Calculating ${}_pF_q$

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For $\mathbf{a} = a_1, \ldots, a_p$ and $\mathbf{b} = b_1, \ldots, b_q$, set $(a)_n := \Gamma(a+n)/\Gamma(a)$ and $(\mathbf{a})_n := \prod_i (a_i)_n$ and define

$$ {}_pF_q\left(\begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array} \middle| z \right) := \sum_{n=0}^{\infty} \frac{(\mathbf{a})_n}{(\mathbf{b})_n} \frac{z^n}{n!} \tag{0.1} $$

The function is undefined when any b_i is $0, -1, -2, \ldots$, i.e. $\Gamma(\mathbf{b}) := \prod_i \Gamma(b_i)$ is infinite. Also, $\hat{\mathbf{a}}_i$ denotes the length $p-1$ vector with the i^{th} entry omitted. The standard quantity $\sigma = \Sigma(\mathbf{b}) - \Sigma(\mathbf{a})$ governs several properties of these functions. Whenever a possibly infinite quantity $\Gamma(a_i - a_j)$ appears in a formula, that formula should be interpret via a limiting cases of the general formula.

1 The case $p < q + 1$ [not implemented]

Just sum the series for any argument. The problem for large $|z|$ is that many terms may be required before the partial sums start to approach the true value. Indeed, in the special case of ${}_1F_1(a_1; b_1|z)$ the ratio of successive terms is

$$ \frac{a_1 + n}{b_1 + n} \cdot \frac{z}{n} \approx \frac{z}{n} $$

and at approximately $|z|$ terms have to be summed before the terms start to decrease. In this case, the formal expansion $(n = q + 1 - p)$

$$ \begin{aligned} {}_pF_q\left(\begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array} \middle| z \right) = & \sum_{i=1}^{p} \frac{\Gamma(\mathbf{b})\Gamma(\hat{\mathbf{a}}_i - a_i)}{\Gamma(\mathbf{b} - a_i)\Gamma(\hat{\mathbf{a}}_i)} (-z)^{-a_i} {}_{q+1}F_{p-1}\left(\begin{array}{c} a_i, 1 + a_i - \mathbf{b} \\ 1 + a_i - \hat{\mathbf{a}}_i \end{array} \middle| \frac{(-1)^n}{z} \right) \\ & + \sum_{\alpha^n = 1} \frac{\Gamma(\mathbf{b})}{n(2\pi)^{\frac{n-1}{2}}\Gamma(\mathbf{a})} e^{n\alpha z^{1/n}} z^{\frac{n-1}{2n} - \frac{\sigma}{n}} \left(1 + \text{ series in } \frac{1}{\alpha z^{1/n}} \right), \end{aligned} $$

which consists of $q + 1$ formal series, is useful. The first p series are hypergeometric and $1/n$–Borel summable. The last n series are ?–Borel summable and the coefficients satisfy recurrences of order ?. TODO: check this.

2 The case $p > q + 1$ [not implemented]

The formal series is divergent except for zero argument or terminating parameters and is $1/(p - q - 1)$–Borel summable in a range of directions. For nonzero arguments this leads to ("The Borel Sum of Divergent Barnes Hypergeometric Series and its Application to a Partial Differential Equation" by Kunio Ichinobe)

$$ {}_pF_q\left(\begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array} \middle| z \right) = \sum_{i=1}^{p} \frac{\Gamma(\mathbf{b})\Gamma(\hat{\mathbf{a}}_i - a_i)}{\Gamma(\mathbf{b} - a_i)\Gamma(\hat{\mathbf{a}}_i)} (-z)^{-a_i} {}_{q+1}F_{p-1}\left(\begin{array}{c} a_i, 1 + a_i - \mathbf{b} \\ 1 + a_i - \hat{\mathbf{a}}_i \end{array} \middle| \frac{(-1)^{p-q-1}}{z} \right) $$

The series on the right are convergent.

The difficulty here is when $|z|$ is so small that the convergent series on the right hand side cannot be summed. In this case, a direct evaluation of the Laplace integral defining the Borel sum should be preferred. This proceeds as follows. For any "direction" ω with $\Re(\omega) > 0$, we have

$$ k! = \int_0^{\infty} e^{-\omega t} (\omega t)^k \omega dt $$

Divide each term of (0.1) by $((p-q-1)n)!$ to obtain the convergent series

$$f(x) := \sum_{n=0}^{\infty} \frac{1}{((p-q-1)n)!} \frac{(\mathbf{a})_n}{(\mathbf{b})_n} \cdot \frac{x^n}{n!}$$

Then, at least formally, with $k = (p-q-1)n$, we have

$${}_pF_q\left(\begin{array}{c}\mathbf{a}\\ \mathbf{b}\end{array}\Big| z\right) = \int_0^{\infty} e^{-\omega t} f(z(\omega t)^{p-q-1})\omega dt,$$

and ω must be chosen so that the integrand does not hit the singularity of $f(x)$ at $x = (p-q-1)^{p-q-1}$.

3 The case $p = q + 1$

3.1 inside unit circle

For arguments sufficiently inside the unit circle, just sum the series.

3.2 outside unit circle

For $z \notin [0,1]$ (?),

$${}_pF_{p-1}\left(\begin{array}{c}\mathbf{a}\\ \mathbf{b}\end{array}\Big| z\right) = \sum_{i=1}^{p} \frac{\Gamma(\mathbf{b})\Gamma(\hat{\mathbf{a}}_i - a_i)}{\Gamma(\mathbf{b}-a_i)\Gamma(\hat{\mathbf{a}}_i)}(-z)^{-a_i}\, {}_pF_{p-1}\left(\begin{array}{c} a_i, 1+a_i-\mathbf{b}\\ 1+a_i-\hat{\mathbf{a}}_i\end{array}\Big| \frac{1}{z}\right)$$

and for arguments sufficiently outside the unit circle we can just sum the series on the right.

