

# Homework 4

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**Problem (1).** The Hamiltonian of the problem is given by

$$H(x_1, x_2, u, p_1, p_2) = \frac{1}{2}u^2 + p_1x_2 + p_2(u - x_2).$$

The adjoint equations are given by

$$\dot{p}_1 = -H_{x_1} = 0$$

$$\dot{p}_2 = -H_{x_2} = p_2 - p_1.$$

The first-order condition demands

$$H_u = u + p_2 = 0$$

$$u = -p_2.$$

Plugging this into the differential equations yield

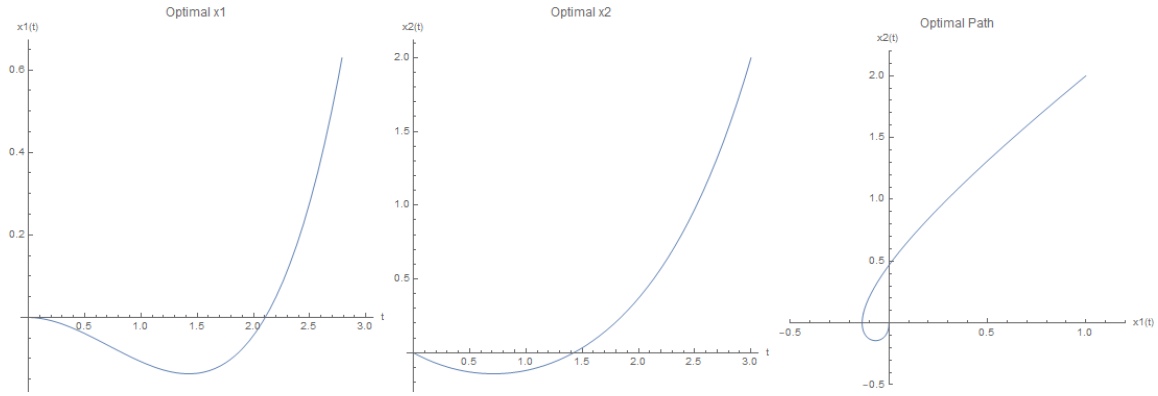
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - p_2.$$

Together with the adjoint equations, we have 4 first-order equations and require 4 boundary conditions.

- (a) Since all initial and final times and states are fixed, we have 4 boundary conditions  $x_1(0) = x_2(0) = 0$ ,  $x_1(3) = 1$ , and  $x_2(3) = 2$ . Mathematica yields

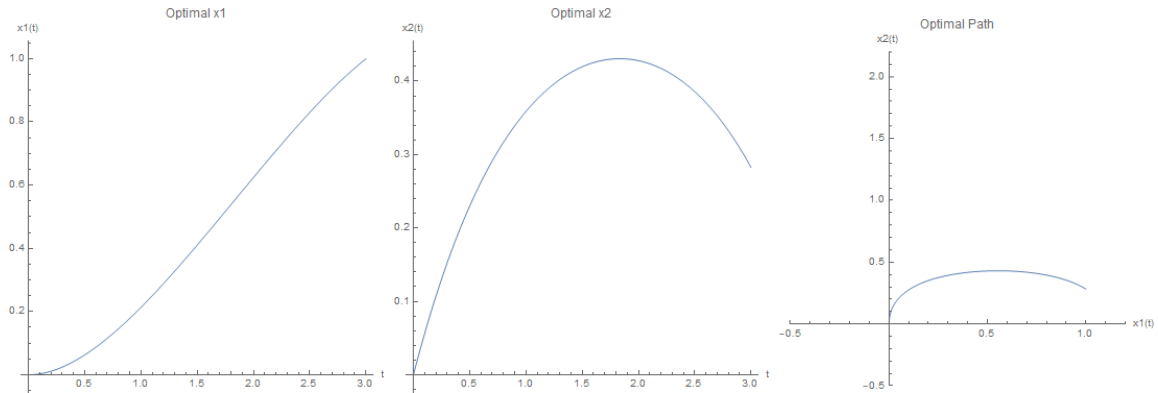
$$\begin{aligned} u(t) &= \frac{6e^{3+t} + 6e^t - e^6 + 4e^3 - 3}{e^6 + 4e^3 - 5} \\ &= -0.6811 + 0.2642e^t. \end{aligned}$$



(b) When  $x_2(3)$  is free, in its place we instead have the transversality condition  $p_2(3) = 0$ .

