Homework 4

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Problem (1)**.** The Hamiltonian of the problem is given by

$$H(x_1, x_2, u, p_1, p_2) = \frac{1}{2}u^2 + p_1 x_2 + p_2(u - x_2).$$

The adjoint equations are given by

$$\begin{aligned} \dot{p_1} &= -H_{x_1} = 0 \\ \dot{p_2} &= -H_{x_2} = p_2 - p_1. \end{aligned}$$

The first-order condition demands

$$\begin{aligned} H_u = u + p_2 &= 0 \\ u &= -p_2. \end{aligned}$$

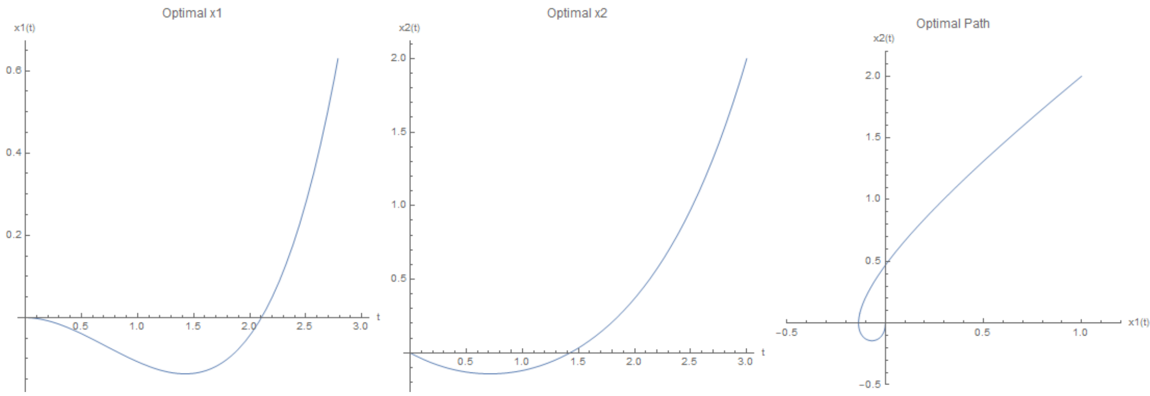
Plugging this into the differential equations yield

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -x_2 - p_2. \end{aligned}$$

Together with the adjoint equations, we have 4 first-order equations and require 4 boundary conditions.

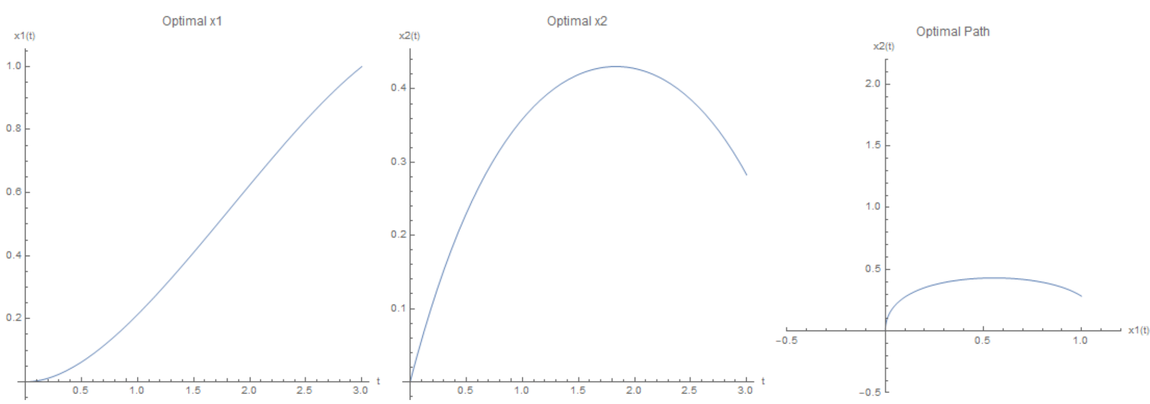
(a) Since all initial and final times and states are fixed, we have 4 boundary conditions $x_1(0) = x_2(0) = 0$, $x_1(3) = 1$, and $x_2(3) = 2$. Mathematica yields

$$\begin{aligned} u(t) &= \frac{6e^{3+t} + 6e^t - e^6 + 4e^3 - 3}{e^6 + 4e^3 - 5} \\ &= -0.6811 + 0.2642e^t. \end{aligned}$$



(b) When $x_2(3)$ is free, in its place we instead have the transversality condition $p_2(3) = 0$. This yields the solution

$$\begin{aligned} u(t) &= -\frac{2e^3(e^t - e^3)}{3e^6 + 4e^3 - 1} \\ &= 0.6256 - 0.0311e^t \end{aligned}$$



(c) I would add a final penalty term Φ:

$$\mathcal{J} = \underbrace{\frac{1}{2}\left((x_1(3) - 1)^2 + (x_2(3) - 2)^2\right)}_{\Phi(3)} + \frac{1}{2}\int_0^3 u^2 dt.$$

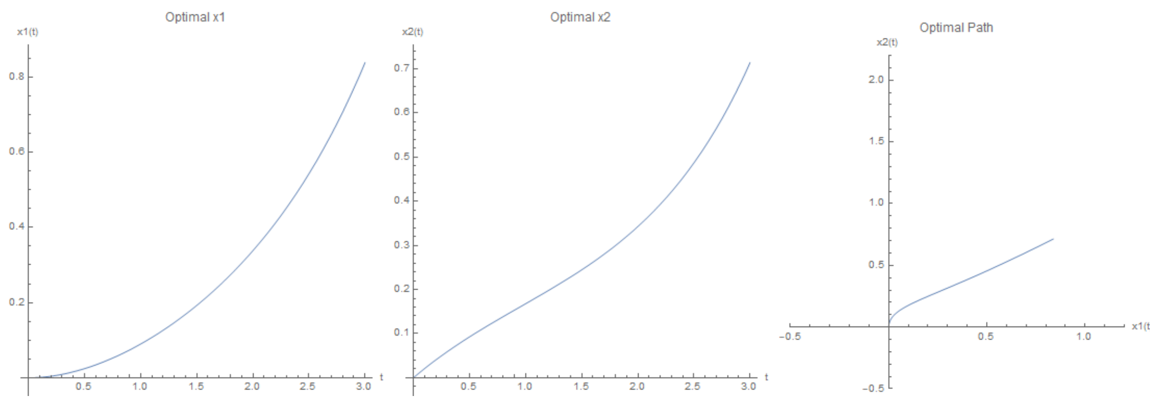
Then instead of $p_1(3) = p_2(3) = 0$, we have

$$p_1(3) = \Phi_{x_1}(3) = x_1(3) - 1$$

$$p_2(3) = \Phi_{x_2}(3) = x_2(3) - 2$$

Then new control is

$$\begin{aligned} u(t) &= \frac{8e^{3+t} + 6e^t + e^6 + 4e^3 - 3}{7e^6 + 8e^3 - 7} \\ &= 0.1615 + 0.056e^t. \end{aligned}$$



We see that $x_1(3) = 0.8385$ and $x_2(3) = 0.7142$, which are not very close to $(1, 2)$. To improve accuracy, I would increase the weight of the penalty term. We see that the cost in part (a) is 4.2859. The cost as a function of the weight coefficient c is shown in the figure below:

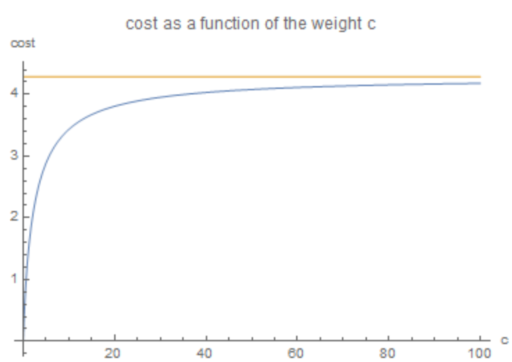


Figure 1: We see that as the weight increases, the cost approaches that of the cost (orange) in part (a) asymptotically.

And we indeed see that the solution ends much closer to $(1, 2)$ when the weight is high:

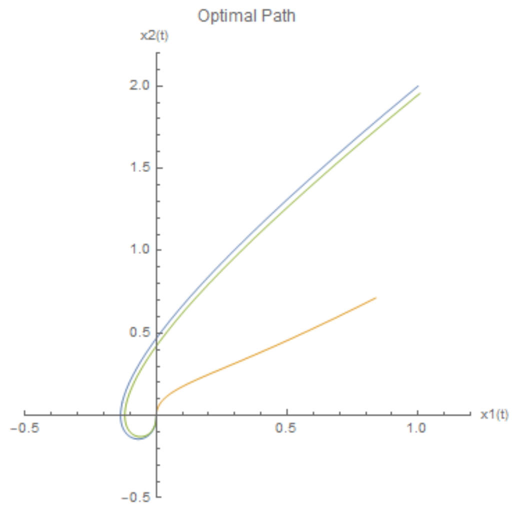


Figure 2: Blue is optimal path from part (a), green is part (c) with weight 100, and orange is part (c) with weight 1.