3.3 near unit circle, away from one

For any argument outside the branch cut $[1,\infty]$, the series on the right hand side of

$$(1+z)^{-2a_p}\, {}_pF_{p-1}\left(\begin{array}{c}\mathbf{a}\\ \mathbf{b}\end{array}\Big| \frac{4z}{(1+z)^2}\right) = \sum_{n=0}^{\infty} u_n z^n, \quad |z| < 1$$

is convergent. However, since computation of the u_n's is a bit expensive, it should only be used when absolutely neccessary. There is a good reason for the prefactor $(1+z)^{-2a_p}$: is present in many quadratic transformation formulas in special cases and has the effect of lowering the order of the recurrence relation for u_n by one.

It is also possible to use Padé approximants here, but do we have useful error bounds?

3.4 near one

This is the most interesting case as the function can fail to be defined at one. The existence of $F(1)$ is determinded by $\Re(\sigma) > 0$. If σ is not an integer we have

$${}_pF_{p-1}\left(\begin{array}{c}\mathbf{a}\\ \mathbf{b}\end{array}\Big| 1-z\right) = \sum_{n=0}^{\infty} u_n \left(\begin{array}{c}\mathbf{a}\\ \mathbf{b}\end{array}\right) z^{\sigma+n} + \sum_{n=0}^{\infty} v_n \left(\begin{array}{c}\mathbf{a}\\ \mathbf{b}\end{array}\right) z^n \tag{3.1}$$

with the $u_1, u_2, \ldots$ determined from recurrences by $u_0 = \Gamma(-\sigma)\Gamma(\mathbf{b})/\Gamma(\mathbf{a})$ and the $v_{p-1}, v_p, \ldots$ are determined from recurrences by $v_0, \ldots, v_{p-2}$. Thus the difficulty is computing these $v_0, \ldots, v_{p-2}$.

If σ is an integer, then at most one $\log(z)$ enters into the series.

3.4.1 generic approach

We simply evaluate Equation (3.1) and its derivatives up to and including order $p-1$ at $z = 1/4$ to solve for the $u_0, v_0, \ldots, v_{p-2}$. The explicit formula for u_0 is surprisingly useless in this approach.

3.4.2 Buehring

Here we sum the first m terms of Equation (0.1) and use a formula derived by Buehring to sum the remaining terms. Since we will generically be dealing with logarithmically convergent series (when $z = 1$) in both sums, it is important to balance the choice of m between the two to ensure a sub-exponential algorithm. We have (Equations (2.7) and (2.9) in "analytic continuation of the generalized hypergeoemtric sereis near unit argument with emphasis on the zero-balanced series" by Buehring and Srivastava)

$$\begin{aligned}\sum_{n=m}^{\infty} \frac{(\mathbf{a})_n}{(\mathbf{b})_n} \frac{z^n}{n!} &= \frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})} z^m \sum_{k=0}^{\infty} \frac{\Gamma(\mathbf{a}+m+k)}{\Gamma(\mathbf{b}+m+k)} \frac{z^k}{\Gamma(1+m+k)} \\ &= \frac{\Gamma(\mathbf{b})(a_p)_m}{\Gamma(\hat{\mathbf{a}}_p)} z^m \sum_{k=0}^{\infty} A_k \left(\begin{array}{c} \hat{\mathbf{a}}_p \\ \mathbf{b} \end{array} \right) {}_2\tilde{F}_1 \left(\begin{array}{c} 1, a_p+m \\ 1+\sigma+a_p+m+k \end{array} \middle| z \right) \end{aligned} \tag{3.2}$$

where the $A_k(\hat{\mathbf{a}}_p; \mathbf{b})$ are independent of m and are polynomials in $a_1, \ldots, a_{p-1}, b_1, \ldots, b_{p-1}$. They can be defined in the base case $p = 2$ as

$$A_k \left(\begin{array}{c} a_1 \\ b_1 \end{array} \right) = \frac{(1-a_1)_k (b_1-a_1)_k}{k!}$$

and inductively for larger p by Hadamard and Cauchy products. After all is said and done, the A_k satisfy an order $p-1$ recurrence and are bounded as

$$\frac{A_k}{k!} \ll \sum_{i<p} k^{\sigma+a_p-1-a_i} \tag{3.3}$$

Now set

$$F_k = {}_2\tilde{F}_1 \left(\begin{array}{c} 1, a_p+m \\ 1+\sigma+a_p+m+k \end{array} \middle| z \right)$$

We have

$$F_k = \frac{(k+\sigma-1-(1-z)(a_p+2k+m+2\sigma-2)) F_{k-1} + (1-z)F_{k-2}}{z(k+\sigma)(a_p+k+m+\sigma-1)}$$

and therefore the bound

$$k! F_k \ll k^{-\sigma} |1-1/z|^k + k^{-m-\sigma-a_p}$$

To ensure convergence of the tail series, we should have $|1-1/z| < 1$ and $m + \Re(a_i) > 0$ for all $i < p$.

In reality the majorant method will probably produce a much worse explicit bound $|A_k/k!| \leq ck^{\mu}$ so we are balancing the sum of the first m terms of a sum whose terms are like $n^{-1-\sigma}$ with another series that we can only prove has terms like $k^{\mu-m-\sigma-a_p}$. Any reasonable overestimation of μ can be compensated by a larger m. Finally, in order to sum in total no more than $O(d)$ terms for d digit accuracy, it probably suffices to take $m \approx d$ for reasonable parameter ranges.

