1 Homotopy groups

Recall that $\pi_n(X) = [S^n, X]_0$. Since S^n is an H'-space (by successive suspension), there is a multiplication on $\pi_n(X)$. What is this product?

Define $f, g : (S^n, p) \to (X, x_0)$.

./figures/product1.png

Sometimes it is useful to see $\pi_n(X)$ as $[(D^n, \partial D^n), (X, x_0)]$. If $\kappa : D^n \to S^n$ collapses ∂D^n to $p \in S^n$, then $\pi_n(X) \to [(D^n, \partial D^n), (X, x_0)], [f] \mapsto [f \circ \kappa]$ is well-defined and injective. It is surjective since any $f : (D^n, \partial D^n) \to (X, x_0)$ factors through (S^n, x_0) (universal property of quotients of pairs?).

Note that we can see $\pi_n(X)$ is abelian for $n \geq 2$ using the multiplication structure.

What is the multiplication structure in $[(D^n, \partial D^n), (X, x_0)]$? We think D^n as $I \times D^{n-1}$.

Given $g, f : I \times D^{n-1} \to X$,

$$f \cdot g(t,x) = \begin{cases} f(2t,x) & t \in [0,\frac{1}{2}] \\ g(2t-1,x) & t \in (\frac{1}{2},1] \end{cases}$$

Now we just move the puzzle pieces to swap them.

Definition 1.1 — We can also define **relative homotopy groups**. Given space X, subspace A and $x_0 \in A$, define

$$\pi_n(X,A) = [(D^n, \partial D^n, s_0), (X, A, x_0)]$$

with $s_0 \in \partial D^n$.

This multiplication does not make sense for $\pi_1(X,A)$. So $\pi_1(X,A)$ is just a set as it doesn't have to preserve base point to get a group structure.

./figures/pi1_bad.png

This definition doesn't help us showing inverses and associativity. So we provide an alternative definition:

Definition 1.2 — Let $D^n = I^n$ and $J = \overline{\partial D^n - (D^{n-1} \times \{1\})} = (D^{n-1} \times \{0\}) \cup (\partial D^{n-1} \times I)$. That is, J is three edges of a square.

Exercise: show $[(D^n, \partial D^n, s_0), (X, A, x_0)]$ is in 1-1 correspondence with $[(D^n, \partial D^n, J), (X, A, x_0)]$. Note $(D^n, \partial D^n, J)/J \cong (D^n, \partial D^n, s_0)$.

Define a multiplication: $f, g \in \pi_n(X, A)$.

$$f \cdot g(x_1, \dots, x_n) = \begin{cases} f(2x_1, x_2, \dots, x_n) & x_1 \in [0, \frac{1}{2}] \\ g(2x_1 - 1, x_2, \dots, x_n) & x_2 \in (\frac{1}{2}, 1] \end{cases}$$

Let $f^{-1}(x_1, \dots, x_n) = f(1 - x_1, \dots, x_n)$.

Exercise:

(1) Show $\pi_n(X, A)$ is a group with identity the constant map for $n \geq 2$.

(2) Show $\pi_n(X, A)$ is abelian for $n \geq 3$ (need the 3rd dimension to move around).

(3) $\pi_n(X, x_0) = \pi_n(X)$.

Lemma 1.3

$f : (D^n, \partial D^n, s_0) \to (X, A, x_0)$ is 0 in $\pi_n(X, A)$ iff it is homotopic rel ∂D^n and s_0 to a map whose image is in A.

Proof. $(\Leftarrow)$: suppose f is homotopic to g with g having image in A. We know D^n deformation retracts to s_0:

$$H : D^n \times I \to D^n, H(x, 0) = x, H(x, 1) = s_0, H(s_0, t) = s_0 \; \forall \; t.$$

Now $g \circ H$ is a homotopy from g to constant map (rel A). Therefore $f = 0 \in \pi_n(X, A)$. □

$(\Rightarrow)$: assume $f = 0$. So there exists a homotopy $H : D^n \times I \to X, H(x, 0) = f, H(x, 1) = x_0, H(x, t) \in A \; \forall \; x \in \partial D^n$. Note $H|_{D^n \times \{1\} \cup \partial D^n \times I}$ is a map $D^n \to A$ ($H|_{D^n \times \{0\}} = f$. We can use H on $D^n \times I$ to give a homotopy f to a map with image in A. Here is the idea: there exists a homeomorphism $D^n \times I \xrightarrow{\phi} D^{n+1}$. There is also a continuous map $D^n \times I \xrightarrow{\psi} D^{n+1}$ that collapses $(\partial D^n \times I)$ to the equator. Now $H \circ \phi^{-1} \circ \psi : D^n \times I \to X$ is the homotopy.

