Homework 2

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Problem (5)**.** Show that $\mathrm{SO}(n+1)/\mathrm{SO}(n) = S^n$.

Notice that we can identify any element $X \in \mathrm{SO}(n)$ by

$$\begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} \in \mathrm{SO}(n+1).$$

Under the quotient, this matrix is in the same equivalence class as the identity matrix.

For a matrix A, let A_i denotes its ith column and let $\overline{A}$ denote the matrix formed by all its columns but the first column. Consider the map $\phi : \mathrm{SO}(n+1)/\mathrm{SO}(n) \to S^n, [A] \mapsto A_1$. Let's show that this map is well-defined. Let $[A] = [B]$. This is equivalent to $[A^{-1}][B] = [I]$. Since $A^T = A^{-1}$, we have $A^T B = C$ where $C = \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix}$ for some $X \in \mathrm{SO}(n)$. Let a_i, b_i denotes the entries of A_1 and B_1 respectively. Then by matching entries, we see that $[A] = [B]$ if and only if the followings hold:

(1) $a_1 b_1 + \cdots + a_n b_n = 1$.

(2) $A_1^T B_j = 0$ and $A_i^T B_1 = 0$ for $i, j \neq 1$.

(3) $\overline{A}^T \overline{B}$ is in $\mathrm{SO}(n)$.

But since $a_1^2 + \cdots + a_n^2 = 1$ and $b_1^2 + \cdots + b_n^2 = 1$, we have

$$a_1^2 + \cdots + a_n^2 + b_1^2 + \cdots + b_n^2 - 2(a_1 b_1 + \cdots a_n b_n) = 1 + 1 - 2 = 0$$
$$(a_1 - b_1)^2 + \cdots + (a_n - b_n)^2 = 0$$

Thus we obtain that $a_i = b_i$, and therefore $A_1 = B_1$, making the map well-defined.

Surjectivity of ϕ is clear as any unit vector in $\mathbb{R}^{n+1}$ can be in the first column of a special orthogonal matrix. Suppose $A_1 = B_1$, then we use the same trick above to obtain 1. Number 2 follows from definition of orthogonal matrix. Since the first entry of all columns are zeros except for the first column, we see that $\overline{A}$ can be identified as the $n \times n$ lower right minor of A (likewise for $\overline{B}$). Since these columns are orthonormal, both $\overline{A}$ and $\overline{B}$ can be identified as elements of $\mathrm{SO}(n)$. Hence $A^T \in \mathrm{SO}(n)$ and therefore $A^T B \in \mathrm{SO}(n)$. Hence we show that

A, B satisfy all three conditions, so $[A] = [B]$, proving injectivity. Thus ϕ is a well-defined bijection.

Problem (6). (1) Suppose $E \xrightarrow{p} B$ is a bundle with fiber F, we want to show that given a map $A \xrightarrow{f} B$, $f^*E \xrightarrow{\pi_1} A$ is a bundle with fiber F, where π_1 is projection onto 1st factor.

Given $a \in A$, we wish to find a neighborhood of a with trivial localization. Consider $b = f(a) \in B$. Then there exists a neighborhood U of b s.t. $p^{-1}(U) \xrightarrow{\phi} U \times F$ is an homeomorphism. Notice $f^{-1}(U)$ is an open neighborhood of a. I claim that $\psi : \pi_1(f^{-1}(U)) \to f^{-1}(U) \times F, (a, e) \mapsto (a, p_2 \circ \phi(e))$ is a homeomorphism with inverse $\psi^{-1} : f^{-1}(U) \times F \to \pi_1^{-1}(f^{-1}(U)), (a, x) \mapsto (a, \phi^{-1}(f(a), x)))$. Continuity of both maps is clear. Let's check bijectivity:

$$\begin{aligned}
\psi^{-1} \circ \psi(a, e) &= \psi^{-1}(a, p_2 \circ \phi(e)) \\
&= (a, \phi^{-1}(f(a), p_2 \circ \phi(e))) \\
&= (a, \phi^{-1}(p(e), p_2 \circ \phi(e))) \\
&= (a, \phi^{-1}(p_1 \circ \phi(e), p_2 \circ \phi(e))) \\
&= (a, e)
\end{aligned}$$

The other direction is similar. Hence $f^*E \xrightarrow{\pi_1} A$ is also a bundle with fiber F.

(2) If f is the inclusion map, then $f^*E = \{(a, e) : p(e) = a\}$. Since $E|_A := p^{-1}(A) = \{e \in E : p(e) \in A\}$. We see that $\pi_2 : f^*E \to E|_A, (a, e) \mapsto e$ and $\pi_2^{-1} : E|_A \to f^*E, e \mapsto (p(e), e)$ are inverses and clearly continuous. Hence $f^*E \cong E|_A$.