3.4.3 hybrid approach for $\sigma \notin \mathbb{Z}$

The necessary coefficients u_0 (respectively $v_0, \ldots, v_{p-2}$) in (3.1) may be evaluated by combining (3.2) (with z replaced by $1-z$) for $m = 0$ (respectively large m) with the expansion

$$\begin{aligned}{}_2\tilde{F}_1 \left(\begin{array}{c} 1, a_p+m \\ 1+\sigma+a_p+m+k \end{array} \middle| 1-z \right) &= \frac{\Gamma(-\sigma-k)}{\Gamma(a_p+m)} \sum_{j=0}^{\infty} \frac{(\sigma+a_p+m+k)_j}{j!} z^{\sigma+k+j} \\ &\quad + \frac{-1}{\Gamma(\sigma+a_p+m+k)} \sum_{j=0}^{\infty} \frac{(a_p+m)_j}{(-\sigma-k)_{j+1}} z^j. \end{aligned} \tag{3.4}$$

The basic idea is that

$$\sum_{n=0}^{m-1} \frac{(\mathbf{a})_n}{(\mathbf{b})_n} \frac{(1-z)^n}{n!} \approx v_0 + v_1 z + v_2 z^2 + \cdots. \tag{3.5}$$

This is no help in evaluating the u_n, but those can be found easily. We have

$$
\begin{aligned}
F(1-z) &= \sum_{n=0}^{m-1} \frac{(\mathbf{a})_n}{(\mathbf{b})_n} \frac{(1-z)^n}{n!} + \frac{\Gamma(\mathbf{b})(a_p)_m}{\Gamma(\hat{\mathbf{a}}_p)} (1-z)^m \sum_{k=0}^{\infty} A_k \left(\begin{array}{c} \hat{\mathbf{a}}_p \\ \mathbf{b} \end{array} \right) {}_2\tilde{F}_1 \left(\begin{array}{c} 1, a_p+m \\ 1+\sigma+a_p+m+k \end{array} \Big| 1-z \right) \\
&= \sum_{n=0}^{m-1} \frac{(\mathbf{a})_n}{(\mathbf{b})_n} \frac{(1-z)^n}{n!} + \frac{\Gamma(\mathbf{b})(a_p)_m}{\Gamma(\hat{\mathbf{a}}_p)} (1-z)^m \sum_{k=0}^{\infty} A_k \left(\begin{array}{c} \hat{\mathbf{a}}_p \\ \mathbf{b} \end{array} \right) \frac{\Gamma(-\sigma-k)}{\Gamma(a_p+m)} \sum_{j=0}^{\infty} \frac{(\sigma+a_p+m+k)_j}{j!} z^{\sigma+k+j} \\
&\qquad + \frac{\Gamma(\mathbf{b})(a_p)_m}{\Gamma(\hat{\mathbf{a}}_p)} (1-z)^m \sum_{k=0}^{\infty} A_k \left(\begin{array}{c} \hat{\mathbf{a}}_p \\ \mathbf{b} \end{array} \right) \frac{-1}{\Gamma(\sigma+a_p+m+k)} \sum_{j=0}^{\infty} \frac{(a_p+m)_j}{(-\sigma-k)_{j+1}} z^j .
\end{aligned}
$$

Taking $m=0$ and equating non-integral powers of z gives

$$
\sum_{n=0}^{\infty} u_n z^{\sigma+n} = \frac{\Gamma(\mathbf{b})}{\Gamma(\hat{\mathbf{a}}_p)} \sum_{k=0}^{\infty} A_k \left(\begin{array}{c} \hat{\mathbf{a}}_p \\ \mathbf{b} \end{array} \right) \frac{\Gamma(-\sigma-k)}{\Gamma(a_p)} \sum_{j=0}^{\infty} \frac{(\sigma+a_p+k)_j}{j!} z^{\sigma+k+j},
$$

so that in particular $u_0 = \Gamma(-\sigma)\Gamma(\mathbf{b})/\Gamma(\mathbf{a})$ and in general

$$
u_n = \frac{\Gamma(\mathbf{b})}{\Gamma(\hat{\mathbf{a}}_p)} \sum_{k+j=n} A_k \left(\begin{array}{c} \hat{\mathbf{a}}_p \\ \mathbf{b} \end{array} \right) \frac{\Gamma(-\sigma-k)}{\Gamma(a_p)} \frac{(\sigma+a_p+k)_j}{j!}.
$$

Equating integral powers of z gives

$$
\sum_{n=0}^{\infty} v_n z^n = \sum_{n=0}^{m-1} \frac{(\mathbf{a})_n}{(\mathbf{b})_n} \frac{(1-z)^n}{n!} + \frac{\Gamma(\mathbf{b})(a_p)_m}{\Gamma(\hat{\mathbf{a}}_p)} \sum_{\ell=0}^{\infty} \frac{(-m)_\ell}{\ell!} z^\ell \sum_{k=0}^{\infty} A_k \left(\begin{array}{c} \hat{\mathbf{a}}_p \\ \mathbf{b} \end{array} \right) \frac{-1}{\Gamma(\sigma+a_p+m+k)} \sum_{j=0}^{\infty} \frac{(a_p+m)_j}{(-\sigma-k)_{j+1}} z^j,
$$

