Midterm

Jaden Wang

Problem (1). (a) First we augment the domain $D = \mathbb{R}^{n+m}$ and let $\overline{f}(\overline{x}) := f(x)$ and $\overline{g}_i(\overline{x}) := g_i(x) + \varepsilon_i$ where $\overline{x} = (x, \varepsilon)$. Let $\overline{F} = \begin{pmatrix} \overline{f} \\ \overline{g} \end{pmatrix}$. Then

$$\nabla \overline{F}(\overline{x}) = \begin{pmatrix} f'(x) & 0 \\ g'(x) & I_m \end{pmatrix}$$

where I_m is the $m \times m$ identity matrix. Let the constrained domain $D_0 := \{(x, \varepsilon) \in \mathbb{R}^n \times \mathbb{R}^m : \varepsilon \succeq 0\} = \mathbb{R}^n \times \mathbb{R}^m_+$. Suppose $\overline{x}_0$ is a local minimizer. We see that $\text{fcone}(D_0, \overline{x}_0) = D_0 = \text{fcone}(\mathbb{R}^n, x) \times \text{fcone}(\mathbb{R}^m_+, \varepsilon)$ (which is already convex). Thus for any $\overline{\xi} \in \text{fcone}(D_0, \overline{x}_0)$, we have $\overline{\xi} = (\eta, \xi) \in \text{fcone}(\mathbb{R}^n, x) \times \text{fcone}(\mathbb{R}^m_+, \varepsilon)$. Notice that $\text{fcone}(\mathbb{R}^m_+, \varepsilon)$ depends on whether $\varepsilon_i = 0$ *i.e.* at the boundary. If ε_i is in the interior *i.e.* $\varepsilon_i > 0$, then $\text{fcone}(\mathbb{R}_+, \varepsilon_i) = \mathbb{R}$. Otherwise, whenever $\varepsilon_i = 0$, the feasible cone of ith factor of $\mathbb{R}$ becomes $\mathbb{R}_+$.

First-order necessarily condition demands that

$$\begin{aligned} \begin{pmatrix} \mu & \lambda^T \end{pmatrix} \nabla \overline{F}(\overline{x}_0) \overline{\xi} &\geq 0 \\ \begin{pmatrix} \mu f'(x_0) + \lambda^T g'(x_0) & \lambda^T \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix} &\geq 0 \\ [\mu f'(x_0) + \lambda^T g'(x_0)]\eta + \lambda^T \xi &\geq 0. \end{aligned}$$

By setting $\eta = 0$, we have $\lambda^T \xi \geq 0$. Notice that whenever $\varepsilon_i > 0$, $\xi_i \in \mathbb{R}$ so by choosing other $\xi_j = 0$, it forces $\lambda_i = 0$. Whenever $\varepsilon_i = 0$ *i.e.* $g_i(x_0) = 0$, $\xi_i \geq 0$ so by choosing other $\xi_j = 0$, it forces $\lambda_i \geq 0$. In either case, we have $\lambda_i g_i(x_0) = 0$ and $\lambda \succeq 0$. Since $\eta \in \mathbb{R}^n$, by setting $\xi = 0$ and plugging in $\eta \neq 0$ and $-\eta$, we have $\mu f'(x_0) + \lambda^T g'(x_0) = 0$. This leads to the first-order necessary condition for the inequality constraint optimization problem: if $x_0 \in \mathbb{R}^n$ is a local minimizer of the problem, there exists $\mu \in \{0, 1\}$ and $\lambda \in \mathbb{R}^m$ such that they are not both zero and such that

$$\mu f'(x_0) + \lambda^T g'(x_0) = 0$$

$$\lambda_i g_i(x_0) = 0$$
$$\lambda \succeq 0.$$

(b) The Lagrangian for the augmented problem is

$$\mathscr{L}(\overline{x}, \mu, \lambda) = \mu \overline{f}(\overline{x}) + \lambda^T \overline{g}(\overline{x})$$

According to Theorem 3.39, for the augmented problem we define

$$J(\overline{x}_0) := \{\zeta = (\eta, \xi) \in \text{fcone}(D_0, \overline{x}_0) : f'(x_0)\eta \leq 0 \text{ and } g'(x_0)\eta + \xi = 0\}.$$

We see that when $\varepsilon_i > 0$, $\xi_i \in \mathbb{R}$, so we can always achieve $g_i'(x)\eta + \xi_i = 0$, so we can remove this inactive constraint $g_i(x)$ in our consideration of J (and thus second-order conditions).

