Homework 6

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Problem (1)**.** From the example, we know that optimal trajectories are clockwise-oriented circles centered at $(\pm 1, 0)$, one for each bang-bang control $u = \pm 1$ respectively. To reach the origin, we must eventually get on the switching curves Γ_+^1 and Γ_-^1 since they are the only circles centered at $(\pm 1, 0)$ that go through the origin. Moreover, the optimal control $u^* = -\text{sign}(\Lambda \cos(\omega t + \phi))$ flips signs and thus must switch every $\frac{\pi}{\omega} = \pi$ except that it might switch sooner at the beginning or in the end.

(a) When $x_1(0) = x_2(0) = 2$, we have the following optimal trajectory:

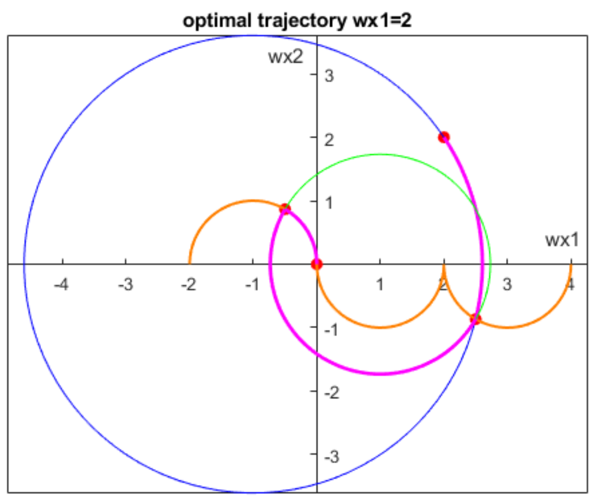


Figure 1: Magenta denotes the optimal trajectory and orange denotes the optimal switching curve. We first find clockwise-oriented circles centered at $(\pm 1, 0)$ that go through $(2, 2)$ and pick the one that reaches the switching surface the fastest. In this case it is the blue circle centered at $(-1, 0)$. Then we switch to the green circle centered at $(1, 0)$ and continue for π unit of time to reach the next switching curve which happens to be Γ_-^1 so we simply follow the singular curve to reach the origin.

Using $\widehat{\Gamma}$, we have the following trajectory:

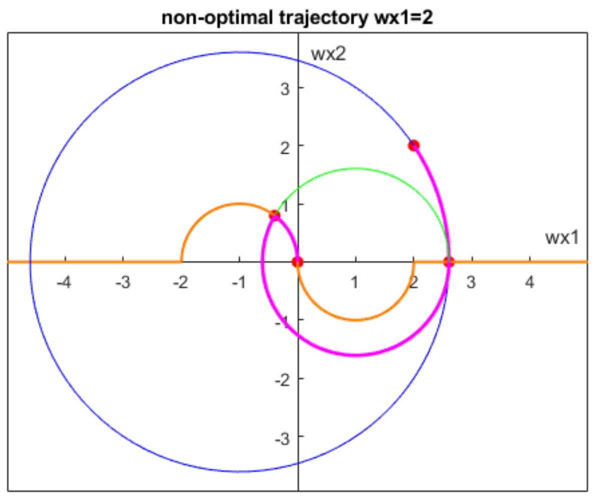
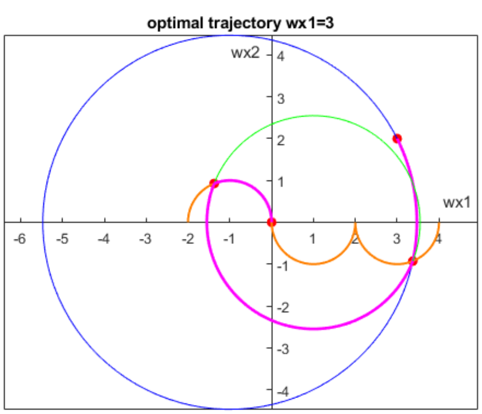
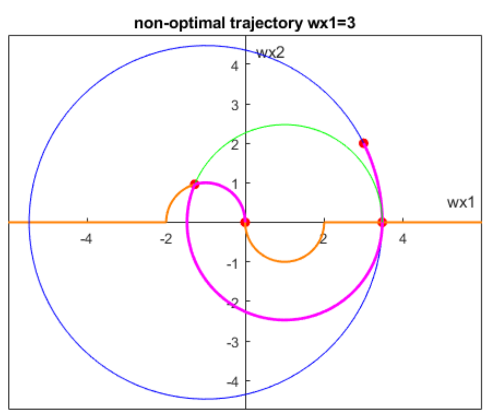


Figure 2: Magenta denotes the non-optimal trajectory and orange denotes the non-optimal switching curve. We repeat the previous first step, reach a new switching curve at $x_2 = 0$, and switch to the other center. This time we don't have to switch every π so we simply reach the next switching curve to switch.