or, for arbitrary m,

$$
\begin{aligned}
v_i - [z^i] \sum_{n=0}^{m-1} \frac{(\mathbf{a})_n}{(\mathbf{b})_n} \frac{(1-z)^n}{n!} &= \sum_{\ell+j=i} \frac{\Gamma(\mathbf{b})(a_p)_m}{\Gamma(\hat{\mathbf{a}}_p)} \frac{(-m)_\ell}{\ell!} \sum_{k=0}^{\infty} A_k \left(\begin{array}{c} \hat{\mathbf{a}}_p \\ \mathbf{b} \end{array} \right) \frac{-1}{\Gamma(\sigma+a_p+m+k)} \frac{(a_p+m)_j}{(-\sigma-k)_{j+1}} \\
&= \frac{\Gamma(\mathbf{b})(a_p)_m}{\Gamma(\hat{\mathbf{a}}_p)\Gamma(\sigma+a_p+m)} \sum_{j=0}^{i} \frac{(a_p+m)_j(-m)_{i-j}}{(i-j)!} \sum_{k=0}^{\infty} A_k \left(\begin{array}{c} \hat{\mathbf{a}}_p \\ \mathbf{b} \end{array} \right) \frac{-1}{(\sigma+a_p+m)_k(-\sigma-k)_{j+1}}
\end{aligned}
$$

This formula for the v_i is as effective as the series in the previous section. Another formula for the v_i follows by noting that both sides of (3.1) can be differentiated, which yields

$$
i!(-1)^i v_i \left(\begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array} \right) = \frac{(\mathbf{a})_i}{(\mathbf{b})_i} v_0 \left(\begin{array}{c} \mathbf{a}+i \\ \mathbf{b}+i \end{array} \right).
$$

Combining this with the above formula for v_0 gives

$$
\begin{aligned}
i!(-1)^i v_i \left(\begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array} \right) &= \frac{(\mathbf{a})_i}{(\mathbf{b})_i} \left(\sum_{n=0}^{m-1} \frac{(\mathbf{a}+i)_n}{(\mathbf{b}+i)_n} \frac{1}{n!} + \frac{\Gamma(\mathbf{b}+i)\Gamma(a_p+i+m)}{\Gamma(\mathbf{a}+i)\Gamma(\sigma+a_p+m)} \sum_{k=0}^{\infty} A_k \left(\begin{array}{c} \hat{\mathbf{a}}_p+i \\ \mathbf{b}+i \end{array} \right) \frac{1}{(\sigma-i+k)(\sigma+a_p+m)_k} \right) \\
&= \sum_{n=0}^{m-1} \frac{(\mathbf{a})_{n+i}}{(\mathbf{b})_{n+i}} \frac{1}{n!} + \frac{\Gamma(\mathbf{b})\Gamma(a_p+i+m)}{\Gamma(\mathbf{a})\Gamma(\sigma+a_p+m)} \sum_{k=0}^{\infty} A_k \left(\begin{array}{c} \hat{\mathbf{a}}_p+i \\ \mathbf{b}+i \end{array} \right) \frac{1}{(\sigma-i+k)(\sigma+a_p+m)_k}.
\end{aligned}
$$

Finally, replacing m by $m-i$ gives the succinct formula

$$
\begin{aligned}
i!(-1)^i v_i \left(\begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array} \right) = &\frac{\Gamma(\mathbf{b})\Gamma(a_p+m)}{\Gamma(\mathbf{a})\Gamma(\sigma-i+a_p+m)} \sum_{k=0}^{\infty} A_k \left(\begin{array}{c} \hat{\mathbf{a}}_p+i \\ \mathbf{b}+i \end{array} \right) \frac{1}{(\sigma-i+k)(\sigma-i+a_p+m)_k} \\
&+ \frac{\partial^i}{\partial z^i}\Big|_{z=1} \sum_{n=0}^{m-1} \frac{(\mathbf{a})_n}{(\mathbf{b})_n} \frac{z^n}{n!}.
\end{aligned}
\tag{3.6}
$$

When $\Re(\sigma) \le i$ so that the coefficient of z^i on the left hand side of (3.5) diverges as m tends to ∞, there is cancelation between the two sums on the right hand side of (3.6). Thus, it should not be used in this case.

3.4.4 hybrid approach for $\sigma \in \mathbb{Z}$

The case of integral σ in (3.4) is in general too messy. Since the final formula is not going to work well for $\sigma < 0$ anyways, let σ be a nonnegative integer. In this case (3.1) takes the shape

$$ {}_pF_{p-1}\left(\begin{array}{c}\mathbf{a}\\ \mathbf{b}\end{array}\middle|1-z\right) = \sum_{n=0}^{\infty} w_n \left(\begin{array}{c}\mathbf{a}\\ \mathbf{b}\end{array}\right) z^{\sigma+n}\log(z) + \sum_{n=0}^{\infty} r_n \left(\begin{array}{c}\mathbf{a}\\ \mathbf{b}\end{array}\right) z^n, $$

and (3.4) becomes

$$ {}_2\tilde{F}_1\left(\begin{array}{c}1, a_p+m\\ 1+\sigma+a_p+m+k\end{array}\middle|1-z\right) = \sum_{j=0}^{\sigma} \frac{(-1)^j z^j}{\Gamma(a_p+m)} \begin{cases} \frac{\Gamma(-j+k+\sigma)\Gamma(j+m+a_p)}{\Gamma(k+\sigma+1)\Gamma(k+m+\sigma+a_p)}, & j<\sigma+k \\ \frac{-\psi^{(0)}(m+\sigma+a_p)-\log(z)-\gamma}{\sigma!}, & j=\sigma+k \end{cases} $$
$$ + O(z^{\sigma+1}). $$

