1 Oriented Degree Theory

Lecture Notes 11.

Definition 1.1 — X is **orientable** if it admits an atlas $\{(U_i, \phi_i)\}$ s.t. $\det d(\phi_i \circ \phi_j^{-1}) > 0$.

Proposition 1.2

If X can be covered with two charts $(U_1, \phi_1), (U_2, \phi_2)$ and $U_1 \cap U_2$ is connected, then X is orientable.

Proof. If

$$\det d(\phi_1 \circ \phi_2^{-1}) < 0$$

we just replace ϕ_2 with $r \circ \phi_2$ where r is reflection about hyperplane. □

Corollary 1.3

S^n is orientable.

Proposition 1.4

Any closed embeded hypersurface in $\mathbb{R}^n$ is orientable.

Proof. Let $p : S^n \to \mathbb{R}P^n$ be the covering map, $r : S^n \to S^n, x \mapsto -x$ be antipodal reflection. Suppose that $\mathbb{R}P^n$ is orientable, then WLOG assume p is orientation preserving. Then r is orientation preserving, since $p = p \circ r$. But

$$r(b_1, \ldots, b_n, N) = (-b_1, \ldots, -b_n, -N)$$

Hence $\det dr = (-1)^{n+1}$. So it is a contradiction when n is even. □

Corollary 1.5

$\mathbb{R}P^2$ cannot be embedded in $\mathbb{R}^3$.

Exercise: $\mathbb{R}P^2 \times S^1$ is not orientable.

Exercise: If X is simply connected, then it is orientable.