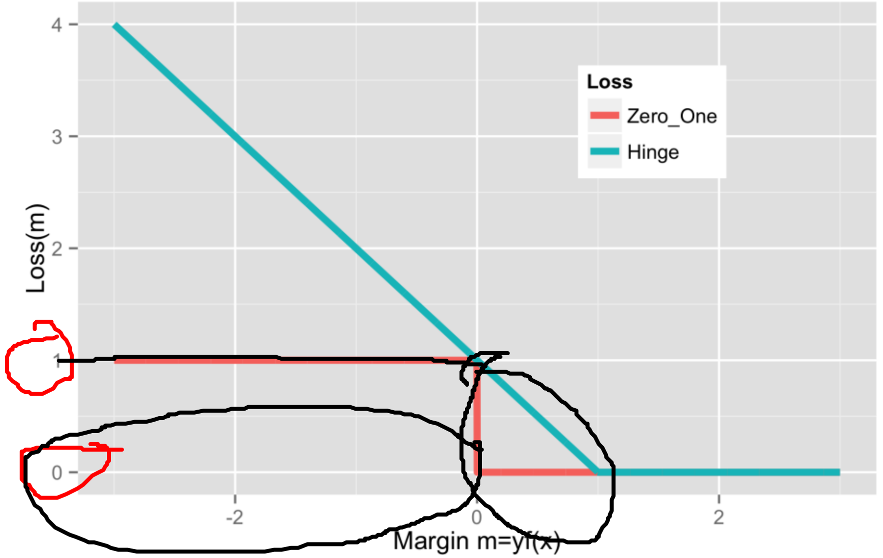
Loss Functions

- **Training Loss / Empirical risk:**

$$R_n(\theta) = \frac{1}{n} \sum_{\text{data } (x,y)} \text{Loss}\big(y(\theta^\top x)\big)$$

- **Zero-one loss:** $\text{Loss}_{0|1}(z) = [\![\underbrace{y(\theta^\top x)}_{z} \leq 0]\!]$

- **Hinge loss:** $\text{Loss}_h(z) = \max\{1 - \underbrace{y(\theta^\top x)}_{z}, 0\}$



CONVEX!
Penalize larger mistakes more.
Penalize near-mistakes, i.e. $0 \leq y(\theta^\top x) \leq 1$.

Hinge Loss (Examples)

$$R_n(\theta) = \frac{1}{n}\sum_{t=1}^{n} \text{Loss}_h(y^{(t)}(\theta \cdot x^{(t)})) = \frac{1}{n}\sum_{t=1}^{n} \max\{1 - y^{(t)}(\theta \cdot x^{(t)}), 0\}$$

Example 1

- original label = -1 and prediction score = 0.4 (this means the model predicted class as 1)
- penalty = max(0, 1+1(0.4)) = $\boxed{1.4}$ which is a very high penalty since the prediction was inaccurate

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1   -0.9

Example 1

- original label = -1 and prediction score = 0.4 (this means the model predicted class as 1)
- penalty = max(0, 1+1(0.4)) = 1.4 which is a very high penalty since the prediction was inaccurate

Example 2

- original label = 1 and prediction score =(- 0.9) (this means the model predicted class as -1)
- penalty = max(0, 1-1(-0.9)) = 1.9 which is a very high penalty since the prediction was inaccurate

Hinge Loss (Examples)

$$R_n(\theta) = \frac{1}{n}\sum_{t=1}^{n} \text{Loss}_h(y^{(t)}(\theta \cdot x^{(t)})) = \frac{1}{n}\sum_{t=1}^{n} \max\{1 - y^{(t)}(\theta \cdot x^{(t)}), 0\}$$

1 0 7

Example 1

- original label = -1 and prediction score = 0.4 (this means the model predicted class as 1)
- penalty = max(0, 1+1(0.4)) = 1.4 which is a very high penalty since the prediction was inaccurate

