Homework 5

Jaden Wang

Problem (5)**.** I claim that for any space X with a CW structure, we can put a CW structure on an $n-$fold cover $\widetilde{X}$ that has n copies of k-cells of X for each k. The base case is clear: the fiber of any 0-cell must have n discrete points, which becomes n 0-cells of $\widetilde{X}$. For each 1-cell of X, they must either leave or return to 0-cells. By local homeomorphisms, for each 0-cell upstairs we must have the same number of leaving and returning branches as downstairs. Therefore, we must have n 1-cells upstairs for each 1-cell downstairs to connect the leaving and returning branches. Now we can proceed with induction. Suppose the $k-1$ skeleton of the covering has been built. We need to figure out how to attach k-cells to build k-skeleton. But we know for each k-cell in the kth skeleton in X, we have a characterstic map $\Phi : e^k \to X$. Since e^k has trivial fundamental group, by the lifting criterion we can lift it to a characteristic map upstairs $\widetilde{\Phi} : e^k \to \widetilde{X}$. Since we get a lift for each base point (0-cell) we choose, and there are n 0-cells upstairs to choose from per 0-cell downstairs, we must get n e^k for each e^k downstairs. Doing this for all k-cells downstairs complete the induction. This yields that $\widetilde{\ell}_k = n\ell_k$ for all k.

Once we establish this, by definition of Euler characteristic, we have

$$\chi(\widetilde{X}) = \sum_{i=0}^{n}(-1)^i \widetilde{\ell}_k = \sum_{i=0}^{n}(-1)^i n \cdot \ell_k = n\chi(X)$$

Problem (6)**.** Recall that for an orientable surface S_g of genus g, we can compute its Euler characteristic using the classical formula $\chi(S_g) = V - E + F$. Using the usual triangulation of 1 vertex, $2g$ edges, and 1 face (corresponding to 1 0-cell, $2g$ 1-cell, and 1 2-cell), we see that $\chi(S_g) = 1 - 2g + 1 = 2 - 2g$.

$(\Rightarrow)$: By previous problem, we have

$$2 - 2g = n(2 - 2h)$$
$$g = n(h-1) + 1$$

$(\Leftarrow)$: First we construct a n-fold covering space for the torus by $p_n : S^1 \times S^1 \to S^1 \times S^1, ((\cos\theta, \sin\theta), y) \mapsto ((\cos n\theta, \sin n\theta), y)$. Then as figure shows, we can simply add genus to the torus, and for each genus added, we would have to add n geni to the covering space. Therefore, using this construction, for a genus-n surface, the genus of this covering space is $n(h-1)+1$.

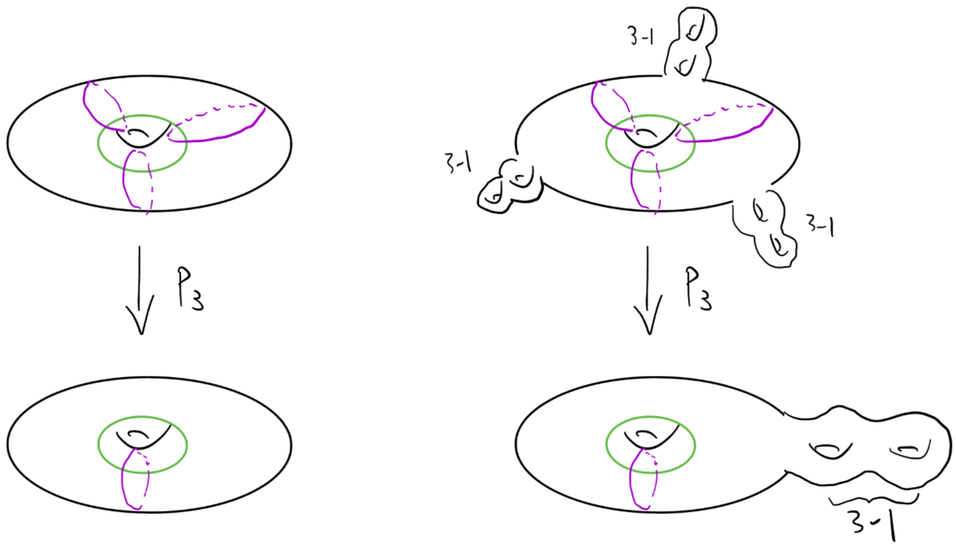

Figure 1: Here we let $g = n = 3$.

