# Homework 3

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**Problem** (3)**.** Note that I used $(x,1) \sim (f(x),0)$ for the problem.

We wish to use SvK (note that it might be easier to figure out via building a CW-complex). Let $v$ be the wedge point of $X$, and let $U_0$ be the thickened $\{v\} \times I$ which is open. Define $A = U_0 \cup X \times I \setminus \{\frac{1}{2}\}, B = X \times \left(\frac{1}{4}, \frac{3}{4}\right)$ as figure shown.

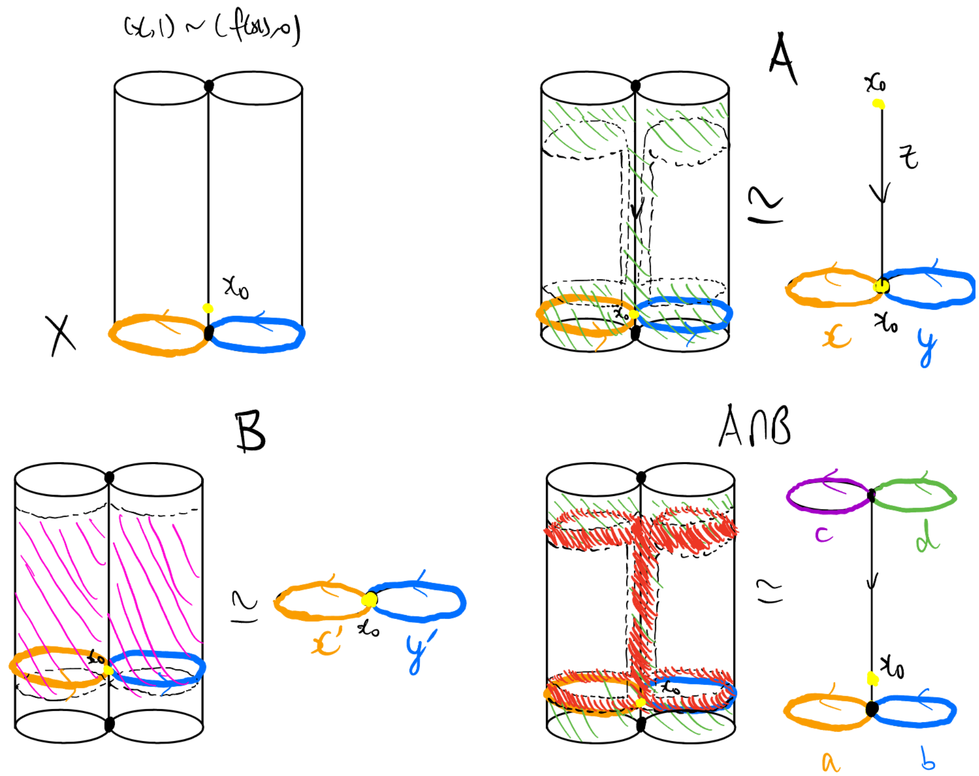


Figure 1: We omit the gluing detail at the top but one should keep it

Choose the base point $x_0 := \{v\} \times \{\frac{2}{5}\}$ (yellow) so that it is contained in $A, B, A \cap B$. By pushing the bottom up (so the base becomes $X \times \{\frac{2}{5}\}$) and applying the identification on the top, *i.e.* we can pretend that $(f(x), 1) \in X \times \{\frac{2}{5}\}$, thus $A$ has the shown homotopy equivalence. Clearly $B$ is homotopic to $X$ by collapsing the height. Lastly, $A \cap B$ is homotopic to a wedge of 4 circles by collapsing to the skeleton. Therefore, we have $\pi_1(A \cap B) = \langle a, b, c, d | \rangle$, $\pi_1(A) = \langle x, y, z | \rangle$ and $\pi_1(B) = \langle x', y' | \rangle$.