Example 2

- original label = 1 and prediction score =(- 0.9) (this means the model predicted class as -1)
- penalty = max(0, 1-1(-0.9)) = 1.9 which is a very high penalty since the prediction was inaccurate

Example 3

- original label = 1 and prediction score = 0.7 (this means the model predicted class as 1)
- penalty = max(0, 1-1(0.7)) = 0.3 (loss is very less but not 0, since the prediction is accurate and has high confidence but not 100%)

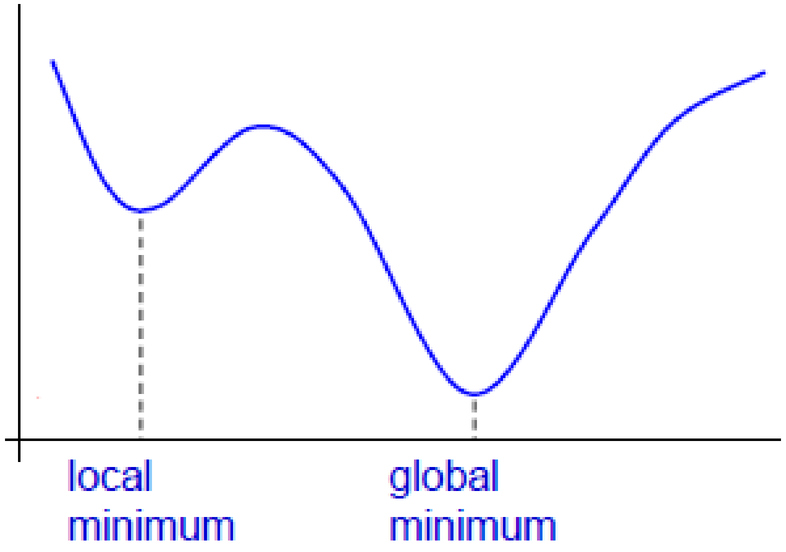
Hinge Loss

- **Empirical risk using hinge loss:**

$$R_n(\theta) = \frac{1}{n}\sum_{t=1}^{n} \text{Loss}_h(y^{(t)}(\theta \cdot x^{(t)})) = \frac{1}{n}\sum_{t=1}^{n} \max\{1 - y^{(t)}(\theta \cdot x^{(t)}), 0\}$$

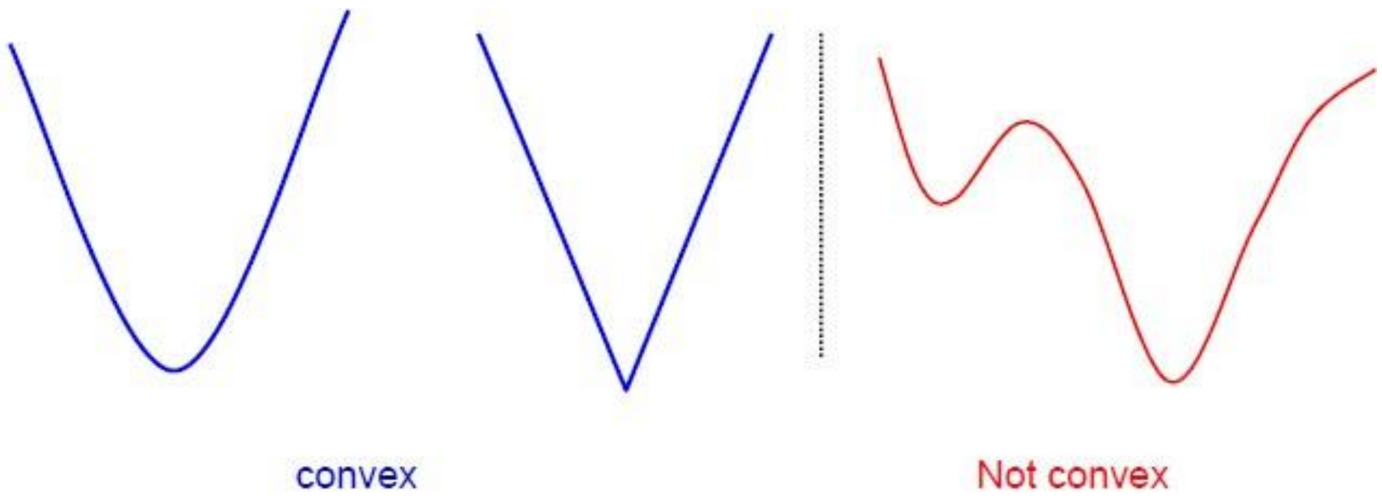
- **Convexity** of empirical risk allows us to find the **minimum** even in **non-realizable** case.