Problem (7)**.** $\mathbb{R}P^n$ is S^n with antipodal points identified. But if we think of S^n as two disks glued along the equator, by the identification of antipodal point, we can forget about one disk and just keep track of how the remaining disk behaves. Then $\mathbb{R}P^n$ is also just D^n with boundary S^{n-1} antipodal points identified. That is, we glue D^n along $\mathbb{R}P^{n-1}$, so the attaching map $f_{\partial D^n}$ is exactly the quotient map $q_{n-1} : S^{n-1} \to \mathbb{R}P^{n-1}$. We can do this inductively to obtain a CW-structure: starting with a single 0-cell, get a circle or $\mathbb{R}P^1$ using a single 1-cell, and glue a single 2-cell via quotient map as attaching map to the 1-skeleton $\mathbb{R}P^1$, then glue a single 3-cell via quotient map to 2-skeleton $\mathbb{R}P^2$, and so on. This way, we have the following chain complex:

$$0 \xrightarrow{\partial_{n+1}} C_n = \langle e^n \rangle \xrightarrow{\partial_n} C_{n-1} = \langle e^{n-1} \rangle \to \cdots \to C_2 = \langle e^2 \rangle \xrightarrow{\partial_2} C_1 = \langle e^1 \rangle \xrightarrow{\partial_1} C_0 = \langle * \rangle \to 0.$$

For any $k < n$, the attaching map $f_{\partial e^{k+1}} = q_k$. We obtain a map $g_k : S^k \to S^k$ by composing

$$\partial e^{k+1} \xrightarrow{q_k} \mathbb{R}P^k \to \mathbb{R}P^k / \mathbb{R}P^{k-1} \cong S^k.$$

Then to compute the degree of g_k, we shall use local degrees. Take $y \in \operatorname{int} S^k$, Then $g_k^{-1}(y)$ is pretty much just the antipodal $x, a_k(x)$ (where $a_k : S^k \to S^k$ is the antipodal map) that got quotient to y. Let $U \ni x, V \ni a_k(x)$ be two disjoint neighborhoods that get identified to the same neighborhood of y, *i.e.* $U = a_k(V)$. Then since $g_k|_V \circ a_k = g_k|_U$, we see that $\deg(g_k|_U) = \deg((g_k|_V) \circ a_k) = \deg g_k|_V \deg a_k = (-1)^{k+1} \deg g_k|_V$.

$$\deg g_k = \deg g_k|_U + \deg g_k|_V = (1 + (-1)^{k+1})(\deg g_k|_U).$$

Since $\deg g_k|_U$ is ± 1, $\deg g_k = 0$ if k is even and ± 2 if k is odd. So $\partial_{k+1}(e^{k+1}) = \deg g_k e^k$. Rather we shift the index so $\partial_k(e^k) = \deg g_{k-1} e^{k-1}$ is 0 for k odd and multiplication by ± 2 for k even. The sign doesn't affect homology, so WLOG assume positive 2. Then for $k < n$, we have $\ker \partial_k = 0$ for k even and $\mathbb{Z}$ for k odd, $\operatorname{im} \partial_{k+1} = 0$ for k even and $2\mathbb{Z}$ for k odd. Thus for $k < n$,

$$\begin{aligned} H_k(\mathbb{R}P^n) &= \ker \partial_k / \operatorname{im} \partial_{k+1} \\ &= \begin{cases} 0 & k \neq 0 \text{ even} \\ \mathbb{Z}/2 & \text{k odd} \\ \mathbb{Z} & \text{k} = 0 \end{cases} \end{aligned}$$

If n is even, then $H_n(\mathbb{R}P^n) = 0/0 = 0$. If n is odd, then $H_n(\mathbb{R}P^n) = \mathbb{Z}/0 = \mathbb{Z}$.

For $\mathbb{Z}/2$ coefficients, the changes are that $\ker \partial_k = \mathbb{Z}/2$ for k odd, and $\operatorname{im} \partial_{k+1} = 0$ for k odd. Thus for $k < n$,

$$\begin{aligned} H_k(\mathbb{R}P^n) &= \ker \partial_k / \operatorname{im} \partial_{k+1} \\ &= \begin{cases} 0 & k \neq 0 \text{ even} \\ \mathbb{Z}/2 & \text{k=0 or odd} \end{cases} \end{aligned}$$

If n is even, then $H_n(\mathbb{R}P^n) = 0/0 = 0$. If n is odd, then $H_n(\mathbb{R}P^n) = \mathbb{Z}/2/0 = \mathbb{Z}/2$.