(d) It is clear that $\Psi = \begin{pmatrix} 2 & 5 \end{pmatrix}$. By transversality condition from Equation 5.234, we have the boundary conditions

$$\begin{pmatrix} -p_1(3) \\ -p_2(3) \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \lambda$$

Together with the initial conditions $x_1(0) = x_2(0) = 0$ and the terminal condition $2x_1(3) + 5x_2(3) = 20$, we have 5 boundary conditions for 4 differential equations and an unknown λ. This allows us to solve by Mathematica. We pre-solve p_1, p_2 for easier code: $p_1(t) \equiv -2\lambda$ and $p_2(t) = -3\lambda e^{-t_f = -2} e^t - 2\lambda$. We obtain the optimal control:

$$u(t) = -p_2(t) = 1.4341 + 0.1071e^t$$

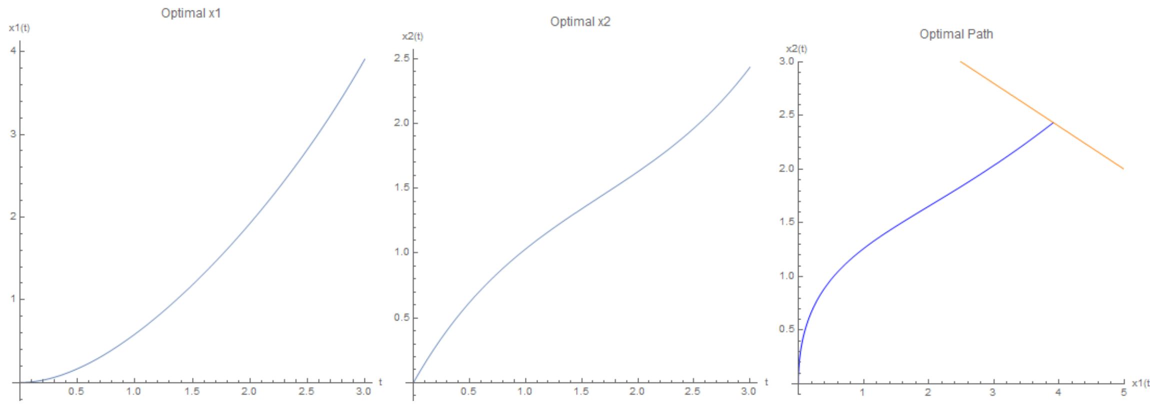


Figure 3: We see that the solution indeed ends at the constraint.

(e) We have the same Ψ but $h(t) = 20 + \frac{t^2}{2}$ so $\dot{h}(t_f) = t_f$. When t_f is also free, based on Equation 5.234 we have the transversality conditions

$$\begin{pmatrix} H(t_f) \\ -p_1(t_f) \\ -p_2(t_f) \end{pmatrix} = \begin{pmatrix} -t_f \\ 2 \\ 5 \end{pmatrix} \lambda$$

Note that by plugging in $u = -p_2$, we have

$$H = -\frac{1}{2}p_2^2 + (p_1 - p_2)x_2$$
$$H(t_f) = -\frac{25}{2}\lambda + 3x_2(t_f) = -t_f$$

Together with two initial conditions and the terminal condition

$$2x_1(t_f) + 5x_2(t_f) = 20 + \frac{t_f^2}{2},$$

we have a total of 6 boundary conditions to match the 4 differential equations and two unknowns λ and t_f. Mathematica yields

$$u(t) = 1.8359 + 0.2547e^t$$

with $t_f = 2.3807$s. The trajectories are

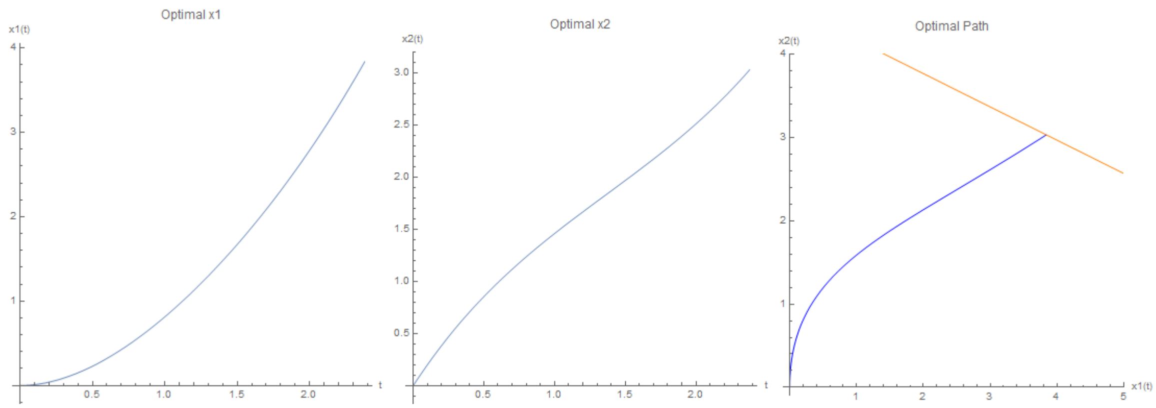


Figure 4: We see the end point hits the constraint.

Problem (2)**.** The Hamiltonian is

$$H = \frac{1}{2}u^2 + p(ax + bu)$$

The adjoint equation is

$$\dot{p} = -H_x = -ap \Rightarrow p(t) = Ce^{-at}.$$

And the first-order condition is

$$H_u = u + bp = 0 \Rightarrow u = -bp$$

Thus

$$\dot{x} = ax - b^2 p = ax - b^2 Ce^{-at}, \quad x(0) = x_0, x(tf) = 0$$

We have $x(t) = \frac{b^2}{2a}Ce^{-at} + Ae^{at}$, $x(0) = \frac{b^2 C}{2a} + A = x_0$ so $A = x_0 - \frac{b^2 C}{2a}$, and

$$x(t_f) = \frac{b^2}{2a}Ce^{-at_f} + \left(x_0 - \frac{b^2 C}{2a}\right)e^{at_f} = 0$$

$$C = \frac{2ax_0 e^{at_f}}{b^2(e^{at_f} - e^{-at_f})}$$

Thus,

$$u(t) = -bp = -b \cdot \left(\frac{2ax_0 e^{at_f}}{b^2(e^{at_f} - e^{-at_f})}\right)e^{-at} = -\frac{2ax_0 e^{at_f}}{b(e^{at_f} - e^{-at_f})}e^{-at}$$

Problem (3)**.** With the mixed constraint $\psi(x_0, x_f) = x_f - x_0 = 0$, we can turn x_0, x_f into free variables by adding a term with Lagrange multiplier. Let $\Phi(x_0, x_f) = \phi(x_f) + \lambda\psi(x_0, x_f) = \frac{1}{2}(x(t_f) - 1)^2 + \lambda(x_f - x_0)$. The Hamiltonian is

$$H = \frac{1}{2}(x^2 + u^2) + pu.$$

The adjoint equation is

$$\dot{p} = -H_x = -x$$

The first-order condition says

$$H_u = u + p = 0 \Rightarrow u = -p.$$

Since both $x(0), x(2)$ is free, from Equation 5.234 we have the transversality condition

$$(\Phi_{x_0} + p^T(t_0))\delta x_0 + (\Phi_{x_f} - p^T(t_f)\delta x_f = 0$$

where δx_0 and δx_f can take any value. This forces that

$$\Phi_{x_0} + p^T(t_0) = 0$$
$$\Phi_{x_f} - p^T(t_f) = 0$$

Thus for this problem, we have

$$p(0) = -\Phi_{x_0} = -(-\lambda) = \lambda$$
$$p(2) = \Phi_{x_f} = x(2) - 1 + \lambda$$

Thus we have two boundary conditions for two differential equations. Mathematica yields

$$x(t) = \frac{1}{2}e^{-2-t}\left(1 - e^{2t}(\lambda - 1) - \lambda + 2e^2\lambda\right).$$

Solving $x(0) = x(2)$, we obtain λ which is a huge number. Mathematica gives

$$u(t) = -p(t) = -0.3491e^{-t} + 0.0472e^{2t}.$$

The associated cost is 0.3018.

