Homework 1

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Problem (1)**.** Show that the mapping cylinder $C_f \simeq Y$.

Proof. We wish to find a homotopy equivalence. Note that any equivalent class in C_f has a single element except for $[(x,0)] = [f(x)]$, so we just implicitly treat it as a single element. Define $\phi : C_f \to Y$ as the following: crush any $(x,t) \in X \times I$ to $(x,0)$ and then project to its equivalent class $[(x,0)]$. Since $(x,0) \sim f(x)$, there is a unique element $f(x) \in Y$ in this equivalence class so we let ϕ map $[f(x)]$ to $f(x)$; for any $y \in Y$, each equivalence class $[y]$ has a single element so $[y] \mapsto y$ is well-defined and agrees with the other definition on the intersection. Define $\iota : Y \to C_f, y \mapsto [y]$. We see that given $(x,t) \in X \times I$,

$$\begin{aligned}
\iota \circ \phi([x,t]) &= \iota(f(x)) \\
&= [f(x)] \\
&= [(x,0)] \\
&= \mathrm{id}_X \times e_0([x,t]) \\
\iota \circ \phi([y]) &= \iota(y) = [y] = \mathrm{id}_{C_f}|_Y
\end{aligned}$$

Since I is contractible, there exists a homotopy G s.t. $\mathrm{id}_{C_f}|_{X\times I} = \mathrm{id}_X \times \mathrm{id}_I \simeq \mathrm{id}_X \times e_0$. By the pasting lemma, we can paste G with the constant homotopy on $\mathrm{id}_{C_f}|_Y$ to get a homotopy between id_{C_f} and $\iota \circ \phi$. Given $y \in Y$, we have

$$\begin{aligned}
\phi \circ \iota(y) &= \phi([y]) \\
&= y \\
&= \mathrm{id}_Y(y)
\end{aligned}$$

Thus ϕ, ι yields a homotopy equivalence and $C_f \simeq Y$. $\square$

Moreover, let j be the injection $X \to C_f, x \mapsto (x,1)$. Show that $j \simeq \iota \circ f$.

Proof. Given $x \in X$, $\iota \circ f : x \mapsto [(x,0)]$ which is an injection from X to C_f as well. Clearly the identity homotopy $(x,t) \mapsto (x,t)$ is a homotopy between the two injections so $j \simeq \iota \circ f$. $\square$

Problem (2)**.** Show that

$$\phi : [X, Y]_0 \times [X, Y]_0 \to [X, Y \times Y]_0, ([f], [g]) \mapsto [f \times g].$$

is well-defined and a bijection.

Surjectivity is clear. Well-definedness and injectivity follows immediately from the following claim:

Claim 0.1. $(f_1, g_1) \sim (f_2, g_2) \Leftrightarrow f_1 \times g_1 \sim' f_2 \times g_2$.

Proof. $(\Rightarrow)$: this direction shows well-definedness. The equivalence relation $(f_1, g_1) \sim (f_2, g_2)$ is defined as $f_1 \sim f_2$, $g_1 \sim g_2$, so let the respective homotopies be H_1 and H_2. Then by mapping into product, we obtain a continuous function $H : X \times I \to Y \times Y$. It's clear that H is a homotopy between $f_1 \times g_1$ and $f_2 \times g_2$.

$(\Leftarrow)$: this direction shows injectivity. Suppose $f_1 \times g_1 \sim' f_2 \times g_2$, *i.e.* $[f_1 \times g_1] = [f_2 \times g_2]$, then we have a homotopy $H : X \times I \to Y \times Y$. Composing H with the projection functions π_1 and π_2 clearly yield homotopies between f_1 and f_2, g_1 and g_2. Thus $[(f_1, g_1)] = [(f_2, g_2)]$.

$\square$

Lemma 0.2

For any pointed (Y, y_0), its suspension $\sum Y$ is an H'-space.