Thus the coefficients of $z^0, z^1, \ldots, z^\sigma$ in $F(1-z)$ as polynomials in $\log(z)$ are equal to those of

$$ \sum_{n=0}^{m-1} \frac{(\mathbf{a})_n}{(\mathbf{b})_n}\frac{(1-z)^n}{n!} + \frac{\Gamma(\mathbf{b})(a_p)_m}{\Gamma(\hat{\mathbf{a}}_p)}(1-z)^m \left({}_2\tilde{F}_1\left(\begin{array}{c}1, a_p+m\\ 1+\sigma+a_p+m+0\end{array}\middle|1-z\right) \right. $$
$$ \left. + \sum_{k=1}^{\infty} A_k \left(\begin{array}{c}\hat{\mathbf{a}}_p\\ \mathbf{b}\end{array}\right) {}_2\tilde{F}_1\left(\begin{array}{c}1, a_p+m\\ 1+\sigma+a_p+m+k\end{array}\middle|1-z\right) \right) $$

or

$$ \sum_{n=0}^{m-1} \frac{(\mathbf{a})_n}{(\mathbf{b})_n}\frac{(1-z)^n}{n!} + \frac{\Gamma(\mathbf{b})(a_p)_m}{\Gamma(\hat{\mathbf{a}}_p)}(1-z)^m \left(\sum_{j=0}^{\sigma-1} z^j \frac{-(m+a_p)_j}{(-\sigma-0)_{j+1}\Gamma\left(0+m+\sigma+a_p\right)} \right. $$
$$ + \frac{(-1)^\sigma z^\sigma}{\Gamma(a_p+m)} \frac{-\psi\left(m+\sigma+a_p\right)-\gamma-\log(z)}{\sigma!} $$
$$ \left. + \sum_{k=1}^{\infty} A_k \left(\begin{array}{c}\hat{\mathbf{a}}_p\\ \mathbf{b}\end{array}\right) \sum_{j=0}^{\sigma} z^j \frac{-(m+a_p)_j}{(-\sigma-k)_{j+1}\Gamma\left(k+m+\sigma+a_p\right)} \right) $$

4 implementation

4.1 Tight ${}_2F_1$ bounds everywhere

The analysis is for real parameters $a, b, c \in \mathbb{R}$, but it should be possible to do something for complex parameters too.

With

$$ f(w) = (1+w)^{-2a}\, {}_2F_1\left(\begin{array}{c}a, b\\ c\end{array}\middle|\frac{4w}{(1+w)^2}\right) = \sum_{n=0}^{\infty} r_n w^n, \quad |w|<1 \tag{4.1} $$

we have $r_0 = 1$, $r_1 = \frac{4ab}{c} - 2a$, and $r_{n+1} = \lambda_0(n) r_n + (1-\lambda_1(n)) r_{n-1}$, where

$$ \lambda_0(n) = \frac{2(2b-c)(n+a)}{(n+1)(n+c)} $$
$$ \lambda_1(n) = \frac{2(1-2a+c)(n+a)}{(n+1)(n+c)} $$

The unit disk $|w|<1$ is mapped into the whole complex z-plane minus $[1,\infty)$ by $z = \frac{4w}{(1+w)^2}$, hence this provides a method for computing the usual branch of ${}_2F_1$ if we can bound the tails of the sum. Note that $\lambda_0, \lambda_1 \to 0$, and for the moment entertain the assumption that $|\lambda_0| \le \lambda_1 \le 1$ for all n:

$$ \begin{aligned} |r_2| &= |\lambda_0 r_1 + (1-\lambda_1) r_0| \\ &\le |\lambda_0||r_1| + (1-\lambda_1)|r_0| \\ &\le (|\lambda_0| + 1 - \lambda_1)\max(|r_0|, |r_1|) \\ &\le \max(|r_0|, |r_1|). \end{aligned} $$

Hence $|r_n| \le \max(|r_0|, |r_1|)$ for all n by induction. For general real parameters a, b, c the inequality $|\lambda_0(n)| \le \lambda_1(n)$ is not possible for all n as singularities (either logarithmetic or algebraic) of the ${}_2F_1$ at $z = \infty$ and $z = 1$ mean that the r_n can grow like an arbitrarily large power of n.

To remedy this, consider $\tilde{r}_n := r_n n^{-\mu}$ for some arbitrary real μ. The transformed recurrence is $\tilde{r}_n = \tilde{\lambda}_0(n)\tilde{r}_{n-1} + (1 - \tilde{\lambda}_1(n))\tilde{r}_{n-2}$ where

$$\begin{aligned}\tilde{\lambda}_0(n) &= \left(\frac{n}{n+1}\right)^{\mu} \lambda_0(n)\\ \tilde{\lambda}_1(n) &= 1 - \left(\frac{n-1}{n+1}\right)^{\mu} (1 - \lambda_1(n))\end{aligned}$$

If $|\tilde{\lambda}_0(n)| \le \tilde{\lambda}_1(n) \le 1$ for all $n \ge n_0$, then it follows as above that $r_n \le \max(|\tilde{r}_{n_0}|, |\tilde{r}_{n_0-1}|)n^{\mu}$ for all $n > n_0$. There are two ways to turn this into an algorithm for bounding the tails. Either choose an n_0 and compute a μ (not recommended), or since

$$\begin{aligned}\tilde{\lambda}_0(n) &= 2(2b - c)n^{-1} + O\left(n^{-2}\right)\\ \tilde{\lambda}_1(n) &= 2(1 - 2a + c + \mu)n^{-1} + O\left(n^{-2}\right)\end{aligned}$$

we can choose any $\mu > -1 + 2a - c + |2b - c|$ and compute an n_0. This is an optimal bound on μ.

