Homework 3

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Problem (1)**.** Recall that since $\lambda_i \geq 0$ and $h_i(x) \leq 0$ for all x and i, we have $\sum \lambda_i h_i(x) \leq 0$. Consider the first inequality when we choose $\lambda_i = 0$ for all i, we have

$$0 = \sum \lambda_i h_i(x^*) \leq \sum \lambda_i^* h_i(x^*) \leq 0$$

This forces $\sum \lambda_i^* h_i(x^*) = 0$. Since each $\lambda_i^* h_i(x^*) \leq 0$, it must be that $\lambda_i^* h_i(x^*) = 0$.

Now consider the second inequality where we choose $\lambda_i \geq 0$ s.t. $\lambda_i h_i(x^*) = \lambda_i^* h_i(x)$. This is possible because $h_i \leq 0$.

$$\begin{aligned}
\mathscr{L}(x^*, \lambda^*) &\leq \mathscr{L}(x, \lambda^*) \\
f(x^*) + \sum \lambda_i^* h_i(x^*) &\leq f(x) + \sum \lambda_i h_i(x) \\
f(x^*) &\leq f(x) + \sum \lambda_i^* (h_i(x) - h_i(x^*)) \\
&\leq f(x) + \underbrace{\sum \lambda_i^* h_i(x) - \sum \lambda_i h_i(x^*)}_{=0} && \text{1st inequality} \\
&\leq f(x)
\end{aligned}$$

Problem (2)**.** Substituting $\lambda^* = (1, 0)$, the Lagrangian becomes

$$\mathscr{L}(x_1, x_2) = x_1^2 - x_1 + x_2^2 + x_2 + 1.$$

Then

$$\nabla \mathscr{L} = \begin{pmatrix} 2x_1 - 1 \\ 2x_2 + 1 \end{pmatrix} \text{ and } \nabla^2 \mathscr{L} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \succ 0$$

Since the Hessian is already positive definite, it satisfies the 2nd order condition on the null space of $h_1' \left(\frac{1}{2}, -\frac{1}{2} \right)$.

Problem (3)**.** To maximize distance, we need to maximize the integral of velocity with respect to time. We are asked to express dt in terms of dm and thrust T in terms of other variables. We see that

$$dt = \frac{m\ dv}{T - D} = -\frac{c\ dm}{T}$$

$$\frac{1}{T} = \frac{1}{D}\left(1 + \frac{m\ dv}{c\ dm}\right)$$

Therefore,

$$\begin{aligned} x(t_f) - x(0) &= \int_0^{t_f} v(t)dt \\ &= \int_{m_0}^{m_f} -\frac{cv}{T}dm \\ &= \int_{m_f}^{m_0} \frac{cv}{T}dm \\ &= \int_{m_f}^{m_0} \frac{cv}{D}\left(1 + \frac{m\ dv}{c\ dm}\right) \end{aligned}$$

Problem (4). (i) By conservation of energy, we have

$$\begin{aligned} K = \frac{1}{2}mv_f^2 &= mgb = T \\ v_f &= \sqrt{2gb} \end{aligned}$$

Since we have constant acceleration induced by gravity, the average velocity must be half of v_f. Thus

$$t_f - 0 = \frac{d}{\frac{1}{2}v_f} = \sqrt{\frac{2(a^2+b^2)}{gb}}.$$

(ii) More generally, the conservation of energy gives

$$\begin{aligned} \frac{1}{2}mv(x) &= mgy(x) \\ v(x) &= \sqrt{2gy(x)} \end{aligned}$$

We also know that arc length has differential $ds = \sqrt{1+\dot{y}(x)^2}dx = v(x(t))dt$. Thus we have

$$\begin{aligned} t &= \int_{t_0}^{t_f} dt \\ t(y) &= \int_0^a \sqrt{\frac{1+\dot{y}(x)^2}{2gy(x)}}dx \end{aligned}$$

