1 Homotopy group and CW-complexes

Recall if A is a top space, and $f : \bigsqcup_{i\in I} S^{n-1} \to A$ then $X = A \cup_f (\bigsqcup_{D^n}) = A \bigsqcup (\bigsqcup D^n)/\sim$ where $x \in \partial(\bigsqcup_{D^n})$ is identified with $f(x) \in A$ is said to be obtained from A by attaching n cells.

A **relative CW-pair** is a pair (X, A) s.t.

(1) X is a top space.

(2) A is a closed subspace.

(3) There exists a sequence of spaces $X^{(n)}, n = -1, 0, 1, \dots$ called n-skeleton s.t.

   (a) $X^{(-1)} = A$.

   (b) $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching n-cells.

   (c) $X = \bigcup_{i=1}^{\infty} X^{(i)}$.

   (d) $B \subseteq X$ is closed iff $B \cap X^{(n)}$ closed for all n.

   If $X^{(n)}$ for some n then we say (X, A) is an ***n*-dimensional CW-pair**. Otherwise infinite. If $A = \emptyset$, then X is a CW-complex. If X has a finite number of cells then (d) is automatically ignored.

exercise: (X, A) a CW-pair then X/A is a CW complex.

Example 1.1 (1) A 1-dimensional CW-complex is a graph.

(2) any surface as a 2-dimensional CW-complex. Any n-manifold is a CW-complex.

(3) If X, Y are CW-complexes, then so is $X \times Y$. Exercise: work out the CW structure on $X \times Y$ from the CW structure on X and Y.

A map $f : X \to Y$ between CW-complexes is **cellular** if $f(X^{(n)}) \subseteq Y^{(n)}\ \forall\ n$.

Theorem 1.2 (cellular approximation)

If $f : X \to Y$ is a map between CW-complexes and f is cellular on $A \subseteq X$ a sub CW-complex. Then f is homotopic rel A to a map $g : X \to Y$ that is cellular on all of X.

Proposition 1.3

$\pi_k(S^n) = 0 \ \forall \ k < n$.

Proof. Given $f : (S^k, s_0) \to (S^n, x_0)$ where s_0, x_0 part of 0-skeleton. We can homotop f to g s.t. $g((S^k)^{(k)}) \subseteq (k\text{-skeleton of } S^n) = \{x_0\}$. So $f \simeq 0$ in $\pi_n(S^n)$. □

What about $\pi_k(S^n)$ for $k > n$. This is very hard in general.

Example 1.4

$\pi_3(S^2) \neq 0$. To see this let $f : S^3 \to S^2$ be the Hopf map. That is, think $S^3 \subseteq \mathbb{C}^2$, $S^1 \subseteq \mathbb{C}$ the unit spheres. S^1 acts on S^3 by multiplication, *i.e.* $\in S^1$, then $\lambda(z_1, z_2) = (\lambda z_1, \lambda z_2) \in S^3$. In fact $S^3/S^1 = \mathbb{C}P^1 \cong S^2$. So the Hopf map is this quotient map. Exercise: $\mathbb{C}P^2 \cong \mathbb{C}P^1 \cup_f D^4$ (glue a 4-cell to S^2 by the Hopf map).

If $f \simeq$ const, then $\mathbb{C}P^2 \cong S^2 \vee S^4$. Easy to see generator $[s^2] \in H^2(S^2 \vee S^4)$. $[s^2] \smile [s^2] = 0$ in $H^4(S^2 \vee S^4)$. Poincare duality says $g \in H^2(\mathbb{C}P^2)$ s.t. $g \smile g \neq 0$ in $H^4(\mathbb{C}P^2)$. So f cannot be trivial in $\pi_3(S^2)$.

Lemma 1.5

X a CW-complex. Let $i : X^{(n)} \to X$ be inclusion then i induces an isomorphism $i_* : \pi_k(X^{(n)}) \to \pi_k(X)$ for $k < n$ and a surjection for $k = n$.

