1 Classifying Spaces

Exercise: $E_n \xrightarrow{p} G_n$ is an n-dimensional vector bundle. Hint: IF $\ell \in G_n$, let $\pi_\ell : \mathbb{R}^\infty \to \ell$ be orthogonal proj. Let $U_\ell = \{\ell' \in G_n : \pi_\ell(\ell') \text{ has dim } n\}$. Show U_ℓ is open and $h : p^{-1}(U_\ell) \to U_\ell \times \ell, (\ell', v) \mapsto (\ell', \pi_\ell(v))$ is a local trivialization.

Theorem 1.1

Let X be paracompact and $E_n = \{(\ell, v) \in G_n \times \mathbb{R}^\infty : v \in \ell\}$. Then $[X, G_n] \to \mathrm{Vect}^n(X), f \mapsto f^* E_n$ is a 1 to 1 correspondence.

Definition 1.2 — For a topological group G, there exists a space BG and a principal G-bundle EG s.t. (EG, BG, G, p) is a bundle and EG is weakly contractible. We call BG the **classifying space for principal G-bundle** and EG the **universal G-bundle**.

Remark 1.3 By the long exact sequence and weakly contractible, $\pi_k(BG) \cong \pi_{k-1}(G) \; \forall \, k \geq 1$.

Theorem 1.4

$[X, BG]$ and principal G-bundles over X is a 1 to 1 correspondence (via $f \mapsto f^* EG$).

Theorem 1.5

The homotopy type of BG is unique.

Example 1.6 (1) G_n is the classifying space of $\mathbb{R}^n$-bundles. In fact, $\mathcal{F}(E_n)$ is an $\mathrm{GL}_n(\mathbb{R})$-bundle and G_n is the $\mathrm{GL}_n(\mathbb{R})$ classifying space. Exercise: $\mathcal{F}(E_n)$ is weakly contractible.

(2) $\mathbb{R} \to S^1$ is a principal $\mathbb{Z}$-bundle with $E\mathbb{Z} = \mathbb{R}$ and $B\mathbb{Z} = S^1$. Principal $\mathbb{Z}$-bundles over X is 1 to 1 correspondence with $[X, S^1] = [X, K(\mathbb{Z}, 1)] \cong H^1(X; \mathbb{Z})$ by Brown representation theorem: $[X, K(\pi, n)] \cong H^n(X; \pi)$.

(3) $S^\infty \to \mathbb{R}P^\infty$ is a principal $\mathbb{Z}/2$-bundle. Note $\mathbb{Z}/2 \cong O(1)$. Then $BO(1) \cong \mathbb{R}P^\infty$

and $EO(1) \cong S^\infty$. Exercise: S^∞ is contractible. So line bundles over X is 1 to 1 with principal $O(1)$-bundles over X is 1 to 1 with $[X, BO(1)] = [X, \mathbb{R}P^\infty] = [X, K(\mathbb{Z}/2, 1)] = H^1(X; \mathbb{Z}/2)$ by Brown.

(4) $S^\infty \to S^\infty/S^1 \cong \mathbb{C}P^\infty$ is a principal S^1-bundle. Note $S^1 = U(1)$. So $BU(1) \cong \mathbb{C}P^\infty$, $EU(1) \cong S^\infty$. Complex line bundles over X 1 to 1 principal $U(1)$-bundles over X 1 to 1 $[X, BU(1)] = [X, \mathbb{C}P^\infty] = [X, K(\mathbb{Z}, 2)] \cong H^2(X; \mathbb{Z})$.

Definition 1.7 — A GCW-complex is a space X with a G-action that is the union of skeleta ...

Exercise:

(1) If X is a GCW-complex, then X/G has a natural CW structure.

(2) If G is a compact Lie group, then any principal G-bundle over a CW-complex is a GCW-complex.

To construct classifying spaces, we need the definition:

Definition 1.8 — Let X, Y be spaces. Their **join** is

$$X * Y = X \times I \times Y / \sim$$

where $(x, 0, y_1) \sim (x, 0, y_2)\ \forall\ y_1, y_2 \in Y$ and $(x_1, 1, y) \sim (x_2, 1, y)\ \forall\ x_1, x_2 \in X$.