The Euler-Lagrange gives $y(1+\dot{y}^2)$ =const. Plugging in the solutions we have $\dot{y}(x) = \frac{dy}{dx} = \frac{dy}{d\psi}\frac{d\psi}{dx} = -\frac{\sin\psi}{1+\cos\psi}$ and

$$\beta(1+\cos\psi)\left(1 + \frac{\sin^2\psi}{(1+\cos\psi)^2}\right) = \beta\left(1 + \frac{\cos\psi + \cos^2\psi + \sin^2\psi}{1+\cos\psi}\right)$$

$$= 2\beta = \text{ const}$$

Thus this is indeed a solution. Since $dx = \beta(1+\cos\psi)d\psi$, we have

$$\begin{aligned} t &= \int_{\psi_1}^{\psi_2} \sqrt{\frac{1+\sin^2\psi/(1+\cos\psi)^2}{2g\beta(1+\cos\psi)}}\beta(1+\cos\psi)d\psi \\ &= \int_{\psi_1}^{\psi_2} \frac{\beta}{\sqrt{g\beta}}d\psi \\ &= \sqrt{\frac{\beta}{g}}(\psi_2-\psi_1) \end{aligned}$$

(iii) The boundary conditions tell us $\beta(1+\cos\psi_1)=0$ so $\cos\psi_1 = -1 \Rightarrow \psi_1 = \pi$. Thus

$$\begin{aligned} \alpha + \beta(\sin\psi_1 + \psi_1) &= 0 \\ \alpha + \beta(0+\pi) &= 0 \\ \pi &= -\frac{\alpha}{\beta} \end{aligned}$$

Then we see that letting $\theta = \psi_2 - \psi_1 = \psi_2 - \pi$, the boundary condition becomes

$$\begin{aligned} \alpha + \beta(\sin(\theta+\pi)+\theta+\pi) &= a \\ \alpha + \beta(-\sin\theta+\theta+\pi) &= a \\ \theta - \sin\theta &= \frac{a-\alpha}{\beta} - \pi \\ &= \frac{a}{\beta} \\ \beta(1+\cos(\theta+\pi)) &= b \\ \beta(1-\cos\theta) &= b \\ (1-\cos\theta) &= \frac{b}{\beta} \end{aligned}$$

Thus it is clear now that θ satisfies

$$(1-\cos\theta) - \frac{b}{a}(\theta - \sin\theta) = 0$$

Thus $\beta = \frac{b}{1-\cos\theta}$ and $\alpha = -\frac{b\pi}{1-\cos\theta}$.

(iv) If $a = 4$ and $b = 2$, then the time for the ramp is $t_1 = 0.79$s and the time for the cycloid is $\theta = 3.50837$ and $t_2 = 0.63$s. That is 0.16s of difference. The distance $d(t)$ of the ramp is $d(t) = \frac{1}{2}g\frac{b}{\sqrt{a^2+b^2}}t^2$. So $d(t_2) = 2.85$ft. It is clear from this formula that if $a \gg b$, $d(t)$ will be much smaller given a fixed time.

Problem (5)**.** We see that $F = \dot{y}(t)^2 + 12ty(t)$. Euler-Lagrange is

$$\begin{aligned} 12t &= \frac{d}{dt}(2\dot{y}) = 2\ddot{y} \\ \ddot{y} &= 6t \\ \dot{y} &= 3t^2 + C \\ y(t) &= t^3 + Ct + D \end{aligned}$$

Since $y(0) = 0 = D$ and $y(1) = 1 + C = 0 \Rightarrow C = -1$, we have the candidate minimizer

$$y^*(t) = t^3 - t.$$

We see that $F_r = 2r$, $F_{rr} = 2 > 0$ so y^* is regular. Also $F_{yy} = 0, F_{yr} = 0$. Let f be the perturbation with $f(0) = f(1) = 0$ and $2\omega(t, f, \dot{f}) = F_{yy}f^2 + 2F_{yr}f\dot{f} + F_{rr}\dot{f}^2$. Then the Jacobi condition requires

$$\begin{aligned} \omega_f &= \omega_{\dot{f}t} + \omega_{\dot{f}f}\dot{f} + \omega_{\dot{f}\dot{f}}\ddot{f} \\ 0 &= \dot{f}\ddot{f} + 0 + \ddot{f} \\ 0 &= (1 + \dot{f})\ddot{f} \\ \dot{f} &= -1 \text{ or } \ddot{f} = 0 \\ f(t) &= -t + C \text{ or } f(t) = C_1 t + D. \end{aligned}$$

