# Homework 6

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**Problem** (LN15 0.4)**.** Show that the bracket satisfies the following properties: $[X, Y] = -[Y, X]$ and $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

*Proof.* The pointwise definition of Lie bracket can be expressed succinctly as $[X, Y] = XY - YX$. Then we have

$$[X, Y] = XY - YX = -(YX - XY) = -[Y, X],$$

and

$$\begin{aligned} [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] =& XYZ - XZY - YZX + ZYX \\ & YZX - ZYX - ZXY + YXZ \\ & ZXY - YXZ - XYZ + XZY \\ =& 0. \end{aligned}$$

□

**Problem** (LN15 0.5)**.** Show that a connection is symmetric iff the corresponding Christoffel symbol satisfy $\Gamma^k_{ii} = \Gamma^k_{ji}$.

*Proof.* Recall that in local coordinates, using Einstein notation we have

$$\nabla_X Y = \left( X(Y^k) + X^j Y^i \Gamma^k_{ij} \right) E_k.$$

Then by unifying $i, j$ indices of $\nabla_X Y$ and $\nabla_Y X$, we obtain

$$\begin{aligned} \nabla_X Y - \nabla_Y X &= \left( X(Y^k) - Y(X^k) + X^i Y^j \left( \Gamma^k_{ji} - \Gamma^k_{ij} \right) \right) E_k \\ &= X(Y^k) E_k - Y(X^k) E_k + X^i Y^j \left( \Gamma^k_{ji} - \Gamma^k_{ij} \right) E_k \\ &= XY - YX + X^i Y^j \left( \Gamma^k_{ji} - \Gamma^k_{ij} \right) E_k \\ &= [X, Y] + X^i Y^j \left( \Gamma^k_{ji} - \Gamma^k_{ij} \right) E_k. \end{aligned}$$

Since $X, Y$ are arbitrary, we see that $\nabla_X Y - \nabla_Y X = [X, Y]$ iff $\Gamma^k_{ji} - \Gamma^k_{ij} = 0$. □

**Problem** (do Carmo 3.1)**.**

*Proof.* First, we compute the Jacobian of $\phi$:

$$\begin{aligned} D\phi(u,v) &= \begin{pmatrix} \frac{\partial \phi}{\partial u}(u,v) & \frac{\partial \phi}{\partial v}(u,v) \end{pmatrix} \\ &= \begin{pmatrix} -f(v)\sin u & f'(v)\cos u \\ f(v)\cos u & f'(v)\sin u \\ 0 & g'(v) \end{pmatrix}. \end{aligned}$$

Since $f'(v)^2 + g'(v)^2 \neq 0$ and $f(v) \neq 0$, we can check

$$\begin{aligned} \langle \frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial u} \rangle &= f(v)^2 \neq 0 \\ \langle \frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v} \rangle &= 0 \\ \langle \frac{\partial \phi}{\partial v}, \frac{\partial \phi}{\partial v} \rangle &= f'(v)^2 + g'(v)^2 \neq 0 \end{aligned}$$

That is, they are nonzero and their dot product is 0, so they are orthogonal. Thus $D\phi$ is injective everywhere, *i.e.* $\phi$ is an immersion.

(a) Recall that the induced metric is just the pairwise dot products of the pushforward basis under $\phi$. Thus by the above computation, we have $g_{11} = f^2$, $g_{12} = 0$, and $g_{22} = (f')^2 + (g')^2$, which are all functions of $v$.

(b) Then $g^{11} = \frac{1}{f^2}$, $g^{12} = 0$, $g^{22} = \frac{1}{(f')^2 + (g')^2}$. Based on the equation $\Gamma^m_{ij} = \frac{1}{2} g^{km} \left( -\frac{\partial g_{ij}}{\partial x_k} + \frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ki}}{\partial x_j} \right)$, we obtain

$$\begin{aligned} \Gamma^1_{11} &= \frac{1}{2f^2}(-0+0+0) + 0 = 0 \\ \Gamma^2_{11} &= 0 + \frac{1}{2((f')^2 + (g')^2)}(-2ff' + 0 + 0) = -\frac{ff'}{(f')^2 + (g')^2} \\ \Gamma^1_{12} &= \frac{1}{2f^2}(-0+0+2ff') + 0 = \frac{ff'}{f^2} \\ \Gamma^2_{12} &= 0 + \frac{1}{2((f')^2 + (g')^2)}(-0+0+0) = 0 \\ \Gamma^1_{21} &= \Gamma^1_{12} = \frac{ff'}{f^2} \\ \Gamma^2_{21} &= \Gamma^2_{12} = 0 \\ \Gamma^1_{22} &= \frac{1}{2f^2}(-0+0+0) + 0 = 0 \end{aligned}$$

$$
\begin{aligned}
\Gamma_{22}^2 &= 0 + \frac{1}{2((f')^2+(g')^2)}(-1+1+1)(2f'f''+2g'g'') \\
&= \frac{f'f''+g'g''}{(f')^2+(g')^2}.
\end{aligned}
$$

The equation of geodesic $\gamma(t) = (u(t), v(t))$ is

$$
\ddot{\gamma}^k + \dot{\gamma}^i\dot{\gamma}^j\Gamma_{ij}^k(\gamma) = 0
$$

$$
\begin{cases}
\ddot{u} + \dot{u}\dot{v}\frac{ff'}{f^2} + \dot{v}\dot{u}\frac{ff'}{f^2} & = 0 \\
\ddot{v} - \dot{u}^2\frac{ff'}{(f')^2+(g')^2} + \dot{v}^2\frac{f'f''+g'g''}{(f')^2+(g')^2} & = 0
\end{cases}
$$

$$
\begin{cases}
\ddot{u} + \frac{2ff'}{f^2}\dot{u}\dot{v} & = 0 \\
\ddot{v} - \frac{ff'}{(f')^2+(g')^2}\dot{u}^2 + \frac{f'f''+g'g''}{(f')^2+(g')^2}\dot{v}^2 & = 0
\end{cases}
$$

