Given $[f] \in \pi_k(X, x_0), f : X^k \to X$, define $h_k([f]) = f_*(1) \in H_k(X)$ where 1 generates $H_k(S^k) \cong \mathbb{Z}$. This gives a well-defined map (homotopic maps go to the same homology class) $h_k : \pi_k(X, x_0) \to H_k(X)$ called the Hurewicz map.

Lemma 0.1

h_k is a homomorphism.

Proof. $[f], [g] \in \pi_k(X, x_0)$, $f, g : S^k \to X$, $h \in [f] \cdot [g]$ is given by

./figures/multi.png

The collapse map $c_* : H_k(S^k) \to H_k(S^k \vee S^k) \cong H_k(S^k) \oplus H_k(S^k), 1 \mapsto (1, 1)$ by counting degrees. Then

$$h_*(1) = (f_*, g_*) \circ c_*(1)$$

$$= (f_*, g_*)(1,1)$$
$$= f_*(1) + g_*(1)$$

□

Theorem 0.2 (Hurewicz)

If X is path-connected, then for $n > 1$, if $\pi_n(X) = 0\ \forall\ k < n$, then $h_n : \pi_n(X) \to \widetilde{H}_n(X)$ is an isomorphism. If $n = 1$, then $\ker(h_1) = [\pi_1(X), \pi_1(X)]$.

Note:

(1) If $n > 1$ is the 1st n s.t. $\pi_n(X) \neq 0$, then h_k is an isomorphism $\ \forall\ k \leq n$.

(2) The theorem is true for $\pi_n(X, A)$ if A is simply connected.

Lemma 0.3

$h_k : \pi_k(S^k) \to H_k(S^k), [f] \mapsto f_*(1)$ is an isomorphism.

Proof. By Lemma 29 we know h_k is a homomorphism. Also, let $f : S^k \to S^k$ be the identity map, then $f_*(1) = 1 \in H_k(S^k) \cong \mathbb{Z}$. So h_k is clearly surjective. For injectivity, note the definition of the degree of a map $f : S^k \to S^k$ is $f_*(1)$ *i.e.* $h_k([f]) = \deg(f)$. We want to show that $\deg f = 0 \Rightarrow f$ is null-homotopic. We do this by induction on k.

Recall from Alg Top 1, $\pi_1(S^1) \cong \mathbb{Z}, H_1(S^1) \cong \mathbb{Z}$ and $h_1 : \pi_1(S^1) \to H_1(S^1)$ is an isomorphism. So the base case is done.

Now assume lemma holds for $k-1$ and $f : S^k \to S^k$ is a function s.t. $\deg(f) = 0$. Homotop f so that it is smooth, take a regular value p of f and codimension is 0 and since p is closed, preimage is closed and S^k is Hausdorff so it is compact, so it is finite points: $f^{-1}(p) = \{x_1, \ldots, x_\ell\}$. At each x_i, $df_{x_i} : T_{x_i}S^k \to T_pS^k$ is either orientating preserving or reversing. From Alg Top 1, $\deg(f) = \sum_{i=1}^{\ell} s(x_i)$ where $s(x_i) = \begin{cases} 1 & df_{x_i} \text{ orientation preserving} \\ -1 & df_{x_i} \text{ orientation reversing} \end{cases}$. Since the degree is zero, we can pair up the points. By inverse function theorem, we know that since df_{x_i} is an isomorphism, that f is a local diffeomorphism around x_i. Thus we can take the intersection where we have local diffeomorphisms and get a neighborhood N of p s.t. $f^{-1}(N) = \bigcup_{i=1}^{\ell} N_i$ where N_i are disjoint neighborhoods of x_i. We want a homotopy from f to constant map, $S^k \times I \to S^k$.

./figures/homotopy_31.png

Let α_i be an arc in $S^k \times [\varepsilon, 1]$ s.t. $\partial\alpha_i$ is a pair of x_i's in $S^k \times \{\varepsilon\}$ with opposite signs and $\operatorname{int}\alpha_i \subseteq S^k \times (\varepsilon, 1)$. All such arcs are disjoint since $k > 1$. Let $T_i = \alpha_i \times D^k$ be a neighborhood of α_i s.t. $T_i \cap (S^k \times \{\varepsilon\}) = N_j$ of $\partial\alpha_i$ and T_i disjoint.

./figures/Ti.png

Let $p' \in \partial N$, and $p_j \in \partial N_j$ s.t. $f(p_j) = g$. Now consider $I \times S^{k-1}$ (this is part of $\partial(\alpha_i \times D^k)0$. Let A be an arc in $I \times S^{k-1}$ from p_1 to p_2. We now have a map

$$(S^{k-1} \times \{0,1\}) \cup A \xrightarrow{\widetilde{f}} \partial N = S^{k-1}$$

where we apply f to the first component and send A to p'. This gives a map $\overline{f} : S^{k-1} \to S^{k-1}$.

