Advanced signal processing techniques
First assignment

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1 Description

Consider $X(k)$, given by

$$X(k) = W(k) - W(k-1), k = \pm 1, \pm 2, \ldots,$$

where $\{W(k)\}$ is a stationary stochastic process with independent, identically distributed (i.i.d) stochastic variables and $E\{W(k)\} = 0$, $E\{W^2(k)\} = 1$ and $E\{W^3(k)\} = 1$. The covariance sequence of $\{X(k)\}$ is given by:

$$\begin{aligned} c_2^x(\tau) = m_2^x(\tau) &= E\{X(k)X(k+\tau)\} \\ &= E\{\big(W(k) - W(k-1)\big)\big(W(k+\tau) - W(k+\tau-1)\big)\} \\ &= 2\delta(\tau) - \delta(\tau-1) - \delta(\tau+1) \end{aligned}$$

where $\delta(\tau)$ is the delta Kronecker function; hence,

$$c_2^x(\tau) = \begin{cases} 2, & \tau = 0 \\ -1, & \tau = 1,\ \tau = -1. \\ 0, & elsewhere \end{cases}$$

The corresponding Power Spectrum is given by

$$C_2^x(\omega) = \sum_{\tau=-1}^{1} c_2^x(\tau) e^{-j\omega\tau} = (2 - 2cos\omega).$$

1. Find the 3rd-order cumulants of $\{X(k)\}$, i.e., $c_3^x(\tau_1, \tau_2)$.
2. Find the skewness $\gamma_3^x = c_3^x(0,0)$. What do you observe?
3. Find the Bispectrum $C_3^x(\omega_1, \omega_2)$ Is it complex, real or imaginary?
4. How the result of 2 affects the result of 3? Can you draw a general comment?

2 Solutions

2.1

$$
\begin{aligned}
c_3^x(\tau_1,\tau_2) &= m_3^x(\tau_1,\tau_2) - \cancelto{0}{m_1^x}[m_2^x(\tau_1) + m_2^x(\tau_2) + m_2^x(\tau_1-\tau_2)] + 2\cancelto{0}{(m_1^x)^3} \\
&= m_3^x(\tau_1,\tau_2) \\
&= E[X(k)X(k+\tau_1)X(k+\tau_2) \\
&= E\{[W(k) - W(k-1)][W(k+\tau_1) - W(k+\tau_1-1)][W(k+\tau_2) - W(k+\tau_2-1)]\} \\
&= E[W(k)W(k+\tau_1)W(k+\tau_2) - W(k)W(k+\tau_1)W(k+\tau_2-1) - W(k)W(k+\tau_1-1)W(k+\tau_2) \\
&\quad + W(k)W(k+\tau_1-1)W(k+\tau_2-1) - W(k-1)w(k+\tau_1)W(k+\tau_2) \\
&\quad + W(k-1)W(k+\tau_1)W(k+\tau_2-1) + W(k-1)W(k+\tau_1-1)W(k+\tau_2) \\
&\quad - W(k-1)W(k+\tau_1-1)W(k+\tau_2-1)]
\end{aligned}
$$

$$
\begin{aligned}
c_3^x(\tau_1,\tau_2) &= -\delta(\tau_1)\delta(\tau_2-1) - \delta(\tau_1-1)\delta(\tau_2) + \delta(\tau_1-1)\delta(\tau_2-1) - \delta(\tau_1+1)\delta(\tau_2+1) \\
&\quad + \delta(\tau_1+1)\delta(\tau_2) + \delta(\tau_1)\delta(\tau_2+1)
\end{aligned}
\tag{1}
$$

2.2

$$c_3^x(0,0) = 0 \tag{2}$$

Since the third order cumulant for zero lags equals to the skewness of the univariate probability density function (pdf) of the stochastic process, we can claim that the pdf is symmetric. Additionally, we could predict in an analogous way with the autocorrelation - variance - integral of power spectrum, that the bispectrum will be zero (third order cumulant - skewness - integral of bispectrum).