(c) We compute

$$
\begin{aligned}
|\dot{\gamma}|^2 &= \begin{pmatrix} \dot{u} & \dot{v} \end{pmatrix} \begin{pmatrix} f^2 & 0 \\ 0 & (f')^2+(g')^2 \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \\
&= f^2\dot{u}^2 + ((f')^2+(g')^2)\dot{v}^2.
\end{aligned}
$$

Taking the time derivative of this equation, using $\dot{f} = f'\dot{v}$, $\dot{f}' = f''\dot{v}$, and the first equation $\ddot{u} = -\frac{2ff'}{f^2}\dot{u}\dot{v}$ we obtain

$$
\begin{aligned}
&2ff'\dot{v}\dot{u}^2 + 2f^2\dot{u}\ddot{u} + 2(f'f''+g'g'')\dot{v}^3 + 2((f')^2+(g')^2)\dot{v}\ddot{v} \\
=&2\dot{v}(-ff'\dot{u}^2 + (f'f''+g'g'')\dot{v}^2 + ((f')^2+(g')^2)\ddot{v}).
\end{aligned}
$$

Since we exclude parallels, $\dot{v} \neq 0$. The energy is constant iff time derivative is 0, where we can simply divide the equation by $2\dot{v}((f')^2+(g')^2) \neq 0$ to obtain the second equation.

Let $P(t) = (a(t), b)$ where $b$ is constant. Then $\dot{P}(t) = (\dot{a}(t), 0)$. Recall that

$$
\begin{aligned}
\cos\beta &= \frac{\langle \dot{\gamma}, \dot{P} \rangle_{(u,v)}}{|\dot{\gamma}||\dot{P}|} \\
&= \frac{f^2\dot{u}\dot{a}}{|\dot{\gamma}||f\dot{a}|} \\
&= \frac{\operatorname{sgn}(f)\operatorname{sgn}(\dot{a})f\dot{u}}{|\dot{\gamma}|}.
\end{aligned}
$$

Using the fact that $\frac{d}{dt}|\dot\gamma| = 0$, the time derivative of $f(v)\cos\beta$ is

$$f'(v)\dot v\cos\beta + f(v)\frac{d}{dt}(\cos\beta) = \frac{\operatorname{sgn}(f)\operatorname{sgn}(\dot a) f f' \dot u \dot v}{|\dot\gamma|} + \frac{\operatorname{sgn}(f)\operatorname{sgn}(\dot a) f(f'\dot v\dot u + f\ddot u)|\dot\gamma| - 0}{|\dot\gamma|^2}$$
$$= \frac{\operatorname{sgn}(f)\operatorname{sgn}(\dot a)}{|\dot\gamma|}(f\ddot u + 2ff'\dot u\dot v).$$

Since $r\cos\beta = |f(v)|\cos\beta$, it is constant iff this time derivative is 0 iff the first equation holds.

(d) Since $r = |v|$, and $|v|\cos\beta$ is constant, the constant $c$ is either zero or nonzero. If $c = 0$, then since we can vary $v$ it must be that $\cos\beta = 0$. Since $\beta < \pi$, it must be that $\beta \equiv \frac{\pi}{2}$, which means the geodesic must intersect parallels at right angle all the time, making it a meridian which we exclude. Thus $c$ must be nonzero and WLOG let $c > 0$. This forces $\cos\beta > 0$. Since we can decrease the radius at will, $\cos\beta$ is forced to increase. But $\cos\beta$ max out at 1, so it must be that $\beta = 0$ precisely when $|v| = c$. Since we can no longer decrease $|v|$ further, and $|v|$ is not allowed to be constant, $|v|$ must increase instead. Therefore, $\gamma$ is going up again. If we can argue that $\gamma$ must always rotate around and is trapped between a minimum $r$ and a maximum $r$, we would complete the proof. But this seems tedious.

□

**Problem** (do Carmo 3.7)**.** Let $M$ be a Riemannian manifold of dimension $n$ and let $p \in M$. Show that there exists a neighborhood $U \subseteq M$ of $p$ and $n$ vector fields $E_1, \dots, E_n \in \mathfrak{X}(U)$, orthonormal at each point of $U$, s.t. at $p$, $\nabla_{E_i} E_j(p) = 0$. This is called a geodesic frame at $p$.

*Proof.* At point $p$, since $\exp_p$ is a local diffeomorphism, there exists open sets $V \subseteq T_pM$ around origin and $U \subseteq M$ around $p$ s.t. $V \cong U$ under $\exp_p$. Since for every point $v$ in $V$, we have a canonical orthonormal frame $F_1(v), \dots, F_n(v) \in T_v(T_pM)$, under the pushforward by diffeomorphism we obtain vector fields $E_i(u) := d(\exp_p)_v(F_i)$ in $\mathfrak{X}(U)$. By Gauss's Lemma, we obtain

$$\langle E_i, E_j\rangle = \langle d(\exp_p)_v(F_i), d(\exp_p)_v(F_j)\rangle = \langle F_i, F_j\rangle = 0.$$

Since the exponential map preserves length, we conclude that $E_i$ is an orthonormal frame of $U$. Moreover, we know that $d(\exp_p)_0$ is the identity, so $E_i(p) = F_i(0)$. In fact, they exactly coincide. Thus, $\nabla_{E_i(p)} E_j(p) = \nabla_{F_i(0)} F_j(0) = 0$, and $U$ is the neighborhood we seek. $\square$