1 Fundamental Theorem of Algebra

Remark 1.1 If $q \notin f(M)$ then q is a regular value.

Example 1.2

$f : M \to N^k$, $\operatorname{rank}(df_p) = k \Rightarrow \operatorname{rank}(df_q) = k \ \forall \ q \in U$, a neighborhood around p. Use the rank theorem?

Example 1.3

$f : M \to N$, $\dim M = \dim N$, q is a regular value, M is compact, then $f^{-1}(q)$ is finite and $\# f^{-1}(q)$ is locally constant. Use the rank theorem to show that locally it is the identity so we can get a neighborhood around each point that disjoint from other points. By compactness it has to be finite. Locally constant comes from finite intersection of open sets is open.

Theorem 1.4 (Fundamental Theorem of Algebra)

Every nonconstant complex polynomial has a root.

Proof. $P : \mathbb{R}^2 \to \mathbb{R}^2, z \mapsto \sum_{i=1}^{0} a_i z^i$, let $\pi_+ : S^2 - N \to \mathbb{R}^2$ be the stereographic projection. Define $f : S^2 \to S^2$ by $f(N) := N$, $f(p) = \pi_+^{-1} \circ P \circ \pi_+(p)$.

Claim 1.5. f is smooth.

□