Homework 7

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Problem (1.5.4)**.** Since $X \pitchfork Z$, $X \cap Z$ is a manifold. Let $y \in X \cap Z$ and given $[\gamma] \in T_y(X \cap Z)$, we know that $\gamma : [-\varepsilon, \varepsilon] \to X \cap Z$, $\gamma(0) = y \in X \cap Z$. Since the base point of this class of smooth curves is in both X and Z, $[\gamma]$ is also an equivalence class of smooth curves in X and Z as well so $[\gamma] \in T_y X$ and $T_y Z$, *i.e.* their intersection. Given $[c] \in T_y X \cap T_y Z$, then $c : [-\varepsilon, \varepsilon] \to Y$ and $c : [-\varepsilon, \varepsilon] \to Z$ and therefore $c : [-\varepsilon, \varepsilon] \to Y \cap Z$. Since the base point y is in both X and C and therefore $y \in X \cap Z$, we have $[c] \in T_y(X \cap Z)$.

Problem (7)**.** We need a fact from Exercise 1.5.5: the tangent space to the preimage of Z is the preimage of the tangent space of Z. The proof is self-evident boring set containment argument similar to 1.5.4, so we leave it as an exercise for the undergrad.

$(\Rightarrow)$: Suppose $f \pitchfork g^{-1}(W)$. Since $g \pitchfork W$, $g^{-1}(W)$ is a submanifold of Y and $dg_y(T_y Y) + T_{g(y)} W = T_{g(y)} Z$. Moreover, $df_x(T_x X) + T_{f(x)} g^{-1}(W) = T_{f(x)} Y$. Applying $dg_{f(x)}$ to both sides yields

$$\begin{aligned}
dg_{f(x)}(df_x(T_x X)) + T_{f(x)} g^{-1}(W)) &= dg_{f(x)}(T_{f(x)} Y) \\
dg_{f(x)}(df_x(T_x X)) + dg_{f(x)}(T_{f(x)} g^{-1}(W)) &= dg_{f(x)}(T_{f(x)} Y) && \text{linearity} \\
dg_{f(x)}(df_x(T_x X)) + T_{g(f(x))} W &= dg_{f(x)}(T_{f(x)} Y) \\
d(g \circ f)_x(T_x X)) + T_{g \circ f(x)} W &= dg_{f(x)}(T_{f(x)} Y) + T_{g \circ f(x)} W && + \text{ means span} \\
d(g \circ f)_x(T_x X)) + T_{g \circ f(x)} W &= T_{g \circ f(x)} Z
\end{aligned}$$

$(\Leftarrow)$: We wish to prove the contrapositive: suppose $f \not\pitchfork g^{-1}(W)$, that is, there exists a vector $[v] \in T_{f(x)} Y$ that is not in $df_x(T_x X) + T_{f(x)} g^{-1}(W)$, then as we apply $dg_{f(x)}([v])$ which is an element of $T_{g \circ f(x)} Z$, then I claim that it is not in $d(g \circ f)_x(T_x X) + T_{g \circ f(x)} W$. We already know that $dg_{f(x)}([v]) \notin T_{g \circ f(x)} W$ because $[v] \notin T_{f(x)} g^{-1}(W)$. It remains to check that it is not in the first term (since any component $dg_{f(x)}([v])$ that is in $T_{g \circ f(x)} W$ comes from $T_{f(x)} g^{-1}(W)$ so WLOG we just need to show the other component is not in the first term).

Suppose to the contrary that $dg_{f(x)}([v]) \in d(g \circ f)_x(T_x X)$, then there exists a vector

$[y] \in df_x(T_xX)$ s.t. $dg_{f(x)}([y]) = dg_{f(x)}([v])$. That means $[v] = [y] + \ker dg_{f(x)}$. But clearly $\ker dg_{f(x)} \subseteq T_{f(x)}g^{-1}(W)$ since $T_{g\circ f(x)}W$ contains 0. Hence $[v] \in df_x(T_xX) + T_{f(x)}g^{-1}(W)$, a contradiction. Therefore, $dg_{f(x)}([v]) \notin d(g \circ f)_x(T_xX) + T_{g\circ f(x)}W$ and thus $g \circ f \not\pitchfork W$.