Problem (4). (a) Since the cost $\mathcal{J} = t_f$, we have $\Phi(t) = t$ and the Hamiltonian is

$$H = p_1(t)\cos\theta(t) + p_2(t)\sin\theta(t) + p_3(t)u(t) + p_4(t)v(t)$$

(b) The adjoint equations are

$$\begin{pmatrix} \dot{p_1} \\ \dot{p_2} \\ \dot{p_3} \\ \dot{p_4} \end{pmatrix} = \begin{pmatrix} -H_u \\ -H_v \\ -H_x \\ -H_y \end{pmatrix} = \begin{pmatrix} p_3 \\ p_4 \\ 0 \\ 0 \end{pmatrix}$$

(c) Since $t_f, u(t_f), v(t_f)$ are free, by Equation 5.234 we have the transversality conditions

$$H(t_f) + \Phi_t(t_f) = 0 \Rightarrow H(t_f) = -1$$
$$p_1(t_f) = \Phi_u = 0$$
$$p_2(t_f) = \Phi_v = 0.$$

(d) Notice that p_3, p_4 are constants, so $p_1(t) = p_3 t$ and $p_2(t) = p_4 t$ by the zero boundary conditions. First-order condition yields

$$H_\theta = -p_1 \sin\theta(t) + p_2\cos\theta(t) = 0$$
$$\tan\theta^*(t) = \frac{p_2(t)}{p_1(t)} = \frac{p_4}{p_3}$$

which is a constant! So are $\cos\theta^*$ and $\sin\theta^*$. Thus by the initial conditions $u(0) = \cos\gamma_0, v(0) = \sin\gamma_0$, we have $u(t) = \cos\theta^* t$ and $v(t) = \sin\theta^*$. By the initial conditions $x(0) = y(0) = 0$, we have $x(t) = \frac{1}{2}\cos\theta^* t^2 + \cos\gamma_0 t$ and $y(t) = \frac{1}{2}\sin\theta^* t^2 + \sin\gamma_0 t$. By the terminal condition $x(t_f) = x_f$ and $y(t_f) = 0$, we have

$$x_f - \cos\gamma_0 t_f = \frac{1}{2}\cos\theta^* t_f^2$$
$$-\sin\gamma_0 t_f = \frac{1}{2}\sin\theta^* t_f^2$$

Squaring both sides and adding the two equations together, we obtain the equation as desired:

$$x_f^2 - 2\cos\gamma_0 t_f + (\cos\gamma_0^2 + \sin\gamma_0^2)t_f^2 = \frac{1}{4}(\cos^2\theta^* + \sin^2\theta^*)t_f^4$$
$$4x_f^2 - 8\cos\gamma_0 t_f + 4t_f^2 - t_f^4 = 0$$

So t_f must be the minimum positive solution of this equation.

(e) By the above equation,

$$\cos\theta^* = 2\left(\frac{x_f}{t_f^2} + \frac{\cos\gamma_0}{t_f}\right)$$
$$\theta^* = \arccos\left(2\left(\frac{x_f}{t_f^2} + \frac{\cos\gamma_0}{t_f}\right)\right)$$

Problem (5)**.** The cost has the Meyer form $J = t_f$ so $\Phi(t) = t$. The Hamiltonian is

$$H(x, y, p_1, p_2) = p_1 r\cos\beta + p_2 r\sin\beta.$$

The adjoint equations are

$$\dot{p_1} = -H_x = -p_1\frac{x}{r}\cos\beta - p_2\frac{x}{r}\sin\beta$$
$$\dot{p_2} = -H_y = p_1\frac{y}{r}\cos\beta - p_2\frac{x}{r}\sin\beta$$

Note that $\dot{p_2} = \frac{y}{x}\dot{p_1}$. First-order condition yields

$$H_\beta = -p_1 r\sin\beta + p_2 r\cos\beta = 0$$
$$\tan\beta = \frac{p_2}{p_1}$$

Thus we obtain $p_2 = \tan\beta p_1$. Moreover, we see that

$$p_2\cos\beta - p_1\sin\beta = \frac{p_2p_1 - p_1p_2}{\sqrt{p_1^2 + p_2^2}}$$
$$= 0.$$

Now we compute

$$0 = \frac{d}{dt}H_\beta = \dot{r}(-p_1\sin\beta + p_2\cos\beta) + r\left(-\frac{d}{dt}(p_1\sin\beta) + \frac{d}{dt}(p_2\cos\beta)\right)$$
$$0 = 0 + r\left(-\dot{p_1}\sin\beta - p1\cos\beta\dot{\beta} + \dot{p_2}\cos\beta - p_2\sin\beta\dot{\beta}\right)$$
$$0 = -\dot{p_1}\sin\beta - p1\cos\beta\dot{\beta} + \frac{y}{x}\dot{p_1}\cos\beta - p_1\tan\beta\sin\beta\dot{\beta}$$
$$p_1(cos\beta + \tan\beta\sin\beta)\dot{\beta} = \dot{p_1}\left(-\sin\beta + \frac{y}{x}\cos\beta\right)$$
$$p_1(cos\beta + \tan\beta\sin\beta)\dot{\beta} = \left(p_1\frac{x}{r}\cos\beta + \tan\beta p_1\frac{x}{r}\sin\beta\right)\left(\sin\beta - \frac{y}{x}\cos\beta\right)$$
$$\dot{\beta} = \frac{x}{r}\left(\sin\beta - \frac{y}{x}\cos\beta\right)$$
$$\dot{\beta} = \frac{x}{r}sin\beta - \frac{y}{r}\cos\beta$$

In[1]:=
```
(*Problem 1*)
(*Part a*)
sol1 =
 DSolve[{x1'[t] == x2[t], x2'[t] == -x2[t] - p2[t], p1'[t] == 0, p2'[t] == p2[t] - p1[t],
   x1[0] == 0, x2[0] == 0, x1[3] == 1, x2[3] == 2}, {x1[t], x2[t], p1[t], p2[t]}, t]
xSol1 = x1[t] /. sol1[[1, 3]]
ySol1 = x2[t] /. sol1[[1, 4]]
cost := Module[{sol, x1s, x2s, u, x13, x23, x1sModified, x2sModified},
  sol = sol1;
  x1s[t_] := x1[t] /. sol[[1, 3]];
  x13 := x1s[t] /. t -> 3;
  x1sModified = x1s[t] /. x1[3] → x13;
  x2s[t_] := x2[t] /. sol[[1, 4]];
  x23 := x2s[t] /. t -> 3;
  x2sModified = x2s[t] /. x2[3] → x23;
  u[t_] := -(p2[t] /. sol[[1, 2]]);
  1/2 ((x13 - 1)^2 + (x23 - 2)^2) + Integrate[1/2 u[t]^2, {t, 0, 3}]]
cost1 = N[cost]
u[t_] = -Simplify[N[p2[t] /. sol1[[1, 2]]]]
```

Out[1]= $\left\{\left\{\text{p1}[t] \to \frac{-3+e^3}{5+e^3},\ \text{p2}[t] \to \frac{3-4e^3+e^6-6e^t-6e^{3+t}}{\left(-1+e^3\right)\left(5+e^3\right)},\right.\right.$

$\text{x1}[t] \to \frac{e^{-t}\left(7e^3-e^6-3e^t+3e^{2t}-10e^{3+t}+e^{6+t}+3e^{3+2t}-3e^t t+4e^{3+t}t-e^{6+t}t\right)}{\left(-1+e^3\right)\left(5+e^3\right)},$

$\left.\left.\text{x2}[t] \to \frac{e^{-t}\left(-7e^3+e^6-3e^t+3e^{2t}+4e^{3+t}-e^{6+t}+3e^{3+2t}\right)}{\left(-1+e^3\right)\left(5+e^3\right)}\right\}\right\}$

Out[2]= $\frac{e^{-t}\left(7e^3-e^6-3e^t+3e^{2t}-10e^{3+t}+e^{6+t}+3e^{3+2t}-3e^t t+4e^{3+t}t-e^{6+t}t\right)}{\left(-1+e^3\right)\left(5+e^3\right)}$

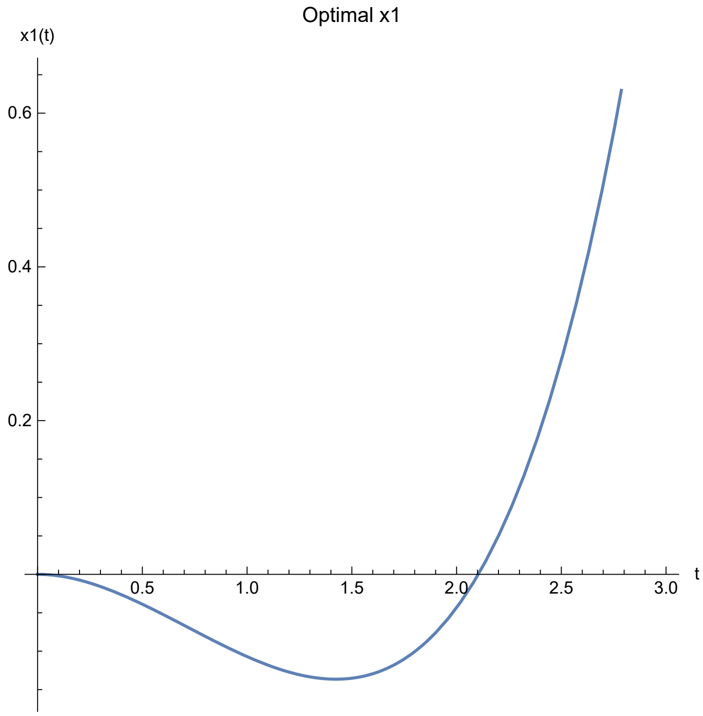
Out[3]= $\frac{e^{-t}\left(-7e^3+e^6-3e^t+3e^{2t}+4e^{3+t}-e^{6+t}+3e^{3+2t}\right)}{\left(-1+e^3\right)\left(5+e^3\right)}$

