Homework

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We first write down the Lagrangian of the system:

$$\begin{aligned}
T &= \frac{1}{2}m\left(\left(L\dot{\theta}\right)^2 + \left(L\sin\theta\dot{\phi}\right)^2\right) \\
U &= mgL\cos\theta \\
\mathcal{L} &= T - U \\
&= \frac{mL^2}{2}\left(\dot{\theta}^2 + \left(\sin\theta\dot{\phi}\right)^2\right) - mgL\cos\theta
\end{aligned}$$

Now we determine the generalized momenta and express $\dot{\boldsymbol{q}}$ in terms of them:

$$\begin{aligned}
p_\theta &= \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mL^2\dot{\theta} \\
\dot{\theta} &= \frac{p_\theta}{mL^2} \\
p_\phi &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mL^2(\sin\theta)^2\dot{\phi} \\
\dot{\phi} &= \frac{p_\phi}{mL^2(\sin\theta)^2}
\end{aligned}$$

The Hamiltonian is given by

$$\begin{aligned}
\mathcal{H}(\theta, p_\theta, \phi, p_\phi) &= p_\theta\dot{\theta} + p_\phi\dot{\phi} - \mathcal{L} \\
&= \frac{p_\theta^2}{mL^2} + \frac{p_\phi^2}{mL^2(\sin\theta)^2} - \frac{mL^2}{2}\left(\frac{p_\theta^2}{(mL^2)^2} + \frac{p_\phi^2}{(mL^2\sin\theta)^2}\right) + mgL\cos\theta \\
&= \frac{1}{2mL^2}\left(p_\theta^2 + \frac{p_\phi^2}{(\sin\theta)^2}\right) + mgL\cos\theta \\
&= T + U
\end{aligned}$$

Since ϕ doesn't not explicitly show up in $\mathcal{H}$, it is an ignorable variable, *i.e.* $\frac{\partial\mathcal{H}}{\partial\phi} = 0$. Hamilton's equations immediately tell us that p_ϕ is a constant. Thus we reduce the degrees of freedom of the system by 1. This contrasts the Lagrangian approach where $\dot{q}$ can remain a variable and prevent us from reducing the degrees of freedom.

Thus the Hamiltonian becomes

$$\mathcal{H}(\theta, p_\theta) = \frac{p_\theta^2}{2\mu} + U_{eff}(\theta),$$

where $\mu = mL^2$ and $U_{eff}(\theta) = \frac{p_\phi^2}{2\mu(\sin\theta)^2} + mgL\cos\theta$ is the effective potential energy that solely depends on θ.

The equations of motions are

$$\dot{\theta} = \frac{p_\theta}{mL^2}$$
$$\dot{p_\theta} = -\frac{\partial \mathcal{H}}{\partial \theta} = \frac{\cos\theta p_\phi^2}{mL^2(\sin\theta)^3} + mgL\sin\theta$$