Homework 4

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Problem (1). The Hamiltonian of the problem is given by

$$H(x_1, x_2, u, p_1, p_2) = \frac{1}{2}u^2 + p_1 x_2 + p_2(u - x_2).$$

The adjoint equations are given by

$$\dot{p}_1 = -H_{x_1} = 0$$

$$\dot{p}_2 = -H_{x_2} = p_2 - p_1.$$

The first-order condition demands

$$H_u = u + p_2 = 0$$

$$u = -p_2.$$

Plugging this into the differential equations yield

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - p_2.$$

Together with the adjoint equations, we have 4 first-order equations and require 4 boundary conditions.

(a) Since all initial and final times and states are fixed, we have 4 boundary conditions $x_1(0) = x_2(0) = 0$, $x_1(3) = 1$, and $x_2(3) = 2$. Mathematica yields
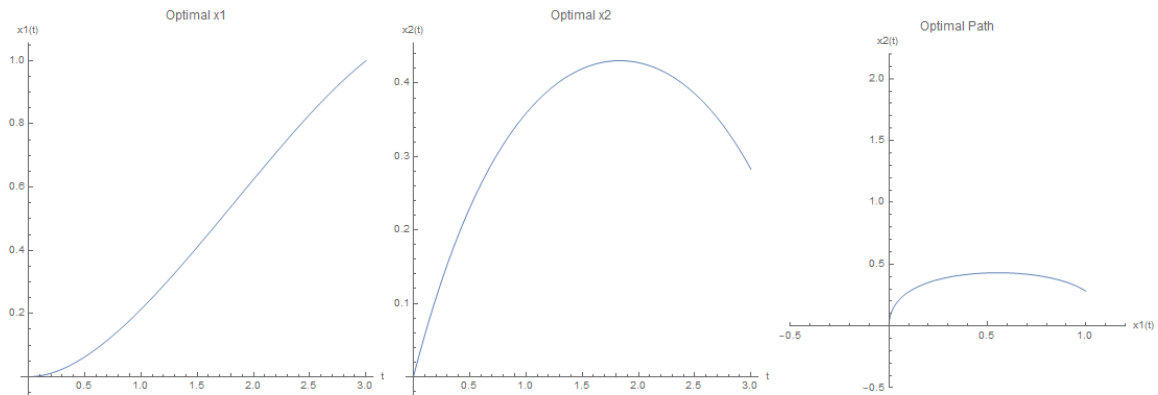
$$u(t) = \frac{6e^{3+t} + 6e^t - e^6 + 4e^3 - 3}{e^6 + 4e^3 - 5}$$

$$= -0.6811 + 0.2642e^t.$$

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Optimal x1   Optimal x2   Optimal Path
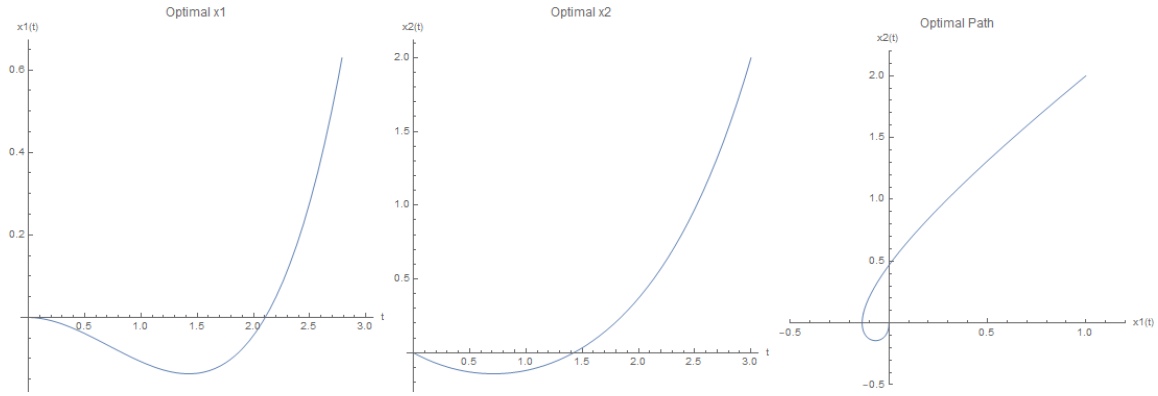
(b) When $x_2(3)$ is free, in its place we instead have the transversality condition $p_2(3) = 0$.
This yields the solution

$$u(t) = -\frac{2e^3(e^t - e^3)}{3e^6 + 4e^3 - 1}$$

$$= 0.6256 - 0.0311e^t$$

Optimal x1   Optimal x2   Optimal Path

(c) I would add a final penalty term Φ:

$$\mathcal{J} = \underbrace{\frac{1}{2}\left((x_1(3) - 1)^2 + (x_2(3) - 2)^2\right)}_{\Phi(3)} + \frac{1}{2}\int_0^3 u^2 dt.$$
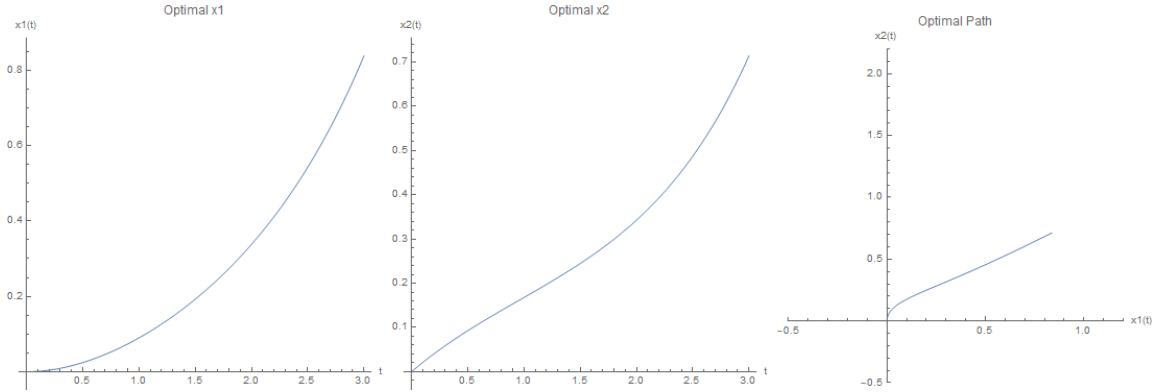
Then instead of $p_1(3) = p_2(3) = 0$, we have

$$p_1(3) = \Phi_{x_1}(3) = x_1(3) - 1$$

2

$$p_2(3) = \Phi_{x_2}(3) = x_2(3) - 2$$

Then new control is

$$u(t) = \frac{8e^{3+t} + 6e^t + e^6 + 4e^3 - 3}{7e^6 + 8e^3 - 7}$$

$$= 0.1615 + 0.056e^t.$$

We see that $x_1(3) = 0.8385$ and $x_2(3) = 0.7142$, which are not very close to $(1, 2)$. To improve accuracy, I would increase the weight of the penalty term. We see that the cost in part (a) is 4.2859. The cost as a function of the weight coefficient c is shown in the figure below:
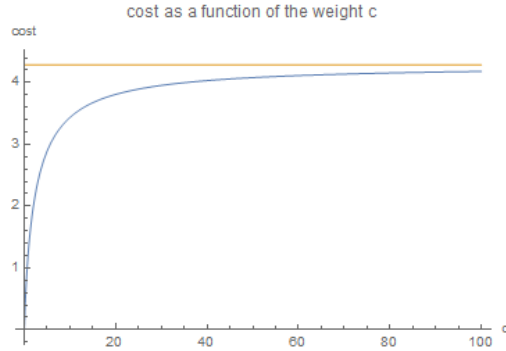

Figure 1: We see that as the weight increases, the cost approaches that of the cost (orange) in part (a) asymptotically.

And we indeed see that the solution ends much closer to $(1, 2)$ when the weight is high:
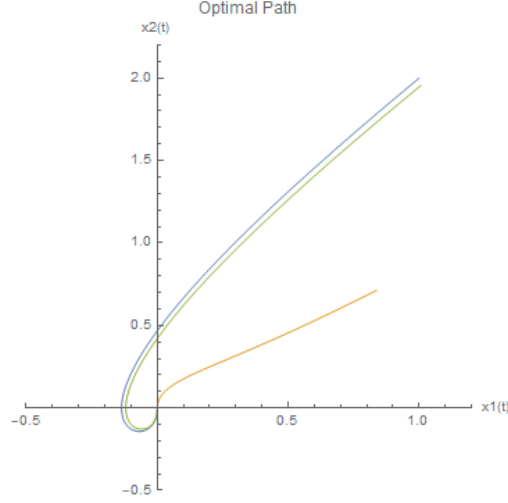
Figure 2: Blue is optimal path from part (a), green is part (c) with weight 100, and orange is part (c) with weight 1.