When $\varepsilon_i = 0$ (active constraint, assume $\lambda_i > 0$), $\xi_i \geq 0$, thus we have $g_i'(x)\eta \leq 0$. Moreover, from previous part we know that by choosing $\xi = 0$, we have

$$[\mu f'(x_0) + \lambda^T g'(x_0)]\eta \geq 0$$

Thus if $f'(x_0)\eta \leq 0$, $\mu f'(x_0)\eta \leq 0$ as well, which forces $\lambda^T g'(x_0)\eta \geq 0$. Since η is any vector in the half-hyperplane that forms obtuse angle from $f'(x_0)$, we must have $\lambda_i g_i'(x_0)\eta \geq 0$ (otherwise we can just tweak to values of η to make the negative entry dominates). Since $\lambda_i > 0$, we have $g_i'(x_0)\eta \geq 0$ and thus $g_i'(x_0)\eta = 0$. Thus we eliminated ξ from the condition so J reduces to

$$J(\overline{x}_0) = J(x_0) = \{\eta \in \text{fcone}(\mathbb{R}^n, x_0) = \mathbb{R}^n : f'(x_0)\eta \leq 0 \text{ and } g_i'(x_0)\eta = 0 \;\forall\; g_i(x_0) = 0, \lambda_i > 0\}.$$

Moreover, we see that

$$\begin{aligned}
\mathscr{L}_{\overline{x}} &= \mu \overline{f}_{\overline{x}}(\overline{x}) + \lambda^T \overline{g}_{\overline{x}}(\overline{x}) \\
&= \mu \begin{pmatrix} f_x(x) & 0 \end{pmatrix} + \lambda^T \begin{pmatrix} g_x(x) & I_m \end{pmatrix} \\
\mathscr{L}_{\overline{x}\overline{x}} &= \mu \begin{pmatrix} f_{xx}(x) & 0 \\ 0 & 0 \end{pmatrix} + \lambda^T \begin{pmatrix} g_{xx}(x) & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \mathscr{L}_{xx} & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

so ξ again becomes irrelevant. Thus the second-order necessary condition from Theorem 3.39 reduces to: let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ be C^2. If x_0 is a local minimizer of the inequality constrained minimization problem, then for all $\eta \in J(x_0)$, there exists nonzero $(\mu, \lambda) \in \{0, 1\} \times \mathbb{R}^m$ such that it satisfies the first-order conditions in (a), and

$$\eta^T \mathscr{L}_{xx}(x_0, \mu, \lambda)\eta \geq 0.$$

Problem (2)**.** From homework and $D = Av^2 + BL^2 = Av^2 + B(mg)^2$, we have

$$\begin{aligned}
F(m, v(m), v'(m)) &= \frac{cv}{Av^2 + B(mg)^2}\left(1 + \frac{m}{c}v'\right) \\
\frac{\partial F}{\partial v} &= \frac{-Acv^2 + Bc(mg)^2}{(Av^2 + B(mg)^2)^2} = \frac{-c(Av^2 - B(mg)^2)}{(Av^2 + B(mg)^2)^2} \\
\frac{\partial F}{\partial v'} &= \frac{mv}{Av^2 + B(mg)^2} \\
\frac{d}{dm}\frac{\partial F}{\partial v'} &= \frac{v + mv'}{Av^2 + B(mg)^2} - \frac{mv(2Avv' + 2Bmg^2)}{(Av^2 + B(mg)^2)^2} \\
&= \frac{v + mv'}{Av^2 + B(mg)^2} - \frac{2Av^2v' + 2B(mg)^2v}{(Av^2 + B(mg)^2)^2} \\
&= \frac{(v - mv')(Av^2 - B(mg)^2)}{(Av^2 + B(mg)^2)^2}
\end{aligned}$$