(3) It is easy to see $\pi_2 : f^*E \to E$ is a bundle map by part 1.

(4) If $f : A \to B, a \mapsto b_0$ is the constant map, then $f^*E = \{(a, e) : p(e) = b_0\} = \{(a, e) : e \in p^{-1}(b_0)\} = A \times p^{-1}(b_0) = \pi_1^{-1}(A)$. Since (f^*E, A, F, π_1) is a bundle, it has local trivialization around $f^{-1}(U)$ where U is the neighborhood of b_0 from part 1. But $f^{-1}(U) = A$. Hence $f^*E = \pi_1^{-1}(A) \cong A \times F$ by part 1.

(5) Since $E = B \times F$ is the trivial bundle, we can take B to be the neighborhood U from part 1. Hence $\pi_1^{-1}(f^{-1}(B)) = \pi_1^{-1}(A) = f^*E \cong A \times F$.

Problem (7). (1) Show that if B is covered by open sets $\{U_\alpha\}$ and we have $\tau_{\alpha\beta} : (U_\alpha \cap U_\beta) \to \text{Homeo}(F)$, then there exists a bundle E over B realizing this data as transition maps.

Define $E = \bigsqcup_\alpha U_\alpha \times F/\sim$ where $U_\alpha \times F \ni (x, y) \sim (x', y') \in U_\beta \times F$ iff $x = x'$ and $\tau_{\beta\alpha} y = y'$. Let $p : E \to B$ be projection onto first coordinate. This is well-defined since the first coordinate is unique in each equivalence class. Then given $x \in B$, it is in some U_α, and $p^{-1}(U_\alpha) = \{[(x', y)] \in E : x' \in U_\alpha\} = U_\alpha \times F/\sim$. But since $U_\alpha \times F \ni (x, y) \sim (x', y') \in U_\alpha \times F$ iff $x = x'$ and $\tau_{\alpha\alpha}(x)y = \text{id}_F(y) = y = y'$, we see that $U_\alpha \times F/\sim \cong U_\alpha \times F$ by ϕ_α mapping any equivalence class to its unique representative in $U_\alpha \times F$. Hence we have local trivialization at U_α. Thus E is a bundle over B.

Notice $\phi_\beta \circ \phi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ is exactly mapping the representative of an equivalence class in $U_\alpha \times F$ to its representative in $U_\beta \times F$. Then we immediately see from the definition of $\sim$ that the map is $(x, y) \mapsto (x, \tau_{\beta\alpha}(x)y)$.

(2) Show that P_E is a principal G-bundle.

Define a right action of G on P_E by

$$[(x, g)].\widetilde{g} = [(x, g\widetilde{g})].$$

First we show it is well-defined. Given representatives $(x, g) \in U_\alpha \times G, (x, g') \in U_\beta \times G$ where $\tau_{\beta\alpha}(x)g = g'$, clearly $\tau_{\beta\alpha}(x)(g) = g'\widetilde{g}$ so $(x, g\widetilde{g}) \sim (x, g'\widetilde{g})$. Next we show it preserves fibers. Given $x \in B$, let $\pi : P_E \to B$ be projection to 1st factor. Then $\pi^{-1}(x) = \{[(x, g)] : g \in G\} = \{x\} \times G/\sim$. Since G acts trivially on the first factor and the action is well-defined on the second factor, it clearly preserves the fiber. Moreover, right multiplication is transitive so given $[(x, g)]$ and $[(x, g')]$, we see that $\widetilde{g} = g^{-1}g'$ would do the job. Finally, given $[(x, g)] \in \{x\} \times G/\sim \subseteq U_\alpha \times G/\sim$ for some α, suppose $[(x, g)].\widetilde{g} = [(x, g\widetilde{g})] = [(x, g)]$. Then

$$\begin{aligned}
\tau_{\alpha\alpha}(g) &= g\widetilde{g} \\
\text{id}_G(g) &= g\widetilde{g} \\
g &= g\widetilde{g}
\end{aligned}$$

$$e_G = \widetilde{g}$$

Hence the action is free. Thus P_E is a principal G-bundle.

Problem (8)**.** (1) A principal G-bundle is trivial iff it admits a section.

($\Rightarrow$) : Suppose $P = B \times G$. Then the inclusion map $i : B \to B \times G, b \mapsto (b, 1_G)$ is a section where $p \circ s = \mathrm{id}_B$.

