Homework 3

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Problem (LN12 0.1.1)**.** Show that the antipodal reflection $a : S^n \to S^n$, $a(x) = -x$ is an isometry.

Proof. The antipodal reflection is clearly smooth and has itself as the smooth inverse, and therefore is a diffeomorphism. For any $p \in S^n, v \in T_pS^n$, and any smooth curve $\gamma : (-\varepsilon, \varepsilon) \to S^n$ s.t. $\gamma(0) = p, \gamma'(0) = v$, its derivative is

$$\begin{aligned} da_p(v) &= (a \circ \gamma)'(0) \\ &= (-\gamma)'(0) \\ &= -\gamma'(0) \\ &= -v. \end{aligned}$$

For any Riemannian metric g on S^n and any $p \in S^n$, we have

$$\begin{aligned} g_p(da_p(v), da_p(w)) &= g_p(-v, -w) \\ &= -g_p(v, -w) && \text{bilinearity} \\ &= g_p(v, w), \end{aligned}$$

which proves that it is an isometry. □

Problem (LN12 0.2.1)**.** Show that inversion $i : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ given by $i(x) = \frac{x}{\|x\|^2}$ is a conformal transformation.

Proof. Let $M = \mathbb{R}^n \setminus \{0\}$. Given $p \in M, v \in T_pM$, and $\gamma : (-\varepsilon, \varepsilon) \to M$ s.t. $\gamma(0) = p$ and $\gamma'(0) = v$, we compute the derivative at p:

$$\begin{aligned} di_p(v) &= (i \circ \gamma)'(0) \\ &= \left(\frac{\gamma}{\|\gamma\|^2} \right)'(0) \\ &= \left(\frac{\gamma'\|\gamma\|^2 - 2\gamma\langle \gamma, \gamma' \rangle}{\|\gamma\|^4} \right)(0) && \text{quotient rule} \end{aligned}$$

$$= \frac{\|p\|^2 v - 2p\langle p, v\rangle}{\|p\|^4}.$$

Given $v, w \in T_pM$, we have

$$\begin{aligned}
g_p(di_p(v), di_p(w)) &= \langle \frac{\|p\|^2 v - 2p\langle p, v\rangle}{\|p\|^4}, \frac{\|p\|^2 w - 2p\langle p, w\rangle}{\|p\|^4} \rangle \\
&= \frac{\langle v, w\rangle}{\|p\|^4} - \frac{4v^T pp^T w}{\|p\|^6} + \frac{4v^T p(p^T p)p^T w}{\|p\|^8} \\
&= \frac{\langle v, w\rangle}{\|p\|^4} - \frac{4v^T pp^T w}{\|p\|^6} + \frac{4v^T pp^T w}{\|p\|^6} \\
&= \frac{\langle v, w\rangle}{\|p\|^4}.
\end{aligned}$$

Then the angle $\theta(di_p(v), di_p(w))$ between $di_p(v), di_p(w)$ is

$$\begin{aligned}
\theta(di_p(v), di_p(w)) &= \arccos\left(\frac{g_p(di_p(v), di_p(w))}{g_p(di_p(v), di_p(v))^{\frac{1}{2}} g_p(di_p(w), di_p(w))^{\frac{1}{2}}}\right) \\
&= \arccos\left(\frac{\frac{\langle v,w\rangle}{\|p\|^4}}{\frac{\|v\|\|w\|}{\|p\|^4}}\right) \\
&= \arccos\left(\frac{\langle v, w\rangle}{\|v\|\|w\|}\right) \\
&= \arccos\left(\frac{g_p(v, w)}{g_p(v, v)^{\frac{1}{2}} g_p(w, w)^{\frac{1}{2}}}\right) \\
&= \theta(v, w).
\end{aligned}$$

That is, i is conformal. $\square$

Problem (LN12 0.2.3)**.** Show that the Poincaré half-plane and disk are isometric.

