Homework 13

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Problem (Do Carmo 8.1)**.** Consider on a neighborhood of $\mathbb{R}^n$, $n > 2$, the metric

$$g_{ij} = \frac{\delta_{ij}}{F^2},$$

where $F \neq 0$ is a function on $\mathbb{R}^n$. Denote $F_i = \frac{\partial F}{\partial x_i}, F_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j}$.

(a) Show that a necessary and sufficient condition for the metric to have constant curvature K is

$$\begin{cases} F_{ij} = 0, \quad i \neq j \\ F(F_{jj} + F_{ii}) = K + \sum_{i=1}^n (F_i)^2. \end{cases}$$

(b) Use above to prove that the metric g_{ij} has constant curvature K iff

$$F = G_1(x_1) + G_2(x_2) + \cdots + G_n(x_n),$$

where $G_i(x_i) = ax_i^2 + b_i x_i + c_i$ and $K = \sum_{i=1}^n (4c_i a - b_i^2)$.

(c) Put $a = \frac{K}{4}, b_i = 0, c_i = \frac{1}{n}$ and obtain the formula of Riemann

$$g_{ij} = \frac{\delta_{ij}}{\left(1 + \frac{K}{4} \|x\|^2\right)^2}$$

for a metric g_{ij} of constant curvature K ($\|\cdot\|$ here denotes Euclidean norm). If $K < 0$ then metric g_{ij} is defined in a ball of radius $\sqrt{\frac{4}{-K}}$.

(d) If $K > 0$, the metric is defined on all of $\mathbb{R}^n$. Show that such a metric on $\mathbb{R}^n$ is not complete.

Proof. (a) We compute

$$f_i = \frac{F_i}{F}, \quad f_{ij} = -\frac{F_i F_j}{F^2} + \frac{F_{ij}}{F}.$$

Based on the formula from book, if any three indices are distinct, then by Chapter 4 Corollary 3.5, constant sectional curvature is equivalent to

$$0 = R_{ijk\ell} = R_{ijk}^{s\ell} g_{s\ell}$$

$$
\begin{aligned}
&= -\delta_{is}(-f_k f_j - f_{kj})g_{i\ell} + \delta_{js}(f_i f_k + f_{ki})g_{j\ell} \\
&= -\frac{1}{F^2}(-F_k F_j + F_k F_j - F F_{kj})\frac{\delta_{i\ell}}{F^2} + \frac{1}{F^2}(F_i F_k - F_i F_k + F F_{ki})\frac{\delta_{j\ell}}{F^2} \\
&= \frac{\delta_{i\ell} F_{kj}}{F^3} - \frac{\delta_{j\ell} F_{ki}}{F^3}.
\end{aligned}
$$

Since $F \neq 0$, it follows that $F_{ij} = 0$ as long as $i \neq j$. Thus we establish equivalence for the first equation.

The second equivalence is a straightforward computation using formula from book:

$$
K = \left(-\sum_\ell \frac{F_\ell^2}{F^2} + \frac{F_i^2}{F^2} + \frac{F_j}{F^2} - \frac{F_i^2}{F^2} + \frac{F_{ii}}{F} - \frac{F_j^2}{F^2} + \frac{F_{jj}}{F} \right) F^2
$$

$$
F(F_{jj} + F_{ii}) = K + \sum_\ell F_\ell^2.
$$

(b) The second partial is zero for $i \neq j$ iff there are no cross terms in F by basic Calculus. Due to this fact and the fact that partials commute in $\mathbb{R}^n$, $(F_{ii})_j = (F_{ij})_i = 0$ whenever $i \neq j$. This is equivalent to $F_{ii} = F_{jj}$ being constants which we call $2a$. Then calculus gives that $G_i(x_i) = ax_i^2 + b_{ixi} + c_i$. And the second equation therefore is equivalent to $K = \sum_{i=1}^n (4c_i a - b_i^2)$.

(c) This is obvious.

(d) Given any point $x \in \mathbb{R}^n$, let $\gamma(t) = tx$ so $\gamma'(t) = x$. Then the distance between the origin 0 and x is upper-bounded by the length of this path:

$$
\begin{aligned}
\int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}dt &= \int_0^1 \frac{\|x\|}{1 + \frac{K}{4}\|tx\|^2} dt \\
&= \frac{4}{K\|x\|} \int_0^1 \frac{1}{\left(\frac{2}{\sqrt{K}\|x\|}\right)^2 + t^2} dt \\
&= \frac{4}{K\|x\|} \frac{\sqrt{K}\|x\|}{2} \arctan\left(\frac{\sqrt{K}\|x\|t}{2}\right)\Bigg|_0^1 \\
&= \frac{2}{\sqrt{K}} \arctan\left(\frac{\sqrt{K}\|x\|}{2}\right) \\
&< \frac{2}{\sqrt{K}} \frac{\pi}{2} \\
&= \frac{\pi}{\sqrt{K}}.
\end{aligned}
$$

Now consider the harmonic series $(x_n) = \left(\sum_{i=1}^{n} \frac{1}{n}, 0, \ldots, 0\right)$. The series is Cauchy because the distance between any element and origin has the same upper bound, and the sequence is monotone increasing in the first entry so as a consequence of Monotone Convergence Theorem, the distance between any two elements x_n, x_m must be less than any given ε when n, m is large enough. However, the series diverges to ∞ so Cauchy sequence does not converge in $\mathbb{R}^n$, and thus the metric is not complete.

$\square$