Out[5]= 4.28588

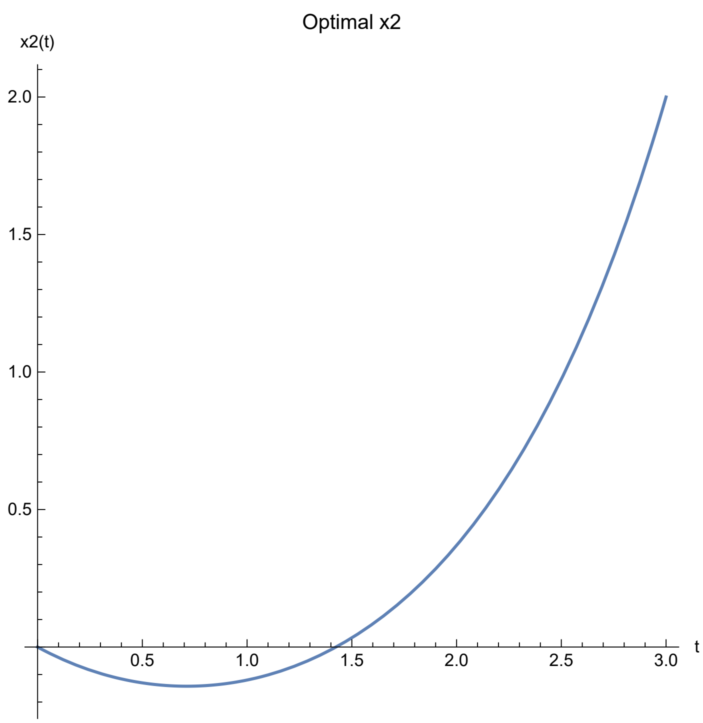
Out[6]= $-0.681091+0.264246\times 2.71828^t$

In[7]:=
```
Plot[xSol1, {t, 0, 3}, PlotLabel → "Optimal x1",
 AxesLabel → {"t", "x1(t)"}, AspectRatio → 1]
Plot[ySol1, {t, 0, 3}, PlotLabel → "Optimal x2",
 AxesLabel → {"t", "x2(t)"}, AspectRatio → 1]
ParametricPlot[{xSol1, ySol1}, {t, 0, 3}, PlotRange → {{-.5, 1.2}, {-.5, 2.2}}
, PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"}, AspectRatio → 1]
```

