Homework 4

Jaden Wang

Problem (1)**.** Suppose $|G| = 24 = 2^3 \cdot 3$. If G is abelian, then it is a product of its Sylow subgroups which are normal in G so G cannot be simple. Suppose G is non-abelian. The only divisor n where $\gcd(n, 24/n) = 1$ are $n = 1, 3, 8, 24$. Since Sylow 2-subgroups have order 8, Sylow 3-subgroups has order 3, and $\{e\}$ and G exist, by Theorem on P105, G is solvable (or use Burnside's Theorem). Then by definition there exists a chain for G:

$$\{e\} \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{s-1} \trianglelefteq G_s = G$$

where G_{i+1}/G_i is abelian. Since G is not abelian but G/G_{s-1} is abelian, $G_{s-1} \neq \{e\}$ so we have found a normal subgroup G_{s-1} of G.

Alternatively : By Sylow we see that $n_2 = 1, 3$ and $n_3 = 1, 4$. Suppose $n_2 = 3$ and $n_3 = 4$. G acts on $P := Syl_2(G)$ by conjugation. This yields a homomorphism $\phi : G \to S_3$. Notice that ϕ is not the trivial action as the action permutes three Sylow 2-subgroups. Hence $|\operatorname{im} \phi| > 1$ and $|\ker \phi| = |G|/|\operatorname{im} \phi| \geq 24/6 = 4$ but $|\ker \phi| = |G|/\operatorname{im} \phi < |G|$. Therefore, $\ker \phi$ is a non-trivial proper normal subgroup of G so G is not simple.

Problem (2)**.**

(a) The ($\Leftarrow$) direction is immediate as cyclic groups are abelian. ($\Rightarrow$) : suppose G is finite and solvable. Then by the FToFGAB, any $G_{i+1}/G_i \cong Z_{p_1}^{\alpha_1} \times \cdots \times Z_{p_k}^{\alpha_k}$. We know $Z_{p_1}^{\alpha_1 - 1} \times \cdots \times Z_{p_k}^{\alpha_k} \trianglelefteq G_{i+1}/G_i$, so by the 4th isomorphism theorem, we can pull back this normal subgroup to a normal subgroup $G_i' \trianglelefteq G_{i+1}$ with index p_1. Therefore $G_{i+1}/G_i' \cong Z_{p_1}$. Continue this process inductively on G_i'/G_i until the quotient is cyclic and do this for all i. This yields a composition series with all cyclic quotients.

(b) First we prove a lemma.

> **Lemma 0.1**
>
> A group G is solvable iff its derived series terminates.

Proof. Denote the commutator subgroup of G as $G^{(1)}$, whose commutator subgroup is denoted as $G^{(2)}$, and so on.

$(\Rightarrow)$: Suppose G is solvable, with the series

$$\{e\} \trianglelefteq N_{s-1} \trianglelefteq \cdots \trianglelefteq N_1 \trianglelefteq N_0 = G.$$

Since G/N_1 is abelian, and $G^{(1)}$ is the smallest normal subgroup that yields abelian quotient, $G^{(1)} \leq N_1$ so we establish the base case. Assume $G^{(i-1)} \leq N_{i-1}$. Notice $G^{(i)} = [G^{(i-1)}, G^{(i-1)}] \leq [N_{i-1}, N_{i-1}]$. Since N_{i-1}/N_i is abelian, $[N_{i-1}, N_{i-1}] \leq N_i$ so $G^{(i)} \leq N_i$ and we complete the induction.

$(\Leftarrow)$: Suppose the derived series of G terminates. Since $G^{(i)} \trianglelefteq G^{(i-1)}$ and $G^{(i-1)}/G^{(i)}$ is abelian by properties of commutator subgroups, the derived series satisfy the definition of solvable group. $\square$

With this equivalent but more concrete definition of solvable groups, we can give elegant proofs to this part.

