Homework 4

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Problem (1)**.** The Hamiltonian of the problem is given by

$$H(x_1, x_2, u, p_1, p_2) = \frac{1}{2}u^2 + p_1 x_2 + p_2(u - x_2).$$

The adjoint equations are given by

$$\begin{aligned} \dot{p_1} &= -H_{x_1} = 0 \\ \dot{p_2} &= -H_{x_2} = p_2 - p_1. \end{aligned}$$

The first-order condition demands

$$\begin{aligned} H_u = u + p_2 &= 0 \\ u &= -p_2. \end{aligned}$$

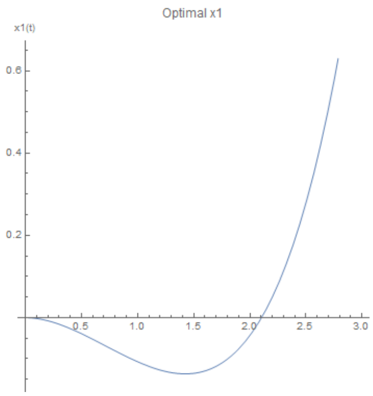
Plugging this into the differential equations yield

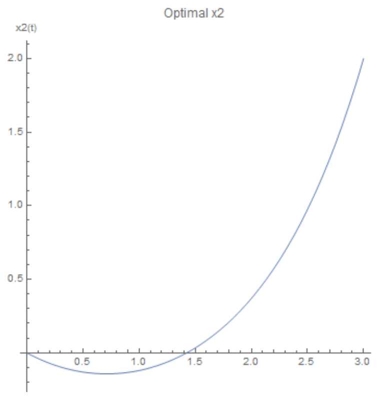
$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -x_2 - p_2. \end{aligned}$$

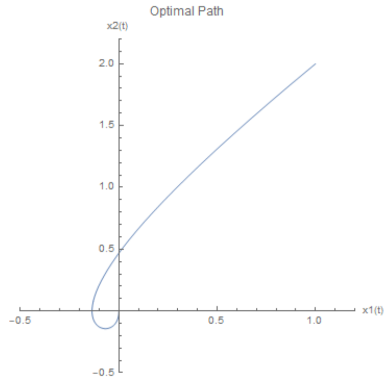
Together with the adjoint equations, we have 4 first-order equations and require 4 boundary conditions.

(a) Since all initial and final times and states are fixed, we have 4 boundary conditions $x_1(0) = x_2(0) = 0$, $x_1(3) = 1$, and $x_2(3) = 2$. Mathematica yields

$$\begin{aligned} u(t) &= \frac{6e^{3+t} + 6e^t - e^6 + 4e^3 - 3}{e^6 + 4e^3 - 5} \\ &= -0.6811 + 0.2642e^t. \end{aligned}$$

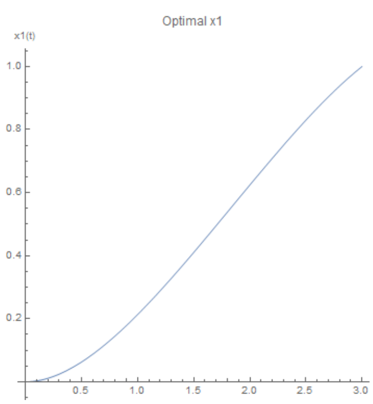


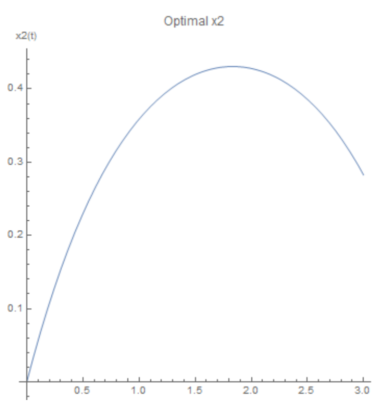


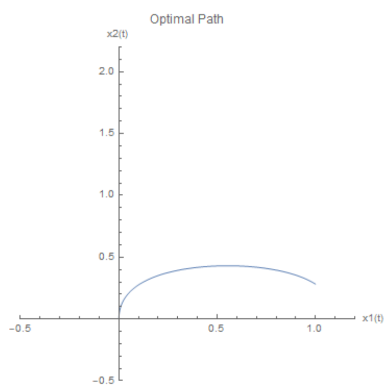


(b) When $x_2(3)$ is free, in its place we instead have the transversality condition $p_2(3) = 0$. This yields the solution

$$
\begin{aligned}
u(t) &= -\frac{2e^3(e^t - e^3)}{3e^6 + 4e^3 - 1} \\
&= 0.6256 - 0.0311e^t
\end{aligned}
$$