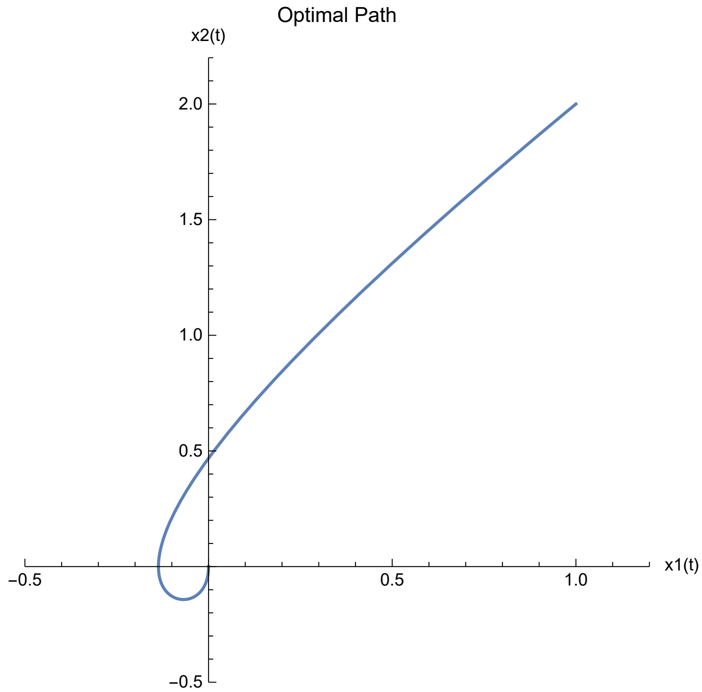
Out[7]=


Optimal x1
x1(t)
0.6
0.4
0.2
0.5
1.0
1.5
2.0
2.5
3.0
t

Out[8]=


Optimal x2
x2(t)
2.0
1.5
1.0
0.5
0.5
1.0
1.5
2.0
2.5
3.0
t

Out[9]=

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(*Part b*)
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In[10]:=
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sol2 = DSolve[{x1'[t] == x2[t], x2'[t] == -x2[t] - p2[t], p1'[t] == 0, p2'[t] == p2[t] - p1[t],
   x1[0] == 0, x2[0] == 0, x1[3] == 1, p2[3] == 0}, {x1[t], x2[t], p1[t], p2[t]}, t]
xSol2 = x1[t] /. sol2[[1, 3]]
ySol2 = x2[t] /. sol2[[1, 4]]
u[t_] = -N[p2[t] /. sol2[[1, 2]]]
Plot[xSol2, {t, 0, 3}, PlotLabel → "Optimal x1",
 AxesLabel → {"t", "x1(t)"}, AspectRatio → 1]
Plot[ySol2, {t, 0, 3}, PlotLabel → "Optimal x2",
 AxesLabel → {"t", "x2(t)"}, AspectRatio → 1]
ParametricPlot[{xSol2, ySol2}, {t, 0, 3}, PlotRange → {{-.5, 1.2}, {-.5, 2.2}}
, PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"}, AspectRatio → 1]
```

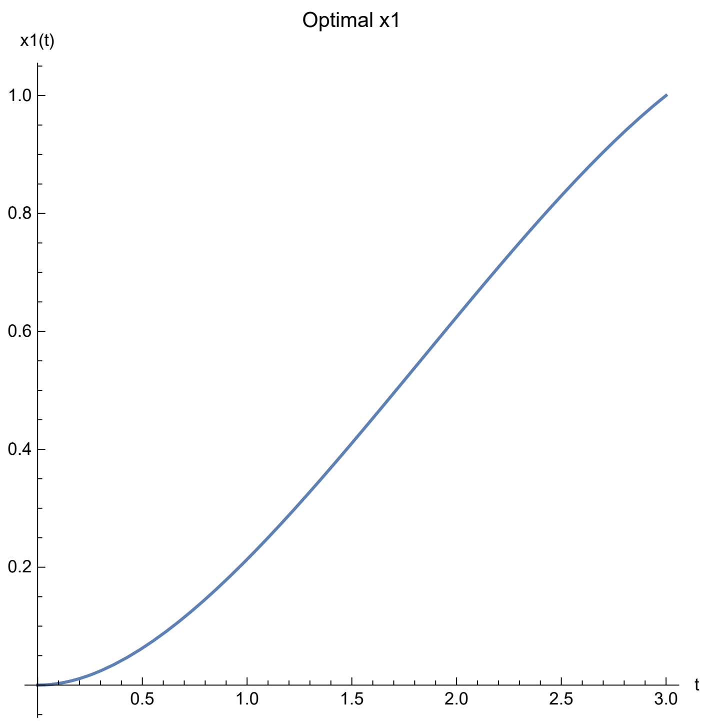
Out[10]= $\left\{\left\{\text{p1}[t] \to -\frac{2 e^6}{-1+4 e^3+3 e^6},\ \text{p2}[t] \to \frac{2 e^3\left(-e^3+e^t\right)}{-1+4 e^3+3 e^6},\ \text{x1}[t] \to -\frac{e^{3-t}\left(1-2 e^3-2 e^t+e^{2 t}+2 e^{3+t}-2 e^{3+t}\, t\right)}{-1+4 e^3+3 e^6},\ \text{x2}[t] \to -\frac{e^{3-t}\left(-1+2 e^3+e^{2 t}-2 e^{3+t}\right)}{-1+4 e^3+3 e^6}\right\}\right\}$

Out[11]= $-\frac{e^{3-t}\left(1-2 e^3-2 e^t+e^{2 t}+2 e^{3+t}-2 e^{3+t}\, t\right)}{-1+4 e^3+3 e^6}$

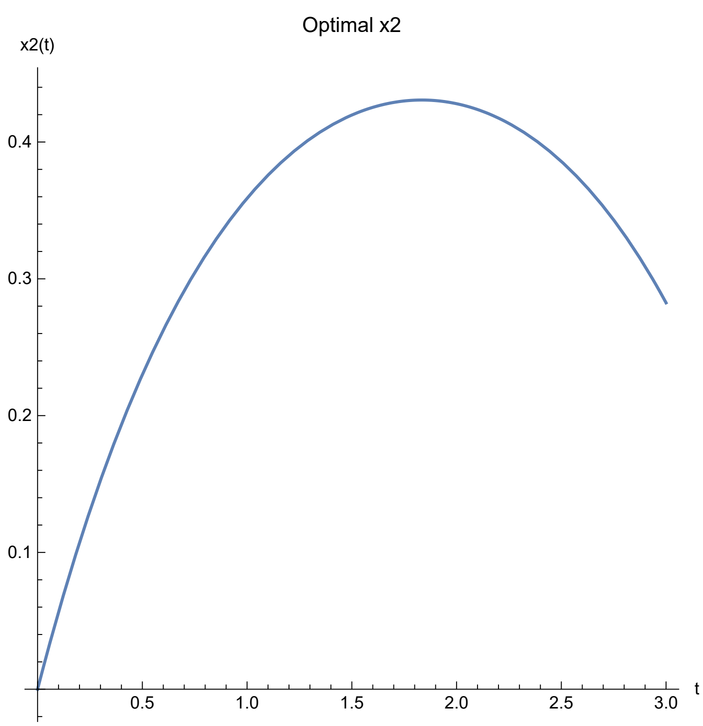
Out[12]= $-\frac{e^{3-t}\left(-1+2 e^3+e^{2 t}-2 e^{3+t}\right)}{-1+4 e^3+3 e^6}$

Out[13]= $-0.0311493\left(-20.0855+2.71828^t\right)$

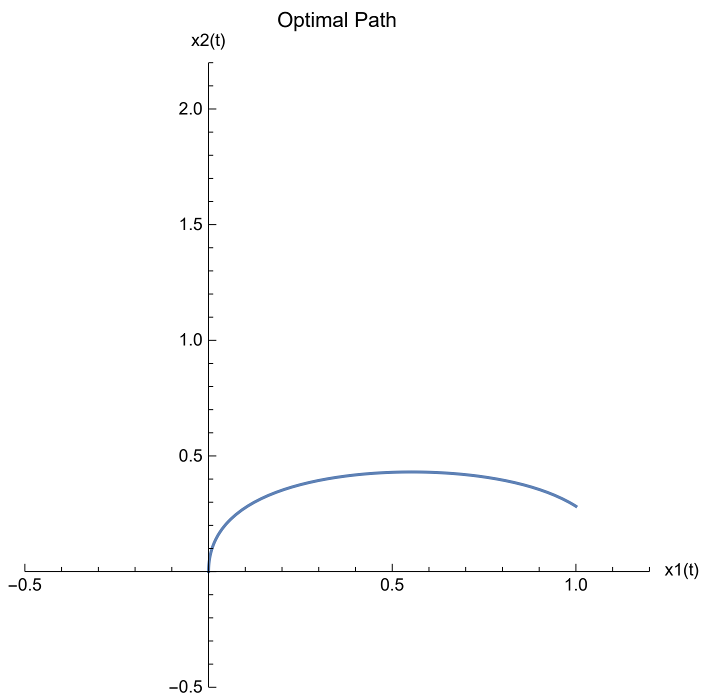
Out[14]=


Optimal x1
x1(t)
1.0
0.8
0.6
0.4
0.2
0.5
1.0
1.5
2.0
2.5
3.0
t

Out[15]=


Optimal x2
x2(t)
0.4
0.3
0.2
0.1
0.5
1.0
1.5
2.0
2.5
3.0
t

Out[16]=

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(*Part c*)
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In[17]:=
```
sol3 = DSolve[
  {x1'[t] == x2[t], x2'[t] == -x2[t] - p2[t], p1'[t] == 0, p2'[t] == p2[t] - p1[t], x1[0] == 0,
   x2[0] == 0, p1[3] == x1[3] - 1, p2[3] == x2[3] - 2}, {x1[t], x2[t], p1[t], p2[t]}, t]
xSol3 = x1[t] /. sol3[[1, 3]]
ySol3 = x2[t] /. sol3[[1, 4]]
u[t_] = -Simplify[N[p2[t] /. sol3[[1, 2]]]]
Plot[xSol3, {t, 0, 3}, PlotLabel → "Optimal x1",
 AxesLabel → {"t", "x1(t)"}, AspectRatio → 1]
Plot[ySol3, {t, 0, 3}, PlotLabel → "Optimal x2",
 AxesLabel → {"t", "x2(t)"}, AspectRatio → 1]
ParametricPlot[{xSol3, ySol3}, {t, 0, 3}, PlotRange → {{-.5, 1.2}, {-.5, 2.2}}
, PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"}, AspectRatio → 1]
```

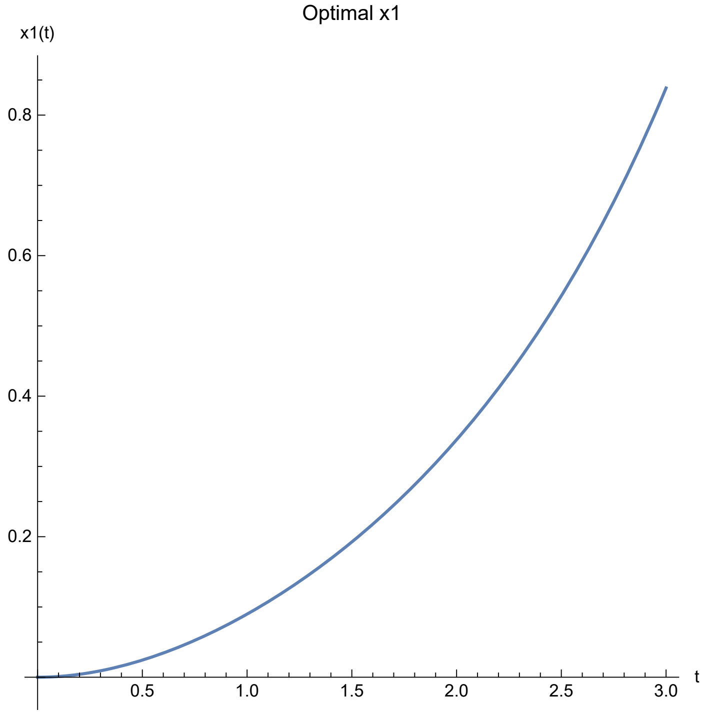
Out[17]=
$$\left\{\left\{p1[t] \to \frac{3 - 4 e^3 - e^6}{-7 + 8 e^3 + 7 e^6},\ p2[t] \to -\frac{-3 + 4 e^3 + e^6 + 6 e^t + 8 e^{3+t}}{-7 + 8 e^3 + 7 e^6},\right.\right.$$
$$x1[t] \to \frac{e^{-t}\left(8 e^3 + e^6 - 3 e^t + 3 e^{2t} - 12 e^{3+t} - e^{6+t} + 4 e^{3+2t} - 3 e^t t + 4 e^{3+t} t + e^{6+t} t\right)}{-7 + 8 e^3 + 7 e^6},$$
$$\left.\left. x2[t] \to \frac{e^{-t}\left(-8 e^3 - e^6 - 3 e^t + 3 e^{2t} + 4 e^{3+t} + e^{6+t} + 4 e^{3+2t}\right)}{-7 + 8 e^3 + 7 e^6}\right\}\right\}$$

Out[18]=
$$\frac{e^{-t}\left(8 e^3 + e^6 - 3 e^t + 3 e^{2t} - 12 e^{3+t} - e^{6+t} + 4 e^{3+2t} - 3 e^t t + 4 e^{3+t} t + e^{6+t} t\right)}{-7 + 8 e^3 + 7 e^6}$$

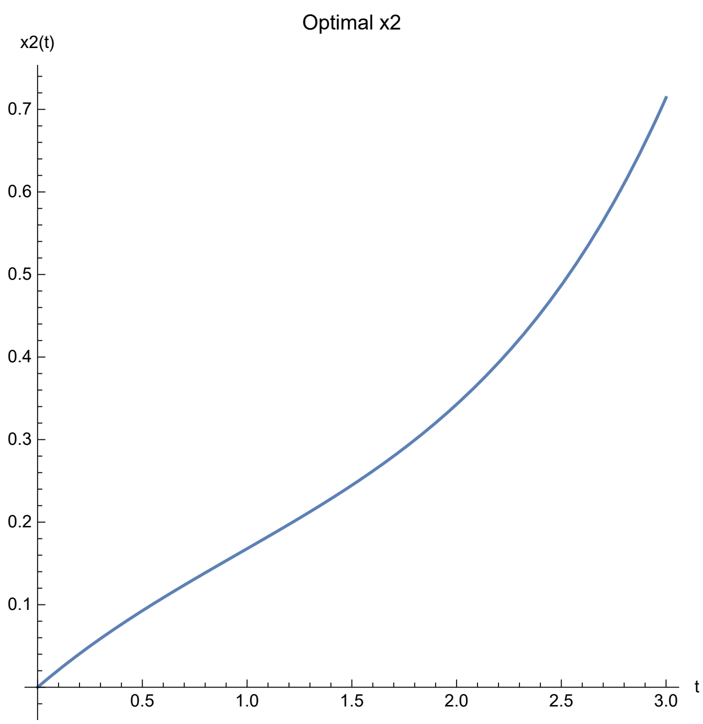
Out[19]=
$$\frac{e^{-t}\left(-8 e^3 - e^6 - 3 e^t + 3 e^{2t} + 4 e^{3+t} + e^{6+t} + 4 e^{3+2t}\right)}{-7 + 8 e^3 + 7 e^6}$$

Out[20]= $0.161458 + 0.0559778 \times 2.71828^{t}$

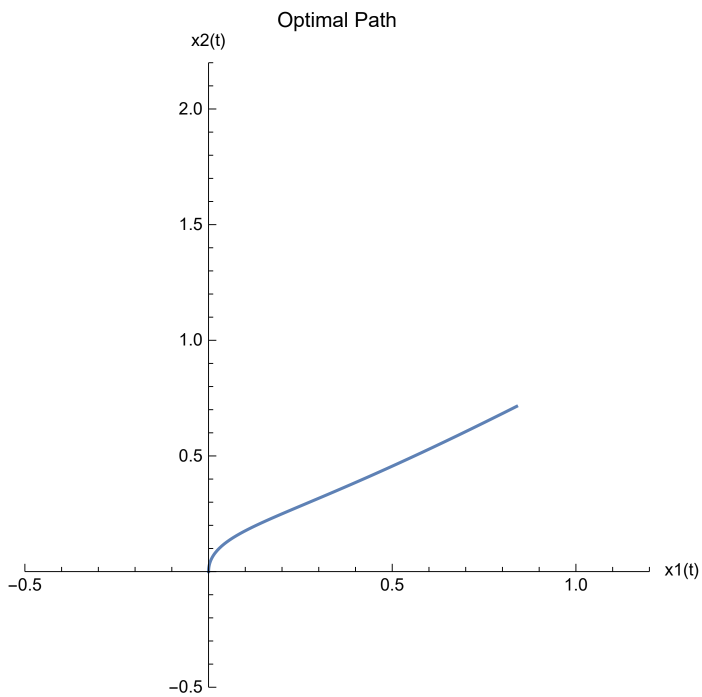
Out[21]=



Out[22]=



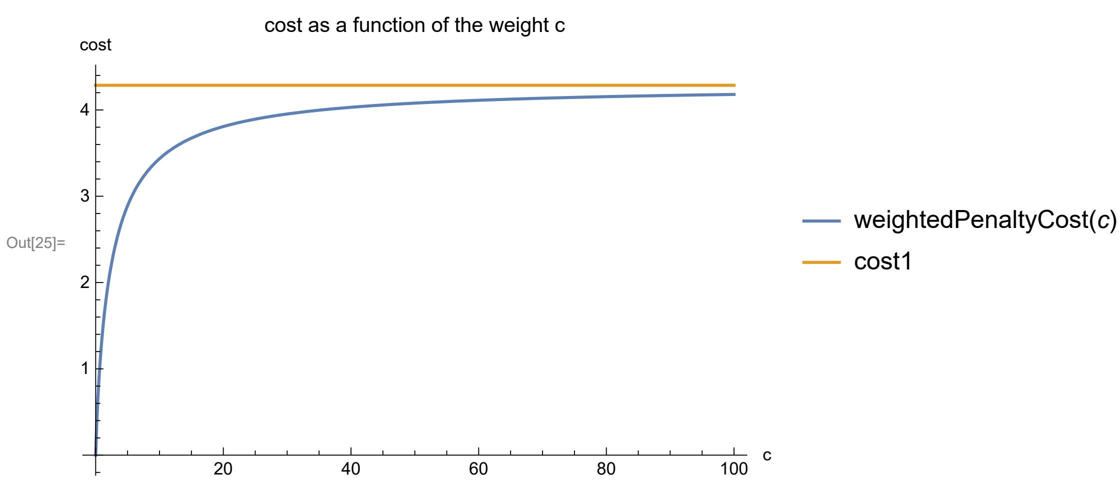
Out[23]=



In[24]:=
```
weightedPenaltyCost[c_] := Module[{sol, x1s, x2s, u, x13, x23, x1sModified, x2sModified},
  sol = DSolve[{x1'[t] == x2[t], x2'[t] == -x2[t] - p2[t], p1'[t] == 0,
     p2'[t] == p2[t] - p1[t], x1[0] == 0, x2[0] == 0, p1[3] == c (x1[3] - 1),
     p2[3] == c (x2[3] - 2)}, {x1[t], x2[t], p1[t], p2[t]}, t];
  x1s[t_] := x1[t] /. sol[[1, 3]];
  x13 := x1s[t] /. t -> 3;
  x1sModified = x1s[t] /. x1[3] → x13;
  x2s[t_] := x2[t] /. sol[[1, 4]];
  x23 := x2s[t] /. t -> 3;
  x2sModified = x2s[t] /. x2[3] → x23;
  u[t_] := -(p2[t] /. sol[[1, 2]]);
  c/2 ((x13 - 1)^2 + (x23 - 2)^2) + Integrate[1/2 u[t]^2, {t, 0, 3}]]

