MATH 6341, Fall 2023

Midterm, November 3, 2023

Name: ______________________________ GT ID#: ______________________________

Guideline: Please read the following carefully.
Print your name first. Remember to show all your work; including all intermediate steps and also explain in words how you are solving a problem. Partial credits are available for most problems. **Correct answers without major steps will only receive minor portion of the credits of the whole problem.** Work by your own, no discussion with others is permitted.

Problem 1 (30 pts) Identify the types of the following equation

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0,$$

and then transfer it into standard form.

(Additional page for Problem 1)

Problem 2 (30 pts) Consider the following initial value probem

$$\begin{cases} u_t + uu_x = -2u, \;\; x \in \mathbf{R}, t > 0, \\ u(x,0) = u_0(x) \in C^1(\mathbf{R}). \end{cases}$$

Here, $u_0(x)$ has bounded C^1-norm.

a) (20 pts) Determine the sufficient and necessary conditions on the initial data $u_0(x)$ for this problem to have a unique global smooth solution.

b) (10 pts) If $u_0(x)$ satisfies the conditions found in part a), prove that the global solution $u(x,t)$ satisfies that $\|u(x,t)\|_{L^\infty(\mathbf{R})}$ converges to zero as t goes to infinity.

Problem 3 (20 pt) Let $B(0,1)$ be the unit ball in $\mathbf{R^3}$ centered at the origin. Find a bounded solution to the following Dirichlet problem outside $B(0,1)$

$$\begin{cases} -\Delta u(x) = 0, \ |x| > 1, \\ u(x) = \dfrac{2}{\sqrt{7+4\sqrt{3}x_3}}, \ for \ |x| = 1. \end{cases}$$

Problem 4 (20 pt) Let $D(0, r)$ be the disk on $\mathbf{R^2}$ centered at the origin with radius r with boundary C . Find the function $u(\rho, \theta)$ in polar coordinates so that it is harmonic on $D(0, r)$ and $u(r, \theta) = 1 + cos^2(\theta)$ on C.

Hint: $cos^2(\theta) = \frac{1}{2}(1 + cos(2\theta))$.