But the initial value condition forces $C = C_1 = D = 0$ and $f(t) = -t$ doesn't satisfies $f(1) = 0$ so it must be that $f(t) = 0$. We see that between 0 and 1, there is no conjugate point to 0 (we don't have corners when we construct the new ϕ in the proof). Thus Jacobi condition is satisfied for y^*.

Finally, we check Weierstrass condition:

$$E(t, y, r, q) = F(t, y, q) - F(t, y, r) - (q - r)F_r(t, y, r)$$

$$= q^2 + 12ty - r^2 - 12ty - (q-r)2r$$
$$= q^2 - r^2 - (q-r)2r$$
$$= (q-r)^2 \geq 0 \ \forall \ q, r$$

Thus y^* (and any other function) satisfies the Weierstrass condition. Since y^* passes all four sufficient conditions, y^* is a strong local minimizer. Since this is the only candidate for a global minimizer, either it is the global minimizer or the solution doesn't exist. Assuming it is the former, we have

$$\begin{aligned}
\min J(y) = J(y^*) &= \int_0^1 (3t^2-1)^2 + 12t(t^3-t)dt \\
&= \int_0^1 (9t^4 - 6t^2 + 1 + 12t^4 - 12t^2)dt \\
&= \int_0^1 (21t^4 - 18t^2 + 1)dt \\
&= \left(\frac{21}{5}t^5 - 6t^3 + t\right)\Big|_0^1 \\
&= -\frac{4}{5}
\end{aligned}$$

Problem (6)**.** We first apply Euler-Lagrange:

$$\begin{aligned}
\frac{\partial F}{\partial x} &= \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} \\
6t^2x + 2t^3\dot{x} &= \frac{d}{dt}(2t^3x) = 6t^2x + 2t^3\dot{x}
\end{aligned}$$

which is trivially satisfied by all $x(t)$. Thus all $x(t)$ are extremals. We have

$$\begin{aligned}
\min J &= \int_{t_0}^{t_1} (3t^2x^2 + 2t^3x\dot{x})dt \\
&= \int_{t_0}^{t_1} \frac{d}{dt}\left(t^3x^2\right)dt \\
&= t^3x^2\Big|_{t_0}^{t_1} \\
&= t_1^3x_1^2 - t_0^3x_0^2
\end{aligned}$$

Problem (7)**.** (i) We have $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial \dot{y}} = (2\dot{y}^2 - 1)\cdot 2\dot{y} = 4\dot{y}^3 - 4\dot{y}$. Thus Euler-Lagrange is

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{y}} = (3\dot{y}^2 - 1)4\ddot{y} = 0$$

$$y(t) = \begin{cases} Ct + D & \ddot{y} = 0 \\ \pm\frac{1}{\sqrt{3}}t + C_1 & \dot{y}^2 = \frac{1}{3} \end{cases}$$

$$y(t) = Ct + D$$

Thus the extremals are line segments.

(ii) We have $F_r = 4r(r^2 - 1)$. Let $p = \dot{y}^*(t^-)$ and $q = \dot{y}^*(t^+)$, and $p \neq q$. Then by strong Erdmann corner condtions, p, q must satisfy

$$p(p^2 - 1) = q(q^2 - 1)$$

and

$$\begin{aligned} F(p) - pF_r(p) &= F(q) - qF_r(q) \\ (p^2-1)^2 - 4p^2(p^2-1) &= (q^2-1)^2 - 4q^2(q^2-1) \\ (p^2-1)(p^2-1-4p^2) &= (q^2-1)(q^2-1-4q^2) \\ (p^2-1)(3p^2+1) &= (q^2-1)(3q^2+1) \end{aligned}$$

