Homework 3

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Problem (1). Since G acts transitively, the orbit of any $a \in A$ is all of A, so $|A| = |G : G_a|$. Since $|A| > 1$, the index of G_a is at least 2. Recall that for any $b = g_b.a$ for some $g_b \in G$, $G_b = g_b G_a g_b^{-1}$. Let $S := \bigcup_{b \in A} g_b G_a g_b^{-1} \subseteq G$. By exercise 2.1.8, the union of subgroups is a subgroup iff they are contained in one of them. If the union is indeed a subgroup, then $S = G_b$ for some $b \in A$. Since the index of G_b is at least 2, there exists some $g \in G \setminus G_b$ which doesn't fix any point by definition of union. If S is not a subgroup, then since G is a group, clearly $S \subsetneq G$ so there exists a $g \in G \setminus S$.

Problem (2).

(a) First we consider all types of symmetries of a cube. As we did in class, there are 24 of them.

Table 1: Cube

type	cycle decomp	# cycles	elements of this type
id	(1)(2)(3)(4)(5)(6)(7)(8)	8	1
90 upright rot	(1234)(5678)	2	6
180 upright rot	(13)(24)(57)(68)	4	3
120 diag rot	(123)(456)(7)(8)	4	8
180 semidiag rot	(17)(28)(34)(56)	4	6

So by Burnside, the total distinct ways are

$$\frac{1}{24}(k^8 + 6k^2 + 3k^4 + 8k^4 + 6k^4) = \frac{1}{24}(k^8 + 17k^4 + 6k^2)$$

Table 2: Tetrahedron

type	cycle decomp	# cycles	elements of this type
id	(1)(2)(3)(4)	4	1
120 upright rot	(123)(4)	2	8
grand rot	(12)(34)	2	3

(b) So total distinct ways are

$$\frac{1}{12}(k^4 + 11k^2)$$

Problem (3)**.** Consider S_n acting on itself by conjugation. Notice that two elements σ, τ commute iff $\sigma\tau\sigma^{-1} = \tau$, that is, τ is a fix point of the action of σ by conjugation. Therefore, the number of fixed points of σ action is the same as the number of elements commuting with σ. By Burnside lemma,

$$|\mathcal{O}| = \frac{1}{|S_n|} \sum_{\sigma \in S_n} f(\sigma)$$

But since conjugacy classes of S_n are simply distinct cycle types, $|\mathcal{O}| = p(n)$. Also notice that the probability p_σ of fixing an σ and picking another element commuting with it is precisely $\frac{f(\sigma)}{|S_n|}$. Since we assume naive probability for picking elements, the probability for picking two such elements is just the average probability of fixing one and picking another. That is,

$$p = \frac{1}{|S_n|} \sum_{\sigma \in S_n} p_\sigma = \frac{1}{n!} \sum_{\sigma \in S_n} \frac{f(\sigma)}{|G|} = \frac{p(n)}{n!}.$$

Problem (4)**.** If $H \trianglelefteq G$, then $gHg^{-1} = H \ \forall \ g \in G$ so $n_H = 1$. Otherwise, G acts on H by conjugation and $n_H = |\mathcal{O}| = |G : G_H|$. Notice that for any $h \in H$, $hHh^{-1} = H$, so $H \leq G_H$. Therefore, $n_H = |G : G_H|$ divides $|G : H| = p$ so it is 1 or p. But if $n_H = 1$, $gHg^{-1} = H \ \forall \ g \in G$ which shows that H is normal, a contradiction. So it must be that $n_H = p$.

Problem (5)**.**

(a) First $|S_5| = 5! = 120 = 2^3 \cdot 3 \cdot 5$. So the Sylow 2-subgroups are conjugates of order 8. By Sylow, n_2 divides 15. We already know that S_4 has three distinct copies of D_8 and there are $\binom{5}{4} = 5$ distinct copies of S_4 in S_5. Thus n_2 is at least $3 \times 5 = 15$. Thus it must be 15. (To form D_8 in S_4, we just need to pick any 4-cycle as the rotation. The reflection comes from one of the transposition in the square of this 4-cycle which is a double transposition. This completely determines D_8. There are $3! = 6$ distinct 4 cycles in S_4. Since each D_8 requires two, we have $6/2 = 3$ copies of D_8 in S_4).

