Homework 3

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Problem (2). (a) Recall from HW2 that $\lim_{t\to\infty} \frac{B_t}{t} = 0$ a.s. so $\lim_{t\to\infty} \frac{B_t}{t} - \frac{1}{2} = -\frac{1}{2}$ a.s. Therefore, we have a.s.

$$\begin{aligned}\lim_{t\to\infty} X_t &= \lim_{t\to\infty} \exp\left(t\left(\frac{B_t}{t} - \frac{1}{2}\right)\right)\\ &= \exp(-\infty) = 0.\end{aligned}$$

(b) No. Suppose that $X_t \to X$ in L^1 for some X, then after rational discretization there exists a subsequence such that $X_t \to X$ a.s., but since $X_t \to 0$ a.s., this forces $X = 0$ a.s. However, we have

$$\mathbb{E}\left[X_t\right] = \mathbb{E}\left[e^{B_t}e^{-\frac{t}{2}}\right] = \mathbb{E}\left[e^{\sqrt{t}B_1}\right]e^{-\frac{t}{2}} = e^{\frac{t}{2}}e^{-\frac{t}{2}} = 1 \neq 0 = \mathbb{E}\left[X\right],$$

a contradiction.

Problem (3). (a) Since each T_n is a stopping time, $\{T_n \leq t\} \in \mathcal{F}_t \ \forall\, n, t$. Then

$$\{\sup_n T_n \leq t\} = \bigcap_n \{T_n \leq t\} \in \mathcal{F}_t.$$

Thus $\sup_n T_n$ is a stopping time.

(b) The case for infimum is different because for infimum to be $\leq t$, it is possible that all $T_n > t$ but the infimum converges to t. Since $\mathcal{F}_t$ is right continuous, we have

$$\{\inf_n T_n \leq t\} = \bigcup_n \underbrace{\{T_n \leq t\}}_{\mathcal{F}_t} \cup \underbrace{\bigcap_{s>t} \{T_n \leq s\}}_{\mathcal{F}_{t+}=\mathcal{F}_t} \in \mathcal{F}_t.$$

Thus $\inf_n T_n$ is a stopping time.

Define $S_n := \sup_{m\geq n} T_m$ and $I_n := \inf_{m\geq n} T_m$. It follows from above that S_n and I_n are stopping times. Then

$$\begin{aligned}\{\limsup_n T_n \leq t\} &= \{\inf_n \sup_{m\geq n} T_m \leq t\} = \{\inf_n S_n \leq t\} \in \mathcal{F}_t\\ \{\liminf_n T_n \leq t\} &= \{\sup_n \inf_{m\geq n} T_m \leq t\} = \{\sup_n I_n \leq t\} \in \mathcal{F}_t.\end{aligned}$$

Problem (4)**.** We first show that (X_n) has uncorrelated increments. Let $m \leq n \in \mathbb{N}$, we repeatedly apply the Tower property:

$$\begin{aligned}
\mathbb{E}\left[(X_n - X_m)^2\right] &= \mathbb{E}\left[\mathbb{E}\left[(X_n - X_m)^2 | \mathcal{F}_m\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[X_n^2 - 2X_n X_m + X_m^2 | \mathcal{F}_m\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[X_n^2 | \mathcal{F}_m\right] - 2\mathbb{E}\left[X_n | \mathcal{F}_m\right] X_m + X_m^2\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[X_n^2 | \mathcal{F}_m\right] - 2X_m^2 + X_m^2\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[X_n^2 | \mathcal{F}_m\right] - X_m^2\right] \\
&= \mathbb{E}\left[X_n^2 - X_m^2\right].
\end{aligned}$$

Let $S := \sup_n \mathbb{E}\left[X_n^2\right] < \infty$. Then $\mathbb{E}\left[X_n^2 - X_m^2\right] \leq 2S$. Consider

$$\begin{aligned}
\mathbb{E}\left[(X_n - X_0)^2\right] &= \mathbb{E}\left[X_n^2 - X_0^2\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{n} (X_{i+1}^2 - X_i^2)\right] && \text{telescope} \\
&= \mathbb{E}\left[\sum_{i=0}^{n} (X_{i+1} - X_i)^2\right] \\
&= \sum_{i=0}^{n} \mathbb{E}\left[(X_{i+1} - X_i)^2\right] \leq 2S.
\end{aligned}$$

Since each term is nonnegative, the partial sum is an monotone increasing sequence, and is bounded above by $2S$, the series converges by MCT, and the tail sum tends to 0. That is, given any $\varepsilon > 0$, by choosing m, n large enough, $\mathbb{E}\left[(X_n - X_m)^2\right] < \varepsilon$. Thus (X_n) is Cauchy and therefore converges in L^2.