Plot[{weightedPenaltyCost[c], cost1}, {c, 0, 100},
 PlotLabel → "cost as a function of the weight c", AxesLabel → {"c", "cost"},
 PlotLegends → "Expressions", PlotPoints → 10, PlotRange → Full]
```

Out[25]=



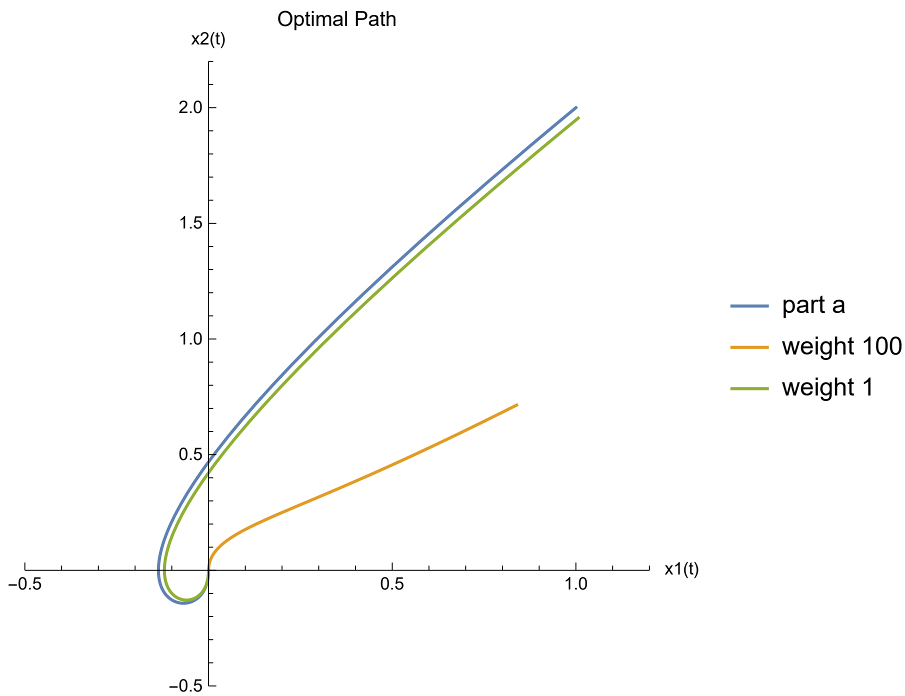
Out[•]= 4.28577

In[26]:=
```
N[weightedPenaltyCost[100000]]
(*we see that for weight 100000 the value is super close to cost from (a)*)
sol3w = DSolve[{x1'[t] == x2[t], x2'[t] == -x2[t] - p2[t], p1'[t] == 0,
    p2'[t] == p2[t] - p1[t], x1[0] == 0, x2[0] == 0, p1[3] == 100 (x1[3] - 1),
    p2[3] == 100 (x2[3] - 2)}, {x1[t], x2[t], p1[t], p2[t]}, t];
x1s[t_] := x1[t] /. sol3w[[1, 3]];
x13 := x1s[t] /. t -> 3;
x1sModified = x1s[t] /. x1[3] → x13;
x2s[t_] := x2[t] /. sol3w[[1, 4]];
x23 := x2s[t] /. t -> 3;
x2sModified = x2s[t] /. x2[3] → x23;
```

Out[26]= 4.28577

In[34]:=
```
ParametricPlot[{{xSol1, ySol1}, {xSol3, ySol3}, {x1sModified, x2sModified}},
 {t, 0, 3}, PlotRange → {{-.5, 1.2}, {-.5, 2.2}}
, PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"},
 PlotLegends → {"part a", "weight 100", "weight 1"}, AspectRatio → 1]
```

Out[34]=