(d) It is clear that $\Psi = \begin{pmatrix} 2 & 5 \end{pmatrix}$. By transversality condition from Equation 5.234, we have the boundary conditions

$$\begin{pmatrix} -p_1(3) \\ -p_2(3) \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \lambda$$

Together with the initial conditions $x_1(0) = x_2(0) = 0$ and the terminal condition $2x_1(3) + 5x_2(3) = 20$, we have 5 boundary conditions for 4 differential equations and an unknown λ. This allows us to solve by Mathematica. We pre-solve p_1, p_2 for easier code: $p_1(t) \equiv -2\lambda$ and $p_2(t) = -3\lambda e^{-t_f = -2} e^t - 2\lambda$. We obtain the optimal control:

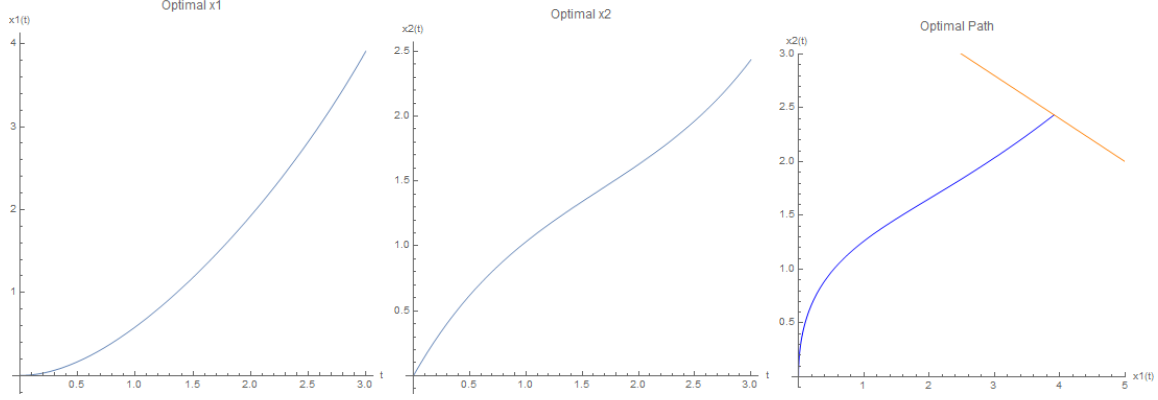$$u(t) = -p_2(t) = 1.4341 + 0.1071 e^t$$

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Figure 3: We see that the solution indeed ends at the constraint.

(e) We have the same Ψ but $h(t) = 20 + \frac{t^2}{2}$ so $\dot{h}(t_f) = t_f$. When t_f is also free, based on Equation 5.234 we have the transversality conditions

$$
\begin{pmatrix} H(t_f) \\ -p_1(t_f) \\ -p_2(t_f) \end{pmatrix} = \begin{pmatrix} -t_f \\ 2 \\ 5 \end{pmatrix} \lambda
$$

Note that by plugging in $u = -p_2$, we have

$$
H = -\frac{1}{2}p_2^2 + (p_1 - p_2)x_2
$$

$$
H(t_f) = -\frac{25}{2}\lambda + 3x_2(t_f) = -t_f
$$

Together with two initial conditions and the terminal condition

$$
2x_1(t_f) + 5x_2(t_f) = 20 + \frac{t_f^2}{2},
$$

we have a total of 6 boundary conditions to match the 4 differential equations and two unknowns λ and t_f. Mathematica yields

$$
u(t) = 1.8359 + 0.2547e^t
$$

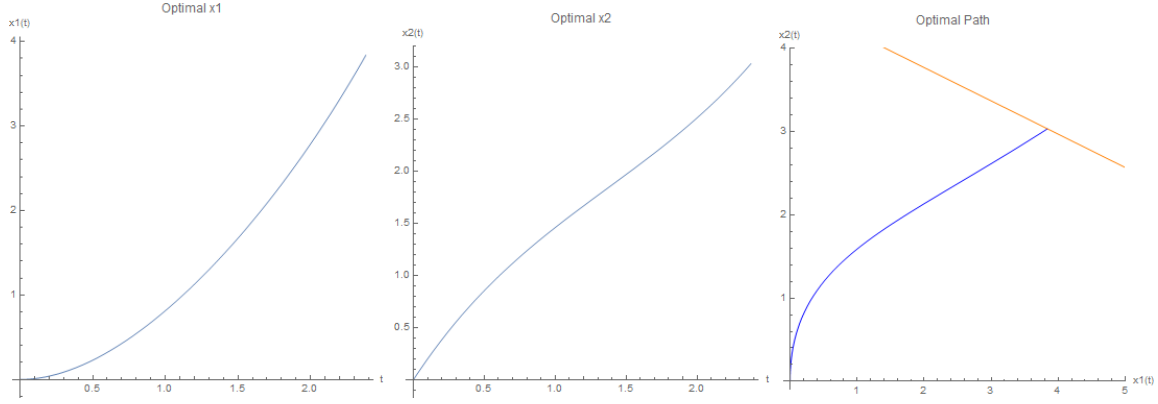with $t_f = 2.3807$s. The trajectories are

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Figure 4: We see the end point hits the constraint.

Problem (2). The Hamiltonian is

$$H = \frac{1}{2}u^2 + p(ax + bu)$$

The adjoint equation is

$$\dot{p} = -H_x = ap \Rightarrow p(t) = Ce^{at}.$$

And the first-order condition is

$$H_u = u + bp = 0 \Rightarrow u = -bp$$

Thus

$$\dot{x} = ax - b^2 p = ax - b^2 Ce^{at}, \quad x(0) = x_0, x(tf) = 0$$

We have $x(t) = b^2 Cte^{at} + Ae^{at}$, $x(0) = A = x_0$, and

$$x(t_f) = b^2 Ct_f e^{at_f} + x_0 e^{at_f} = 0$$
$$C = -\frac{x_0}{b^2 t_f}$$

Thus,

$$u(t) = -bp = -b \cdot \left(-\frac{x_0}{b^2 t_f} \right) e^{at} = \frac{x_0}{bt_f} e^{at}$$

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Problem (3). With the mixed constraint $\psi(x_0, x_f) = x_f - x_0 = 0$, we can turn x_0, x_f into free variables by adding a term with Lagrange multiplier. Let $\Phi(x_0, x_f) = \phi(x_f) + \lambda\psi(x_0, x_f) = \frac{1}{2}(x(t_f) - 1)^2 + \lambda(x_f - x_0)$. The Hamiltonian is

$$H = \frac{1}{2}(x^2 + u^2) + pu.$$

The adjoint equation is

$$\dot{p} = -H_x = x$$

The first-order condition says

$$H_u = u + p = 0 \Rightarrow u = -p.$$

Since both $x(0), x(2)$ is free, from Equation 5.234 we have the transversality condition

$$(\Phi_{x_0} + p^T(t_0))\delta x_0 + (\Phi_{x_f} - p^T(t_f))\delta x_f = 0$$

where δx_0 and δx_f can take any value. This forces that

$$\Phi_{x_0} + p^T(t_0) = 0$$
$$\Phi_{x_f} - p^T(t_f) = 0$$

Thus for this problem, we have

$$p(0) = -\Phi_{x_0} = -(-\lambda) = \lambda$$
$$p(2) = \Phi_{x_f} = x(2) - 1 + \lambda$$

Thus we have two boundary conditions for two differential equations. Mathematica yields

$$x(t) = \frac{\lambda\cos(2-t) + \cos t - \lambda\cos t + \lambda\sin(2-t)}{\cos 2 - \sin 2}.$$

Solving $x(0) = x(2)$, we obtain $\lambda = \frac{\cos(2)-1}{2\cos 2 + \sin 2 - 2} \approx 0.7364$.

$$u(t) = -p(t) = \frac{\lambda(\cos(2-t) - \sin(2-t) - \sin t) + \sin t}{\cos 2 - \sin 2}$$
$$= 0.5556(\cos(2-t) - \sin(2-t)) + 0.1989\sin t$$

The associated cost is 1.8506.

Problem (4). (a) Since the cost $\mathcal{J} = t_f$, we have $\Phi(t) = t$ and the Hamiltonian is

$$H = p_1(t)\cos\theta(t) + p_2(t)\sin\theta(t) + p_3(t)u(t) + p_4(t)v(t)$$

(b) The adjoint equations are

$$\begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \end{pmatrix} = \begin{pmatrix} -H_u \\ -H_v \\ -H_x \\ -H_y \end{pmatrix} = \begin{pmatrix} p_3 \\ p_4 \\ 0 \\ 0 \end{pmatrix}$$

(c) Since $t_f, u(t_f), v(t_f)$ are free, by Equation 5.234 we have the transversality conditions

$$H(t_f) + \Phi_t(t_f) = 0 \Rightarrow H(t_f) = -1$$

$$p_1(t_f) = \Phi_u = 0$$

$$p_2(t_f) = \Phi_v = 0.$$

(d) Notice that p_3, p_4 are constants, so $p_1(t) = p_3 t$ and $p_2(t) = p_4 t$ by the zero boundary conditions. First-order condition yields

$$H_\theta = -p_1\sin\theta(t) + p_2\cos\theta(t) = 0$$

$$\tan\theta^*(t) = \frac{p_2(t)}{p_1(t)} = \frac{p_4}{p_3}$$

which is a constant! So are $\cos\theta^*$ and $\sin\theta^*$. Thus by the initial conditions $u(0) = \cos\gamma_0, v(0) = \sin\gamma_0$, we have $u(t) = \cos\theta^* t$ and $v(t) = \sin\theta^*$. By the initial conditions $x(0) = y(0) = 0$, we have $x(t) = \frac{1}{2}\cos\theta^* t^2 + \cos\gamma_0 t$ and $y(t) = \frac{1}{2}\sin\theta^* t^2 + \sin\gamma_0 t$. By the terminal condition $x(t_f) = x_f$ and $y(t_f) = 0$, we have

$$x_f - \cos\gamma_0 t_f = \frac{1}{2}\cos\theta^* t_f^2$$

$$-\sin\gamma_0 t_f = \frac{1}{2}\sin\theta^* t_f^2$$

Squaring both sides and adding the two equations together, we obtain the equation as desired:

$$x_f^2 - 2\cos\gamma_0 t_f + (\cos\gamma_0^2 + \sin\gamma_0^2)t_f^2 = \frac{1}{4}(\cos^2\theta^* + \sin^2\theta^*)t_f^4$$

$$4x_f^2 - 8\cos\gamma_0 t_f + 4t_f^2 - t_f^4 = 0$$

So t_f must be the minimum positive solution of this equation.