Euler-Lagrange demands that

$$\begin{aligned}
F_v &= \frac{d}{dm}F_{v'} \\
(v - mv' + c)(Av^2 - B(mg)^2) &= 0 && \text{denominator} > 0 \\
\frac{dv}{v + c} &= \frac{dm}{m} \text{ or } Av^2 = B(mg)^2 \\
v(m) &= C_1 m - c \text{ or } v(m) = mg\sqrt{\frac{B}{A}}
\end{aligned}$$

Since c is a positive constant, the first equation would imply that the velocity is negative when mass is zero, which makes no physical sense. It follows that the extremal is $v(m) = mg\sqrt{\frac{B}{A}}$. Since the problem intuitively should have a maximum, and this is the only candidate, this must be the maximizer.

Problem (3)**.** For simplicity, write $T(v) = Av^2 + \frac{B}{Cv^2}$ where $A, B, C > 0$ are corresponding constants. We see that the domain of T is implicitly $\mathbb{R}_{++}$ which is a convex set. Since Av^2

and $\frac{B}{Cv^2}$ are clearly strictly convex in this domain based on their epigraphs, their sum which is T is also strictly convex (clear from definition of convex functions): if $f = g + h$ where g, h are strictly convex functions, then

$$\begin{aligned} f(tx + (1-t)y) &= g(tx + (1-t)y) + h(tx + (1-t)y) \\ &< tg(x) + (1-t)g(y) + th(x) + (1-t)h(y) \\ &= t(g+h)(x) + (1-t)(g+h)(y) = tf(x) + (1-t)f(y) \end{aligned}$$

First order condition is

$$\begin{aligned} T'(v) = 2Av - \frac{2B}{Cv^3} &= 0 \\ 2ACv^4 - 2B &= 0 \qquad\qquad C > 0, v > 0 \\ v^* &= \sqrt[4]{\frac{B}{AC}} \end{aligned}$$

Since $T''(v) = 2A + \frac{6B}{Cv^4} > 0$ for all v, we see that v^* is a strict local minimizer. Since T is strictly convex, Theorem 1.30 gives that v^* is a global minimizer and Proposition 1.31 states that this is the unique global minimizer. Let $C_p := C_{D_{par}}$, we have

$$\begin{aligned} T(v^*) &= A\sqrt{\frac{B}{AC}} + \frac{B}{C\sqrt{\frac{B}{AC}}} = \sqrt{\frac{AB}{C}} + \sqrt{\frac{AB}{C}} = 2\sqrt{C_p K W^2} \\ C_L &= \frac{W}{Cv^2} = \sqrt{\frac{C_p}{K}} \\ C_D &= C_p + KC_L^2 = 2C_p \\ C_L/C_D &= \frac{1}{2\sqrt{KC_p}} \end{aligned}$$

Problem (4)**.** First, if $a_1 = a_2 = 0$, the function is constantly zero so every (x_1, x_2) that satisfies the constraint is a global minimizer. So we assume at least one $a_i \neq 0$.

Suppose $\mu = 1$. If $\lambda = 0$, the constraint vanishes so the nonzero linear function goes to $-\infty$ and has no minimum, so we need $\lambda \neq 0$. When $\mu = 0$, $\lambda \neq 0$ by assumption. Thus in either case, $\lambda \neq 0$.

