Midterm

Jaden Wang

Problem (1)**.** Let $t > 0$. First, we perform a change of variable so that we integrate from 0 to 1 to match the range of supremum. Let $r = \frac{1}{t}s$, then $dr = \frac{1}{t}ds$. Define $W_r := B_s^{\sqrt{t}} = \frac{1}{\sqrt{t}}B_{ts}$, then we have

$$\begin{aligned}\int_0^t \exp(B_s)\mathrm{d}s &= \int_0^1 \exp(B_{ts})t\mathrm{d}r \\ &= t\int_0^1 \exp\left(\sqrt{t}\frac{1}{\sqrt{t}}B_{ts}\right)\mathrm{d}r \\ &= t\int_0^1 \exp\left(\sqrt{t}W_r\right)\mathrm{d}r.\end{aligned}$$

Denote $\|\cdot\|_{\sqrt{t}}$ to be the $L^{\sqrt{t}}([0,1])$-norm. Then we have

$$\begin{aligned}\frac{1}{\sqrt{t}}\log\left(t\int_0^1 \exp\left(\sqrt{t}W_r\right)\mathrm{d}r\right) &= \frac{1}{\sqrt{t}}\log t + \frac{1}{\sqrt{t}}\log\left(\int_0^1 \exp\left(\sqrt{t}W_r\right)\mathrm{d}r\right) \\ &= \frac{\log t}{\sqrt{t}} + \log\left(\left(\int_0^1 \exp(W_r)^{\sqrt{t}}\mathrm{d}r\right)^{\frac{1}{\sqrt{t}}}\right) \\ &= \frac{\log t}{\sqrt{t}} + \log\|\exp(W_r)\|_{\sqrt{t}}.\end{aligned}$$

As $t \to \infty$, the RHS becomes

$$\begin{aligned}0 + \log\|\exp(W_r)\|_\infty &= \log\sup_{r\in[0,1]}\exp(W_r) \\ &= \sup_{r\in[0,1]} W_r \\ &\sim \sup_{r\in[0,1]} B_r.\end{aligned}$$

where by scaling invariance of BM, W_r has the same distribution as B_r.

Problem (2)**.** To show that (Y_t) does not satisfy the assumptions of the Kolmogorov continuity theorem, it suffices to show that the conclusion does not hold for (Y_t), i.e. (Y_t) does not have a continuous modification. Suppose for a contradiction that (Y_t) has a continuous modification $(\widetilde{Y}_t)$, this means $Y_t = \widetilde{Y}_t$ a.s. $\forall\ t \geq 0$ and $\widetilde{Y}_t(\omega)$ is continuous for any $\omega \in \Omega$. Since $\widetilde{Y}_t$ can a.s. only take on values in the natural numbers, if we choose $\varepsilon = \frac{1}{2}$, then by continuity there exists a $\delta > 0$ such that whenever $|t-s| < \delta$, we have $\widetilde{Y}_t = \widetilde{Y}_s$ a.s. We wish to show that no such δ exists, which would complete the contradiction.

Since (X_k) is a sequence of iid exponentially distributed r.v. with rate $\lambda > 0$, S_n has Gamma distribution $\Gamma(n, \lambda)$. Let $0 \le s < t$, since (Y_t) is a Poisson process, $\mathbb{E}[Y_t] = \lambda t < \infty$. Then we have

$$\begin{aligned}
\mathbb{E}\left[\widetilde{Y}_t - \widetilde{Y}_s\right] &= \mathbb{E}\left[\sum_{n=1}^{\infty}(\mathbb{1}_{\{S_n \le t\}} - \mathbb{1}_{\{S_n \le s\}})\right] \\
&= \mathbb{E}\left[\sum_{n=1}^{\infty}\mathbb{1}_{\{s < S_n \le t\}}\right] \\
&= \sum_{n=1}^{\infty}\mathbb{P}\left(s < S_n \le t\right) && \text{DCT} \\
&> 0,
\end{aligned}$$

where the last inequality comes from that $s < t$ and S_n has density function. This implies that $\mathbb{P}\left(\widetilde{Y}_t = \widetilde{Y}_s\right) < 1$, because if they equal a.s. it would force their expectation difference to be 0. Therefore, no matter how small $\delta > 0$ is, there exists $s < t$ such that $|t - s| < \delta$ and $\mathbb{P}\left(\widetilde{Y}_t = \widetilde{Y}_s\right) < 1$. This concludes the proof.