Problem (8)**.** We start with a single 0-cell which generates $\mathbb{Z}$. Suppose by induction we have built the $(i-1)$-skeleton that gives the desired homology. For G_i, since it is finitely generated abelian group, it is the direct sum of cyclic groups, either free as $\mathbb{Z}$ or torsion as $\mathbb{Z}/k$ for some k. For each copy of $\mathbb{Z}$, we attach e^i by the constant map to the 0-cell, *i.e.* it gives an S^i. For each $\mathbb{Z}/k$ for some k, do the same but also attached an e^{i+1} to the S^i by wrapping around k-times. This way, we see that $C_i(X)$ is generated by one e^i per component of G_i, plus one additional e^i per torsion component of G_{i-1}. This is a wedge of S^i spheres (possibly filled). Clearly $\ker\partial_i$ is the e^i that only come from G_i, since the ones from G_{i-1} are all multiplication by k map which is injective. Similarly, $\operatorname{im}\partial_{i+1}$ is the image of multiplication by k maps into e^i from torsion components, whereas the e^i from free part is not the boundary of any higher-dimensional cell so it remains free. That is, $H_i(X) = G_i$.

Problem (9)**.** It goes without saying that CW superscript is omitted. The cellular chain complex of X is

$$0 \xrightarrow{\partial_3} C_2 = \langle \sigma, \tau \rangle \xrightarrow{\partial_2} C_1 = \langle \gamma \rangle \xrightarrow{\partial_1} C_0 = \langle * \rangle \to 0.$$

Clearly X is path-connected so $H_0(X) = \mathbb{Z}$, and $\partial_1 = 0$ so $\ker\partial_1 = \langle \gamma \rangle$. We see that

$$\partial_2(\sigma) = 2\gamma, \partial_2(\tau) = 3\gamma$$

Thus $\partial_2(\tau - \sigma) = \gamma$ meaning ∂_2 is surjective. Thus $H_1(X) = 0$. Finally, by the split short exact sequence, $\ker\partial_2 \cong \mathbb{Z}$. Since $\operatorname{im}\partial_3 = 0$, we have $H_2(X) = \mathbb{Z}$.

Problem (10)**.**

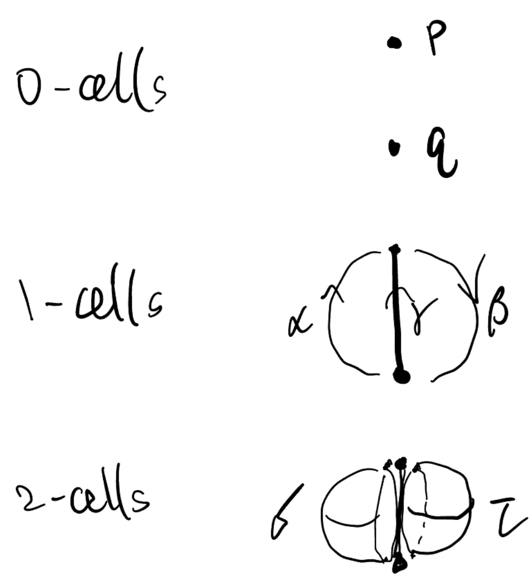

Figure 2: Cell structure of X.

The cellular chain complex of X is

$$0 \xrightarrow{\partial_3} \langle \sigma, \tau \rangle \xrightarrow{\partial_2} \langle \alpha, \beta, \gamma \rangle \xrightarrow{\partial_1} \langle p, q \rangle \to 0$$

Again X is path-connected so $H_0(X) = \mathbb{Z}$. By taking the boundary with the orientations shown in the figure, we have $\partial_1(\alpha) = -\partial_1(\beta) = \partial_1(\gamma) = p - q$. It follows that $\ker \partial_1 = \langle \alpha + \beta, \beta + \gamma \rangle$. As shown in the figure, we see that $\partial_2(\sigma) = \alpha + \beta$ and $\partial_2(\tau) = -\alpha - \beta$, so $\operatorname{im} \partial_2 = \langle \alpha + \beta \rangle$ and $\ker \partial_2 \cong \mathbb{Z}$. It follows that $H_1(X) = \langle \beta + \gamma \rangle \cong \mathbb{Z}$ and $H_2(X) = \mathbb{Z}$.