Homework 4

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Problem (1: VIII Theorem 2)**.** If (M, ξ) is tight and L is a Legendrian knot with $L = \partial\Sigma$, then prove $tb(L) - r(L) \leq -\chi(\Sigma)$.

Proof. We can WLOG assume $tb(L) < 0$ since any negative stablization of L decreases both $tb(L)$ and $r(L)$ by 1 so $tb(L) - r(L)$ remains constant. Moreover, $\chi(\Sigma_-) \leq -tb(L)$. Indeed, the only surface with boundary that gives positive Euler characteristic is a disk with $\chi(D) = 1$, and by Giroux Criterion, dividing curves cannot bound a disk, so the only possible disks appear on the boundary L. The dividing curves must intersect L $-2tb(L)$ times by Theorem VII.9. That means at most $-tb(L)$ number of disks can be formed in Σ_-. Thus $\chi(\Sigma_-)$ is upper bounded by $-tb(L)$. Then

$$\begin{aligned} tb(L) - r(L) &\leq tb(L) - r(L) - 2\chi(\Sigma_-) - 2tb(L) \\ &= -tb(L) - (\chi(\Sigma_+) - \chi(\Sigma_-)) - 2\chi(\Sigma_-) \\ &= -tb(L) - \chi(\Sigma_+) - \chi(\Sigma_-) \\ &= -\chi(\Sigma) \end{aligned}$$

by Claim 1. □

Problem (2: IV Lemma 4)**.** If γ_0, γ_1 are cobordant via $\Sigma \subset M \times I$, we project Σ to M and triangulate $\pi(\Sigma)$ to obtain a 2-chain c in $C_2(M)$. Show that $\partial c = \gamma_1 - \gamma_0$.

Proof. Let us triangulate Σ and obtain a 2-chain c' in $C_2(M \times I)$. I claim that $\partial(c') = \gamma_1 - \gamma_0$. This is because the boundary operator in a chain complex is exactly defined so that when a manifold with boundary is triangulated into a chain complex, its triangulated boundary is the image of the boundary operator. Let $\pi_\# : C_2(M \times I) \to C_2(M)$ be the induced chain map, then we have

$$\partial c = \partial \pi_\#(c') = \pi_\# \partial(c') = \pi_\#(\gamma_1 - \gamma_0) = \gamma_1 - \gamma_0,$$

where we use the commutativity of chain map and boundary map, and the fact that projection to M does not change γ_i. □

Problem (3: IV Lemma 5). Let y be a free generator of $H_1(M)$ (since we assume $d(x) \neq 0$ and $x = d(x)y$ is thus non-torsion). Show that there exists a surface α in M s.t. $y \cdot \alpha = 1$. Note M is compact.

Proof. By Poincare duality, $PD(y)$ is a free generator in $H^2(M)$. By the universal coefficient theorem, $H^2(M) \cong \mathrm{Hom}(H_2(M), \mathbb{Z}) \oplus \mathrm{Ext}(H_1(M), \mathbb{Z})$, where the Hom term contains the free part and the Ext term contains the torsion part. Thus $PD(y)$ can be viewed as a free generator in $\mathrm{Hom}(H_2(M), \mathbb{Z})$ which is free. Choose a basis (containing $PD(y)$) of $\mathrm{Hom}(H_2(M), \mathbb{Z})$, which is dual to $\mathrm{Free}(H_2(M))$. Then dualize $\mathrm{Hom}(H_2(M), \mathbb{Z})$ to get the double-dual under the dual basis containing $PD(y)^* : \mathrm{Hom}(M, \mathbb{Z}) \to \mathbb{Z}$ that sends $PD(y)$ to 1 and all other generators to 0. By the canonical isomorphism of a finite-rank free module and its double-dual, we know that the generator $PD(y)^*$ is the evaluation map where $PD(y)$ evaluated at some generator $[\alpha] \in \mathrm{Free}(H_2(M))$ is 1. *i.e.* $\langle PD(y), [\alpha] \rangle = 1$. Let α be a surface representing this corresponding generator. Then $y \cdot \alpha = \langle PD(y), [\alpha] \rangle = 1$. □

Problem (4: VIII Theorem 1). Show that Theorem VIII.1 implies that there exists only finitely many classes in $H^2(M)$ that can be the Euler class of a tight contact structure.

Proof. By universal coefficient theorem, $H^2(M) \cong \mathrm{Hom}(H_2(M), \mathbb{Z}) \oplus \mathrm{Ext}(H_1(M), \mathbb{Z})$, where $\mathrm{Ext}(H_1(M), \mathbb{Z})$ is isomorphic to the torsion part of $H_1(M)$ and $\mathrm{Hom}(H_2(M), \mathbb{Z})$ is isomorphic to the free part of $H_2(M)$. Choose any basis of the free part of $H_2(M)$, which we can represent via surfaces in M. For any such basis surface Σ, if $e(\xi)$ is the Euler class of ξ, then Theorem 1 says that $|\langle e(\xi), [\Sigma] \rangle| \leq -\chi(\Sigma)$ or 0, meaning that $e(\xi)$ can only map basis to finitely many integers. This gives finitely many choices to define $e(\xi)$ on $\mathrm{Hom}(H_2(M), \mathbb{Z})$. Now choose a generator set for the torsion part of $H_1(M)$. Again we only have finitely many integers to map the generators by definition of torsion. Thus we completely define $e(\xi)$ by where it maps the generators of $H_1(M)$ and basis of $H_2(M)$ but only have finitely many choices altogether. This proves the claim. □