Homework 8

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Problem (9.14)**.** Since h_u is linear, its derivative $dh_{up} : T_p\Gamma \to T_p\mathbb{R}$ is just itself $\langle p, u \rangle$. The only critical points of h_u are those points s.t. $dh_{up} = 0 = \langle p, u \rangle$, *i.e.* points that are orthogonal to u. So it suffices to show that there are finitely many points on M that are orthogonal to u.

Let $f : \Gamma \to S^1$ be the function that maps any point on the curve to a vector orthogonal to it (by 90 degree clockwise rotation). Then f is linear and therefore smooth. Then by Sard's Theorem, almost all $u \in S^1$ are regular values of f. That is, for almost every $u \in S^1$, $f^{-1}(u)$ is a 0-dimensional manifold, *i.e.* a set of points in Γ. Since S^1 is compact, so is Γ. Then by homework we know that $f^{-1}(u)$ is a finite number of points. Thus we show that there are only finitely many points of Γ that is orthogonal to u.

Problem (10.1)**.** Sard's Theorem says that all except for measure zero set of elements in N are regular values of f. But since $\dim M < \dim N$, df_p can at most have rank $\dim M$ so none of the elements in $f(M)$ can be regular value. Hence they must be in the measure zero set.

Problem (10.2)**.** Given $[f] \in \pi_1(S^2)$, where $f : S^1 \to S^2$ is continuous, we can always choose a smooth representative by Weierstrass Approximation Theorem, *i.e.* perturbing f into a polynomial $\overline{f} : S^1 \to S^2$, which is smooth. (This is because we can take an 2ε tube around the curve $f(S^1)$ and we know $\overline{f}$ is in this 2ε tube. Since this 2ε tube is an hyper-annulus, it is homotopic to a hyper-circle so we can always homotop f to $\overline{f}$. If $\overline{f}$ lies outside of S^n, then we just project it to S^n which is still in the 2ε tube. Since projection is smooth, composition of smooth functions are smooth so we get a smooth function on S^n.) So we can assume f is smooth. Then by 10.1, since $\dim S^1 < \dim S^n$, $f(S^1)$ has measure zero in S^n and thus misses at least one point in S^2. But $S^n - \{p\}$ is diffeomorphic to $\mathbb{R}^n$ via stereographic projection. Since $\mathbb{R}^n$ is contractible, so is $f(S^1) \subseteq S^n - \{p\}$. Therefore, the loop f is homotopic to the constant loop, *i.e.* $\pi_1(S^n) = 0$. Since S^n is path-connected, we see that it is simply connected.

Problem (10.3)**.** Every hyperplane is completely identified by its outward normal vector and the offset from origin.

Consider the unit normal vector field $\nu : M \to S^n$. Since M is a smooth manifold, its unit normal vector field is also smooth. Then by Sard's Theore, almost every $u \in S^n$ is a regular value of f. That is, almost every $\nu^{-1}(u)$ is a 0-dimensional manifold (as $\dim M = \dim S^n = n$), *i.e.* a set of points. Since M is compact, $\nu^{-1}(u)$ is a finite set of points. That is, only a finite number of points in M that have tangent planes parallel to H. If H happens to be tangent to M then a small perturbation would give us a hyperplane that either has a normal vector that is not tangent or an offset that doesn't land on M. Either way this hyperplane will be transversal. So almost all hyperplanes are transversal.

Problem (10.5)**.** For any smooth map $f : M \to N$ and a critical point $p \in M$, then df_p has rank less than n. We can choose charts (U, ϕ) of M around p, (V, ψ) of N around $q := f(p)$. Let $g := \psi \circ f \circ \phi^{-1} : \phi(U) \subseteq \mathbb{R}^m \to \mathbb{R}^n$, it is clearly smooth. Then $dg_{\phi^{-1}(p)} = d\psi \circ df_p d\phi_p^{-1}$ also has rank less than n. Thus $\phi^{-1}(p)$ is a critical point and $g(p)$ is a critical value. Therefore, any critical value of f yields a critical value of g. Since the set of critical value is measure zero, since ψ^{-1} is smooth, it follows that the critical value of f is also measure zero. Thus almost all points of N are regular values of f.