Homework 6

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1 Lagrangian Mechanics

Disclaimer: most derivatives and equations are computed by Mathematica. We have the following equantities where subscripts represent relating to (center of mass of) 1st or 2nd rod.

$$\boldsymbol{r}_1 = \frac{L}{2}\begin{pmatrix} \sin\theta_1 \\ -\cos\theta_1 \\ 0 \end{pmatrix},\ \boldsymbol{r}_2 = \begin{pmatrix} L\sin\theta_1 + \frac{L}{2}\sin\theta_2 \\ -L\cos\theta_1 - \frac{L}{2}\cos\theta_2 \\ 0 \end{pmatrix}$$

$$\boldsymbol{v}_1 = \frac{L}{2}\begin{pmatrix} \cos\theta_1\dot{\theta}_1 \\ \sin\theta_1\dot{\theta}_1 \\ 0 \end{pmatrix},\ \boldsymbol{v}_2 = \begin{pmatrix} L\cos\theta_1\dot{\theta}_1 + \frac{L}{2}\cos\theta_2\dot{\theta}_2 \\ L\sin\theta_1\dot{\theta}_1 + \frac{L}{2}\sin\theta_2\dot{\theta}_2 \\ 0 \end{pmatrix}$$

$$\boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix}\ \boldsymbol{\omega}_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix},\ \mathcal{I}_1 = \mathcal{I}_2 = \begin{pmatrix} \mathbf{0}_2 & 0 \\ 0 & \frac{1}{12}mL^2 \end{pmatrix}$$

We can now compute the kinetic and potential energy of the system and therefore the Lagrangian and its derivatives:

$$\begin{aligned}
T &= \frac{1}{2}m(\boldsymbol{v}_1\cdot\boldsymbol{v}_1 + \boldsymbol{v}_2\cdot\boldsymbol{v}_2) + \frac{1}{2}\boldsymbol{\omega}_1^T\mathcal{I}_1\boldsymbol{\omega}_1 + \frac{1}{2}\boldsymbol{\omega}_2^T\mathcal{I}_2\boldsymbol{\omega}_2 \\
&= \frac{1}{6}mL^2\left(4\dot{\theta}_1^2 + 3\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right) \\
V &= -mg\frac{3L}{2}\cos\theta_1 - mg\frac{L}{2}\cos\theta_2 \\
\mathscr{L} &:= T - V \\
\frac{\partial\mathscr{L}}{\partial\theta_1} &= -\frac{1}{2}mL\left(L\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 + 3g\sin\theta_1\right) \\
\frac{\partial\mathscr{L}}{\partial\dot{\theta}_1} &= \frac{1}{6}mL^2\left(8\dot{\theta}_1 + 3\cos(\theta_1-\theta_2)\dot{\theta}_2\right) \\
\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{\theta}_1} &= \frac{1}{6}mL^2\left(8\ddot{\theta}_1 - 3\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2)\dot{\theta}_2 + 3\cos(\theta_1-\theta_2)\ddot{\theta}_2\right) \\
\frac{\partial\mathscr{L}}{\partial\theta_2} &= \frac{1}{2}mL\left(L\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 - g\sin\theta_2\right) \\
\frac{\partial\mathscr{L}}{\partial\dot{\theta}_2} &= \frac{1}{6}mL^2\left(2\dot{\theta}_2 + 3\cos(\theta_1-\theta_2)\dot{\theta}_1\right) \\
\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{\theta}_2} &= \frac{1}{6}mL^2\left(2\ddot{\theta}_2 - 3\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2)\dot{\theta}_1 + 3\cos(\theta_1-\theta_2)\ddot{\theta}_1\right)
\end{aligned}$$

Therefore, the Euler-Lagrange equations can be simplified as

$$mL\left(9g\sin\theta_1 + 3L\sin(\theta_1-\theta_2)\dot{\theta}_2^2 + 3L\cos(\theta_1-\theta_2)\ddot{\theta}_2 + 8L\ddot{\theta}_1\right) = 0 \tag{1}$$

$$mL\left(3g\sin\theta_2 - 3L\sin(\theta_1-\theta_2)\dot{\theta}_1^2 + 3L\cos(\theta_1-\theta_2)\ddot{\theta}_1 + 2L\ddot{\theta}_2\right) = 0. \tag{2}$$

2 Newton-Euler

Denote the gravity of two rods by $\boldsymbol{G}_1 = \boldsymbol{G}_2 = \begin{pmatrix} 0 \\ -mg \\ 0 \end{pmatrix}$. Consider the 1st rod rotating about O which is fixed in the inertial frame. By parallel axis theorem, in the zz component we have $\mathcal{I}_1^O = \mathcal{I}_1 + m\frac{L^2}{4} = \frac{1}{3}mL^2$ (and 0 elsewhere). We also have $\boldsymbol{r}_P = 2\boldsymbol{r}_1$. Let $\boldsymbol{F} = \begin{pmatrix} F_x \\ F_y \\ 0 \end{pmatrix}$ be the tension force experienced by the 2nd rod at P. Thus the tension force experienced by the 1st rod at P is $-\boldsymbol{F}$. Then Euler's 2nd law about O states

$$\begin{aligned}\frac{d\boldsymbol{L}^O}{dt} &= \boldsymbol{M}_{net}^O \\ \mathcal{I}_1^O\ddot{\theta}_1 &= \boldsymbol{r}_1 \times \boldsymbol{G}_1 + \boldsymbol{r}_P \times (-\boldsymbol{F})\end{aligned}$$

Only z-components survive. Simplifying the equation yields

$$mL\left(9g\sin\theta_1 + 3L\sin(\theta_1-\theta_2)\dot{\theta}_2^2 + 3L\cos(\theta_1-\theta_2)\ddot{\theta}_2 + 8L\ddot{\theta}_1\right) = 0. \tag{3}$$

We see that Equation 3 matches Equation 1!