Optimization



- Does this loss function have a unique solution?
- If the loss function is convex, then a locally optimal point is globally optimal (provided the optimization is over a convex set, which it is in our case)

Convex Function Examples



A non-negative sum of convex functions is convex

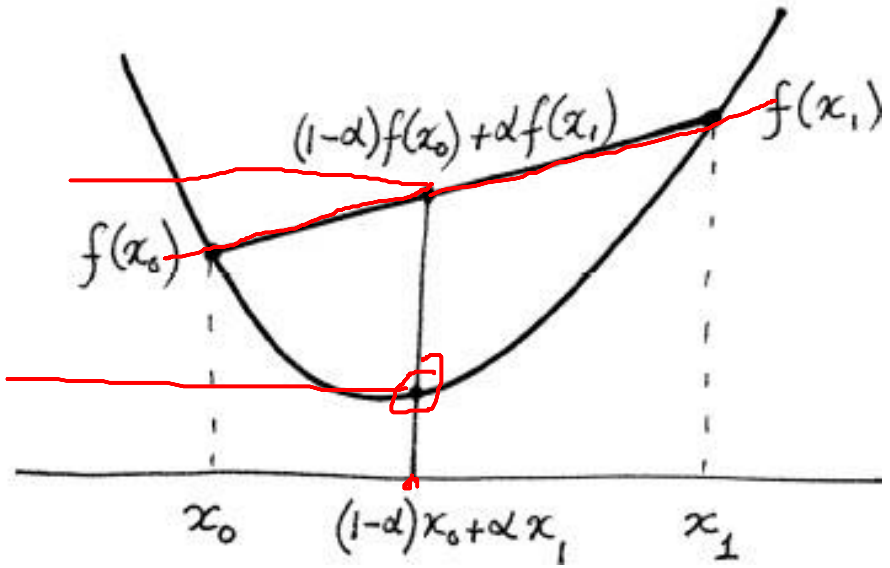
Convex Functions

D – a domain in $\mathbb{R}^n$.

A convex function $f : D \to \mathbb{R}$ is one that satisfies, for any $\mathbf{x}_0$ and $\mathbf{x}_1$ in D:

$$f((1-\alpha)\mathbf{x}_0 + \alpha\mathbf{x}_1) \le (1-\alpha)f(\mathbf{x}_0) + \alpha f(\mathbf{x}_1) \quad .$$

Line joining $(\mathbf{x}_0, f(\mathbf{x}_0))$ and $(\mathbf{x}_1, f(\mathbf{x}_1))$ lies above the function graph.



Hinge Loss

- **Empirical risk using hinge loss:**

$$R_n(\theta) = \frac{1}{n}\sum_{t=1}^{n} \text{Loss}_h(y^{(t)}(\theta \cdot x^{(t)})) = \frac{1}{n}\sum_{t=1}^{n} \max\{1 - y^{(t)}(\theta \cdot x^{(t)}), 0\}$$

$\theta_1, \theta_2, \theta_3, \ldots, \theta_d$

- **Convexity** of empirical risk allows us to find the **minimum** even in **non-realizable** case.

