Homework 1

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Problem (1.4.3)**.** The coefficient matrix of second order terms is

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

Spectral theorem yields

$$\begin{aligned} O^T A O &= \operatorname{diag}\{4,3,1\}, \text{ where } O = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \\ U^{-1} A U &= \operatorname{diag}\{1,1,1\}, \text{ where } U = \begin{pmatrix} \frac{2}{\sqrt{3}} & -\frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \end{aligned}$$

Thus by a change of basis via U, under the change of variables $\begin{pmatrix} r \\ s \\ t \end{pmatrix} = U^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, where

$$U^{-1} = \begin{pmatrix} \frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \end{pmatrix}.$$

By the chain rule, the derivative of u under basis u_x, u_y, u_z is Du under basis u_r, u_s, u_t composed with Jacobian of this change of variable, which is evidently U^{-1}. That is

$$\begin{aligned} \begin{pmatrix} u_x & u_y & u_z \end{pmatrix} &= U^{-1} \begin{pmatrix} u_r & u_s & u_t \end{pmatrix} \\ u_x &= \frac{\sqrt{3}}{6} u_r - \frac{\sqrt{3}}{6} u_s + \frac{\sqrt{3}}{6} u_t \\ u_y &= -\frac{\sqrt{6}}{6} u_r + \frac{\sqrt{6}}{6} u_t \end{aligned}$$

Thus, the standard form is

$$u_{rr} + u_{ss} + u_{tt} - \frac{5\sqrt{6}+\sqrt{3}}{6} u_r + \frac{\sqrt{3}}{6} u_s + \frac{5\sqrt{6}-\sqrt{3}}{6} u_t + 10u\left(U(r,s,t)\right) = 0$$

Problem (1.4.7)**.** We see that $a = 1, b = 0, c = y$, so $d = ac - b^2 = y$.

Case (1). If $y < 0$, then we have the hyperbolic case. Let $s = s(x, y)$ and $t = t(x, y)$ s.t. the Jacobian of this change of variable is invertible. Then the linear principal part becomes $L_0 u = a^* u_{ss} + 2b^* u_{st} + c^* u_{tt}$, where

$$\begin{cases} a^* = s_x^2 + y s_y^2 \\ b^* = s_x t_x + y s_y t_y \\ c^* = t_x^2 + y t_y^2 \end{cases}$$

To convert to standard form, we want $a^* = c^* = 0$ and $b^* \neq 0$, which indeed makes the equation hyperbolic. Then s, t has to satisfy the following equation

$$\phi_x^2 + y\phi_y^2 = 0$$

Suppose $\phi(x, y)$ =constant, *i.e.* a level set, then y can locally be expressed as a function of x (or the other way around). Thus if $\phi_y \neq 0$, by implicit differentiation we have

$$\begin{aligned} D\phi = \phi_x + \phi_y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{\phi_x}{\phi_y} \end{aligned}$$

Thus we have the following ODE:

$$\begin{aligned} \left(\frac{dy}{dx}\right)^2 + y &= 0 \\ \frac{dy}{\sqrt{-y}} &= \pm dx \\ 2\sqrt{-y} &= \mp x + C \end{aligned}$$

Thus we have

$$\phi_1(x, y) = x + 2\sqrt{-y} = c_1, \ \phi_2(x, y) = -x + 2\sqrt{-y} = c_2.$$

The Jacobian of the change of variable is invertible, so we let $s = \phi_1$ and $t = \phi_2$. Then $s_x = 1$, $s_y = t_y = -(-y)^{-\frac{1}{2}}$, $t_x = -1, s_{yy} = t_{yy} = -\frac{1}{2}(-y)^{-\frac{3}{2}}$. Then $u_x = u_s s_x + u_t t_x = u_s - u_t$

and $u_y = u_s s_y + u_t t_y = -(-y)^{-\frac{1}{2}}(u_s + u_t)$. $u_{xx} = u_{ss}s_x^2 + 2u_{st}s_x t_x + u_s s_{xx} + u_{tt}t_x^2 + u_t s_{xx} = u_{ss} + u_{tt} - 2u_{st}$ and $u_{yy} = u_{ss}s_y^2 + 2u_{st}s_y t_y + u_{tt}t_y^2 + u_s s_{yy} + u_t t_{yy} = (-y)^{-1}(u_{ss} + u_{tt} + 2u_{st}) - \frac{1}{2}(-y)^{-\frac{3}{2}}(u_s + u_t)$. Therefore, the equation becomes

$$\begin{aligned}
u_{ss} + u_{tt} - 2u_{st} + y(-y)^{-1}(u_{ss} + u_{tt} + 2u_{st}) - \frac{1}{2}y(-y)^{-\frac{3}{2}}(u_s + u_t) - \frac{1}{2\sqrt{-y}}(u_s + u_t) &= 0 \\
-4u_{st} + \left(\frac{1}{2\sqrt{-y}} - \frac{1}{2\sqrt{-y}}\right)(u_s + u_t) &= 0 \\
u_{st} &= 0 \\
u_t &= g(t) \\
u(s,t) &= f(s) + g(t)
\end{aligned}$$

for some function $f(s)$ and $g(t)$.

