Homework 8

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Problem (1)**.** Compute the coefficients of the Riemann curvature tensor R in terms of the Christoffel symbols Γ_{ij}^k.

Proof. We use Einstein notation throughout.

$$\begin{aligned}
R_{ijk}^{\ell}\frac{\partial}{\partial x_\ell} &= R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} \\
&= \left(\nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}} - \nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}} + \nabla_{\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]}\right)\frac{\partial}{\partial x_k} \\
&= \nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_k} - \nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k} \\
&= \nabla_{\frac{\partial}{\partial x_j}}\left(\Gamma_{ki}^s\frac{\partial}{\partial x_s}\right) - \nabla_{\frac{\partial}{\partial x_i}}\left(\Gamma_{kj}^s\frac{\partial}{\partial x_s}\right) \\
&= \frac{\partial \Gamma_{ki}^s}{\partial x_j}\frac{\partial}{\partial x_s} + \Gamma_{ki}^s\Gamma_{sj}^\ell\frac{\partial}{\partial x_\ell} - \frac{\partial \Gamma_{kj}^s}{\partial x_i}\frac{\partial}{\partial x_s} - \Gamma_{kj}^s\Gamma_{si}^\ell\frac{\partial}{\partial x_\ell} && \text{Leibniz rule} \\
&= \frac{\partial \Gamma_{ki}^\ell}{\partial x_j}\frac{\partial}{\partial x_\ell} + \Gamma_{ki}^s\Gamma_{sj}^\ell\frac{\partial}{\partial x_\ell} - \frac{\partial \Gamma_{kj}^\ell}{\partial x_i}\frac{\partial}{\partial x_\ell} - \Gamma_{kj}^s\Gamma_{si}^\ell\frac{\partial}{\partial x_\ell} && \text{reindexing} \\
&= \left(\frac{\partial \Gamma_{ik}^\ell}{\partial x_j} - \frac{\partial \Gamma_{jk}^\ell}{\partial x_i} + \Gamma_{ik}^s\Gamma_{js}^\ell - \Gamma_{jk}^s\Gamma_{is}^\ell\right)\frac{\partial}{\partial x_\ell}. && \nabla \text{ symmetric}
\end{aligned}$$

Hence, the coefficients of the Riemann curvature tensor is

$$R_{ijk}^\ell = \frac{\partial \Gamma_{ik}^\ell}{\partial x_j} - \frac{\partial \Gamma_{jk}^\ell}{\partial x_i} + \Gamma_{ik}^s\Gamma_{js}^\ell - \Gamma_{jk}^s\Gamma_{is}^\ell.$$

□

Problem (2)**.** Show that the curvature of $\mathbb{R}^n$ is zero by (i) using the formula from the last exercise, and (ii) using the abstract definition of R in terms of the covariant derivative ∇.

Proof. (i) Recall that

$$\Gamma_{ij}^k = \frac{1}{2}g^{sk}\left(-\frac{\partial g_{ij}}{\partial x_s} + \frac{\partial g_{js}}{\partial x_i} + \frac{\partial g_{si}}{\partial x_j}\right).$$

In $\mathbb{R}^n$, $g^{sk} = g_{sk}^{-1} = \delta_{sk}$. Thus

$$\Gamma_{ij}^k = \frac{1}{2}\left(-\frac{\partial g_{ii}}{\partial x_i} + \frac{\partial g_{ii}}{\partial x_i} + \frac{\partial g_{ii}}{\partial x_i}\right) = 0.$$

We immediately have

$$R^{\ell}_{ijk} = 0.$$

Therefore, the curvature is zero.

(ii) In $\mathbb{R}^n$, a global canonical basis enables $\nabla_X Z = (X(Z^1, \ldots, X(Z^n)) =: X(Z)$. Thus we have

$$\begin{aligned} R(X,Y)Z &= (\nabla_Y \nabla_X - \nabla_X \nabla_Y + \nabla_{[X,Y]})Z \\ &= YX(Z) - XY(Z) + XY(Z) - YX(Z) = 0. \end{aligned}$$

$\square$

Problem (3)**.** Compute the curvature of the hyperbolic plane H^2 using the formula from the first exercise.