Gradient Descent

- Use **gradient descent** to minimize $R_n(\theta)$

$$\nabla_\theta R_n(\theta) = \left[\frac{\partial R_n(\theta)}{\partial \theta_1}, \ldots, \frac{\partial R_n(\theta)}{\partial \theta_d}\right]^T$$

- Positive gradient is in the direction where $R_n(\theta)$ **increases**.
- Need to update the weight in the **opposite direction** to minimize error.

$$\theta^{(k+1)} = \theta^{(k)} - \eta_k \nabla_\theta R_n(\theta)_{\theta=\theta^{(k)}}$$

Gradient Descent

$$R_n(\theta) = \frac{1}{n}\sum_{t=1}^{n} \text{Loss}(y^{(t)}(\theta \cdot x^{(t)}))$$

$$R_n(\theta) = \frac{1}{n}\sum_{t=1}^{n} \max\{1 - y^{(t)}(\theta \cdot x^{(t)}), 0\}$$

- Need to update the weight in the **opposite direction**.

$$\theta^{(k+1)} = \theta^{(k)} - \eta_k \nabla_\theta R_n(\theta)_{\theta=\theta^{(k)}}$$

$$\theta^{(k+1)} = \theta^{(k)} - \eta_k \nabla_\theta \text{ Loss }_h(y^{(t)}\theta \cdot x^{(t)})_{|\theta=\theta^{(k)}}$$

$$\nabla_\theta \text{ Loss }_h(y^{(t)}\theta \cdot x^{(t)})_{|\theta=\theta^{(k)}} = \nabla_\theta(1 - y^{(t)}\theta \cdot x^{(t)})_{|\theta=\theta^{(k)}} = -y^{(t)}x^{(t)}$$

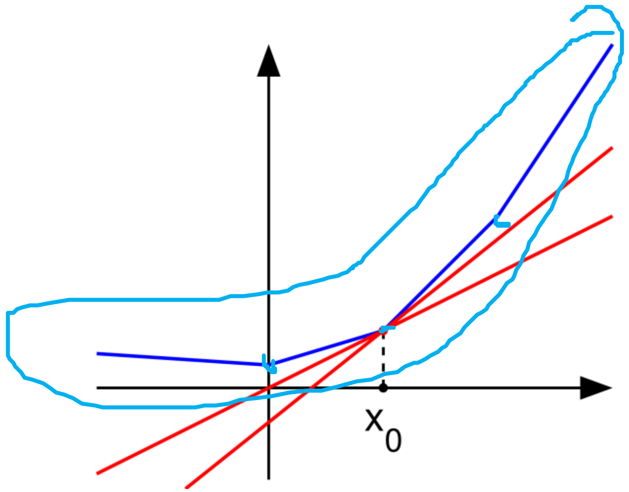
$$\theta^{(k+1)} = \theta^{(k)} + \eta_k y^{(t)} x^{(t)}$$

Gradient Descent

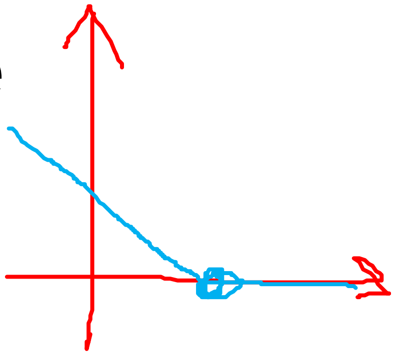
$$R_n(\theta) = \frac{1}{n}\sum_{t=1}^{n} \max\{1 - y^{(t)}(\theta \cdot x^{(t)}), 0\}$$

- $R_n(\theta)$ is **not differentiable** everywhere as **hinge loss functions** are **piece-wise linear**.

- There are several **possible gradients** at the kinks which are collectively defined as **sub-differential**.



- To minimize $R_n(\theta)$, we need to select one possible gradient at any point.