Now let's see what inclusion map induces on the fundamental groups. When we include $a, b$ in $A$, they are precisely $x, y$ respectively. When we include $c, d$ in $A$, we push them to the top so they are identified with loops postcomposed with $f$. Note that for $c, d$ to be expressed by $x, y$, they must travel down first along $z$ in the $A$ skeleton to get the correct orientation. That is, $c \mapsto zf_*(x)z^{-1}$ and $d \mapsto zf_*(y)z^{-1}$. It is clear that for $A \cap B \to B$ we just map $a, c \mapsto x', b, d \mapsto y'$. Thus the fibered coproduct is

$$\begin{aligned}\pi_1(T_f) &= \langle x, y, z, x', y' | x(x')^{-1}, y(y')^{-1}, zf_*(x)z^{-1}x', zf_*(y)z^{-1}y' \rangle \\ &= \langle x, y, z | zf_*(x)z^{-1}x^{-1}, zf_*(y)z^{-1}y^{-1} \rangle.\end{aligned}$$

**Problem** (6)**.**

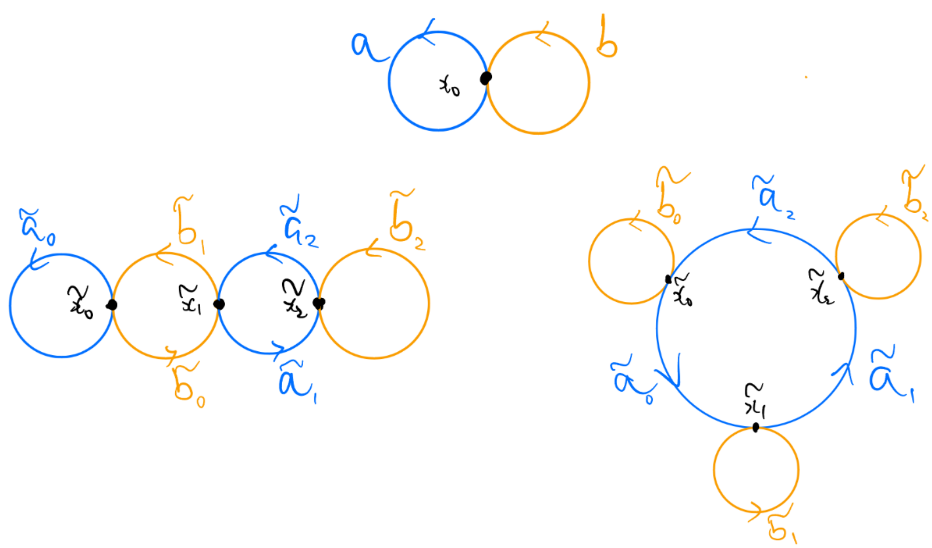


Figure 2

Since $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = F_2$, and by the correspondence theorem, index $n$ subgroups of $F_2$ corresponds to $n$-fold covering spaces of $S^1 \vee S^1$, it suffices find a normal and non-normal 3-fold covers of $S^1 \vee S^1$. In the figure, it is straightforward to verify that both are 3-fold covers. The right figure is clearly normal as it has $\frac{\pi}{3}$ rotational symmetry, which is the automorphism that maps between points in $p^{-1}(x_0)$. It corresponds to a normal subgroup $N = \langle a, b^2, ba^2b, babab \rangle$. I claim that the left figure is not normal. In particular, there is no deck transformation that maps $\tilde{x}_0$ to $\tilde{x}_1$, since the lift of $a$ based at $\tilde{x}_0$ is a loop. and the lift of $a$ based at $\widetilde{x}_1$ is a path, and they cannot be homeomorphic so no deck transformation sends $\widetilde{x}_0$ to $\widetilde{x}_1$. It corresponds to $H = \langle b, a^3, aba^2, a^2ba, ababa \rangle$.