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(e) By the above equation,

$$\cos \theta^* = 2 \left(\frac{x_f}{t_f^2} + \frac{\cos \gamma_0}{t_f} \right)$$

$$\theta^* = \arccos \left(2 \left(\frac{x_f}{t_f^2} + \frac{\cos \gamma_0}{t_f} \right) \right)$$

Problem (5). The cost has the Meyer form $J = t_f$ so $\Phi(t) = t$. The Hamiltonian is

$$H(x, y, p_1, p_2) = p_1 r \cos \beta + p_2 r \sin \beta.$$

The adjoint equations are

$$\dot{p}_1 = -H_x = -p_1 \frac{x}{r} \cos \beta - p_2 \frac{x}{r} \sin \beta$$

$$\dot{p}_2 = -H_y = p_1 \frac{y}{r} \cos \beta - p_2 \frac{x}{r} \sin \beta$$

Note that $\dot{p}_2 = \frac{y}{x} \dot{p}_1$. First-order condition yields

$$H_\beta = -p_1 r \sin \beta + p_2 r \cos \beta = 0$$

$$\tan \beta = \frac{p_2}{p_1}$$

Thus we obtain $p_2 = \tan \beta p_1$. Moreover, we see that

$$p_2 \cos \beta - p_1 \sin \beta = \frac{p_2 p_1 - p_1 p_2}{\sqrt{p_1^2 + p_2^2}}$$

$$= 0.$$

Now we compute

$$0 = \frac{d}{dt} H_\beta = \dot{r}(-p_1 \sin \beta + p_2 \cos \beta) + r \left(-\frac{d}{dt}(p_1 \sin \beta) + \frac{d}{dt}(p_2 \cos \beta) \right)$$

$$0 = 0 + r \left(-\dot{p}_1 \sin \beta - p1 \cos \beta \dot{\beta} + \dot{p}_2 \cos \beta - p_2 \sin \beta \dot{\beta} \right)$$

$$0 = -\dot{p}_1 \sin \beta - p1 \cos \beta \dot{\beta} + \frac{y}{x} \dot{p}_1 \cos \beta - p_1 \tan \beta \sin \beta \dot{\beta}$$

$$p_1 (cos\beta + \tan \beta \sin \beta) \dot{\beta} = \dot{p}_1 \left(-\sin \beta + \frac{y}{x} \cos \beta \right)$$

$$p_1 (cos\beta + \tan \beta \sin \beta) \dot{\beta} = \left(p_1 \frac{x}{r} \cos \beta + \tan \beta p_1 \frac{x}{r} \sin \beta \right) \left(\sin \beta - \frac{y}{x} \cos \beta \right)$$

$$\dot{\beta} = \frac{x}{r} \left(\sin \beta - \frac{y}{x} \cos \beta \right)$$

$$\dot{\beta} = \frac{x}{r} sin\beta - \frac{y}{r} \cos \beta$$

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In[1]:= `(*Problem 1*)`
`(*Part a*)`
`sol1 =`
  `DSolve[{x1'[t] == x2[t], x2'[t] == -x2[t] - p2[t], p1'[t] == 0, p2'[t] == p2[t] - p1[t],`
    `x1[0] == 0, x2[0] == 0, x1[3] == 1, x2[3] == 2}, {x1[t], x2[t], p1[t], p2[t]}, t]`
`xSol1 = x1[t] /. sol1[[1, 3]]`
`ySol1 = x2[t] /. sol1[[1, 4]]`
`cost := Module[{sol, x1s, x2s, u, x13, x23, x1sModified, x2sModified},`
  `sol = sol1;`
  `x1s[t_] := x1[t] /. sol[[1, 3]];`
  `x13 := x1s[t] /. t -> 3;`
  `x1sModified = x1s[t] /. x1[3] → x13;`
  `x2s[t_] := x2[t] /. sol[[1, 4]];`
  `x23 := x2s[t] /. t -> 3;`
  `x2sModified = x2s[t] /. x2[3] → x23;`
  `u[t_] := -(p2[t] /. sol[[1, 2]]);`
  `1/2 ((x13 - 1)^2 + (x23 - 2)^2) + Integrate[1/2 u[t]^2, {t, 0, 3}]]`
`cost1 = N[cost]`
`u[t_] = -Simplify[N[p2[t] /. sol1[[1, 2]]]]`

Out[1]= $\left\{\left\{ p1[t] \rightarrow \frac{-3 + e^3}{5 + e^3}, p2[t] \rightarrow \frac{3 - 4 e^3 + e^6 - 6 e^t - 6 e^{3+t}}{\left(-1 + e^3\right) \left(5 + e^3\right)}, \right. \right.$

$x1[t] \rightarrow \frac{e^{-t} \left(7 e^3 - e^6 - 3 e^t + 3 e^{2t} - 10 e^{3+t} + e^{6+t} + 3 e^{3+2t} - 3 e^t t + 4 e^{3+t} t - e^{6+t} t\right)}{\left(-1 + e^3\right) \left(5 + e^3\right)},$

$x2[t] \rightarrow \left. \left. \frac{e^{-t} \left(-7 e^3 + e^6 - 3 e^t + 3 e^{2t} + 4 e^{3+t} - e^{6+t} + 3 e^{3+2t}\right)}{\left(-1 + e^3\right) \left(5 + e^3\right)}\right\}\right\}$

Out[2]= $\frac{e^{-t} \left(7 e^3 - e^6 - 3 e^t + 3 e^{2t} - 10 e^{3+t} + e^{6+t} + 3 e^{3+2t} - 3 e^t t + 4 e^{3+t} t - e^{6+t} t\right)}{\left(-1 + e^3\right) \left(5 + e^3\right)}$

Out[3]= $\frac{e^{-t} \left(-7 e^3 + e^6 - 3 e^t + 3 e^{2t} + 4 e^{3+t} - e^{6+t} + 3 e^{3+2t}\right)}{\left(-1 + e^3\right) \left(5 + e^3\right)}$

Out[5]= 4.28588

Out[6]= $-0.681091 + 0.264246 \times 2.71828^t$

In[7]:= `Plot[xSol1, {t, 0, 3}, PlotLabel → "Optimal x1",`
  `AxesLabel → {"t", "x1(t)"}, AspectRatio → 1]`
`Plot[ySol1, {t, 0, 3}, PlotLabel → "Optimal x2",`
  `AxesLabel → {"t", "x2(t)"}, AspectRatio → 1]`
`ParametricPlot[{xSol1, ySol1}, {t, 0, 3}, PlotRange → {{-.5, 1.2}, {-.5, 2.2}}`
  `, PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"}, AspectRatio → 1]`

Optimal x1

x1(t)

Out[7]=

Optimal x2

x2(t)

Out[8]=

Optimal Path

x2(t)

