Homework 4

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Problem (1)**.** f is 1-1 continuous so it is an immersion and is bijective onto its image $f(X)$. By Theorem 26.6 of Munkres, f is a homeomorphism onto $f(X)$ so it is an embedding.

Problem (4)**.** Since f is a product map of polynomials, it is continuous. Let $\widetilde{f}$ denotes the map with domain restricted to the unit sphere in $\mathbb{R}^3$ and codomain restricted to its image, which remains continuous. Suppose $(x_1y_1, y_1z_1, x_1z_1, x_1^2+2y_1^2+3z_1^2) = (x_2y_2, y_2z_2, x_2z_2, x_2^2+2y_2^2+3z_2^2)$. In the case when none of the entries is zero, since $x_1 = \frac{x_2y_2}{y_1} = \frac{x_2z_2}{z_1}$, combining with $y_1z_1 = y_2z_2$ we obtain that $z_1^2 = z_2^2$ so $z_1 = \pm z_2$. Then $y_1 = \pm y_2$ and $x_1 = \pm x_2$ (the signs must be all + or all -). These solutions are consistent with the 4th entry so they are all valid. Whenever there is at least one entry that is zero, WLOG assume $x_1 = 0$, this forces that either x_2 or y_2 to be zero and either x_2 or z_2 to be zero. Suppose $x_2 \neq 0$, then $y_2 = z_2 = 0$, which forces that either y_1 or z_1 to be zero. WLOG suppose $y_1 = 0$, then by the relation that $x^2+y^2+z^2 = 1$, we obtain $z_1 = x_2 = 1$. Thus in the 4th entry we have

$$0^2 + 2\cdot 0^2 + 3z_1^2 = x_2^2 + 2\cdot 0^2 + 3\cdot 0^2$$
$$3 = 1,$$

a contradiction. Thus $x_1 = 0$ forces $x_2 = 0 = \pm x_1$.

Taken both cases together, for any $q \in \widetilde{f}(S^2)$, $\widetilde{f}^{-1}(q) = \{\pm p\}$ for some p in S^2. That is, if we quotient S^2 by the antipodal action of $\mathbb{Z}_2$, we obtain a map $g : S^2/\sim \to \widetilde{f}(S^2)$, which again is continuous by the universal property of quotient topology. But $\mathbb{R}P^2 = S^2/\sim$ so we have found a well-defined continuous map from $\mathbb{R}P^2$ into $\mathbb{R}^4$. It is clearly injective and surjective onto its image per the argument above, so it remains to show that g is a homeomorphism. By Corollary 22.3 of Munkres, it suffices to show that $\widetilde{f}$ is a quotient map. By Exercise 22.2(b), it suffices to show that there exists a continuous map $h : \widetilde{f}(S^2) \to S^2$ s.t. $\widetilde{f} \circ h = \mathrm{id}_{\widetilde{f}(S^2)}$. Given $(a,b,c,d) \in \widetilde{f}(S^2)$, we want to recover x, y, z in terms of a, b, c, d. Define $A = \sqrt{d - 1 + 2\sqrt{2}b}$ and $B = \sqrt{3 - d + 2\sqrt{2}a}$. Then we can check that

$$x = \frac{\sqrt{2}(d-2)}{2(A-B)} - \frac{A-B}{2\sqrt{2}}$$

$$
\begin{aligned}
y &= \frac{a}{x} \\
z &= \frac{A-B}{2\sqrt{2}} + \frac{\sqrt{2}(d-2)}{2(A-B)}
\end{aligned}
$$

does the job (using the relation that $x^2+y^2+z^2=1$). Then the map $h:(a,b,c,d)\mapsto(x,y,z)$ is clearly continuous as the composition of continuous functions and gives us the identity (no need to worry about dividing by 0 since we know such x, y, z exist as (a,b,c,d) is in the image). It follows that g is a homemomorphism onto its image so it is an embedding.

Alternatively, we could use the fact that since $\mathbb{R}P^2$ is the quotient of a compact space and therefore compact, and $\mathbb{R}^4$ is Hausdorff so any subspace is Hausdorff, since g is already bijective continuous, it is a homeomorphism by Munkres Theorem 26.6.

Problem (7.16)**.** I claim that 1 is a regular value of $g: TM\to\mathbb{R}$. Given $p\in g^{-1}(1)$, $\operatorname{rank} dg_p = 0$ or 1. Suppose $\operatorname{rank} dg_p = 0$, then it has to send everything to 0, but since g_p is an inner product, it is positive definite and cannot be constant so the derivative cannot be 0. Hence $\operatorname{rank} dg_p = 1$ and 1 is a regular value of g. Thus $g^{-1}(1)$ is a $2n-1$ manifold which is the unit tangent bundle.

Problem (8.5)**.** Suppose $X\subseteq M$ is measure zero. Then for any $p\in M$, denote the chart given by the definition (U_p,ϕ_p) s.t. $\phi_p(U_p\cap X)$ has measure zero. Clearly these charts form an atlas for M. By Exercise 8, we can obtain a countable atlas (U_i,ϕ_i). Given any arbitrary chart (V,ψ) of M, $V=\bigcup_{i\in\mathbb{N}} U_i$. Then $\psi\circ\phi_i^{-1}:\phi_i(U_i)\cap\psi(V)\subseteq\mathbb{R}^n\to\mathbb{R}^n$ is smooth. Thus we can apply Lemma 4 and obtain:

$$\psi\circ\phi_i^{-1}(\phi_i(U_i\cap X)\cap\phi_i(U_i\cap V)) = \psi(U_i\cap X\cap V)$$

has measure zero. Then we see that

$$\psi(V\cap X)\subseteq\bigcup_{i\in\mathbb{N}}\psi(U_i\cap X\cap V)$$

Since countable union of measure zero sets are measure zero, $\psi(V\cap X)$ is also measure zero by monotonicty.

Problem (8.8)**.** Given an arbitrary atlas $\mathcal{A}=\{(U_\alpha,\phi_\alpha):\alpha\in J\}$ of a manifold M, since M

is second-countable, by Theorem 30.3 of Munkres, it is a Lindelof space. Therefore, there exists a countable subcover $\mathcal{A}'$ of $\mathcal{A}$ that serves as a countable atlas.

Problem (8.11)**.** ($\Rightarrow$): The derivative of $\pi_u(x)$ at p is the unique linear operator d_{π_u} : $T_pM \to \mathbb{R}^{n-1}$ satisfying the following

$$\lim_{t\to 0} \frac{\pi_u(x+tv) - (\pi_u(x) + td_{\pi_u}(v))}{t} = 0$$

where $v \in T_pM \cong \mathbb{R}^n \subseteq \mathbb{R}^N$. Notice

$$\begin{aligned}\frac{1}{t}(x + tv - \langle x+tv, u\rangle u - x - \langle x, u\rangle u) &= \frac{x}{t} + v - \langle \frac{x}{t}, u\rangle u - \langle v, u\rangle u - \frac{x}{t} - \langle \frac{x}{t}, u\rangle u \\ &= v - \langle v, u\rangle u\end{aligned}$$

Alternatively, projection is a linear operator so its derivative is just itself. Therefore, it's easy to see that $d_{\pi_u}(v) = v - \langle v, u\rangle u$. Now we have $u \notin T_pM$ iff u and v are linearly independent iff $tu - v \neq 0, v \neq 0,\ \forall\ t \in \mathbb{R}$ iff d_{π_u} doesn't send any nonzero vector in T_pM to 0 iff $\ker d_{\pi_u}$ iff it is not singular at p onto its image. ($\Leftarrow$) :