**Problem** (7)**.** Since $S^n$ is simply connected, any $f : S^n \to X$ satisfies the lifting criterion $f_*(\pi_1(S^n)) = p_*(\widetilde{X}) = 0$. Thus we obtain a lift $\widetilde{f} : S^n \to X$. Since $\widetilde{X}$ is contractible, $\widetilde{f}$ is homotopic to the constant map via $H : S^n \times I \to \widetilde{X}$. As we see from the figure, since $H(x, 1)$ is constant, we can quotient it out to $H' : S^n \times I/S^n \times \{1\} \to \widetilde{X}$. The quotient is a cone of $S^n$, clearly homeomorphic to $D^{n+1}$ with $\widetilde{f}$ on the boundary. Denote this modified $H'$ as $\widetilde{H} : D^{n+1} \to \widetilde{X}$. Thus we obtain a map $F := p \circ \widetilde{H} : D^{n+1} \to X$, and $F|_{\partial D^{n+1}} = p \circ \widetilde{f} = f$ so it is the extension we seek.

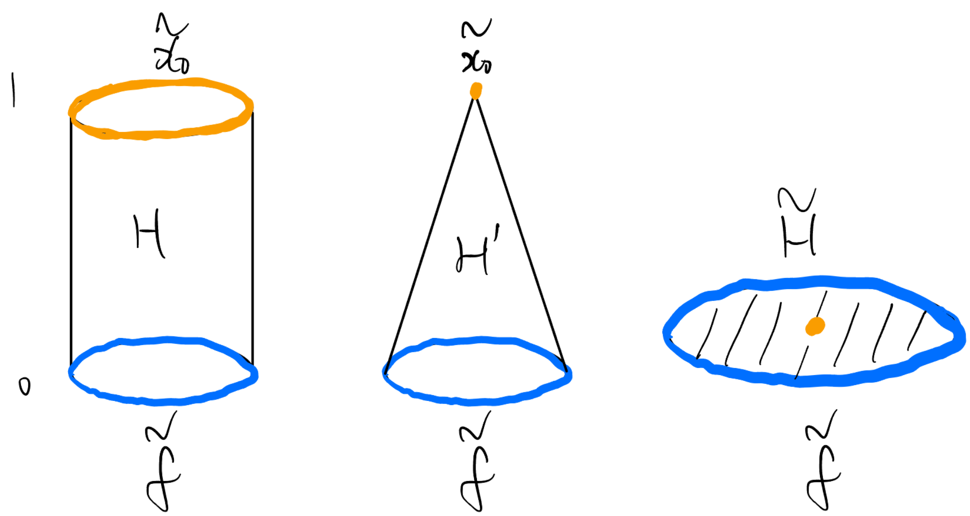


Figure 3

**Problem** (8)**.** We shall define $f$ inductively on the skeletons of $Y$. Since $Y^{(0)}$ is a discrete set of points, define $f_0 : Y^{(0)} \to X, y \mapsto x_0$, and it is a constant map so it is continuous. Since $Y$ is path-connected, $Y^{(1)}$ must be path-connected too, since the paths between 0-cells must traverse the 1-cells. We can quotient all 0-cells and paths between them so that $Y^{(1)}$ becomes a wedge of circles. Clearly we can trace out each circle via a loop, . Then take any representative $\eta \in \phi([\gamma]) : S^1 \to X$ and define that to be $f$ on that circle (think of circle as the quotient of an interval via attaching maps). Doing this for all circles yields a continuous map via the universal property of quotient map map defined for all circles $f_1 : Y^{(1)} \to X$ where $f_1$ is constant on the 0-cells and 1-cells connecting the 0-cells, so $f_1$ clearly extends

$f_0$.

Next, we consider $Y^{(2)}$, which is obtained from $Y^{(1)}$ by attaching $D^2$. Since any attaching map $a : S^1 \to Y^{(1)}$ extends to a characteristic map bounding a disk $D^2 \to Y^{(1)}$, $D^2$ is contractible and $Y^{(1)}$ path-connected, $a$ is nullhomotopic. Since $a$ introduces a relation in $\pi_1(Y)$, there exists some words in $\pi_1(Y)$ that gets killed. Since $\phi$ is a homomorphism, we see that $\phi = f_{1*}$ takes the words to constant as well, making $f_1 \circ a : S^1 \to X$ also nullhomotopic. Thus it extends to a map $D^2 \to X$, and we are allowed to define $f_2$ to be this map on the 2-cell using universal property of quotient map. Doing this for all 2-cells yield $f_2$ that extends $f_1$.