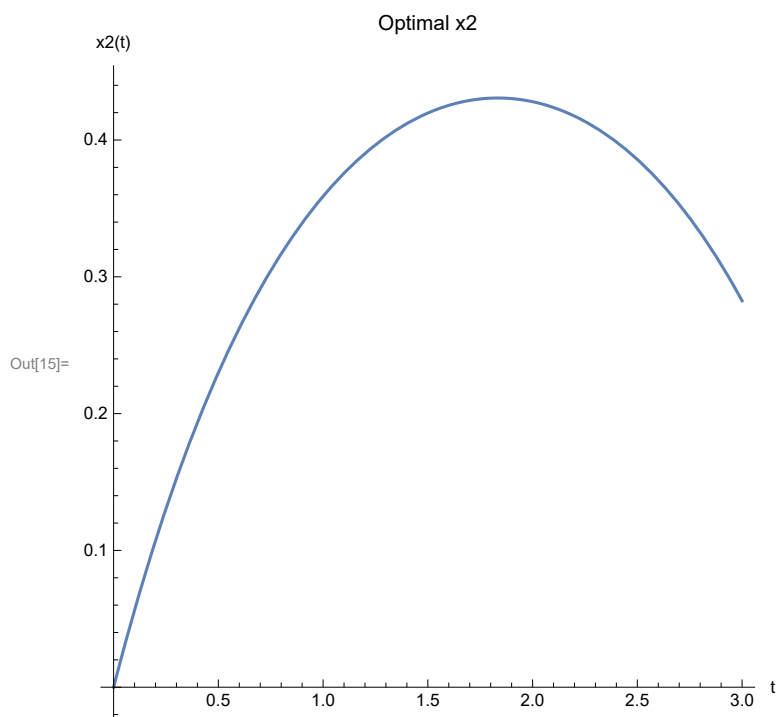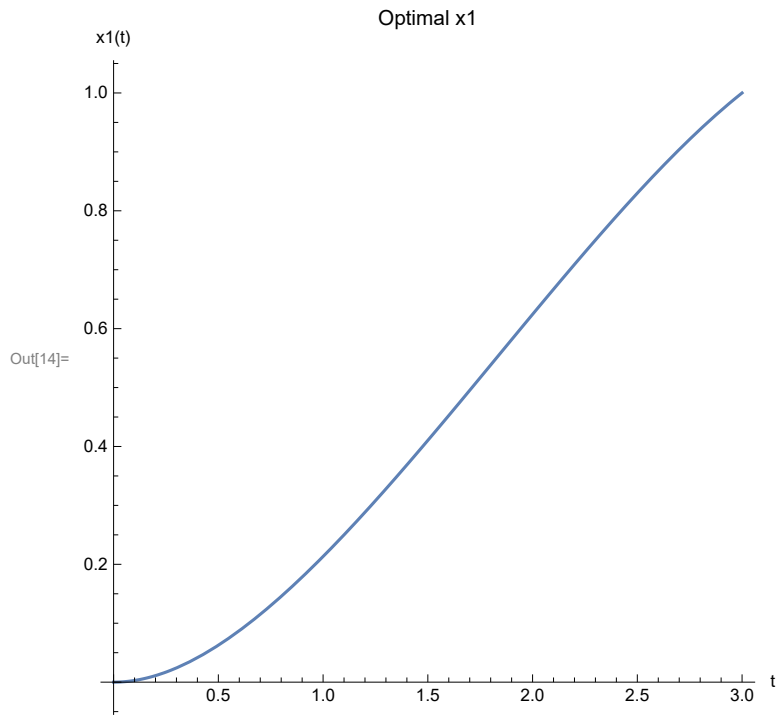

Out[9]=
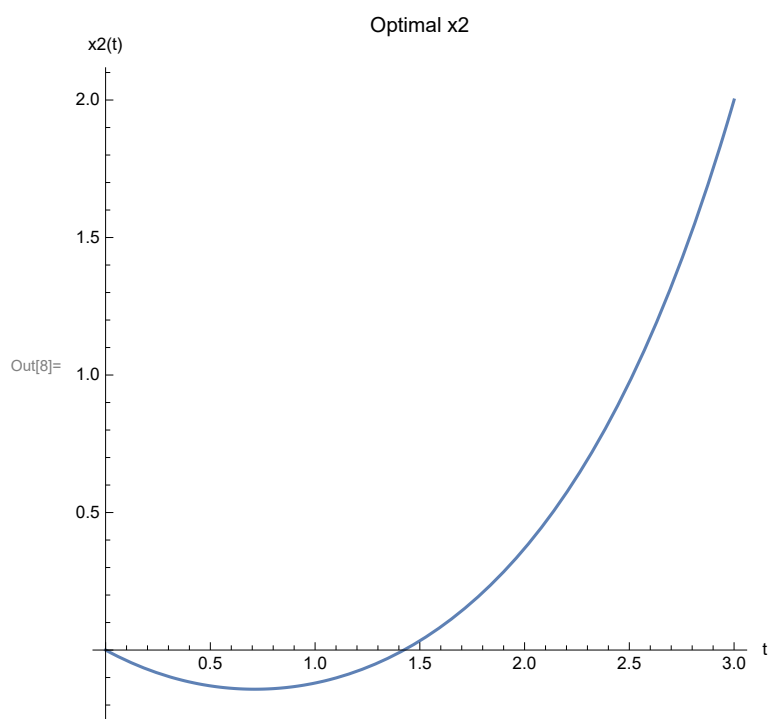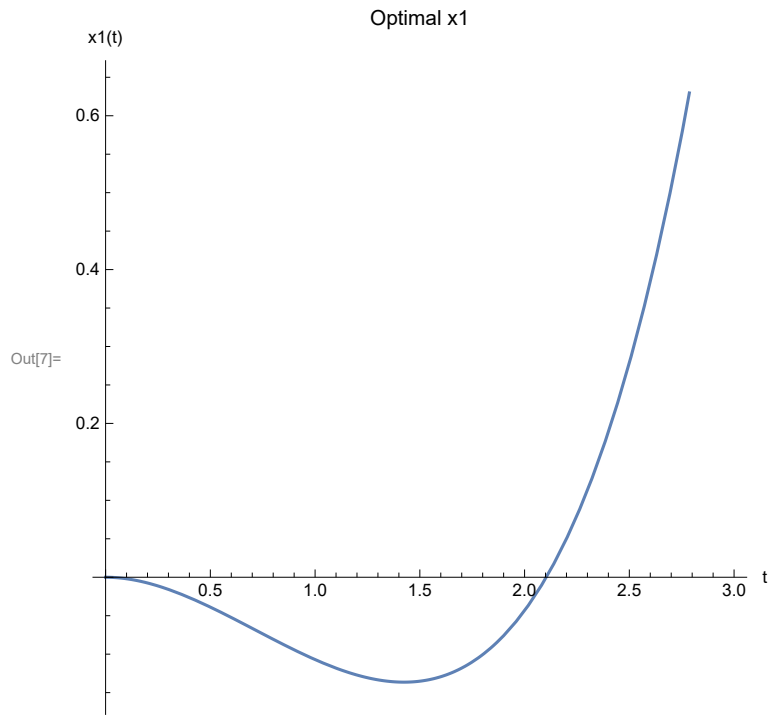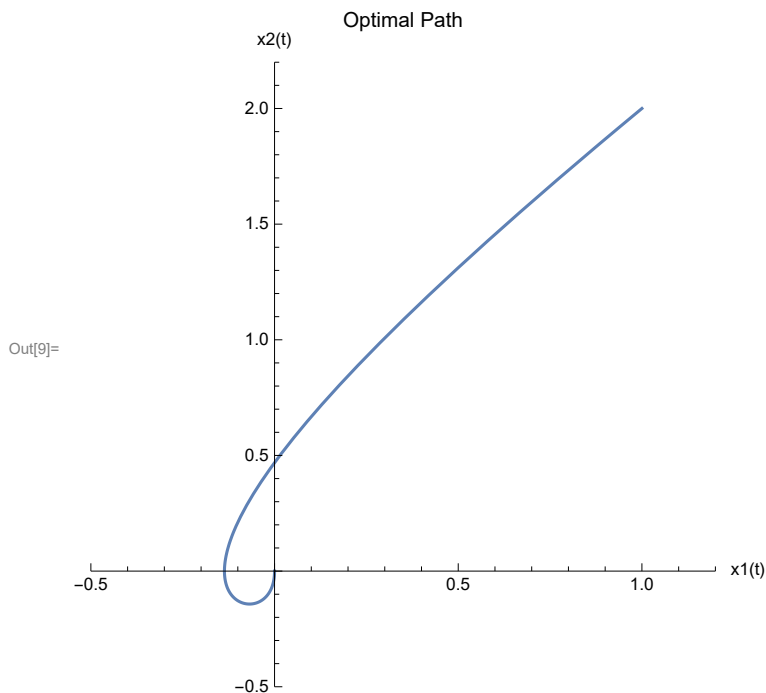
(*Part b*)

```
In[10]:= sol2 = DSolve[{x1'[t] == x2[t], x2'[t] == -x2[t] - p2[t], p1'[t] == 0, p2'[t] == p2[t] - p1[t],
        x1[0] == 0, x2[0] == 0, x1[3] == 1, p2[3] == 0}, {x1[t], x2[t], p1[t], p2[t]}, t]
    xSol2 = x1[t] /. sol2[[1, 3]]
    ySol2 = x2[t] /. sol2[[1, 4]]
    u[t_] = -N[p2[t] /. sol2[[1, 2]]]
    Plot[xSol2, {t, 0, 3}, PlotLabel → "Optimal x1",
     AxesLabel → {"t", "x1(t)"}, AspectRatio → 1]
    Plot[ySol2, {t, 0, 3}, PlotLabel → "Optimal x2",
     AxesLabel → {"t", "x2(t)"}, AspectRatio → 1]
    ParametricPlot[{xSol2, ySol2}, {t, 0, 3}, PlotRange → {{-.5, 1.2}, {-.5, 2.2}}
    , PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"}, AspectRatio → 1]
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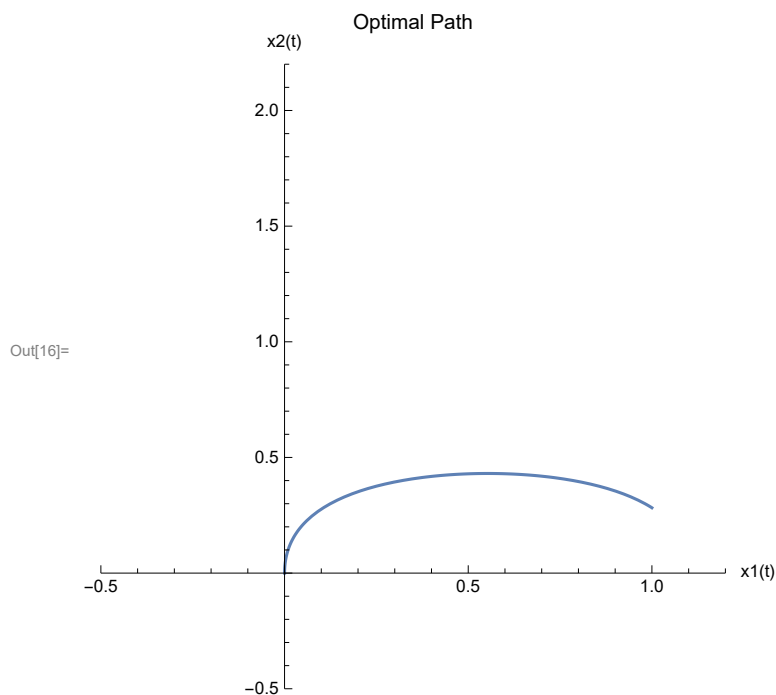
Out[10]= $\left\{\left\{ p1[t] \rightarrow -\frac{2\,e^6}{-1 + 4\,e^3 + 3\,e^6},\ p2[t] \rightarrow \frac{2\,e^3\,\left(-e^3 + e^t\right)}{-1 + 4\,e^3 + 3\,e^6}, \right.\right.$

$\left.\left. x1[t] \rightarrow -\frac{e^{3-t}\,\left(1 - 2\,e^3 - 2\,e^t + e^{2\,t} + 2\,e^{3+t} - 2\,e^{3+t}\,t\right)}{-1 + 4\,e^3 + 3\,e^6},\ x2[t] \rightarrow -\frac{e^{3-t}\,\left(-1 + 2\,e^3 + e^{2\,t} - 2\,e^{3+t}\right)}{-1 + 4\,e^3 + 3\,e^6} \right\}\right\}$

Out[11]= $-\frac{e^{3-t}\,\left(1 - 2\,e^3 - 2\,e^t + e^{2\,t} + 2\,e^{3+t} - 2\,e^{3+t}\,t\right)}{-1 + 4\,e^3 + 3\,e^6}$

Out[12]= $-\frac{e^{3-t}\,\left(-1 + 2\,e^3 + e^{2\,t} - 2\,e^{3+t}\right)}{-1 + 4\,e^3 + 3\,e^6}$

Out[13]= $-0.0311493\,\left(-20.0855 + 2.71828^t\right)$

Optimal x1

Out[14]=

Optimal x2

Out[15]=

Out[16]=

(*Part c*)