This yields the solution

$$\begin{aligned} u(t) &= -\frac{2e^3(e^t - e^3)}{3e^6 + 4e^3 - 1} \\ &= 0.6256 - 0.0311e^t \end{aligned}$$



(c) I would add a final penalty term  $\Phi$ :

$$\mathcal{J} = \underbrace{\frac{1}{2} \left( (x_1(3) - 1)^2 + (x_2(3) - 2)^2 \right)}_{\Phi(3)} + \frac{1}{2} \int_0^3 u^2 dt.$$

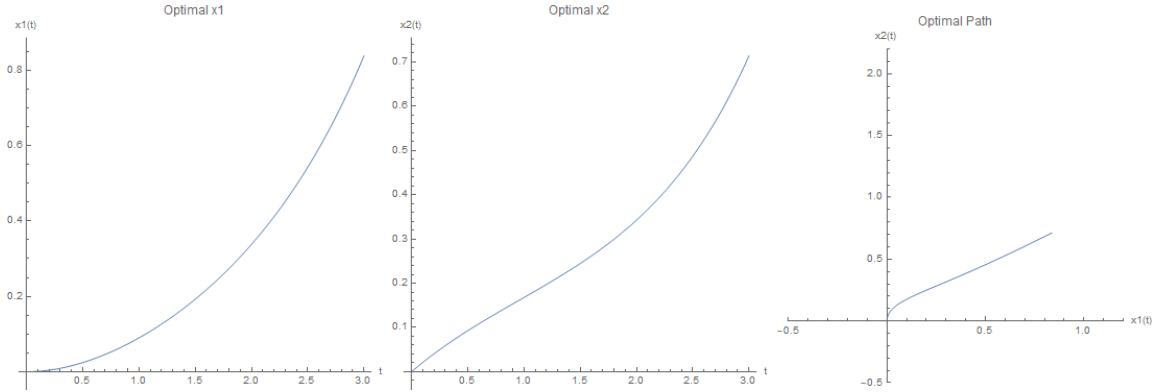
Then instead of  $p_1(3) = p_2(3) = 0$ , we have

$$p_1(3) = \Phi_{x_1}(3) = x_1(3) - 1$$

$$p_2(3) = \Phi_{x_2}(3) = x_2(3) - 2$$

Then new control is

$$\begin{aligned} u(t) &= \frac{8e^{3+t} + 6e^t + e^6 + 4e^3 - 3}{7e^6 + 8e^3 - 7} \\ &= 0.1615 + 0.056e^t. \end{aligned}$$



We see that  $x_1(3) = 0.8385$  and  $x_2(3) = 0.7142$ , which are not very close to  $(1, 2)$ . To improve accuracy, I would increase the weight of the penalty term. We see that the cost in part (a) is 4.2859. The cost as a function of the weight coefficient  $c$  is shown in the figure below:

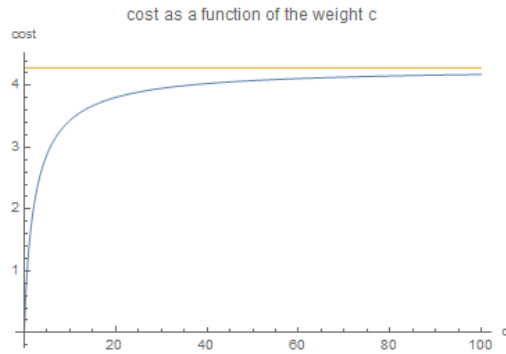


Figure 1: We see that as the weight increases, the cost approaches that of the cost (orange) in part (a) asymptotically.

And we indeed see that the solution ends much closer to  $(1, 2)$  when the weight is high:

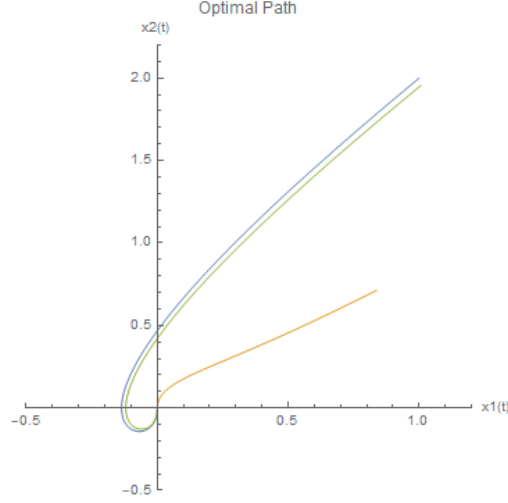


Figure 2: Blue is optimal path from part (a), green is part (c) with weight 100, and orange is part (c) with weight 1.

