Problem (1)**.** Since every F has only two ideals, namely $0 = \langle 0 \rangle$ and $F = \langle 1 \rangle$, clearly every ideal of F is finitely generated so it is Noetherian. Therefore by repeated application of Hilbert basis theorem, $F[x_1, \ldots, x_n]$ is Noetherian. Hence $I = \langle a_1, \ldots, a_n \rangle$ is finitely generated.

FIX: Since $I = \langle S \rangle$, $a_i \in \langle S \rangle$ implies that a_i is a finite sum of elements of S. Hence I is generated by finitely elements of S.

Problem (2)**.**

(a) (i) reflexive: given $(r, s) \in R \times S$, clearly $1(rs - rs) = 0$ so $(r, s) \sim (r, s)$.

(ii) symmetric: suppose $(r_1, s_1) \sim (r_2, s_2) \in R \times S$, *i.e.* there exists $t \in S$ s.t. $t(r_1 s_2 - r_2 s_1) = 0$. Then

$$\begin{aligned} t(r_1 s_2 - r_2 s_1) &= -t(r_2 s_1 - r_1 s_2) \\ &= t(r_1 s_2 - r_2 s_1) \\ &= 0 \end{aligned}$$

Thus $(r_2, r_2) \sim (r_1, s_1)$.

(iii) transitive: suppose additionally that $(r_2, s_2) \sim (r_3, s_3)$, *i.e.* there exists $t' \in S$ s.t. $t'(r_2 s_3 - r_3 s_2)$. Then since S is closed under multiplication, $tt's_2 \in S$, so

$$\begin{aligned} tt's_2(r_1 s_3 - r_3 s_1) &= tt'(r_1 s_2 s_3 - r_3 s_1 s_2) \\ &= tt'(r_1 s_2 s_3 - r_2 s_1 s_3 + r_2 s_1 s_3 - r_3 s_1 s_2) \\ &= tt'(s_3(r_1 s_2 - r_2 s_1) + s_1(r_2 s_3 - r_3 s_2)) \\ &= s_3 t'(t(r_1 s_2 - r_2 s_1)) + s_1 t(t'(r_2 s_3 - r_3 s_2)) \\ &= 0 + 0 = 0 \end{aligned}$$

Take $\frac{r_1}{s_1}, \frac{r_2}{s_2}$, we want to check that the obvious addition and multiplication are well-defined. WLOG we just check independence of representatives for one term: let $\frac{r_1}{s_1} \sim \frac{r_3}{s_3}$ with t, then

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1 s_2 + r_2 s_1}{s_1 s_2}$$

$$\frac{r_3}{s_3} + \frac{r_2}{s_2} = \frac{r_3 s_2 + r_2 s_3}{s_2 s_3}$$

Notice $ts_2^2 \in S$, so

$$\begin{aligned} t((r_1 s_2 + r_2 s_1)(s_2 s_3) - (r_3 s_2 + r_2 s_3)(s_1 s_2)) &= t(r_1 s_2^2 s_3 - r_3 s_2^2 s_1) \\ &= s_2^2 t(r_1 s_3 - r_3 s_1) \\ &= 0 \end{aligned}$$

So addition is well-defined and clearly associative and commutative. Similarly,

$$\begin{aligned} t(r_1 r_2 s_3 s_2 - r_3 r_2 s_1 s_2) &= r_2 s_2 t(r_1 s_3 - r_3 s_1) \\ &= 0 \end{aligned}$$

so multiplication is well-defined and clearly associative and commutative. The identity is obviously $\frac{1}{1}$ as it satisfies the axiom.

(b) $(\Rightarrow)$: We prove the contrapositive. Suppose that there is a zero-divisor $t \in S$ of some $r \in R$, *i.e.* $tr = 0$ but $t, r \neq 0$. Then $t(r - 0) = 0$ so $\frac{r}{1} = \frac{0}{1}$, *i.e.* $r \in \ker j$. Since the kernel is nontrivial, j is not injective.

$(\Leftarrow)$: suppose S contains no zero-divisors of R, then whenever $\frac{r}{1} = \frac{0}{1}$, *i.e.* $tr = 0$ for some $t \in S$, since t is not a zero-divisor, by definition of zero-divisor $tr = 0 \Leftrightarrow r = 0$, which proves injectivity.

Problem (3)**.** Since R is an integral domain, it doesn't contain any zero divisors. We also see that for any prime ideal P of R and $S_P := R - P$, $1 \in S_P$, and if $a, b \in S_P$, $ab \notin P$ so $ab \in S_P$. Thus S_P is closed under multiplication. Therefore by Problem 2, $j : R \to R_P, r \mapsto \frac{r}{1}$ is injective. That is, there is a canonical identification of R in each R_P, *i.e.* $R \subseteq R_P$. Hence $R \subseteq \bigcap_P R_P$.