Since $\omega = 1$, the elapsed time is the same as the angle the trajectory traced out. By adding the angles together, we obtain that $t^* = 5.0194$ and $\hat{t} = 5.1685$.

(b) We repeat the procedure for $\omega x_1 = 3, 4, 5$.





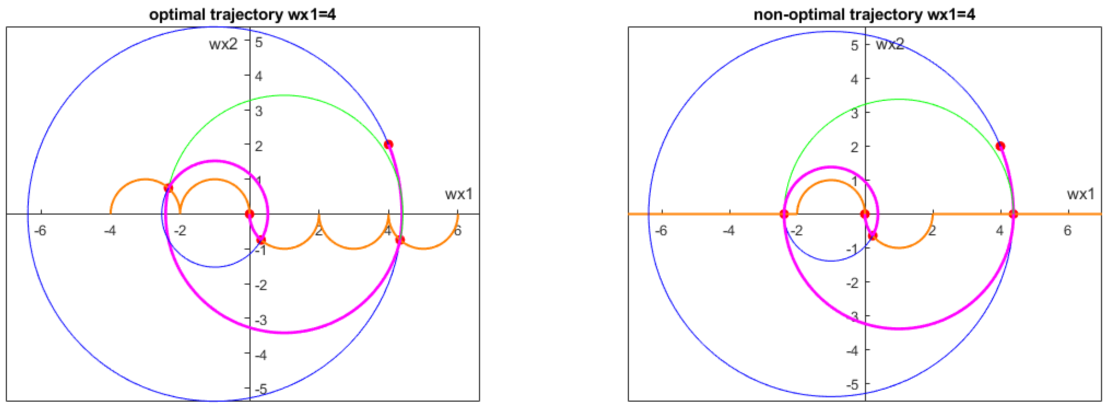
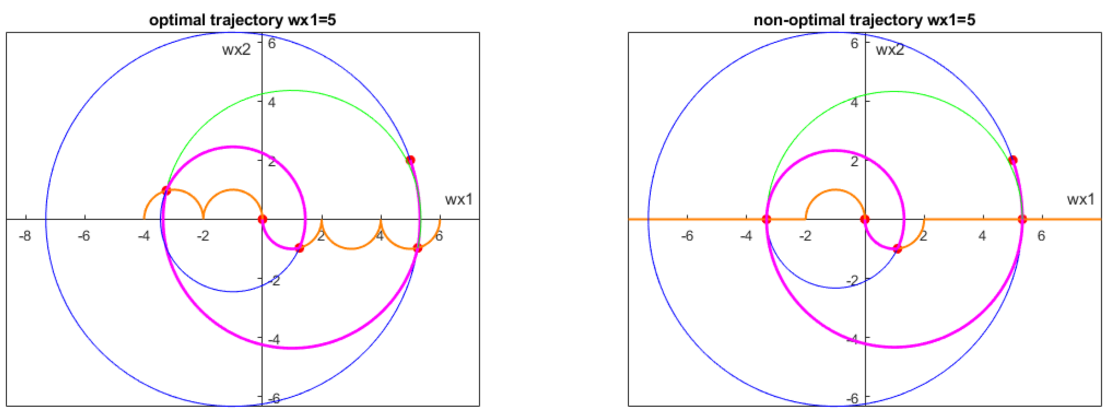


Figure 3: Both trajectories becomes more complicated. We have four arcs corresponding to three switchings.



From the figures, we can roughly see that the optimal trajectories and their non-optimal counterparts largely resembles each other. Moreover, it is not hard to imagine that as ωx_1 gets larger, the total angles traced out by both trajectories increase but their differences don't increase much. Thus the ratio $\hat{t}/t^*$ trends down to 1. There is not enough sample size in the following plot to fully illustrate that but it is a start:

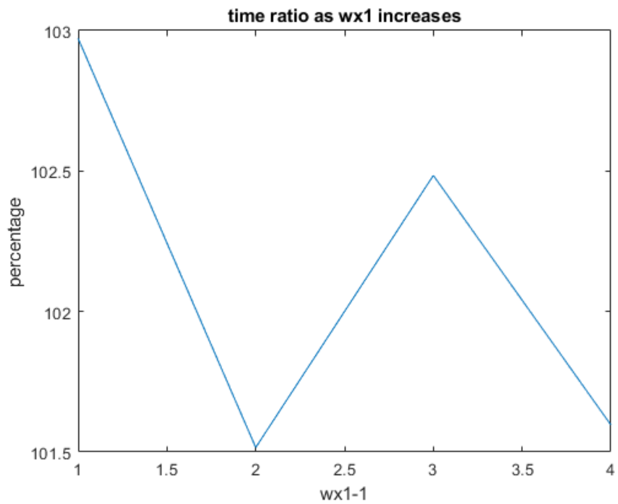


Figure 4: The ratio fluctuates but trends down.

Problem (2)**.**