Proof. i_* is surjective for $k = n$ by similar argument to previous proposition. Given $[f] \in \pi_n(X)$, we have $f : S^n \to X$. By cellular approximation theorem, we can homotop f to a cellular map g s.t. $g(S^n) \subseteq X^{(n)}$. Then viewing g as a map from S^n to $X^{(n)}$, we see that $[g] \in \pi_n(X^{(n)})$ is the element that maps to $[f]$ under i_*.

If $k < n$ then i_* is injective. suppose $f : S^k \to X^{(n)}, g : S^k \to X^{(n)}$ and $[f] = [g]$ in $\pi_k(X)$. By cellular approximation, we can assume f, g map into $X^{(k)}$. Let $H : S^k \times I \to X$ be the homotopy. Note: H is cellular on $(S^k \times I) \cup (s_0 \times I)$. Exercise: $S^k \times I$ has a CW structure of $\dim k+1$. Cellular approximation says we can homotop H and $S^k \times I$ and $s_0 \times I$ so its image is in $X^{(k+1)} \subseteq X^{(n)}$. Therefore, $f \simeq g$ in $X^{(n)}$. □

Lemma 1.6 (Homotopy extension theorem)

Given a relative CW-complex (X, A) a map $f : X \to Y$ and a homotopy $H : A \times I \to Y$ of $f|_A$, then there exists an extension of H to $G : X \times I \to Y$ s.t. $G(x, t) = H(x, t)$ on $A \times I$ and $G(x, 0) = f(x)$.

Exercise: prove theorem 21 and 24 directly using this lemma.

Proof. For any D^n there is a deformation retraction of $D^n \times I$ to $D^n \times \{0\} \cup (\partial D^n \times I) =: B$. To see this, $D^n \subseteq \mathbb{R}^n = \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{n+1}$. Also $D^n \times I \subseteq \mathbb{R}^{n+1}$. Let $p = (0, \dots, 0, 2)$. For any $x \in D^n \times I$, let ℓ_x be the line through p, x and it is going to intersect B at a unique point $\tilde{r}(x)$. Then we have a deformation retract $\tilde{r}_t(x) = t\tilde{r}(x) + (1-t)x$.

Now suppose $X - A$ has one cell D^n. We know $\partial D^n \subseteq A$, by hypothesis of the lemma, we have a map $\overline{H} : X \times \{0\} \cup (A \times I) =: C \to Y, (x, 0) \mapsto f(x), (x, t) \mapsto H(x, t)$. Now let

$$G : X \times I \to Y, G(x, t) = \begin{cases} \overline{H}(x, t) & x \in C \\ \overline{H} \circ \tilde{r}(x, t) & x \in D^n \times I \end{cases}$$

This is an extension, we can do this cell by cell. □

Lemma 1.7

If (X, A) a relative CW-complex and A contractible, then $X/A \simeq X$.

Proof. Since A is contractible, we have a homotopy $f : A \times I \to A$ s.t. $f(x, 0) = x, f_1$ is constant, $f_t(x) := f(x, t)$. Note that $f_0 = F_0|_A$ where $F_0 = \mathrm{id}_X$. So HET yields a homotopy $F : X \times I \to X$ by extending f. Note that $F_t(A) \subseteq A$. Therefore, there are induced maps $\overline{F}_t : X/A \to X/A$ since everything in A gets sent to the same equivalence class, and everything outside A is untouched by F_t so the diagram commutes. Also $F_1(A) =$pt. So F_1 also induces a map $h : X/A \to X$. By commutative diagram, $h \circ q = F_1$, $q \circ h = \overline{F}_1$. But $h \circ q = F_1 \simeq F_0 = \mathrm{id}_X$ and $q \circ h = \overline{F}_1 \simeq \overline{F}_0 = \mathrm{id}_{X/A}$ so h, q are homotopy equivalences. □

Definition 1.8 — A space X is ***k*-connected** if $\pi_\ell(X) = 0 \;\forall\; \ell \leq k$.

Theorem 1.9

If X is a k-connected CW-complex, then $X \simeq X'$ where X' is a CW-complex containing a single vertex and no cells of dimension 1 through k.