4.2 Tight ${}_3F_2$ bounds near 1

Series expansions of solutions around $z = 1$ can be constructed as

$$\sum_{n=0}^{\infty} r_n (1 - z)^{n+\lambda}$$

where $\lambda = 0$ or $\lambda = b_1 + b_2 - a_1 - a_2 - a_3$ and $r_{n+2} + \kappa_1(n)r_{n+1} + \kappa_0(n)r_n = 0$ where

$$\begin{aligned}\kappa_0(n) &= \frac{(a_1 + \lambda + n)(a_2 + \lambda + n)(a_3 + \lambda + n)}{(\lambda + n + 1)(\lambda + n + 2)(a_1 + a_2 + a_3 - b_1 - b_2 + \lambda + n + 2)}\\ &= 1 + (b_1 + b_2 - 5)\,n^{-1} + O(n^{-2})\\ \kappa_1(n) &= -2 - (b_1 + b_2 - 5)\,n^{-1} + O(n^{-2})\end{aligned}$$

For $\lambda = b_1 + b_2 - a_1 - a_2 - a_3$ the r_n are determined once r_0 is fixed, while for $\lambda = 0$, the r_n depend freely on r_0 and r_1. This gives 3 solutions.

By the substitution $r_n = \tilde{r}_n n^{\mu}$ where $\mu = -2 + \max(b_1, b_2)$, this equation can be brought to the form

$$\tilde{r}_{n+2} + \left(-2 + \frac{d_1}{n} + \frac{d_2}{n^2} + O(\frac{1}{n^3})\right)\tilde{r}_{n+1} + \left(1 - \frac{d_1}{n} - \frac{d_2}{n^2} + O(\frac{1}{n^3})\right)\tilde{r}_n = 0$$

where crutially $d_1 = 1 + |b_1 - b_2|$ is positive. This equation can be rewritten as

$$\tilde{r}_{n+2} - \tilde{r}_{n+1} = \left(1 - \frac{d_1}{n} - \frac{d_2}{n^2}\right)(\tilde{r}_{n+1} - \tilde{r}_n) + O(\frac{\max(|\tilde{r}_{n+1}|, |\tilde{r}_n|)}{n^3})$$

All constants hidden by the O notation are effective and depend only on the parameters b_i, a_i. We would like to show that $\tilde{r}_n = O(n^{\epsilon})$ for every $\epsilon > 0$.

4.3 majorant method

This is a terse summary of Messarobba. We would like to study the various functions

$$F\left(z\right), \quad F\left(\frac{1}{z}\right), \quad F\left(1-z\right), \quad (1+z)^{-2a_1}F\left(\frac{4z}{(1+z)^2}\right), \quad \ldots$$

as convergent power series for $|z| < 1$ as this allows for the computation of F everywhere. In order to evaluate these power series, we need bounds on the coefficients, and tight bounds are already difficult to prove for ${}_2F_1$ and ${}_3F_2$. If we are not near the radius of convergence of these series, an overestimation of the coefficients is acceptable if it allows us to actually get proven bounds.

Each of these functions $f(z)$ satisfies a homogeneous linear differential equation $P(f(z)) = 0$ which will we write in terms of $\theta = z\frac{d}{dz}$. Since $z\theta = \theta z - z$, we can write the operator P with θ *on the left*. When θ is on the left and z is on the right, it is easy to transform the differential equation to a recurrsion on the coefficients. For example, for $F(z) = {}_2F_1\left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix} \middle| z\right) = \sum_{n=0}^{\infty} u_n z^n$, we have

$$P = (\theta + b_1 - 1)(\theta) - (\theta + a_1 - 1)(\theta + a_2 - 1)z \Leftrightarrow \frac{u_n}{u_{n-1}} = \frac{(n + a_1 - 1)(n + a_2 - 1)}{(n + b_1 - 1)(n)}$$

4.3.1 coefficient recursions

Write the differential operator as $P(z, \theta) = \theta^r p_r(z) + \cdots + \theta p_1(z) + p_0(z) = P_s(\theta)z^s + \cdots + P_1(\theta)z + P_0(\theta) \in \mathbb{F}[z, \theta]$ with θ on the left and assume that $P_0(0) \neq 0$. Define the operator $L(z, \theta) = P(z, \theta)p_r(z)^{-1} = \sum_{j=0}^{\infty} Q_j(\theta)z^j$ and note that $\deg(Q_0(\theta)) = r$ and $\deg(Q_j(\theta)) < r$ for $j > 0$. Let $\lambda \in \overline{\mathbb{F}}$ denote a fixed root of Q_0 such that none of $\lambda - 1, \lambda - 2, \ldots$ is a root of Q_0. Let $\mu(\nu)$ denote the multiplicity of ν as a root of Q_0 (or as a root of P_0). For a double sequence $\{u_{\lambda+n,k}\}_{n,k\geq 0}$, let

$$u(z) = \sum_{\substack{n=0 \\ \nu=\lambda+n}}^{\infty} \sum_{k=0}^{\infty} u_{\nu,k} z^{\nu} \frac{\log^k z}{k!},$$

be a solution to $P(z, \theta)(u(z)) = 0$. This is actually a polynomial in $\log z$, so let $\tau(n)$ be a nondecreasing integer-valued function of n satisfying $u_{\lambda+n,k} = 0$ for $k \geq \tau(n)$. We will see shortly that we can take $\tau(0) \leq \mu(\lambda + 0)$ and $\tau(n) \leq \tau(n - 1) + \mu(\lambda + n)$. In terms of the operator S_k, which shifts a sequence $\{a_k\}_{k\geq 0}$ to $\{a_{k+1}\}_{k\geq 0}$, the differential equation says that

$$P_0(\nu + S_k)u_{\nu} = -\sum_{j=1}^{s} P_j(\nu + S_k)u_{\nu-j}$$

Since $P_0(\nu + S_k) = S_k^{\mu(\nu)}(c_0 + c_1 S_k + \cdots)$, this equation allows us to determine all $u_{\lambda+n,k}$ with $k \geq \mu(\lambda + n)$ once the initial values $E_\lambda = \{u_{\lambda+n,k} \,|\, 0 \leq k < \mu(\lambda + n)\}$ are determined. Considering all possible λ gives r linearly independent solutions to $P = 0$.

