1 Characteristic Classes

There is another way to think of Steifel-Whitney classes:

Theorem 1.1

There exists a unique function $w_i : \text{Vect}(M) \to H^i(M; \mathbb{Z}/2)$, where $\text{Vect}(M)$ is the set of vector bundles over M, for all M and i satisfying

(1) $w_i(f^*E) = f^*(w_i(E)) \;\forall\; f : M \to N$

(2) $w_0(E) = 1, w_1(E) = 0 \;\forall\; i >$ fiber dimension of E.

(3) $w(E_1 \oplus E_2) = w(E_1) \smile w(E_2)$ where $w(E) = 1 + w_2(E) + w_2(E) + \ldots$. $E_1 \oplus E_2$ has fibers the direct sum of fibers of E_1 and E_2. That is, given $(E, M, \mathbb{R}^m), (E, N, \mathbb{R}^n)$ we get $(E_1 \times E_2, M \times N, \mathbb{R}^m \times \mathbb{R}^n)$. If $M = N$, let $\Delta : M \to M \times M$ be the diagonal map. Define $E_1 \oplus E_2 = \Delta^*(E_1 \times E_2)$.

(4) $w_1(\gamma_n) \neq 0$ where γ_n is the universal line bundle over $\mathbb{R}P^n$.

For 4 recall

$$\gamma_n = \{(\ell, v) \in \mathbb{R}P^n + \mathbb{R}^{n+1} : v \in \ell\}.$$

Exercise: γ_n is a line bundle over $\mathbb{R}P^n$.

Recall $H^i(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2 \;\forall\; 0 \leq i \leq n$. So 4) implies $w_1(\gamma_n)$ generates $H^1(\mathbb{R}P^n l \mathbb{Z}/2)$.

get γ over $\mathbb{R}P^\infty$ and 4) implies $w_i(\gamma) \neq 0$.

Exercise:

(1) $i : \mathbb{R}P^n \to \mathbb{R}P^m$ we get $i^*(\gamma_m) = \gamma_n$. If $w_1(\gamma_1) \neq 0$ then true for $\gamma_n \;\forall\; n$.

(2) Show $\mathbb{R}P^1 \cong S^1$ and γ_1 is infinite Mobius band. This is non-orientable, so from above $w_1(\gamma_1) \neq 0$.

It remains to prove part 3 and uniqueness in the theorem. First we look at some consequences.

Easy consequences:

(1) If $E_1 \cong E_2$ then $w_i(E_1) = w_i(E_2) \;\forall\; i$ by 1.

(2) If E is a trivial bundle, then $w_i(E) = 0 \; \forall \; i > 0$. This follows from obstruction theory. Let's check using Theorem 5. If $E \to M$ is trivial, let $f : M to \{x_0\}$, then $f^*(\{x_0\} \times \mathbb{R}^n) = E$. So $w_i(E) = f^*(w_i(\{x_0\} \times \mathbb{R}^n)) = 0$.

(3) If E' is a trivial bundle and E any vector bundle then

$$w_i(E \oplus E') = w_i(E)$$

from 3.

Recall Whitney showed that any n-manifold embeds in $\mathbb{R}^{2n}$ and immerses in $\mathbb{R}^{2n-1}$.

Theorem 1.2

If $\mathbb{R}P^{2^r}$ is immersed in $\mathbb{R}^{2^r+k}$ then k must be at least $2^r - 1$.

That is, Whitney's Theorem cannot be improved for all manifolds.

Note: if $f : M^n \to \mathbb{R}^k$ is an embedding, then we have the normal bundle of $f(M^n)$:

$$\nu(M) = \{v \in T_x\mathbb{R}^k : v \perp T_xM \; \forall \; x \in f(M)\}.$$

And $TM \oplus \nu(M) = T\mathbb{R}^k|_M = M \times \mathbb{R}^k$.

Exercise: show $\nu(M)$ is well-defined if f is just an immersion and we still have

$$TM \oplus \nu(M) = f^*T\mathbb{R}^k = M \times \mathbb{R}^k.$$

Theorem 1.3

If M is the boundary of a compact manifold W, then all Stiefel-Whitney numbers are zero.

Remark 1.4 The converse is also true by Thom.

Definition 1.5 — Given two unoriented manifolds M_1 and M_2, we say they are **unoriented cobordant** if there exists a compact manifold W s.t. $\partial W = M_1 \cup M_2$.

Corollary 1.6

Two closed, connected manifolds M_1 and M_2 are unoriented cobordant iff they have the same Stiefel-Whitney numbers.