1 Homotopy and Stability

We require homotopy to be smooth maps in the category of smooth manifolds.

Definition 1.1 — Let $f : X \to Y$, a **perturbation** of f is a homotopy $F : X \times [-\varepsilon, \varepsilon] \to Y$ s.t. $F_0 = f$.

Definition 1.2 — A class of mappings $X \to Y$ is **stable** if it is preserved under perturbations.

Theorem 1.3

The following mappings are stable:

(1) Immersions

(2) Submersions

(3) local diffeomorphisms

(4) transversal maps

(5) embeddings

Proof. The first 4 came from semi-continuity of linear independence (rank). □

Theorem 1.4

Let X be compact, then any embedding $f : X \to Y$ is stable.

Proof. Since X is compact, f is an 1-1 immersion. Let F be a perturbation of f. Define $G : X \times I \to X \times I, G(x,t) = (F_t(x), t)$. Suppose that the injectivity of f is not stable wrt F. Then for all $\varepsilon > 0$ we should be able to find distinct x and y s.t. $G(x,t) = G(y,t)$. Then let $t_i \to 0$, we get a sequence x_i and y_i that satisfy above condition. Since X is compact, both sequences have a subsequence that converges to x_0 and y_0. Then

$$G(x_0, 0) = G(y_0, 0)$$

by continuity. □

2 Morse Functions

Lemma 2.1 (Morse)

$$f(x_1, \ldots, x_n) = f() + x^T Hx$$

Definition 2.2 — If $df_p = 0$, then the **index** $i(f, p)$ is the number of negative eigenvalues of Hessian of f at p.

Remark 2.3 Any function $f : M \to \mathbb{R}$ became Morse after a perturbation.

Theorem 2.4

Any Morse function has only finitely many critical points on compact manifold.

Proof. Let $g = \left(\frac{d^2 f}{dx_1{}^2}, \frac{d^2 f}{dx_n{}^2}\right)$. Then $dg_p = H_p f$. If p is not degenerate, then rank $dg_p = n$ so g is locally 1-1, so $df_q \neq 0$, for q near 0. □

Theorem 2.5

Let $f : M \to \mathbb{R}$ be a Morse function. Then

$$(M) = \sum_{i=0}^{n} (-1)^i C_i$$

where C_i is the number of critical points of index i.