All rings are commutative, associative, and with 1. All ring homomorphisms preserve 1. All modules are unital.

Definition 0.1 — A ring is **Noetherian** if every ideal is finitely generated.

Remark 0.2 We will mostly consider Noetherian rings in this course.

1 Algebraic sets

Let K be a field, usually algebraically closed. A polynomial in $K[x_1, \ldots, x_n]$ may be thought of as a function $K^n \to K$.

Definition 1.1 — Given polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$, a set of the form

$$\mathcal{V}(f_1, \ldots, f_m) = \{(a_1, \ldots, a_n) \in K^n : f_i(a_1, \ldots, a_n) = 0 \ \forall \ i\}.$$

is called an **algebraic set in $\boldsymbol{K^n}$**.

Example 1.2

$K = \mathbb{R}$, $K[x, y]$, then $\mathcal{V}(y - x^2)$ is the parabola $y = x^2$.

Question 1.3. What is the dimension of an algebraic set?

Recall from linear algebra,

$$\dim(V \cap W) = \dim V + \dim W - \dim(V + W)$$
$$\geq \dim V + \dim W - n$$

Example 1.4

Intersection of 2 planes in 3-space, has dimension at least 2+2-3=1.

Remark 1.5 Suppose $\mathcal{V}, \mathcal{W} \subseteq K^n$ are algebraic sets, then $\mathcal{V} \cap \mathcal{W}$ may consist of a union of smaller, algebraic sets called (irreducible) components. The components may have different dimensions. Let $a \in \mathcal{V}$. Then $\dim_a \mathcal{V}$ is the dimension of the largest component of $\mathcal{V}$ containing a. If $a\mathcal{V} \cap \mathcal{W}$, then

$$\dim_a(\mathcal{V} \cap \mathcal{W}) \geq \dim_a(\mathcal{V}) + \dim_a(\mathcal{W}) - n.$$

Question 1.6. Given an algebraic set $\mathcal{V}(f_1, \ldots, f_m)$, what is the ideal in $K[x_1, \ldots, x_n]$ of all polynomials vanishing on that set? Call this ideal J.

Example 1.7

$I = (x^2)$ then $J = \langle x \rangle$.

Example 1.8 (the commutator scheme)

See lecture notes.

Question 1.9. Given an algebraic set $\mathcal{V}(f_1, \ldots, f_m)$, what is the least number k such that

Question 1.10. What is the geometry associated to a commutative ring? See Nullstellensatz.