1 Overview

Remark 1.1 Linearity in Banach spaces no longer comes with continuity. We have to specify bounded linear functions.

We have four big analysis theorems: Hahn-Banach, contraction mapping, open mapping, and closed graph theorems.

Definition 1.2 — The **spectrum** of a bounded linear operator L is the set $\{\lambda \in \mathbb{C} : L - \lambda \text{ id is not invertible}\}$.

A major open problem of functional analysis is: given a continuous linear endomorphism L of a complex Hilbert space, does L have any invariant subspace?

Examples: ℓ^p spaces, L^p spaces, continuity spaces $\mathcal{C}^{k,\alpha}$, $W^{k,p}$ Sobolev spaces.

Definition 1.3 — An **inner product** $\langle \cdot, \cdot \rangle : X \times X \to F$ satisfies

(1) $\langle y, x \rangle = \overline{\langle x, y \rangle}$.

(2) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for $\alpha, \beta \in F$ and $x, y, z \in X$.

(3) $\langle x, x \rangle \geq 0$ for $x \in X$ with equality iff $x = 0$.

Notice that for $\langle z, \alpha x + \beta y \rangle$, we can use the first axiom:

$$\begin{aligned}
\langle z, \alpha x + \beta y \rangle &= \overline{\langle \alpha x + \beta y, z \rangle} \\
&= \overline{\alpha \langle x, z \rangle + \beta \langle y, z \rangle} \\
&= \overline{\alpha} \langle z, x \rangle + \overline{\beta} \langle z, y \rangle
\end{aligned}$$

So inner products are not linear in the second argument.

Norms satisfy absolute homogeneity, positive definiteness, and triangle inequality.

Any inner product yields a norm: $\|x\| = \sqrt{\langle x, x \rangle}$. Triangle inequality is shown by Cauchy-Schwarz inequality

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle}$$

To prove Cauchy-Schwarz on complex numbers, for any $\alpha \in \mathbb{C}$,

$$0 \leq \|x + \alpha y\|^2 = \|x\|^2 + \overline{\alpha}\langle x, y\rangle + \alpha\langle y, x\rangle + |\alpha^2|\|y\|^2$$

Take $\alpha = -\frac{\langle x,y\rangle}{\|y\|^2}$, we have

$$\begin{aligned} \|x\|^2 &= \frac{\overline{\langle x, y\rangle}}{\|y\|^2}\langle x, y\rangle - \frac{\langle x, y\rangle}{\|y\|^2}\langle y, x\rangle + \frac{|\langle x, y\rangle|^2}{\|y\|^4}\|y\|^2 \\ &= \|x\|^2 - \frac{|\langle x, y\rangle|}{\|y\|^2} \geq 0 \end{aligned}$$

Remark 1.4 The absolute homogeneity of the norm yields the symmetry of the induced metric by taking $\alpha = -1$.