In[17]:= **sol3 = DSolve[**
    {x1'[t] == x2[t], x2'[t] == -x2[t] - p2[t], p1'[t] == 0, p2'[t] == p2[t] - p1[t], x1[0] == 0,
     x2[0] == 0, p1[3] == x1[3] - 1, p2[3] == x2[3] - 2}, {x1[t], x2[t], p1[t], p2[t]}, t]
  xSol3 = x1[t] /. sol3[[1, 3]]
  ySol3 = x2[t] /. sol3[[1, 4]]
  u[t_] = -Simplify[N[p2[t] /. sol3[[1, 2]]]]
  Plot[xSol3, {t, 0, 3}, PlotLabel → "Optimal x1",
   AxesLabel → {"t", "x1(t)"}, AspectRatio → 1]
  Plot[ySol3, {t, 0, 3}, PlotLabel → "Optimal x2",
   AxesLabel → {"t", "x2(t)"}, AspectRatio → 1]
  ParametricPlot[{xSol3, ySol3}, {t, 0, 3}, PlotRange → {{-.5, 1.2}, {-.5, 2.2}}
  , PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"}, AspectRatio → 1]
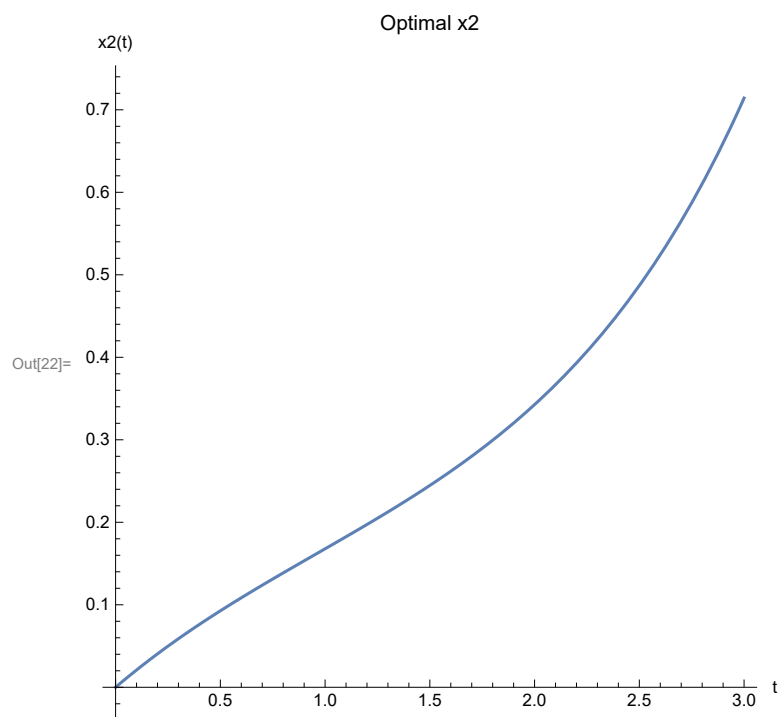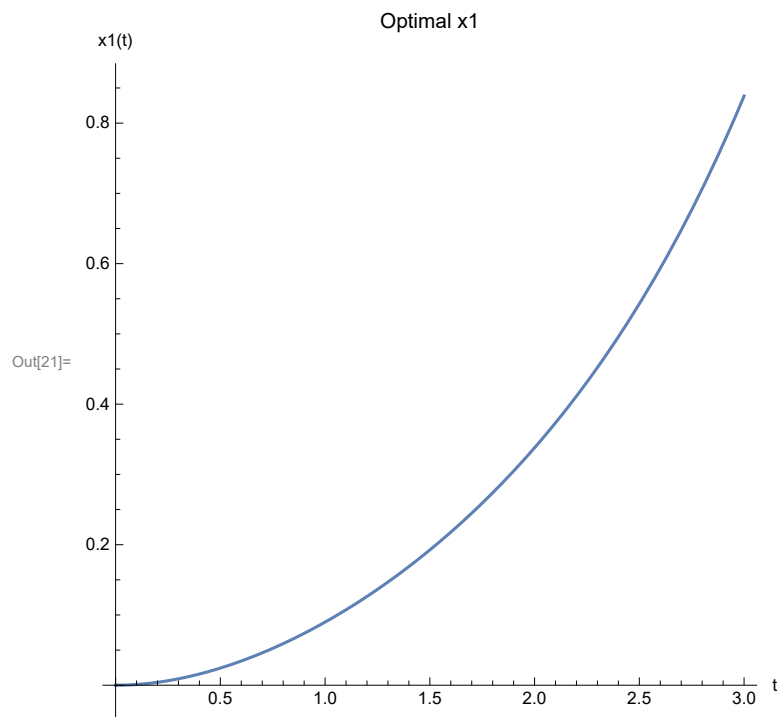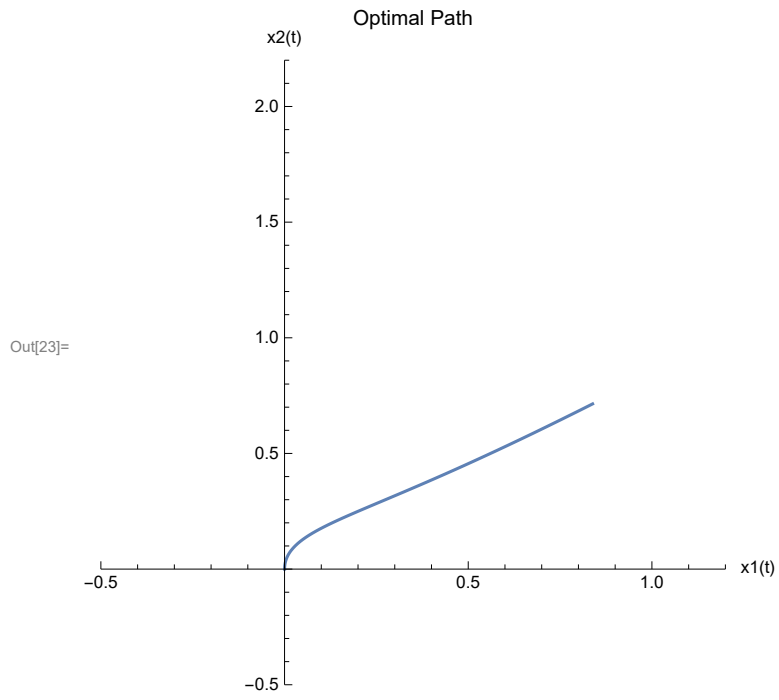
Out[17]= $\left\{ \left\{ p1[t] \rightarrow \dfrac{3 - 4\,e^3 - e^6}{-7 + 8\,e^3 + 7\,e^6}, \; p2[t] \rightarrow -\dfrac{-3 + 4\,e^3 + e^6 + 6\,e^t + 8\,e^{3+t}}{-7 + 8\,e^3 + 7\,e^6}, \right. \right.$

$x1[t] \rightarrow \dfrac{e^{-t}\left(8\,e^3 + e^6 - 3\,e^t + 3\,e^{2t} - 12\,e^{3+t} - e^{6+t} + 4\,e^{3+2t} - 3\,e^t\,t + 4\,e^{3+t}\,t + e^{6+t}\,t\right)}{-7 + 8\,e^3 + 7\,e^6},$

$\left. \left. x2[t] \rightarrow \dfrac{e^{-t}\left(-8\,e^3 - e^6 - 3\,e^t + 3\,e^{2t} + 4\,e^{3+t} + e^{6+t} + 4\,e^{3+2t}\right)}{-7 + 8\,e^3 + 7\,e^6} \right\} \right\}$

Out[18]= $\dfrac{e^{-t}\left(8\,e^3 + e^6 - 3\,e^t + 3\,e^{2t} - 12\,e^{3+t} - e^{6+t} + 4\,e^{3+2t} - 3\,e^t\,t + 4\,e^{3+t}\,t + e^{6+t}\,t\right)}{-7 + 8\,e^3 + 7\,e^6}$

Out[19]= $\dfrac{e^{-t}\left(-8\,e^3 - e^6 - 3\,e^t + 3\,e^{2t} + 4\,e^{3+t} + e^{6+t} + 4\,e^{3+2t}\right)}{-7 + 8\,e^3 + 7\,e^6}$

Out[20]= $0.161458 + 0.0559778 \times 2.71828^t$

Optimal x1

x1(t)

Out[21]=

Optimal x2

x2(t)

Out[22]=

Out[23]=

Optimal Path

