1 Homotopy Theory

1.1 Homotopy classes of maps

Recall homotopy, denoted by $\simeq$.

Example 1.1

X is any space, $f : X \to I := [0,1]$ is homotopic to the constant map $g(x) = 0$ as I is convex. The homotopy is the straight-line homotopy $\Phi(x,t) = (1-t)f(x)$.

Let $C(X,Y) = \{\text{continuous functions from } X \to Y\}$. Denote $[X,Y] = C(X,Y)/\simeq$, so homotopic maps are identified.

Example 1.2

$[X,I] = \{g(x) = 0\}$.

In a pointed space, denote $[X,Y]_0$ to be the homotopy classes of morphisms from pointed spaces (X,x_0) to (Y,y_0). If $f : X \to X'$ a continuous function, then this induces a functor $f^* : [X',Y]_0 \to [X,Y]_0$ by precomposition. Likewise for postcomposition.

We have the covariant functor $[X,-] : \mathsf{Top} \to \mathsf{Set}$ and the contravariant functor $[-,X] : \mathsf{Top} \to \mathsf{Set}$.

Recall homotopy equivalence, also denoted by $\simeq$.

Example 1.3

$X = S^1$ and $Y = S^1 \times [0,1]$. These are homotopy equivalent. We have $f : X \to Y, \theta \mapsto (\theta, 0)$ and $g : Y \to X, (\theta,t) \mapsto \theta$.

Example 1.4

X, Y are any spaces, morphism $f : X \to Y$. The **mapping cylinder** is

$$C_f = ((X \times I) \cup Y)/\sim$$

where $(x,0) \sim f(x)$.

This is homotopy equivalent to Y. Exercise.

Show $\pi : C_f \to Y, (x,t) \in X \times [0,1] \to f(x), y \in Y \mapsto y$ has homotopy inverse. Note there is an inclusion $j : X \to C_f, x \mapsto (x,1)$.

Show $j \cong i \circ f$.

Moral: any map is an inclusion up to homotopy.

Definition 1.5 — A pointed space (Y, y_0) is called an **H-space** if there exists maps $\mu : Y \times Y \to Y$ and $\nu : Y \to Y$ s.t.

(1) for $i_1 : y \mapsto (y, y_0)$ and $i_2 : y \mapsto (y_0, y)$, we have

$$\mu \circ i_1 \simeq \mathrm{id}_Y, \mu \circ i_2 \simeq \mathrm{id}_Y.$$

(2) The compositions

$$Y \times (Y \times Y) \xrightarrow{\mathrm{id}_Y \times \mu} Y \times Y \xrightarrow{\mu} Y$$

and

$$(Y \times Y) \times Y \xrightarrow{\mu \times \mathrm{id}_Y} Y \times Y \xrightarrow{\mu} Y$$

are homotopic.

(3) The composition

$$Y \xrightarrow{\mathrm{id}_Y \times \nu} Y \times Y \xrightarrow{\mu} Y$$

$$Y \xrightarrow{\nu \times \mathrm{id}_Y} \times Y \times Y \xrightarrow{\mu} Y$$

are homotopic to constant maps.

Remark 1.6 This definition should remind us of group axioms: identity, association, and inverses. We see that μ hints at multiplication whereas ν hints at inversion.

Example 1.7

If G is a topological group (group with topology s.t. multiplication and inverses are continuous maps).

Exercise: (G, e) is an H-space.

Theorem 1.8

The set $[X, Y]_0$ has a natural group structure for all pointed spaces X iff Y is an H-space.

Natural means if $f : X \to X'$, then the induced map $f^* : [X', Y]_0 \to [X, Y]_0$ is a homomorphism. That is, $[-, Y]_0$ is a functor.

