The general setting for this section is the abelian category. However, it is out of the scope for this project, and by the embedding theorem, restricting the setting to where R is a commutative ring with 1 is not too much of a loss. Moreover, any functor F in this setting is assumed to be **additive**. That is, given $f, g \in \operatorname{Hom}_R(A, B)$, we have $Ff + Fg = F(f + g)$.

A functor is called **exact** if it sends short exact sequences to short exact sequences. A functor is **left/right-exact** if it preserves kernels/cokernels. That is, if F is a left-exact functor, given $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$, we have $0 \to F(A) \xrightarrow{Fi} F(B) \xrightarrow{Fp} F(C)$. In the interest of conciseness, we shall restrict the discussion to left-exact functors and leave the dual case to the reader (also covered by the book).

The failure of exactness on the right raises a natural question: is there a way to use a longer sequence to patch this failure? This immediately reminds us of long exact sequence of cohomology, but to obtain such long exact sequence, we must start with a short exact sequence involving the image of F. It appears that we are stuck, but it turns out we can circumvent this conundrum using injective resolutions.

We have shown that there are enough injectives in CREF. That means we can find an injective resolution Q for C:

$$0 \to C = \ker f^0 \rightarrowtail Q_0 \xrightarrow{f^0} Q_1 \xrightarrow{f^1} Q_2 \to \cdots$$

Applying F to this resolution, we obtain a chain complex FQ

$$0 \to FC = \ker Ff^0 \rightarrowtail FQ_0 \xrightarrow{Ff^0} FQ_1 \xrightarrow{Ff^1} FQ_2 \to \cdots$$

which is no longer necessarily exact, as F doesn't necessarily preserve cokernels. The in-exactness gives rise to non-trivial cohomology, therefore we can *derive* interesting functors called the ***i*th right-derived functor** from F by $RF^i(C) = H^i FQ := \ker Ff^i / \operatorname{im} Ff^{i-1}$.

Proposition 0.1

The right-derived functors of F are well-defined (does not depend on of the choice of resolutions) and satisfy the following properties:

(1) $RF^0 = F$.

(2) If C is an injective module, then $RF^i(C) = 0$ for all $i > 0$.

(3) Suppose $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is a short exact sequence, then there exists a long exact sequence of cohomology:

(4) The connecting homomorphisms δ^i in the long exact sequence are natural.

Proof. By Corollary CREF, RF^i is well-defined.

(1) $RF^0(C) = H^0(FQ) = \ker Ff^0 / \operatorname{im} Ff^1 = FC/0 = F(C)$.

(2) Since C is an injective module, let $0 \to C \to C \xrightarrow{f^0} 0 \xrightarrow{f^1} 0 \to \cdots$ be its injective resolution. It is clear that $RF^i(C) = \ker f^i / \operatorname{im} f^{i-1} = 0/0 = 0$ if $i > 0$.

(3) From the given short exact sequence, we obtain a short exact sequence of chain complexes by taking injective resolutions described in CREF:

$$0 \to Q_A \xrightarrow{j} Q_B \to Q_C \to 0$$

Since Q_A is injective, since j is a monomorphism, 1_{Q_A} lifts to a map $s : Q_B \to Q_A$ such that $1_{Q_A} = s \circ j$. Thus, the short exact sequence splits. It follows that $0 \to FQ_A \to FQ_B \to FQ_C \to 0$ is a split exact sequence by CREF. Applying CREF to this short exact sequence, we obtain the long exact sequence on cohomology as desired.

(4) This is a standard exercise.

$\square$

Let $F = \operatorname{Hom}_R(M, -)$ covariant, we define $\operatorname{Ext}^i(M, -) := RF^i$.

Let $F = \operatorname{Hom}_R(-, N)$ contravariant, we define $\operatorname{Ext}^i(-, N) := RF^i$.

Therefore, to compute $\operatorname{Ext}^i(M, N)$, we can either take a projective resolution P_M of M or an injective resolution Q_N of N.

Let $F = - \otimes_R M$ covariant, we define $\operatorname{Tor}_i(-, M) = \operatorname{Tor}_i(M, -) := LF_i$.