```
In[24]:= weightedPenaltyCost[c_] := Module[{sol, x1s, x2s, u, x13, x23, x1sModified, x2sModified},
          sol = DSolve[{x1'[t] == x2[t], x2'[t] == -x2[t] - p2[t], p1'[t] == 0,
              p2'[t] == p2[t] - p1[t], x1[0] == 0, x2[0] == 0, p1[3] == c (x1[3] - 1),
              p2[3] == c (x2[3] - 2)}, {x1[t], x2[t], p1[t], p2[t]}, t];
          x1s[t_] := x1[t] /. sol[[1, 3]];
          x13 := x1s[t] /. t -> 3;
          x1sModified = x1s[t] /. x1[3] → x13;
          x2s[t_] := x2[t] /. sol[[1, 4]];
          x23 := x2s[t] /. t -> 3;
          x2sModified = x2s[t] /. x2[3] → x23;
          u[t_] := -(p2[t] /. sol[[1, 2]]);
          c / 2 ((x13 - 1)^2 + (x23 - 2)^2) + Integrate[1 / 2 u[t]^2, {t, 0, 3}]]

        Plot[{weightedPenaltyCost[c], cost1}, {c, 0, 100},
         PlotLabel → "cost as a function of the weight c", AxesLabel → {"c", "cost"},
         PlotLegends → "Expressions", PlotPoints → 10, PlotRange → Full]
```
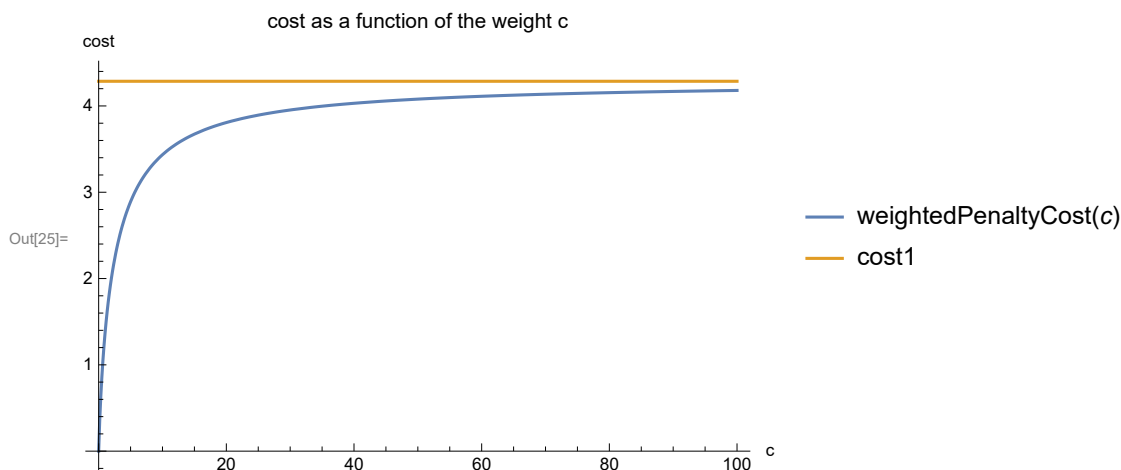


