1 Serre fibration

Definition 1.1 — A continuous map $p : E \to B$ is called a **fibration** (or a **Serre fibration**) if it has the homotopy lifting property (HLP). That is, given a function $\tilde{g} : Y \to E$ and a homotopy $G : Y \times I \to B$ of $p \circ \tilde{g}$. Then there exists a homotopy $\widetilde{G} : Y \times I \to E$ s.t. $p \circ \widetilde{G} = G$. In other words, the following diagram commutes:

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "\tilde{g}"] \arrow[d, hook] & E \arrow[d, "p"] \\
Y \times I \arrow[r, "G"'] \arrow[ur, dashed, "\tilde{G}"'] & B
\end{tikzcd}$$

Remark 1.2 A locally trivial fibration is a fibration because $Y \to Y, y \mapsto y$ is a bundle with fiber a point.

$$\begin{tikzcd}
E \arrow[r, "\tilde{h}_0"] \arrow[d, "p"'] & Y \arrow[d, "\mathrm{id}_Y"] \\
B \arrow[r, "h_0"] & Y
\end{tikzcd}$$

So we get a homotopy lifting by Theorem 2.

Theorem 1.3

If $p : E \to B$ is a Serre fibration, and $x_0, x_1 \in B$ are in the same path component, then $p^{-1}(x_0) \simeq p^{-1}(x_1)$.

Proof. Let $F_i = p^{-1}(x_i)$ and γ a path from x_0 to x_1. Diagram. So HLP gives a homotopy $A^\gamma : F_0 \times I \to E$ and $A_1^\gamma : F_0 \to F_1$.

Claim 1.4. If γ_0, γ_1 are homotopic rel end points, then A^{γ_0} and A^{γ_1} are homotopic and hence $A_1^{\gamma_0} \simeq A_1^{\gamma_1}$.

Let $H : I \times I \to B$ be homotopy γ_0 to γ_1. Consider $\Lambda : F_0 \times I \times I \to B, (e, s, t) \mapsto H(s, t)$.

./figures/homotopy_claim.png

Define $F_0 \times I\{0\} = A^{\gamma_0}$, $F_0 \times I \times \{1\} = A^{\gamma_1}$, and $F_0 \times \{0\} \times I = (e, 0, s) \mapsto e$. Let $C = (I \times \{0,1\}) \cup (\{0\} \times I) \subseteq I \times I$, there exists a homeo taking C to $I \times \{0\}$. Diagram. Compose with $\mathrm{id}_{F_0} \times f^{-1}$ to get $\widetilde{\Lambda}$. Diagrams. So $\widetilde{\Lambda}$ is a homotopy from A^{γ_0} to A^{γ_1}. Thus $\widetilde{\Lambda}|_{F_0 \times \{1\} \times I}$ is a homotopy $A_1^{\gamma_0}$ to $A_1^{\gamma_1}$.

Now consider $A_1^{\gamma}, A_1^{\gamma^{-1}} : F_1 \to F_0$. Note that $A_1^{\gamma} \circ A_1^{\gamma^{-1}} : F_1 \to F_1$ is a lifting of homotopy $\gamma * \overline{\gamma} \simeq$ constant path rel end points. Hence

$$A_1^{\gamma} \circ A_1^{\overline{\gamma}} \simeq \mathrm{id}_{F_0}.$$

The other direction follows similarly so we prove the theorem. $\square$

Remark 1.5 This theorem says that although the lifted homotopies aren't unique, they are homotopic.

Example 1.6

Let (X, x_0) be a based topological space. Set $P(X) = C((I, \{0\}), (X, x_0))$ (all paths that starts with x_0), often called the **path space**, and $p : P(X) \to X, \gamma \mapsto \gamma(1)$.

Lemma 1.7

In the case above, $p : P(X) \to X$ is a fibration and $P(X)$ is contractible.

Proof. We need to check HLP so the diagram commutes.