(b) We know that for the S_4 copy on $\{1,2,3,4\}$, the V_4 of double transpositions $e,(1,3)(2,4)$ $(1,2)(3,4),(1,4)(2,3)$, a 4 cycle and its inverse $(1,2,3,4),(1,4,3,2)$, and two single transpositions $(1,3),(2,4)$. Similarly, S_4 on $\{1,2,3,5\}$ yields another D_8. We simply conjugate one by $(4,5)$ to obtain the other.

Problem (6)**.**

(a) Since p is an odd prime, any $P \in Syl_p(D_{2n})$ cannot contain any reflection or 2 would divide p by Lagrange. Therefore, any element in P is a power of rotation r. The smallest such power is clearly a generator of P so P is cyclic. Since r commutes with its powers, $rPr^{-1} = P$. Since $sr^i s = r^{-i}$, $s$acts on P as inversion which is clearly an automorphism so $sPs^{-1} = P$. Since r, s generates D_{2n}, we see that $P \trianglelefteq D_{2n}$.

(b) Let $n = 2^k m$ so given $P \in Syl_2(D_{2n})$, $|P| = 2^{k+1}$. Since $o(r) = n = 2^k m$, we see that $o(r^m) = 2^k$ so r^m must be in some Sylow 2-groups and so does its powers. Moreover, all reflections must also be evenly distributed in Sylow 2-groups FIX: since conjugation preserves cycle type). We see that $\langle s, r^m \rangle$ consists of elements of the form $s^i(r^m)^j$ where $i = 0, 1$, $0 \leq j \leq 2^k - 1$. That is, the order of the group is 2^k rotations and 2^k reflections with a total order of 2^{k+1}. This is precisely the order of a Sylow 2-subgroup. Thus, each Sylow 2-subgroup contains 2^k reflections, so $n_2 = 2^k m / 2^k = m$.

Problem (7)**.**

(a) $|G| = pqr, p < q < r$. By Sylow, $n_r | pq$ and $n_r = 1 \bmod r$. Since $p < q < 1 + kr$ for $k > 0$, it must be that $n_r = 1$ or pq. Suppose $n_r = pq$. Then by Lagrange we know these prime order groups all intersect trivially so we have $pq(r-1)$ distinct nontrivial elements from these groups. Now consider $n_q | pr$ and $n_q = 1 \bmod q$. Again since $p < 1 + kq$ for $k > 0$, $n_q = 1, r, pr$. Suppose $n_q = r$. Then we have $(q-1)r \geq pr > pq$ additional nontrivial distinct elements. That is, there are $pq(r-1) + (q-1)r > pq(r-1) + pq = pqr$ distinct elements from these two Sylow subgroups alone, a contradiction. Hence at least one of them must unique and thus is normal, which implies that G is not simple.

(b) Since at least one of them is normal, if R is normal we are done. If $P \trianglelefteq G$, then G/P is a quotient group. Let $\overline{S} \in Syl_r(G/P)$. Then $\overline{n}_r = 1$ or p and $\overline{n}_r = 1 \bmod r$.

Since $p \neq 1 \bmod r$ as $p < r$, we have $\overline{n}_r = 1$ so $\overline{S} \trianglelefteq G/P$. By the 4th isomorphism theorem, there exists a corresponding subgroup $S \trianglelefteq G$ s.t. $S/P = \overline{S}$. We see that $|S| = |\overline{S}||P| = pr$. Therefore, $P \leq S$ and there exists an $R \in Syl_r(G)$ s.t. $R \leq S$. Since $P \trianglelefteq G$, PR is a subgroup and $PR \leq S$. Since $P \cap R = \{e\}$, by prime and Lagrange $|PR| = pr/1 = pr = |S|$ so $PR = S$ and thus $PR \trianglelefteq G$. Now, let n_r' be the number of Sylow r-subgroups in PR. Then $n_r' = 1$ or p and $n_r' = 1 \bmod r$ so $p < r$ forces $n_r' = 1$. That is, R is the unique subgroup of order r of PR and therefore characteristic in PR. By "transitivity" of characteristic subgroups of a normal subgroup, $R \trianglelefteq G$ as desired. The case when $Q \trianglelefteq G$ is similar.