4.3.2 tail bounds

Let $K < \tau(\infty)$ denote the higest power of $\log z$ occuring in $u(z)$, and consider the truncation

$$\tilde{u}(z) = \sum_{n=0}^{N-1} \sum_{k=0}^{K} u_{\lambda+n,k} z^{\lambda+n} \frac{\log^k z}{k!},$$

and the normalized residual $q(z)$ defined by $P(z, \theta)(\tilde{u}(z)) = Q_0(\theta)q(z)$. This has the form

$$q(z) = \sum_{j=0}^{s-1} \sum_{k=0}^{K} q_{\lambda+N+j,k} z^{\lambda+N+j} \frac{\log^k z}{k!}$$

where the $q_{\lambda+N}, \ldots, q_{\lambda+N+s-1}$ can be computed from $P(z, \theta)$ and $u_{\lambda+N-1}, \ldots, u_{\lambda+N-s}$.

Consider $y(z) = p_r(z)(\tilde{u}(z) - u(z))$ as a solution of $L(z, \theta)(y(z)) = Q_0(\theta)(q(z))$. Suppose that for some $n_0 > 0$ we have constructed power series $\hat{a}(z) = \sum_{j>0} \hat{a}_j z^j$, $\hat{q}(z) = \sum_{n>0} \hat{q}_n z^n$, and $\hat{y}(z) = \sum_{n\geq 0} \hat{y}_n z^n$ with nonnegative coefficients satisfying

1. For all $j > 0$ and $n \geq n_0$,

$$n \sum_{t=0}^{\tau(n)-1} \left| [X^t] \frac{Q_j(\lambda + n + X)}{X^{-\mu(\lambda+n)} Q_0(\lambda + n + X)} \right| \leq \hat{a}_j.$$

2. For all $n \geq n_0$ and $k \geq 0$, $|q_{\lambda+n,k}| \leq \hat{q}_n$.
3. $|y_{\lambda+n,k}| \leq \hat{y}_n$ for all $n < n_0$ and $k \geq 0$.

4. $|y_{\lambda+n,k}| \leq \hat{y}_n$ for all $n \geq n_0$ and $k < \mu(\lambda+n)$.

5. $\hat{y}(z)$ satsifies
$$z\hat{y}'(z) = \hat{a}(z)\hat{y}(z) + \hat{q}(z).$$

If all of these are true, we have $|z^{-\lambda}y(z)| \leq \hat{y}(z)$. The reason for dividing the differential equation by $p_r(z)$ on the right is that $\deg Q_j < \deg Q_0$, so we can expect finite values for the $\hat{a}_j$.

Now, we have
$$\sum_{j=1}^{\infty} Q_j(\theta)z^j = \frac{P(z,\theta)}{p_r(z)} - Q_0(\theta) = \frac{P(z,\theta)}{p_r(z)} - \frac{P(0,\theta)}{p_r(0)}.$$

For all differential equations arising from hypergeometric functions considered here, $\sum_{j=1}^{\infty} Q_j(\theta)z^j$ will be a finite linear combination of functions of the form $(i, k \geq 0)$
$$z^i, \quad z\frac{\partial}{\partial z}\frac{z^i}{(1-z)^k}, \quad z\frac{\partial}{\partial z}\log\left(\frac{1}{1-z}\right),$$
$$z\frac{\partial}{\partial z}\frac{z^i}{(1-z^2)^k}, \quad z\frac{\partial}{\partial z}\log\left(\frac{1}{1-z^2}\right), \quad z\frac{\partial}{\partial z}\log\left(\frac{1+z}{1-z}\right),$$
all with nonnegative coefficients as power series in z.

Remark 4.1. *This is not accurate for equations arising from Borel resummation, where the list needs to be augmented by ?.*

The coefficients of the linear combination, say $f_j(\theta)$, will be polynomials in θ. Bounding the combinations
$$n\sum_{t=0}^{\tau(n)-1}\left|[X^t]\frac{f_j(\lambda+n+X)}{X^{-\mu(\lambda+n)}Q_0(\lambda+n+X)}\right|$$
for each j and for all $n \geq n_0$ will give a valid $\hat{a}(z)$ and a nice formula for $\hat{h}(z) = \exp\int_0^z \hat{a}(z)/z dz$. It now suffices to choose a $\hat{q}(z)$ so that
$$\hat{y}(z) = \hat{h}(z)\int_0^z \frac{\hat{q}(z)/z}{\hat{h}(z)}dz$$
satisfies conditions 2 and 4.