Stochastic Gradient Descent

1. Initialize the **weight** ($\theta^{(0)} = 0$).

2. Select $t \in \{1, \ldots, n\}$ at random

- If $y^{(t)}(\theta^{(k)} \cdot x^{(t)}) \leq 1$, then update the weight

$$\theta^{(k+1)} = \theta^{(k)} + \eta_k y^{(t)} x^{(t)}$$

t 101

3. Repeat Step (2) until stopping criterion is met.
(e.g. when improvement in $R_n(\theta)$ is small enough;
when k is big enough)

Stochastic Gradient Descent

Differences from Perceptron algorithm:

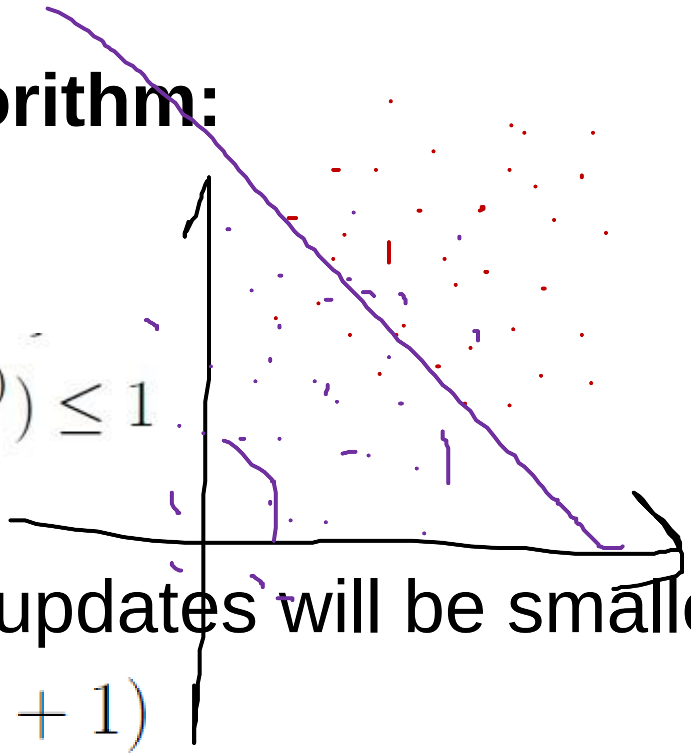
- Near mistakes are also penalized

$$y^{(t)}(\theta^{(k)} \cdot x^{(t)}) \leq 1$$

- Learning rate is decreasing (later updates will be smaller)

$$\eta_k = 1/(k+1)$$

- Random samples avoid oscillation



$R_n(\Theta) =$

Stochastic Gradient Descent

- Keep track of best solution (weight), $\theta^{(i_k)}$, where, $i_k = \text{argmin}_{i=1,\ldots,k} R_n(\theta^{(i)})$ is monotonically decreasing.

- **Note: asymptotically empirical risk does not become zero as the points may not be linearly separable.**

Hinge Loss SGD Algorithm

1. Training Set (Not Necessarily Linearly Separable)

$$(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \ldots, (x^{(n)}, y^{(n)})$$

2. Model (Set of Perceptrons)

$$h(x;\theta) = \text{sign}(\theta_1 x_1 + \cdots + \theta_d x_d)$$

3. Training Loss (Hinge Loss)

$$R_n(\theta) = \frac{1}{n} \sigma_{(x,y)\in \mathcal{S}_n} \max\{1 - y(\theta^\top x), 0\}$$

4. Algorithm (Gradient Descent)

$$\theta^{(k+1)} = \theta^{(k)} + \eta_k y^{(t)} x^{(t)}$$

Summary

- **Linear Classification**
  - Decision Boundary
  - Linearly Separable
- **Perceptron Algorithm**
  - Mistake-Driven
  - Zero-One Loss
  - Only for Linearly Separable Data
- **Hinge Loss**
  - Gradient Descent
  - Hinge Loss SGD Algorithm
  - Differences with Perceptron Algorithm
  - OK for Non Linearly Separable Data

Intended Learning Outcome

Hinge Loss

- Write down the hinge loss, and plot its graph. Write down the training loss and its gradient.
- List differences between the hinge loss SGD algorithm and the perceptron algorithm.
- Explain why the hinge loss SGD applies to non linearly separable data while the perceptron algorithm does not.