1 Locally trivial fibrations

Definition 1.1 — A **fiber bundle** (or a **locally trivial fibration** or a **twisted product** or a **fibration**) is a 4-tuple (E, B, F, p) where E, B, F are topological spaces and $p : E \to B$ continuous s.t. $\forall\, x \in B$, there exists an open set $U \subseteq B$ containing x and a homeomorphism

$$\phi : p^{-1}(U) \to U \times F$$

s.t. $\pi_1 \circ \phi = p$ where π_1 is projection onto first factor. Here E is called the **total space**, B is **base space**, F is the **fiber**, p is the **projection**, ϕ is a **local trivialization**.

Example 1.2

(1) $E = B \times F$.

(2) Mobius band: $M = \mathbb{R}^2/(x, y) \sim (x+1, -y)$. Let $q : \mathbb{R}^2 \to M$ be the quotient map.

(3) S^{2n-1} be the unit sphere in $\mathbb{C}^n$. Recall S^1 the unit circle in $\mathbb{C}$ acts on $\mathbb{C}^{2n-1}$ by

$$S^1 \times S^{2n-1} \to S^{2n-1}, (\lambda(z_1, \dots, z_n)) \mapsto (\lambda z_1, \dots, \lambda z_n).$$

Exercise: $S^{2n-1}/S^1 \cong \mathbb{C}P^{n-1}$.

Exercise: show $(S^{2n-1}, \mathbb{C}P^{n-1}, S^1, p)$ is a fiber bundle.

(4) If G is a Lie group, and H a compact subgroup of G, then $(G, G/H, H, p)$ is a fiber bundle where p is canonical projection. Exercise.

Example 1.3 (1) $O(n) = \{A \in GL(n, \mathbb{R}) : \langle Av, Aw\rangle = \langle v, w\rangle\} = \{A \in GL(n, \mathbb{R}) : A^T = A^{-1}\}$. And $SO(n) = \{A \in O(n) : \det A = 1\}$. Recall from diff top that they are smooth manifold of dimension $n(n-1)/2$. $O(n)$ has two components and $SO(n)$ is the component containing the identity. Exercise: $SO(1) = \{1\}$. $SO(2) \cong S^1$. $SO(3) \cong \mathbb{R}P^3$.

Notice $SO(n) \leq SO(n+1)$. Exercise: prove that $SO(n+1)/SO(n) \cong S^n$. Hint:

note the 1st column of $B \in SO(n+1)$ is a unit vector in $\mathbb{R}^{n+1}$.

(2) Let $V_{n,k}$ be orthonormal k-frames (ordered k vectors) in $\mathbb{R}^n$. Exercise: Steifel manifold $V_{n,k} = O(n)/O(n-k)$. So $V_{n,n} \cong O(n)$. $V_{n,1} = S^{n-1}$. $V_{n,n-1} \cong SO(n)$. Exercise: if $k < n$, then $V_{n,k} \cong SO(n)/SO(n-k)$.

(3) $G_{n,k}$ is the k-dimensional subspaces in $\mathbb{R}^n$. Exercise: $G_{n,k} = O(n)/O(n-k) \times O(k)$.

(4) Recall the unitary group $U(n) = \{A \in GL_{n,\mathbb{C}} : \langle Av, Au\rangle = \langle v, u\rangle\}$ where $\langle v, u\rangle = \overline{v} \cdot u$. Alternatively, $U(n) = \{A \in GL(n, \mathbb{C}) : \overline{A}^T = A^{-1}\}$. The special unitary group is $SU(n) = \{A \in U(n) : \det A = 1\}$. From diff top, these are manifolds and $\dim U(n) = n^2$ and $\dim SU(n) = n^2 - 1$. Exercise: $U(n)/SU(n) \cong S^1$. Exercise: $U(1) \cong S^1$, $SU(2) \cong S^3$, $U(2) \cong S^3 \times S^1$. $SU(n+1)/SU(n) \cong S^{2n+1}$.

(5) $V_{n,k}(\mathbb{C}) \cong U(n)/U(n-k)$.

(6) $G_{n,k}(\mathbb{C}) \cong= U(n)/U(k) \times U(n-k)$.

(7) If $f : M \to N$ a smooth map, s.t.

   (i) f is surjective

   (ii) f is a submersion

   (iii) f is proper *i.e.* preimage of compact set is compact.

   Then $f^{-1}(p)$ where p is any point is a fiber bundle. This is Ehresmann's Theorem.

(8) vector bundles are fiber bundles with fiber $\mathbb{R}^k$ or $\mathbb{C}^k$ with extra structure on the fibers. This includes the tangent bundles, cotangent bundles, normal bundles.

(9) covering space is a bundle with discrete fiber.

Definition 1.4 — Given a fiber bundle $E \xrightarrow{p} B$ and a map $f : A \to B$, the **pull-back** of E to A is

$$f^*(E) = \{(a, e) \in A \times E : f(a) = p(e)\}.$$

$$p : f^*E \to A : (a, e) \mapsto a.$$

Exercise:

(1) Show $f^*E \to A$ is a fiber bundle with the same fiber as $E \to B$.

(2) If A is a subset of B and $f : A \to B$ is inclusion, then show $f^*(E) \cong E|_A$ *i.e.* $E|_A = p^{-1}(A)$.

(3) $\widetilde{f} : f^*E \to E, (a, e) \mapsto e$ is a bundle map so the diagram commutes.

(4) If $f : A \to B$ is constant and fiber of E is F, then $f^*E \cong A \times F$.

(5) If $E = B \times F$ then $f^*E \cong A \times F$.

