1 Hopf Degree Theorem

Proof. Let $\overline{X} := X \times \{0,1\}$, $\overline{f}_0 : X \times \{0\} \to S^n, \overline{f}_1 : X \times \{1\}$. So $\overline{f} : \overline{X} \to S^n$ by putting them together. Let $W = X \times I$ then $\overline{X} = \partial W$. So $f_0 \sim f_1 \Leftrightarrow \overline{f}$ extends to F on W (so F is the homotopy). Recall $\deg(\partial F) = \deg f_1 - \deg f_0 = 0$. So Hopf Theorem is a corollary of the Extension Theorem. □

Theorem 1.1

Let $f : X^n \to S^n$, $X^n = \partial W$, then $\deg f = 0 \Leftrightarrow f$ can be extended to W.

Remark 1.2 It is important that the codomain is S^n.

Proof. The theorem holds if the codomain is $\mathbb{R}^{n+1}$ by the tubular neighborhood theorem. In this case, f extends to a neighborhood U of X in W: $x \in U$, $f(x) := f(\overline{x})$ where $\overline{x}$ is the unique closest point to x in X. Let $\rho : W \to I$ s.t. $\rho = 0$ on $W - U$ and $\rho = 1$ near X. Now define f on W by

$$f(x) := \begin{cases} 0, & x \notin U \\ \rho(x), & x \in U \end{cases}$$

So as long as the image misses one point (origin) in $\mathbb{R}^{n+1}$, we can always project the extension to S^n by projecting along the lines through origin.

□

Lemma 1.3

$f : \mathbb{R}^n \to \mathbb{R}^n$, $x \in \mathbb{R}^n$, if $f^{-1}(x)$ is finite and $\sum_{y \in f^{-1}(x)} \operatorname{sgn}(y) = 0$, then $f \sim g : \mathbb{R}^n \to \mathbb{R}^n \setminus \{0\}$.