```
(*Part d*)
```

In[35]:=
```
sol42 =
 Simplify[DSolve[{x2'[t] == -x2[t] - (-3 lam Exp[-3] Exp[t] - 2 lam), x2[0] == 0}, x2[t], t]]
 (*pre-solve p1,p2*)
sol4x2[t_] = x2[t] /. sol42[[1]]
sol41 = Simplify[DSolve[{x1'[t] == sol4x2[t], x1[0] == 0}, x1[t], t]]
sol4x1[t_] = x1[t] /. sol41[[1]]
lamSol = N[Solve[2 sol4x1[3] + 5 sol4x2[3] == 20, lam]] (*solve for lambda*)
lamS = lam /. lamSol[[1]]
u[t_] = 3 lamS Exp[-3] Exp[t] + 2 lamS
```

Out[35]= $\left\{\left\{x2[t] \to \frac{1}{2} e^{-3-t} \left(-3 - 4 e^{3} + 3 e^{2t} + 4 e^{3+t}\right) \text{lam}\right\}\right\}$

Out[36]= $\frac{1}{2} e^{-3-t} \left(-3 - 4 e^{3} + 3 e^{2t} + 4 e^{3+t}\right) \text{lam}$

Out[37]= $\left\{\left\{x1[t] \to \frac{1}{2} e^{-3-t} \text{lam} \left(3 + 4 e^{3} - 6 e^{t} + 3 e^{2t} + 4 e^{3+t} (-1 + t)\right)\right\}\right\}$

Out[38]= $\frac{1}{2} e^{-3-t} \text{lam} \left(3 + 4 e^{3} - 6 e^{t} + 3 e^{2t} + 4 e^{3+t} (-1 + t)\right)$

Out[39]= {{lam → 0.717067}}

Out[40]= 0.717067

Out[41]= $1.43413 + 0.107102\, e^{t}$

In[42]:=
```
xSol4[t_] = sol4x1[t] /. lam → lamS
ySol4[t_] = sol4x2[t] /. lam → lamS
xSol4[3]
ySol4[3]
Plot[xSol4[t], {t, 0, 3}, PlotLabel → "Optimal x1",
 AxesLabel → {"t", "x1(t)"}, AspectRatio → 1]
Plot[ySol4[t], {t, 0, 3}, PlotLabel → "Optimal x2",
 AxesLabel → {"t", "x2(t)"}, AspectRatio → 1]
x2[x1_] = 1/5 (20 - 2 x1);
plot = Plot[x2[x1], {x1, -10, 10}, PlotStyle → Orange];
parametric = ParametricPlot[{xSol4[t], ySol4[t]},
   {t, 0, 3}, PlotRange -> {{-.1, 5}, {-.1, 3}}, PlotStyle → Blue,
   PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"}, AspectRatio → 1];
Show[parametric, plot, PlotLabel → "Optimal Path"]
```

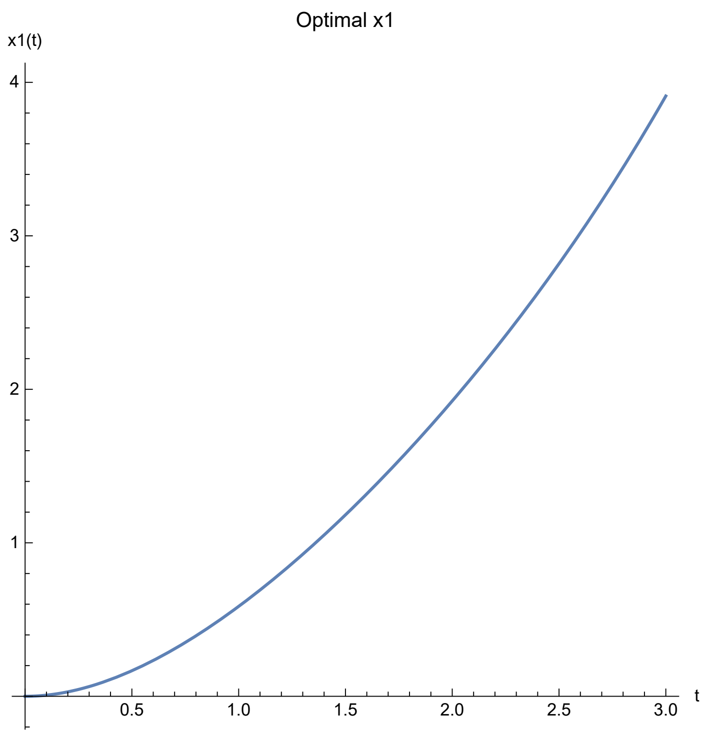
Out[42]= $0.358533\, e^{-3-t} \left(3 + 4 e^{3} - 6 e^{t} + 3 e^{2t} + 4 e^{3+t} (-1 + t)\right)$

Out[43]= $0.358533\, e^{-3-t} \left(-3 - 4 e^{3} + 3 e^{2t} + 4 e^{3+t}\right)$

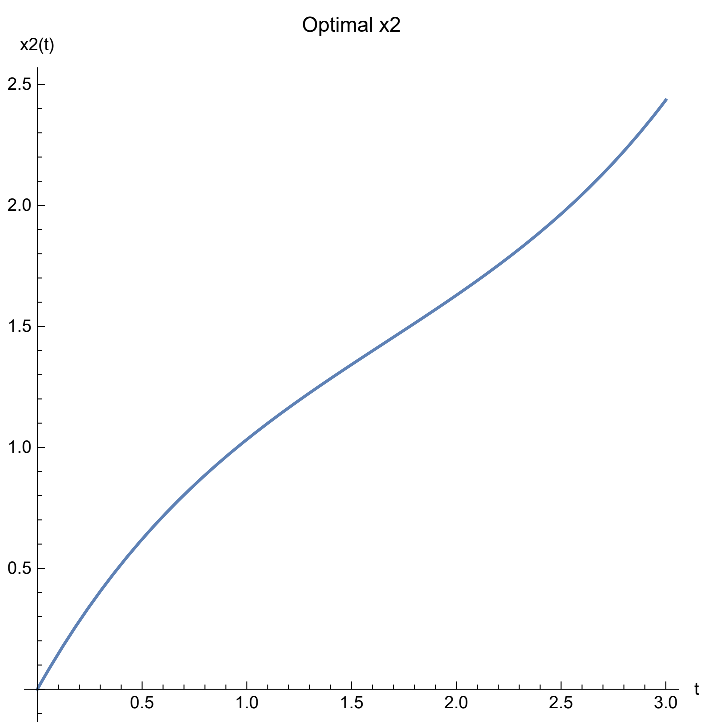
Out[44]= 3.91083

Out[45]= 2.43567

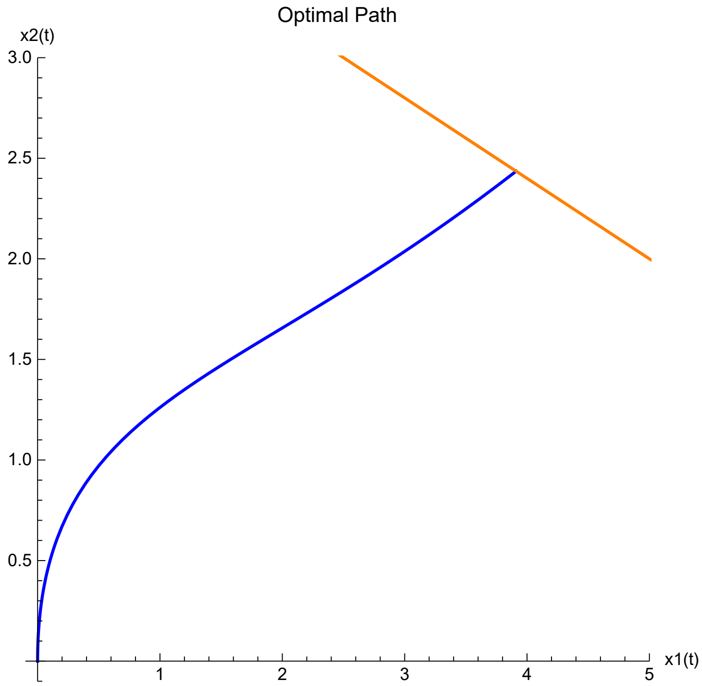
Out[46]=


Optimal x1
x1(t)
4
3
2
1
0.5
1.0
1.5
2.0
2.5
3.0
t

Out[47]=


Optimal x2
x2(t)
2.5
2.0
1.5
1.0
0.5
0.5
1.0
1.5
2.0
2.5
3.0
t

Out[51]=



In[69]:=