(c) I would add a final penalty term Φ:

$$
\mathcal{J} = \underbrace{\frac{1}{2}\left((x_1(3) - 1)^2 + (x_2(3) - 2)^2\right)}_{\Phi(3)} + \frac{1}{2}\int_0^3 u^2 dt.
$$

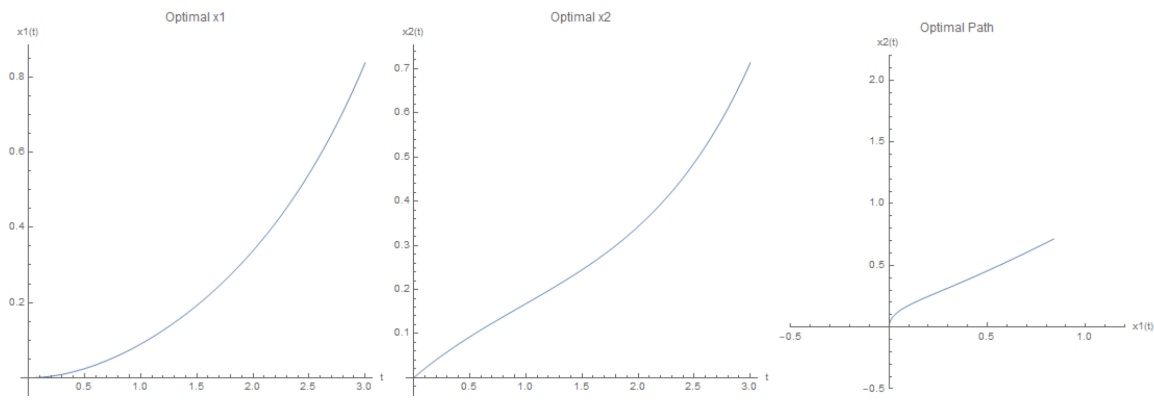
Then instead of $p_1(3) = p_2(3) = 0$, we have

$$
p_1(3) = \Phi_{x_1}(3) = x_1(3) - 1
$$

$$p_2(3) = \Phi_{x_2}(3) = x_2(3) - 2$$

Then new control is

$$u(t) = \frac{8e^{3+t} + 6e^t + e^6 + 4e^3 - 3}{7e^6 + 8e^3 - 7}$$
$$= 0.1615 + 0.056e^t.$$



We see that $x_1(3) = 0.8385$ and $x_2(3) = 0.7142$, which are not very close to $(1, 2)$. To improve accuracy, I would increase the weight of the penalty term. We see that the cost in part (a) is 4.2859. The cost as a function of the weight coefficient c is shown in the figure below:

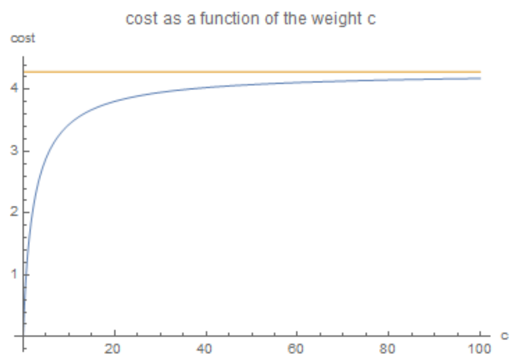


Figure 1: We see that as the weight increases, the cost approaches that of the cost (orange) in part (a) asymptotically.

And we indeed see that the solution ends much closer to $(1, 2)$ when the weight is high:

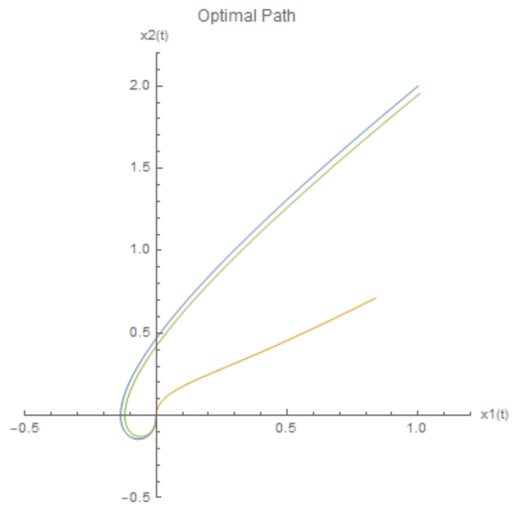


Figure 2: Blue is optimal path from part (a), green is part (c) with weight 100, and orange is part (c) with weight 1.

