Homework 10

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Problem (2.5.1)**.** Since $F(x) \neq z \ \forall \ x \in D$, we see that $u : X \to S^{n-1}$ can be extended to all of D. Hence by the boundary theorem, $W_2(f, z) = \deg_2(u) = 0$.

Problem (2.5.2)**.** Since $y_i \in \operatorname{int} B_i$, $u_i : \partial B_i \to S^{n-1}, x \mapsto \frac{f_i(x)-z}{\|f_i(x)-z\|}$ is well-defined. And $W_2(f_i, z) = \deg_2(u_i)$. Now consider the compact manifold (removing disjoint open balls remains closed and hence compact as a subspace) with boundary $\widetilde{D} := D - \bigcup_{i=1}^{\ell} \operatorname{int} B_i$. Let $\widetilde{f} : \partial \widetilde{D} = X \sqcup \bigsqcup_{i=1}^{\ell} \partial B_i \to \mathbb{R}^n$ be defined by f and f_i. Then define $\widetilde{u}$ as usual using $\widetilde{f}$ and let $\widetilde{F} = F|_{\widetilde{D}}$. Since $\widetilde{F}$ clearly doesn't hit z, we have

$$W_2(\widetilde{f}, z) = \deg_2(\widetilde{u}) = I_2(\widetilde{u}, z) = 0$$

Recall that $I_2(\widetilde{u}, z)$ is the number mod 2 of points in $\widetilde{u}^{-1}(z)$. Since $\partial \widetilde{D}$ is a disjoint unions of boundaries, we can compute the intersection number by summing the intersection numbers on each disjoint boundary (mod 2), as

$$\widetilde{u}^{-1}(z) = (\widetilde{u}|_X)^{-1}(z) \sqcup \bigsqcup_{i=1}^{n} (\widetilde{u}|_{\partial B_i})^{-1}(z) = u^{-1}(z) \sqcup \bigsqcup_{i=1}^{\ell} u_i^{-1}(z)$$

Thus,

$$\begin{aligned} I_2(u, z) + \sum_{i=1}^{\ell} I_2(u_i, z) = 0 \bmod 2 \\ I_2(u, z) &= -\sum_{i=1}^{\ell} I_2(u_i, z) \bmod 2 \\ I_2(u, z) &= \sum_{i=1}^{\ell} I_2(u_i, z) \bmod 2 \\ W_2(f, z) &= \sum_{i=1}^{\ell} W_2(f_i, z) \bmod 2 \end{aligned}$$

Problem (2.5.12)**.** By 8 we have a relation $W_2(X, z_0) = W_2(X, z_1) + \ell \bmod 2$ where ℓ is the intersection number of the ray from z_0 passing through z_1. By 9 we know that the winding number mod 2 of X around z completely determines the component that z is in. By 10 we know that the outside component has $W_2(X, z) = 0$. This implies that the points inside X

has $W_2(X,z)=1$. For any z and a ray from z transversal to X, let z' be a point on the ray super far away that it is certainly outside X and there is no intersection with X from z' to infinity. This allows ℓ to be exactly the number of intersections of the ray with X.

$(\Rightarrow)$: If z is inside X, then . We see that $\ell = W_2(X,z) - W_2(X,z') = 1 - 0 = 1 \bmod 2$. So ℓ is odd.

$(\Leftarrow)$: If ℓ is odd, then $W_2(X,z) = W_2(X,z') + \ell \bmod 2 = 0 + \ell \bmod 2 = 1$. So z is inside.

Problem (2.6.1)**.** Recall that $u : S^k \to S^k, \frac{f(x)}{\|f(x)\|}$ is well-defined iff $f : S^k \to \mathbb{R}^{k+1}$ doesn't hit the origin. Moreover, $f(-x) = -f(x)$ iff $u(-x) = -u(x)$ *i.e.* u carries antipodal points to antipodal points. Then $W_2(f,0) = \deg_2(u) = 1$. So they are equivalent statements.

Problem (2.6.2)**.** Let $f : S^1 \to S^1$ satisfy $f(-x) = -f(x)$. So

$$\begin{aligned}
(\cos g(t+\pi), \sin g(t+\pi)) &= f(\cos(t+\pi), \sin(t+\pi)) \\
&= f(-\cos t, -\sin t) \\
&= -f(\cos t, \sin t) \\
&= (-\cos g(t), -\sin g(t))
\end{aligned}$$

Hence we have $\cos g(t+\pi) = -\cos g(t) = \cos(g(t)+q\pi)$ where q is odd. Therefore, $g(t+\pi) = g(t) + q\pi$. The sin part yields the same relation. We see that $g(t+2\pi) = g(t+\pi+\pi) = g(t+\pi) + q\pi = g(t) + 2q\pi$ so this is exactly the q described by Exercise 2.4.8. It follows that $\deg_2(f) = q \bmod 2 = 1$ since q is odd.

Problem (2.6.3)**.** Since $p_1, \ldots, p_n$ are homogeneous polynomials of odd degree, their polynomial functions $f_1, \ldots, f_n : \mathbb{R}^{n+1} \to \mathbb{R}$ are odd functions, *i.e.* $f_i(-x) = -f_i(x)$. Define $g_i := f_i|_{S^n}$. Then by corollary of Boruk-Ulam, g_i's must possess a common root, say v, in S^n. But if v is a root of g_i, it is a root of f_i, and any scalar multiple tv of v is also a root of f_i due to homogeneous degrees. Hence they share a line of roots tv.