1 Homotopy Groups of Spheres

Theorem 1.1

$\pi_4(S^3) \cong \mathbb{Z}/2$.

Remark 1.2 Recall $\pi_3(S^2) \cong \mathbb{Z}$ so $\pi_{n+1}(S^n)$ is not fixed. So we need new ideas for this computations.

Definition 1.3 — If X is a CW complex, then for each n there exists a sequence of fibrations $K(\pi_q(X), q) \to Y_q \xrightarrow{p_q} Y_{q-1}$ for $q = 1, \ldots, n$ and maps $f_q : X \to Y_q$ s.t.

(1) $\pi_k(X) \xrightarrow{(f_q)_*} \pi_k(Y_q)$ is an isomorphism for all $k \leq q$.

(2) $\pi_k(Y_q) = 0 \ \forall\ k > q$.

(3) the diagram commutes.

This is called a **Postnikov tower** and the Y_q are called **Postnikov approximation of $\boldsymbol{X}$**.

Lemma 1.4

For each n, every CW complex has a Postnikov tower.

Lemma 1.5

$\pi_q(S^3) \cong H_{q+1}(Y_{q-1})$ for $q > 3$ where Y_{q-1} is the $(q-1)$st term in Postnikov tower for S^3.

Corollary 1.6

$\pi_4(S^3) = H_5(K(\mathbb{Z}, 3))$.

Proof.

$$\pi_4(S^3) = H_5(Y_3) = H_5(K(\pi_3(S^3), 3)) = H_5(K(\mathbb{Z}, 3)).$$

□

Theorem 1.7

$H_5(K(\mathbb{Z}, 3)) \cong \mathbb{Z}/2$.

Theorem 1.8

$\pi_5(S^3) \cong \mathbb{Z}/2$.

The same trick doesn't work here. Given a CW complex X, we can construct a sequence of fibrations s.t.

(1) X_n is n-connected, *i.e.* $\pi_k(X_n) = 0 \; \forall \; k \leq n$.

(2) $\pi_k(X_n) \cong \pi_k(X) \; \forall \; k > n$.

(3) $X_n \to X_{n-1}$ has fiber $K(\pi_n(X), n-1)$.

This is called the **Whitehead tower** of X. It generalizes the universal cover and this is kind of dual to Portnikov tower.

Lemma 1.9

Every CW complex has a Whitehead tower.

Lemma 1.10

$H_4(K(\mathbb{Z}/2.3)) = 0$ and $H_5(K(\mathbb{Z}/2, 3)) = \mathbb{Z}/2$.