Homework 9

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Problem (4.6)**.** Let M be a Riemannian manifold. M is a locally symmetric space if $\nabla R = 0$, where R is the curvature tensor of M.

(a) Let $\gamma : [0, \ell) \to M$ be a geodesic of M. Let X, Y, Z be parallel vector fields along γ. Prove that $R(X, Y)Z$ is a parallel field along γ.

(b) Prove that if M is locally symmetric, connected, and has dimension two, then M has constant sectional curvature.

(c) Prove that if M has constant (sectional) curvature, then M is a locally symmetric space.

Proof. (a) Let $V := \dot{\gamma}$ be the velocity field along γ. Since X, Y, Z are parallel along γ, we have $\nabla_V X = \nabla_V Y = \nabla_V Z = 0$ along γ. Let W be any vector field, then tensor derivative yields

$$\begin{aligned}
\nabla R(X, Y, Z, W, V) &= V(R(X, Y, Z, W)) - R(\nabla_V X, Y, Z, W) - R(X, \nabla_V Y, Z, W) \\
&\quad - R(X, Y, \nabla_V Z, W) - R(X, Y, Z, \nabla_V W) \\
&= V(R(X, Y, Z, W)) - R(0, Y, Z, W) - R(X, 0, Z, W) \\
&\quad - R(X, Y, 0, W) - R(X, Y, Z, \nabla_V W) \\
&= V(R(X, Y, Z, W)) - R(X, Y, Z, \nabla_V W) = 0.
\end{aligned}$$

This yields $V(R(X, Y, Z, W)) = R(X, Y, Z, \nabla_V W)$. Recall the Leibniz rule for the metric:

$$V(\langle R(X, Y)Z, W\rangle) = \langle \nabla_V(R(X, Y)Z), W\rangle + \langle R(X, Y)Z, \nabla_V W\rangle .$$

It follows that $\langle \nabla_V(R(X, Y)Z), W\rangle = 0$. Since W is arbitrary, it must be that $\nabla_V(R(X, Y)Z) = 0$, proving that $R(X, Y)Z$ is parallel along γ.

(b) First, let $c = K(p)$ for any $p \in M$. Now define $A = \{p \in M : K(p) = c\}$. Thus we know that A is not empty. Since K is a smooth function on M, A is the preimage of a single point c and therefore is closed in M. To prove that M has constant sectional curvature, it suffices to show that A is open as well so we must have $A = M$. That is, we wish to show that any $p \in A$ has an open neighborhood U such that $U \subseteq A$.

Consider the normal neighborhood U of p with geodesic frame $\{E_1, E_2\}$. Since M has dimension two, this frame spans the entire TU so we only need to show K is constant in U under this frame. Let $q \in U$ and let γ be a geodesic connecting p and q. By definition of geodesic frame, E_1 and E_2 are parallel fields along γ. Since M is locally symmetric, it follows from part (a) that $R(E_1, E_2)E_1$ is a parallel field along γ as well. Since $K(p) = c$, we have

$$\begin{aligned} K(p) &= \frac{\langle R(E_1, E_2)E_1(p), E_2(p)\rangle}{\|E_1(p)\|^2\|E_2(p)\|^2 - \langle E_1(p), E_2(p)\rangle^2} \\ &= \frac{\langle R(E_1, E_2)E_1(q), E_2(q)\rangle}{\|E_1(q)\|^2\|E_2(q)\|^2 - \langle E_1(q), E_2(q)\rangle^2} \qquad \text{parallel transport is isometry} \\ &= K(q) = c. \end{aligned}$$

We conclude that $U \subseteq A$ and thus A is open as desired.

(c) Since M has constant sectional curvature K_0, by Lemma 3.4, $R = K_0 R'$, where

$$\langle R'(X, Y, Z), W\rangle = \langle X, Z\rangle \langle Y, W\rangle - \langle Y, Z\rangle \langle X, W\rangle .$$

Claim 0.1. For any $V \in \mathfrak{X}(M)$, $\nabla_V R' = 0$.

The proof is a straightforward computation:

$$\begin{aligned} V(R'(X, Y, Z, W)) &= V[\langle X, Z\rangle \langle Y, W\rangle - \langle Y, Z\rangle \langle X, W\rangle] \\ &= [\langle \nabla_V X, Z\rangle + \langle X, \nabla_V Z\rangle] \langle Y, W\rangle + \langle X, Z\rangle [\langle \nabla_V Y, W\rangle + \langle Y, \nabla_V W\rangle] \\ &\quad - [\langle \nabla_V Y, Z\rangle + \langle Y, \nabla_V Z\rangle] \langle X, W\rangle - \langle Y, Z\rangle [\langle \nabla_V X, W\rangle + \langle X, \nabla_V W\rangle] \\ &= \langle \nabla_V X, Z\rangle \langle Y, W\rangle + \langle X, \nabla_V Z\rangle \langle Y, W\rangle \\ &\quad + \langle X, Z\rangle \langle \nabla_V Y, W\rangle + \langle X, Z\rangle \langle Y, \nabla_V W\rangle \\ &\quad - \langle \nabla_V Y, Z\rangle \langle X, W\rangle - \langle Y, \nabla_V Z\rangle \langle X, W\rangle \\ &\quad - \langle Y, Z\rangle \langle \nabla_V X, W\rangle - \langle Y, Z\rangle \langle X, \nabla_V W\rangle \end{aligned}$$

$$
\begin{aligned}
&= R'(\nabla_V X, Y, Z, W) + R'(X, \nabla_V Y, Z, W) \\
&\quad + R'(X, Y, \nabla_V Z, W) + R'(X, Y, Z, \nabla_V W).
\end{aligned}
$$