```
Out[•]= 4.28577
```

```
In[26]:= N[weightedPenaltyCost[100 000]]
        (*we see that for weight 100000 the value is super close to cost from (a)*)
        sol3w = DSolve[{x1'[t] == x2[t], x2'[t] == -x2[t] - p2[t], p1'[t] == 0,
            p2'[t] == p2[t] - p1[t], x1[0] == 0, x2[0] == 0, p1[3] == 100 (x1[3] - 1),
            p2[3] == 100 (x2[3] - 2)}, {x1[t], x2[t], p1[t], p2[t]}, t];
        x1s[t_] := x1[t] /. sol3w[[1, 3]];
        x13 := x1s[t] /. t -> 3;
        x1sModified = x1s[t] /. x1[3] → x13;
        x2s[t_] := x2[t] /. sol3w[[1, 4]];
        x23 := x2s[t] /. t -> 3;
        x2sModified = x2s[t] /. x2[3] → x23;
```
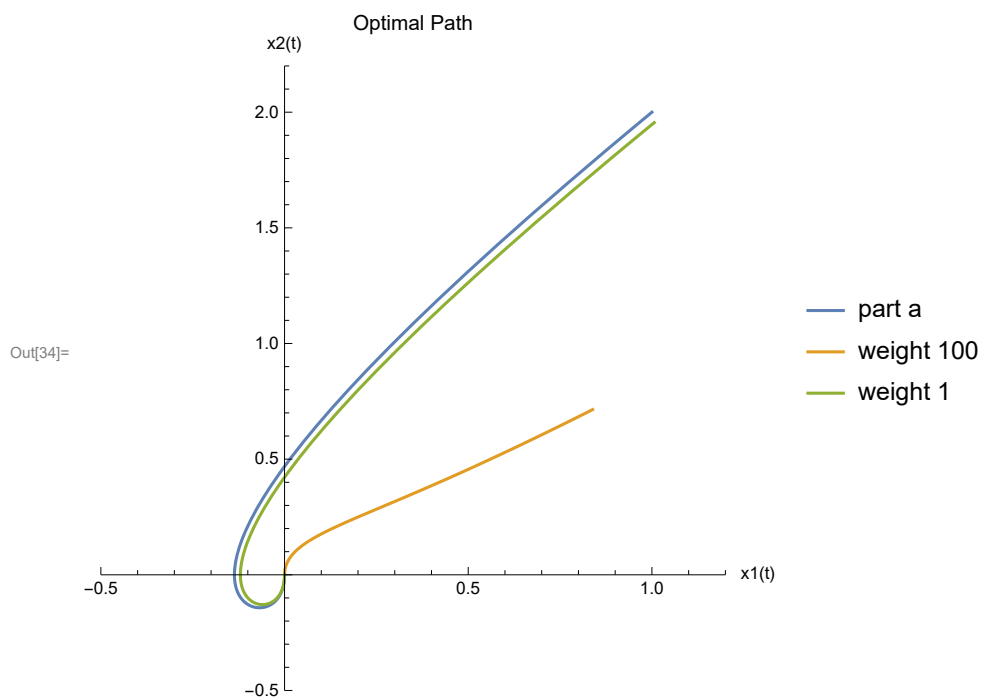
```
Out[26]= 4.28577
```

In[34]:= `ParametricPlot[{{xSol1, ySol1}, {xSol3, ySol3}, {x1sModified, x2sModified}},`
`{t, 0, 3}, PlotRange → {{-.5, 1.2}, {-.5, 2.2}}`
`, PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"},`
`PlotLegends → {"part a", "weight 100", "weight 1"}, AspectRatio → 1]`

Out[34]=

(*Part d*)

In[35]:=
```
sol42 =
  Simplify[DSolve[{x2'[t] == -x2[t] - (-3 lam Exp[-3] Exp[t] - 2 lam), x2[0] == 0}, x2[t], t]]
   (*pre-solve p1,p2*)
sol4x2[t_] = x2[t] /. sol42[[1]]
sol41 = Simplify[DSolve[{x1'[t] == sol4x2[t], x1[0] == 0}, x1[t], t]]
sol4x1[t_] = x1[t] /. sol41[[1]]
lamSol = N[Solve[2 sol4x1[3] + 5 sol4x2[3] == 20, lam]] (*solve for lambda*)
lamS = lam /. lamSol[[1]]
u[t_] = 3 lamS Exp[-3] Exp[t] + 2 lamS
```

Out[35]= $\left\{\left\{x2[t] \rightarrow \frac{1}{2} e^{-3-t} \left(-3 - 4 e^3 + 3 e^{2t} + 4 e^{3+t}\right) \text{lam}\right\}\right\}$

Out[36]= $\frac{1}{2} e^{-3-t} \left(-3 - 4 e^3 + 3 e^{2t} + 4 e^{3+t}\right) \text{lam}$

Out[37]= $\left\{\left\{x1[t] \rightarrow \frac{1}{2} e^{-3-t} \text{lam} \left(3 + 4 e^3 - 6 e^t + 3 e^{2t} + 4 e^{3+t} \left(-1 + t\right)\right)\right\}\right\}$

Out[38]= $\frac{1}{2} e^{-3-t} \text{lam} \left(3 + 4 e^3 - 6 e^t + 3 e^{2t} + 4 e^{3+t} \left(-1 + t\right)\right)$

Out[39]= $\{\{\text{lam} \rightarrow 0.717067\}\}$

Out[40]= 0.717067

Out[41]= $1.43413 + 0.107102 \, e^t$

In[42]:=
```
xSol4[t_] = sol4x1[t] /. lam → lamS
ySol4[t_] = sol4x2[t] /. lam → lamS
xSol4[3]
ySol4[3]
Plot[xSol4[t], {t, 0, 3}, PlotLabel → "Optimal x1",
 AxesLabel → {"t", "x1(t)"}, AspectRatio → 1]
Plot[ySol4[t], {t, 0, 3}, PlotLabel → "Optimal x2",
 AxesLabel → {"t", "x2(t)"}, AspectRatio → 1]
x2[x1_] = 1/5 (20 - 2 x1);
plot = Plot[x2[x1], {x1, -10, 10}, PlotStyle → Orange];
parametric = ParametricPlot[{xSol4[t], ySol4[t]},
    {t, 0, 3}, PlotRange -> {{-.1, 5}, {-.1, 3}}, PlotStyle → Blue,
    PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"}, AspectRatio → 1];
Show[parametric, plot, PlotLabel → "Optimal Path"]
```
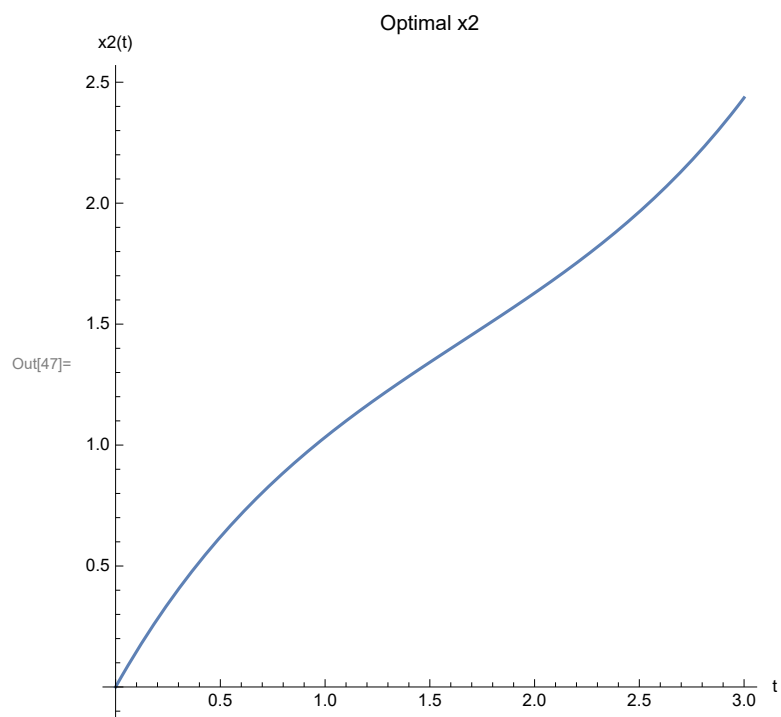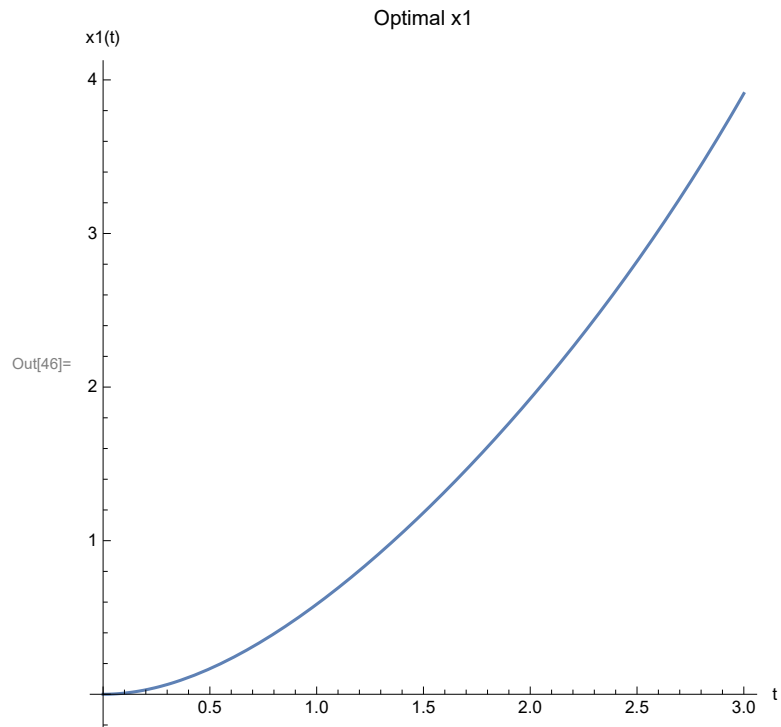
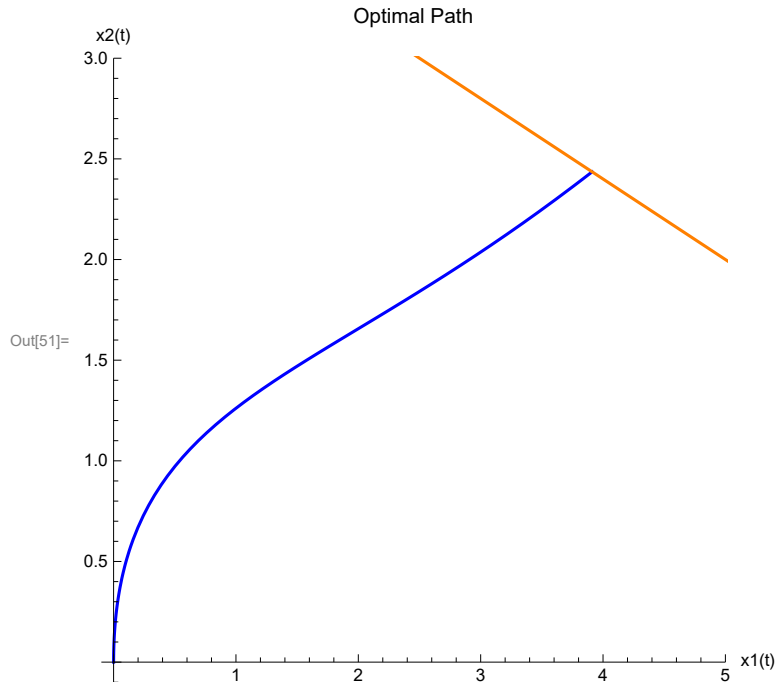Out[42]= $0.358533 \, e^{-3-t} \left(3 + 4 e^3 - 6 e^t + 3 e^{2t} + 4 e^{3+t} \left(-1 + t\right)\right)$

Out[43]= $0.358533 \, e^{-3-t} \left(-3 - 4 e^3 + 3 e^{2t} + 4 e^{3+t}\right)$

Out[44]= 3.91083

Out[45]= 2.43567

Out[46]=

Optimal x1

Out[47]=

Optimal x2

Out[51]=

