

Homework 2

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Problem (2.25). (1) First, we show that (W_t) is a pre-BM. We shall use the second definition and show that it is a centered Gaussian process with $K(s, t) = s \wedge t$.

Uniform scaling doesn't affect the mean so W_t is still zero-mean. Since (B_t) is a centered Gaussian process, then clearly $\frac{1}{t}B_{\frac{1}{t}}$ are scalar multiple of centered Gaussians from the same centered Gaussian space and therefore a centered Gaussian process. Let $0 < s < t$, then we have $0 < \frac{1}{t} < \frac{1}{s}$ and

$$K(s, t) = \mathbb{E}[W_t W_s] = \mathbb{E}\left[t B_{\frac{1}{t}} s B_{\frac{1}{s}}\right] = ts \left(\frac{1}{t} \wedge \frac{1}{s}\right) = ts \frac{1}{t} = s = s \wedge t.$$

With $W_0 := 0$ with $K(0, 0) = 0 \wedge 0 = 0$, this proves that (W_t) is a pre-BM on $[0, \infty)$.

Now we wish to show that (W_t) has continuous sample path on $[0, \infty)$. For $(0, \infty)$, we know inversion $i : (0, \infty) \rightarrow (0, \infty), t \mapsto \frac{1}{t}$ is continuous, $t \mapsto B_t(\omega)$ is continuous for a fixed ω , and multiplication by t is continuous, so the composition $t B_{\frac{1}{t}}(\omega)$ is continuous on $(0, \infty)$. It remains to show that W_t is right continuous at $t = 0$.

Recall that in the proof of Komogorov lemma, we applied the analytic lemma to show that for a pre-BM (W_t) (since it satisfies the assumption of Komogorov), we have

$$\mathbb{P}(W_s(\omega), W_t(\omega)) \leq \frac{2k(\omega)}{1 - 2^{-\alpha}} |t - s|^\alpha,$$

for $t, s \in D$ (dyadic rationals that partitions the interval $[0, 1]$) some $k(\omega) < \infty$ a.s. and $\alpha \in \left(0, \frac{1}{2} - \frac{1}{q}\right)$ with $q > 2$. This implies that as $t \rightarrow 0$, over D we have a.s. $W_t \rightarrow W_0 := 0$. Since W_t is continuous on $(0, \infty)$ and D is dense in $[0, 1]$, we can extend this result to the entire $[0, 1]$ a.s. by the typical density argument. That is, $W_t(\omega)$ is continuous a.s. on $[0, \infty)$. Hence, (W_t) is indistinguishable of a BM.

(2) Since time inversion is an involution, we have $B_t = t W_{\frac{1}{t}}$ a.s., so we have a.s.

$$\lim_{t \rightarrow \infty} \frac{B_t}{t} = \lim_{t \rightarrow \infty} W_{\frac{1}{t}} = W_0 = 0.$$

Problem (2.28). We wish to use a fact from Theorem 2.21: for every $t > 0$, $\sup_{s \leq t} B_s$ has the same distribution as $|B_t|$. Since (B_t) is a Gaussian process, $|B_t|$ has continuous distribution

(wrt Σ_t) as well. It follows that $\sup_{s \leq t} B_s$ has continuous distribution for BM (B_t) wrt to Σ_t .

We want to get to the above sup form. Under sup we can WLOG subtract B_r from both LHS and RHS, which yields the LHS

$$\begin{aligned} \sup_{p \leq t \leq q} B_t - B_r &= \sup_{p \leq t \leq q} (B_t - B_p) + (B_p - B_r) \\ &= \sup_{t \leq q-p} B_t^{(p)} + (B_p - B_r). \end{aligned}$$

and the RHS

$$\sup_{r \leq t \leq s} B_t - B_r = \sup_{t \leq s-r} B_t^{(r)}.$$

Since $(B_t^{(p)})$ and $(B_t^{(r)})$ are also BMs with continuous distributions (wrt Σ_r and Σ_s), by the fact the sup terms have continuous distributions, and $B_p - B_r$ also have continuous distributions wrt to Σ_r . Since $p < q < r < s$, by Markov property the RHS is independent of Σ_r , which is the sigma algebra that makes LHS distributions continuous. Thus, the probability that two independent r.v.s with continuous distributions equal is 0 (which follows from separating the double integral where the inner integral is over a single point and thus 0). That is, $\mathbb{P}\left(\sup_{p \leq t \leq q} B_t \neq \sup_{r \leq t \leq s} B_t\right) = 1$.

Problem (2.29). The first event is equivalent to $A = \bigcap_{n=1}^{\infty} \sup_{0 < t \leq \frac{1}{n}} \left\{ \frac{B_t}{\sqrt{t}} = +\infty \right\}$. We wish to show that $A \in \Sigma_{0+}$ and $\mathbb{P}(A) > 0$ to conclude that $\mathbb{P}(A) = 1$ by Blumenthal's 0/1 law. The infimum case follows from symmetry.

Since A is a tail event, the first finitely many events don't matter in the intersection, so for all $r > 0$, $n \in \mathbb{N}$, we have

$$A = A_r := \bigcap_{n \geq \frac{1}{r}} \sup_{0 < t < \frac{1}{n}} \left\{ \frac{B_t}{\sqrt{t}} = +\infty \right\} =: \bigcap_{n \geq \frac{1}{r}} A_n.$$

Since $\frac{1}{n} < r$, by the Markov property, $A_n \in \Sigma_r$ so the countable intersection A_r is in Σ_r . Intersecting over all $r > 0$ yields $A \in \Sigma_{0+}$.

For $M > 0$ and $N \in \mathbb{N}$, we have

$$\mathbb{P}(A) \geq \limsup_{n \rightarrow \infty} \mathbb{P}(A_n = +\infty) \quad \text{Fatou}$$

$$\begin{aligned}
&\geq \limsup_{n \rightarrow \infty} \mathbb{P}(A_n > M) \\
&\geq \mathbb{P}(A_N > M) \\
&= \mathbb{P}\left(\frac{B_{\frac{1}{N}}}{\sqrt{\frac{1}{N}}} > M\right) \\
&= 1 - \mathbb{P}\left(\frac{B_{\frac{1}{N}}}{\sqrt{\frac{1}{N}}} \leq M\right) \\
&> 0,
\end{aligned}$$

where the last inequality comes from fact of the CDF of Gaussian distribution. It follows that $\mathbb{P}(A) = 1$.

To show that the right derivative doesn't exist, it suffices to show that the following limsup and liminf do not agree. For $s \in [0, \infty)$, we have

$$\begin{aligned}
\limsup_{t \rightarrow 0} \frac{B_{s+t} - B_s}{t} &= \limsup_{t \rightarrow 0} \frac{1}{\sqrt{t}} \frac{B_t^{(s)}}{\sqrt{t}} \\
&\geq \limsup_{t \rightarrow 0} \frac{B_t^{(s)}}{\sqrt{t}} = +\infty.
\end{aligned}$$

However,

$$\liminf_{t \rightarrow 0} \frac{1}{\sqrt{t}} \frac{B_t^{(s)}}{\sqrt{t}} < 0,$$

since the first term is positive and the second term tends to $-\infty$. Therefore, they disagree.

Alternatively, I believe we can use the same time rescaling proof technique we used for the non-monotone property to prove the statement directly (use rescaling $t \mapsto t\lambda^{-1}$, where $\lambda = \frac{\delta}{M}$). But since it doesn't directly use 0/1 law and instead use its consequence involving the sup, I omit it.