Homework 12

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Problem (4.2.1)**.** First note that since T is alternating, if any two of the entries coincide *i.e.* $v_i = v_j$, then

$$\begin{aligned} T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) &= -T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n) && \text{alternating} \\ &= -T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) && v_i = v_j \\ &= 0 && x = -x \Rightarrow x = 0 \end{aligned}$$

by the definition of alternating. Since $v_1, \ldots, v_n$ are linearly dependent, there exists $a_1, \ldots, a_n \in \mathbb{R}$ not all zeros (WLOG $a_1 \neq 0$) such that $a_1 v_1 + \cdots + a_n v_n = 0$. This yields $v_1 = \frac{a_2}{a_1} v_2 + \cdots + \frac{a_n}{a_1} v_n$. Thus we have

$$\begin{aligned} T(v_1, \ldots, v_n) &= T\left(\frac{a_2}{a_1} v_2 + \cdots + \frac{a_n}{a_1} v_n, v_2, \ldots, v_n\right) \\ &= \frac{a_2}{a_1} T\left(v_2, v_2, \ldots, v_n\right) + \cdots + \frac{a_n}{a_1} T\left(v_n, v_2, \ldots, v_n\right) \\ &= 0 + \cdots + 0 = 0 \end{aligned}$$

Problem (4.2.3)**.** If ϕ_i are linearly dependent, then the dependence relation would make the matrix $[\phi_i(v_j)]$ having linearly dependent columns, so det is 0, matching the result of Exercise 2. If ϕ_i are linear independent. First consider the standard dual basis $x_1, \ldots, x_k$ where $x_i(e_j) = \delta_{ij}$. Let $M = (v_1, v_2, .., v_n)$. It is easy to see that $M = [x_i(v_j)]$. Thus we have

$$\begin{aligned} x_1 \wedge \cdots \wedge x_k(M) &= M^*(x_1 \wedge \cdots \wedge x_k(I)) \\ &= \det M \operatorname{Alt}(x_1 \otimes \cdots \otimes x_k)(I) \\ &= \det M \left(\frac{1}{k!} \sum_{\sigma \in S_k} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}\right)(I) \\ &= \frac{1}{k!} \det M (x_1 \otimes \cdots \otimes x_k)(I) \\ &= \frac{1}{k!} \det M (1 \cdots 1) \\ &= \frac{1}{k!} \det [x_i(e_j)] \end{aligned}$$

Now we check that $\det[\phi_i]$ is an alternating tensor. We have that det is multilinear in the columns, and the matrix where each entry ϕ_i is linear is clearly multilinear in all the columns. Thus the composition $\det[\phi_i]$ is multilinear. Moreover, det is alternating; since swapping v_j, v_k leads to swapping of $\phi_i(v_j)$ and $\phi_i(v_k)$, we obtain the negative of the original determinant so $\det[\phi_i]$ is alternating. Hence $\det[\phi_i] \in \Lambda^k(\mathbb{R}^{k*})$. Since its dimension is one, $\phi_1 \wedge \cdots \wedge \phi_k(v_1, \ldots, v_k) = \lambda \det[\phi_i(v_j)]$. Since $\phi_i = a_1^i x_1 + a_n^i x_n$, define $w_j = \frac{1}{a_1^j} e_1 + \cdots \frac{1}{a_n^j} e_n$ wherever $a_k^j \neq 0$. Then it is easy to see that $\phi_i(w_j) = \delta_{ij}$. By the same argument as in the dual basis case, $\lambda = \frac{1}{k!}$, it follows that

$$\phi_1 \wedge \cdots \wedge \phi_k(v_1, \ldots, v_k) = \frac{1}{k!} \det[\phi_i(v_j)]$$

Problem (4.2.5)**.**

$$\begin{aligned}\mathrm{Alt}(\phi_1 \otimes \phi_2 \otimes \phi_3) = \frac{1}{6}(&\phi_1 \otimes \phi_2 \otimes \phi_3 - \phi_1 \otimes \phi_3 \otimes \phi_2 + \phi_3 \otimes \phi_1 \otimes \phi_2 \\ &- \phi_3 \otimes \phi_2 \otimes \phi_1 + \phi_2 \otimes \phi_3 \otimes \phi_1 - \phi_2 \otimes \phi_1 \otimes \phi_3)\end{aligned}$$

Problem (4.2.6)**.** (a) Two ordered bases are equivalently oriented iff the linear map defined by mapping between them has positive determinant. Define $A : V \to V, v_i \mapsto v_i'$. Then notice

$$\begin{aligned}T(v_1', \ldots, v_n') &= T(Av_1, \ldots, Av_n) \\ &= A^* T(v_1, \ldots, v_n) \\ &= \det A T(v_1, \ldots, v_n)\end{aligned}$$

It follows that $\det A > 0 \Leftrightarrow T(v_1, \ldots, v_n)$ and $T(v_1', \ldots, v_n')$ have the same sign.

(b) By part a), the sign of T is well-defined independent of the choice of positively oriented basis.

(c) We define the orientation of V by the orientation of its ordered basis. An ordered basis $\{v_1, \ldots, v_n\}$ is positively oriented if the sign of $T(v_1, \ldots, v_n)$ is positive. This is again well-defined by part a). Given an orientation on $\Lambda^n(V^*)$, any two alternating tensors are scalar multiples of each other, so their sign difference will also be passed to the orientation of the vectors $v_1, \ldots, v_n$, making it well-defined.

Problem (4.2.7)**.** Define $D(A) := \det A^T$ so $D \in \Lambda^k(\mathbb{R}^{k*})$. We wish to show that D is multilinear, alternating, and $D(I) = 1$. Since det is multilinear in the row, we see that D is multilinear in the columns. Let P be the permutation matrix that swap i, jth row of A. Then

$$\begin{aligned}
D(PA) &= \det(PA)^T \\
&= \det A^T \det P^T \\
&= \det A^T(-1) && \text{swap row viewed as swap col} \\
&= -D(A)
\end{aligned}$$

So D is alternating. Finally $D(I) = \det(I^T) = \det I = 1$. Hence by uniqueness of det, $D = \det$ so $D(A) = \det A^T = \det A$.

Problem (P165 Exercise)**.** Since $\phi : Y \to \mathbb{R}$, $f^*\phi = \phi \circ f$. Therefore,

$$\begin{aligned}
f^*(d\phi) &= d\phi \circ df \\
&= d(\phi \circ f) && \text{chain rule} \\
&= d(f^*\phi)
\end{aligned}$$