Homework 1

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Problem (1.1.1). (a)

$$\begin{aligned} u^* &= -Q^{-1}S \\ &= -\begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

Since $Q < 0$, u^* is a global maximum. Then

$$\begin{aligned} L^* &= -\frac{1}{2}S^TQ^{-1}S = \frac{1}{2}S^Tu^* \\ &= \frac{1}{2}\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} L_u &= Qu + S \\ &= \begin{pmatrix} -u_1 + u_2 \\ u_1 - 2u_2 + 1 \end{pmatrix} \end{aligned}$$

(b)

$$\begin{aligned} u^* &= -\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \end{aligned}$$

Since Q is indefinite, u^* is a saddle point.

$$\begin{aligned} L^* &= \frac{1}{2}S^Tu^* \\ &= \frac{1}{2}\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \end{aligned}$$

$$= \frac{1}{6}$$

$$\begin{aligned} L_u &= Qu + S \\ &= \begin{pmatrix} -u_1 + u_2 \\ u_1 + 2u_2 + 1 \end{pmatrix} \end{aligned}$$

TODO

Problem (1.1.2)**.** We see that

$$Q = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, S = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

So

$$\begin{aligned} x^* &= -Q^{-1}S = -\frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ -1 \end{pmatrix} \end{aligned}$$

So

Problem (1.1.3)**.**

$$\nabla f(x, y) = \begin{pmatrix} 2x \\ 4y^3 \end{pmatrix}$$

So $\nabla f(0, 0) = 0$, indeed a critical point. The Hessian is

$$\nabla^2 f(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & 12y^2 \end{pmatrix}$$

which is positive semidefinite and thus singular at the origin.

Since $x^2 \geq 0$ and $y^4 \geq 0$, it is clear that $f(x, y) \geq 0$. So $f(0, 0) = 0$ is indeed a minimum of f.

Problem (1.2.1)**.** The cost function is $L(x, y) = \frac{1}{2}(x - 20)^2 + \frac{1}{2}(y - 30)^2$ and the constraint is $F(x, y) = y - \sqrt{3}x = 0$. The Hamiltonian is

$$H(x, y) = L(x, y) + \lambda F(x, y).$$

Then

$$
\begin{aligned}
H_x &= x - 20 - \sqrt{3}\lambda = 0 \\
H_y &= y - 30 + \lambda = 0 \\
H_\lambda &= y - \sqrt{3}x = 0
\end{aligned}
$$

Solving this yields $x = \frac{1}{2}(15\sqrt{3} + 10)$ and $y = \frac{1}{2}(45 + 10\sqrt{3})$. The distance is then $d^* = \sqrt{x^2 + y^2} = 36$ miles. Thus the time is about $36/10 = 3.6$ hours.

Problem (1.2.2)**.** The cost function is $L(x_3, y_3) = \frac{1}{2}(x_3 - x_1)^2 + \frac{1}{2}(y_3 - y_1)^2 + \frac{1}{2}(x_3 - x_2)^2 + \frac{1}{2}(y_3 - y_2)^2$. The constraint is $F(x_3, y_3) = \frac{1}{2}(x_3 - x_1)^2 + \frac{1}{2}(y_3 - y_1)^2 - \frac{1}{2}(x_3 - x_2)^2 - \frac{1}{2}(y_3 - y_2)^2 = 0$. Therefore,

$$
\begin{aligned}
H_{x_3} &= x_3 - x_1 + x_3 - x_2 + \lambda(x_3 - x_1) - \lambda(x_3 - x_2) = 0 \\
H_{y_3} &= y_3 - y_1 + y_3 - y_2 + \lambda(y_3 - y_1) - \lambda(y_3 - y_2) \\
H_\lambda &= 0
\end{aligned}
$$

Solving this yields $x_3 =$