(d) It is clear that $\Psi = \begin{pmatrix} 2 & 5 \end{pmatrix}$. By transversality condition from Equation 5.234, we have the boundary conditions

$$\begin{pmatrix} -p_1(3) \\ -p_2(3) \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \lambda$$

Together with the initial conditions $x_1(0) = x_2(0) = 0$ and the terminal condition $2x_1(3) + 5x_2(3) = 20$, we have 5 boundary conditions for 4 differential equations and an unknown λ. This allows us to solve by Mathematica and obtain the optimal control:

$$u(t) = -p_2(t) = 1.4341 + 0.1071e^t$$

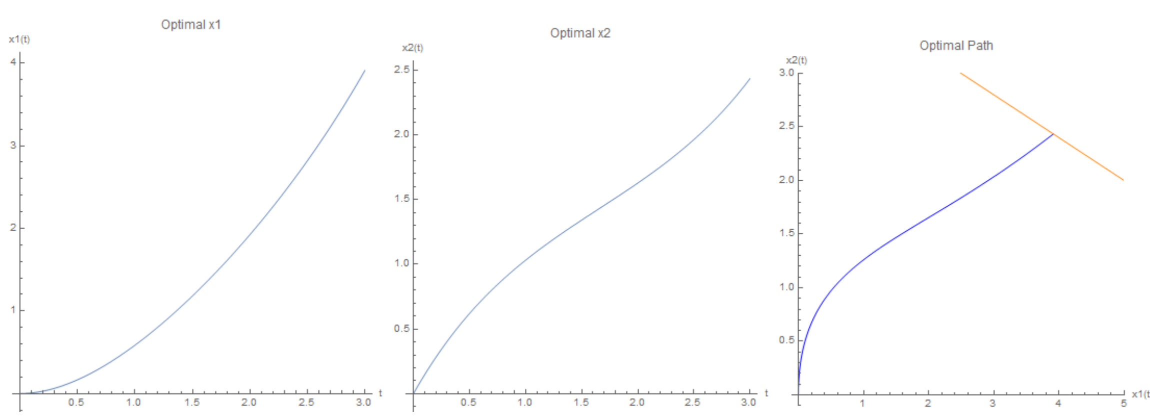


Figure 3: We see that the solution indeed ends at the constraint.

(e) We have the same Ψ but $h(t) = 20 + \frac{t^2}{2}$ so $\dot{h}(t_f) = t_f$. When t_f is also free, based on Equation 5.234 we have the transversality conditions

$$\begin{pmatrix} H(t_f) \\ -p_1(t_f) \\ -p_2(t_f) \end{pmatrix} = \begin{pmatrix} -t_f \\ 2 \\ 5 \end{pmatrix} \lambda$$

Note that by plugging in $u = -p_2$, we have

$$\begin{aligned} H &= -\frac{1}{2}p_2^2 + (p_1 - p_2)x_2 \\ H(t_f) &= -\frac{25}{2}\lambda + 3x_2(t_f) = -t_f \end{aligned}$$

Together with two initial conditions and the terminal condition

$$2x_1(t_f) + 5x_2(t_f) = 20 + \frac{t_f^2}{2},$$

we have a total of 6 boundary conditions to match the 4 differential equations and two unknowns λ and t_f. Mathematica yields

$$u(t) = 1.8359 + 0.2547e^t.$$

The trajectories are

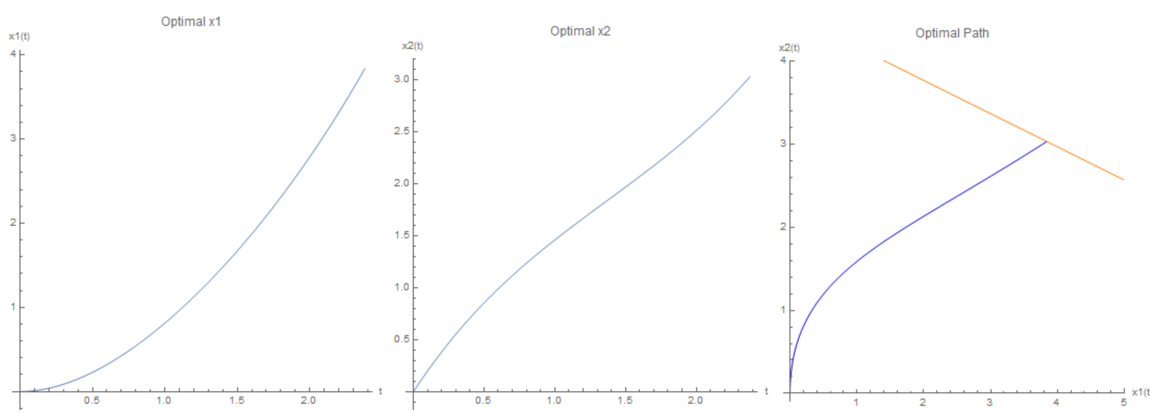


Figure 4: We see the end point hits the constraint.

