Homework 1

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Problem (1)**.** Show that all plane fields can be locally written as $\ker \alpha$ for some α.

Proof. Fix a Riemannian metric on M so we have the notion of orthogonality. Given $p \in M$, take a contractible neighborhood U of p and let $\ell := (\xi|_U)^\perp$ be the line bundle formed by orthogonal complements of ξ over U. Since U is contractible, $\ell \cong U \times \mathbb{R}$. Take any nonzero smooth section s of ℓ, say $s : x \mapsto (x, 1)$. Now define a 1-form α of U by

$$\alpha : U \to T^*U = (\xi|_U)^* \oplus \ell^*, x \mapsto (0, s),$$

where $0 : \xi|_U \to \mathbb{R}$ is the zero function. From this definition, it is clear that α is a smooth section of T^*U and thus a smooth 1-form and $\xi|_U = \ker \alpha$ as desired. □

Problem (2)**.** Let M be an orientable manifold. Show that TFAE:

(1) ξ can be written as $\ker \alpha$ for some α.

(2) There exists a vector field v transverse to ξ for all $p \in M$.

(3) ξ is orientable.

Proof. (1) $\Rightarrow$ (2) : Let $\ell := \xi^\perp$ be the global line bundle. Then $\alpha|_{\ell^*}$ is a nonzero smooth section of ℓ^*. It follows that $(\alpha|_{\ell^*})_p$ is a linear isomorphism from $\ell_p \to \mathbb{R}$. Therefore, there is a unique vector in ℓ_p that is mapped to 1. By thinking $\alpha|_{\ell^*}$ as a smooth function from $\ell \to \mathbb{R}$ with trivial kernel, we have that $v := (\alpha|_{\ell^*})^{-1}(1)$ is a smooth vector field in ℓ, which is transversal to ξ.

(2) $\Rightarrow$ (1): Such vector field v gives a basis for ℓ. Construct α by $\alpha(\xi) = 0$ and $\alpha(v) = 1$ (which determines where ℓ is mapped). This is clearly a smooth section with $\ker \alpha = \xi$ globally.

(2) $\Rightarrow$ (3) : Such vector field v gives a smoothly varying basis for ℓ, *i.e.* an orientation. Since M is orientable, we have TM orientable. As $\xi^\perp$ is also orientable, we can thus orient ξ.

(3) $\Rightarrow$ (2) : If ξ is orientable, and M is orientable by assumption, then ℓ is also orientable.

Fix an atlas of M. Given an orientation of ℓ and $p \in M$, we have a nonzero vector v_i for each chart U_i containing p, *i.e.* the basis vector for that chart. Since M is orientable, all the transition functions between charts are orientation preserving. That is, if p is in both U_i, U_j, then v_i and v_j differ by a positive scalar. Let $\{\phi_i\}$ be a partition of unity for the atlas. Then for each point p, we obtain a vector $v = \sum \phi_i v_i$ which is never zero since v_i's all positive scalar multiples of each other. Smoothness is provided by partition of unity, so we have the desired nonvanishing vector field. □

Problem (3)**.** Let $\alpha_3 = \cos r dz + r \sin r d\theta$. Show that $\alpha_3 \wedge d\alpha_3 > 0$.

Proof.

$$\begin{aligned} d\alpha_3 &= d(\cos r) \wedge dz + d(r \sin r) \wedge d\theta \\ &= -\sin r dr \wedge dz + (\sin r + r \cos r) dr \wedge d\theta \end{aligned}$$

$$\begin{aligned} \alpha_3 \wedge d\alpha_3 &= (\cos r dz + r \sin r d\theta) \wedge (-\sin r dr \wedge dz + (\sin r + r \cos r) dr \wedge d\theta) \\ &= (\cos r \sin r + r \cos^2 r) dz \wedge dr \wedge d\theta - r \sin^2 r d\theta \wedge dr \wedge dz \\ &= (\cos r \sin r + r) dz \wedge dr \wedge d\theta \\ &= \left(\frac{\sin 2r}{2r} + 1 \right) r dr \wedge d\theta \wedge dz \end{aligned}$$

Note that this is already in volume form. We can check that $\frac{\sin 2r}{2r} + 1 > 0$ for all $r > 0$. First, if $r > \frac{1}{2}$, then since $|\sin 2r| \leq 1$, we have positivity. If $0 < r \leq \frac{1}{2}$, then notice that let $x := 2r$ and by Taylor expansion,

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \dots$$

Clearly each negative term is dominated by the preceding positive term for $0 < x \leq 1$. The expression thus remains positive. □

Problem (4)**.** Prove Theorem I.4: Legendrian knots have contactomorphic neighborhoods.

Proof. Given two Legendrian knots $K_1 \subset (M_1, \xi_1), K_2 \subset (M_2, \xi_2)$, we wish to find a diffeomorphism from a neighborhood U_1 of K_1 to a neighborhood U_2 of K_2 that maps $\xi_1|_{K_1}$ to

$\xi_2|_{K_2}$. Then we finish the proof by wiggling U_2 using the isotopy from Theorem II.1 so that after appropriate shrinking of neighborhoods, K_1, K_2 have contactomorphic neighborhoods.

Take any diffeomorphism $f : K_1 \to K_2$ (both are circles). Since K_i are Legendrian, $T_{K_i}M_i \subset \xi_i$. Fix Riemannian metrics on M_i, then the normal bundles $\nu(K_i) = \ell_i \oplus \xi_i^{\perp}$, where ℓ_i is the orthogonal complement of $T_{K_i}M_i$ within ξ_i. Since $T_{K_i}M_i$ is an orientable S^1 vector bundle, it must be trivial so $T_{K_i}M_i \cong S^1 \times \mathbb{R}^3$. In particular, the fiber can be canonically identified as $TK_i \oplus \ell_i \oplus \xi_i^{\perp}$. Choose L to be a fiberwise linear isomorphism that maps ℓ_1 to ℓ_2, $\xi_1^{\perp}$ to $\xi_2^{\perp}$. Define $F : T_{K_1}M_1 \to T_{K_2}M_2, (x, (v, w, z)) \mapsto (f(x), (df_x(v), L(w, z)))$ which is a bundle map.

Here is a fact: the exponential map exp yields a diffeomorphism from a neighborhood of the zero section of any submanifold N in $\nu(N)$ to a neighborhood of N. This way, we obtain a neighborhood U_i of K_i that is diffeomorphic to a neighborhood of the zero section of $\nu(K_i)$. Then $\exp|_{U_2} \circ F \circ \exp|_{U_1}^{-1} : U_1 \to U_2$ is a diffeomorphism that takes $\xi_1|_{U_1}$ to $\xi_2|_{U_2}$ (after shrinking neighborhoods appropriately). $\square$