Homework 6

Jaden Wang

Problem (1)**.**

(1) Suppose R is a local ring with the unique maximal ideal M. Given $r \in R - M$, we see that $\langle r \rangle$ cannot be a proper ideal, otherwise $\langle r \rangle \leq M$ since all proper ideals must be contained in this unique maximal ideal, contradicting that $r \notin M$. This forces $\langle r \rangle = \langle 1 \rangle = R$. Hence r is a unit.

(2) Let M be the set of non-units of R where M is an ideal. Suppose $M \leq J \leq R$. If there exists an $r \in J - M$, that means that r is a unit. Then $R = \langle r \rangle \leq J$ so $J = R$. Hence M is maximal. Uniqueness follows from the fact that any ideal not contained in M must have an element not in M so it cannot be proper (by argument above).

(3) We wish to show that $\langle 2 \rangle$ is precisely the set of non-units M in R. Given $2\frac{p}{q} \in \langle 2 \rangle$ where $\frac{p}{q}$ is a reduced rational number with q odd, then $\frac{2p}{q}r = 1$ yields that $r = \frac{q}{2p}$ which clearly has even denominator. So $2\frac{p}{q}$ cannot be a unit so $\langle 2 \rangle \leq M$. Given any non-unit (reduced) $\frac{m}{n} \in M$, it must be that $m = 2k$ for some $k \in \mathbb{Z}$, otherwise if m is odd then $\frac{n}{m} \in R$ is the inverse. So $\frac{m}{n} \in \langle 2 \rangle$. Thus $\langle 2 \rangle = M$. Then by part b, we obtain that R is a local ring with M as its maximal ideal.

Problem (2)**.**

(a) Suppose $x^m = 0$ for some $m \in \mathbb{Z}^+$ and that $yr = 1$ for some $r \in R$. Then it suffices to show that $-(-y)^m \in \langle x + y \rangle$ since $-(-y)^m$ is a unit (with inverse $-(-r)^m$). Recall that $x^m - (-y)^m = (x - (-y))A$ for some polynomial A in x, y. Then $A \in R$ so $x^m - (-y)^m = -(-y)^m \in \langle 1 \rangle$.

(b) First $N(R)$ is not empty since 0 is nilpotent. Given $a, b \in N(R)$, where $a^m = b^n = 0$. Then $(a+b)^{mn+m+n} = 0$ by binomial theorem so it's closed under addition. It is clearly closed under negation. Given $r \in R$, we see that $(ar)^m = a^m r^m = 0 \cdot r^m = 0$ so $N(R)$ is an ideal.

Consider $R = M_2(\mathbb{R})$. Clearly $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are nilpotent but their sum is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is full rank and invertible. So $N(R)$ is not closed under addition so not an ideal.

(c) Suppose there exists $rN(R) \in R/N(R)$ s.t. it is nilpotent, *i.e.* $r^m \in N(R)$ for some positive integer m. Then $r^m = a$ for some nilpotent a s.t. $a^n = 0$. Hence $r^{mn} = a^n = 0$, so r is nilpotent and $rN(R) = N(R)$. Hence the only nilpotent element of $R/N(R)$ is the identity $N(R)$.

(d) (collab with Ari, Griffin, Will) Suppose $a \in R$ is not nilpotent and let Σ be the set of all ideals not containing a-positive powers. Given any chain in this set $I_1 \subseteq I_2 \subseteq \cdots$. Define $I = \bigcup_i I_i$. I claim that I is an ideal not containing a-positive powers. The fact that I is an ideal is routine. Since none of the I_i contains a-positive power, the union doesn't contain a-positive power either. Thus I is clearly an upper bound of the chain. Then by Zorn's lemma, we have a maximal element P of Σ. Note that P is an ideal not containing a-positive power. Then suppose $x, y \notin P$, since P is maximal, $\langle P, x\rangle$ and $\langle P, y\rangle$ must contain a-positive powers. That is, there exists $a, b, c, d \in R$ s.t.

