PDE Midterm

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Problem (1)**.** We see that $a = x^2, b = xy, c = y^2$. So $d = ac - b^2 = x^2y^2 - (xy)^2 = 0$, and the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is clearly not identically zero. So the equation is parabolic on $\mathbb{R}^2$.

To change into the standard form, we wish to do an invertible change of coordinate using $s(x, y)$ and $t(x, y)$. Under this transformation, the principal linear part becomes $L_0u = a^*u_{ss} + 2b^*u_{st} + c^*u_{tt}$, where $a^*(x, y), b^*(x, y), c^*(x, y)$ are the coefficients after applying chain rules. In particular, $a^*(x, y) = as_x^2 + 2bs_xs_y + cs_y^2$. If we wish the u_{ss} term to vanish, we must set $a^* = 0$. This means that $s(x, y)$ satisfies the characteristic equation of L_0u. Let us find all solutions of the characteristic equation. If $\phi(x, y)$ is such solution, this means that each level set of $\phi(x, y)$ gives a characteristic curve. Locally, the level set of $\phi(x, y)$ can be expressed as $y(x)$ (or $x(y)$ if $\phi_y = 0$) by the implicit function theorem.

$$\begin{aligned} 0 = \frac{d\phi}{dx} &= \phi_x + \frac{dy}{dx}\phi_y \\ \frac{dy}{dx} &= -\frac{\phi_x}{\phi_y} \end{aligned}$$

We can then substitute this into a^*. If $x = 0$, then $a = 0$ which forces $b = 0$. Then for points $(0, y)$, the equation is already in the standard form. Likewise for $y = 0$. WLOG assume $x > 0$ and $y > 0$. We have

$$\begin{aligned} a\left(\frac{dy}{dx}\right)^2 + 2b\frac{dy}{dx} + c = 0 \\ \frac{dy}{dx} &= \frac{2b \pm \sqrt{4b^2 - 4ac}}{2a} \\ &= \frac{b \pm \sqrt{b^2 - ac}}{a} \\ &= \frac{b}{a} \\ &= \frac{\sqrt{ac}}{a} \\ &= \sqrt{\frac{c}{a}} \\ &= \frac{y}{x} \end{aligned}$$

Solving this gives one family of real solutions $y = Cx$ for each level set $\phi(x, y) = C$. Thus define $s(x, y) = \frac{y}{x}$ for $x \neq 0$. Notice that since $d^* = a^*c^* - b^{*2} = -b^{*2}$ is the determinant, which is invariant under a change of basis, $-b^{*2} = d = 0$ which forces $b^* = 0$, which also forces $c^* \neq 0$ since A is not identically zero and shall remain so under change of basis. Thus we just need to choose an $t(x, y)$ that is independent from $s(x, y)$ so the Jacobian is nonsingular. We see that $t(x, y) = x$ will do, since

$$\det J = \det \begin{pmatrix} s_x & s_y \\ t_x & t_y \end{pmatrix} = \begin{pmatrix} s_x & \frac{1}{x} \\ 1 & 0 \end{pmatrix} = -\frac{1}{x} \neq 0.$$

It follows that under this change of coordinates, the equation reduces to

$$\begin{aligned} c^* u_{tt} &= 0 \\ u_{tt} &= 0 \end{aligned}$$

which is in the standard form.

Problem (2). (a) This is Burger's equation so the intuition from homework is that a global solution exists iff u_x doesn't blow up in finite time. To make this precise, we solve the problem using the method of characteristic first.

The initial curve S_0 is parametrized by $t_0 = 0, x_0 = s, z_0 = u_0(s)$. It is the graph of $u_0(x)$ over the domain $\{0\} \times \mathbb{R}$. From the equation we have $a = 1, b = u, c = -2u$. Since

$$\det \begin{pmatrix} a & b \\ t_0'(s) & x_0'(s) \end{pmatrix} = \det \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = 1 \neq 0,$$

the projection of S_0 onto (t, x)-plane C is not characteristic so u can flow out of the initial curve. Now we can solve the characteristic curves using the characteristic equation:

$$\begin{cases} \frac{dt}{d\tau} = 1, & t(0) = 0 \qquad \Rightarrow t = \tau \\ \frac{dx}{dt} = z, & x(0) = s \\ \frac{dz}{dt} = -2z, & z(0) = u_0(s) \qquad \Rightarrow u(s, t) = z(s, t) = u_0(s)e^{-2t} \end{cases}$$