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "f_0"] \arrow[d, hook] & P(X) \arrow[d, "p"] \\ Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\tilde{F}"'] & X \end{tikzcd}$$

We need to define $\widetilde{F} : Y \times I \to P(X)$.

For $(y, s) \in Y \times I$,

$$\widetilde{F}(y, s) : I \to X, t \mapsto \begin{cases} f_0(y)\left(\frac{2t}{2-s}\right) & t \in \left[0, \frac{2-s}{2}\right] \\ F(y, 2t - 2 + s) & t \in \left[\frac{2-s}{2}, 1\right] \end{cases}$$

(1) Note this path is well-defined:

$$f_0(y)\left(\frac{2(2-s)/2}{2-s}\right) = f_0(y)(1)$$

$$F(y, 2(2-s)/2 - 2 + s) = F(y, 0)$$

and since $p \circ f_0 = F$ they are the same.

(2) $\widetilde{F}(y, 0)(t) = f_0(y)(t)$.

(3) $\widetilde{F}(y, s)(0) = f_0(y)(0) = x_0$.

(4) $p \circ \widetilde{F}(y, s) = \widetilde{F}(y, s)(1) = F(y, s)$.

So $\widetilde{F}$ is a lift of F. Now for contractibility, we have

$$H : P(X) \times I \to P(X), (\gamma, s) \mapsto \gamma((1-s)t).$$

This is a strong deformation retraction to one point. $\square$

Remark 1.8 Since $p^{-1}(x_0)$ is all paths that also end with x_0, $p^{-1}(x_0) = \Omega(X)$ the loop space. So by Theorem 3, $p^{-1}(x) \simeq \Omega(X)\ \forall\ x \in X$ if X is path-connected. Diagram.

Example 1.9

Given $f : X \to Y$, we saw earlier that f is homotopic to an inclusion. Recall if $C_f = X \times I \cup Y/(x, 0) \sim f(x)$ the mapping cylinder, then $Y \sim C_f$. And diagram. So up to homotopy we can assume $X \subseteq Y$. Now let $E = (C(I, \{0\}), (Y, X))$ which are all paths in Y that starts in X. Let $B = C(\{0, 1\}, \{0\}, (Y, X)) = X \times Y$.

Exercise: show that $E \to Y, \gamma \mapsto \gamma(1)$ is a fibration (almost the same as lemma 5). Note $E \simeq X$ (same as $P(X)$ contractible). So the diagram holds. Hence $f \simeq j \simeq p$ a fibration. Hence we have the slogan:

Any map is a fibration upto homotopy.

Lemma 1.10

If (E, B, F, p) is a fibration, then $\pi_n(E, F) \cong \pi_n(B)$.

Proof. Let b_0 be a base point in B where $F = p^{-1}(b_0)$ and $e_0 \in F \subseteq E$. Given $f : (D^n, \partial D^n) \to (E, F)$, we have $p \circ f : (D^n, \partial D^n) \to (B, b_0)$. So p induces a map $p_* : \pi_n(E, F) \to \pi_n(B)$. Exercise: p_* is well-defined and a homomorphism.

Claim 1.11. p_* is surjective.

Given $g \in \pi_n(B)$, think of $D^n = D^{n-1} \times I$. Define

$$\widetilde{g}_0 : (D^{n-1} \times \{0\}) \to E, x \mapsto e_0$$

So g is a homotopy of $p \circ \widetilde{g}_0$ so HLP implies there exists $\widetilde{g} : D^{n-1} \times I \to E$ lifting g. Since $p \circ \widetilde{g}(\partial(D^{n-1} \times I)) = \{b_0\}$, so $\widetilde{g}(\partial(D^{n-1} \times I)) \subseteq F = p^{-1}(b_0)$. So $\widetilde{g} \in \pi_n(E, F)$. Clearly $p \circ \widetilde{g} = g$.

Claim 1.12. p_* is injective.