Problem (3)**.** It suffices to show that the covariance function $\Gamma(s,t) = \mathbb{1}_{s=t}$ is symmetric and has positive type. Symmetry is clear. Given any $c : T \to \mathbb{R}$ with finite support,

$$\sum_{s,t\in T} c(s)c(t)\Gamma(s,t) = \sum_{s,t\in T} c(s)c(t)\mathbb{1}_{s=t} = \sum_{t\in T} c(t)^2 \ge 0.$$

Thus, by the converse theorem, there exists a centered Gaussian process with this covariance function.

For a contradiction, suppose (X_t) has a jointly measurable modification $(\widetilde{X}_t)$, that is, $(\omega, s) \mapsto \widetilde{X}_s(\omega)$ is $\mathcal{F} \otimes \mathcal{B}([0,t])$ for all $t \ge 0$. Define $Y_t = \int_0^t \widetilde{X}_s \mathrm{d}s$. Since $\widetilde{X}_t$ has finite moments, assumptions of Fubini are satisfied, and we have

$$\begin{aligned}
\mathbb{E}\left[Y_t^2\right] &= \mathbb{E}\left[\int_0^t \widetilde{X}_s \mathrm{d}s \int_0^t \widetilde{X}_r \mathrm{d}r\right] \\
&= \int_0^t\int_0^t \mathbb{E}\left[\widetilde{X}_s\widetilde{X}_r\right]\mathrm{d}s\mathrm{d}r && \text{Fubini} \\
&= \int_0^t\int_0^t \mathbb{1}_{s=r}\mathrm{d}s\mathrm{d}r \\
&= \int_0^t 0\mathrm{d}r \\
&= 0.
\end{aligned}$$

This implies that $Y_t^2 = 0$ a.s. which implies $\int_0^t X_s \mathrm{d}s = 0$ a.s. Since this is true for any $t \geq 0$, we must have $X_t = 0$ a.s. That is, $\mathbb{E}\left[\int_0^t X_s^2 \mathrm{d}s\right] = 0$. However, using Fubini we obtain

$$\begin{aligned}\mathbb{E}\left[\int_0^t X_s^2 \mathrm{d}s\right] &= \int_0^t \mathbb{E}\left[X_s^2\right] \mathrm{d}s \\ &= \int_0^t 1 \mathrm{d}s = t,\end{aligned}$$

a contradiction.

Problem (4). (a) We wish to use independent increment to get the bound with t in the exponent. To do that we need to discretize t. Let $n = \lfloor t \rfloor$. Consider $B_k^1 = B_{k+1} - B_k$ so $|B_k^1| \leq |B_{k+1}| + |B_k|$ and $B_k^1 \sim N(0,1)$ is independent of B_k. Then we have

$$\begin{aligned}\mathbb{P}\left(U_1 \geq t\right) &= \mathbb{P}\left(\inf\{t \geq 0 : |B_t| = 1\}\right) \\ &= \mathbb{P}\left(|B_s| < 1 \ \forall \ s \in [0,t)\right) \\ &\leq \mathbb{P}\left(|B_k| < 1 \ \forall \ 1 \leq k \leq n\right) \\ &\leq \mathbb{P}\left(|B_k^1| \leq |B_{k+1}| + |B_k| < 2 \ \forall \ 0 \leq k \leq n-1\right) \\ &= \mathbb{P}\left(|B_k^1| < 2\right)^n && \text{independence} \\ &= \left(\int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x\right)^n.\end{aligned}$$

Since $t \geq 1$ thus $n \geq 1$, we can apply Jensen's inequality to move the exponent to the integrand:

$$\begin{aligned}&\leq \int_{-2}^{2} \frac{1}{\sqrt{2\pi}^n} e^{-\frac{x^2 n}{2}} \mathrm{d}x \\ &< \int_{-2}^{2} \frac{1}{\sqrt{2\pi}^n} e^{-x^2 t} \mathrm{d}x,\end{aligned}$$

as $1 \leq \frac{t}{n} < 2$. Since $e^{-x^2 t}$ is exponential decay in t, the integral is something like the average of exponential decay at different rates, therefore the RHS can be majorized by e^{-ct} for some $c > 0$. Thus, $\mathbb{P}\left(U_1 \geq t\right) \leq e^{-ct}$. Then we have

$$\begin{aligned}\mathbb{E}\left[U_1^p\right] &= \int_0^\infty t^p e^{-ct} \mathrm{d}t \\ &< \infty,\end{aligned}$$

since exponential decay dominates polynomial.

(b) I feel like we could use similar strategy as above. The intuition is that B_t grows at the rate $\sqrt{t}$ so it should outgrows $t^{\frac{1}{3}}$. The tricky thing is that now t would appear in the integration range, making it difficult to find a constant $c > 0$.