Homework 2

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Problem (2.1(i))**.** The initial curve can be parameterized: $x_0 = s, y_0 = s^2, z_0 = u(x_0, y_0) = s^2$. Let us check it is not characteristic. The characteristic equation is $a\sigma_0 + b\sigma_1 = 0$, where (σ_0, σ_1) is the outward normal to the characteristic curve which has tangent (a, b). Thus we don't want the initial curve to have tangent parallel to (a, b). And (a, b) for initial curve is parameterized by (s, s^2). We check

$$\det \begin{pmatrix} s & s^2 \\ 1 & 2s \end{pmatrix} = 2s^2 - s^2 = s^2 \neq 0 \Leftrightarrow s \neq 0,$$

which is always true or we wouldn't have an initial curve. We can safely proceed to solve

$$\begin{cases} \frac{dx}{dt} = x, x(0) = s & \Rightarrow x = se^t \\ \frac{dy}{dt} = y, y(0) = s^2 & \Rightarrow y = s^2 e^t \\ \frac{dz}{dt} = c = z + 1, z(0) = s^2 & \Rightarrow z = (s^2 + 1)e^t \end{cases}$$

We have $s = \frac{y}{x}$ and $e^t = \frac{x^2}{y}$. So the solution is $u(x, y) = \left(\frac{y^2}{x^2} + 1\right)\frac{x^2}{y} - 1 = y + \frac{x^2}{y} - 1$.

Problem (2)**.** We shall use the method of integrating factor. Multiplying the PDE by e^{ct} yields

$$\begin{aligned} e^{ct}u_t + e^{ct}b \cdot D_x u + ce^{ct}u &= 0 \\ \left(e^{ct}u\right)_t + e^{ct}b \cdot D_x u &= 0 && \text{product rule} \\ \left(e^{ct}u\right)_t + b \cdot D_x\left(e^{ct}u\right) &= 0 && e^{ct} \text{ can be treated as constant} \\ e^{ct}u &= g(x - tb) && \text{linear transport solution} \\ u &= g(x - tb)e^{-ct} && e^{ct} \neq 0 \end{aligned}$$

Problem (3)**.** This is Burger's equation. The initial curve is parameterized as $t_0 = 0, x_0 = s, z_0 = \frac{1}{1+s^2}$. We see that

$$\det \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \neq 0$$

Then ODEs are

$$\frac{dt}{d\tau} = 1, t(0) = 0 \Rightarrow t = \tau$$
$$\frac{dx}{d\tau} = u, x(0) = s \Rightarrow x(t) = s + \frac{1}{1+s^2}t = x_0 + tu(x_0, 0)$$
$$\frac{dz}{d\tau} = 0, z(0) = \frac{1}{1+s^2} \Rightarrow z(t) = \frac{1}{1+s^2} = u(x_0, 0)$$

Therefore, $u(x,t)$ is completely determined by x_0. The solution blows up when $w := u_x$ tends to infinity. We see

$$\begin{aligned} 0 = (u_t + uu_x)_x &= u_{xt} + u_x^2 + uu_{xx} \\ &= w_t + w^2 + uw_x \end{aligned}$$

Moreover, on the characteristic line $x(t) = x_0 + tu(x_0, 0)$, we have

$$\begin{aligned} \frac{d}{dt}(w) &= w_t + w_x \frac{dx}{dt} \\ &= w_t + w_x u(x_0, 0) \end{aligned}$$

Combining the two equations, we have the following ODE on the characteristic line:

$$\begin{cases} \dot{w} & = -w^2 \\ w(x_0, 0) & = \left(\frac{1}{1+x_0^2}\right)' = -\frac{2x_0}{(1+x_0^2)^2} \end{cases}$$

Solving this yields

$$w^{-1} = t - \frac{(1+x_0^2)^2}{2x_0}$$

so w blows up when $t = \frac{(1+x_0^2)^2}{2x_0}$. To find the first time it blows up, *i.e.* minimum time, we set

$$\begin{aligned} t' = \frac{8x_0^2(1+x_0^2) - 2(1+x_0^2)^2}{4x_0^2} &= 0 \\ 2(3x_0^4 + 2x_0^2 - 1) &= 0 \\ x_0 &= \pm\frac{1}{\sqrt{3}} \\ t &= \pm\frac{8}{3\sqrt{3}} \end{aligned}$$

We can easily see from the graph that minimum is achieved at $t = \frac{8}{3\sqrt{3}}$.