For the other direction, given $\frac{r}{s} \in \bigcap_P R_P = \{\frac{r}{s} : r \in R, s \in R - P \ \forall \ P\}$. I claim that s is a unit of R. Suppose not, then $\langle s \rangle$ is a proper ideal of R so it must be contained in some maximal ideal P which is also a prime ideal. But then $s \notin R - P$, a contradiction. Thus we see that $(r - (rs^{-1})s) = 0$ so $(r, s) \sim (rs^{-1}, 1)$ so $\frac{r}{s} \in R$. Hence $R = \bigcap_P R_P$.

Problem (4)**.**

(a) Let $a + b\sqrt{-n}$ be a factor of 2 which implies that at least one of a, b is not 0.

$$\begin{aligned}\frac{2}{a+b\sqrt{-n}} &= \frac{2(a-b\sqrt{-n})}{a^2+b^2n} \\ &= \frac{2a}{a^2+b^2n} - \frac{2b}{a^2+b^2n}\sqrt{-n}\end{aligned}$$

If $b \neq 0$, then $\frac{2b}{b^2n} = \frac{2}{bn}$ can never be an integer since $n > 3$ and $|b| \geq 1$. Thus $\frac{2b}{a^2+b^2n}$ can also never be an integer since $a^2 \geq 0$.

If $b = 0$, then $a \neq 0$, so $\frac{2a}{a^2} = \frac{2}{a}$ is an integer only if $a = \pm 1$ or ± 2. Hence we found factorizations $2 = \pm 1 \cdot \pm 2$ so 2 is irreducible.

Alternatively, $4 = N(2) = N(a)N(b)$ so $N(a)$ and $N(b)$ can only be 1,2, or 4. But the norm is greater than 4 unless the real part is 1 or 2 and imaginary part is 0 (since $n > 3$). So $2 = 1 \cdot 2$ is the only possible factorization, hence 2 is irreducible.

Similarly, we see that

$$\begin{aligned}\frac{\sqrt{-n}}{a+b\sqrt{-n}} &= \frac{a\sqrt{-n}+bn}{a^2+b^2n} \\ &= \frac{bn}{a^2+b^2n} + \frac{a}{a^2+b^2n}\sqrt{-n}\end{aligned}$$

If $b \neq 0$, then $\frac{bn}{b^2n} = \frac{1}{b}$ is integer only if $b = \pm 1$. In that case $\frac{a}{a^2+n}$ can never be integer unless $a = 0$. So we found factorizations $\sqrt{-n} = \pm 1 \cdot \pm\sqrt{-n}$.

If $b = 0$, then $a \neq 0$, so $\frac{a}{a^2+0} = \frac{1}{a}$ is an integer only if $a = \pm$ which yields the case above. Hence $\sqrt{-n}$ is an irreducible.

$$\frac{1+\sqrt{-n}}{a+b\sqrt{-n}} = \frac{a+bn}{a^2+b^2n} + \frac{a-b}{a^2+b^2n}\sqrt{-n}$$

Notice

$$|a-b| \leq |a| + |b| \leq a^2 + b^2 \leq a^2 + b^2n$$

Thus $\frac{a-b}{a^2+b^2n}$ can be an integer only if $a - b = 0$ or if equality is achieved throughout the inequality chain.

In the latter case, For the last inequality to be equality, it forces $b = 0$. The middle equality forces $a = \pm 1$. Thus we have the factorization $1 + \sqrt{-n} = \pm 1 \cdot \pm(1 + \sqrt{-n})$.

If $a - b = 0$ *i.e.* $a = b \neq 0$, then $\frac{a+bn}{a^2+b^2n} = \frac{a(n+1)}{a^2(n+1} = \frac{1}{a}$ is an integer only if $a = b = \pm 1$ so we recover the factorization above. Hence $1 + \sqrt{-n}$ is irreducible.

(b) If n is odd, then $1 + n$ is even so $1 + n = (1 + \sqrt{-n})(1 - \sqrt{-n}) = 2 \cdot \frac{n+1}{2}$. We already know $1 + \sqrt{-n}$ doesn't divide 2 by irreducibility. Moreover,

$$\frac{(n+1)/2}{1+\sqrt{-n}} = \frac{(n+1)/2}{n+1} - \frac{(n+1)/2}{n+1}\sqrt{-n} = \frac{1}{2} - \frac{1}{2}\sqrt{-n}$$

which are not integer coefficients so $1 + \sqrt{-n}$ is not prime and $\mathbb{Z}[\sqrt{-n}]$ is not a UFD when n is odd.

If n is even, then $-n = \sqrt{-n}^2 = 2 \cdot \frac{-n}{2}$. We know $\sqrt{-n}$ doesn't divide 2. Moreover,

$$\frac{-n/2}{\sqrt{-n}} = \frac{1}{2}\sqrt{-n}$$

which are not integer coefficients so $\sqrt{-n}$ is not prime and $\mathbb{Z}[\sqrt{-n}]$ is not a UFD when n is even. That's all the cases.

