Homework 12

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Problem (7.1)**.** If M, N are Riemannian manifolds such that the inclusion $i : M \subset N$ is an isometric immersion, show by an example that the strict inequality $d_M(x, y) > d_N(x, y)$ can occur.

Proof. Let $N = \mathbb{R}^2$ with the Euclidean metric and $M = \mathbb{R}^2 \setminus D^2$ where D^2 is the closed unit disk centered at the origin. Then the inclusion is clearly an isometric immersion. Notice $d_M((-1.1, 0), (1.1, 0)) > \pi$ since any path has to go around half of the unit disk. But $d_N((-1.1, 0), (1.1, 0)) = 2.2 < \pi$. □

Problem (7.2)**.** Let $\widetilde{M}$ be a covering space of a Riemannian manifold M. Show that it is possible to give $\widetilde{M}$ a Riemannian structure such that the covering map $\pi : \widetilde{M} \to M$ is a local isometry. Show that $\widetilde{M}$ is complete in the covering metric if and only if M is complete.

Proof. Let g be the metric on M. Define the pullback metric $\pi^* g_{\widetilde{p}} = g_{\pi(p)}$ for any $\widetilde{p} \in \widetilde{M}$. Since π is already a local diffeomorphism, by definition of pullback metric it is a local isometry. Recall that in a covering space, any path downstairs can be lifted to a unique path upstairs once we choose an initial point, and any path upstairs can be projected to a unique path downstairs. Since geodesics are defined using local notions (covariant derivative), and since the covering map is a local isometry, any geodesic downstairs can be lifted to a unique geodesic upstairs with chosen initial point and vice versa. Thus if M is complete, for any point $\widetilde{p} \in \widetilde{M}$ we can take any geodesic $\widetilde{\gamma}$ emanating from $\widetilde{p}$, project it to a geodesic in M and extend it for all time and then lift it back to the same starting point $\widetilde{p}$, which by uniqueness guarantees to coincide with the original geodesic $\widetilde{\gamma}$ at original time interval. The reverse direction is pretty much identical. By Hopf–Rinow, geodesic completeness and completeness are equivalent so we are done with the proof. □

Problem (7.9)**.** Consider the upper half-plane with the metric $g_{11} = 1, g_{12} = 0, g_{22} = \frac{1}{y}$. Show that the length of the vertical segment $x = 0$, $\varepsilon \leq y \leq 1$ with $\varepsilon > 0$ tends to 2 as $\varepsilon \to 0$. Conclude from this that such a metric is not complete.

Proof. Let $\gamma(t) = (0, t)$ so $\gamma'(t) = (0, 1)$. Then the desired length is

$$\begin{aligned}\int_{\varepsilon}^{1} \sqrt{g_{(0,t)}((0,1),(0,1))}\, dt &= \int_{\varepsilon}^{1} t^{-\frac{1}{2}}\, dt \\ &= 2t^{\frac{1}{2}}\Big|_{\varepsilon}^{1} \\ &= 2 - 2\sqrt{\varepsilon}.\end{aligned}$$

Clearly as $\varepsilon \to 0$, the length tends to 2. Now we wish to prove the negation of statement (e) in Hopf–Rinow: for all sequences of compact subsets $K_n \subset M$, $K_n \subset K_{n+1}$ and $\bigcup_n K_n = M$, there exists a $q_n \notin K_n$ such that $\lim\limits_{n\to\infty} d(p, q_n) < \infty$.

Given any such sequence (K_n), and WLOG assume the intersection between K_n and the y-axis is nonempty since eventually K_n has to tend to the entire $\mathbb{R}^2_+$ and we only care about tail behavior. Since K_n is compact and the projection function is continuous, there exists a point $(0, y_n) \in K_n$ with a minimum y-value $y_n > 0$ in K_n. Define $q_n = \left(0, \frac{y_n}{2}\right)$. By minimality of y_n, we have $q_n \notin K_n$ and $q_n \to (0, 0)$. However, $d((0, 1), q_n) \to 2 < \infty$, proving the negation. □

Problem (7.10)**.** Prove that the upper half-pane $\mathbb{R}^2_+$ with the Lobatchevski metric $g_{11} = g_{22} = \frac{1}{y^2}$, $g_{12} = 0$ is complete.

Proof. From Example 3.10 of Chapter 3, we know that its geodesics are vertical rays from the x-axis and semicircles with center on the x-axis. Given any $p = (x, y) \in \mathbb{R}^2_+$ and a semicircle or ray emanating from p, clearly it can always be extended upward as there is no boundary or holes when y-value increases. But as y-value decreases, we just need to check that it doesn't reach the boundary x-axis in finite time. In the case of a ray, as before we can integrate its length which is $\ln y - \ln \varepsilon$. As $\varepsilon \to 0$, the length goes to infinity. In the case of a semicircle, we can parameterize it as $\gamma(t) = (y \cos t + x, y \sin t)$ with radius y and center at $(x, 0)$.

$$\begin{aligned}L(\gamma) &= \int_{\varepsilon}^{\frac{\pi}{2}} \frac{1}{y \sin t} \|(-y \sin t, y \cos t)\| dt \\ &= \int_{\varepsilon}^{\frac{\pi}{2}} \frac{dt}{\sin t} \\ &= \ln \tan \frac{t}{2}\Big|_{\varepsilon}^{\frac{\pi}{2}}\end{aligned}$$

$$= \ln \tan \frac{\pi}{4} - \ln \varepsilon.$$

Again the arc length approaches ∞ as $\varepsilon \to 0$. Since the geodesic moves at constant speed, it cannot cover arbitrarily large arc length in finite time. That it, it can be extended to all time. By Hopf–Rinow, the Lobatchevsky plane is complete. □

Problem (7.12)**.** A Riemannian manifold is *homogeneous* if given $p, q \in M$ there exists an isometry of M which takes p into q. Prove that any homogeneous manifold is complete.

Proof. Let $B \subset T_pM$ be the largest closed ball with radius $r < \infty$ centered at 0 where $\exp_p$ is still defined. Take any $v \in \partial B$ so it has length r. Define $q = \exp_p(v)$ which is also $\gamma(1)$ where γ is the geodesic emanating from p with $\dot{\gamma}(0) = v$. Let $w = \dot{\gamma}(1)$ which also has length r since geodesic has constant speed (parallel translation is an isometry). Let f be the isometry that takes p to q and define $u = df_q^{-1}(w)$ which again has length r and therefore $u \in B$. Let $c(t)$ be the geodesic emanating from p with $\dot{c}(0) = u$ and we have $c(1) = \exp_p(u)$. Since isometry maps geodesics to geodesics, we see that $f \circ c$ becomes a geodesic emanating from q with initial velocity w. By local uniqueness of geodesic, we can concatenate $\gamma(t)$ and $f \circ c(t)$ at q into a single geodesic. But notice that $\exp_p(2v) = f \circ c(1)$. Since v is on the boundary of B and is arbitrary, it is a contradiction that $2v \in B$. Thus, we must have $r = \infty$. Thus $\exp_p$ is defined for all of T_pM and therefore by Hopf-Rinow, the manifold is complete. □