Problem (5)**.** We first prove the hints.

(a) $(\Rightarrow)$ is a straightforward computation using LOTUS.

$(\Leftarrow)$: If the MGF $\mathbb{E}\left[e^{\lambda X}\right] = e^{\lambda^2/2}$ for every real λ, then since both sides are analytic functions, by analytic continuation we can extend the equality to the complex plane and obtain the characteristic function $\mathbb{E}\left[e^{i\lambda X}\right] = e^{-\lambda^2/2}$. Since characteristic function uniquely determines the distribution, X must be standard Gaussian.

(b) If the conditional MGF equals unconditional MGF, then by the characteristic equation argument above, the conditional and unconditional probability distributions of X must be the same. That is, for any $x \in \mathbb{R}$ and $B \in \mathcal{B}$ such that $\mathbb{P}(B) > 0$, we have

$$\begin{aligned}
\mathbb{P}(X \leq x|B) &= \mathbb{P}(X \leq x) \\
\frac{\mathbb{P}(X \leq x \cap B)}{\mathbb{P}(B)} &= \mathbb{P}(X \leq x) \\
\mathbb{P}(X \leq x \cap B) &= \mathbb{P}(X \leq x)\,\mathbb{P}(B).
\end{aligned}$$

When B is a null-set, *i.e.* $\mathbb{P}(B) = 0$, the equality trivially holds. This proves independence since $X \leq x$ and B are generators of $\sigma(X)$ and $\mathcal{B}$.

Since X_t already has continuous paths, it remains to show that it is a pre-BM. We shall use definition 3. We already have $X_0 = 0$ a.s. Notice that since $M_t^{(1)}$ is $\mathcal{F}_t$-measurable, $X_t = \log M_t^{(1)} + \frac{t}{2}$ is also $\mathcal{F}_t$-measurable.

Let $0 \leq s \leq t$, so $\mathcal{F}_s \subset \mathcal{F}_t$. Next we show for $X_t - X_s \sim N(0, t-s)$. Consider

$$\begin{aligned}
\mathbb{E}\left[e^{X_t - X_s}\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{X_t - X_s}|\mathcal{F}_s\right]\right] && \text{tower rule} \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{X_t}|\mathcal{F}_s\right] e^{-X_s}\right] && X_s \in \mathcal{F}_s \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{X_t - \frac{1}{2}t}|\mathcal{F}_s\right] e^{-X_s + \frac{1}{2}s}\right] e^{\frac{1}{2}(t-s)} \\
&= \mathbb{E}\left[\mathbb{E}\left[M_t^{(1)}|\mathcal{F}_s\right] \frac{1}{M_s^{(1)}}\right] e^{\frac{1}{2}(t-s)} \\
&= \mathbb{E}\left[M_s^{(1)} \frac{1}{M_s^{(1)}}\right] e^{\frac{1}{2}(t-s)} && \text{Martingale} \\
&= e^{\frac{1}{2}(t-s)}.
\end{aligned}$$

Then the result follows from hint (a).

Finally, we show independent increment. Since $\mathcal{F}_r \subset \mathcal{F}_s$ for all $0 \leq r \leq s$, it suffices to show that $X_t - X_s$ is independent of $\mathcal{F}_s$. Using similar computation from above, we obtain

$$\begin{aligned}
\mathbb{E}\left[e^{X_t - X_s}|\mathcal{F}_s\right] &= e^{\frac{1}{2}(t-s)} \\
&= \mathbb{E}\left[e^{X_t - X_s}\right].
\end{aligned}$$

The result follows from hint (b). Therefore, X_t is a BM.