Proof. Let x_0 be a vertex, and $v_1, \ldots, v_\ell$ be all the vertices. Since $k > 0$, $\pi_0(X) = 0$ so X is path-connected, so there exists a path γ_i from x_0 to v_i. By cellular approximation we can assume $\operatorname{im}\gamma_i \subseteq X^{(1)}$. Attach D^2 to X as follows:

./figures/kconnect_disk.png

Call result $\widetilde{X}'$. Note: $\widetilde{X}'$ is a CW-complex where for each i we add a 1-cell and a 2-cell. Also $\widetilde{X}' \simeq X$ since we can just push the disk down into the boundary. Let $e = \overline{\widetilde{X}' - X}$. Note that e is a contractible subcomplex of $\widetilde{X}'$ (push down to path and then retract along the paths to x_0). Now set $\widetilde{X} = \widetilde{X}'/e$ then lemma 20 says $\widetilde{X} \simeq \widetilde{X}'$ since e is contractible. So $X \simeq \widetilde{X}$ which has one vertex. More generally, let T be a tree in $X^{(1)}$ so $\widetilde{X} = X/T \simeq X$.

Assume $X \simeq \widehat{X}$ where $\widehat{X}$ is a CW-complex with one vertex and no cells of dim $1, \ldots, \ell$ for $\ell < k$. For each $\ell + 1$ cell, $e^{\ell+1}$, the attaching map is $\partial e^{\ell+1} \xrightarrow{f} X^{(\ell)} = \{e_0\}$. This attaches a $\ell + 1$-sphere to $\widehat{X}$. So $e^{\ell+1}$ is an element of $\pi_{\ell+1}(\widehat{X}) = 0$, so there must exist a disk $\alpha : D^{\ell+2} \to \widehat{X}$ s.t. $\alpha(\partial D^{\ell+2}) = e^{\ell+1}$???. We can assume $\alpha(D^{\ell+2}) \subseteq \widehat{X}^{(\ell+2)}$ by cellular approximation. Now glue $D^{\ell+3}$ to $\widehat{X}$ by

./figures/kconnect_disk2.png

call result $\widetilde{X} := \widehat{X}$ with a $\ell + 2$ cell e and a $\ell + 3$ cell e'.

Since e' is homotopic to $\overline{\partial e' - e}$ so $\widetilde{X}' \simeq \widehat{X}$. Since e is contractible, $\widehat{X}' = \widetilde{X}'/e \simeq \widetilde{X}' \simeq \widehat{X}$. NOw $\widehat{tX}'$ has one less $\ell + 1$ cells and we repeat to get rid of all of them. $\square$

Corollary 1.10

If X is a CW-complex with $\pi_i(X) = 0 \ \forall\ i$, then X is contractible.

Proof. If X is a finite dimensional CW-complex, then theorem above says $X \simeq \{\text{pt}\}$. If X is infinite, use weak topology. $\square$

Corollary 1.11

If X is a k-connected CW-complex, then $\widetilde{H}_\ell(X) = 0 \ \forall \ \ell \leq k$.

That is, $\pi_\ell(X) = 0$ for all $\ell \leq k$ implies that $\widetilde{H}_\ell(X) = 0 \ \forall \ \ell \leq k$. Recall that we remove a $\mathbb{Z}$ from 0th homology to get reduced homology.

Proof. Compute $\widetilde{H}_\ell(X)$ using cellular homology. Recall $C_\ell^{\mathrm{CW}}(X)$ is the free abelian group generated by the ℓ-cells. We can assume no ℓ-cells for $\ell = 1, \dots, k$ and for $\ell = 0$. So $H_\ell(X) = 0 \ \forall \ \ell = 1, \dots, k$. Also $H_0(X) = \mathbb{Z}$ since it is path-connected so $\widetilde{H}_0(X) = 0$. $\square$

Theorem 1.12

If (X, A) is a CW pair and $\pi_n(X, A) = 0 \ \forall \ n$ then X deformation retracts to A, *i.e.* $X \simeq A$.