For a global solution to exist, intuitively we want u to flow along characteristic curve for forever. Thus along the characteristic curve, we cannot have u_x goes to infinity in finite time, or the flow would terminate. Thus we investigate $w := u_x$. Taking partial derivative on both sides of the Burger's equation, we have

$$\begin{aligned}
(u_t + uu_x)_x &= -2u_x \\
u_{xt} + u_x^2 + u_{xx} &= -2u_x \\
w_t + uw_x &= -2w - w^2 \\
\dot{w} &= -2w - w^2
\end{aligned}$$

Moreover, since $u_0(s) \in C^1(\mathbb{R})$, we have $w(s,0) = \left(u_0(s)e^{-2\cdot 0}\right)_x = u_0'(s)$. Using the method of characteristic again, we see that along the characteristic curve, w satisfies

$$\begin{cases} \frac{dw}{dt} = -2w - w^2 \\ w(s,0) = u_0'(s) \end{cases}$$

We compute

$$\begin{aligned}
-\frac{dw}{w(w+2)} &= dt \\
-\frac{1}{2}\left(\frac{1}{w} - \frac{1}{w+2}\right) dw &= dt \\
\frac{1}{2}\ln|w+2| - \frac{1}{2}\ln|w| &= t + C' \\
\frac{1}{2}\ln\left|\frac{w+2}{w}\right| &= t + C' \\
1 + \frac{2}{w} &= Ce^{2t} \\
w &= \frac{2}{Ce^{2t} - 1} \\
w(s,0) = \frac{2}{C-1} &= u_0'(s) \\
C &= \frac{2}{u_0'(s)} + 1 \\
w(s,t) &= \frac{2}{\left(\frac{2}{u_0'(s)} + 1\right)e^{2t} - 1}
\end{aligned}$$

We see that for a fixed s (*i.e.* on the characteristic curve), $w(s,t)$ blows up when the denominator approaches 0. This happens when

$$\left(\frac{2}{u_0'(s)} + 1\right)e^{2t} = 1$$

$$
\begin{aligned}
e^{2t} &= \frac{u_0'(s)}{2+u_0'(s)} \\
t &= \frac{1}{2}\ln\left(\frac{u_0'(s)}{2+u_0'(s)}\right)
\end{aligned}
$$

This is the time of w blowing up. But if $\frac{u_0'(s)}{2+u_0'(s)} \leq 0 \Leftrightarrow -2 \leq u_0'(s) \leq 0$, or if $t < 0 \Leftrightarrow u_0'(s) \geq 0$, then t would not have a solution in the range $t \geq 0$ so w would not blow up. Therefore, the necessary condition for u_0 is that for all $x \in \mathbb{R}$, $u_0'(x) \geq -2$.

We now show that this is also the sufficient condition. Suppose the condition is true. Notice that since $u_0(x)$ has bounded C^1-norm, $u(x,0)$ is Lipschitz continuous in x. Along each characteristic curve, for $t > 0$ we have $u(x,t) = u(x,0)e^{-2t}$ which is also Lipschitz continuous in x. Now consider $\frac{dx}{dt} = u$. Picard's theorem guarantees the uniqueness of $x(t)$ that passes through each (x,t). That is, no two characteristic curve collide in the (x,t)-plane. Since the determinant condition is always satisfied, as long as u_x doesn't blow up, we can let $u(x,t)$ flow out via the formula $u(x,t) = u_0(x)e^{-2t}$ unobstructedly and obtain a unique global solution. We have shown that u_x doesn't blow up exactly when $u_0'(x) \geq -2$. Since $u_0(x)$ is smooth, ODE theory yields that the global solution $u(x,t)$ is smooth as well.

We remark that this result can also be obtained via the phase line diagram of $\frac{dw}{dt} = -w(w+2)$ along a characteristic curve. We see that the critical points are 0 and 2, where the solution is constant (by uniqueness). If the initial value is at $w < -2$, we see that $\frac{dw}{dt} < 0$ so $w \to \infty$ as $t \to \infty$, *i.e.* u_x blows up. If the initial value is at $-2 < w < 0$ or $w > 0$, we see that $w \to 0$ as $t \to \infty$. Taken together, u_x doesn't blow up when the initial value $w(0) = u_0'(x) \geq -2$, which agrees with the above analysis.

(b) We see that any $u(x,t)$ is the result of flow along some characteristic curve originated from the initial surface. Recall along any characteristic curve, $u(x,t) = u_0(x)e^{-2t}$. Since $u_0(x)$ is bounded, $|u_0(x)| \leq M$ for some $M > 0$. Then

$$
\begin{aligned}
\sup_{x\in\mathbb{R}} |u(x,t)| &= \sup_{x\in\mathbb{R}} |u_0(x)e^{-2t}| \\
&= \sup_{x\in\mathbb{R}} |u_0(x)||e^{-2t}| \\
&\leq Me^{-2t}
\end{aligned}
$$

Thus we see that as $t \to \infty$, $\sup_{x\in\mathbb{R}} |u(x,t)| \to 0$, as desired.

