Homework 10

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Problem (Do Carmo 4.4)**.** Let M be a Riemannian manifold with the following property: given any $p, q \in M$, the parallel transport from p to q does not depend on the curve. Prove that the curvature of M is identically zero, *i.e.* for all $X, Y, Z \in \mathfrak{X}(M)$, $R(X,Y)Z = 0$.

Proof. Consider a parametrized surface $f : U \subset \mathbb{R}^2 \to M$, where U is the ε-neighorhood of the unit square in $\mathbb{R}^2$ and $f(s,0) = f(0,0)$ for all s. Given $V_0 \in T_{(0,0)}M$, define a vector field V along f by setting $V(s,0) = V_0$ and if $t \neq 0$, $V(s,t)$ is the parallel transport of V_0 along the curve $t \mapsto f(s,t)$. By Lemma 4.1, any vector field along a parametrized surface f satisfies

$$\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right).$$

Since $V(s,t)$ is parallel transported along t-curves, $\frac{D}{\partial t}V \equiv 0$. Since parallel transport in M does not depend on the curve, $V(s,1)$ is also the parallel transport of $V(0,1)$ along an s-curve. Therefore, $\frac{D}{\partial s}V(s,1) \equiv 0$. It follows that

$$R_{f(s,1)}\left(\frac{\partial f}{\partial s}(s,1), \frac{\partial f}{\partial t}(s,1)\right)V(s,1) = 0.$$

In particular this is true for $s = 0$. Since f and V_0 are arbitrary, given $p \in M$, $X, Y, Z \in \mathfrak{X}(M)$, we can let $f(0,1) = p$, $\frac{\partial f}{\partial s}(0,1) = X(p)$, $\frac{\partial f}{\partial t}(0,1) = Y(p)$, and V_0 equal the parallel transport from $V(0,1) := Z(p)$. The result follows pointwise.

□

Problem (5.1)**.** Let M be a Riemannian manifold with sectional curvature identically zero. Show that, for every $p \in M$, the mapping $\exp_p : B_\varepsilon(0) \subset T_pM \to B_\varepsilon(p)$ is an isometry, where $B_\varepsilon(p)$ is a normal ball at p.

Proof. By the polarization trick, it suffices to show that for any $v \in B_\varepsilon(0)$ and $w \in T_v(T_pM)$, we have

$$\left\| d(\exp_p)_v(w) \right\| = \|w\|.$$

It in turn suffices to show that

$$\left\|d(\exp_p)_{tv}(tw)\right\| = \|tw\|.$$

Suppose $V(0) = v$, $V'(0) = w$, then we have the Jacobi field along the geodesic $\gamma(t) = \exp_p(tv)$:

$$\begin{aligned} J(t) &= \frac{\partial}{\partial s}\left(\exp_p(tV(s))\right)(t,0) \\ &= d\left(\exp_p\right)_{tV(0)}(tV'(0)) \\ &= d\left(\exp_p\right)_{tv}(tw). \end{aligned}$$

Let E_i be an orthonormal parallel frame along γ and let $J(t) = f_i(t)E_i(t)$. Since the frame is parallel along the geodesic, we have $\frac{D}{\partial t}E_i(t) \equiv 0$ so $J''(t) = f_i''(t)E_i(t)$. Since the sectional curvature $K \equiv 0$, the only potentially nonzero curvatures of the form $R_{ijij} = K$ must be zero as well. Thus the Jacobi equation reduces to $f_i''(t) = 0$, yielding $f_i(t) = a_it + b_i$. Since $J(0) = 0$, we have $f_i(t) = a_it$. Also recall $J'(0) = V'(0) = w = f_i'(0)E_i(0) = a_iE_i(0)$. Therefore, we obtain

$$\begin{aligned} \left\|d(\exp_p)_{tv}(tw)\right\| &= \|J(t)\| \\ &= \|ta_iE_i(t)\| \\ &= t\sum_{i=1}^{n} a_i^2 && E_i \text{ is orthonormal} \\ &= \|tw\|. \end{aligned}$$

$\square$

Problem (5.6)**.** Let M be a Riemannian manifold of dimension 2. Let $B_\delta(p)$ be a normal ball around the point $p \in M$ and consider the parametrized surface

$$f(\rho,\theta) = \exp_p(\rho v(\theta)), \qquad 0 < \rho < \delta, -\pi < \theta < \theta,$$

where $v(\theta)$ is a circle of radius 1 in T_pM parametrized by the angle.

(a) Show that (ρ,θ) are coordinates in an open set $U \subset M$ formed by the open ball $B_\delta(p)$ minus the ray $\exp_p(-\rho v(0))$. This is the polar coordinates at p.

(b) Show that g_{ij} of the Riemannian metric in these coordinates are $g_{12} = 0, g_{11} = \left\| \frac{\partial f}{\partial \rho} \right\|^2 = \|v(\theta)\|^2 = 1, g_{22} = \left\| \frac{\partial f}{\partial \theta} \right\|^2$.