Suppose $p_*([f]) = [0] \in \pi_n(B)$, *i.e.* $p \circ f \simeq$ constant b_0 map. Let $H : (D^n, \partial D^n) \times I \to (B, b_0)$

be the homotopy where $H(x,0) = p \circ f(x)$. So by HLP, there exists $\widetilde{H} : (D^n, \partial D^n) \times ItoE$. As previous, $\widetilde{H}(\partial D^{n-1} \times I) \subseteq F$ and $\widetilde{H}(D^n \times \{1\}) \subseteq F$. So $\widetilde{H}$ is a homotopy from f to a map with image in F, so $[f] = [0] \in \pi_n(E, F)$ by lemma I.16. $\square$

Corollary 1.13

If (E, B, F, p) is a fibration, then we get a long exact sequence

$$\cdots \to \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \to \cdots$$

where i is inclusion and $\pi_n(B) \cong \pi_n(E, F) \xrightarrow{\partial} \pi_{n-1}(F)$.

Proof. Theorem I.17 gives the long exact sequence and we simply replace $\pi_n(E, F)$ with $\pi_n(B)$. $\square$

Corollary 1.14

$\pi_k(S^{2n+1}) \cong \pi_k(\mathbb{C}P^n)$ for $k > 2$. In particular, $\pi_3(S^2 = \mathbb{C}P^1) \cong \pi_3(S^3) \cong \mathbb{Z}$.

Proof. Recall we have the Hopf fibrations. So

$$\pi_k(S^1) \to \pi_k(S^{2n+1}) \to \pi_k(\mathbb{C}P^n) \to \pi_{k-1}(S^1)$$

Since $\mathbb{R}$ is the universal cover of S^1, we know $\pi_k(S^1) \cong \pi_k(\mathbb{R}) = 0$ for $k \geq 2$. So for $k > 0$ we have $k - 1 > 1$ so

$$0 \to \pi_k(S^{2n+1}) \to \pi_k(\mathbb{C}P^n) \to 0$$

$\square$

Note

$$0 = \pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) = \mathbb{Z} \to \pi_1(S^3) = 0$$

So we know this without Hurewicz.

Corollary 1.15

X is path connected, then

$$\pi_k(X) \cong \pi_{k-1}(\Omega(X)).$$

Note: we already know this from Cor I.8.

Proof. Since $(P(X), X, \Omega(X),)$ is a fibration and $P(X)$ is contractible so $\pi_k(P(X)) = 0$. Hence

$$\to \pi_k(P(X)) \to \pi_k(X) \to \pi_{k-1}(\Omega(X)) \to \pi_{k-1}(P(X))$$

$\square$

Corollary 1.16

$\pi_k(O(n-1)) \cong \pi_k(O(n))$ for $k < n-2$. $\pi_k(U(n)) \cong \pi_k(U(n-1))$ for $k < 2n-2$.

Proof. Recall $(O(n), S^{n-1}, O(n-1))$ is a fibration.

$$\pi_{k+1}(S^{n-1}) \to \pi_k(O(n-1)) \to \pi_k(O(n)) \to \pi_k(S^{n-1})$$

since $k+1 < n-1$ so we have iso. Similar for $U(n)$. $\square$

Remark 1.17 This corollary implies that for large n, $\pi_k(O(n))$ is independent of k for k small. Can we compute this?

We have inclusions $O(1) \to O(2) \to \cdots$. Let $O = \lim_{n\to\infty} O(n) = \bigcup_{n=1}^{\infty} O(n)$. Similar for U. Then the corollary yields $\pi_k(O) \cong \pi_k(O(n))$ if $n > k+2$ and $\pi_k(U) \cong \pi_k(U(n))$ if $n > k + 2/2$.

Theorem 1.18 (Bott Perodicity)

$\pi_k(O) \cong \pi_{k+8}(O)$. $\pi_k(U) \cong \pi_{k+2}(U)$.