- (d) It is clear that  $\Psi = \begin{pmatrix} 2 & 5 \end{pmatrix}$ . By transversality condition from Equation 5.234, we have the boundary conditions

$$\begin{pmatrix} -p_1(3) \\ -p_2(3) \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \lambda$$

Together with the initial conditions  $x_1(0) = x_2(0) = 0$  and the terminal condition  $2x_1(3) + 5x_2(3) = 20$ , we have 5 boundary conditions for 4 differential equations and an unknown  $\lambda$ . This allows us to solve by Mathematica and obtain the optimal control:

$$u(t) = -p_2(t) = 1.4341 + 0.1071e^t$$

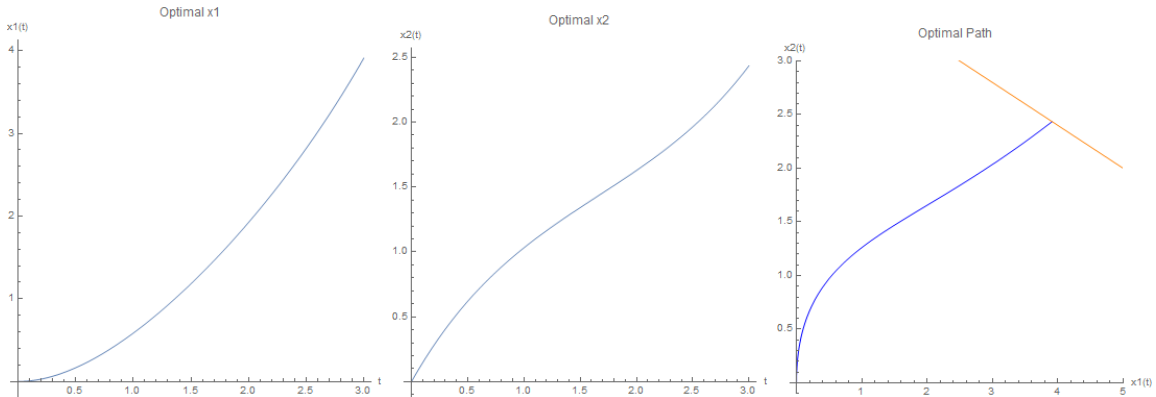


Figure 3: We see that the solution indeed ends at the constraint.

- (e) We have the same  $\Psi$  but  $h(t) = 20 + \frac{t^2}{2}$  so  $\dot{h}(t_f) = t_f$ . When  $t_f$  is also free, based on Equation 5.234 we have the transversality conditions

$$\begin{pmatrix} H(t_f) \\ -p_1(t_f) \\ -p_2(t_f) \end{pmatrix} = \begin{pmatrix} -t_f \\ 2 \\ 5 \end{pmatrix} \lambda$$

Note that by plugging in  $u = -p_2$ , we have

$$\begin{aligned} H &= -\frac{1}{2}p_2^2 + (p_1 - p_2)x_2 \\ H(t_f) &= -\frac{25}{2}\lambda + 3x_2(t_f) = -t_f \end{aligned}$$

Together with two initial conditions and the terminal condition

$$2x_1(t_f) + 5x_2(t_f) = 20 + \frac{t_f^2}{2},$$

we have a total of 6 boundary conditions to match the 4 differential equations and two unknowns  $\lambda$  and  $t_f$ . Mathematica yields

$$u(t) = 1.8359 + 0.2547e^t$$

with  $t_f = 2.3807$ s. The trajectories are

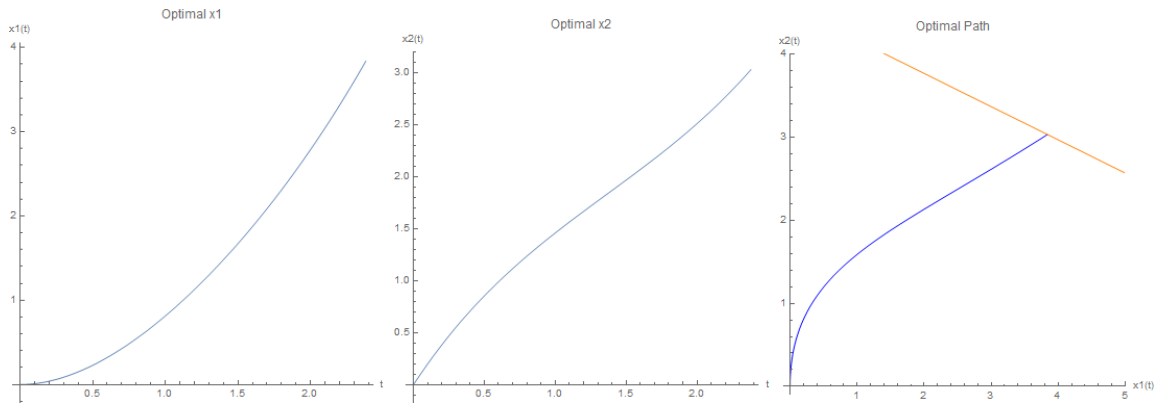


Figure 4: We see the end point hits the constraint.

