# 1 Manifold with boundary

Exercise: is the unit square $S := I \times I \subseteq \mathbb{R}^2$ a smooth manifold with $\partial$?

Take $v_1, v_2$ at the corner $p$ along the edges, they are in the tangent space of $p$. Then $d\phi_p(v_1), d\phi_p(v_2) \subseteq T_{\phi(p)}\partial H \subseteq \mathbb{R}$ so $d\phi_p(v_1)$ and $d\phi_p(v_2)$ are linearly dependent. So are $v_1, v_2$ as $\phi$ is a diffeomorphism, $d\phi_p$ is full rank, a contradiction.

**Proposition 1.1**

Let $p \in \partial M$ be a regular point of a $f : M^n \to N^n$. Then $p$ is a regular point of $\partial f$.

*Proof.* $\operatorname{rank} df_p = \dim N = \dim M$ so $df_p$ preserves linear independence. Notice $T_p\partial M \subseteq T_pM$. TODO $\square$

**Theorem 1.2** (Sard's)

$f : M \to N$ where $N$ doesn't have $\partial$. Almost every $q \in N$ is a regular value of both $f$ and $\partial f$.

*Proof.* By regular Sard's Theorem, $\mu(\partial f(\{\text{critical points of} \partial f\})) = 0$. $\mu(f(\{\text{critical points of } f \in \operatorname{int} M\})) = 0$. $\mu(f(\{\text{critical points of } f \in \partial M\}))$.

Prove by contrapositive: let $p$ be a regular point of $\partial f$, then $\operatorname{rank} d\partial f_p = \dim N$, but $\operatorname{rank} df_p \geq d\partial f_p$, so $p$ is a regular point of $f$. So $\{$ critical point of $\partial f\} \subseteq \{$ critical points of $f$ on $\partial M\}$.

$\square$