0.1 Invariant EKF

TODO: add group-affine definition.

Let $\eta = X^{-1}\widehat{X}$ be the left-invariant error. In EKF, $\widehat{X}$ represents estimated state and X is the true state, where they share the same deterministic dynamics and only differ in initial state. In the context of uncertainty propagation, X is modeled by a stochastic process governed by an SDE, and $\widehat{X}$ is the propagated mean.

When the dynamics f_u is group-affine, the evolution of the error η satisfies

$$\frac{d}{dt}\eta = g_u(\eta)$$

where g_u satisfies $g_u(\eta) = f_u(\eta) - f_u(Id)\eta$. It turns out that g_u can be converted to a linear dynamic under the exponential coordinates. Let $(A\xi)^\wedge$ be the first-order approximation of $g_u(\exp(\xi))$. Then we have $\eta = \exp(\xi)$ for all time where

$$\frac{d}{dt}\xi = A\xi.$$

This means that the error η is trajectory-independent and can be linearized in the Lie algebra without approximation.

Note that if f_u is affine (*i.e.* $f_u(x) = f_0 + Ax + Bu$), we know that the error has the linear dynamics $\dot{e} = Ae$, independent of the state-trajectory. This is a generalization of that.

So in a stochastic setting, if the drift of SDE is group-affine, then no matter how state deviates from the mean, the drift error is trajectory-independent and can be propagated using the same ODE in the Lie algebra. If the drift is not group-affine, then using the drift equation (Ricatti?) along the mean trajectory should give additional error.

Euler-Poincare Equation:

$$\mathbb{I}\dot{\xi} =^*_\xi (\mathbb{I}\xi) + u$$