Homework 5

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Problem (1)**.**

$$J(u) = \frac{1}{2}cx^2(t_f) + \frac{1}{2}\int_0^{t_f} u^2(t)dt$$

Let $\phi(x_f) = \frac{1}{2}cx_f^2$. The Hamiltonian is

$$H = \frac{1}{2}u^2 + pu$$

Problem (3)**.** The Hamiltonian is

$$H = u + p(-\alpha x + u) = (p+1)u - \alpha px$$

So the switching function is $p+1$. Since the Hamiltonian is linear in u, we want to use the Pontryagin maximum principle. To minimize $H(x^*, u, p^*, t)$, we want to make $(p+1)u$ as negative as possible. Thus

$$u^* = \begin{cases} m & p^* < -1 \\ 0 & p^* > -1 \end{cases}$$

I claim that $p^* \neq -1$. Suppose $p^* = 1$. Since T is free, transversality yields $H(T) = 0$. But since H is time-independent, H is constant so $H \equiv 0$. However, we see that

$$H(0) = 0 \cdot u - \alpha(-1)x(0) = \alpha a > 0,$$

a contradiciton.

Now solving $\dot{p} = -H_x = \alpha p$, we get $p(t) = Ke^{\widehat{p}}$. Notice that this is a monotone function and therefore can at most cross the line $p = -1$ once. That is, the control will switch at most once.

Case (1). If $K \geq 0$, $p(t) > -1$ for all t so we have no switching and $u^* = 0$. Then solving $\dot{x} = -\alpha x$ yields $x(t) = K_1 e^{-\alpha t}$ and $x(0) = K_1 = a$. Moreover, we have

$$x(T) = ae^{-\alpha T} = c$$

$$
\begin{aligned}
e^{-\alpha T} &= \frac{c}{a} > 0 \\
-\alpha T &= \ln \frac{c}{a} \\
T &= \frac{1}{\alpha} \ln \frac{a}{c}
\end{aligned}
$$

If $a > c$, then $\frac{a}{c} > 1$ and $T > 0$. Thus, $J^* = 0$. But if $a \leq c$, then $T \leq 0$ which is unphysical. Thus in the case when $K \geq 0$, $a > c$, we have $u^* \equiv 0$.

Case (2). If $K < 0$, p might cross the line $p = -1$ once from above. Since $p(0) = K$ starts in the $p > -1$ regime, we have $u^*(0) = 0$. But

$$
\begin{aligned}
H(0) = (p(0) + 1)u(0) - \alpha p(0)x(0) &= 0 \\
-\alpha a p(0) &= 0 \\
K = p(0) &= 0 \qquad\qquad \alpha, a > 0,
\end{aligned}
$$

a contradiction. Thus in this case we also have no switching and $u^* \equiv m$. Then we solve

$$
\begin{aligned}
\dot{x} &= -\alpha x + m \\
\frac{dx}{m - \alpha x} &= dt \\
x(t) &= \frac{m}{\alpha} - K_2 e^{-\alpha t} \\
x(0) &= \frac{m}{\alpha} - K_2 = a \\
x(t) &= \frac{m}{\alpha} - \left(\frac{m}{\alpha} - a\right) e^{-\alpha t}
\end{aligned}
$$

We have

$$
\begin{aligned}
x(T) = \frac{m}{\alpha} - \left(\frac{m}{\alpha} - a\right) e^{-\alpha T} &= c \\
T &= \frac{1}{\alpha} \ln \left(\frac{m - \alpha a}{m - \alpha c}\right)
\end{aligned}
$$

If $a < c$, we see that $T > 0$, and $J^* = mT = \frac{m}{\alpha} \ln \left(\frac{m-\alpha a}{m-\alpha c}\right)$. If $a \geq c$, we get $T \leq 0$ which is unphysical. Thus in the case when $K < 0$, $a < c$, we have $u^* \equiv m$.