2.3

$$
\begin{aligned}
C_3^x(\omega_1,\omega_2) &= \sum_{\tau_1=-\infty}^{\infty}\sum_{\tau_2=-\infty}^{\infty} c_x^3(\tau_1,\tau_2)e^{-j(\omega_1\tau_1+\omega_2\tau_2)} \\
&= \sum_{\tau_1=-\infty}^{\infty}\sum_{\tau_2=-\infty}^{\infty} [-\delta(\tau_1)\delta(\tau_2-1) - \delta(\tau_1-1)\delta(\tau_2) + \delta(\tau_1-1)\delta(\tau_2-1) - \delta(\tau_1+1)\delta(\tau_2+1) \\
&\quad + \delta(\tau_1+1)\delta(\tau_2) + \delta(\tau_1)\delta(\tau_2+1)]e^{-j(\omega_1\tau_1+\omega_2\tau_2)} \\
&= e^{j\omega_2} - e^{-j\omega_2} + e^{j\omega_1} - e^{-j\omega_1} + e^{-j(\omega_1+\omega_2)} - e^{j(\omega_1+\omega_2)} \\
&= 2j(\sin\omega_1 + \sin\omega_2 - \sin(\omega_1+\omega_2))
\end{aligned}
\tag{3}
$$

We can observe that the bipsectrum has only the imaginary part so our prediction that the bispectrum will be zero was false. We should claim that the real part of the bispectrum will be zero. This is reasonable, since now bispectrum can be complex opposite to the real power spectrum. Thus, in the inverse 2d fourier transform of the bispectrum, in order to calculate the skewness we have the following:

$$\gamma_3^x = \sum_{\omega_2}\sum_{\omega_2} C_3^x(\omega_1,\omega_2) = 0 \tag{4}$$

Taking into account the bispectrum symmetries, we can conclude that in order for the above equation to hold, the real part should be indeed zero with no restriction regrading the imaginary part, since there is cancellation due to conjugate symmetry.

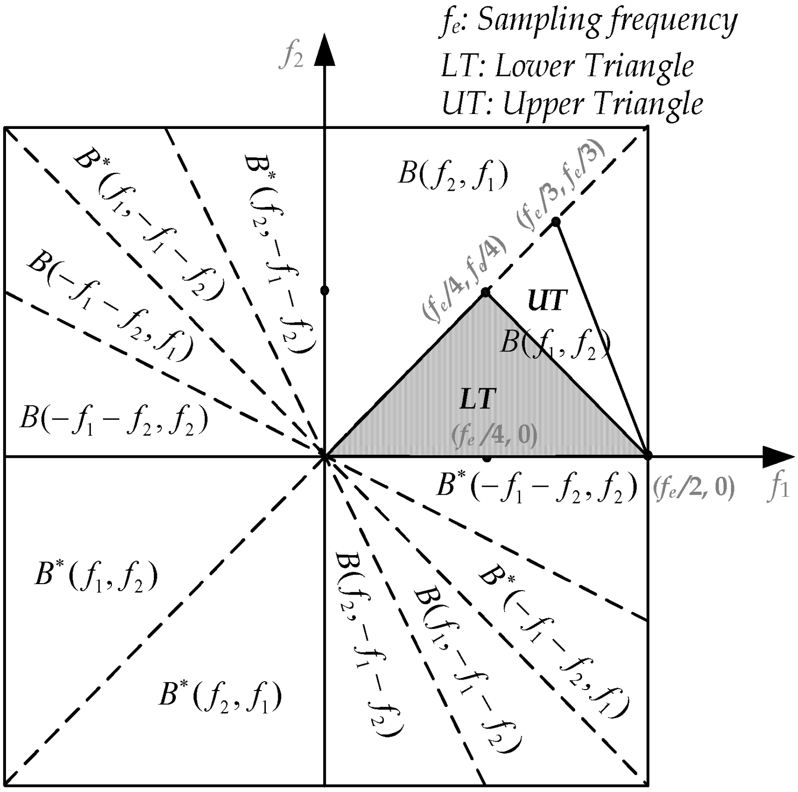

Figure 1: Bispectrum 12 symmetries illustrated
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