Proof. ($\Leftarrow$): suppose Y is an H-space. Given (X, x_0) and notice $\mu : Y \times Y \to Y$ induces $\mu^* : [X, Y \times Y]_0 \to [X, Y]_0$. There is also a canonical function

$$\phi : [X, Y]_0 \times [X, Y]_0 \to [X, Y \times Y]_0, ([f], [g]) \mapsto [f \times g].$$

Exercise: ϕ is well-defined and a bijection. Define multiplication

$$m = \mu^* \circ \phi : [X, Y]_0 \times [X, Y]_0 \to [X, Y]_0.$$

This is clearly well-defined since ϕ is well-defined and post-composing homotopic functions are still homotopic. Denote $m([f], [g])$ by $[f] \cdot [g]$. Denote $\nu_x([f])$ by $[f]^{-1}$. Let $e(x) = y_0$ be the constant map. Now we check the group axioms:

Identity: Tracking the representatives of $[e] \cdot [g]$ yields

$$\begin{aligned} \mu(y_0, g(x)) &= (\mu \circ i_1) \circ g(x) \\ &\cong \mathrm{id}_Y \circ g(x) = g(x) \end{aligned}$$

The other direction follows from using i_2. Thus $[e] \cdot [g] = [g] = [g] \cdot [e]$.

Associativity: Given $[f], [g], [h] \in [X, Y]_0$, we see that

$$\begin{aligned} ([f] \cdot [g]) \cdot [h] &= \mu^* \circ \phi([f], [g]) \cdot [h] \\ &= \mu^*([f \times g]) \cdot [h] \end{aligned}$$

$$
\begin{aligned}
&= [\mu \circ (f \times g)] \cdot [h] \\
&= [\mu \circ ((\mu \circ (f \times g)) \times h)] \\
&= [\mu \circ (\mu \times \mathrm{id}_Y) \circ f \times g \times h] \\
&= [\mu \circ (\mathrm{id}_Y \times \mu) \circ f \times g \times h] && \text{condition 2} \\
&= [\mu \circ (f \times (\mu \circ (g \times h)))] \\
&= [f] \cdot ([g] \cdot [h])
\end{aligned}
$$

Inverse: Given $[f] \in [X, Y]_0$, we have

$$
\begin{aligned}
[f] \cdot [f]^{-1} &= [\mu \circ (f \times (\nu \circ f))] \\
&= [\mu \circ (\mathrm{id}_Y \times \nu) \circ f] \\
&= [e]
\end{aligned}
$$

The other direction follows similarly.

($\Rightarrow$): Suppose $[X, Y]_0$ has a natural group structure for all pointed spaces X. Take $X = Y \times Y$, and p_1, p_2 be the projections onto 1st and 2nd factors respectively. This yields $[p_1], [p_2] \in [Y \times Y, Y]_0$. Let μ be a representative of $[p_1] \cdot [p_2]$. Let ν be a representative of $[\mathrm{id}_Y]^{-1}$.

Now check condition 1. $i_1 : Y \to Y \times Y, y \mapsto (y, y_0)$ induces i_1^* so that

$$
\begin{aligned}
i_1^*([p_1]) &= [p_1 \circ i_1] = [\mathrm{id}_Y] \\
i_1^*([p_2]) &= [e].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
i_1^*([\mu]) &= i^*([p_1] \cdot [p_2]) \\
&= [\mathrm{id}_Y] \cdot [e].
\end{aligned}
$$

Since $[e]$ is the identity on $[Y, Y]_0$, $[\mu \circ i_1] = [\ \mathrm{id}_Y]$.

2 and 3 are similar so left as exercises. $\square$

Definition 1.9 — If (Y, y_0) is a point space, then the **loop space** of Y is

$$\Omega(Y) = C^0((I, \{0,1\}), (Y, y_0)) = C^0((S^1, x_0), (Y, y_0)).$$

Lemma 1.10

$\Omega(Y)$ is an H-space.

Proof. Same as the proof for fundamental group. □

Definition 1.11 — Given pointed spaces $(X, x_0), (Y, y_0)$, the **wedge product** is $X \vee Y = (X \times \{y_0\}) \cup (\{x_0\} \times Y) \subseteq X \times Y$ with base point (x_0, y_0).

Definition 1.12 — A pointed space (Y, y_0) is an **H'-space** if there are maps $\mu : Y \to Y \vee Y$ and $\nu : Y \to Y$ s.t.

(1) $p_1 \circ \mu \simeq \mathrm{id}_Y$ and $p_2 \circ \mu \simeq \mathrm{id}_Y$ where $p_1, p_2 : Y \vee Y \to Y$ are projections onto the 1st and 2nd factors.

(2) The compositions

$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{\mathrm{id}_Y \vee \mu} Y \vee (Y \vee Y)$$

and

$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{\mu \vee \mathrm{id}_Y} (Y \vee Y) \vee Y$$

are homotopic.