Problem (4)**.** First, since (x, y) is the outward normal vector of the unit circle, the condition $\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} > 0$ means that (a, b) also points outward from the tangent line at (x, y). This means that for a small t, $(x - ta, y - tb) \in \operatorname{int} \Omega$.

Since Ω is compact, u achieves maximum and minimum. Suppose the minimum is achieved at $\operatorname{int} \Omega$. Then necessary condition says $\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which yields $u(x, y) = au_x + bu_y = 0$. Thus by definition of minimum, $u(x, y) \geq 0$ for all x, y. Likewise, if maximum is achieved at the interior, then $u(x, y) \leq 0$. We have four cases:

Case (1). If both maximum and minimum are achieved in the interior, we have $u \equiv 0$ immediately.

Case (2). If only minimum is in the interior, then $u(x, y) \geq 0$. Let $(x^*, y^*) \in \partial\Omega$ denotes the maximizer. If $u(x^*, y^*) = 0$, then $u \equiv 0$. Suppose $u(x^*, y^*) > 0$, then by Taylor's theorem, since $u \in C^1$,

$$\begin{aligned} u(x^* - ta, y^* - tb) &= u(x^*, y^*) - t \begin{pmatrix} a(x^*, y^*) & b(x^*, y^*) \end{pmatrix} \begin{pmatrix} u_x(x^*, y^*) \\ u_y(x^*, y^*) \end{pmatrix} + r_1(t) \\ &= u(x^*, y^*) + t \underbrace{u(x^*, y^*)}_{>0} + r_1(t) \end{aligned}$$

where $\lim\limits_{t \to 0} \frac{r_1(t)}{t} = 0$. Thus we can find a small enough t s.t. $tu(x^*, y^*) + r_1(t) > 0$, contradicting that $u(x^*, y^*)$ is the maximum.

Case (3). Similarly, if only maximum is in the interior, then $u(x, y) \leq 0$. Let $(x_*, y_*) \in \partial\Omega$ denotes the minimizer. If $u(x_*, y_*) = 0$, then $u \equiv 0$. Suppose $u(x_*, y_*) < 0$, then by Taylor's theorem, since $u \in C^1$,

$$\begin{aligned} u(x_* - ta, y_* - tb) &= u(x_*, y_*) - t \begin{pmatrix} a(x_*, y_*) & b(x_*, y_*) \end{pmatrix} \begin{pmatrix} u_x(x_*, y_*) \\ u_y(x_*, y_*) \end{pmatrix} + r_1(t) \\ &= u(x_*, y_*) + t \underbrace{u(x_*, y_*)}_{<0} + r_1(t) \end{aligned}$$

where $\lim\limits_{t \to 0} \frac{r_1(t)}{t} = 0$. Thus we can find a small enough t s.t. $tu(x_*, y_*) + r_1(t) < 0$, contradicting that $u(x_*, y_*)$ is the minimum.

Case (4). Suppose both maximum and minimum are achieved on $\partial\Omega$. Let (x_*, y_*) denotes the minimizer and (x^*, y^*) denotes the maximizer. Then we have two subcases. If $u(x_*, y_*) \geq 0$, this reduces to case 2. The remaining case is $u(x_*, y_*) < 0$. But this is exactly the condition we need for the Taylor argument to work in case 3.

Hence, for all cases, we have $u \equiv 0$.

I am aware of another proof where if a maximum is on the boundary, then the function is non-decreasing near the maximizer, so by continuity the derivative of u along outward tangent of the characteristic curve (which is exactly $-u$) must be non-negative, so the maximum must be non-positive. Likewise for the minimum. But I already did it using Taylor's theorem so I omit it.

Problem (5)**.** The initial curve γ_0 is $x_0^2+y_0^2 = a^2$, and $z_0 = y$. The problem is not well-posed but we shall solve it anyway. The ODEs are

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = x \\ \frac{dz}{dt} = 0 \end{cases}$$

The first two ODEs yield

$$\begin{aligned} x\frac{dx}{dt} + y\frac{dy}{dt} &= xy - xy = 0 \\ \frac{d}{dt}\left(\frac{1}{2}\left(x^2+y^2\right)\right) &= 0 \\ x(t)^2 + y(t)^2 &= C \end{aligned}$$

Thus by the initial condition, $C = a^2$. Moreover, the third ODE yields $z = u = C_1$. By the initial condition, $u \equiv y$. However, at point $(a, 0)$, we see that

$$yu_x - xu_y = y \cdot 0 - x \cdot 1 = -x = -a \neq 0,$$

a contradiction. Therefore, such solution doesn't exist.