The claim follows immediately. Using this claim, we obtain

$$
\begin{aligned}
V(R(X,Y,Z,W)) &= K_0 V(R'(X,Y,Z,W)) \\
&= K_0[R'(\nabla_V X, Y, Z, W) + R'(X, \nabla_V Y, Z, W) \\
&\quad + R'(X, Y, \nabla_V Z, W) + R'(X, Y, Z, \nabla_V W)] \\
&= R(\nabla_V X, Y, Z, W) + R(X, \nabla_V Y, Z, W) \\
&\quad + R(X, Y, \nabla_V Z, W) + R(X, Y, Z, \nabla_V W).
\end{aligned}
$$

It follows that $\nabla R(X,Y,Z,W,V) = V(R(X,Y,Z,W)) - (R(\nabla_V X, Y, Z, W) + R(X, \nabla_V Y, Z, W) + R(X, Y, \nabla_V Z, W) + R(X, Y, Z, \nabla_V W)) = 0$. That is, M is locally symmetric.

$\square$

Problem (4.8 (Schur's Theorem))**.** Let M^n be a connected Riemmannian manifold with $n \geq 3$. Suppose that M is isotropic, that is, for each $p \in M$, the sectional curvature $K(p,\sigma)$ does not depend on $\sigma \subseteq T_pM$. Prove that M has constant sectional curvature, that is, $K(p,\sigma)$ also does not depend on p.

Proof. By the proof of Lemma 3.4, since $K(p)$ is independent of $\sigma \subset T_pM$, we have $R(p) = K(p)R'(p)$. By the claim from previous problem, we know that for any $V \in \mathfrak{X}(M)$, $\nabla_V R' = 0$. It follows that $\nabla_V(R) = \nabla_V(KR') = V(K)R' + K\nabla_V R' = V(K)R'$. Now consider the 2nd Bianchi identity:

$$
\begin{aligned}
0 &= \nabla R(X,Y,Z,W,V) + \nabla R(X,Y,W,V,Z) + \nabla R(X,Y,V,Z,W) \\
0 &= V(K)R'(X,Y,Z,W) + Z(K)R'(X,Y,W,V) + W(K)R'(X,Y,V,Z) \\
0 &= V(K)[\langle X,Z\rangle \langle Y,W\rangle - \langle Y,Z\rangle \langle X,W\rangle] \\
&\quad + Z(K)[\langle X,W\rangle \langle Y,V\rangle - \langle Y,W\rangle \langle X,V\rangle] \\
&\quad + W(K)[\langle X,V\rangle \langle Y,Z\rangle - \langle Y,V\rangle \langle X,Z\rangle].
\end{aligned}
$$

Since $n \geq 3$, for any Z we can find W, Y s.t. $\langle Z,W\rangle = \langle Y,W\rangle = \langle Y,Z\rangle = 0$ (*i.e.* Z, W, Y

are linearly independent) and $\langle Y, Y\rangle \equiv 1$. The equation becomes

$$
\begin{aligned}
0 &= Z(K)\langle X, W\rangle - W(K)\langle X, Z\rangle \\
0 &= \langle Z(K)W - W(K)Z, X\rangle \\
0 &= Z(K)W - W(K)Z && X \text{ is arbitrary} \\
0 &= Z(K) := dK(Z) && Z, W \text{ linearly independent.}
\end{aligned}
$$

Since Z is arbitrary, we conclude that $dK \equiv 0$. That is, K is constant everywhere. □

Problem (4.9)**.** Prove that the scalar curvature $K(p)$ at $p \in M$ is given by

$$K(p) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \mathrm{Ric}_p(x) dS^{n-1},$$

where ω_{n-1} is the area of the sphere S^{n-1} in T_pM and dS^{n-1} is the area elements on S^{n-1}.

Proof. Fix a $p \in M$. Recall the symmetric bilinear form $Q(x, y)$ that is the trace of the linear map $z \mapsto R_p(x, z)y$. We know that there is a real symmetric matrix A s.t. $\frac{1}{n-1}Q(x, y) = \langle Ax, y\rangle$. Spectral Theorem yields an orthonormal eigenbasis $\{e_i\}$ of A with corresponding real eigenvalues λ_i. Thus, if a unit vector $x = x^i e_i$, we have $\mathrm{Ric}_p(x) = \frac{1}{n-1}Q(x, x) = \lambda^i x_i^2$. Moreover, x is a unit normal vector to S^{n-1}. Let $V = \lambda^i x^i e_i$, with $\operatorname{div} V = \sum_{i=1}^n \lambda_i$, we obtain

$$
\begin{aligned}
\frac{1}{\omega_{n-1}} \int_{S^{n-1}} \mathrm{Ric}_p(x) dS^{n-1} &= \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \lambda^i x_i^2 dS^{n-1} \\
&= \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \langle V, x\rangle\, dS^{n-1} \\
&= \frac{1}{\omega_{n-1}} \int_{B^n} \operatorname{div} V \; dB^n && \text{Stokes Theorem} \\
&= \frac{\operatorname{div} V}{\omega_{n-1}} \int_{B^n} dB^n \\
&= \frac{\sum_{i=1}^n \lambda_i}{n} && area(S^{n-1}) = \left.\frac{d(R^n vol(B^n))}{dR}\right|_{R=1} = n \cdot vol(B^n) \\
&= \frac{\sum_{i=1}^n \mathrm{Ric}_p(e_i)}{n} \\
&= K(p).
\end{aligned}
$$

□