Problem (4.7.1)**.** Since $\partial[a,b] = b - a$, we see that by Stokes Theorem,

$$\begin{aligned}\int_{[a,b]} df &= \int_{\partial[a,b]} f \\ &= \int_{b-a} f \\ &= f(b) - f(a)\end{aligned}$$

Problem (4.7.2)**.** First note that

$$\begin{aligned}d(fdx + gdy) &= d(fdx) + d(gdy) \\ &= df \wedge dx + 0 + dg \wedge dy + 0 \\ &= \frac{\partial f}{\partial x} dx \wedge dx + \frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial g}{\partial x} dx \wedge dy + \frac{\partial g}{\partial y} dy \wedge dy \\ &= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy\end{aligned}$$

Therefore, by Stokes Theorem,

$$\begin{aligned}\int_{\gamma} fdx + gdy &= \int_W d(fdx + gdy) \\ &= \int_W \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy\end{aligned}$$

Problem (4.7.3)**.** Let ω be the one from Exercise 4.4.14, then

$$\begin{aligned}d\omega &= df_1 \wedge dx_2 \wedge dx_3 + 0 + 0 + df_2 \wedge dx_3 \wedge dx_1 + 0 + 0 + df_3 \wedge dx_1 \wedge dx_2 + 0 + 0 \\ &= \frac{\partial f_1}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial f_2}{\partial x_2} dx_2 \wedge dx_3 \wedge dx_1 + \frac{\partial f_3}{\partial x_3} dx_3 \wedge dx_1 \wedge dx_2 \\ &= \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3 \\ &= \operatorname{div} \mathbf{F} dx_1 \wedge dx_2 \wedge dx_3\end{aligned}$$

By Exercise 4.4.14, we immediately have

$$d(\mathbf{F} \cdot \mathbf{n}\; dA) = d\omega.$$

By Stokes Theorem,

$$\begin{aligned}\int_{\partial W} (\mathbf{F} \cdot \mathbf{n}) dA &= \int_W d\omega \\ &= \int_W \operatorname{div} \mathbf{F} dx_1 dx_2 dx_3\end{aligned}$$

Problem (4.7.4)**.** Recall that

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2}, \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}, \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}\right) =: (g_1, g_2, g_3)$$

Again by Exercise 4.4.14,

$$(\operatorname{curl}\mathbf{F}\cdot\mathbf{n})dA = g_1 dx_2 \wedge dx_3 + g_2 dx_3 \wedge dx_1 \wedge g_3 dx_1 \wedge dx_2 =: \omega.$$

Moreover, we see that

$$\begin{aligned} d(f_1 dx_1 + f_2 dx_2 + f_3 dx_3) &= df_1 \wedge dx_1 + df_2 \wedge dx_2 + df_3 \wedge dx_3 \\ &= \frac{\partial f_1}{\partial x_2} dx_2 \wedge dx_1 + \frac{\partial f_1}{\partial x_3} dx_3 \wedge dx_1 + \cdots \\ &= \omega \end{aligned}$$

Therefore by Stokes,

$$\begin{aligned} \int_S (\operatorname{curl}\mathbf{F}\cdot\mathbf{n})dA &= \int_S \omega \\ &= \int_{\partial S} f_1 dx_1 + f_2 dx_2 + f_3 dx_3 \end{aligned}$$

Problem (4.7.7)**.** Since ω is exact, there exists a $(k-1)$-form ω' s.t. $d\omega' = \omega$. By Stokes Theorem,

$$\int_X \omega = \int_{\partial X} \omega' = \int_{\emptyset} \omega' = 0.$$

Problem (4.7.8)**.** Since ω is a closed k-form, $d\omega = 0$. Let $F : W \to Y$ be the extension of f. Then $\partial F(W) = F(\partial W) = f(X)$. Thus we have

$$\begin{aligned} \int_X f^*\omega &= \int_{f(X)} \omega \\ &= \int_{\partial F(W)} \omega \\ &= \int_{F(W)} d\omega && \text{Stokes} \\ &= \int_{F(W)} 0 = 0 \end{aligned}$$

Problem (4.7.9)**.** Let $H : X \times I \to Y$ be the homotopy between f_0 and f_1. Since X is boundaryless, $\partial(X \times I) = X \times \{1\} - X \times \{0\} =: X_1 - X_0$. Define $f = f_0$ on X_0 and $f = f_1$ on X_1. Since f extends to H, by Exercise 8 we have

$$\begin{aligned}
\int_{\partial(X\times I)} f^*\omega &= 0 \\
\int_{X_1 - X_0} f^*\omega &= 0 \\
\int_{X_1} f^*\omega - \int_{X_0} f^*\omega &= 0 && X_0, X_1 \text{ are disjoint} \\
\int_{X_1} f^*\omega &= \int_{X_0} f^*\omega \\
\int_X f_1^*\omega &= \int_X f_0^*\omega
\end{aligned}$$