Problem (2)**.** The Hamiltonian is

$$H = \frac{1}{2}u^2 + p(ax + bu)$$

The adjoint equation is

$$\dot{p} = -H_x = ap \Rightarrow p(t) = Ce^{at}.$$

And the first-order condition is

$$H_u = u + bp = 0 \Rightarrow u = -bp$$

Thus

$$\dot{x} = ax - b^2 p = ax - b^2 C e^{at}, \quad x(0) = x_0, x(tf) = 0$$

We have $x(t) = b^2 C t e^{at} + A e^{at}$, $x(0) = A = x_0$, and

$$x(t_f) = b^2 C t_f e^{at_f} + x_0 e^{at_f} = 0$$
$$C = -\frac{x_0}{b^2 t_f}$$

Thus,

$$u(t) = -bp = -b \cdot \left(-\frac{x_0}{b^2 t_f} \right) e^{at} = \frac{x_0}{bt_f} e^{at}$$

Problem (3)**.** With the mixed constraint $\psi(x_0, x_f) = x_f - x_0 = 0$, we can turn x_0, x_f into free variables by adding a term with Lagrange multiplier. Let $\Phi(x_0, x_f) = \phi(x_f) + \lambda\psi(x_0, x_f) = \frac{1}{2}(x(t_f) - 1)^2 + \lambda(x_f - x_0)$. The Hamiltonian is

$$H = \frac{1}{2}(x^2 + u^2) + pu.$$

The adjoint equation is

$$\dot{p} = -H_x = x$$

The first-order condition says

$$H_u = u + p = 0 \Rightarrow u = -p.$$

Since both $x(0), x(2)$ is free, from Equation 5.234 we have the transversality condition

$$(\Phi_{x_0} + p^T(t_0))\delta x_0 + (\Phi_{x_f} - p^T(t_f)\delta x_f = 0$$

where δx_0 and δx_f can take any value. This forces that

$$\Phi_{x_0} + p^T(t_0) = 0$$
$$\Phi_{x_f} - p^T(t_f) = 0$$

Thus for this problem, we have

$$p(0) = -\Phi_{x_0} = -(-\lambda) = \lambda$$
$$p(2) = \Phi_{x_f} = x(2) - 1 + \lambda$$

Thus we have two boundary conditions for two differential equations. Mathematica yields

$$x(t) = \frac{\lambda\cos(2-t) + \cos t - \lambda t + \lambda\sin(2-t)}{\cos 2 - \sin 2}.$$

Solving $x(0) = x(2)$, we obtain $\lambda = \frac{\cos(2)-1}{2\cos 2 + \sin 2 - 2} \approx 0.7364$.

$$\begin{aligned} u(t) = -p(t) &= \frac{\lambda(\cos(2-t) - \sin(2-t) - \sin t) + \sin t}{\cos 2 - \sin 2} \\ &= 0.5556(\cos(2-t) - \sin(2-t)) + 0.1989\sin t \end{aligned}$$

The associated cost is 1.8506.

Problem (4). (a) Since the cost $\mathcal{J} = t_f$, we have $\Phi(t) = t$ and the Hamiltonian is

$$H = p_1(t)\cos\theta(t) + p_2(t)\sin\theta(t) + p_3(t)u(t) + p_4(t)v(t)$$

(b) The adjoint equations are

$$\begin{pmatrix} \dot{p_1} \\ \dot{p_2} \\ \dot{p_3} \\ \dot{p_4} \end{pmatrix} = \begin{pmatrix} -H_u \\ -H_v \\ -H_x \\ -H_y \end{pmatrix} = \begin{pmatrix} p_3 \\ p_4 \\ 0 \\ 0 \end{pmatrix}$$

(c) Since $t_f, u(t_f), v(t_f)$ are free, by Equation 5.234 we have the transversality conditions

$$H(t_f) + \Phi_t(t_f) = 0 \Rightarrow H(t_f) = -1$$
$$p_1(t_f) = \Phi_u = 0$$
$$p_2(t_f) = \Phi_v = 0.$$