$$\mathscr{L}(x, \mu, \lambda) = \mu(a_1x_1 + a_2x_2) + \lambda(b_1x_1^2 + b_2x_2^2) = 0$$

$$\mathscr{L}_x = \begin{pmatrix} \mu a_1 + 2\lambda b_1 x_1 \\ \mu a_2 + 2\lambda b_2 x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Assume that $b_1 \neq 0$. Then we can use the constraint to solve $x_1^2 = -\frac{b_2}{b_1}x_2^2$. If $-\frac{b_2}{b_1} \geq 0$, then $x_1 = \pm\sqrt{-\frac{b_2}{b_1}}x_2$ and f becomes a linear equation in x_2 with no constraint. Since at least one $a_i \neq 0$, this unconstrained linear function goes to $-\infty$, there is no solution. Now suppose $-\frac{b_2}{b_1} < 0$, then $x_1^2 = -\frac{b_2}{b_1}x_2^2 \leq 0$ so $x_1^2 = 0$. This forces $x_1 = x_2 = 0$. Since $g'(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the solution is abnormal. This candidate satisfies the first-order conditions for $\mu = 0$. Thus λ can be any nonzero number. The second order condition requires that $\mathscr{L}_{xx}(x, 0, \lambda) \succ 0$ since null space of g' is $\mathbb{R}^2$. We see that

$$\mathscr{L}_{xx} = \begin{pmatrix} 2b_1\lambda & 0 \\ 0 & 2b_2\lambda \end{pmatrix}$$

which can be made positive definite since $\frac{b_2}{b_1} > 0$ and we can choose $\lambda \neq 0$ so that $\lambda b_1 > 0$. It is thus a strict local minimizer. Then the minimum of f is 0 in this case.

By symmetry of the problem, the case when $b_2 \neq 0$ is the same. It remains to check when $b_1 = b_2 = 0$. But this means that the nonzero linear function is unconstrained and goes to $-\infty$.

Hence, the only solution to the nontrivial minimization problem is when $b_1 \neq 0$ or $b_2 \neq 0$ with $f(0,0) = 0$.

Problem (5). (a) We have

$$F(t, x, \dot{x}) = \frac{1}{2}((\dot{x} - x)^2 - \alpha x^2)$$
$$F_x = -\dot{x} + (1 - \alpha)x$$
$$F_{\dot{x}} = \dot{x} - x$$

Euler-Lagrange demands

$$\frac{d}{dt}F_{\dot{x}} = \ddot{x} - \dot{x} = -\dot{x} + (1 - \alpha)x$$
$$\ddot{x} = -(\alpha - 1)x$$
$$x(t) = A\cos\sqrt{\alpha - 1}t + B\sin\sqrt{\alpha - 1}t \qquad \alpha > 1$$

$$x(0) = A + 0 = 0$$

Thus the extremal trajectory is $x(t) = B \sin \sqrt{\alpha - 1}t, B \in \mathbb{R}$. It can be verified by direct integration that J is 0 for any B.

(b) We see that $F_{\dot{x}\dot{x}} = 1 > 0$ so the accessory minimization problem is regular. For $\alpha = 2$ and $T = \pi$, we have $F_x = -x - r$, $F_{xx} = -1$, $F_r = r - x$, $F_{rr} = 1$, and $F_{xr} = -1$. Let the perturbation be f. Then

$$\omega(t, x, r) = -\frac{1}{2} f^2 - f\dot{f} + \frac{1}{2}\dot{f}^2$$

The Legendre condition requires

$$\begin{aligned}
\omega_f &= \omega_{rt} + \omega(rf)\dot{f} + \omega_{rr}\ddot{f} \\
-f - \dot{f} &= -\dot{f} + \ddot{f} - \dot{f} + \ddot{f} \\
2\ddot{f} - \dot{f} + f &= 0 \\
f(t) &= e^{\frac{t}{4}} \left(C \cos \frac{\sqrt{7}}{4} t + D \sin \frac{\sqrt{7}}{4} t \right) \\
f(0) &= C = 0 \\
f(\pi) &= De^{\frac{\pi}{4}} \sin \frac{\sqrt{7}\pi}{4} = 0 \Rightarrow D = 0
\end{aligned}$$

Thus $f(t) \equiv 0$ and there is no conjugate point between $[0, \pi]$ as everything vanishes in that interval.