./figures/fbar.png

Note $\overline{f}|_{S^k\times\{0\}}$ is orientation preserving diffeomorphism and $\overline{f}|_{S^k\times\{1\}}$ is reversing. Hence for any regular value $p'' \in \partial N$, $\overline{f}^{-1}(p'') = \{\widetilde{p_1}, \widetilde{p_2}\}$ so $\deg \overline{f} = 0$. So by induction $\overline{f}$ is null-homotopic. Therefore, $\overline{f}$ extends to a map $\overline{F} : D^k \to \partial N$ with $\overline{F}|_{\partial D^k} = \overline{f}$.

Note: $(I \times S^{k-1}) - A \cong D^k$. Use $\overline{F}$ to give a map $S^k \cong \partial T_i \to N \cong D^k$. This map takes $\partial T_i - (N_1 \cup N_2)$ into ∂N. Then any map $S^k \to D^k$ can be extended to a map $D^k \to D^k$ (since D^k is contractible), *i.e.* $g : S^k \to D^k$,

$$D^{k+1} = S^k \times I/S^{k-1} \times \{0\} \to D^k, (x,t) \mapsto tg(x).$$

Therefore, we get a map defined on T_i:

$$(S^k \times [0,\varepsilon]) \cup (\bigcup_{i=1}^{\ell} T_i) =: M \to S^k$$

$\partial M = (S^k \times \{0\}) \cup Y$ and $G(Y) \subseteq S^k - \{p\}$. Since $S^k - \{p\}$ is contractible, $G|_Y$ is homotopic to a constant map q. Let $Y \times I$ be a neighborhood of Y in $(S^k \times I) - M$ and extend G over $Y \times I$ using $G|_Y$ to q. Now extend G the rest of $S^k \times I$ by sending everything to q. This is null-homotopy of f. □

Note: $\pi_n(S^k) \cong \mathbb{Z}$ and if $g : S^k \to S^k$ then $[g] \in \pi_k(S^k)$ is the degree of g.

Also $f, g : S^k \to S^k$ are homotopic iff $\deg f = \deg g$.

Corollary 0.4

$h_k : \pi_k(\bigvee_n S^k) \to H_k(\bigvee_n S^k)$ is an isomorphism.

Proof. $\pi_k(\bigvee_n S^k) = \bigoplus_n \pi_k(S^k)$. Exercise: prove this (be careful it is not true that $\pi_k(X \vee Y) \cong \pi_k(X) \oplus \pi_k(Y)$. Counterexample: $\pi_2(S^1 \vee S^2)$. The universal cover of $S^1 \vee S^2$ is $\bigvee_\infty S^2$.) □

Proposition 0.5

If base points in X, Y are NDR pairs respectively, then

$$H_k(X \vee Y) \cong H_k(X) \oplus H_k(Y).$$

So the proof follows from this and the lemma.

Proof of Hurewicz. We prove for CW-complexes (true in general but harder). Let X be a CW-complex s.t. $\pi_k(X) = 0 \ \forall\ k < n$. Corollary 23 says $\widetilde{H}_k(X) = 0 \ \forall\ k < n$. Theorem 21 says we can assume $X^{(n-1)} = \{e^0\}$. So $X^{(n)} = \bigvee_{i \in J} S^n_i$ where J is any indexing set, one for each n-cell e^n_i. The cellular chain groups are $C_n(X) = \bigoplus_{i \in J} \mathbb{Z}$, e^n_i generates ith factor. $\partial^{CW}_n : C_n(X) \to C_{n-1}(X) = 0$. So $\ker \partial^{CW}_n = C_n(X)$ so $H_n(X) = C_n(X)/\operatorname{im}(\partial^{CW} : C_{n+1}(X) \to C_n(X))$. Given $\beta_j : \partial D^{n+1} \to X^{(n)}$ attaching map for an $(n+1)$-cell. Recall $\partial^{CW}\beta_j$ is defined as follows:

./figures/boundary.png

$\partial^{CW}\beta = \sum_{i\in J}(\deg({}_i\circ\beta_j))e_i^n$. The sum is finite because S^n is compact.

By Theorem 27, $\pi_n(X, e_0) = \langle e_i^n | [\beta_j] \rangle$ (free abelian group). Now $h_n : \pi_n(X, e_0) \to H_n(X), [e_i^n] \mapsto (e_i^n)_*(1) = e_i^n$. So h_n sends generators to generators and

$$
\begin{aligned}
h_n(\beta_j) &= h_n(\prod [e_i^n]^{\deg({}_i\circ\beta_j)}) \\
&= \sum_{i\in J} \deg(p_i \circ \beta_j) h_n([e_j^n]) \\
&= \sum_{i\in J} \deg(p_i \circ \beta_j) e_n^i \\
&= \partial^{CW}(\beta_j)
\end{aligned}
$$

so h_n also sends relations to relations. So h_n is an isomorphism.

