1 Fibrations

Theorem 1.1 (Leray-Serre spectral sequence)

If $E \xrightarrow{p} B$ is a Serre fibration and B is a simply connected CW complex (only need $\pi_1(B)$ acts trivially on $H_*(F)$). Then there exists a spectral sequence that converges to $G(H_*(E))$ with $E^2_{s,t} = H_s(B; H_t(F))$.

Remark 1.2 There is a similar cohomology spectral sequence with $E_2^{s,t} = H^s(B; H^t(F))$.

Proof. Exercise: show d^1 is the boundary map for chain complex $C_*^{CW}(B; H_t(F))$. □

Lemma 1.3

$H_{s+t}(E^s, E^{s-1}) \cong C_s(B; H_t(F))$.

Proof. Exercise: for $p > 0$, $H_p(Y) \cong H_{p+1}(\Sigma Y)$.

Recall $\Sigma Y \cong S^1 \wedge Y = S^1 \times Y / S^1 \vee Y$. Also $S^p = S^1 \wedge \cdots \wedge S^1$ p times. There exists an S^s in $D^s \times F / S^{s-1} \times F$ by taking a point in F.

Exercise: for any spaces X, Y, Z with $Y \subseteq x$

$$\frac{X \times Z}{(Y \times Z) \cup (X \times \{*\})} \cong \frac{X/Y \times Z}{(\{*\} \times Z) \vee (X \times \{*\}/Y \times \{*\})}.$$

Exercise: think about $t = 0, 1$.

□

Theorem 1.4

For $k \geq 2$,

$$H_q(\Omega S^k) = \begin{cases} \mathbb{Z} & q = a(k-1), a \geq 0 \\ 0 & \text{else} \end{cases}$$

Proof. Apply Theorem 4 and Lemma 1. □

Theorem 1.5 (Gysin sequence)

Let $E \xrightarrow{p} B$ be a fibration with fiber S^n and B a CW complex. Assume $\pi_1(B)$ acts trivially on $H_*(S^n)$, there exists an exact sequence

$$\dots H_r(E) \xrightarrow{p_*} H_r(B) \to H_{r-n-1}(B) \to H_{r-1}(E) \xrightarrow{p_*} H_{r-1}(B) \dots$$

for $k \geq n+1$.

Proof. Apply Theorem 4 and Lemma 1. □

Exercise: If $E \xrightarrow{p} S^n$ is a fibration with fiber F, show there exists an exact sequence

$$\dots H_r(F) \to H_r(E) \to H_{r-n}(F) \to H_{r-1}(F) \to \dots$$

called the Wang sequence.

Let's consider a cohomology version:

Theorem 1.6 (Leray-Serre for cohomology)

Let $E \xrightarrow{p} B$ be a Serre fibration with B a CW complex where $\pi_1(B)$ acts trivially on $H^*(F)$. There exists a spectral sequence converging to $G(H^*(F))^{s,t}$ with $E_2^{s,t} = H^s(B; H^t(F))$ and

(1) $\{E_r^{s,t}\}$ is a bigraded algebra, *i.e.* there exists a product $E_r^{s,t} \times E_r^{p,q} \to E_r^{s+p,t+q}$

(2) $d_r : E_r \to E_r$ is a derivation $(r, -r+1)$, *i.e.*

$$d_r(a \cdot b) = (d_r a) \cdot b + (-1)^{p+q} a \cdot d_r b$$

(3) $E_2^{*,0} \cong H^*(B)$ as rings and $E_2^{0,*} \cong H^*(F)$ as rings.

Remark 1.7 The product structure on $E_2^{s,t}$ is

$$H^p(B; H^q(F)) \times H^s(B; H^t(F)) \to H^{p+s}(B; H^q(F) \otimes H^t(F))$$

and compose with the cup product on $H^q(F) \otimes H^t(F)$.

Example 1.8

$\mathbb{C}P^n$.

Theorem 1.9

$H^*(U(n)) \cong \Lambda(x_1, x_3, \ldots, x_{2n+1})$ with $\deg x_i = i$.

Remark 1.10 From this we can compate $H^*(BU(n)) \cong \mathbb{Z}[c_1, \ldots, c_n]$ where c_i has degree $2i$ (this is Theorem II.17).