$$ap + bx = a^m$$

$$cp' + dy = a^n$$

$$\underbrace{acpp' + bxcp + dyap}_{\in P} + bdxy = \underbrace{a^{m+n}}_{\notin P}$$

This implies that $(bd)xy \notin P$. Since P is an ideal, $xy \notin P$ either. This proves that P is prime. Since P doesn't contain a-positive power it doesn't contain a. Thus we show that if a is not nilpotent, then there exists a prime ideal of R not containing a. Thus the contrapositive is true: if all prime ideals of R contains some element x, then x is nilpotent. That is, $\bigcap_i P_i \subseteq N(R)$.

Given $x \in N(R)$, then $x^m = xx^{m-1} = 0 \in \bigcap_i P_i$. Since P_i are prime, either $x \in P_i$ or $x^{m-1} \in P_i \ \forall\ i$. If $x^{m-1} \in P_i$, we can rewrite it as $xx^{m-2} \in P_i$ and repeat this process, which terminates because m is finite. Eventually we must have $x \in P_i \ \forall\ i$ so $N(R) \subseteq \bigcap_i P_i$.

Problem (3)**.** $(i) \Rightarrow (ii)$: Suppose R has a unique prime ideal. By 2(d), $N(R)$ must be the unique prime ideal P. Moreover, any maximal ideal is a prime ideal, so R has a unique maximal ideal. By 1(a), we see that every element in $R - N(R)$ is a unit. Thus any element in R is either nilpotent or a unit.

$(ii) \Rightarrow (iii)$: Notice the set of non-units in R is simply $N(R)$ which is an ideal by 2(b). Then by 1(b), $N(R)$ is the unique maximal ideal of R so $R/N(R)$ is a field.

$(iii) \Rightarrow (i)$: Since $R/N(R)$ is a field, $N(R)$ is maximal. Since $N(R)$ is in the intersection of all prime ideals, given any prime ideal P, we have $N(R) \subseteq P$. Since $N(R)$ is maximal, and prime ideals are proper, $P = N(R)$. Hence $N(R)$ is the unique prime ideal of R.

Problem (4)**.** (collab with Ari, Will, Griffin): Suppose to the contrary that $I \subseteq \bigcup_i P_i$ but $I \nsubseteq P_i \; \forall \; i$. We wish to prove by induction. If $k = 1$, then $I \subseteq P_1$ trivially holds. Suppose if $I \subseteq \bigcup_{i=1}^{k-1} P_i$, then $I \subseteq P_i$ for some i when there are $k-1$ prime ideals. To prove the inductive step, suppose that $I \subseteq \bigcup_{j=1}^{k} P_j$ but $I \nsubseteq \bigcup_{j \neq i} P_j$ and $I \nsubseteq P_i$ for all $1 \leq i \leq k$. This implies that for every $1 \leq i \leq k$, there exists an $a_i \in I$ s.t. $a_i \notin \bigcup_{j \neq i} P_j$ which forces $a_i \in P_i$. Denote $\widehat{a}_i = \prod_{j \neq i} a_j$. Notice that $\widehat{a}_i \notin P_i$ by construction. Now consider

$$x := \sum_{i=1}^{k} \prod_{j \neq i} a_j.$$

Since $x \in I$, we have $x \in P_n$ for some $1 \leq n \leq k$. Since P_n is an ideal, $ra_n \in P_n \; \forall \; r \in R$, so

$$\begin{aligned} x &= \underbrace{\left(\sum_{i \neq n} \widehat{a}_{i,n} \right) a_n}_{\in P_n} + \underbrace{\widehat{a}_n}_{\notin P_n} \\ &\notin P_n. \end{aligned}$$

This is a contradiction. So it must be that $I \subseteq \bigcup_{j \neq i} P_j$ or $I \subseteq P_j$ for some j. If it's the former we are done by inductive hypothesis. If it is the latter we are done immediately. Therefore by induction, for any k, if $I \subseteq \bigcup_{i=1}^{k} P_i$, then $I \subseteq P_i$ for some i.