($\Leftarrow$) : Suppose P is a principal G-bundle and admits a section $s : B \to P$ s.t. $p \circ s = \mathrm{id}_B$. Then I claim that $\phi : B \times G \to P, (b, g) \mapsto s(b).g$ and $\phi^{-1} : P \to B \times G, e \mapsto (p(e), g)$ where $g = (\pi_2(s \circ p(e)))^{-1}\pi_2(e)$ (*i.e.* $s \circ p(e).g = e$) are well-defined inverses and are continuous.

First, $s(b) \in p^{-1}(b)$ as $p(s(b)) = b$. Since the action preserves fiber, $s(b).g \in p^{-1}(b)$ so $p(s(b).g) = b$. Since $e, s \circ p(e) \in p^{-1}(p(e))$, and the action is transitive on $p^{-1}(p(e))$, there exists a $g \in G$ s.t. $s \circ p(e).g = e$.

$$\begin{aligned}
\phi^{-1} \circ \phi(b, g) &= \phi^{-1}(s(b).g) \\
&= (p(s(b).g), g') \\
&= (b, g')
\end{aligned}$$

where $s(p(s(b).g)).g' = s(b).g' = s(b).g$. Since the action is free on $p^{-1}(b)$, we have $g' = g$. It follows that $\phi^{-1} \circ \phi = \mathrm{id}_{B \times G}$. The other direction is similar. Since s is continuous and G is a topological group of homeomorphisms (*i.e.* group operation and inversion are continuous), ϕ and ϕ^{-1} are continuous.

(2) If E is a vector bundle, show that sections of E is the same as $\mathrm{GL}_n(\mathbb{R})$-equivariant maps $v : \mathcal{F}(E) \to \mathbb{R}^n$, *i.e.* $v(y.g) = g^{-1}(v(y))$.

Given $s : B \to E$ and $y \in E$ which is a point $x \in B$ with a full-rank matrix or n-frame attached. Define $v(y) = s(p(y))$, which is a vector in $\mathbb{R}^n$ attached to x expressed under the n-frame basis, so that $v : \mathcal{F}(E) \to \mathbb{R}^n$. Then we see that $y.g$ is simply changing the n-frame to a new one by multiplying g on the right, then the coordinates of $v(y)$ under this new basis must change contravariantly, *i.e.* $v(y).g = g^{-1}(v(y))$. This is exactly $v(y.g) = g^{-1}(v(y))$.

Given a $\mathrm{GL}_n(\mathbb{R})$-equivariant map $v : \mathcal{F}(E) \to \mathbb{R}^n$ where $v(y.g) = g^{-1}(v(y))$. We wish to define a section $s : B \to E$, *i.e.* $p \circ s = \mathrm{id}_B$.

I claim that $E \cong \mathcal{F}(E) \times \mathbb{R}^n/G$, where G acts on the product componentiwse as above. First notice given any $x \in B$, $\pi^{-1}(x)/G$ has a single orbit since G acts transitively on the fiber. Let $\phi : E \to \mathcal{F}(E) \times \mathbb{R}^n/G, (x, a) \mapsto ([r], v([r]))$, where r is any representative of the single equivalence class $\pi^{-1}(p(e))/G$. Since v is G-equivariant, v is well-defined on $\pi^{-1}(x)/G$. Also define $\phi^{-1} : \mathcal{F}(E) \times \mathbb{R}^n/G \to E, [([x, g], a)] \mapsto (x, ga)$. This is well-defined because for another representative $([x, g], a).g' = ([x, g].g', a.g') = ([x, gg'], g^{-1\prime}a)$, it is sent to $(x, gg'g'^{-1}a) = (x, ga)$ (the inner equivalence class has a unique representative in the fiber). We can check that they are indeed inverses. Given $e = (x, a) \in E$, let $r = (x, g') \in \pi^{-1}(x)$. Then $r.g'^{-1} = (x, g'g^{-1}) = (x, e_G) \in [r]$. Hence we can always use (x, e_G) as the representative for $[r]$. Since $v(r)$ and a are both vectors of $\mathbb{R}^n$, there exists an invertible matrix $g \in G$ s.t. $ga = v(r)$ or $a = g^{-1}v(r)$. Now we have

$$\begin{aligned}
\phi^{-1} \circ \phi(x, a) &= \phi^{-1}([r], v([r]) \\
&= (x, e_G v([r])) \\
&= (x, v([r.g]) \\
&= (x, g^{-1}v([r]) \\
&= (x, a)
\end{aligned}$$

This way, we simply define $s : B \to E, x \mapsto \phi^{-1}([x, g], a)$ where we pick any $g \in G$, $a \in \mathbb{R}^n$. Then

$$\begin{aligned}
p \circ s(x) &= p \circ \phi^{-1}([x, g], a) \\
&= p(x, ga) \\
&= x
\end{aligned}$$

So s is indeed a section.