4.4 series evaluation

For large enough τ, the solution takes the form
$$f(z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\tau-1} u_{i,j}z^{\lambda+i}\log(z)^j/j!,$$
and the coefficients $u_{i,j}$ satsify $u_{i,j} = 0$ for $j \geq \tau$. Therefore, write $u_i = \sum_{j=0}^{\tau-1} u_{i,j}\Lambda^{\tau-1-j}$ where *everything is modulo* Λ^τ. Eventually the coefficients u_i satisfy a relation of the form
$$u_n = a_1u_{n-1} + \cdots + a_su_{n-s}, \quad a_i \in \mathbb{F}(n)[\Lambda] \tag{4.2}$$
Let $M_n \in \mathbb{F}(n)[\Lambda]^{s\times s}$ be the companion matrix (with the a_i on the first row) such that
$$\begin{pmatrix} u_n \\ \vdots \\ u_{n-(s-1)} \end{pmatrix} = M_n \begin{pmatrix} u_{n-1} \\ \vdots \\ u_{n-s} \end{pmatrix}$$
Set $f_{[N_0,N_1)}(z) = \sum_{i=N_0}^{N_1-1} z^{\lambda+i}u_i(\log(z))$ where $u_i(\log(z))$ denotes $u_i \in \mathbb{F}[\Lambda]$ with $\Lambda^{\tau-1-j}$ replaced by $\log(z)^j/j!$. For the derivative $f^{(d)}(z)$ of order d we have
$$\begin{pmatrix} f^{(d)}_{[N_0,N_1)}(z) \\ ? \\ \vdots \end{pmatrix} = \sum_{i=N_0}^{N_1-1} z^{\lambda+i-d}(\Lambda+\lambda+i)^{(d)}\prod_{N_0\leq\ell\leq i} M_\ell \begin{pmatrix} u_{N_0-1} \\ \vdots \\ u_{N_0-s} \end{pmatrix}(\log(z))$$

where $x^{(d)} := x(x-1)\cdots(x-(d-1))$ on the right hand side denotes the falling factorial. Therefore, to evaluate several derivatives of f, it suffices to take the first entry of the right hand side for several values of d, where the products $\prod_{N_0 \le \ell < i} M_\ell$ can be reused. Furthermore, the final product $\prod_{N_0 \le \ell < N_1} M(\ell)$, when multiplied by the initial values $u_{N_0-1}, \dots u_{N_0-s}$, gives the final $u_{N_1-1}, \dots, u_{N_1-s}$, which are needed for the estimation of the tail $\sum_{i=N_1}^{\infty} z^i u_i(\log(z))$.

To avoid either a catastrophic linear loss of precision when the a_i are approximate quantities or a slow algorithm when the a_i are "small" exact quantities, the above sum should be evaluated via binary splitting: that is, for example

$$\sum_{i=0}^{7} z^i \prod_{0 \le \ell \le i} M_\ell = (M_0 + zM_1M_0 + z^2(M_2 + zM_3M_2)M_1M_0) \\ + z^4(M_4 + zM_5M_4 + z^2(M_6 + zM_7M_6)M_5M_4)M_3M_2M_1M_0.$$

4.5 putting everything together

This section discusses the reliable evaluation of the solution and its derivatives $f(z), f'(z), \dots, f^{(\delta-1)}(z)$, which can be writen as

$$f^{(d)}(z) = f^{(d)}_{[0,N_0)}(z) + f^{(d)}_{[N_0,N)}(z) + f^{(d)}_{[N,\infty)}(z)$$

where

$$f_{[N_0,N_1)}(z) = \sum_{N_0 \le i < N_1, j} u_{i,j} z^{\lambda+i} \log(z)^j / j!$$

The the quantities $z^{\lambda+i}\log(z)^j/j!$ for integers i and j need to be evaluated reliably. This is a problem when z is zero or a ball containing zero.

The first block $f^{(d)}_{[0,N_0)}(z)$ includes those "problematic" terms $u_{i,j} z^{\lambda+i} \log(z)^j/j!$ where $\lambda + i$ is a root of $Q_0(\theta)$. These terms are problematic because the denominators of (4.2) could vanish, thus they should be dealt with separately. For each of these terms we just evaluate $z^{\lambda+i}\log(z)^j/j!$ directly and take care when z contains zero, where the sign of $\Re(\lambda + i)$ is relevent.

The next block also requires care with respect to the evaluation of $\log(z)$. What we actually get out of the previous section is a reliable evaluation of

$$z^{d-N_0-\lambda} f^{(d)}_{[N_0,N)}(z) = \sum_{j=0}^{\tau-1} e_j \Lambda^{\tau-1-j} \in \mathbb{C}[\Lambda],$$

that is, we still have to evaluate $\sum_{j=0}^{\tau-1} e_j z^{\lambda+N_0-d} \log(z)^j/j!$ as reliably as possible. For this, it is helpful if $\Re(\lambda + N_0 - d) > 0$ which is why it is a good ideal to at least choose an $N_0 \ge \delta$.

For the final block, the majorant method produces a power series $\hat{B}(z) = \sum_{i=0}^{\infty} b_i z^i \in \mathbb{R}_{\ge 0}[z]$ with $z^N \hat{B}(z)$ majorizing the tail $\sum_{i=N}^{\infty} u_{i,j} z^i$ for all $j < \tau$. Thus to bound $|f(z)| + |f'(z)|\epsilon + \cdots + |f^{\delta-1}(z)|/(\delta-1)!\epsilon^{\delta-1}$, we need to calculate, while working in ϵ modulo ϵ^δ, a majorant (in ϵ) of $(z+\epsilon)^{\lambda+N}\log(z+\epsilon)^j/j!$ for each $j < \tau$, add these up, and multiply the sum by $B(z+\epsilon)$. Since the derivatives of $z^\delta \log(z)^j$ up to and including order $\delta - 1$ are continuous at $z = 0$, it suffices to steal z^δ from the $z^{\lambda+N}$. If the deficit $z^{\lambda+N-\delta}$ is not continuous at $z = 0$, the situation is hopeless anyways. For fixed δ we have

$$(z+\epsilon)^\delta \log(z+\epsilon)^j = \sum_{k=0}^{\delta-1} c_{j,k}(\log(z)) z^{\delta-k}\epsilon^k + O(\epsilon^\delta)$$

for certain polynomials $c_{j,k}$ of degree j satisfying $c_{j+1,k} = \log(z)c_{j,k} + \sum_{\ell=1}^{k}(-1)^{\ell-1}c_{k-\ell}/\ell$.