For $n = 1$, since $H_1(X)$ is abelian. We know $[\pi_1(X), \pi_1(X)] \subseteq \ker h_1$, therefore h_1 induces a map

$$\overline{h}_1 : \pi_1(X)/[\pi_1(X), \pi_1(X)] \to H_1(X)$$

as above $\overline{h}_1$ takes generators to generators and relations to relations so it is an isomorphism.

□

Lemma 0.6

If $\pi_1(X) = 1$ and $f : X \to Y$ induces an isomorphism $H_k(X) \to H_k(Y)\ \forall\ k \leq n$, then it induces an isomorphism $\pi_k(X) \to \pi_k(Y)\ \forall\ k < n$ and is surjective for $k = n$.

Remark 0.7 The lemma is true with H_k, π_k swapped.

Proof. Let $C_f = Y \cup X \times I/(x,0) \sim f(x)$ be the mapping cylinder of f. Since $Y \to C_f$ is a retract of C_f, so $\pi_k(C_f) \cong \pi_k(Y)$ and $H_k(c_f) \cong H_k(Y)\ \forall\ k$ and there exists an inclusion $i_X : X \to C_f$ s.t. the diagram commutes.

So $(i_X)_* : H_k(X) \to H_k(C_f)$ is an isomorphism $\ \forall\ k \leq n$. Exercise: If C_f and X are simply connected, then $\pi_1(C_f, X) = 0$. Recall the long exact sequence of homology:

$$H_i(X) \to H_i(C_f) \to H_i(C_f, X) \to H_{i-1}(X) \to H_{i-1}(C_f)$$

Exercise: the diagram commutes. By exactness, we can show that $H_i(C_f, X) = 0$ for $i \leq n$. The relative Hurewicz theorem, $\pi_i(C_f, X) = 0\ \forall\ i \leq n$. Hence $f_* : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i < n$ and surjective for $i = n$. □

Theorem 0.8

If X, Y are simply connected CW complexes and $f : X \to Y$ induces isomorphisms $f_* : H_k(X) \to H_k(Y)\ \forall\ k$, then f is a homotopy equivalence.

Proof. Lemma 33 implies that f_* is isomorphism on all homotopy groups, so by Whitehead we obtain the claim. □

Definition 0.9 — Given a group Π and a positive integer n, if $n > 1$, then Π needs to be abelian. Then a topological space X is called an **Elienberg-MacLane** space of type (Π, n) or also called a $K(\Pi, n)$ if

$$\pi_k(X) \cong \begin{cases} 0 & k \neq n \\ \Pi & k = n \end{cases}$$

Example 0.10

Note S^1 is a $K(\mathbb{Z}, 1)$ since $\pi_1(S^1) \cong \mathbb{Z}$. Theorem 18 says for $k > 1$, $\pi_k(S^1) = \pi_k(\mathbb{R}) = 0$.

Theorem 0.11

Given any group Π and positive integer n as above, then there exists a CW complex that is a $K(\Pi, n)$ and it is unique up to homotopy equivalence.

Proof. Exercise: show $n = 1$ case. Assume $n > 1$, let $\{\alpha_i\}_{i\in I}$ be generators for Π. Set $\widehat{X} = \{e^0\} \cap \{e_i^n\}_{i\in I}$, where each e_i^n attached to $\widehat{X}^{(0)} = e^0$ by constant map. This is a wedge of n-spheres $\bigvee_{i\in I} S^n$. By previous lemma, $\pi_k(\widehat{X}) = 0$ for $k < n$ and $\pi_n(\widehat{X}) = \bigoplus_{i\in I} \mathbb{Z}$. Let $\{r_j\}_{jiinJ}$ be the relations for Π. For each r_j, there exists a map $f_j : S^n \to \widehat{X}$ s.t.

$$r_j = [f_j] \in \pi_n(\widehat{X}).$$

Exercise: show this. Since $\widehat{X}$ is simply connected, by Theorem 27 attaching e^{n+1} to $\widehat{X}$ by f_j will add the relation r_j to π_n and not change π_k, $k < n$. Thus let $\overline{X} = \widehat{X} \cap \{e_j^{n+1}\}$ using f_j, then $\pi_k(\overline{X}) \cong \begin{cases} 0 & k < n \\ \Pi & k = n \end{cases}$ Now $\pi_{n+1}(\overline{X})$ generated by some maps $g_i : S^{n+1} \to \overline{X}$, add an $(n+2)$-cell to $\overline{X}$ using g_i to get $\widetilde{X}$. Now

$$\pi_k(\widetilde{X}) = \begin{cases} 0 & k < n, k = n+1 \\ \Pi & k = n \end{cases}$$

