Homework 8

Jaden Wang

Problem (2.1.3)**.** Let s be a corner of the square and let f be the diffeomorphism that maps a neighborhood U of s to H^2, with the boundary(two edges) mapped to the boundary of H^2 which is the real line. Then v_1, v_2 depicted in Figure 2-4 are two smooth half curves based at s so they are in T_sS. They are also clearly linearly independent. However, since they are along the boundary, they are mapped into the real line, becoming linearly dependent. Thus df_s is singular so f cannot be a diffeomorphism, a contradiction. Thus S is not a manifold with boundary.

Problem (2.1.10)**.** Let $h : H^n \to \mathbb{R}$ be the height function which is clearly smooth as it is linear. It is easy to see that $h(z) = 0$ iff $z \in \partial H^n$ which is the real line. Now since X is a manifold with boundary, take $x \in \partial X$, take a chart (U, ϕ) around x. Then define $f = h \circ \phi$. Since ϕ maps boundary to boundary, we see that for any boundary point $z \in \partial U$, $\phi(z) \in \partial H^n$, and $f(z) = h(\phi(z)) = 0$. Moreover, if $f(z) = h(\phi(z)) = 0$, then $\phi(z)$ must be a boundary point in H^n, so $z = \phi^{-1}\phi(z) \in \partial U$.

If $z \in \partial U$, then the outward normal $n(z)$. It corresponds to a vector $v = (0, \ldots, 0, -a)$ where $a > 0$ pointing straight down in H^n, *i.e.* then the curve is vt. So

$$\begin{aligned} df_z(n(z)) &= (f \circ \phi^{-1}(vt))'(0) \\ &= (h(vt))'(0) \\ &= (t(-a))'(0) \\ &= -a < 0. \end{aligned}$$

Problem (2.2.3)**.** You can simply rotate the solid torus by an angle that isn't a multiple of 2π. The proof fails at the fact that the ray from $f(x)$ to x can hit the boundary more than once as the solid torus isn't convex, so $g(x)$ isn't well-defined. Even if we arbitrarily pick a point of intersection, some boundary points can be mapped to other boundary points, so $g(x)$ may not be identity on the boundary so we cannot apply the no-retract theorem. Moreover, $g(x)$ won't be continuous.

Problem (2.2.7)**.** Assume A is nonsingular with nonnegative entries. Consider the map $f : S^{n-1} \to S^{n-1}, v \mapsto Av/\|Av\|$. Suppose v is in Q (*i.e.* unit vector with nonnegative entries), then clearly Av with each entry being the sum of nonnegative numbers would remain in the first quadrant. Hence $Av/\|Av\|$ is in Q. So $f|_Q : Q \to Q$. Since $Q \cong B^{n-1}$, we obtain a continuous map $g : B^{n-1} \to B^{n-1}$. By Brouwer Fixed-Point Theorem for continuous maps, g has a fix point: there exists an $x \in B^{n-1}$ s.t. $g(x) = x$. The homemorphism yields that there exists a $v \in Q$ s.t. $f|_Q(v) = Av/\|Av\| = v$. So $Av = \|Av\|v$. By positive definiteness of norm, A has an nonnegative eigenvalue $\|Av\|$.

Problem (2.3.4)**.** Consider $F : X \times \mathbb{R}^n \to \mathbb{R}^n, (x, a) \mapsto x + a$. I claim that F is transversal to Y. First notice that F is linear so dF is just F. Furthermore notice that dF is surjective since given any $v \in \mathbb{R}^n$, just choose $x = 0$ and $a = v$ and we have $dF(0, v) = v$. This yields $dF_p(T_p(X \times \mathbb{R}^n)) = \mathbb{R}^n$. Thus we have

$$dF_p(T_p(X \times \mathbb{R}^n)) + T_{F(p)}Y = T_{F(p)}\mathbb{R}^n = \mathbb{R}^n.$$

By the Transversality Theorem, for almost every $a \in \mathbb{R}^n$, $f_a : X \times \{a\} \to x + a$ is transversal to Y. Since $f_a(X \times \{a\}) = X + a$, we show that $X + a$ is transversal to Y.

Problem (2.3.5)**.** Given $\varepsilon > 0$. Consider the inclusion map $i : X \to Y$. It is an embedding of X. By the corollary Tubular Neighborhood Theorem, there exists a $F : X \times B^n \to Y$ where $n = \dim Y$ s.t. $F_0 = i(x)$ and F is a submersion, *i.e.* $F \pitchfork Z$. Then by Transversality Theorem, for almost all $s \in B^n$, $F_s \pitchfork Z$. Since X is compact, by the generalized stability theorem of Exercise 1.6.11, there exists an $\varepsilon_1 > 0$ s.t. F_s is also an embedding if $|s| < \varepsilon_1$. Moreover, since F is continuous, we can take the closure of B^n to compactify the domain, so F is uniformly continuous. That is, there exists a $\delta > 0$ s.t. if $|s| < \delta$, for every $x \in X$ we have $|F(x, 0) - F(x, s)| = |x - i_s(x)| < \varepsilon$. Finally, we set $\varepsilon_2 = \min\{\delta, \varepsilon_1\}$ and choose any $|s_1| < \varepsilon_2$ that makes $F_{s_1} \pitchfork Z$. This means that X_s is a manifold and is transversal to Z, but since $\dim X = \dim X_s$ and $\dim X + \dim Z < \dim Y$, we see that X_s and Z can be transversal only if they do not intersect. Hence X_s is the deformation of X that doesn't intersect Z.

Problem (2.3.9)**.** Consider $F : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k, (x, a) \mapsto \left(\frac{\partial f}{\partial x_1} + a_1, \dots, \frac{\partial f}{\partial x_k} + a_k\right)$. I claim that F is a submersion. Notice F is linear in the second argument so dF with a fixed x is just

F with a fixed x. Given $v \in \mathbb{R}^k$, we can set $x = 0$ and $a = v$ so that $dF(0, v) = 0 + v = v$. So F is a submersion. Hence $F \pitchfork \{0\}$. Then by Transversality Theorem, $F_a : \mathbb{R}^k \times \{a\} \to \mathbb{R}^k$ is transversal to $\{0\}$ as well. Denote the hessian of f_a as H. Notice that $df_a = F_a$ so $H = dF_a$. Since F_a is transversal to $\{0\}$, dF_a is surjective so does H. Then H as a $k \times k$ matrix must be invertible and hence is nondegenerate. Therefore, any critical point of f_a must be nondegenerate so f_a is a Morse function.