We establish the base case for $n = 2$. Now for induction ($n \geq 2$), assume that the desired $f_n : Y^{(n)} \to X$ is defined. For each $D^{n+1}$ we attach to $Y^{(n)}$ via the attaching map $a : S^n \to Y$. Then $f_n \circ a : S^n \to X$ extends to $g : D^{n+1} \to X$ by Problem 7. Thus we can define a map $h := (f_n, g) : Y^{(n)} \sqcup D^{n+1} \to X$. Since $g(x) = f_n(a(x))$ at the gluing site, the map is constant on each equivalent class of $Y^{(n)} \sqcup_a D^{n+1}$ so we obtain a continuous map $h' : Y^{(n)} \sqcup_a D^{n+1} \to X$. Constructing such map for all $(n+1)$-cells simultaneously yields a map $f_{n+1} : Y^{(n+1)} \to X$, completing the inductive step.

**Problem** (11).

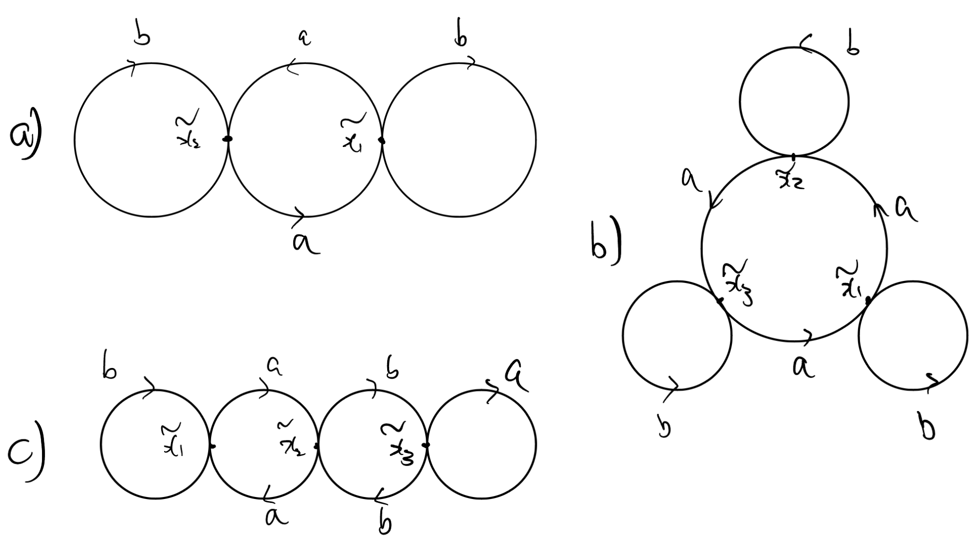


Figure 4: monodromy.png

(a) $a \mapsto (12), b \mapsto \text{id}$.

(b) $a \mapsto (123), b \mapsto \text{id}$.

(c) $a \mapsto (12), b \mapsto (23)$.

**Problem** (13)**.** Given $f : X \to S^1$, since $\pi_1(X)$ is finite, $f_*(\pi_1(X))$ is also finite, and the only finite subgroup of $\pi_1(S^1) \cong \mathbb{Z}$ is the trivial subgroup. Thus $f_*(\pi_1(X)) = 0 = p_*(\pi_1(\mathbb{R}))$, satisfying the lifting criterion. Thus we have a lift $\widetilde{f} : X \to \mathbb{R}$. Since $\mathbb{R}$ is contractible, $\widetilde{f}$ is nullhomotopic upstairs via $\widetilde{H}$, and $f$ is nullhomotopic downstairs via $p \circ \widetilde{H}$.