(3) The compositions

$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{\nu \vee \mathrm{id}_Y} Y$$

and

$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{\mathrm{id}_Y \vee \nu} Y$$

are homotopic to the constant map.

Theorem 1.13

The set $[Y, X]_0$ has a natural group structure for all (X, x_0) iff Y is an H'-space.

Proof. Exercise. □

Definition 1.14 — Given a space X, its **suspension** is

$$\Sigma X = X \times I / \sim,$$

where $X \times \{0\}$ and $X \times \{1\}$ are collapsed to two distinct points.

Remark 1.15 If (X, x_0) is pointed, then $\sum X = X \times I/\{X \times \{0\}, X \times \{1\}, \{x_0\} \times I\}$ with base point $[\{x_0\}]$.

Example 1.16

Collapsing the two bases of a cylinder yields

(1) $S^n = \Sigma S^{n-1}$.

(2) $(S^n, x_0) = \Sigma(S^{n-1}, x_0)$.

Exercise: if M is any manifold and C is an arc in M, then prove M/C is homeomorphic to M. (hint: prove it for n-disk.)

Lemma 1.17

For any pointed space (Y, y_0), its suspension ΣY is an H'-space.

Proof. Define $\mu : \Sigma Y \to \Sigma Y \vee \Sigma Y$. See ipad. Exercise. $\square$

Theorem 1.18

If X is an H'-space and Y is an H-space, then the corresponding binary operations on $[X, Y]_0$ agree and are commutative.

Proof. Denote the binary operation from H'-space by $+$ and the other by $\cdot$. Let f_1, f_2 be maps representing elements in $[X, Y]_0$. See ipad for diagram. Let $\Delta : X \to X \times X$ be the diagonal map, $\nabla : Y \times Y \to Y$,Note that

$$[f_1] \cdot [f_2] = [\mu_Y \circ (f_1 \times f_2) \circ \Delta], [f_1] + [f_2] = [\nabla \circ (f_1 \vee f_2) \circ \mu_X].$$

Condition 1 of H'-space says $i \circ \mu_X \simeq \Delta$. Condition 1 of H-space says $\mu_Y \circ j \simeq \nabla$. Now $(x, x_0) \in X \vee X$, then $i(x, x_0) = (x, x_0) \in X \times X$. So

$$(f_1 \times f_2) \circ i(x, x_0) = (f_1(x), y_0)$$
$$f_1 \vee f_2(x, x_0) = (f_1(x), y_0)$$

$$j \circ (f_1 \vee f_2)(x, x_0) = (f_1(x), y_0)$$

Similarly for (x_0, x) so center square in the diagram commutes.

$$\begin{aligned}
\nabla \circ (f_1 \vee f_2) \circ \mu_X &\simeq \mu_Y \circ j \circ (f_1 \vee f_2) \circ \mu_X \\
&= \mu_Y \circ (f_1 \times f_2) \circ i \circ \mu_X \\
&\simeq \mu_Y \circ (f_1 \times f_2) \circ \Delta
\end{aligned}$$

It remains to show abelian. Fact: if $\rho : G \times G \to G, (g, h) \mapsto gh$ is a homomorphism then G is abelian. To see this, notice

$$\begin{aligned}
\rho((g, h)(g^{-1}, h^{-1})) &= \rho((gg^{-1}, hh^{-1})) \\
&= \rho(e, e) = e \\
\rho(g, h)\rho(g^{-1}, h^{-1}) &= ghg^{-1}h^{-1}
\end{aligned}$$

$\mu_Y : Y \times Y \to Y$ induces a homomorphism $[X, Y \times Y]_0 \xrightarrow{\mu_Y} [X, Y]_0$. We also have the bijection

$$\phi : [X, Y]_0 \times [X, Y]_0 \to [X, Y \times Y]_0, ([f], [g]) \mapsto [f \times g].$$

Claim 1.19. ϕ is a homomorphism.

Let $p_1 : Y \times Y \to Y$ be projection to 1st factor, which induces homomorphisms $p_i : [X, Y \times Y]_0 \to [X, Y]_0$. Clearly ϕ is the inverse of $(p_1)_* \times (p_2)_*$ so ϕ is a homomorphism. Then

$$(\mu_Y)_* \circ \phi : [X, Y]_0 \times [X, Y]_0 \to [X, Y]_0, ([f], [g]) \mapsto [f] \cdot [g]$$

So the group is abelian by the fact. □

Definition 1.20 — A space is **locally compact** if every point has a neighborhood that is contained in a compact set.

