1 Smooth Manifolds

We can define smooth functions from any set $X \subseteq \mathbb{R}^n$ if we can extend domain of f to an open set U s.t. $X \subseteq U$ so that $\overline{f}$ is smooth.

Transition maps need to be smooth in order to define smooth maps between manifolds. It has to be independent of choice of charts. Exercise: show this.

Lemma 1.1

$\psi \circ f \circ \phi^{-1}$ is smooth iff $\widetilde{\psi} \circ f \circ \widetilde{\phi}^{-1}$.

Proof.

$$\widetilde{\psi} \circ f \circ \widetilde{\phi}^{-1} = \widetilde{\psi} \circ \psi^{-1} \circ (\psi \circ f \circ \phi^{-1}) \circ \phi \circ \widetilde{\phi}^{-1}$$

which is a composition of smooth functions. □

Example 1.2

S^{n-1} with atlas stereographic projection. $U^+ = S^{n-1} - N$ and $U^- = S^{n-1} - S$. We have $\pi^+ : U^+ \to \mathbb{R}^{n-1}$ and $\pi^- : U^- \to \mathbb{R}^{n-1}$. We can compute that the transition map $\pi^- \circ \pi^{+-1}(z) = \frac{z}{|z|^2}$ which is the inversion map. Or we can show that $y = \frac{1}{x}$ using elementary geometry.

Facts:

(1) Open subset of manifold is a manifold.

(2) If M, N are manifolds, so are their product.

Example 1.3 (1) $\mathcal{M}^{n\times m}$ the space of $n \times m$ matrices.

(2) GL_n is an open subset of $\mathcal{M}^{n\times n}$, which has dimension n^2.

(3) $T^n = S^1 \times \cdots \times S^1$ the n-dimensional torus.

(4) RP^n. $U \subseteq RP^n$ is open if $U \cap S^n$ is open. Let $U_i \subseteq RP^n$ be the lines which intersect $x_i = 1$. Define $\phi_i : U_i \to \mathbb{R}^n, \phi_i(\ell) = \ell \cap \{x_i = 1\}$ and drop the last

coordinate. Another way is to identify antipodal points.

Theorem 1.4

IF G is a group of homeomorphisms acting freely and properly discontinuously etc (nicely), then M/G is again a manifold.