If $p = 0$ then $q = \pm 1 \neq p$. But then $(p^2-1)(3p^2+1) \neq 0$, a contradiction so $p \neq 0$. If we assume that $q^2 - 1 \neq 0$, then we have

$$\begin{aligned} \frac{p^2-1}{q^2-1} = \frac{q}{p} &= \frac{3q^2+1}{3p^2+1} \\ 3p^2q + q &= 3pq^2 + p \\ (p-q)(3pq-1) &= 0 \\ pq &= \frac{1}{3} \qquad\qquad p \neq q \end{aligned}$$

But this is symmetric, so substituting this back to the first condition would give $p = q$, a contradiction. Therefore, it must be that $q^2 - 1 = 0 \Rightarrow q = \pm 1$. But the two conditions forces $p = \pm 1$ as well so the slope must be $p = -q = \pm 1$.

(iii) We know any local minimum must be PWS where each piece has the form $y(t) = \pm t + d$. Then the figure shows a function y^* with such form. Then $F(y^*, \dot{y}^*, t) = 0$ and thus

$J(y^*) = 0$. But since $F \geq 0$, $J(y) \geq 0$ always. So this y^* indeed achieves the global minimum.

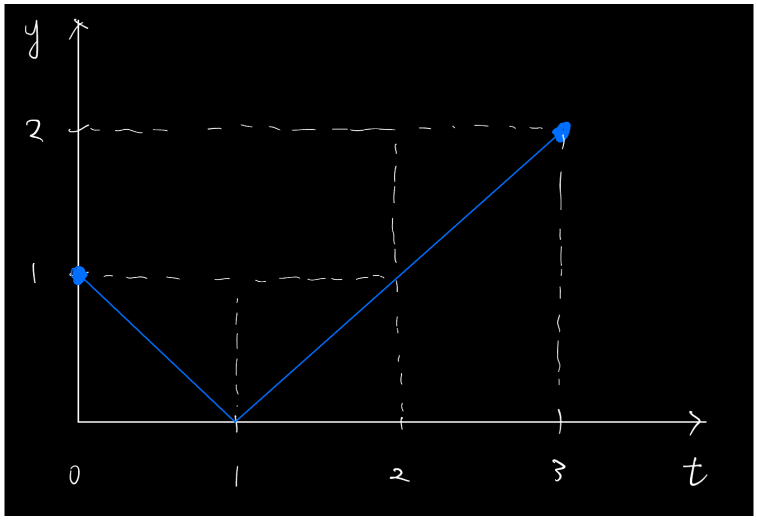

Figure 1: A PWS solution with a corner.

(iv) In this case, there is no solution with corners as the figure clearly shows we cannot maintain a slope of ± 1 to reach $y(1) = 2$. In the smooth case, the only extremal that satisfies the boundary conditions is $y^* = 2t$. Then $\dot{y} = 2$ and $F_{rr} = 12r - 4 = 20 > 0$ so strong Legendre is satisfied. For the Jacobi conditions, we have $F_{yy} = F_{yr} = 0$ and let f be the perturbation with $f(t_0) = f(t_1) = 0$ and $2\omega = 20\dot{f}^2$. Then

$$
\begin{aligned}
0 &= 20\ddot{f} + 0 + 20\ddot{f} \\
\ddot{f} &= 0 \\
f &\equiv 0
\end{aligned}
$$

by the initial conditions. Thus there is no conjugate points as Problem 5.

Finally, Weierstrass conditions says

$$
\begin{aligned}
E(t, y, q, r) &= (q^2 - 1)^2 - (2^2 - 1)^2 - (q - 2)4 \cdot 2(2^2 - 1) \\
&= q^4 - 2q^2 - 24q + 39
\end{aligned}
$$

We see that when $q = 2$, $E(q) = -1 < 0$ so y^* failed the Weierstrass test. By the sufficient conditions, it is a weak local minimum.