Proof. Exercise. Much like 21 and 22. $\square$

Theorem 1.13 (Whitehead)

If X, Y are CW complexes, with base points $x_0 \in X^{(0)}, y_0 \in Y^{(0)}$ with Y connected, and $f : (X, x_0) \to (Y, y_0)$ is a map s.t. $f_* \pi_k(X, x_0) \to \pi_k(Y, y_0)$ is an isomorphism for all k, then $f : X \to Y$ is a homotopy equivalence.

Remark 1.14 (1) f satisfying the hypothesis is called a **weak homotopy equivalence**. So theorem says for CW-complexes, a weak homotopy equivalence is a homotopy equivalence.

(2) 2 spaces can have isomorphic π_n $\forall$ n but not be homotopy equivalence. We do need this map.

Example 1.15

Let $X = \mathbb{R}P^2 \times S^3$, $Y = S^2 \times \mathbb{R}P^2$. Note $S^2 \times S^3$ is the universal cover of X and Y, by lemma 18, $\pi_n(X) \cong \pi_n(S^2 \times S^3) \cong \pi_n(Y)$ $\forall$ $k \geq 2$. So $\pi_1(X) = \mathbb{Z}/2 = \pi_1(Y)$. They are path-connected so they have isomorphic π_0. But X is not homotopy equivalence to Y, because X is not orientable but Y is so $H_5(X) = 0, H_5(Y) \cong \mathbb{Z}$.

(3) If X, Y are not CW-complexes, then $f : XtoY$ inducing isomorphisms on all homotopy groups, then f doesn't not need to be a homotopy equivalence. Consider topologist's comb and a point at the top of first bar.

Proof. Given $f : X \to Y$ we can make it cellular, consider the mapping cylinder $C_f = (X \times I) \sqcup Y/(x, 0) \sim f(x)$. Exercise: C_f has the structure of a CW-complex where $X \times \{1\}$ is a subcomplex. Recall $C_f \simeq Y$ given by j which has a homotopy inverse $i : Y \to C_f$. Let $i_x : X \to C_f, x \mapsto (x, 1)$, then $j \circ i_X \simeq f$. Since $f_* : \pi_n(X) \to \pi_n(Y)$ is an iso for all n, so is $(i_X)_*$. By long exact sequence in lemma 17,

By Theorem 24, $C_f \simeq X$. □

Let's go back to computing π_k. Recall by lemma 10, $\pi_1(X, x_0)$ acts on $\pi_n(X, x_0)$. Given $[\gamma] \in \pi_1(X, x_0), [f] \in \pi_n(X, x_0)$. Define $[\gamma].[f]$ by

Exercise: this makes $\pi_n(X, x_0)$ into a $\mathbb{Z}[\pi_1(X, x_0)]$-module (group ring).

Theorem 1.16

Given (X, x_0), $f : \partial D^n \to X$ a map s.t. $f(y_0) = x_0$. Let $\widehat{X} = X \cup_f D^n$. Let $i : X \to \widehat{X}$ be inclusion. Then $i_* : \pi_k(X, x_0) \to \pi_k(\widehat{X}, x_0)$ is an isomorphism for $k < n-1$ and surjective for $k = n-1$ with kernel generated by $[f]$ and $[\gamma].[f]$ for all $[\gamma] \in \pi_1(X, x_0)$.

Proof. Given $g : S^k \to \widehat{X}$ s.t. $[g] \in \pi_k(\widehat{X})$, we want to find an element in $\pi_k(X)$ that maps to it. Consider $\int(D^n)$ this is a smooth open manifold. So $g^{-1}(\int D^n)$ is a smooth open submanifold of S^k (open subset of smooth manifold). We can homotop $g|_{g^{-1}(\int D^n)}$ to be smooth. Choose a regular value p of $g|_{g^{-1}(\int D^n)}$ by Sard's Theorem. If $k < n$ then $g^{-1}(p) = 0$ by dimension < 0. Since $D^n - p$ deformation retracts to ∂D^n, we can homotop g to $\widehat{g}$ s.t. $\operatorname{im} \widehat{g} \cap \int D^n = \emptyset$. So $\widehat{g} \in \pi_n(X)$ and $i_*([\widehat{g}]) = [g]$. So i_* is surjective if $k \le n-1$.