Definition 1.5 — If $E \xrightarrow{p} B$ are $E' \xrightarrow{p'} B$ are bundles, we say they are **bundle isomorphic** if there exists a homeomorphism $h : E \to E'$ s.t. the diagram commutes. We denote $E \cong E'$.

Theorem 1.6

If $f_i : A \to B$, $i = 0, 1$ are homotopic and A is locally compact and normal (*e.g.* a CW complex), then $f_0^*E \cong f_1^*E$.

Proof. Let $f_i : A \to B$ and homotopy $H : A \times I \to B$. Diagrams. Theorem 2 says there exists a homotopy $\widetilde{H}$:

$H^*E = \{(x, t, e) \in A \times I \times E : H(x, t) = p(e)\}$. Define

$$\overline{H}((x, e), t) = (x, t, \widetilde{H}(x, e, t)).$$

$$\begin{tikzcd} f_0^*E \times I \arrow[r, "\tilde{H}"] \arrow[d, "\pi_1 \times \mathrm{id}_I"'] & E \arrow[d, "p"] \\ A \times I \arrow[r, "H"] & B \end{tikzcd} \qquad \begin{tikzcd} f_0^*E \times I \arrow[r, "\overline{H}"] \arrow[d, "\pi_1 \times \mathrm{id}_I"'] & H^*E \arrow[dl, "\pi_1"] \\ A \times I & \end{tikzcd}$$

Exercise: $\overline{H}$ is a bundle isomorphism.

Restricting $\overline{H}$ to $f_0^*E \times \{1\}$ yields a bundle isomorphism. Notice $\overline{H}(f_0^*E \times \{1\}) = \{(x, 1, e) \in A \times I \times E : H(x, 1) = f_1(x) = p(e)\} = f_1^*E$. Hence $f_0^*E \cong f_1^*E$. $\square$

Theorem 1.7 (covering homotopy property)

Let $p_0 : E \to B$ and $q : Z \to Y$ be fiber bundles with the same fiber. Suppose B is locally compact and normal. Given $\widetilde{h}_0 : E \to Z, h_0 : B \to Y$ s.t. the diagram commutes, and a homotopy $H : B \times I \to Y$ of h_0, then there exists a homotopy $\widetilde{H} : E \times I \to Z$ of bundle maps covering H.

Proof. We assume B is compact (locally compact case is an exercise). Idea: break E into pieces where bundle is trivial $U \times F$. Here the theorem is clear. Then we put the homotopies together.

Let $\{V_\beta\}$ be a cover of Y by locally trivializing charts so we have an isomorphism

$$q^{-1}(V_\beta) \xrightarrow{\phi_\beta} V_\beta \times F$$

$\{H^{-1}(V_\beta)\}$ is an open cover of $B \times I$ since $B \times I$ is compact, we have a finite subcover $\{U_\alpha \times I_j\}$ covering $B \times I$ s.t. $H(U_\alpha \times I_j) \subseteq V_\beta$ for some β. Note: H^*Z is trivial over $U_\alpha \times I_j$ since Z is trivial over V_β. We can take the I_j to be segements. We will inductively lift H to $\widetilde{H} : E \times [0, t_k] \to Z$. For each $x \in B$ there exists neighborhoods W, W' s.t. $x \in W \subseteq \overline{W} \subseteq W'$ and $\overline{W}' \subseteq U_i$ for some i by normal. There are finite number of $\{W_i, W_i'\}_{i=1}^s$ s.t. $\{W_i\}$ cover B. By Urysohn's lemma, there exist maps $u_i : B \to [t_k, t_{k+1}]$ s.t. $u_i(\overline{W}_i) = t_{k+1}$ and $u_i(B - W_i') = t_k$. Set $\tau_0(x) = t_k \ \forall\ x$ and $\tau_i(x) = \max\{u_1(x), \ldots, u_i(x)\}$. So $t_k = \tau_0(x) \le \tau_1(x) \le \ldots \le \tau_s(x) = t_{k+1}$. Set $B_i = \{(x,t) \in B \times I : t_k \le t \le \tau_i(x)\}$. Let E_i be the part of $E \times I$ above B_i. So $E_0 = E \times \{t_k\} \subseteq E_1 \subseteq \ldots \subseteq E_s = E \times [t_k, t_{k+1}]$. Assume we have $\widetilde{H}$ defined on $E \times [0, t_k]$ so it is defined on $E \times \{t_k\} = E_0$. We inductively extend $\widetilde{H}$ over E_i. Note if $(x,t) \in B_i - B_{i-1}$, then $\tau_{i-1} < t \le \tau_i(x)$. So $u_i(x) > \tau_{i-1}(x)$. Thus $x(,t) \in W_i' \times \{t_k, t_{k+1}\}$. By definition $W_i' \times [t_k, t_{k+1}] \subseteq U_\alpha \times I_j$. So $H(B_i - B_{i-1}) \subseteq V_\beta$ for some β and $q^{-1}(V_\beta) \xrightarrow{\phi_\beta} V_\beta \times F$. Let $\rho_\beta : q^{-1}(V_\beta) \to F$ be ϕ_β composed with projection. For $(e,t) \in E_i - E_{i-1}$, let $p(e) = x \in B$. Set $\widetilde{H}(e,t) = \phi_\beta^{-1}(H(x,t), \rho_\beta(\widetilde{H}(e, \tau_{i-1}(x))))$. Exercise: show this extends $\widetilde{H}$ over E_i. $\square$

Corollary 1.8

If X is contractible and locally compact and normal, then any fiber bundle over X is trivial, *i.e.* $E \cong X \times F$.

Proof. X contractible means the identity map f_0 is homotopy to the constant map f_1. Therefore, $f_0^* E \cong E \cong f_1^* E \cong X \times F$. □