Now let us consider the 2nd rod. By Euler's 1st law, we have

$$\begin{aligned}m\ddot{\boldsymbol{r}}_2 &= \boldsymbol{F}_{net} = \boldsymbol{G}_2 + \boldsymbol{F} \\ \boldsymbol{F} &= m\ddot{\boldsymbol{r}}_2 - \boldsymbol{G}_2 \\ &= \begin{pmatrix} -\frac{1}{2}mL\left(2\sin\theta_1\dot{\theta}_1^2 + \sin\theta_2\dot{\theta}_2^2 - 2\cos\theta_1\ddot{\theta}_1 - \cos\theta_2\ddot{\theta}_2\right) \\ m\left(g + L\cos\theta_1\dot{\theta}_1^2 + \frac{1}{2}L\cos\theta_2\dot{\theta}_2^2 + L\sin\theta_1\ddot{\theta}_1 + \frac{1}{2}L\sin\theta_2\ddot{\theta}_2\right) \\ 0 \end{pmatrix}\end{aligned}$$

Euler's 2nd law on the 2nd rod about center of mass gives

$$\begin{aligned}\frac{d\boldsymbol{L}^{C_2}}{dt} &= \boldsymbol{M}_{net}^{C_2} \\ \mathcal{I}_2\ddot{\theta}_2 &= (\boldsymbol{r}_p - \boldsymbol{r}_2) \times \boldsymbol{F}\end{aligned}$$

$$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{12}mL^2\ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}L\sin\theta_2 \\ \frac{1}{2}L\cos\theta_2 \\ 0 \end{pmatrix} \times \boldsymbol{F}$$

Only z-components survive. Simplifying the equation yields

$$mL\left(3g\sin\theta_2 - 3L\sin(\theta_1-\theta_2)\dot{\theta}_1^2 + 3L\cos(\theta_1-\theta_2)\ddot{\theta}_1 + 2L\ddot{\theta}_2\right) = 0. \tag{4}$$

We see that Equation 4 exactly matches Equation 2! Therefore, we have analytically shown that the two methods give equivalent equations of motion. Using $m = 10kg, L = 2m, g = 9.8m/s^2, t_f = 20s$ and the initial conditions of $\theta_1(0) = 1, \theta_2(0) = 0, \dot{\theta}(0) = 0, \dot{\theta}(0) = 0$, we have the following plots:

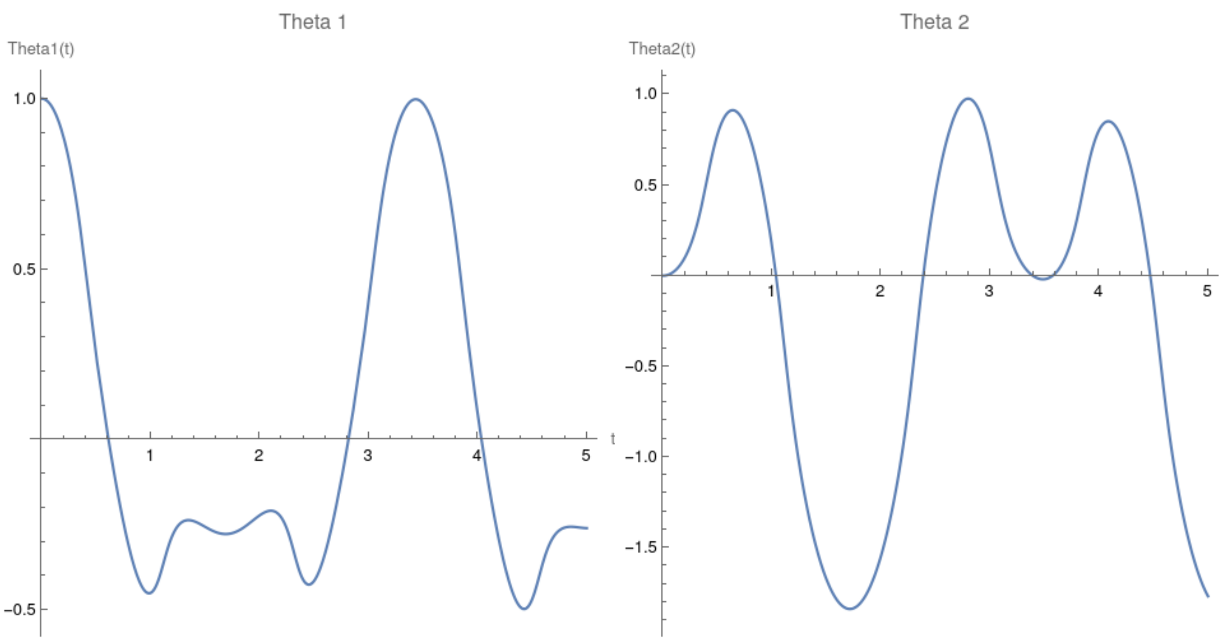

Figure 1: Since the system start from being stationary and θ_2 being vertical, we expect θ_1 to drop and swing to the other side while θ_2 increasing initially. The plots match our expectations.