1 Brouwer's Fixed Point Theorem

Theorem 1.1

Let $f: B^n \to B^n$ be a continuous map, then there exists a $p \in B^n$ s.t. $f(p) = p$.

Definition 1.2 — A **manifold with boundary** M models after halfspaces H^n : $\{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_n \geq 0\}$.

Exercise: If $\dim(M) = n$, then ∂M is an $(n-1)$-manifold (therefore $\partial(\partial M) = \emptyset$.

The tangent space are one-sided derivatives of half-curves and has the same dimension as M.

Every n-manifold with ∂ is diffeomorphic to a subset of a n-manifold without ∂. (Doubling of a manifold by identifying boundaries of two manifolds).

Exercise: $f: M^m \to \mathbb{R}$ and 0 is a regular value of f, then $f^{-1}([0, \infty))$ is an m-manifold with boundary and $\partial(f^{-1}[0, \infty)) = f^{-1}(0)$.

Theorem 1.3

$f: M^m \to N^n$, M is a manifold with ∂, $q \in N$ is a regular value of both f and $f|_{\partial M}$ and $f^{-1}(q) \neq \emptyset$, then $f^{-1}(q)$ is $(m-n)$-manifold with ∂, and

$$\partial(f^{-1}(q)) = f^{-1}(q) \cap \partial M$$

Exercise 9: $T_p H^m \cong T_p \mathbb{R}^m$ since we can extend any half-curve to a curve.

Exercise 13: take $v \in T_p f^{-1}(q)$, $v = \alpha'(0), \alpha : (-\varepsilon, \varepsilon) \to f^{-1}(q), \alpha(0) = p$. Since $f \circ \alpha(t) = q$ so $df_p(v) = (f' \circ \alpha)'(0) = 0$.

Proof. WLOG $M = H^m$. Take $pf^{-1}(q) \cap \partial M$. Take a neighborhood V of p, by smoothness of f we can extend it to $\widetilde{f} : V \to N$ where $\widetilde{f} = f$ on $H^m \cap V$. By $d\widetilde{f}_p = df_p$, we know that p is a regular point of $\widetilde{f}$. This implies that q is a regular value of $\widetilde{f}$ as we can make V small so $\{p\} = \widetilde{f}^{-1}(q)$. So $\widetilde{f}^{-1}(q)$ is a manifold in V. Define $g : \widetilde{f}^{-1}(q) \to \mathbb{R}, (x_1, \ldots, x_n) \mapsto x_n$. Then $g(p) = 0$ and $V \cap f^{-1}(q) = H \cap \widetilde{f}^{-1}(q) = g^{-1}([0, \infty))$. Suppose 0 is not a regular value, then rank is 0 so $T_p\widetilde{f}^{-1}(q) = \ker \deg_p \subseteq T_p \partial H$. □

Theorem 1.4 (Sard's)

Let $f : M \to N$ be a smooth map, almost every (except for a set of measure zero)$g \in N$ is a regular value of f.

Proof of Brouwer. First we may assume that f is smooth by approximation theorem. Suppose to the contrary that $f : B^n \to B^n$ has no fixed point. Then there exists a smooth retraction $r : B^n \to \partial B^n = S^{n-1}$ by a ray at $f(p)$ through p. Notice that $r(p) = p \; \forall \; p \in \partial B^n$. By Sard's Theorem, there exists a $q \in S^{n-1}$ which is a regular value of r and $r^{-1}(q) \neq \emptyset$. By the regular value theorem, $r^{-1}(q)$ is a 1-dim manifold with ∂. Recall $\partial(r^{-1}(q)) = r^{-1}(q) \cap S^{n-1}$. Since $q \in r^{-1}(q)$. So $r^{-1}(q)$ must be an interval with distinct endpoints on the boundary. But this says that $r(q') = q$ yet $r(q') = q'$, a contradiction. □