Homework 4

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Problem (1)**.** The Hamiltonian of the problem is given by

$$H(x_1, x_2, u, p_1, p_2) = \frac{1}{2}u^2 + p_1 x_2 + p_2(u - x_2).$$

The adjoint equations are given by

$$\dot{p_1} = -H_{x_1} = 0$$
$$\dot{p_2} = -H_{x_2} = p_2 - p_1$$

The first-order condition demands

$$\begin{aligned} H_u = u + p_2 &= 0 \\ u &= -p_2 \end{aligned}$$

Plugging this into the differential equations yield

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -x_2 - p_2$$

Together with the adjoint equations, we have 4 first-order equations and require 4 boundary conditions.

(a) Since all initial and final times and states are fixed, we have 4 boundary conditions $x_1(0) = x_2(0) = 0$, $x_1(3) = 1$, and $x_2(3) = 2$. Mathematica yields

$$\begin{aligned} u(t) &= \frac{6e^{3+t} + 6e^t - e^6 + 4e^3 - 3}{e^6 + 4e^3 - 5} \\ &= -0.6811 + 0.2642e^t. \end{aligned}$$

(b) When $x_2(3)$ is free, in its place we instead have the transversality condition $p_2(3) = 0$. This yields the solution

$$u(t) = -\frac{2e^3(e^t - e^3)}{3e^6 + 4e^3 - 1}$$

$$= 0.6256 - 0.0311e^t$$

(c) I would add a final penalty term Φ:

$$\mathcal{J} = \underbrace{\frac{1}{2}\left((x_1(3)-1)^2 + (x_2(3)-2)^2\right)}_{\Phi(3)} + \frac{1}{2}\int_0^3 u^2 dt.$$

Then instead of $p_1(3) = p_2(3) = 0$, we have

$$p_1(3) = \Phi_{x_1}(3) = x_1(3) - 1$$
$$p_2(3) = \Phi_{x_2}(3) = x_2(3) - 2$$

Then new control is

$$u(t) = \frac{8e^{3+t} + 6e^t + e^6 + 4e^3 - 3}{7e^6 + 8e^3 - 7}$$
$$= 0.1615 + 0.056e^t.$$

We see that $x_1(3) = 0.8385$ and $x_2(3) = 0.7142$, which are not very close to $(1, 2)$. To improve accuracy, I would increase the weight of the penalty term. We see that the cost in part (a) is 4.2859. The cost as a function of the weight coefficient c is shown in the figure below:

(d) It is clear that $\Psi = \begin{pmatrix} 2 & 5 \end{pmatrix}$. By transversality condition from Equation 5.234, we have the boundary conditions

$$\begin{pmatrix} -p_1(3) \\ -p_2(3) \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \lambda$$

Together with the initial conditions $x_1(0) = x_2(0) = 0$ and the terminal condition $2x_1(3) + 5x_2(3) = 20$, we have 5 boundary conditions for 4 differential equations and an unknown λ. This allows us to solve by Mathematica and obtain the optimal control:

$$u(t) = -p_2(t) = 1.4341 + 0.1071e^t$$

(e) We have the same Ψ but $h(t) = 20 + \frac{t^2}{2}$ so $\dot{h}(t_f) = t_f$. When t_f is also free, based on Equation 5.234 we have the transversality conditions

$$\begin{pmatrix} H(t_f) \\ -p_1(t_f) \\ -p_2(t_f) \end{pmatrix} = \begin{pmatrix} -t_f \\ 2 \\ 5 \end{pmatrix} \lambda$$

Note that by plugging in $u = -p_2$, we have

$$H = -\frac{1}{2}p_2^2 + (p_1 - p_2)x_2$$
$$H(t_f) = -\frac{25}{2}\lambda + 3x_2(t_f) = -t_f$$

Together with two initial conditions and the terminal condition

$$2x_1(t_f) + 5x_2(t_f) = 20 + \frac{t_f^2}{2},$$

we have a total of 6 boundary conditions to match the 4 differential equations and two unknowns λ and t_f. Mathematica yields

Problem (2)**.** The Hamiltonian is

$$H = \frac{1}{2}u^2 + p(ax + bu)$$

The adjoint equation is

$$\dot{p} = -H_x = ap \Rightarrow p(t) = Ce^{at}.$$

And the first-order condition is

$$H_u = u + bp = 0 \Rightarrow u = -bp$$

Thus

$$\dot{x} = ax - b^2p = ax - b^2Ce^{at}, \quad x(0) = x_0, x(tf) = 0$$

We have $x(t) = b^2Cte^{at} + Ae^{at}$, $x(0) = A = x_0$, and

$$x(t_f) = b^2Ct_fe^{at_f} + x_0e^{at_f} = 0$$
$$C = -\frac{x_0}{b^2t_f}$$

Thus,

$$u(t) = -bp = -b \cdot \left(-\frac{x_0}{b^2t_f}\right)e^{at} = \frac{x_0}{bt_f}e^{at}$$

Problem (3)**.** We see that $\Phi(t) = \frac{1}{2}(x(t)-1)^2$ and the Hamiltonian is

$$H = \frac{1}{2}(x^2 + u^2) + pu.$$

The adjoint equation is

$$\dot{p} = -H_x = x$$

The first-order condition says

$$H_u = u + p = 0 \Rightarrow u = -p.$$

Since $x(2)$ is free, we have the transversality condition

$$p(2) = \Phi_x(2) = x(2) - 1.$$

Thus we have two boundary conditions for two differential equations. Mathematica yields

$$u(t) = -p(t) = \frac{\sin(1-t)}{\cos 1 - \sin 1}.$$

The associated cost is 14.9283.

Problem (5)**.** The cost has the Meyer form $J = t_f$ so $\Phi(t) = t$. The Hamiltonian is

$$H(x, y, p_1, p_2) = p_1 r \cos\beta + p_2 r \sin\beta.$$

The adjoint equations are

$$\begin{aligned}\dot{p_1} &= -H_x = -p_1 \frac{x}{r}\cos\beta - p_2 \frac{x}{r}\sin\beta \\ \dot{p_2} &= -H_y = p_1 \frac{y}{r}\cos\beta - p_2 \frac{x}{r}\sin\beta\end{aligned}$$

First-order condition yields

$$\begin{aligned}H_\beta = -p_1 r \sin\beta + p_2 r \cos\beta &= 0 \\ \tan\beta &= \frac{p_2}{p_1}\end{aligned}$$

Since t_f is free, transversality demands $H(t_f) = -\Phi_t(t_f) = -1$ and