Note. (1) The inclusion maps $(A, x_0) \subseteq (X, x_0) \subseteq (X, A)$ yield

$$\pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A).$$

(2) if $f : (D^n, \partial D^n, J) \to (X, A, x_0)$ then define $\partial f : (\partial D^n, J) \to (A, x_0)$. This induces a map $\pi_n(X, A) \to \pi_{n-1}(X, A)$. Note $\pi_{n-1}(A) = [(\partial D^n, (A, x_0), Y]_0$. Exercise: show it's well-defined.

Theorem 1.4

Given (X, A, x_0) we have a long exact sequence

$$\cdots \to \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \to \cdots$$

is equivariant under $\pi_1(A)$ action.

Proof. First $j_* i_* = 0$ by lemma 16. If $[f] \in \ker j_*$ then $f : (D^n, \partial D^n) \to (X, x_0)$ and homotopy $H : D^n \times I \to X$ s.t.

(1) $H(x, 0) = j \circ f(x) = f(x)$

(2) $H(x, 1) \in A$ by lemma 16.

(3) $H(x, t) \in A \ \forall\ t$ and $x \in \partial D^n$.

(4) $H(s_0, t) = x_0 \ \forall\ t$.

Note $D' := (D^n \times \{1\}) \cup (\partial D^n \times I)$ is a disk and $H|_{D'} : D' \to A$ s.t. $H(\partial D') = x_0$. So $g = H|_{D'} : D^n \to A$ this is in $\pi_n(A)$ and as in the proof of lemma 16, f is homotopic to g, so $i_*([g]) = [f]$ so $\operatorname{im} i_* = \ker j_*$ yielding exactness.

Now we show $\operatorname{im} j_* \subseteq \ker \partial$. Given $[f] \in \pi_n(X, A)$, s.t. $\partial f = 0$ in $\pi_{n-1}(A)$. There exists homotopy $H : S^{n-1} \times I \to A$ s.t. $H(x, 0) = f(x)$, $H(x, 1) = x_0$, and $H(s_0, t) = x_0$. Let $D' = D^n \cup (S^{n-1} \times I)$ is a disk.

$f' : D' \to X : x \mapsto \begin{cases} f(x) & x \in D^n \\ H(x) & x \in S^{n-1} \times I \end{cases}$. Note $[f'] \in \pi_n(X)$. Exercise $f' \sim f$ in $\pi_n(X, A)$ (because it can't see what's in A intuitively). Then $j_*([f']) = [f]$ so $\ker \partial \subseteq \operatorname{im} j_*$.

Exercise: show $\operatorname{im} \partial = \ker i_*$. $\square$

Theorem 1.5

Let $p : \widetilde{X} \to X$ be a covering space, then $p_* : \pi_n(\widetilde{X}, \widetilde{x_0}) \to \pi_n(X, p(\widetilde{x_0}))$ is an isomorphism $\forall\ n \geq 2$.

Proof. Recall the lifting criterion: given a $f : Y \to X$ s.t. $f(y_0) = x_0$, f lifts to a map $\widetilde{f} : Y \to \widetilde{X}$ with $\widetilde{f}(y_0) = \widetilde{x_0}$ iff $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$.

From this, we see that p_* is surjective for $n \geq 2$. Given $[f] \in \pi_n(X, x_0)$, then

$$f_*(\pi_1(S^n)) = \{e\} < p_*(pi_1(\widetilde{X})).$$

So there exists a lift $\widetilde{f} : S^n \to \widetilde{X}$ s.t. $p_*([\widetilde{f}]) = p \circ \widetilde{f} = f$.

It is also injective for $n \geq 2$. Given $[f] \in \pi_n(\widetilde{X})$, suppose $p_*([f]) = 0$ in $\pi_n(X)$. So there exists a homotopy $H : S^n \times I \to X$ s.t. $H(x, 0) = p \circ f(x)$, $H(x, 1) = x_0$, and $H(s_0, t) = x_0$. Recall covering spacess satisfying the homotopy lifting property. So there exists $\widetilde{H} : S^n \times I \to \widetilde{X}$ s.t. $\widetilde{H}(x, 0) = f(x)$, $\widetilde{H}(x, 1) = \widetilde{x_0}$, and $\widetilde{H}(s_0, t) = \widetilde{x_0}$. So $[f] = 0$ in $\pi_n(\widetilde{X})$. $\square$