Suppose G is solvable and let $H \leq G$. We wish to show that the derived series of H terminates. First notice that intersecting the derived series of G with H factorwise yields a terminating series

$$\{e\} \trianglelefteq H \cap G^{(s-1)} \trianglelefteq \cdots \trianglelefteq H \cap G^{(1)} \trianglelefteq H$$

Moreover, we see that for every $h \in H^{(i)}$, clearly $h \in H$ and $h \in G^{(i)}$, so $H^{(i)} \leq H \cap G^{(i)}$. Therefore, we establish factorwise containment of the derived series of H with the terminating series above. This shows that the derived series of H also terminates so H is solvable.

Proposition 0.2

Let G be a solvable group and $\phi : G \to H$ be a group homomorphism, then $\ker \phi$ and $\operatorname{im} \phi$ are solvable.

Proof. Denote $N := \ker \phi$ and $K := \operatorname{im} \phi \cong G/N$. Since $N \trianglelefteq G$, it is solvable by argument above. Since ϕ is a homomorphism, it maps generators $[g_1, g_2]$ of $G^{(1)}$ to $[\phi(g_1), \phi(g_2)]$ which are generators of $K^{(1)}$. Since ϕ is surjective onto its image K, all generators of $K^{(1)}$ can be hit this way so we have the base case $K^{(1)} \leq \phi(G^{(1)})$. Assume $K^{(i-1)} \leq \phi(G^{(i-1)})$. Then

$$K^{(i)} = [K^{(i-1)}, K^{(i-1)}] \leq [\phi(G^{(i-1)}), \phi(G^{(i-1)})] = \phi([G^{(i-1)}, G^{(i-1)}]) = \phi(G^{(i)}).$$

Hence we complete the induction. Since the factorwise image of derived series of G terminates, so does the derived series of K. Hence K is solvable. □

It follows immediately that for any $N \trianglelefteq G$, G/N is solvable.

Problem (3)**.** (collab with Will and Ari) Suppose $|G| = 200 = 2^3 \cdot 5^2$. By Sylow, $n_5|8$ and $n_5 = 1 \bmod 5$ so $n_5 = 1$. That is, Sylow 5-subgroup P_5 is normal. Moreover, P_5 has order 25 so it is a p-group and therefore solvable. Consider G/P_5 which has order 8. It is also a p-group so it is solvable. By Theorem, G is therefore solvable.

Problem (4)**.** $(\Rightarrow)$: Suppose every group of odd order is solvable. Then any simple group of odd order G must be abelian since the only possible solvable series is $\{e\} \trianglelefteq G$ by simplicity so $G/\{e\}$ is abelian. Since G is finite, by the FToFGAB, $G \cong Z_{p_1}^{\alpha_1} \times \cdots \times Z_{p_k}^{\alpha_k}$. Since G is simple, it cannot have more than one product and it cannot be trivial so $G \cong Z_{p_j}$, where p_j is a prime. Then $|G| = p_j$ which forces p_j to be odd.

$(\Leftarrow)$: Suppose that every simple group of odd order is Z_p for some odd prime p. Then given a group G of odd order and its decomposition series, the quotients G_{i+1}/G_i is simple and must have odd order by Lagrange, so $G_{i+1}/G_i \cong Z_{p_i}$ by assumption, which is abelian. Hence G is solvable.

Problem (5)**.**

(a) First, any $hk \in HK$ can be expressed as $hkh^{-1}h = (hkh^{-1})h \in KH$ since $K \trianglelefteq G$. Likewise any $kh = hh^{-1}kh \in HK$. So $KH = HK$. HK is clearly non-empty. Given $h_1k_1, h_2k_2 \in HK$, we see that $h_1k_1h_2k_2 = (h_1h_2)(h_2^{-1}k_1h_2k_2) \in HK$ since $h_2^{-1}k_1h_2 \in K$ by normality so HK is closed under operation. Clearly $(hk)^{-1} = k^{-1}h^{-1} \in KH = HK$ so it is closed under inverses. Hence $KH = HK \leq G$.

This part claims that H acts on K by left conjugation. Given $h_1, h_2 \in H$, we see that

$$\gamma(h_1h_2)(k) = \gamma_{h_1h_2}(k)$$

$$
\begin{aligned}
&= h_2^{-1} h_1^{-1} k h_1 h_2 \\
&= h_2^{-1} \gamma_{h_1}(k) h_2 \\
&= \gamma_{h_2} \circ \gamma_{h_1}(k) \\
&= \phi(h_1) \circ \phi(h_2)(k)
\end{aligned}
$$

So γ is a homomorphism.

