When does a bundle have a section?

Lemma 0.1

$\delta\widetilde{\sigma}(s_k) = 0$.

Proof. Recall $\partial^{CW} : C_{k+1}(M) \to C_k(M)$ is as follows: for e_i^{k+1},

$$\partial e_i^{k+1} = S^k \xrightarrow{a_i} M^{(k)} \to M^{k+1}/M^k \cong \bigvee S^k \xrightarrow{p_j} S^k$$
$$\partial^{CW} e_i^{k+1} = \sum_i (\deg g_{ij}) e_j^k$$

where g_{ij} is the composition map above. Then

$$\delta : \mathrm{Hom}(C_{k-1}(M); G) \to \mathrm{Hom}(C_k(M); G)$$
$$(\delta h)(e_i^k) := h(\partial^{CW} e_i^k)$$

$\delta(\widetilde{\sigma}(s_k)) : C_{k+2}(M) \to \pi_k(F), e^{k+2} \mapsto \widetilde{\sigma}(s_k)(\partial e^{k+2})$. Let $a : \partial e^{k+2} \to M^{(k+1)}$ be the attaching map and $I : e^{k+2} \to M$ be the "inclusion". We homotop a as in exercise so

$$(\delta\widetilde{\sigma}(s_k))(e^{k+2}) = \widetilde{\sigma}(s_k)(\sum d_j[e_j^{k+1}])$$

where d_j are degrees of maps. As above $I^*E \cong e^{k+2} \times F$ arrows into e^{k+2} and s_k gives a section above $a(\partial d^{k+2} - \bigcup_n D_n^{k+1})$ from exercise. And $a|_{\partial D_n^{k+1}}$ is the attaching map for some e_i^{k+1}. We can use $p_2 \circ a|_{\pm\partial D_n^{k+2}}$ to define $\widetilde{\sigma}(s_k)(e_i^{k+1})$. Hence the maps used in definition of $\sum d_j \widetilde{\sigma}(s_k)(e_j^{k+1})$ can be extended over $\partial e^{k+2} - \bigcup D_n^{k+1}$.

Exercise: show this means $\sum d_j \widetilde{\sigma}(s_k)(e_j^{k+1})$ is 0 in $\pi_k(F)$. Hint: first consider the case when there is only one D_n^{k+1}. So this finishes the proof. $\square$

Now suppose s_k, s_k' are two sections over $M^{(k)}$ that agree on $M^{(k-1)}$, then their **difference class** in $C^k(M, \pi_k(F))$ is defined as follows:

$$D(s_k, s_k') : C_k(M) \to \pi_k(F)$$

Let $I_i : e_i^k \to M$ be inclusion of a k-cell. Then $I_i^* E \cong e_i^k \times F$. Now $s_k|_{\partial e_i^k} = s_k'|_{\partial e_i^k}$. So putting them together as upper and lower hemispheres, we have a map $p_2 \circ (s_k|_{e_i^k} - s_k'|_{e_i^k}) : S^k \to F$. Define $D(s_k, s_k')(e_i^k) := p_2 \circ (s_k|_{e_i^k} - s_k'|_{e_i^k}) \in \pi_k(F)$.

Lemma 0.2 (1) $\delta(D(s_k, s'_k)) = \widetilde{\sigma}(s_k) - \widetilde{\sigma}(s'_k)$.

(2) given any s_k and $h \in C^k(M; \pi_k(F))$, there exists s'_k s.t. $D(s_k, s'_k) = h$.

Proof. 1 is similar to proof of lemma 1. Exercise.

Let s_k be a section over M, for fix a k-cell e^k, define

$$h(e^k) := [g] \in \pi_k(F)$$

and $h = 0$ on other k-cells. If we can find s'_k s.t. $D(s_k, s'_k) = h$, then we are done (by doing it cell by cell). Let $I : e^k \to M$ be the inclusion so pullback bundle is trivial. We choose a disk $D^k \subseteq \int e^k$ and homotop s_k on e^k so $p \circ s_k(D^k) = x_0 \in F$.

Let $s'_k = s_k$ on $M^{(k)} - D^k$ and on D^k let it be $-g \in \pi_k(F)$. Clearly $D(s_k, s'_k) = h$. □

The two lemmas above give

Theorem 0.3

Given a bundle (E, M, F, p) satisfying the three assumptions and a section $s_k : M^{(k)} \to E$ then $s_k|_{M^{(k-1)}}$ extends to $M^{(k+1)} \Leftrightarrow$

$$\sigma(s_k) = [\widetilde{\sigma}(s_k)] = 0 \in H^{k+1}(M; \pi_k(F)).$$

Remark 0.4 If $\pi_k(F) = 0$ for all $k < \dim M$, then the above shows there exists a section of (E, M, F). In particular, if F is contractible then any bundle with fiber F has a section.

Remark 0.5 $\sigma(s_k)$ depends on $s_k|_{M^{(k-1)}}$ so it is not an obstruction to the existence of a section of E over $M^{(k+1)}$ but only an obstruction to the existence of an extension of $s_k|_{M^{(k-1)}}$ to $M^{(k+1)}$.

But the “first obstruction" is independent of any choices and is “natural".

Theorem 0.6

Given a bundle satisfying the three assumptions, if $\pi_k(F) = 0$ for $k < n$, then there exists a section $s_n : M^{(n)} \to E$ and the obstruction $\sigma(s_n)$ does not depend on s_n, *i.e.* it is well-defined independent of choices. Denote $\sigma(s_n)$ by $\gamma^{n+1}(E)$, called the **primary obstruction**. And if $f : N \to M$ is a map then

$$\gamma^{n+1}(f^*E) = f^*(\gamma^{n+1}(E)).$$

Definition 0.7 — γ^{n+1} is called a **characteristic class**.

Proof. The discussion above says s_n exists since all obstructions vanish. You can develop an obstruction theory to homotoping one section to another: given s, s' agreeing on $k-1$-skeleton, then $s|_{M^{(k)}}$ is homotopic to $s'|_{M^{(k)}} \Leftrightarrow \sigma(s, s') \in H^k(M; \pi_k(F))$ vanishes (by the same argument before). Exercise.

So there is a unique (up to homotopy) section of E over $M^{(n-1)}$. Thus $\sigma(s_n)$ is independent of s_n.

For naturality, WLOG suppose $f : N \to M$ is a cellular map. Now a section s of $E \to M$ gives a section $f^*(S)$ of $f^*(E)$. Exercise: check this. For any $\Phi : (D^{n+1}, \partial D^{n+1}) \to (N^{(n+1), N^{(n)}}$ we see

$$\pi_{n+1}(N^{(n+1)}, N^{(n)}) \xrightarrow{f_*} \pi_{n+1}(M^{(n+1)}, M^{(n)}) \to \pi_n(F)$$
$$[\Phi] \mapsto [f \circ \Phi] \mapsto [p_2 \circ s \circ f \circ \Phi_{\partial D^{n+1}}]$$

is essentially both $\sigma(f^*s)(\Phi)$ and $(f^*\sigma(s))(\Phi)$. Now $pi_{n+1}(N^{(n+1)}, N^{(n)} \cong H_{n+1}(N^{(n+1)}, N^{(n)}) \cong C^{CW}_{n+1}(N)$. Similarly for M. So the cocycle $\gamma^{n+1}(f^*E)$ is $f^*(\gamma^{n+1}(E))$. $\square$