Problem (1.5.11)**.** Let C be a closed set of $\mathbb{R}^k$, then there exists a smooth function $f : \mathbb{R}^k \to \mathbb{R}$ s.t. $C = f^{-1}(0)$. Consider the function $g : \mathbb{R}^{k+1} \to \mathbb{R}, (x_1, \dots, x_k, x_{k+1}) \mapsto f(x_1, \dots, x^k) + x_{k+1}$. I claim that 0 is a regular value of g. Indeed, the Jacobian of this function is $(df, 1)$ which has full rank for any point. Therefore, $M := g^{-1}(0)$ is a submanifold of $\mathbb{R}^{k+1}$. Now consider

$$\begin{aligned}
M \cap \mathbb{R}^k &= \{x \in \mathbb{R}^{k+1} : x_{k+1} = 0, g(x) = f(x_1, \dots, x_k) + x_{k+1} = 0\} \\
&= \{x \in \mathbb{R}^{k+1} : f(x_1, \dots, x_k) = 0\} \\
&= \{f^{-1}(0)\} \\
&= C
\end{aligned}$$

as desired.

Problem (1.6.7)**.** We prove by induction. Denote I_n to be the $n \times n$ identity matrix. For $k = 1$, we see that by choosing a basis of $\mathbb{R}^2$, the antipodal map $S^1 \to S^1, x \mapsto -x$ can be described by $-I$. Then

$$\begin{pmatrix} \cos(\pi t) & \sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix}$$

is a homotopy between the identity map I and antipodal map $-I$.

Now assume that the identity I_k is homotopic to the antipodal map $-I_k$ for odd k via some homotopy $H : S^k \times I \to S^k$. Then for the next odd number $k + 2$,

$$\begin{pmatrix} H_t & 0 & 0 \\ 0 & \cos(\pi t) & \sin(\pi t) \\ 0 & \sin(\pi t) & \cos(\pi t) \end{pmatrix}$$

is a homotopy between I_{k+2} and $-I_{k+2}$. Hence we show that identity and antipodal maps are homotopic for all odd k.

Problem (1.6.8)**.** First, we consider the case where the manifold is path-connected. Note that since any manifold is locally diffeomorphic to $\mathbb{R}^n$, it is locally path-connected since $\mathbb{R}^n$

is. This allows us to use path-connectedness and connectedness interchangably by Munkres Theorem 25.5. Since f is a diffeomorphism, it is also a local diffeomorphism and an embedding onto N. Therefore for any homotopy of f, there exist $\varepsilon_1, \varepsilon_2 > 0$ that preserve these properties under perturbation. Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$, then $f_t : M \to N$ is a local diffeomorphism and an embedding for any $t \in [0, \varepsilon)$. Given any open set U of M, let U_m denotes the neighborhood of m where f_t is a local diffeomorphism. Then $U = \bigcup_{m \in U} U \cap U_m$. Notice f_t is also a diffeomorphism restricted to $U \cap U_m$ which is open so $f_t(U)$ is a union of open sets which is open. That is, f_t is an open map. In particular, $f_t(M)$ is open. Since M is compact, $f_t(M)$ is also compact. Since N is Hausdorff, $f_t(M)$ is closed so $N - f_t(M)$ is open. This implies that $f_t(M)$ and $N - f_t(M)$ form a separation of N. But since N is connected, and clearly $f_t(M)$ as an embedding is not empty so this forces $f_t(M) = N$. That is, $f_t(M)$is a surjective embedding so it is a diffeomorphism onto N.

Now suppose M has multiple path components. Then since f is a diffeomorphism, it must map path components of M to path components of N in a bijective way. Together with the fact that homotopy can never cross path components, we are allowed to consider one pair of components at a time so the above result applies. Since M is compact, the path components must be finite (or each component would require at least one open set in any covering and we would not have finite subcover). , it suffices to take the minimum of the ε yielded from each path component from above and we are done.