(c) Consider $\langle 2, \sqrt{-n} \rangle$. Suppose to the contrary that $\langle 2, \sqrt{-n} \rangle = \langle a + b\sqrt{-n} \rangle$. That means $2 = (a + b\sqrt{-n})(c + d\sqrt{-n})$ which by irreducibility forces $a = \pm 2, b = 0$ or $a = \pm 1, b = 0$. But then $\sqrt{-n} = (a + b\sqrt{-n})(x + y\sqrt{-n})$ doesn't have factors with such values of a, b as we showed above. This is a contradiction so $\langle 2, \sqrt{-n} \rangle$ is not a principal ideal.

Problem (5)**.**

(a) Let $N(a + b\sqrt{-2}) = a^2 + 2b^2$ be the norm on $\mathbb{Z}[\sqrt{-2}]$. Then for $x, y \in \mathbb{Z}[\sqrt{-2}]$, $y \neq 0$, it suffices to find a $q, r \in \mathbb{Z}[\sqrt{-2}]$ s.t. $x = qy + r$ with $N(r) < N(y)$. Notice

$$\begin{aligned} q + \frac{r}{y} &= \frac{x}{y} \\ &= \frac{a + b\sqrt{-2}}{c + d\sqrt{-2}} \\ &= \frac{(a + b\sqrt{-2})(c - d\sqrt{-2})}{c^2 + 2d^2} \\ &= \frac{ac + 2bd + (ad + bc)\sqrt{-2}}{c^2 + 2d^2} \\ &= \frac{ac + 2bd}{c^2 + 2d^2} + \frac{(ad + bc)}{c^2 + 2d^2}\sqrt{-2} \end{aligned}$$

$$=: \alpha + \beta\sqrt{-2}$$

Choose q to be $m + n\sqrt{-2}$, where m, n are the closest integers to α, β respectively. That is $|\operatorname{Re} \frac{r}{y}| = |m - \alpha| \leq \frac{1}{2}$ and $|\operatorname{Im} \frac{r}{y}| = |n - \beta| \leq \frac{1}{2}$. Thus $N\left(\frac{r}{y}\right) = \frac{1}{4} + 2 \cdot \frac{1}{4} = \frac{3}{4} < 1$ so $N(r) < N(y)$ as desired.

(b) Rewrite $x^3 - y^2 = 2$ as $y^2 + 2 = x^3$. Factoring it in $\mathbb{Z}[\sqrt{-2}]$ yields

$$(y - \sqrt{-2})(y + \sqrt{-2}) = x^3$$

I claim that $y - \sqrt{-2}$ and $y + \sqrt{-2}$ are relatively prime. First I claim that $\sqrt{-2}$ is irreducible. To see this, suppose $\sqrt{-2} = ab$, then $2 = N(\sqrt{-2}) = N(a)N(b)$, so one factor must have norm 1. It is easy to see that the only elements with norm 1 in $\mathbb{Z}[\sqrt{-2}]$ are ± 1, which are units. Hence $\sqrt{-2}$ is an irreducible. Now suppose $y - \sqrt{-2}$ and $y + \sqrt{-2}$ have a common irreducible factor p. Then p must also divide the sum and difference of the two, *i.e.* $p|2y$ and $p|2\sqrt{-2}$. Notice that $2\sqrt{-2} = -\sqrt{-2}^3$ is a product of irreducibles. Since we are in a Euclidean domain, it is also a UFD. So this factorization is unique. Therefore, the only irreducibles dividing $2\sqrt{-2}$ is $\sqrt{-2}$, which also divides $2y$. However, since x^3 is a cube, we must have $\sqrt{-2}^3$ dividing x^3. This forces that at least one more $\sqrt{-2}$ has to divide $y - \sqrt{-2}/\sqrt{2}$. But then we wouldn't have integer coefficients. So $\sqrt{-2}$ cannot be the common factor. Hence this forces the two to be relatively prime.

Since they share no common factor, and their product is a perfect cube, in a UFD it must be that each of them is a perfect cube. Hence

$$\begin{aligned} y + \sqrt{-2} &= (a + b\sqrt{-2})^3 \\ &= a^3 - 6ab^2 + (3a^2b - 2b^3)\sqrt{-2} \end{aligned}$$

$$a(a^2 - 6b^2) = y, \qquad b(3a^2 - 2b^2) = 1$$

This says that b must a unit of $\mathbb{Z}$, *i.e.* $b = \pm 1$. If $b = 1$, then $3a^2 - 2 = 1$ and $a = \pm 1$. If $b = -1$. then $3a^2 - 2 = -1$ so a has no integer solution. Thus $y = a(a^2 - 6b^2) = \pm 5$ and $x^3 = 25 + 2$ so $x = 3$. Hence $(3, 5)$ and $(3, -5)$ are the only solutions for $x^3 - y^2 = 2$.