(d) Notice that p_3, p_4 are constants, so $p_1(t) = p_3 t$ and $p_2(t) = p_4 t$ by the zero boundary conditions. First-order condition yields

$$H_\theta = -p_1 \sin\theta(t) + p_2 \cos\theta(t) = 0$$
$$\tan\theta^*(t) = \frac{p_2(t)}{p_1(t)} = \frac{p_4}{p_3}$$

which is a constant! So are $\cos\theta^*$ and $\sin\theta^*$. Thus by the initial conditions $u(0) = \cos\gamma_0, v(0) = \sin\gamma_0$, we have $u(t) = \cos\theta^* t$ and $v(t) = \sin\theta^*$. By the initial conditions $x(0) = y(0) = 0$, we have $x(t) = \frac{1}{2}\cos\theta^* t^2 + \cos\gamma_0 t$ and $y(t) = \frac{1}{2}\sin\theta^* t^2 + \sin\gamma_0 t$. By the terminal condition $x(t_f) = x_f$ and $y(t_f) = 0$, we have

$$x_f - \cos\gamma_0 t_f = \frac{1}{2}\cos\theta^* t_f^2$$
$$-\sin\gamma_0 t_f = \frac{1}{2}\sin\theta^* t_f^2$$

Squaring both sides and adding the two equations together, we obtain the equation as desired:

$$x_f^2 - 2\cos\gamma_0 t_f + (\cos\gamma_0^2 + \sin\gamma_0^2)t_f^2 = \frac{1}{4}(\cos^2\theta^* + \sin^2\theta^*)t_f^4$$
$$4x_f^2 - 8\cos\gamma_0 t_f + 4t_f^2 - t_f^4 = 0$$

So t_f must be the minimum positive solution of this equation.

(e) By the above equation,

$$\cos\theta^* = 2\left(\frac{x_f}{t_f^2} + \frac{\cos\gamma_0}{t_f}\right)$$
$$\theta^* = \arccos\left(2\left(\frac{x_f}{t_f^2} + \frac{\cos\gamma_0}{t_f}\right)\right)$$

Problem (5)**.** The cost has the Meyer form $J = t_f$ so $\Phi(t) = t$. The Hamiltonian is

$$H(x, y, p_1, p_2) = p_1 r\cos\beta + p_2 r\sin\beta.$$

The adjoint equations are

$$\dot{p_1} = -H_x = -p_1\frac{x}{r}\cos\beta - p_2\frac{x}{r}\sin\beta$$

$$\dot{p_2} = -H_y = p_1 \frac{y}{r} \cos\beta - p_2 \frac{x}{r} \sin\beta$$

Note that $\dot{p_2} = \frac{y}{x}\dot{p_1}$. First-order condition yields

$$H_\beta = -p_1 r \sin\beta + p_2 r \cos\beta = 0$$
$$\tan\beta = \frac{p_2}{p_1}$$

Thus we obtain $p_2 = \tan\beta p_1$. Moreover, we see that

$$\begin{aligned} p_2 \cos\beta - p_1 \sin\beta &= \frac{p_2 p_1 - p_1 p_2}{\sqrt{p_1^2 + p_2^2}} \\ &= 0. \end{aligned}$$

Now we compute

$$\begin{aligned} 0 = \frac{d}{dt} H_\beta &= \dot{r}(-p_1 \sin\beta + p_2 \cos\beta) + r\left(-\frac{d}{dt}(p_1 \sin\beta) + \frac{d}{dt}(p_2 \cos\beta)\right) \\ 0 &= 0 + r\left(-\dot{p_1} \sin\beta - p1 \cos\beta\dot{\beta} + \dot{p_2} \cos\beta - p_2 \sin\beta\dot{\beta}\right) \\ 0 &= -\dot{p_1} \sin\beta - p1 \cos\beta\dot{\beta} + \frac{y}{x}\dot{p_1} \cos\beta - p_1 \tan\beta \sin\beta\dot{\beta} \\ p_1(cos\beta + \tan\beta \sin\beta)\dot{\beta} &= \dot{p_1}\left(-\sin\beta + \frac{y}{x}\cos\beta\right) \\ p_1(cos\beta + \tan\beta \sin\beta)\dot{\beta} &= \left(p_1 \frac{x}{r} \cos\beta + \tan\beta p_1 \frac{x}{r} \sin\beta\right)\left(\sin\beta - \frac{y}{x}\cos\beta\right) \\ \dot{\beta} &= \frac{x}{r}\left(\sin\beta - \frac{y}{x}\cos\beta\right) \\ \dot{\beta} &= \frac{x}{r} sin\beta - \frac{y}{r}\cos\beta \end{aligned}$$