Remark 1.19 Use $(O(n), \{\pm 1\}, \mathrm{SO}(n), \det)$ is a bundle so $\pi_k(\mathrm{SO}(n)) \cong \pi_k(O(n))\ \forall\ k > 0$. Similarly $(U(n), S^1, \mathrm{SU}(n),)$. So $\pi_k(\mathrm{SU}(n)) \cong \pi_k(U(n))\ \forall\ k > 1$.

Recall $V_{n,k} \cong O(n)/O(n-k)$ are the k-frames in $\mathbb{R}^n$ and $V_{n,k}(\mathbb{C}) \cong U(n)/U(n-k)$.

Corollary 1.20

$$\pi_j(V_{n,k}) \cong \begin{cases} 0 & j < n-k \\ \mathbb{Z} & j = n-k \text{ even or } k = 1 \\ \mathbb{Z}/2 & j = n-k \text{ odd} \end{cases} \pi_j V_{n,k}(\mathbb{C}) \cong \begin{cases} 0 & j \leq 2(n-k) \\ \mathbb{Z} & j = 2(n-k)+1 \end{cases}$$

Proof. Recall $V_{n+1,k+1} = O(n+1)/O(n-k) = \mathrm{SO}((n+1)/\mathrm{SO}(n-k)$. Since $\mathrm{SO}(n) \subseteq \mathrm{SO}(n+1)$, we have $V_{n,k} \subseteq V_{n+1,k+1}$ as quotient groups. Diagram.

Let's start with $k = 1$. Diagram.

$$\pi_j(S^n) \xrightarrow{\partial} \pi_{j-1}(S^{n-1}) \to \pi_{j-1}(V_{n+1,2}) \to \pi_{j-1}(S^n)$$

If $j \leq n-1$ then $\pi_j(S^n) = 0 = \pi_{j-1}(S^n)$ so $\pi_{j-1}(V_{n+1,2}) \cong \pi_{j-1}(S^{n-1}) = 0$. For $j = n$ we get

$$\pi_n(S^n) \cong \mathbb{Z} \xrightarrow{\partial} \pi_{n-1}(S^{n-1}) \cong \mathbb{Z} \to \pi_{n-1}(V_{n+1,2}) \to 0$$

So $\pi_{n-1}(V_{n+1,2}) \cong \pi_{n-1}(S^{n-1})/\operatorname{im}\partial$. Recall we define ∂ by taking $f : (D^n, \partial D^n) \to (S^n, s_0) \in \pi_n(S^n)$ lfiting to get $\widetilde{f} : (D^n, \partial D^n) \to (V_{n+1,2}, F)$ taking $\widetilde{f}|_{\partial D^n} : \partial D^n \to S^{n-1}$. So we have

$$\partial([f]) = \widetilde{f}|_{\partial D^n} : \partial D^n \to S^{n-1}.$$

Fact:

(1) There exists a vector field v on S^n with a single zero at s_0, its index is 0 if n odd and 2 if n even. Index: for an isolated zero of a vector field v, take a small sphere S^{n-1}_ε. Then we have a map $S^{n-1}_\varepsilon \to S^{n-1}, x \mapsto \frac{v(x)}{|v(x)|}$. Then the index is just the degree of this map.

(2) If $f : (D^n, \partial D^n) \to S^n$ is the quotient map, then it generates $\pi_n(S^n)$, and $\widetilde{f} : S^n - \{s_0\} \to V_{n+1,2}$,

$$\widetilde{f}(x) = \left(x, \frac{v(x)}{|v(x)|}\right)$$

is a lift of f to $V_{n+1,2}$. Note $p \circ \widetilde{f} = f$.

(3) index of v is the degree of $\widetilde{f}|_{\partial D^n} : \partial D^n \to S^{n-1}$, so

$$\partial[f] = \deg(\widetilde{f}|_{\partial D^n})[g]$$

where $[g]$ is generator of $\pi_{n-1}(S^{n-1})$.

Hence we prove $k = 1$ case.

Assume this is true for k and we show $k + 1$. □