```
(*Part e*)
ClearAll[lam, x2, lamS, tf, tfS, u]
sol52 = Simplify[
  DSolve[{x2'[t] == -x2[t] - (-3 lam Exp[-tf] Exp[t] - 2 lam), x2[0] == 0}, x2[t], t]]
sol5x2[t_] = x2[t] /. sol52[[1]]
sol51 = Simplify[DSolve[{x1'[t] == sol5x2[t], x1[0] == 0}, x1[t], t]]
sol5x1[t_] = x1[t] /. sol51[[1]]
sol5x2[tf]
(*sol5=Solve[
  {2*sol5x1[tf]+5 *sol5x2[tf] == 20+tf^2/2,-25/2 lam+3*sol5x2[tf]== tf},{lam,tf}]*)
(*lamS = lam /.sol5[[1,1]]*)
(*tfS= tf /.sol5[[1,2]]*)
lamS = 0.917966088109113799928811368357O6;
(*computed from MATLAB since Mathematica cannot compute this*)
tfS = 2.38067401962272398753992308724S;
(*computed from MATLAB since Mathematica cannot compute this*)
xSol5[t_] = sol5x1[t] /. {lam → lamS, tf → tfS};
ySol5[t_] = sol5x2[t] /. {lam → lamS, tf → tfS};
Plot[xSol5[t], {t, 0, tfS}, PlotLabel → "Optimal x1",
 AxesLabel → {"t", "x1(t)"}, AspectRatio → 1]
Plot[ySol5[t], {t, 0, tfS}, PlotLabel → "Optimal x2",
 AxesLabel → {"t", "x2(t)"}, AspectRatio → 1]
x2[x1_] = 1/5 (20 + tfS^2/2 - 2 x1);
plot = Plot[x2[x1], {x1, -10, 10}, PlotStyle → Orange];
parametric = ParametricPlot[{xSol5[t], ySol5[t]},
   {t, 0, tfS}, PlotRange -> {{-.1, 5}, {-.1, 4}}, PlotStyle → Blue,
   PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"}, AspectRatio → 1];
Show[parametric, plot, PlotLabel → "Optimal Path"]
u[t_] = 3 lamS Exp[-tfS] Exp[t] + 2 lamS
```

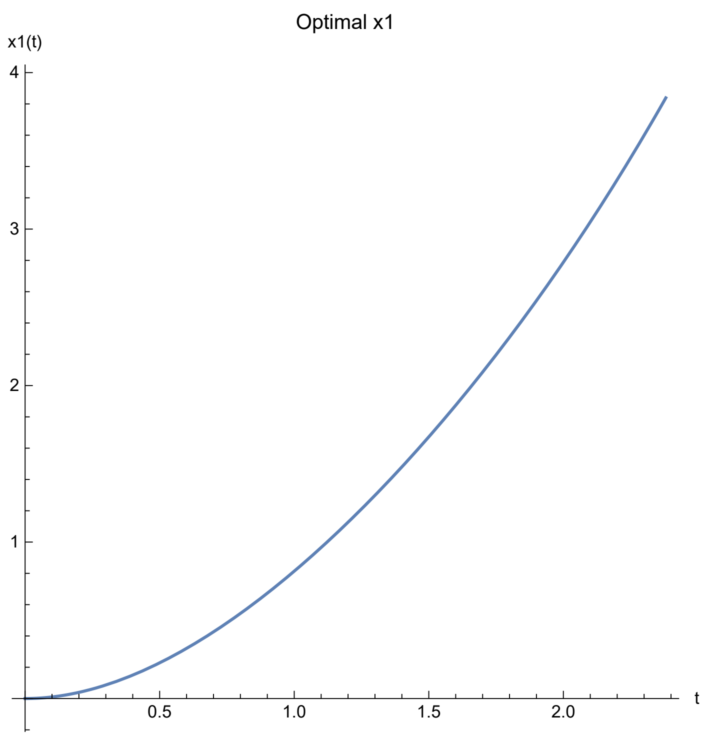
Out[70]= $\left\{\left\{x2[t] \to \frac{1}{2} e^{-t-tf} \left(-1 + e^{t}\right) \left(3 + 3 e^{t} + 4 e^{tf}\right) lam\right\}\right\}$

Out[71]= $\frac{1}{2} e^{-t-tf} \left(-1 + e^{t}\right) \left(3 + 3 e^{t} + 4 e^{tf}\right) lam$

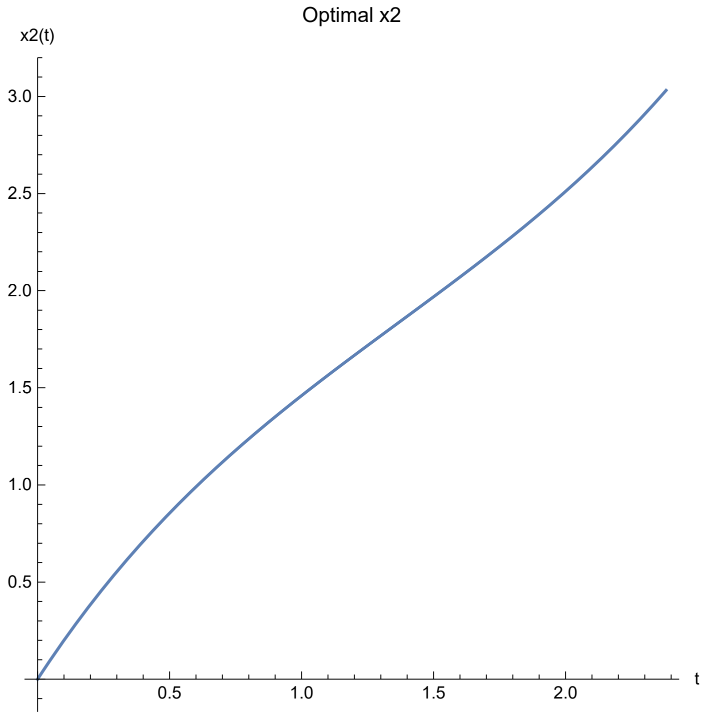
Out[72]= $\left\{\left\{x1[t] \to \frac{1}{2} e^{-t-tf} lam \left(3 - 6 e^{t} + 3 e^{2 t} + 4 e^{tf} + 4 e^{t+tf} \left(-1 + t\right)\right)\right\}\right\}$

Out[73]= $\frac{1}{2} e^{-t-tf} lam \left(3 - 6 e^{t} + 3 e^{2 t} + 4 e^{tf} + 4 e^{t+tf} \left(-1 + t\right)\right)$

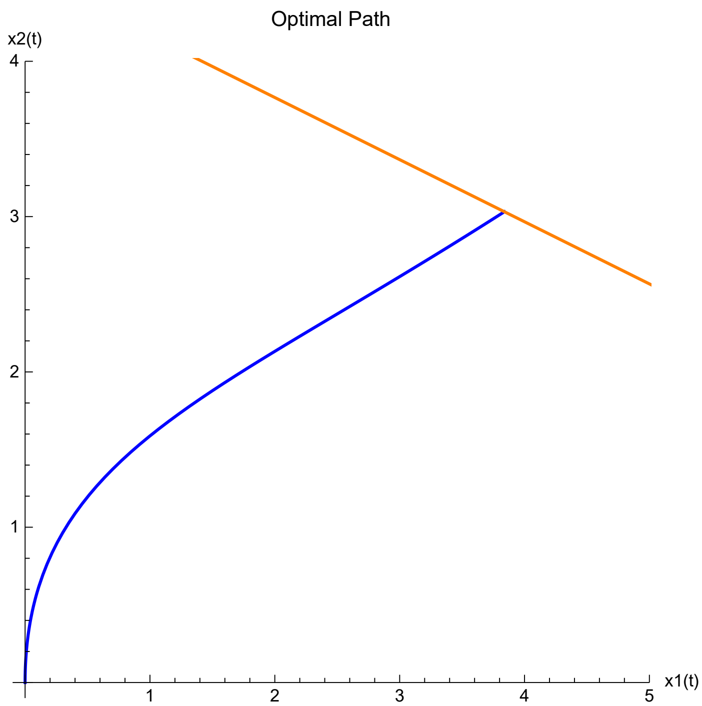
Out[74]= $\frac{1}{2} e^{-2 tf} \left(-1 + e^{tf}\right) \left(3 + 7 e^{tf}\right) lam$

Out[79]=


Out[80]=



Out[84]=



Out[85]= $1.8359321762182275998576227367141 + 0.25470314198342928298964400299 7\, e^{t}$

(*Problem 3*)

In[136]:=
```
sol = Simplify[DSolve[
   {x'[t] == -p[t], p'[t] == -x[t], p[2] == x[2] - 1 + lam, p[0] == lam}, {x[t], p[t]}, t]]
xSol[t_] = x[t] /. sol[[1, 2]]
pSol[t_] = p[t] /. sol[[1, 1]]
xSol0[t_] = xSol[t] /.
   lam → 15 164 043 596 883 160 878 601 741 802 992 / 50 239 011 507 324 117 710 738 448 669 621;
(*lambda solved by MATLAB*)
pSol0[t_] = pSol[t] /.
   lam → 15 164 043 596 883 160 878 601 741 802 992 / 50 239 011 507 324 117 710 738 448 669 621;
u[t_] = Simplify[-N[pSol0[t]]]
cost = N[1/2 (xSol0[2] - 1)^2 + 1/2 Integrate[xSol0[t]^2 + pSol0[t]^2, {t, 0, 2}]]
hamiltonian[t_] = 1/2 (xSol0[t]^2 - pSol0[t]^2);
Plot[hamiltonian[t], {t, 0, 2}]
```

Out[136]= $\left\{\left\{p[t] \to \frac{1}{2} e^{-2-t} \left(1 + e^{2t} (-1 + lam) - lam + 2 e^{2} lam\right), x[t] \to \frac{1}{2} e^{-2-t} \left(1 - e^{2t} (-1 + lam) - lam + 2 e^{2} lam\right)\right\}\right\}$

Out[137]= $\frac{1}{2} e^{-2-t} \left(1 - e^{2t} (-1 + lam) - lam + 2 e^{2} lam\right)$

Out[138]= $\frac{1}{2} e^{-2-t} \left(1 + e^{2t} (-1 + lam) - lam + 2 e^{2} lam\right)$

Out[141]= $e^{-1.\, t} \left(-0.349081 + 0.047243\, e^{2.\, t}\right)$

Out[142]= 0.301838

Out[144]=