Case (2). If $y > 0$, then the equation is elliptic. Suppose $\phi(x,y) = \phi_1(x,y) + i\phi_2(x,y) =$constant, then we have

$$\begin{aligned}
\frac{dy}{dx} &= \pm i\sqrt{y} \\
\sqrt{y} &= \pm\frac{1}{2}ix + C
\end{aligned}$$

Thus $\phi_1(x,y) = 2\sqrt{y}$ and $\phi_2(x,y) = x$. The Jacobian is invertible so let $s = 2\sqrt{y}$ and $t = x$. Then we have $s_x = t_y = 0$, $s_y = y^{-\frac{1}{2}} = \frac{2}{s}$, and $t_x = 1$. Thus

$$\begin{aligned}
a^* &= 0 + yy^{-1} = 1 \\
c^* &= 1^2 + 0 = 1
\end{aligned}$$

Moreover, $u_y = u_s s_y + u_t t_y = \frac{2}{s}u_s$. Thus, the standard form is

$$u_{ss} + u_{tt} + \frac{2}{s}u_s = 0$$

Case (3). If $y = 0$, the equation is parabolic. The equation becomes

$$u_{xx} + \frac{1}{2}u_y = 0$$

which is already in standard form.

Problem (1.4.9a)**.** We see that $a = 1$, $b = 2$, and $c = 3$. Then $d = ac - b^2 = -1$ which means the equation is hyperbolic.

$$\begin{cases} a^* = s_x^2 + 4s_x s_y + 3s_y^2 = 0 \\ b^* = s_x t_x + 2(s_x t_y + s_y t_x) + 3s_y t_y \\ c^* = t_x^2 + 4t_x t_y + 3t_y^2 = 0 \end{cases}$$

As before, we solve the ODE

$$\left(\frac{dy}{dx}\right)^2 + 4\frac{dy}{dx} + 3 = 0$$
$$\frac{dy}{dx} = 3 \text{ or } 1$$
$$y = 3x + C \text{ or } y = x + C$$

Thus $\phi_1(x, y) = -3x + y = c_1$ and $\phi_2(x, y) = -x + y = c_2$. The Jacobian is invertible so we let $s = -3x + y$ and $t = -x + y$. Then $s_x = -3$, $s_y = t_y = 1$, and $t_x = -1$. Thus $b^* = 3 + 2(-3 - 1) + 3 = -2$. Moreover, $u_x = u_s s_x + u_t t_x = -3u_x - u_t$ and $u_y = u_s s_y + u_t t_y = u_s + u_t$. Thus the standard form is

$$-4u_{st} + 3(-3u_s - u_t) - (u_s + u_t) + 2u = 0$$
$$u_{st} - \frac{5}{2}u_s + u_t - \frac{1}{2}u = 0$$

Now suppose that $u = ve^{\lambda s + \mu t} =: vw$. Then

$$u_s = v_s w + \lambda vw$$
$$u_t = v_t w + \mu vw$$
$$u_{st} = v_{st} w + \mu v_s w + \lambda v_t w + \lambda\mu vw$$

To eliminate first order terms, the sum of their coefficients must be zero. That is,

$$\mu - \frac{5}{2} = 0$$
$$\lambda + 1 = 0$$

So $\mu = \frac{5}{2}$ and $\lambda = -1$. Thus the standard form becomes

$$v_{st} w - \frac{5}{2}vw + \frac{5}{2}vw + \frac{5}{2}vw - \frac{1}{2}vw = 0$$

$$(v_{st} - 2vw)w = 0$$
$$v_{st} - 2vw = 0 \qquad\qquad w > 0$$

Problem (1.4.11)**.** We see that

$$u_t = v_t w + v w_t$$
$$u_x = v_x w + v w_x$$
$$u_{xx} = v_{xx} w + 2v_x w_x + v w_{xx}$$

The equation becomes

$$v_t w + v w_t - v_{xx} w - 2v_x w_x - v w_{xx} + a v_x w + a v w_x + b v w = f(x,t)$$
$$(v_t - v_{xx})w + v(w_t - w_{xx} + a w_x + b w) + v_x(a w - 2w_x) = f(x,t)$$

To obtain the desired form, the following must be true:

$$w_t - w_{xx} + a w_x + b w = 0$$
$$a w - 2w_x = 0$$
$$w \neq 0$$

Thus we have $w_x = \frac{a}{2} w$ which yields $w = g(t)e^{\frac{a}{2}x}$. Then $w_{xx} = \frac{a^2}{4} g(t) e^{\frac{a}{2}x}$. Thus

$$w_t = \left(\frac{a^2}{4} - \frac{a^2}{2} - b\right) w$$
$$w(x,t) = \exp\left(\left(-\frac{a^2}{4} - b\right) t + \frac{a}{2} x\right) \neq 0$$

Problem (1.4.12)**.** Since $u_t = \widetilde{u}' \cdot \left(-\frac{x}{2}\right) t^{-\frac{3}{2}}$ and $u_{xx} = \frac{1}{t}\widetilde{u}''$. Let $y := \frac{x}{\sqrt{t}}$, then the ODE is

$$-\frac{x}{t^{\frac{3}{2}}}\widetilde{u}' - \frac{a^2}{t}\widetilde{u}'' = 0$$
$$-\frac{1}{t}(y\widetilde{u}' + a^2\widetilde{u}'') = 0$$
$$a^2\widetilde{u}'' + y\widetilde{u}' = 0$$
$$u(y) = c_1\sqrt{\frac{\pi}{2}}a \ \mathrm{erf}\left(\frac{y}{\sqrt{2}a}\right) + c_2$$

via substitution and integration by parts. Since $u(0,t) = \widetilde{u}(0) = 0$, we have that $c_2 = 0$. Since $u(x,0) = u_0$, and $\lim_{t\to 0} \widetilde{u} = c_1\sqrt{\frac{\pi}{2}}a \cdot 1$, we have $c_1 = \frac{u_0}{a}\sqrt{\frac{2}{\pi}}$. Thus

$$u(y) = u_0 \operatorname{erf}\left(\frac{y}{\sqrt{2}a}\right).$$