Suppose $[g_0], [g_1] \in \pi_k(X)$ s.t. $i_*([g_0]) = i_*([g_1])$, that is, there exists $H : S^k \times I \to \widehat{X}$ between g_0 and g_1. Note $S^k \times I$ is a smooth manifold of $\dim k+1$. So if $k+1 \le n-1$, then the argument above (for surjectivity) says we can homotop H to $\widehat{H}$ s.t. $\widehat{H} : S^k \times I \to X$ is a homotop of g_0 to g_1 in X. So i_* is injective for $k \le n-2$.

Now for $i_* : \pi_{n-1}(X) \to \pi_{n-1}(\widehat{X})$, clearly $[f]$ and $[\gamma] \cdot [f]$ are in $\ker i_*$. So it remains to show $[g] \in \ker i_*$ is in the subgroup generated by $[f]$ and $[\gamma] \cdot [f]$. We have $G : D^n \to \widehat{X}$ s.t. $G|_{\partial D^n} = g$. We can assume there exists $p \in \int(D^n)$ (the cell we added to get $\widehat{X}$) s.t. $G^{-1}(p) = \{p_1, \ldots, p_\ell\}$ by codimension. So there exists open balls N_i around p_i s.t. $G|_{N_i}$ embeds N_i into $\int D^n$. Note that $G|_{D^n - \cup N_i}$ misses p so we can deformation retract to boundary, so homotopic to G' with image in X and each boundary component of $\partial[(D^n - \cup N_i) - \partial D^n]$ has image equal to f. So there exists $p_i \in \partial N_i$ s.t. $G'(p_i) = x_0$. Let $\alpha_i : I \to D^n - \cup N_i$ be a path from p_i' to x_0. $\square$

Theorem 1.17

Any topological space is weakly homotopy equivalent to a CW-complex.

Proof. Given a topological space X, WLOG path-connected with base point x_0, set $Y_0 = \{e^0\}$ and $f_0 : Y_0 \to X, e^0 \mapsto x_0$. We see that f_0 is an isomorphism on π_0.

Let $\alpha_1, \ldots, \alpha_k : I \to X$ generate $\pi_1(X, x_0)$. Set $Y_1 = Y_0 \cup e_1^1 \cup \cdots \cup e_k^1$ which is a wedge of circles. Extend f_0 to $f_1' : Y_1' \to X$ by α_i on each e_i^1. Clearly f_1' is an isomorphism on π_n for $n < 1$ and surjective on π_1. Let β_1, β_ℓ generate $\ker(f_1')_*$ on π_1, *i.e.* $f_1' \circ \beta_i : I \to X$ are null-homotopic. So we have $F_i : D^2 \to X$ s.t. $F_2|_{\partial D^2} = f_1' \circ \beta_i$??? Let $Y_1 = Y_1' \cup \bigsqcup_{i=1}^{\ell} \bar{e}_1^2$, glue $\bar{e}_i^2$ to Y_1' by β_i. Extend f_1' to $f_1 : Y_1 \to X$ by F_i on $\bar{e}_i^2$.

Exercise: $\pi_1(Y_1) \cong \pi_1(Y_1')/\langle \beta_1, \ldots, \beta_\ell \rangle$ so f_1 is an isomorphism on π_n for $n \leq 1$.

Now let $\alpha_1, \ldots, \alpha_k : D^2 \to X$, generate $\pi_2(X)$, $Y_2' = Y_1 \cup e_1^2 \cup \cdots \cup e_k^2$ which each 2-cell is attached by the constant map (a wedge of spheres). Extend f_1 on Y_1 to $f_2' : Y_2' \to X$ by α_i on each e_i^2. Clearly f_2' induces an isomorphism on π_n for $n \leq 1$ and a surjection on π_2. Let $\beta_1, \ldots, \beta_\ell$ generate $\ker(f_2')_*$. $\square$