1 Obstruction Theory Revisited

Suppose $A \subseteq X$ and a map $f : A \to Y$, can we extend f to a map $X \to Y$?

As usual we assume

(a) (X, A) is a relative CW-complex *i.e.* $X^{(-1)} = A$.

(b) Y is n-simple for all n, *i.e.* $\pi_1(Y)$ acts trivially on $\pi_n(Y)$ which simplies $\pi_n(Y) = [S^n, Y]$ (unbased).

Theorem 1.1

Given (X, A) satisfying a) and Y satisfying b) and $f : X^{(n)} \to Y$. Then

(1) there exists a cocycle

$$\widetilde{\sigma}(f) \in C^{n+1}(X, A; \pi_n(Y))$$

which vanishes iff f extends to $X^{(n+1)}$.

(2) $\sigma(f) = [\widetilde{\sigma}(f)] \in H^{n+1}(X, A; \pi_n(Y))$ vanishes $\Leftrightarrow$ $f|_{x^{(n-1)}}$ extends to $X^{(n+1)}$.

Proof. JUst like in Section A,

$$\widetilde{\sigma}(f) : C^{CW}_{n+1}(X, A) \to \pi_n(Y)$$

is defined as follows: e_i^{n+1} is attached to $X^{(n)}$ by $\phi_i : S^n \to Y$ so

$$\widetilde{\sigma}(e_i^{n+1}) = [f \circ \phi_i] \in [S^n, Y] = \pi_n(Y)$$

Exercise:

(1) $\widetilde{\sigma}(f) = 0 \Leftrightarrow f$ extends to $X^{(n+1)}$.

(2) $\widetilde{\sigma}(f)$ is unchanged if you homotop f.

(3) $\delta\widetilde{\sigma}(f) = 0$.

(4) Given $f, g : X^{(n)} toY$ s.t. $f = g$ on $X^{(n-1)}$ then there exists $\tau(f, g) iin C^n(X, A; \pi_n(Y))$ s.t.

$$\delta\tau(f, g) = \widetilde{\sigma}(f) - \widetilde{\sigma}(g)$$

(5) By varying the homotopy class of f on $X^{(n)}$ relative to $X^{(n-1)}$, we can change $\widetilde{\sigma}(f)$ by an arbitrary coboundary.

The theorem follows.

$\square$

Theorem 1.2

Let $f, g : X \to Y$ be given (satisfying a), b)), and $H : X^{(n)} \times I \to Y$ a homotopy from $f|_{X^{(n)}} \to g|_{X^{(n)}}$. Then the obstruction to extending H to $X^{(n+1)} \times I \to Y$ lies in $H^n(X, A; \pi_n(Y))$.

Proof. Theorem 19 says we get an obstruction in $H^{n+1}(X \times I, ((A \times I) \cup (X \times \{0,1\})); \pi_n(Y))$.

Let $U_1 = X \times [0, \frac{3}{4}]$ and $V_1 = (X \times \{0\}) \cup (A \times [0, \frac{3}{4}])$, $U_2 = X \times [\frac{1}{4}, 1]$, $V_2 = (A \times [\frac{1}{4}, 1] \cup (X \times \{1\}))$.

By Lemma I.9, since (X, A) is a NDR-pair, we know V_1 is a retract of U_1. So $H^n(U_1, V_1) = 0$.

$$0 = H^n(U_1, V_1) \oplus H^n(U_2, V_2) \to H^n(U_1 \cap U_2, V_1 \cap V_2) \to H^{n+1}(U_1 \cup U_2, V_1 \cup V_2) \to H^{n+1}(U_1, V_1) \oplus H^{n+1}(U$$

So $H^n(X, A) \cong H^n(X \times [\frac{1}{4}, \frac{3}{4}], A \times [\frac{1}{4}, \frac{3}{4}]) \cong H^{n+1}(U_1 \cup U_2, V_1 \cup V_2)$. $\square$

Theorem 1.3

Let (X, A) be a relative CW complex and Y n-simple space $\forall\, n$. If $\pi_k(Y)\ \forall\, k < n-1$, then for any $f : A \to Y$, there exists an extension $\widetilde{f} : X^{(n)} \to Y$ and the obstruction $[\widetilde{\sigma}(\widetilde{f})]$ only depends on f and is denoted $\gamma^{n+1}(f)$, the **primary obstruction**.

Moreover, if $g : (X', A') \to (X, A)$, then

$$g^*(\gamma^{n+1}(f)) = \gamma^{n+1}(f \circ g).$$

Proof. Same as proof of Theorem 4. $\square$

Theorem 1.4 (Brown Representation Theorem)

Let (X, A) be a relative CW-pair, there is a natural bijection

$$[(X, A), K(\pi, n), x_0] \longleftrightarrow H^n(X, A; \pi)$$

Proof. BY Hurewicz, $H_k(K(\pi, n)) = 0 \ \forall \ k < n$ and $H_n(K(\pi, n)) = \pi$. By the Universal Coefficients Theorem,

$$H^n(K(\pi, n); \pi) \cong \mathrm{Hom}(H_n(K(\pi, n)), \pi) \oplus (H_{n-1}(K(\pi, n)), \pi) = \mathrm{Hom}(\pi, \pi).$$

as is 0. Let $\iota \in H^n(K(\pi, n); \pi)$ corresponds to 1_π. Define $\psi : [(X, A), (K(\pi, n); x_0)] \to H^n(X, A; \pi), f \mapsto f^*\iota$.

Note since $\pi_n(K(\pi, n)) = 0$ for $k < n$. The first obstruction to homotopying a map $f : (X, A) \to (K(\pi, n), x_0)$ to be constant lives in $H^n((X, A); \pi)$.

Claim 1.5. This obstruction is $\psi(f)$.

Proof. By naturality, it suffices to check that ι is the primary obstruction to homotopying the identity map on $K(\pi, n)$ to a constant map.

We know $(K(\pi, n))^{(n-1)} = \{x_0\}$. So the identity and constant map agree on $n-1$ skeleton. The n-cell e_i^n corresponding to a generator of $\pi = \pi_n(K(\pi, n))$.

Claim 1.6. ψ is onto.

Let $\alpha \in H^n(X, A; \pi)$ so there exists $\widetilde{\alpha} \in C^n(X, A; \pi)$ s.t. $\alpha = [\widetilde{\alpha}]$, $\widetilde{\alpha} : C_n(X, A) \to \pi$, define f_α to be constant on $X^{(n-1)}$ and for each n-cell e_i^n of X. Let $f_\alpha : e_i^n to K(\pi, n)$ represents $[f_\alpha(e_i^n)] = \widetilde{\alpha}(e_i^n) \in \pi = \pi_n(K(\pi, n))$. □

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