Proof. Translating the half-plane up by 1 to get $H = \{(x, y) \in \mathbb{R}^2 : y > 1\}$ is clearly an isometry. Let $D = \{(x, y) : x^2 + (y - 0.5)^2 < 0.25\}$ be the open disk centered at $(0, 0.5)$ with radius 0.5. Inversion is a composition of smooth maps and thus smooth. First we show that the inversion map is a dffeomorphism between H and D. Given $p = (x, y) \in H$, observe

$$\begin{aligned}
\left\| \frac{p}{\|p\|^2} - \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} \frac{x}{x^2+y^2} \\ \frac{2y-(x^2+y^2)}{2(x^2+y^2)} \end{pmatrix} \right\|^2 \\
&= \frac{4x^2 + 4y^2 - 4(x^2 + y^2)y + (x^2 + y^2)^2}{4(x^2 + y^2)^2}
\end{aligned}$$

$$= \frac{1}{4} + 4(x^2 + y^2)(1 - y)$$
$$< \frac{1}{4}$$

since $y > 1$. Thus the restricted inversion $i : H \to D$ is well-defined and remains smooth. I claim that its inverse is itself, which we name $j : D \to H$ due to differences in domain and codomain. We check j is well-defined: if we have $p = (x, y) \in D$, then

$$\begin{aligned}
x^2 + (y - 0.5)^2 = x^2 + y^2 - y + \frac{1}{4} &< \frac{1}{4} \\
y &> x^2 + y^2 \\
\frac{y}{x^2 + y^2} &> 1 && p \neq 0 \\
y_{j(p)} &> 1.
\end{aligned}$$

Since it is clear that $i \circ j(p) = j \circ i(p) = p$, inversion is a diffeomorphism. Now endow H and D with the modified metrics g and h respectively, where $g_p(v, w) = \frac{\langle v,w \rangle}{(y-1)^2}$ and $h_p(v, w) = \frac{\langle v,w \rangle}{(0.25 - \|p-(0,0.5)\|^2)^2}$ due to the translation and scaling of the plane and disk. We observe

$$\begin{aligned}
h_{i(p)}(di_p(v), di_p(w)) &= \frac{\langle di_p(v), di_p(w) \rangle}{(0.25 - \|i(p) - (0, 0.5)\|^2)^2} \\
&= \frac{\langle v, w \rangle}{\|p\|^4 (0.25 - \left\| \frac{p}{\|p\|^2} - (0, 0.5) \right\|^2)^2} \\
&= \frac{\langle v, w \rangle}{\|p\|^4 \left(0.25 - \frac{1-y}{\|p\|^2} - 0.25\right)^2} \\
&= \frac{\langle v, w \rangle}{(y - 1)^2} \\
&= g_p(v, w).
\end{aligned}$$

Hence (H, g) and (D, h) are isometric. □

Problem (LN12 0.3.1)**.** Compute the metric of S^2 in terms of spherical coordinates θ and ϕ.

The parametric equation is $f(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$. Then

$$\frac{\partial f}{\partial \theta} = (-\sin\theta \sin\phi, \cos\theta \sin\phi, 0)$$

$$\frac{\partial f}{\partial \phi} = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi).$$

Since S^2 is endowed with the ambience Euclidean metric $\langle\cdot,\cdot\rangle$, the pullback metric on the parameter space is $g_{ij} = \langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}\rangle$, which is

$$\begin{aligned}
G(\theta,\phi) &= \begin{pmatrix} \sin^2\theta\sin^2\phi + \cos^2\theta\sin^2\phi & 0 \\ 0 & \cos^2\theta\cos^2\theta + \sin^2\theta\cos^2\phi + \sin^2\phi \end{pmatrix} \\
&= \begin{pmatrix} \sin^2\phi & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Problem (LN12 0.4.1)**.** Compute the length of the radius of the Poincaré disk (with respect to the Poincaré metric).

Consider the unit open disk D with the metric $g_p(v,w) = \frac{\langle v,w\rangle}{(1-\|p\|^2)^2}$ and the curve $\gamma : [0,1) \to D, t \mapsto (0,t)$ which traces out the radius. Then

$$\begin{aligned}
L[\gamma] &= \int_0^1 g_{\gamma(t)}(\gamma'(t),\gamma'(t))^{\frac{1}{2}}\,dt \\
&= \int_0^1 g_{\gamma(t)}((0,1),(0,1))^{\frac{1}{2}}\,dt \\
&= \int_0^1 \frac{1}{1-\|(0,t)\|^2}dt \\
&= \int_0^1 \frac{1}{1-t^2}dt \\
&= \int_0^1 \frac{dt}{1-t} + \int_0^1 \frac{dt}{1+t} \\
&= (\ln|1-t| + \ln|1+t|)\Big|_0^1,
\end{aligned}$$

which diverges. Thus loosely speaking, the radius is infinite.