1 Introduction to Spectral Sequences

Definition 1.1 — A **bigraded module** is an indexed collection of modules $E_{s,t}$ for every pair of integers s, t.

A **differential of bidegree $(-r, r-1)$** is a collection of homomorphisms $d : E_{s,t} \to E_{s-r,t+r-1}$ s.t. $d^2 = 0$ (where composition makes sense).

The **homology of d** is

$$H_{s,t}(E, d) := \frac{\ker : E_{x,t} \to E_{s-r,t+r-1}}{\operatorname{im}(d : E_{s+r,t-r+1 s,t})}$$

If $E_q = \bigoplus_{s+t=q} E_{s,t}$ then d induces a homomorphism $\partial : E_q \to E_{q-1}$). so (E_q, ∂) is a chain complex. Moreover, $H_q(E_*, \partial) = \bigoplus_{s+t=q} H_{s,t}(E, d)$.

Definition 1.2 — An **E^k-spectral sequence** is a sequence $\{E^r, d^r\}$ for $r \geq k$ s.t.

(a) E^r is a bigraded module and d^r is a differential of bidegree $(-r, r-1)$,

(b) $E^{r+1} = H(E^r, d^r)\ \forall\ r > k$.

Example 1.3

E^1.

Definition 1.4 — Suppose for every s, t there exists a $\#r(s,t)$ s.t. $\forall\ r > r(s,t)$

$$d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$$

is the zero map, then $E^{r+1}_{s,t}$ is just a quotient of $E^r_{s,t}$ we can define $E^\infty_{s,t}$ to be the direct limit of $E^r_{s,t}$. In this situation we say the spectral sequence **coverges** to $E^\infty_{s,t}$.

Remark 1.5 If $E^r_{s,t} = 0, s < 0, t < 0$ (first quadrant ss), then for each s, t there is some r s.t. $E^r_{s,t}$ is constant in r.

So where do spectral sequences come from? Filtrations.

Definition 1.6 — A **filtration F on a module A** is a sequence of submodules $\{F_sA\}$ of A s.t.

$$A \supseteq \ldots \supseteq F_{s+1}A \supseteq F_sA \supseteq \ldots$$

s is the **filtration degree**, t is the **complementary degree**, and $s+t$ is the **total degree**. A filtration is **convergent** if

$$\bigcap_s F_sA = 0$$

and

$$\bigcup_s F_sA = A$$

We will usually have a finite filtration where $F_{-1}(A) = 0$.

If A is graded $\{A_k\}$ and F respects the grading, then the filtration inherits a grading $F_sA = \{F_sA_k\}$. The **associated graded module** is

$$G(A)_s = F_sA/F_{s-1}A$$
$$G(A) = \bigoplus_s G(A)_s$$

If A is graded then $G(A)$ is bigraded

$$G(A)_{s,t} = F_sA_{s+t}/F_{s-1}A_{s+t}$$

Example 1.7 (1) $A = F_1A = \mathbb{Z}/4$, $F_0A = \mathbb{Z}/2$, $F_{-1}A = 0$, so

$$G(A)_s = \begin{cases} \mathbb{Z}/2 & s = 1, 0 \\ 0 & s \neq 0, 1 \end{cases}$$

So $G(A) = \bigoplus_s G(A) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

(2) $A = F_1A = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $F_0A = \mathbb{Z}_2$, $F_{-1}A = 0$. Then

$$G(A)_s = \begin{cases} \mathbb{Z}/2 & i = 1, 0 \\ 0 & i \neq 0, 1 \end{cases}$$

(3) $A = A_1 = \mathbb{Z}$, $A_0 = 2\mathbb{Z}$, $A_{-1} = 0$, then

$$\begin{aligned} G(A) &= A_1/A_0 \oplus A_0/A_{-1} \\ &\cong \mathbb{Z}_2 \oplus \mathbb{Z} \ncong A \end{aligned}$$

But $G(A)$ is "close" to A.

Lemma 1.8

If F is a finite filtration of A_1 then

(1) If $G(A)_s$ is free for all s then $G(A) \cong A$.

(2) If $G(A)_s$ is a vector space over a field then $G(A) \cong A$.

(3) If $G(A)_s$ is finite for all s, then A is finite and order (A) =order $G(A)$.

(4) If $G(A)_s$ is finitely generated $\forall\ s$, then A is finitely generated and rank A = rank$(G(A))$.

(5) If $G(A)_s = 0$ for all but one s, then $A \cong G(A)$.

(6) If $G(A)_s = 0$ for all but two s, say $G(A)_k, G(A)_\ell$ with $k < \ell$ then

$$0 \to G(A)_k \to A \to G(A)_\ell \to 0$$

is exact.

Proof. (1) We have

$$0 \to F_{n-1}A \to F_nA = A \to F_nA/F_{n-1}A = G(A)_n \to 0$$

Exercise: if $0 \to A \to B \to C \to 0$ is exact and C is free, then $B = A \oplus C$. So

$A \cong G(A)_n \oplus F_{n-1}(A)$. Similarly,

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(2) this is a corollary of 1.

(3)

(4)

(5)

$$G(A)_s = \begin{cases} F_s(A) & s = k \\ 0 & s \neq k \end{cases}$$

So $G(A) = G(A)_k$ and

$$A = F_n(A) = F_{n-1}(A) = \ldots = F_k(A) \supseteq F_{k-1}(A) = \ldots = F_{-1}A$$

so $A = F_k(A) = F_k(A)/F_{k-1}(A) = G(A)_k = G(A)$.