Problem (3)**.** *Proof.* Define $\mu : \Sigma Y \to \Sigma Y \vee \Sigma Y$ by collapsing $Y \times \{\frac{1}{2}\}$ in ΣY. Then $p_1 \circ \mu \simeq \mathrm{id}_{\Sigma Y}$ because we can use straight-line homotopy on $Y \times I$ to move $Y \times [\frac{1}{2}, 1]$ to $Y \times \{1\}$ and then collapse it to a point. Likewise for $p_2 \circ \mu$.

Similarly, for $(\mu \vee \mathrm{id}_{\Sigma Y}) \circ \mu \simeq (\mathrm{id}_{\Sigma Y} \vee \mu) \circ \mu$, one collapses $Y \times \{\frac{1}{2}\}$ and $Y \times \{\frac{1}{4}\}$, the other collapses $Y \times \{\frac{1}{2}\}$ and $Y \times \{\frac{3}{4}\}$. Straight-line homotopy moving $Y \times \{\frac{1}{2}\}$ to $Y \times \{\frac{3}{4}\}$ and moving $Y \times \{\frac{1}{4}\}$ to $Y \times \{\frac{1}{2}\}$ would do.

Define $\nu : \Sigma Y \to \Sigma Y, (y, t) \mapsto (y, 1 - t)$. Then $f := (\mathrm{id}_{\Sigma Y} \vee \nu) \circ \mu \simeq e_0$ where $e_0 : \Sigma Y \to \Sigma Y, (y, t) \mapsto \{y_0\} \times I \cup Y \times \{0, 1\}$. This is because fixing any y, we see that ν forces $\{y\} \times I$ in the cylinder to have $f(y, 0) = f(y, 1)$. That is, $f(\{y\} \times I)$ yields a loop which we can continuously shrink to a constant loop. This allows it to remain in the base $Y \times \{0\}$ which we then collapse to the base point, yielding the constant map.

$\square$

Problem (4)**.** Compact-open topology:

(1) This is skipped as it is not necessary for the proof of lemma.

(2) $(\Rightarrow)$: Suppose $f : X \times Y \to Z$ is continuous. We wish to use the local continuity definition to show that $F : X \to C^0(X, Y)$ is also continuous. Given $x \in X$, take a neighborhood of f_x WLOG we use a subbasis element $S(C_x, U_x)$ around f_x instead for convenience, where C_x is compact in Y and U_x is open in Z. Then $f^{-1}(U_x)$ is an open set in $X \times Y$ so it is also open in $X \times C_x$. Since $f_x(C_x) \subseteq U_x$, $\{x\} \times C_x \in f^{-1}(U_x) \cap (X \times C_x)$, by the tube lemma, there exists a neighborhood W of X s.t. $\{x\} \times C_x \subseteq W \times C_x \subseteq f^{-1}(U_x) \cap (X \times C_x)$. Then we see that

$$\begin{aligned} F(W) &= \{f_w : w \in W | f_w(C_x) \subseteq U_x\} \\ &\subseteq S(C_x, U_x) \end{aligned}$$

Hence F is continuous.

$(\Leftarrow)$: Suppose F is continuous and Y is locally-compact Hausdorff. Take $(x, y) \in X \times Y$ and let U be a neighborhood of $f(x, y)$ in Z. We wish to find a neighborhood W of (x, y) s.t. $f(W) \subseteq U$. First let U_y be any neighborhood around y. Since Y is locally compact Hausdorff, by Theorem 29.2 of Munkres, U_y admits an neighborhood V_y around y s.t. $C_y := \overline{V_y} \subseteq U_y$ and is compact. Since F is continuous, $F^{-1}(S(C_y, U))$ is open in X. Then define $W := F^{-1}(S(C_y, U)) \times V_y$ which is a product of open sets so it is open in $X \times Y$. Then

$$\begin{aligned} f(W) &= \{f(a, b) : (a, b) \in W\} \\ &= \{f(a, b) : (a, b) \in F^{-1}(S(C_y, U)) \times V_y\} \\ &= \{f(a, b) : f_a(C_y) \subseteq U, b \in V_y\} \\ &\subseteq U && V_y \subseteq C_y \end{aligned}$$

So f is continuous.

(3) By part 2 and 3, both maps in the lemma are well-defined. The bijection follows easily.