Inductively add j cells for $j > n$ to kill $\pi_k(\widetilde{X})$, $k > n$ get X s.t.

$$\pi_k(X) \cong \begin{cases} 0 & k \neq n \\ \Pi & k = n \end{cases}$$

If X, Y are 2 such CW-complexes, then we build a map $f : X \to Y$ s.t. f_* is an isomorphism on π_k $\forall$ k. By Whitehead's Theorem, f is a homotopy equivalence. We can build f as in the proof below. $\square$

Theorem 0.12

If X, Y are connected CW-complexes, X is a $K(\Pi, 1)$, then there exists a 1-1 correspondence

$$[(Y, y_0), (X, x_0)]_0 \to \operatorname{Hom}(\pi_1(Y, y_0), \pi_1(X, x_0))$$

Definition 0.13 — A connected space with $\pi_k = 0$ $\forall$ $k > 1$ is called **aspherical**.

Proof. Assume $X^{(0)} = \{x_0\}, Y^{(0)} = \{y_0\}$. Given any $[f] \in [Y, X]_0$ then we get

$$f_* : \pi_1(Y) \to \pi_1(X)$$

in $\operatorname{Hom}(\pi_1(Y), \pi_1(X))$.

Claim: this map is onto. Given $h : \pi_1(Y) \to \pi_1(X)$, we inductively build $f : Y \to X$ s.t. $f_* = h$. This is obvious for 0-skeleton $f : y_0 \mapsto x_0$. For each 1-cell e^1 in Y, note it is a loop in Y, in particular, $[e^1] \in \pi_1(X)$. So $h([e^1]) \in \pi_1(X)$. Let $\gamma \in h([e^1]), \gamma : I \to X$ a loop/ Extend f over e^1 using γ. This gives $f : Y^{(1)} \to X$. If e^2 is a 2-cell in $Y^{(2)}$ then ∂e^2 is a loop η in $Y^{(1)}$ and $[\eta] = 0 \in \pi_1(Y)$. So $f_*([\eta]) = h([\eta]) = 0 \in \pi_1(X)$. So $f \circ \eta$ is null-homotopic. Thus we can extend this to a map $F : D^2 \to X$, *i.e.* $F|_{\partial D^2} = f \circ \eta$. Use F to extend f over e^2. Do this for all 2-cells to get $f : Y^{(2)} \to X$. Inductively assume $f : Y^{(k)} \to X$ is defined. Assume e^{k+1} is a $k+1$-cell in $Y^{(k+1)}$. Then $\partial e^{k+1} \subseteq Y^{(k)}$. So $f(\partial e^{k+1})$ is a k-sphere in X. Since $\pi_k(X) = 0$ we know it extends to a $k+1$-disk

$$F : D^{k+1} \to X, F|_{\partial D^{n+1}} = f|_{\partial e^{k+1}}.$$

So use F to extend f over e^{k+1} to get $f : Y^{(k+1)} \to X$. Since f_* and h do same thing on generators, $f_* = h$.

Claim: this is injective. Given $f, g : Y \to X$ s.t. $f_* = g_*$ on π_1, need to show $f \simeq g$. We need $H : Y \times I \to X$ that is a based homotopy from f to g. Notice that $Y \times I$ has a product CW-structure. We get two i-cells: $e^i \times \{0\}, e^i \times \{1\}$ and one $i+1$-cell: $\widetilde{e}^i = e^i \times I$ attached in obvious way. So we have $H : Y \times I \to X$ defined by f on $Y \times \{0\}$, $X \times I$, x_0 on $\{y_0\} \times I$. Now inductively extend over $\widetilde{e}^i$ for $\widetilde{e}^1$ in $(Y \times I)^{(2)}$. Note $\partial \widetilde{e}^1 = \overline{e^1 \times \{0\}} * (e^0 \times I) * (e^1 \times \{1\}) * (e^0 \times I)$. So

$$H(\partial \widetilde{e}^1) = \overline{f(e^1)} * x_0 * g(e^1) * x_0 \simeq \overline{f(e^1)} * g(e^1)$$

This is 0 in $\pi_1(X)$. So there exists a map $F : D^2 \to X$ s.t. $F|_{\partial D^2} = H(\partial \widetilde{e}^1)$. Extend H over $\widetilde{e}^1$ by F. Extend H defined on $(Y \times I)^{(2)} \cup Y \times \{0\} \cup Y \times \{1\}$. Extend H over higher skeleton as above, this gives H so $f \simeq g$. $\square$