(b) Consider the map $\phi : HK \to H \times K, hk \mapsto (h, hkh^{-1})$. Then

$$
\begin{aligned}
\phi(h_1 k_1 h_2 k_2) &= \phi(h_1 h_2 h_2^{-1} k_1 h_2 k_2) \\
&= \phi((h_1 h_2)(h_2^{-1} k_1 h_2 k_2) \\
&= (h_1 h_2, h_2^{-1} k_1 h_2 k_2) && h_2^{-1} k_1 h_2 \in K \\
&= (h_1, k_1) * (h_2, k_2) \\
&= \phi(h_1 k_1) \phi(h_2 k_2)
\end{aligned}
$$

shows that ϕ is a homomorphism. Surjectivity is clear. Clearly if $(h, k) = 0$, then $h, k = 0$ so $hk = 0$. Hence ϕ is injective and therefore an isomorphism. That is, $HK = G \cong H \times K$.

(c) Closure under operation is clear from definition of the operation.

$$
\begin{aligned}
((x_1, y_1) * (x_2, y_2)) * (x_3, y_3) &= (x_1 x_2, \phi_{x_2^{-1}}(y_1) y_2) * (x_3, y_3) \\
&= (x_1 x_2 x_3, \phi_{x_3^{-1}}(\phi_{x_2^{-1}}(y_1) y_2) y_3) \\
&= (x_1 x_2 x_3, (\phi_{x_3^{-1}} \circ \phi_{x_2^{-1}})(y_1) \phi_{x_3^{-1}}(y_2) y_3) \\
&= (x_1 x_2 x_3, (\phi_{(x_2 x_3)^{-1}})(y_1) \phi_{x_3^{-1}}(y_2) y_3) \\
&= (x_1, y_1) * (x_2 x_3, \phi_{x_3^{-1}}(y_2) y_3) \\
&= (x_1, y_1) * ((x_2, y_2) * (x_3, y_3))
\end{aligned}
$$

I claim that (e_H, e_K) is the identity. Not surprisingly, $(h, k) * (e_H, e_K) = (he_H, \phi_{e_H^{-1}}(k) e_K) = (h, \phi_{e_H}(k)) = (h, k)$. The other direction is similar. Finally, $(h^{-1}, \phi_h(k^{-1}))$ is the inverse of (h, k):

$$
(h, k)(h^{-1}, \phi_h(k^{-1})) = (hh^{-1}, \phi_h(k) \phi_h(k^{-1})
$$

$$= (e_H, \phi_h(kk^{-1}))$$
$$= (e_H, \phi_h(e_K))$$
$$= (e_H, e_K)$$

And the other direction is similar. Hence G is a group.

(d) Given $(h,k) \in G, (e,k') \in K$, notice

$$\begin{aligned}(h,k) * (e,k') * (h,k)^{-1} &= (h,kk') * (h^{-1}, \phi_h(k^{-1})) \\ &= (e, \phi_h(kk'k^{-1}) \in K\end{aligned}$$

So $K \trianglelefteq G$. Clearly $H \cap K = (e,e)$. Then by counting, $|HK| = |H||K|/|H \cap K| = |H||K| = |H \times K|$ so $G = HK$.

Problem (6)**.** Clearly $G \cong G \times \{e\} \leq H$. Also $G \times Z_2 \cong Z_2 \times G$. Denote e_i to be 1 in the ith entry and zero elsewhere. Then clearly e_i for all $i \in \mathbb{N}_+$ are the generators of G and H. Define $\phi : Z_2 \times G \to G, (1,0,0,\ldots)) \mapsto (2,0,0,\ldots)$ and then $e_i \mapsto e_{i+1}$ for $i > 1$. Then ϕ is clearly a homomorphism. It is also injective as none of the generators is mapped to 0 so the kernel is trivial. Hence $H \cong \phi(H) \leq G$. However, H and G are not isomorphic because G has no generators of order 2, but H does. This is a structural difference.