**Problem (2).** The Hamiltonian is

$$H = \frac{1}{2}u^2 + p(ax + bu)$$

The adjoint equation is

$$\dot{p} = -H_x = ap \Rightarrow p(t) = Ce^{at}.$$

And the first-order condition is

$$H_u = u + bp = 0 \Rightarrow u = -bp$$

Thus

$$\dot{x} = ax - b^2p = ax - b^2Ce^{at}, \quad x(0) = x_0, x(t_f) = 0$$

We have  $x(t) = b^2Cte^{at} + Ae^{at}$ ,  $x(0) = A = x_0$ , and

$$\begin{aligned} x(t_f) &= b^2Ct_f e^{at_f} + x_0 e^{at_f} = 0 \\ C &= -\frac{x_0}{b^2t_f} \end{aligned}$$

Thus,

$$u(t) = -bp = -b \cdot \left( -\frac{x_0}{b^2t_f} \right) e^{at} = \frac{x_0}{bt_f} e^{at}$$

**Problem (3).** With the mixed constraint  $\psi(x_0, x_f) = x_f - x_0 = 0$ , we can turn  $x_0, x_f$  into free variables by adding a term with Lagrange multiplier. Let  $\Phi(x_0, x_f) = \phi(x_f) + \lambda\psi(x_0, x_f) = \frac{1}{2}(x(t_f) - 1)^2 + \lambda(x_f - x_0)$ . The Hamiltonian is

$$H = \frac{1}{2}(x^2 + u^2) + pu.$$

The adjoint equation is

$$\dot{p} = -H_x = x$$

The first-order condition says

$$H_u = u + p = 0 \Rightarrow u = -p.$$

Since both  $x(0), x(2)$  is free, from Equation 5.234 we have the transversality condition

$$(\Phi_{x_0} + p^T(t_0))\delta x_0 + (\Phi_{x_f} - p^T(t_f))\delta x_f = 0$$

where  $\delta x_0$  and  $\delta x_f$  can take any value. This forces that

$$\Phi_{x_0} + p^T(t_0) = 0$$

$$\Phi_{x_f} - p^T(t_f) = 0$$

Thus for this problem, we have

$$p(0) = -\Phi_{x_0} = -(-\lambda) = \lambda$$

$$p(2) = \Phi_{x_f} = x(2) - 1 + \lambda$$

Thus we have two boundary conditions for two differential equations. Mathematica yields

$$x(t) = \frac{\lambda \cos(2-t) + \cos t - \lambda t + \lambda \sin(2-t)}{\cos 2 - \sin 2}.$$

Solving  $x(0) = x(2)$ , we obtain  $\lambda = \frac{\cos(2)-1}{2\cos 2+\sin 2-2} \approx 0.7364$ .

$$\begin{aligned} u(t) = -p(t) &= \frac{\lambda(\cos(2-t) - \sin(2-t) - \sin t) + \sin t}{\cos 2 - \sin 2} \\ &= 0.5556(\cos(2-t) - \sin(2-t)) + 0.1989 \sin t \end{aligned}$$

The associated cost is 1.8506.

**Problem (4).** (a) Since the cost  $\mathcal{J} = t_f$ , we have  $\Phi(t) = t$  and the Hamiltonian is

$$H = p_1(t) \cos \theta(t) + p_2(t) \sin \theta(t) + p_3(t)u(t) + p_4(t)v(t)$$

(b) The adjoint equations are

$$\begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \end{pmatrix} = \begin{pmatrix} -H_u \\ -H_v \\ -H_x \\ -H_y \end{pmatrix} = \begin{pmatrix} p_3 \\ p_4 \\ 0 \\ 0 \end{pmatrix}$$

