# Homework 4

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**Problem** (LN12 0.2.2)**.** Show that the stereograpphic projection $\pi : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$ is a conformal transformation.

To show that $\pi$ preserves angle, we use the Euclidean metric. Recall that by treating $S^2$ as the unit sphere with north pole $N = (0,0,1)$, the stereographic projection is $\pi : (\overline{p}, p_3) \mapsto \frac{\overline{p}}{1-p_3}$, where $\overline{p} = (p_1, p_2)$. Notice for any $p \in S^2$, we have $\|\overline{p}\|^2 = p_1^2 + p_2^2 = 1 - p_3^2$; for any $v \in T_p S^2$, we have $\langle v, p \rangle = 0$ so $\langle \overline{v}, \overline{p} \rangle = -v_3 p_3$. Let $\gamma$ be a curve s.t. $\gamma(0) = p$ and $\gamma'(0) = v$. Then we have

$$
\begin{aligned}
d\pi_p(v) &= (\pi \circ \gamma)'(0) \\
&= \left( \frac{\overline{\gamma}}{1 - \gamma_3} \right)'(0) \\
&= \left( \frac{\overline{\gamma}'(1 - \gamma_3) - \overline{\gamma}(-\gamma_3')}{(1 - \gamma_3)^2} \right)(0) \\
&= \frac{\overline{v}(1 - p_3) + v_3 \overline{p}}{(1 - p_3)^2}.
\end{aligned}
$$

Then we compute

$$
\begin{aligned}
g_p(d\pi_p(v), d\pi_p(w)) &= \langle d\pi_p(v), d\pi_p(w) \rangle \\
&= \frac{\langle \overline{v}, \overline{w} \rangle}{(1-p_3)^2} + \frac{v_3 \langle \overline{w}, \overline{p} \rangle + w_3 \langle \overline{v}, \overline{p} \rangle}{(1-p_3)^3} + \frac{\|\overline{p}\|^2 v_3 w_3}{(1-p_3)^4} \\
&= \frac{\langle \overline{v}, \overline{w} \rangle}{(1-p_3)^2} - \frac{2 v_3 w_3 p_3}{(1-p_3)^3} + \frac{\|\overline{p}\|^2 v_3 w_3}{(1-p_3)^4} \\
&= \frac{\langle \overline{v}, \overline{w} \rangle}{(1-p_3)^2} + \frac{v_3 w_3 (2p_3^2 - 2p_3 + \|\overline{p}\|^2)}{(1-p_3)^4} \\
&= \frac{\langle \overline{v}, \overline{w} \rangle}{(1-p_3)^2} + \frac{v_3 w_3 (p_3^2 - 2p_3 + 1)}{(1-p_3)^4} \\
&= \frac{\langle \overline{v}, \overline{w} \rangle}{(1-p_3)^2} + \frac{v_3 w_3 (p_3 - 1)^2}{(1-p_3)^4} \\
&= \frac{v_1 w_1 + v_2 w_2}{(1-p_3)^2} + \frac{v_3 w_3}{(1-p_3)^2} \\
&= \frac{\langle v, w \rangle}{(1-p_3)^2}
\end{aligned}
$$

$$= \frac{g_p(v,w)}{(1-p_3)^2}.$$

It follows that

$$\begin{aligned}\theta(d\pi_p(v), d\pi_p(w)) &= \arccos\left(\frac{g_p(d\pi_p(v), d\pi_p(w))}{g_p(d\pi_p(v), d\pi_p(v))^{\frac{1}{2}} g_p(d\pi_p(w), d\pi_p(w))^{\frac{1}{2}}}\right) \\ &= \arccos\left(\frac{g_p(v,w)}{g_p(v,v)^{\frac{1}{2}} g_p(w,w)^{\frac{1}{2}}}\right) \\ &= \theta(v,w).\end{aligned}$$

**Problem** (LN12 0.3.2)**.** Compute the metric of the surface given by the graph of a function $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$.

Let $F(x,y) = (x, y, f(x,y))$ be the graph of $f$. We compute

$$\frac{\partial F}{\partial x} = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x} \end{pmatrix}$$

$$\frac{\partial F}{\partial y} = \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y} \end{pmatrix}.$$

Then then pullback metric is

$$G = \begin{pmatrix} 1 + \left(\frac{\partial f}{\partial x}\right)^2 & \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} & 1 + \left(\frac{\partial f}{\partial y}\right)^2 \end{pmatrix}.$$

**Problem** (LN13 0.2)**.** Compute the area of a torus of revolution in $\mathbb{R}^3$.

Let $R$ be the radius of the longitudinal circle and $r$ be that of the meridian circle. Any point on the initial meridian can be parameterized by $(x(\phi), z(\phi)) := (R + r\cos\phi, r\sin\phi)$, where $\phi \in (0, 2\pi)$. Then we can parameterize the torus using

$$f(\theta, \phi) = \begin{pmatrix} \cos\theta x(\phi) \\ \sin\theta x(\phi) \\ z(\phi) \end{pmatrix},$$

where $\theta \in (0, 2\pi)$. We compute

$$\frac{\partial f}{\partial \theta} = \begin{pmatrix} -\sin\theta x \\ \cos\theta x \\ 0 \end{pmatrix}$$

$$\frac{\partial f}{\partial \phi} = \begin{pmatrix} -r\cos\theta\sin\phi \\ -r\sin\theta\sin\phi \\ r\cos\phi \end{pmatrix}.$$