6): If

$$G(A)_s = \begin{cases} C, & s = \ell \\ B, & s = k \\ 0, & \text{else} \end{cases}$$

Then

$$A = F_n(A) = \ldots F_\ell(A) \supseteq F_{\ell-1}(A) = \ldots F_k(A) \supseteq F_{k-1}(A) = \ldots = F_{-1}(A) = 0$$

So $G(A)_\ell = A/F_{\ell-1}(A) \cong C$. and

$$B = G(A)_k = (F_{l-1}A = F_k(A))/(F_{k-1}A = 0) = F_{\ell-1}A$$

So $A/B = C$ iff

$$0 \to B \to A \to C \to 0$$

Exercise: prove 3-4. $\square$

If F is a filtration of a chain complex $\{C_*, \partial\}$ s.t. $(F_s C_*, \partial)$ is a subcomplex of (C_*, ∂), then F induces a filtration on the homology $H_*(C_*, \partial)$

$$F_s H(C_*, \partial) = \mathrm{im}(H(F_s C_*, \partial) \to H_*(C_*, \partial))$$

If F is finite, then it is easy to see

$$\begin{aligned} H(C_*, \partial) &= \bigcup F_s H(C_*, \partial) \\ \bigcap (F_s H(C_*, \partial)) &= 0 \end{aligned}$$

Theorem 1.9

Let F be a finite filtration of a chain complex (c_*, ∂), then there is an E^1 spectral sequence with

(1) $E^1_{s,t} = H_{s+t}(F_s C / F_{s-1} C)$

(2) d^1 is the connecting homomorphism of the triple $(F_s C, F_{s-1} C, F_{s-2} C)$.

(3) $G(H(C_*, \partial))_{s,t} = E^\infty_{s,t}$.

Proof.

$$E^r_{s,t} = \frac{\{c \in F_s(C_{s,t}) : \partial c \in F_{s-r}(C_{s+t+1})\}}{F_{s-1} C_{s+t} + \partial(F_{s+r-1} C_{s+t-1})}$$

and $d^n = \partial$ applied to representatives of $E^r_{s,t}$. $\square$

For 2, recall

$$0 \to F_{s-1} C / F_{s-2} C \to F_s C / F_{s-2} c \to F_s C / F_{s-1} C \to 0$$

Third isomorphism theorem, induces a long exact sequence on homology.

$$H_k(F_s C / F_{s-1} C) \xrightarrow{d^1} H_{k-1}(F_{s-1} C / F_{s-2} C)$$

Example 1.10

Computing the homology of T^2.

Example 1.11

Let $A \subseteq X$ a subspace we get a filtration of $C_*(X)$

$$F_1C = C_*(X)$$
$$F_0C = C_*(A)$$
$$F_{-1}C = 0$$

So we get a spectral sequence with

$$E^1_{s,t} = H_{s+t}\left(\frac{F_sC_*}{F_{s-1}C_*}\right) \cong \begin{cases} H_{s+t}(X,A) & s=1 \\ H_{s+t}(A) & s=1 \\ 0 & s \neq 0,1 \end{cases}$$

Note $E^2 = E^{2+k} = E^\infty$. So $G(H(C_*(X)))_k = \bigoplus_{s+t=k} E^\infty_{s,t} = E^\infty_{0,k} \oplus E^\infty_{1,k-1}$. Lemma 1 says we get

$$0 \to H_k(A)/\operatorname{im}\partial \to H_k(X) \to \ker\partial \to 0$$

Note this is equivalent to

$$0 \to \operatorname{im}(\partial : H_{k+1}(X,A) \to H_k(A)) \to H_k(A) \xrightarrow{i_8} H_k(X) \to \ker(\partial : H_k(X,A) \to H_{k-1}(A)) \to 0$$

If $i : A \to X$ and $j : X \to (X,A)$ inclusions. The above says $\operatorname{im}\partial = \ker i_*$ and $\ker\partial = \operatorname{im} j_*$. This proves the hard part of the long exact sequence for the pair (X,A). So the spectral sequence generalizes long exact sequence.

Example 1.12

We will show for X CW complex , $H_*^{CW}(X) \cong H_*^{Sing}(X)$.

Let $F_kC_*(X) = C_*(X^{(k)})$. This is a finite filtration so there exists a spectral sequence that converges to $E^\infty_{s,t}$.

$$\begin{aligned}
E^1_{s,t} = H_{s+t}(F_sC/F_{s-1}C) \\
&= H_{s+t}(C_*(X^{(s)})/C_*(X^{(s-1)})) \\
&= H_{s+t}(X^{(s)}, X^{(s-1)}) && \text{by defn} \\
&= \widetilde{H}_{s,t}(X^{(s)}/X^{(s-1)}) \\
&= \widetilde{H}_{s+t}(\text{wedge of s-spheres}) \\
&= \begin{cases} \bigoplus_{s-\text{cells}} \mathbb{Z} & t = 0 \\ 0 & t \neq 0 \end{cases} \\
&= \begin{cases} C^{CW}_s(X) & t = 0 \\ 0 & t \neq 0 \end{cases}
\end{aligned}$$

and in sequence $d^1 = \partial$ map of the LES of (F_s, F_{s-1}, F_{s-2}) *i.e.* $H_s(X^{(s)}, X^{(s-1)}) \to H_{s-1}(X^{(s-1)}, X^{(s-2)})$. We know $d^1 = \partial^{CW}$. So

$$E^2 = E^\infty_{s,t} = \begin{cases} H^{CW}_s(X) & t = 0 \\ 0 & t \neq 0 \end{cases}$$

By Lemma 1 part 5, we have $H_p(X) \cong H^{CW}_p(X)$.