```
In[69]:= (*Part e*)
ClearAll[lam, x2, lamS, tf, tfS, u]
sol52 = Simplify[
    DSolve[{x2'[t] == -x2[t] - (-3 lam Exp[-tf] Exp[t] - 2 lam), x2[0] == 0}, x2[t], t]]
sol5x2[t_] = x2[t] /. sol52[[1]]
sol51 = Simplify[DSolve[{x1'[t] == sol5x2[t], x1[0] == 0}, x1[t], t]]
sol5x1[t_] = x1[t] /. sol51[[1]]
sol5x2[tf]
(*sol5=Solve[
    {2*sol5x1[tf]+5 *sol5x2[tf] == 20+tf^2/2,-25/2 lam+3*sol5x2[tf]== tf},{lam,tf}]*)
(*lamS = lam /.sol5[[1,1]]*)
(*tfS= tf /.sol5[[1,2]]*)
lamS = 0.91796608810911379992881136835706;
(*computed from MATLAB since Mathematica cannot compute this*)
tfS = 2.3806740196227239875399923087245;
(*computed from MATLAB since Mathematica cannot compute this*)
xSol5[t_] = sol5x1[t] /. {lam → lamS, tf → tfS};
ySol5[t_] = sol5x2[t] /. {lam → lamS, tf → tfS};
Plot[xSol5[t], {t, 0, tfS}, PlotLabel → "Optimal x1",
 AxesLabel → {"t", "x1(t)"}, AspectRatio → 1]
Plot[ySol5[t], {t, 0, tfS}, PlotLabel → "Optimal x2",
 AxesLabel → {"t", "x2(t)"}, AspectRatio → 1]
x2[x1_] = 1/5 (20 + tfS^2/2 - 2 x1);
plot = Plot[x2[x1], {x1, -10, 10}, PlotStyle → Orange];
parametric = ParametricPlot[{xSol5[t], ySol5[t]},
    {t, 0, tfS}, PlotRange -> {{-.1, 5}, {-.1, 4}}, PlotStyle → Blue,
    PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"}, AspectRatio → 1];
Show[parametric, plot, PlotLabel → "Optimal Path"]
u[t_] = 3 lamS Exp[-tfS] Exp[t] + 2 lamS
```
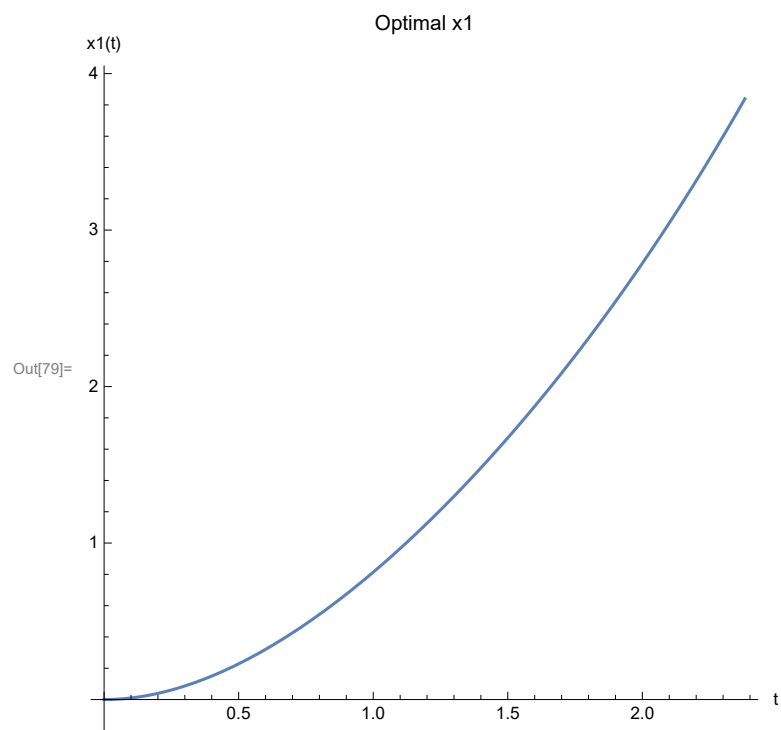
Out[70]= $\left\{\left\{x2[t] \rightarrow \frac{1}{2} e^{-t-tf} \left(-1 + e^t\right) \left(3 + 3 e^t + 4 e^{tf}\right) lam\right\}\right\}$

Out[71]= $\frac{1}{2} e^{-t-tf} \left(-1 + e^t\right) \left(3 + 3 e^t + 4 e^{tf}\right) lam$

Out[72]= $\left\{\left\{x1[t] \rightarrow \frac{1}{2} e^{-t-tf} lam \left(3 - 6 e^t + 3 e^{2t} + 4 e^{tf} + 4 e^{t+tf} \left(-1 + t\right)\right)\right\}\right\}$

Out[73]= $\frac{1}{2} e^{-t-tf} lam \left(3 - 6 e^t + 3 e^{2t} + 4 e^{tf} + 4 e^{t+tf} \left(-1 + t\right)\right)$

Out[74]= $\frac{1}{2} e^{-2tf} \left(-1 + e^{tf}\right) \left(3 + 7 e^{tf}\right) lam$

Out[79]=

Optimal x1

Optimal x2

Out[80]=

Optimal Path

Out[84]=
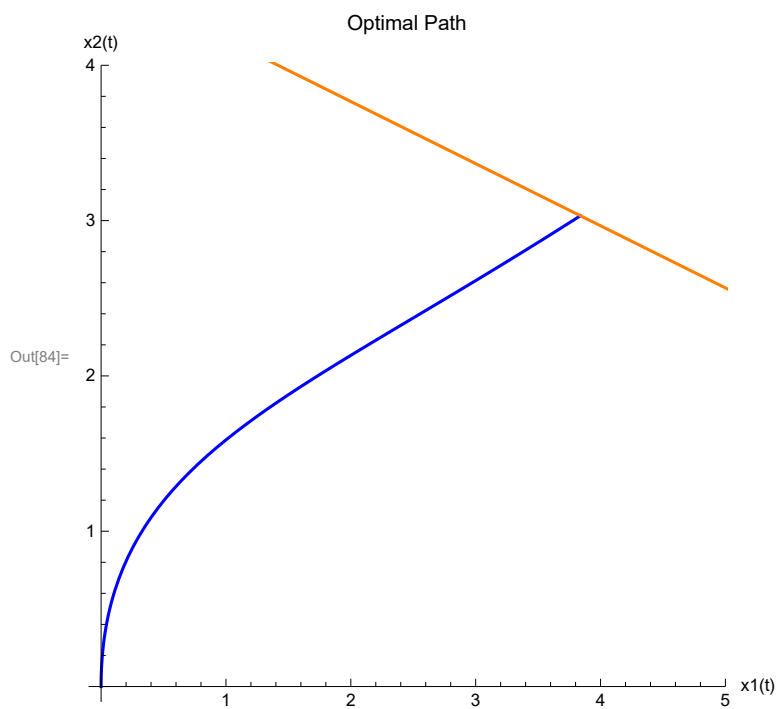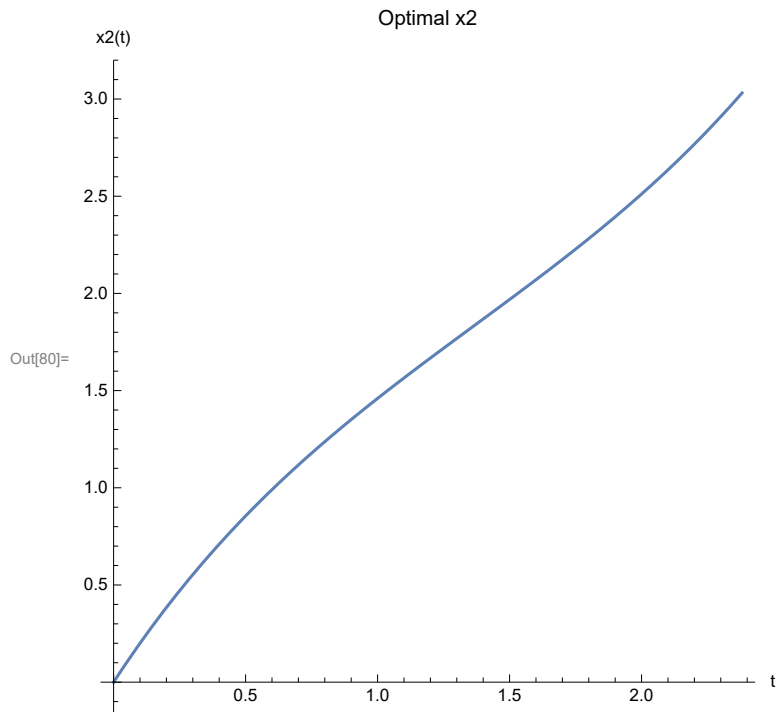
Out[85]= $1.835932176218227599857622736714 + 0.2547031419834292829896440029974 \, e^t$

(*Problem 3*)

In[86]:= `sol = Simplify[DSolve[`
      `{x'[t] == -p[t], p'[t] == x[t], p[2] == x[2] - 1 + lam, p[0] == lam}, {x[t], p[t]}, t]]`
    `xSol[t_] = x[t] /. sol[[1, 2]]`
    `pSol[t_] = p[t] /. sol[[1, 1]]`
    `xSol0[t_] = xSol[t] /. lam → (Cos[2] - 1) / (2 Cos[2] + Sin[2] - 2)`
    `pSol0[t_] = pSol[t] /. lam → (Cos[2] - 1) / (2 Cos[2] + Sin[2] - 2)`
    `N[(Cos[2] - 1) / (2 Cos[2] + Sin[2] - 2)]`
    `Simplify[-N[pSol0[t]]]`
    `cost = N[1/2 (xSol0[2] - 1)^2 + 1/2 Integrate[xSol0[t]^2 + pSol0[t]^2, {t, 0, 2}]]`

Out[86]= $\left\{\left\{p[t] \to \dfrac{\text{lam} \cos[2-t] - \text{lam} \sin[2-t] + \sin[t] - \text{lam} \sin[t]}{\cos[2] - \sin[2]}\right.\right.$,

$\left.\left.x[t] \to \dfrac{\text{lam} \cos[2-t] + \cos[t] - \text{lam} \cos[t] + \text{lam} \sin[2-t]}{\cos[2] - \sin[2]}\right\}\right\}$

Out[87]= $\dfrac{\text{lam} \cos[2-t] + \cos[t] - \text{lam} \cos[t] + \text{lam} \sin[2-t]}{\cos[2] - \sin[2]}$

Out[88]= $\dfrac{\text{lam} \cos[2-t] - \text{lam} \sin[2-t] + \sin[t] - \text{lam} \sin[t]}{\cos[2] - \sin[2]}$

Out[89]= $\dfrac{\cos[t] + \dfrac{(-1+\cos[2]) \cos[2-t]}{-2+2\cos[2]+\sin[2]} - \dfrac{(-1+\cos[2]) \cos[t]}{-2+2\cos[2]+\sin[2]} + \dfrac{(-1+\cos[2]) \sin[2-t]}{-2+2\cos[2]+\sin[2]}}{\cos[2] - \sin[2]}$

Out[90]= $\dfrac{\dfrac{(-1+\cos[2]) \cos[2-t]}{-2+2\cos[2]+\sin[2]} - \dfrac{(-1+\cos[2]) \sin[2-t]}{-2+2\cos[2]+\sin[2]} + \sin[t] - \dfrac{(-1+\cos[2]) \sin[t]}{-2+2\cos[2]+\sin[2]}}{\cos[2] - \sin[2]}$

Out[91]= `0.736427`

Out[92]= `0.555608 Cos[2. - 1. t] - 0.555608 Sin[2. - 1. t] + 0.198856 Sin[t]`

Out[93]= `1.85057`