Thus the pullback metric is

$$G(\theta,\phi) = \begin{pmatrix} x(\phi)^2 & 0 \\ 0 & r^2 \end{pmatrix}$$

Then

$$\begin{aligned} A &= \int_U \sqrt{g} \\ &= \int_0^{2\pi}\int_0^{2\pi} \sqrt{\det G}\ d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{2\pi} xr\ d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{2\pi} (R + r\cos\phi)r\ d\phi d\theta \\ &= \int_0^{2\pi} (Rr\phi - r^2\sin\phi)|_0^{2\pi} d\theta \\ &= \int_0^{2\pi} 2\pi Rr\ d\theta \\ &= 4\pi^2 Rr. \end{aligned}$$

**Problem** (LN13 0.9)**.** Show that every manifold is normal, *i.e.* for every disjoint closed sets $A_1, A_2$ in $M$, there exists a pair of disjoint open subsets $U_1, U_2$ of $M$ s.t. $A_1 \subset U_1, A_2 \subset U_2$.

*Proof.* WLOG suppose $M$ is path-connected since we can just take the union of disjoint open sets from all path components. Since every manifold admits a metric, let $g$ be a Riemannian metric of $M$. Therefore, for every point $x \in A_1, y \in A_2$, let $\Gamma$ be the set of all possible paths between $x$ and $y$, we can define the distance between them $d(x,y) = \inf_{\gamma\in\Gamma} L[\gamma]$, where $L[\gamma]$ is computed the usual way using $g$. This distance can be checked to be a metric and therefore endows $M$ with a metric space structure. Every metric space is normal. We prove this below.

Define the distance between a point $x$ and a set $A$ to be $d(x, A) = \inf\{d(x,y) : y \in A\}$. Since $A_1, A_2$ are closed so they contain all their limit points, for any $y \in M \setminus A_1$ and $x \in M \setminus A_2$, we must have $d(y, A_1) = \varepsilon_y > 0$ and $d(x, A_2) = \varepsilon_x > 0$. Let $r_x = \frac{\varepsilon_x}{3}$ and $r_y = \frac{\varepsilon_y}{3}$. Then take

$$U_1 := \bigcup_{x \in A_1} B_{r_x}(x)$$
$$U_2 := \bigcup_{y \in A_2} B_{r_y}(y),$$

which are unions of open balls and thus open. Now suppose there exists a $p \in U_1 \cap U_2$. Then by definition $p \in B_{r_x}(x) \cap B_{r_y}(y)$ for some $x \in A_1$ and $y \in A_2$. It follows that

$$\begin{aligned} \frac{\varepsilon_x + \varepsilon_y}{2} &\leq d(x, y) && \text{definition of } \varepsilon_x, \varepsilon_y \\ &\leq d(x, p) + d(p, y) && \Delta \text{ inequality} \\ &\leq r_x + r_y \\ &= \frac{\varepsilon_x + \varepsilon_y}{3}, \end{aligned}$$

a contradiction since $\varepsilon_x + \varepsilon_y > 0$. Hence $U_1, U_2$ are disjoint and are the open sets we seek. $\square$

**Problem** (LN13 0.12)**.** Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold which may be parameteried by $f : U \to \mathbb{R}^n$, for some open set $U \subseteq \mathbb{R}^m$, *i.e.* , $f$ is a 1-to-1 smooth immersion and $f(U) = M$. Show that then $vol(M) = \int_U \sqrt{\det(J_x(f)^T J_x(f))}dx$, where $J_x(f)$ is the Jacobian matrix of $f$ at $x$. (Note that since we define Jacobian differently than the lecture notes, the order is flipped.)

*Proof.* Since $f$ is a 1-to-1 immersion onto $M$, $f$ is a diffeomorphism onto its image and thus a parameterization of $M$. Since $M$ is an embedded submanifold of $\mathbb{R}^n$, it is endowed with the ambient Euclidean metric. So we can endow the pullback metric on $U$ which is just $g_{ij_x} = \langle \frac{\partial f}{\partial x_i}(x), \frac{\partial f}{\partial x_j}(x) \rangle$. Recall that the $i$th column of the Jacobian $J_x(f)_i = \frac{\partial f}{\partial x_i}$. Therefore, the $ij$th entry of $J_x(f)^T J_x(f)$ is precisely $\langle \frac{\partial f}{\partial x_i}(x), \frac{\partial f}{\partial x_j}(x) \rangle = g_{ij_x}$. Then we have

$$\begin{aligned} vol(M) &= \int_U \sqrt{\det G_x} dx \\ &= \int_U \sqrt{\det(J_x(f)^T J_x(f))} dx. \end{aligned}$$

$\square$