Problem (3)**.** We guess that the solution has the form

$$u(x) = \frac{1}{|x-y|},$$

where $y \in B(0,1)$ is chosen to satisfy the initial data. We observe that

$$\frac{2}{\sqrt{7+4\sqrt{3}x_3}} = \frac{1}{\sqrt{\frac{7}{4}+\sqrt{3}x_3}}$$

whose denominator looks like a norm. So for $x \in \partial B(0,1)$, we solve

$$\frac{7}{4} + \sqrt{3}x_3 = |x-y|^2 = |x|^2 - 2x\cdot y + |y|^2 = 1 - 2x\cdot y + |y|^2$$

$$\begin{cases} 1 + y_1^2 + y_2^2 + y_3^2 = \frac{7}{4} \\ -2x\cdot y = \sqrt{3}x_3 \end{cases}$$

Thus we obtain $y_1 = y_2 = 0$ and $y_3 = -\frac{\sqrt{3}}{2}$, which indeed lies inside $B(0,1)$. Thus,

$$u(x) = \frac{1}{\sqrt{x_1^2 + x_2^2 + (x_3 + \sqrt{3}/2)^2}}.$$

Since $u(x) > 0$, the bound depends on how small the denominator gets. Reverse triangle equality yields $|x-y| \geq ||x|-|y|| = ||x| - \sqrt{3}/2| \geq 1 - \sqrt{3}/2$ since $|x| \geq 1$. Thus the denominator is lower bounded by a constant, which means $u(x)$ is upper-bounded by a constant as well. Thus it is bounded.

Problem (4)**.** We know that $\rho^2 \cos 2\theta$ is harmonic, multiplying a harmonic function by a scalar keeps it harmonic (linearity of differentiation), and adding it by a constant keeps it harmonic. Thus,

$$u(\rho,\theta) = 1 + \frac{1}{2}\left(1 + \left(\frac{\rho}{r}\right)^2 \cos 2\theta\right)$$

is harmonic in $D(0,r)$. When $\rho = r$, we have

$$u(r,\theta) = 1 + \frac{1}{2}(1+\cos 2\theta) = 1 + \cos^2\theta.$$

Thus $u(\rho,\theta)$ is the solution.

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Name: ______________________________ GT ID#: ______________________________

Guideline: Please read the following carefully.
Print your name first. Remember to show all your work; including all intermediate steps and also explain in words how you are solving a problem. Partial credits are available for most problems. **Correct answers without major steps will only receive minor portion of the credits of the whole problem.** Work by your own, no discussion with others is permitted.

Problem 1 (30 pts) Identify the types of the following equation

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0,$$

and then transfer it into standard form.

(Additional page for Problem 1)

Problem 2 (30 pts) Consider the following initial value probem

$$\begin{cases} u_t + uu_x = -2u, \ \ x \in \mathbf{R}, t > 0, \\ u(x,0) = u_0(x) \in C^1(\mathbf{R}). \end{cases}$$

Here, $u_0(x)$ has bounded C^1-norm.

a) (20 pts) Determine the sufficient and necessary conditions on the initial data $u_0(x)$ for this problem to have a unique global smooth solution.

b) (10 pts) If $u_0(x)$ satisfies the conditions found in part a), prove that the global solution $u(x,t)$ satisfies that $\|u(x,t)\|_{L^\infty(\mathbf{R})}$ converges to zero as t goes to infinity.

Problem 3 (20 pt) Let $B(0,1)$ be the unit ball in $\mathbf{R^3}$ centered at the origin. Find a bounded solution to the following Dirichlet problem outside $B(0,1)$

$$\begin{cases} -\Delta u(x) = 0, \ |x| > 1, \\ u(x) = \dfrac{2}{\sqrt{7 + 4\sqrt{3}x_3}}, \ for \ |x| = 1. \end{cases}$$

Problem 4 (20 pt) Let $D(0, r)$ be the disk on $\mathbf{R^2}$ centered at the origin with radius r with boundary C . Find the function $u(\rho, \theta)$ in polar coordinates so that it is harmonic on $D(0, r)$ and $u(r, \theta) = 1 + cos^2(\theta)$ on C.

Hint: $cos^2(\theta) = \frac{1}{2}(1 + cos(2\theta))$.