1 Tangent Spaces

There is no linear structure on manifolds.

Example 1.1 (tangent space on $\mathbb{R}^n$)

$T_p\mathbb{R}^n = \{(p,q) : q \in \mathbb{R}^n\}$. This is a vector space. Note that $T_p\mathbb{R}^n \cong \mathbb{R}^n, (p,q) \mapsto q - p$. This is canonical. We lose this canonical isomorphism in other manifolds.

From the submanifold perspective:

Claim 1.2. $\dim(T_pM) = n$.

Proof.

$$\alpha = \phi^{-1} \circ (\phi \circ \alpha)$$
$$\alpha'(0) = D_0\phi^{-1}(\phi \circ \alpha)'(0)$$

Claim 1.3. $\operatorname{rank} D_0\phi^{-1} = n$.

Since $\overline{\phi} \circ \phi^{-1} = \mathrm{id}_{\phi(U)}$, by chain rule,

$$D\overline{\phi} \circ D\phi^{-1} = I$$

which yields that $\operatorname{rank} D\overline{\phi} \circ D\phi^{-1} = n$. It follows that $\operatorname{rank} D\phi_{-1 \geq n}$ which yields equality. $\square$

From the intrinsic perspective, we define $df_p : T_pM \to T_{f(p)}N$. Given $v \in T_pM$, where $v = \alpha'(0)$ for some curve α that passes through p. Then define

$$df_p(v) := (f \circ \alpha)'(0)$$

Exercise: If $M = \mathbb{R}^n$ and $N = \mathbb{R}^m$, then $df_p(v)$ reduces to the Jacobian at p multiplying by v (directional derivative).

Define $\gamma'(0)$ by operating on functions $M \to \mathbb{R}$:

$$[\gamma'(0)](f) := (f \circ \gamma)'(0)$$

So $\gamma \sim \widetilde{\gamma}$ iff $[\gamma(0)]f = [\gamma'(0)]f$ for all $f \in \mathcal{F}(M)$ space of smooth functions on M.

Proposition 1.4

T_pM is a vector space with natural operations and has the same dimension as that of the manifold.

Proof.

$$\begin{aligned}[\alpha'(0)]f &= (f \circ \alpha)'(0) \\ &= (f \circ x \circ x^{-1} \circ \alpha)'(0) \\ &= \sum_{i=1}^{n} \frac{\partial(f \circ x)}{\partial x_i}((x^{-1} \circ \alpha)^i)'(0)\end{aligned}$$

Define local coordinates via $x : \mathbb{R}^n \to M.x_i(t) = x(0, \ldots, 0, t, 0, \ldots 0)$ at ith entry. Then

$$[x_i'(0)]f = (f \circ x_i)'(0) = \frac{\partial(f \circ x)}{\partial x_i}(0) = \sum \lambda_i [x_i'(0)](f).$$

So $\alpha'(0)$ is a linear combination of $(x_i)'(0)$. It remains to show they are linearly independent. Suppose $\sum_i x_i'(0) = 0$. $\square$

Definition 1.5 — Let $f : M \to N$, $p \in M$, define $df_p(\alpha'(0)) = (f \circ \alpha)'(0)$.

Exercise: this is linear. If $M = \mathbb{R}^n$, $N = \mathbb{R}^m$, df_p is the Jacobian.

Theorem 1.6 (Chain Rule)

Let $f : M \to N$ and $g : N \to L$. Then

$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

Proof.

$$\begin{aligned}d(g \circ f)(\alpha'(0)) &= (g \circ f \circ \alpha)'(0) \\ &= dg((f \circ \alpha)'(0)) \\ &= dg \circ df(\alpha'(0))\end{aligned}$$

$\square$

Corollary 1.7

If M, N are diffeomorphic, then $\dim M = \dim N$.

Proof. Let $f : M \to N$ be the diffeomorphism, then

$$\begin{aligned} f \circ f^{-1} &= \mathrm{id}_M \\ df \circ df^{-1} &= I \end{aligned}$$

Thus df is invertible and $\dim M = \dim T_p M = \dim T_{f(p)} N = \dim N$. $\square$