1 Structure gorups of fiber bundles

Given a local trivial fibration Diagram.

$$\phi_2 \circ \phi_1^{-1} : (U_1 \cap U_2) \times F \to (U_1 \cap U_2) \times F, (x, y) \to (x, \tau_{21}(x)(y))$$

where $\tau_{21} : (U_1 \cap U_2) \to \text{Homeo}(F)$ (compact-open), which is called a **transition (or clutching) function**.

Remark 1.1 If $\{(U_\alpha, \phi_\alpha)\}$ is a collection of local trivializations s.t. $B = \bigcup_\alpha U_\alpha$. Then the transition maps satisfy $(*)$:

$$\tau_{\alpha\alpha}(x) = \text{id}_F$$
$$(\tau_{\alpha\beta(x)})^{-1} = \tau_{\beta\alpha}(x)$$
$$\tau_{\gamma\beta}(x) \circ \tau_{\beta\alpha}(x) = \tau_{\gamma\alpha}(x)$$

Exercise: show that if $\{U_\alpha\}$ is a cover of B by open sets and $\tau_{\alpha\beta} : (U_\alpha \cap U_\beta) \to \text{Homeo}(F)$ are maps satisfying $(*)$, then there exists a bundle E over B realizing this data as transition maps.

Hint: let $E = \bigsqcup_{U_\alpha \times F} / \sim$ where $(x, y) \in U_\alpha \times F \sim (x', y') \in U_\beta \times F \Leftrightarrow x = x'$ and $\tau_{\beta\alpha}(x)(y) = y'$. There is an obvious projection $p : E \to B$. Prove this is a bundle.

Exercise: find the transition maps for Diagram.

Definition 1.2 — Suppose $G \subseteq \text{Homeo}(F)$ is a topological group. If diagram has a collection of transition functions

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to G,$$

then we say E has **structure group** G.

If the transition functions do not map into G but can be homtoped through functions satisfying $(*)$ to one with image in G, then we say the structure group of E **reduces** to G.

Remark 1.3 If G preserves some structure on F, then the fibers of $p : E \to B$ will have this structure.

Example 1.4 (1) If $F = \mathbb{R}^n$ and $G = \mathrm{GL}_n(R) \subseteq \mathrm{Homeo}(\mathbb{R}^n)$, then each fiber of a bundle with structure group G has a linear structure.

(2) If $F = \mathbb{R}^n$ and $G = \mathrm{GL}_n^+(\mathbb{R})$, then fibers of F are oriented vector spaces so E is an **oriented vector bundle**.

(3) If $F = \mathbb{R}^n$ and $G = O(n)$. Then E is a vector bundle with a metric (inner product). Note $O(n) \to \mathrm{GL}_n(\mathbb{R})$ (inclusion) is a homotopy equivalence. Hence all vector bundles admit metrics.

(4) If $F = \mathbb{R}^{2n} = \mathbb{C}^n$, then $G = \mathrm{GL}_n(\mathbb{C}) \Leftrightarrow E$ has a complex structure. $G = U(n) \Leftrightarrow E$ has a Hermitian structure.

(5) If $F = \mathbb{R}^n$ and $G = \mathrm{GL}_k(\mathbb{R}) \times \mathrm{GL}_{n-k}(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R}), (A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, then E has a G-structure $\Leftrightarrow E \cong E_1 \oplus E_2$ where E_1 is an $\mathbb{R}^k$-bundle, E_2 is an $\mathbb{R}^{n-k}$-bundle. Similarly of $G = \mathrm{GL}_{n-k}(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$, then E has a G-structure $\Leftrightarrow E \cong E^1 \oplus \mathbb{R}^k$ where E^1 is an $\mathbb{R}^{n-k}$-bundle.

Question: when can we reduce the structure group?

Definition 1.5 — If G is a Lie group (topological group), then a bundle diagram is a **principal G-bundle** if there exists a smooth (or continuous) right G-action $P \times G \to P$ s.t.

(1) action preserves fibers, *i.e.* $y \in p^{-1}(x) \Rightarrow y.g \in p^{-1}(x)\ \forall\ x, y, g$.

(2) G acts freely and transitively on $p^{-1}(x)\ \forall\ x$.

Remark 1.6 This can also be defined as a smooth manifold P with a smooth right G-action that is free and proper, *i.e.* for the map $P \times G \to P \times P, (p, g) \mapsto (p.g, p)$ preimages of compact sets are compact.

Example 1.7 (1) If (E, B, F, p) is a bundle with structure group G, then there is a cover of B by local trivialization $\{(U_\alpha, \phi_\alpha\}$ with transition functions $\tau_{\alpha\beta} : (U_\alpha \cap U_\beta) \to G$ we can construct a principal G-bundle as follows

$$P_E = \bigsqcup_\alpha U_\alpha \times G / \sim$$

when $(x, g) \in U_\alpha \sim (x', g') \in U_\beta \times G \Leftrightarrow x = x'$ and $\tau_{\beta\alpha}(x)g = g'$. Exercise: show this is a principal G-bundle.

If E is a vector bundle then P_E is a principal $\mathrm{GL}_n(\mathbb{R})$-bundle. It is called the **frame bundle** because you can think of points in the fibers of P_E as frames for the fibers of E. Exercise: think through this. We denote this $\mathcal{F}(E)$. Note $O(n) \simeq \mathrm{GL}_n(\mathbb{R})$ so we could take $\mathcal{F}(E)$ to be a principal $O(n)$-bundle.

(2) diagram is a principal S^1-bundle.

(3) Regular covering spaces are of a manifold are principal bundles. Exercise: check this and what are fibers? Can an irregular cover be a principal bundle?

Exercise:

(1) Show a prinicpal G-bundle is trivial iff it admits a section.

(2) If E is a vector bundle, the sections of E are the same as $\mathrm{GL}_n(\mathbb{R})$-equivariant maps $v : \mathcal{F}(E) \to \mathbb{R}^n$, *i.e.* $v(y.g) = g^{-1}v(y)$. Hint: given $s : B \to E$ then for each $y \in \mathcal{F}(E)$, let $v(y) = s(p(y))$ expressed in the frame y. Then $p : \mathcal{F}(E) \to B$. This allows us to turn sections into functions which are easier to work with.

Given $P \to B$ a principal G-bundle, and $\rho : G \to G'$ is a homomorphism, where $G' \subseteq \mathrm{Homeo}(F)$. Then we can construct an F-bundle with structure group G' as follows

$$P \times_\rho F = P \times F/(p.g, f) \sim (p, \rho(g)f)$$