(c) Since  $t_f, u(t_f), v(t_f)$  are free, by Equation 5.234 we have the transversality conditions

$$H(t_f) + \Phi_t(t_f) = 0 \Rightarrow H(t_f) = -1$$

$$p_1(t_f) = \Phi_u = 0$$

$$p_2(t_f) = \Phi_v = 0.$$

- (d) Notice that  $p_3, p_4$  are constants, so  $p_1(t) = p_3 t$  and  $p_2(t) = p_4 t$  by the zero boundary conditions. First-order condition yields

$$H_\theta = -p_1 \sin \theta(t) + p_2 \cos \theta(t) = 0$$

$$\tan \theta^*(t) = \frac{p_2(t)}{p_1(t)} = \frac{p_4}{p_3}$$

which is a constant! So are  $\cos \theta^*$  and  $\sin \theta^*$ . Thus by the initial conditions  $u(0) = \cos \gamma_0, v(0) = \sin \gamma_0$ , we have  $u(t) = \cos \theta^* t$  and  $v(t) = \sin \theta^* t$ . By the initial conditions  $x(0) = y(0) = 0$ , we have  $x(t) = \frac{1}{2} \cos \theta^* t^2 + \cos \gamma_0 t$  and  $y(t) = \frac{1}{2} \sin \theta^* t^2 + \sin \gamma_0 t$ . By the terminal condition  $x(t_f) = x_f$  and  $y(t_f) = 0$ , we have

$$\begin{aligned} x_f - \cos \gamma_0 t_f &= \frac{1}{2} \cos \theta^* t_f^2 \\ -\sin \gamma_0 t_f &= \frac{1}{2} \sin \theta^* t_f^2 \end{aligned}$$

Squaring both sides and adding the two equations together, we obtain the equation as desired:

$$\begin{aligned} x_f^2 - 2 \cos \gamma_0 t_f + (\cos \gamma_0^2 + \sin \gamma_0^2) t_f^2 &= \frac{1}{4} (\cos^2 \theta^* + \sin^2 \theta^*) t_f^4 \\ 4x_f^2 - 8 \cos \gamma_0 t_f + 4t_f^2 - t_f^4 &= 0 \end{aligned}$$

So  $t_f$  must be the minimum positive solution of this equation.

- (e) By the above equation,

$$\begin{aligned} \cos \theta^* &= 2 \left( \frac{x_f}{t_f^2} + \frac{\cos \gamma_0}{t_f} \right) \\ \theta^* &= \arccos \left( 2 \left( \frac{x_f}{t_f^2} + \frac{\cos \gamma_0}{t_f} \right) \right) \end{aligned}$$

**Problem (5).** The cost has the Meyer form  $J = t_f$  so  $\Phi(t) = t$ . The Hamiltonian is

$$H(x, y, p_1, p_2) = p_1 r \cos \beta + p_2 r \sin \beta.$$

The adjoint equations are

$$\dot{p}_1 = -H_x = -p_1 \frac{x}{r} \cos \beta - p_2 \frac{x}{r} \sin \beta$$



$$\dot{p}_2 = -H_y = p_1 \frac{y}{r} \cos \beta - p_2 \frac{x}{r} \sin \beta$$

Note that  $\dot{p}_2 = \frac{y}{x} \dot{p}_1$ . First-order condition yields

$$\begin{aligned} H_\beta &= -p_1 r \sin \beta + p_2 r \cos \beta = 0 \\ \tan \beta &= \frac{p_2}{p_1} \end{aligned}$$

Thus we obtain  $p_2 = \tan \beta p_1$ . Moreover, we see that

$$\begin{aligned} p_2 \cos \beta - p_1 \sin \beta &= \frac{p_2 p_1 - p_1 p_2}{\sqrt{p_1^2 + p_2^2}} \\ &= 0. \end{aligned}$$

Now we compute

$$\begin{aligned} 0 &= \frac{d}{dt} H_\beta = \dot{r}(-p_1 \sin \beta + p_2 \cos \beta) + r \left( -\frac{d}{dt}(p_1 \sin \beta) + \frac{d}{dt}(p_2 \cos \beta) \right) \\ 0 &= 0 + r \left( -\dot{p}_1 \sin \beta - p_1 \cos \beta \dot{\beta} + \dot{p}_2 \cos \beta - p_2 \sin \beta \dot{\beta} \right) \\ 0 &= -\dot{p}_1 \sin \beta - p_1 \cos \beta \dot{\beta} + \frac{y}{x} \dot{p}_1 \cos \beta - p_1 \tan \beta \sin \beta \dot{\beta} \\ p_1 (\cos \beta + \tan \beta \sin \beta) \dot{\beta} &= \dot{p}_1 \left( -\sin \beta + \frac{y}{x} \cos \beta \right) \\ p_1 (\cos \beta + \tan \beta \sin \beta) \dot{\beta} &= \left( p_1 \frac{x}{r} \cos \beta + \tan \beta p_1 \frac{x}{r} \sin \beta \right) \left( \sin \beta - \frac{y}{x} \cos \beta \right) \\ \dot{\beta} &= \frac{x}{r} \left( \sin \beta - \frac{y}{x} \cos \beta \right) \\ \dot{\beta} &= \frac{x}{r} \sin \beta - \frac{y}{r} \cos \beta \end{aligned}$$