Lemma 1.21

If Y is locally compact and Hausdorff, there is a bijection

$$C^0(X \times Y, Z) \to C^0(X, C^0(Y, Z)).$$

If X is also Hausdorff, this is a homeomorphism. Note that C^0 is simply the Hom functor.

Remark 1.22 Any manifold or CW-complexes are locally compact.

Remark 1.23 The lemma implies that $[X \times Y, Z] = [X, C^0(Y, Z)]$.

Definition 1.24 — Given C a compact set in X, W an open set in Y, let $U(C, W) = \{f \in C^0(X, Y) : f(C) \subseteq W)\}$. This forms a subbasis for a topology on $C^0(X, Y)$, called **compact-open topology**.

Exercises:

(1) If Y is a metric space, show that this topology is the topology of compact convergence, *i.e.* $f_n \to f$ iff for all compact sets $C \subseteq X$, $f_n|_C \to f|_C$ uniformly.

(2) If $f : X \times Y \to Z$ is continuous, then so is

$$F : X \to C^0(Y, Z), x \mapsto f_x : Y \to Z, y \mapsto f(x, y)$$

(3) The converse is true if Y is locally compact.

(4) Prove the theorem.

Proof. We need a topology on C^0: the compact-open topology. $\square$

Definition 1.25 — The **smash product** of two pointed spaces is

$$X \wedge Y = X \times Y / X \vee Y = X \times Y / X \times \{y_0\} \cup \{x_0\} \times Y.$$

Recall that the **reduced suspension** is

$$\Sigma X = S^1 \wedge X = S^1 \times X / S^1 \times \{x_0\} \cup \{e_0\} \times X.$$

See ipad.

Corollary 1.26

If Y is locally compact, then

$$[X \wedge Y, Z]_0 = [X, C^0_{\text{based}}(Y, Z)]_0.$$

Proof. If $f \in C^0_{\text{based}}(X, C^0_{\text{based}}(Y, Z))$ then it has to send base point to base point: $f(x_0) =$ constant map $Y \to Z, y_0 \mapsto z_0$. So $F : X \times Y \to Z$, $(x, y) \mapsto f(x)(y)$ sends $\{x_0\} \times Y \to z_0$ by the lemma. As $f(x) : Y \to Z$ sends $y_0 \to z_0$, F induces a map on $X \wedge Y = X \times Y / X \vee Y \xrightarrow{F} Z$. So $F \in [X \wedge Y, Z]_0$.

We can similarly define an inverse: exercise. □

Recall that a loop space is $\Omega(X) = C^0_{\text{based}}(S^1, X)$.

Corollary 1.27

$[\Sigma X, Y]_0 = [X, \Omega(Y)]_0$. That is, suspension is the adjoint of looping.

Proof.

$$\begin{aligned} [\Sigma X, Y]_0 &= [S^1 \wedge X, Y]_0 \\ &= [X, C^0_{\text{based}}(S^1, Y)]_0 \\ &= [X, \Omega(Y)]_0 \end{aligned}$$

□

Remark 1.28 They are isomorphic as groups TODO

Definition 1.29 — The $\boldsymbol{n}$**th homotopy group** of (X, x_0) is

$$\pi_n(X) = [S^n, X]_0.$$

Note that

(1)

$$\begin{aligned}\pi_n(X) &= [S^n, X]_0 \\ &= [\Sigma S^{n-1}, X]_0 \\ &= [S^{n-1}, \Omega(X)]_0 \\ &= [S^0, \Omega^n(X)]_0\end{aligned}$$

So $\pi_n(X) = \pi_0(\Omega^n(X))$, the path components of $\Omega(X)$.

(2) $\pi_n(X) = [S^{n-1}, \Omega(X)]_0$. If $n \geq 2$, $\Omega(X)$ is H-space, S^{n-1} is H'-space, so by Theorem $\pi_n(X)$ is abelian for $n \geq 2$.

(3) If X is a Lie group, the it is an H-space so $\pi_1(X)$ is abelian.