Proof. Recall that the metric tensor of H^2 is $g_{ij} = \delta_{ij}/y^2$. Thus $g^{ij} = \delta_{ij} y^2$, and

$$\begin{aligned} &\Gamma^1_{11} = 0 \\ &\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2} y^2 \left(-0 - \frac{2}{y^3} + 0 \right) + 0 = -\frac{1}{y} \\ &\Gamma^1_{22} = \frac{1}{2} y^2 \left(-0 + 0 + 0 \right) = 0 \\ &\Gamma^2_{11} = 0 + \frac{1}{2} y^2 \left(\frac{2}{y^3} + 0 + 0 \right) = \frac{1}{y} \\ &\Gamma^2_{12} = \Gamma^2_{21} = 0 + \frac{1}{2} y^2 \left(-0 + 0 + 0 \right) = 0 \\ &\Gamma^2_{22} = 0 + \frac{1}{2} y^2 \left(\frac{2}{y^3} - \frac{2}{y^3} - \frac{2}{y^3} \right) = -\frac{1}{y}. \end{aligned}$$

We define $R_{ijk\ell} := R^s_{ijk} g_{s\ell}$. Using identities, we obtain

$$\begin{aligned} R_{1111} &= -R_{1111} \Leftrightarrow R_{1111} = 0 \\ R_{1112} &= R_{1211} = -R_{1121} = -R_{2111} \\ &= -R_{1112} \Leftrightarrow R_{1112} = 0 \\ R_{1122} &= R_{2211} \\ &= -R_{1122} \Leftrightarrow R_{1122} = 0 \end{aligned}$$

$$
\begin{aligned}
R_{1221} &= R_{2112} = -R_{2121} = -R_{1212} \\
&= R^1_{122}g_{11} + R^2_{122}g_{21} \\
&= \left(\frac{1}{y^2} - 0 + \frac{1}{y^2} + 0 - 0 - \frac{1}{y^2}\right)\frac{1}{y^2} + 0 = \frac{1}{y^4} \\
R_{1222} &= R_{2212} = -R_{2122} = -R_{2221} \\
&= -R_{1222} \Leftrightarrow R_{1222} = 0 \\
R_{2222} &= -R_{2222} \Leftrightarrow R_{2222} = 0.
\end{aligned}
$$

There are a total of $2^4 = 16$ terms so we have computed all of them. Therefore, we have

$$
\begin{aligned}
R_{1212} &= \langle R(E_1, E_2)E_1, E_2\rangle = -\frac{1}{y^4} \\
K(E_1, E_2) &= \frac{R_{1212}}{\|E_1 \wedge E_2\|^2} \\
&= \frac{R_{1212}}{\|E_1\|^2\|E_2\|^2 - \langle E_1, E_2\rangle^2} \\
&= \frac{-\frac{1}{y^4}}{\frac{1}{y^2}\frac{1}{y^2} - 0} \\
&= -1.
\end{aligned}
$$

□

Problem (4)**.** Compute the curvature of the unit sphere S^2 using the formula from the first exercise.

Proof. Recall that the unit sphere under spherical basis $\{\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\theta}\}$ (longitude, latitude) has metric tensor $G = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\phi \end{pmatrix}$.

$$
\begin{aligned}
\Gamma^1_{11} &= 0 \\
\Gamma^1_{12} &= \Gamma^1_{21} = 0 \\
\Gamma^1_{22} &= \frac{1}{2}\left(-2\sin\phi\cos\phi + 0 + 0\right) = -\sin\phi\cos\phi \\
\Gamma^2_{11} &= 0 \\
\Gamma^2_{12} &= \Gamma^2_{21} = \frac{1}{2}\frac{1}{\sin^2\phi}\left(-0 + 0 + 2\sin\phi\cos\phi\right) = \frac{\cos\phi}{\sin\phi} = \cot\phi
\end{aligned}
$$

$$\Gamma^2_{22} = \frac{1}{2} \sin^2 \phi \, (-0 + 0 + 0) = 0.$$

As before, curvature is 0 except for the following terms:

$$\begin{aligned}
R_{\phi\theta\theta\phi} = R_{1221} &= R_{2112} = -R_{2121} = -R_{1212} \\
&= R^1_{122} g_{11} + R^2_{122} g_{21} \\
&= \left(0 + \cos^2 \phi - \sin^2 \phi + 0 - \frac{\cos \phi}{\sin \phi} \sin \phi \cos \phi - 0 - 0 \right) \cdot 1 + 0 \\
&= -\sin^2 \phi.
\end{aligned}$$

Thus $R_{1212} = \sin^2 \phi$ and the sectional curvature is

$$\begin{aligned}
K(E_1, E_2) &= \frac{R_{1212}}{\|E_1\|^2 \|E_2\|^2} \\
&= \frac{\sin^2 \phi}{\sin^2 \phi} \\
&= 1.
\end{aligned}$$

□