Problem (1.7.11)**.** Since a is a nondegenerate critical point of f, by Morse Lemma, there exists a local coordinate system $x = (x_1, \ldots, x_n)^T$ s.t.

$$f = f(a) + x^T H x,$$

where H is the nondegenerate Hessian of f under this coordinate system. Notice that H is symmetric so it is also diagonalizable, *i.e.* $H = P^{-1}DP = P^T DP$ where D has nonzero eigenvalues $\lambda_1, \ldots, \lambda_n$ on the diagonal. Let $\varepsilon_i = \operatorname{sgn}(\lambda_i)$. Then

$$H = P^T \begin{pmatrix} \sqrt{|\lambda_1|} & 0 & \cdots & 0 \\ 0 & \sqrt{|\lambda_2|} & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \end{pmatrix} \begin{pmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix} \begin{pmatrix} \sqrt{|\lambda_1|} & 0 & \cdots & 0 \\ 0 & \sqrt{|\lambda_2|} & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \end{pmatrix} P$$

$$
\begin{aligned}
&=: (P^T\Lambda)E(\Lambda P) \\
&= (\Lambda P)^T E(\Lambda P) \\
&=: \widetilde{P}^T E\widetilde{P}
\end{aligned}
$$

Since each $|\lambda_i| > 0$, the diagonal matrix Λ and thus $\widetilde{P}$ is invertible. Let $\widetilde{x} = \widetilde{P}x$. Since $\widetilde{P}$ is a linear isomorphism, $\widetilde{x}$ is also a local coordinate. Under this local coordinate, we see that

$$f = f(a) + \widetilde{x}^T E\widetilde{x}$$

as desired.

Problem (1.7.14)**.** We wish to use the lemma on page 42 of the book to show that the poles are the only critical points and they are nondegenerate (and therefore the height function h is a Morse function on S^{k-1}).

First we consider the south pole S. View S^{k-1} as a unit sphere inside $\mathbb{R}^k$ with the south pole at the origin so $S = 0$. Consider the height function restricted to $S^{k-1} - N$ which is $h_S : S^{k-1} - N \to \mathbb{R}$, and the inverse stereographic projection $g_S : \mathbb{R}^{k-1} \to S^{k-1} - N$. We see that $h_S = (h_S \circ g_S) \circ g_S^{-1}$. Since $g_S^{-1}(0) = 0$ (maps the south pole to the origin), by lemma it suffices to consider the critical points of $h_S \circ g_S : \mathbb{R}^{k-1} \to \mathbb{R}, x \mapsto \frac{4+\|x\|^2}{4-\|x\|^2}$. By the quotient rule, its derivative is

$$
\begin{aligned}
d(h_S \circ g_S)(x) &= \frac{2\|x\|(4 - \|x\|^2) - (4 + \|x\|^2)(-2\|x\|)}{(4 - \|x\|^2)^2} \\
&= \frac{16\|x\|}{(4 - \|x\|^2)^2}.
\end{aligned}
$$

By basic geometry, $\|x\|^2 = 4 \Leftrightarrow x = (0, \ldots, 0, 2) = N$, the derivative is well-defined everywhere. It equals zero iff the numerator is zero iff $\|x\| = 0$ iff $x = 0 = S$ by positive definiteness of the norm. Hence S is the unique critical point. Again by the positive definiteness of the norm, the Jacobian of the derivative, *i.e.* the Hessian of $h_S \circ g$, must also be positive definite. Thus S is nondegenerate as well. Hence by lemma, S is the unique critical point of h restricted to one chart and it is nondegenerate.

By a similar argument, we can view S^{k-1} as the unit sphere with normal pole at the origin. Consider $h_N : S^{k-1} - S \to \mathbb{R}$ and inverse stereographic projection $g_N : \mathbb{R}^{k-1} \to S^{k-1} - S$.

The composition $h_N \circ g_N : \mathbb{R}^{k-1} \to \mathbb{R}, x \mapsto \frac{4-\|x\|^2}{4+\|x\|^2}$. This time we have negative definition Hessian. So N is the unique critical point of h restricted to the other chart and it is also nondegenerate. That is all the critical points of h and both are nondegenerate so h is a Morse function of S^{k-1}.