1 Sard's Theorem

Example 1.1

Consider $T : P \to S^1$, take $p \in S^1$, by Sard's Theorem and rank theorem, $T^{-1}(q)$ is finite points.

Lemma 1.2 (Fubini for measure zero)

Let $A \subseteq \mathbb{R}^n$, closed, $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$, $S_x := \{x\} \times \mathbb{R}^{n-1}, x \in \mathbb{R}$. If $\mu(A \cap S_x) = 0 \; \forall \; x \in \mathbb{R}$, then $\mu(A) = 0$.

Proof. We may assume A is compact (by countable). If $A \cap S_x$ lies in an open set $U \subseteq S_x$, denote $S_{(x-\varepsilon, x+\varepsilon)} := \bigcup_{x' \in (x-\varepsilon, x+\varepsilon) S_{x'}}^n$ (thickened slice), then $A \cap S_{x-\varepsilon, x+\varepsilon} \subseteq U \times (x-\varepsilon, x+\varepsilon)$. Then

$$A \subseteq \bigcup_{i=1}^n U_i \times (x_i - \varepsilon_i, x_i + \varepsilon_i) =: N.$$

We may also assume that $\sum_{i=1}^n 2\varepsilon_i \leq 2(b-a)$. Then by subadditivity,

$$\begin{aligned} \mu(N) &\leq \sum_{i=1}^n \mu(U_i \times (x_i - \varepsilon_i, x_i + \varepsilon_i)) \\ &\leq \sum_{i=1}^n \mu(U_i) \times 2 \max\{\varepsilon_i\} \end{aligned}$$

□

Proof. Baby case: $f : M \to N$, $\dim M \leq \dim N$. The case when $\dim M = \dim N$ is implicitly shown in Lecture 8 (lemma 4). The rest is shown in Lemma 7.

May assume that $M = \mathbb{R}^n$ and $N = \mathbb{R}^p$ because we are proving a local property and countable union of measure zero sets is measure zero. When $n = 0$, it has measure zero. Assume theorem holds for $n-1$. $f : \mathbb{R}^n \to \mathbb{R}^p$. Let C be the set of critical points in $\mathbb{R}^n$. Define $C_i := \{x \in U : \text{ all partial derivatives up to order } k = 0\}$. Then $C \supseteq C_1 \supseteq C_2 \supseteq \ldots$. Then it suffices to show that

(1) $\mu(f(C - C_1)) = 0$

(2) $\mu(f(C_k - C_{k+1})) = 0$

(3) $\mu(f(C_k)) = 0$ for some large k.

For 1, it suffices to show that there exists an open neighborhood V of $x \in \mathbb{R}^n$ s.t. $\mu(f(V \cap C)) = 0$ by countable basis of $\mathbb{R}^n$. Suppose $x \notin C_1$, then WLOG assume $\frac{\partial f^1}{\partial x_1} \neq 0$. Define $h : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ by $h(x) = (f^1(x), x_2, \ldots, x_n)$ so rank $dh_x = n$ since the Jacobian is

$$dh_x = \begin{pmatrix} \frac{\partial f^1}{\partial x_1} & 0 \\ 0 & I \end{pmatrix}$$

So h is a local diffeomorphism by IFT. Let $g := f \circ h^{-1}$. I claim that locally g and f share the same critical points. Moreover, g fixes the first coordinate of any point by definition of h and g. Thus $t \times \mathbb{R}^{n-1} \xrightarrow{g} t \times \mathbb{R}^{p-1}$. Define $g^t = g|_{t \times \mathbb{R}^{n-1}}$. Then

$$dg = \begin{pmatrix} 1 & 0 \\ 0 & dg^t \end{pmatrix}$$

Then critical points of g^t are also critical points of g since the matrix has 0 determinant iff $\det dg^t = 0$. Now we've reduced the dimension and can apply induction hypothesis. So the critical points of g has measure zero and thus same goes for f.

2 is very similar.

□

Example 1.3

Application: S^n is simply connected for $n \geq 2$. Any curve will miss a point by Sard's.

Example 1.4

Let M be a smooth closed (compact, connected, without boundary) hypersurface in $\mathbb{R}^n$ so $\dim M = n - 1$. Then we have the Gauss map $\nu : M to S^{n-1}$, where we map each point to its outward unit normal vector. Then $\#\nu^{-1}(u) < \infty$ for almost every $u \in S^{n-1}$. Since regular point would have codimension 0 and surface is compact. This implies that for almost every $u \in S^{n-1}$, there exists finitely many tangent hyperplanes H of

M that are orthogonal to u. Any hyperplane $H \subseteq \mathbb{R}^n$ will be transversal to M, after a perturbation. The set of transversal hyperplanes to M is open and dense. This is a metric space with distance of unit normal vectors and the offset from origin as the metric. It is diffeomorphic to $S^{n-1} \times [0, \infty)$.