Homework 5

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Problem (9.1)**.** Suppose df_p has rank k, this means that df_p maps T_pM to a k-dimensional subspace of T_pN. Then the matrix representation of df_p would have k-pivots after row reduction so the $k \times k$ upper left submatrix has nonzero determinant. So take an open interval I between this number and 0.

Let (U, ϕ) be a chart around p and (V, ψ) be a chart around $f(p)$. Then $F := \psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$. Then $dF_p : T_p\mathbb{R}^m \to T_p\mathbb{R}^n$ is just the derivative of F at p and can be identified as an element of $L(\mathbb{R}^m, \mathbb{R}^n)$. It's easy to see that $\operatorname{rank} dF_p = \operatorname{rank} df_p = k$. Since f, ϕ^{-1}, ψ are smooth, F is smooth and so are all of its derivatives. In particular, $F' : \mathbb{R}^m \to L(\mathbb{R}^m, \mathbb{R}^n), p \mapsto dF_p$ is continuous. Then by identifying dF_p with a matrix representation where the first k-columns are linearly independent (since it has rank k), we can take the determinant of its upper $k \times k$ minor. Denote this function by $\det_k$. Now we have a chain of composing continuous functions, denoted by $g : U \to \mathbb{R}$:

$$M \supseteq U \xrightarrow{\phi} \mathbb{R}^m \xrightarrow{F'} L(\mathbb{R}^m, \mathbb{R}^n) \xrightarrow{\det_k} \mathbb{R}.$$

Since dF_p has rank k, $\det_k(dF_p) = a \neq 0$. WLOG suppose $a > 0$ and let $I := (0, a)$. Therefore, $g^{-1}(I)$ is open in U. That is, there exists an open set $V \subseteq M$ s.t. $V \cap U = g^{-1}(I)$. Since intersection of open sets is open, $g^{-1}(I)$ is open in M. By the way we define g, we see that the differential has rank k everywhere in $g^{-1}(I)$, as desired.

Problem (9.3)**.** Since q is a regular value of f, by the regular value theorem, $f^{-1}(q)$ is a submanifold of dimension $\dim M - \dim N = 0$ so it is a set of points. Since N is Hausdorff, $\{q\}$ is closed in N so by continuity of f, $f^{-1}(q)$ is also closed. Since M is compact, the closed subspace $f^{-1}(q)$ must also be compact (Theorem 26.2 of Munkres). Take any $p \in f^{-1}(q)$, since $\operatorname{rank} df_p = \dim N = \dim M$, df_p is a linear isomorphism so f is a local diffeomorphism at p. That is, each p has a neighborhood that is diffeomorphic to a neighborhood of q. We see that each of these neighborhood contains exactly one point (otherwise the diffeomorphism wouldn't be 1-1) and all of them together form a cover of $f^{-1}(q)$. Since $f^{-1}(q)$ is compact, we can find a finite subcover. Since each neighborhood contains exactly one point, $f^{-1}(q)$

must only have finite number of points.

Denote these points as $\{p_i\}_{i=1}^n$ and the respective disjoint neighborhoods as $\{V_i\}_{i=1}^n$. Let $\{U_i\}_{i=1}^n$ be the corresponding diffeomorphic neighborhood of q and set $U := \bigcap_{i=1}^n U_i$, which is still an open neighborhood of q. Now denote $C := M - \bigcup_{i=1}^n V_i$ which is closed. Since M is compact, C is also compact. Then $f(C)$ is compact and therefore closed since N is Hausdorff. Hence $N - C$ is open so $\widetilde{U} := U \cap (N - C)$ is open. Take any $q' \in \widetilde{U}$, we see that $f^{-1}(q')$ must have one point in each V_i by local diffeomorphism and nowhere else outside V_i. Therefore we have $\# f^{-1}(q') = \# f^{-1}(q) = n$.

Problem (9.5)**.** In exercise 8 of Lecture Notes 5, we have already shown that $dP_z(w) = \theta_z^{-1}(P'(z)\theta_z(w))$ where $P'(z)$ can be expressed as the Jacobian of P at z.

$(\Rightarrow)$: if z is a singular point of P, then $\operatorname{rank} dP_z = 0, 1$. But by the Cauchy-Riemann equations, $\det dP_z = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$, which is positive definite. Hence $\det dP_z = 0$ iff $dP_z = 0$ which is equivalent to $P'(z) = 0$ so z is a root of $P'(z)$.

$(\Leftarrow)$: suppose z is a root of $P'(z)$. Then $dP_z(w) = P'(z)w = 0w = 0$ so $\operatorname{rank} dP_z = 0 \neq 2$, so z is a singular point.

Problem (9.6)**.** Since any boundary point p of M cannot be mapped to the interior of H^m, p must have a neighborhood U_p that is mapped homeomorphically by ϕ into the boundary $\mathbb{R}^{m-1} \times \{0\} \cong \mathbb{R}^{m-1}$. This yields a well-defined homeomorphism $\phi|_{U_p} : U_p \to \phi(U_p) \subseteq \mathbb{R}^{m-1}$. That is, ∂M is locally homeomorphic to an open subset of $\mathbb{R}^{m-1}$. Since ∂M as a subspace of M inherits Hausdorff and second-countable, we show that ∂M is a $(m-1)$-manifold. Since no point in ∂M is locally homeomorphic to the boundary of H^{m-1}, we see that ∂M has no boundary point and therefore no boundary.

Problem (9.9)**.** Notice that any half-curve $\gamma : [0, \varepsilon) \to M$ or $\gamma : (-\varepsilon, 0]$ with $\gamma(0) = p$ can be smoothly extended to a curve $\tilde{\gamma} : (-\varepsilon, \varepsilon) \to M$ with $\tilde{\gamma}(0) = p$ (by just going straight along the tangent vector direction). It's therefore easy to see that equivalent classes of half-curves (elements of T_pH^m) yield the same tangent space at p as the equivalent classes of curves (elements of T_pU), *i.e.* $T_pH^m = T_pU$. Observe $df_p : T_pH^m = T_pU \to T_pM$, $\frac{d}{dt}\Big|_{t=0} p + tv \mapsto$

$\frac{d}{dt}\Big|_{t=0} f(p+tv)$ and $d\widetilde{f}_p : T_pU \to T_pM$, $\frac{d}{dt}\Big|_{t=0} p+tv \mapsto \frac{d}{dt}\Big|_{t=0} \widetilde{f}(p+tv)$. Since we already know that $f = \widetilde{f}$ on $U \cap H^m$ of p, by taking ε small enough s.t. $p+tv \in U \cap H^m$, $\widetilde{f}(p+tv) = f(p+tv)$ $\forall\ t \in [0, \varepsilon)$, so they have the same derivative at p (by extension of half-curve this is well-defined). Thus we have $df_p([v]) = d\widetilde{f}_p([v])$ and they thus equal as functions.

Problem (9.13)**.** First, $f^{-1}(q)$ is a submanifold of $\dim(m-n)$. $df_p : T_pM \to T_pN$ has rank n so its null space has dimension $(m-n)$. Since every point in $f^{-1}(q)$ maps to q, this is a constant map so its derivative is 0, *i.e.* $df_p|_{f^{-1}(q)} = 0$. Thus $T_pf^{-1}(q) \subseteq \ker df_p$. But since they have the same dimension, they must be isomorphic so we achieve equality $T_pf^{-1}(q) = \ker df_p$.