(c) Show that, along the geodesic $f(\rho, 0)$, we have

$$(\sqrt{g_{22}})_{\rho\rho} = -K(p)\rho + R(p),$$

where $\lim_{\rho \to 0} \frac{R(p)}{\rho} = 0$.

(d) Prove that

$$\lim_{\rho \to 0} \frac{(\sqrt{g_{22}}_{\rho\rho})}{\sqrt{g_{22}}} = -K(p).$$

Proof. (a) Since the image of f is contained in a normal ball, $\exp_p$ has a smooth inverse. Using the norm and inverse tangent (which is only bijective if we remove the ray) we can compose a smooth inverse of f.

(b) Notice $v(\theta)$ is tangent to the curve $\rho v(\theta)$. To compute the pullback metric, we have:

$$\begin{aligned}
g_{11}(\rho, \theta) &= \left\langle \frac{\partial}{\partial \rho}(\rho, \theta), \frac{\partial}{\partial \rho}(\rho, \theta) \right\rangle \\
&= \left\langle \frac{\partial f}{\partial \rho}(\rho, \theta), \frac{\partial f}{\partial \rho}(\rho, \theta) \right\rangle \\
&= \left| \frac{\partial f}{\partial \rho}(\rho, \theta) \right|^2 && \text{proof of Gauss's lemma} \\
&= \left\| \frac{\partial}{\partial \rho} \exp_p(\rho v(\theta)) \right\|^2 \\
&= \left\| d(\exp_p)_{\rho v(\theta)}[v(\theta)] \right\|^2 \\
&= \|v(\theta)\|^2 && \text{geodesic preserves tangent vector length} \\
&= 1 \\
g_{12} &= \left\langle \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \theta} \right\rangle \\
&= \left\langle \frac{\partial f}{\partial \rho}, \frac{\partial f}{\partial \theta} \right\rangle \\
&= 0 && \text{proof of Gauss's lemma} \\
g_{22} &= \left\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right\rangle
\end{aligned}$$

$$= \left\langle \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \theta} \right\rangle$$
$$= \left| \frac{\partial f}{\partial \theta} \right|^2 \qquad \text{proof of Gauss's lemma.}$$

(c) Since $|J(\rho)| = \left|\frac{\partial f}{\partial \theta}(\rho, 0)\right| = \sqrt{g_{22}}(\rho)$ by part (b), by Corollary 5.2.10 we obtain

$$\sqrt{g_{22}}(\rho) = \rho - \frac{1}{6}K(p)\rho^3 + \widetilde{R}(\rho),$$

where $\lim_{\rho\to 0} \frac{\widetilde{R}(\rho)}{\rho^3} = 0$. Differentiating twice with respect to ρ, we obtain

$$\sqrt{g_{22}}_{\rho\rho} = -K(p)\rho + R(\rho),$$

where $\lim_{\rho\to 0} \frac{R(\rho)}{\rho} = 0$.

(d)

$$\begin{aligned}
\lim_{\rho\to 0} \frac{(\sqrt{g_{22}}_{\rho\rho})}{\sqrt{g_{22}}} &= \lim_{\rho\to 0} \frac{-K(p)\rho + R(\rho)}{\rho - \frac{1}{6}K(p)\rho^3 + \widetilde{R}(\rho)} \\
&= \lim_{\rho\to 0} \frac{-K(p)\rho + O(\rho^2)}{\rho + O(\rho^3)} \\
&= -K(p).
\end{aligned}$$

□

Problem (5.7)**.** Let M be Riemannian manifold with dimension 2. Let $p \in M$ and $V \subset T_pM$ be a normal neighborhood. Let $S_r(0) \subset V$ be a circle of radius r centered at the origin, and let L_r be the length of the curve $\exp_p(S_r)$ in M. Prove that the sectional curvature at p is

$$K(p) = \lim_{r\to 0} \frac{3}{\pi} \frac{2\pi r - L_r}{r^3}.$$

Proof. First, using result of Exercise 5.6, we obtain

$$\begin{aligned}
L_r &= \int_{-\pi}^{\pi} \sqrt{\left\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right\rangle} d\theta \\
&= \int_{-\pi}^{\pi} \sqrt{g_{22}(r,\theta)} d\theta.
\end{aligned}$$

Thus by Corollary 5.2.10 we have

$$2\pi r - L_r = \int_{-\pi}^{\pi} \left(r - \left(r - \frac{1}{6}K(p)r^3 + \widetilde{R}(\rho) + r \right) \right) d\theta$$

$$
\begin{aligned}
&= \int_{-\pi}^{\pi} \left(\frac{1}{6} K(p) - \frac{\widetilde{R}(r)}{r^3} \right) d\theta \\
&= 2\pi \left(\frac{1}{6} K(p) - \frac{\widetilde{R}(r)}{r^3} \right).
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
\lim_{r \to 0} \frac{3}{\pi} \frac{2\pi r - L_r}{r^3} &= \lim_{r \to 0} \frac{3}{\pi} 2\pi \left(\frac{1}{6} K(p) - \frac{\widetilde{R}(r)}{r^3} \right) \\
&= K(p).
\end{aligned}
$$

$\square$