Examples:

(1) $X * \{*\} \cong \text{Cone}(X)$.

(2) $X * \{p_1, p_2\} \cong \Sigma X$.

(3) $\{x_0\} * \cdots * \{x_k\}$ is a k-simplex.

(4) Exercise: $S^n * S^m \cong S^{n+m+1}$. Start with $S^1 * S^1 \cong S^3$, $\mathbb{R}^2 * \mathbb{R}^2, \mathbb{R}^4$. DO THIS.

Remark 1.9 The join generalizes the cone.

There exists inclusions $X \xrightarrow{i} X * Y, x \mapsto (x, 0, y)$ for any y and $Y \xrightarrow{j} X * Y, y \mapsto (x, 1, y)$ for any x.

Lemma 1.10

The inclusion $i : X \to X * Y$ and $j : Y \to X * Y$ are nullhomotopic.

Proof. For any $y_0 \in Y$, i factors through $X \to X * \{y_0\} = C(X)$ and hence $X \to X * \{y_0\}$ is nullhomotopic. The claim follows. $\square$

Given G a topological group, let $G^{*(k+1)} = \underbrace{G * G * \cdots * G}_{k+1}$. This has a G-action:

$$(g_0, t_1, g_1, t_2, \ldots, t_k, g_k).g = (g_0 g, t_1, g_1 g, t_2, \ldots, t_k, g_k g).$$

Exercise:

(1) There exists a natural G-equivariant map $\Delta^k \times G^{k+1} \to G^{*(k+1)}$ that is a homeomorphism when restricted to interior of $\Delta^k \times G^{k+1}$. (think simplex example).

(2) Use 1 to show $G^{*(k+1)}$ has the structure of a GCW-complex.

Let $\mathcal{J}(G) = \lim_{k\to\infty} G^{*(k+1)}$.

Theorem 1.11

The quotient map $p : \mathcal{J}(G) = EG \to J(G)/G = BG$ is a universal principal G-bundle.

Proof. Exercise: show p is a principal G-bundle.

We are done if $\mathcal{J}(G)$ is weakly contractible. For any $\alpha : S^n \to \mathcal{J}(G)$, there exists some k s.t. $\alpha(S^n) \subseteq G^{*(k+1)} \subseteq \mathcal{J}(G)$ and $G^{*(k+1)} \to G^{*(k+1)} \subseteq \mathcal{J}(G)$ is nullhomotopic. So $\alpha : S^n to G^{*(k+1)} \subseteq G^{*(k+2)}$ is nullhomotopic. $\square$

From the construction, given $f : H to G$ a homo, then we get an induced map $Ef : EH = \mathcal{J}(H) \to EG = \mathcal{J}(G)$ and $Bf : BH \to BG$.

Exercise:

(1) Bf is the classifying map for the bundle $BH \times_f G$, *i.e.* $(Bf)^* EG \cong BH \times_f G$.

(2) If $H \leq G$ and $P \to M$ is a principal G-bundle, then structure group of P reduces to H iff the classifying map $f : M \to BG$ (after homotopy) factors through $M \to BH$.

A different view of characteristic classes:

Theorem 1.12

$H^*(BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \ldots, w_n]$ where w_i has degree i.

$H^*(BU(n); \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_n]$ where c_i has degree $2i$.

We can use this theorem to define characteristic classes of an $\mathbb{R}^n$-bundle $E \to M$. There exists an associated $O(n)$-bundle. By theorem 13 there exists a map $f : M \to BO(n)$ s.t. $\mathcal{F}(E) \cong f^*EO(n)$. Define the ith Steifel-Whitney class of E to be

$$w_i(E) = f^* w_i.$$

Simiarly for Chern classes.

Theorem 1.13

$H^*(BSO(2n+1); \mathbb{Z}) \cong \mathbb{Z}[p_1, \ldots, p_n] \oplus$ Torsion where Torsion is $\beta(H^n(BSO(2n+1); \mathbb{Z}/2))$.

$H^*(BSO(2n); \mathbb{Z}) \cong \mathbb{Z}[p_1, \ldots, p_n, e]/\langle e^2 = p_n \rangle \oplus$ Torsion.