Homework 4

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Problem (1). (a) No. Here, jointly measurable means that $(\omega, t) \mapsto Z_t(\omega)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}([0,\infty))$ measurable. That implies that any slice is measurable, *i.e.* if we fix an ω, then $Z_t(\omega) = \omega(t)$ should be $\mathcal{B}([0,\infty))$ measurable. That is, ω should be a $\mathcal{B}([0,\infty))$-measurable function. However, setting ω to be 0 on a Vitali subset of $[0,\infty)$ and 1 on its complement makes it not measurable, a contradiction.

(b) Versions only care about same distribution whereas modifications care about a.s. pointwise equality, *i.e.* $\mathbb{P}\left(X_t = \widetilde{X}_t\right) = 1$ for all t. Consider (X_t) to be an iid standard Gaussian process and $\widetilde{X}_t := -X_t$. Since standard Gaussian is symmetric about 0, they clearly have the same distribution. However, they only equal each other at exactly one point (0), so they are not modifications of each other.

Problem (2). Since f is smooth, it has bounded f' in $[0,1]$. Similarly, $\sqrt{t}$ is also bounded. Therefore, we have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_0^1 |f'(t)|\sqrt{t}xe^{-\frac{x^2}{2}} \mathrm{d}x\mathrm{d}t < \infty.$$

This allows us to apply Fubini:

$$\begin{aligned}
\mathbb{E}\left[\int_0^1 f'(t)B_t \mathrm{d}t\right] &= \mathbb{E}\left[\int_0^1 f'(t)\sqrt{t}B_1 \mathrm{d}t\right] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_0^1 f'(t)\sqrt{t}xe^{-\frac{x^2}{2}} \mathrm{d}x\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}} \int_0^1 f'(t)\sqrt{t} \left(\int_{-\infty}^{\infty} xe^{-\frac{x^2}{2}} \mathrm{d}x\right) \mathrm{d}t \\
&= 0.
\end{aligned}$$

So the mean is 0.

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^1 f'(t)B_t \mathrm{d}t\right)\right] &= \mathbb{E}\left[\int_0^1 f'(t) \int_0^1 f'(s)B_tB_s \mathrm{d}s\mathrm{d}t\right] \\
&= \int_0^1 f'(t) \int_0^1 f'(s)\mathbb{E}\left[B_tB_s\right] \mathrm{d}s\mathrm{d}t \\
&= \int_0^1 f'(t) \int_0^1 f'(s) s \wedge t \mathrm{d}s\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 f'(t)\left(\int_0^t f'(s)s\mathrm{d}s + t\int_t^1 f'(s)\mathrm{d}s\right) \\
&= \int_0^1 f'(t)\left(\left.(f(s)s)\right|_0^t - \int_0^t f(s)\mathrm{d}s + \left.tf(s)\right|_t^1\right)\mathrm{d}t \\
&= \int_0^1 f'(t)\left(f(t)t - \int_0^t f(s)\mathrm{d}s - f(t)t\right)\mathrm{d}t \\
&= -\int_0^1 f'(t)\int_0^t f(s)\mathrm{d}s\mathrm{d}t \\
&= \int_0^1 f(s)\int_1^s f'(t)\mathrm{d}t\mathrm{d}s && \text{Fubini} \\
&= \int_0^1 f(s)\left.f(t)\right|_1^s\mathrm{d}s \\
&= \int_0^1 f(s)^2\mathrm{d}s.
\end{aligned}
$$

Problem (3)**.** Notice that since $T_{k+1} \geq T_k$, we can rewrite

$$
\begin{aligned}
T_{k+1} - T_k &= \inf\{t \geq T_k : B_t = 1 + B_{T_k}\} - T_k \\
&= \inf\{t - T_k \geq 0 : B_t - B_{T_k} = 1\} \\
&= \inf\{s \geq 0 : B_{s+T_k} - B_{T_k} = 1\} \\
&= \inf\{s \geq 0 : B_s^{(T_k)} = 1\}.
\end{aligned}
$$

By the strong Markov property, we know that $B_t^{(T_k)} \sim B_t$ and is independent of $\Sigma_{T_k} \supset \Sigma_{T_{k-1}} \supset \cdots \supset \Sigma_{T_1}$. Thus it is iid with $T_1, T_2 - T_1, \ldots, T_{k-1} - T_{k-2}$ by simple induction.

Problem (4)**.** Take any limit point t of E with sequence $(t_k) \subset E$ such that $t_k \to t$, by continuity $B_t = \lim_{k\to\infty} B_{t_k} = \lim_{k\to\infty} 1 = 1$, ie any limit point of E is in E. Hence E is closed. Suppose E is finite, then let $s := \max E < \infty$. By the Markov property, $B_t^{(s)} = B_{t+s} - B_s$ is a BM. Therefore it crosses $B_s = 1$ a.s. at least once after time s by the fact of BM that $\sup B_t^{(s)} > 0$ and $\inf B_t^{(s)} < 0$ for any time after s, a contradiction that s is the maximum. Thus, E must be infinite.

Next, notice that for a fixed t, we have $\mathbb{P}(B_t = 1) = 0$. Since $(t, \omega) \mapsto B_t(\omega)$ is jointly measurable, we can apply Fubini on the product Lebesgue measure:

$$
\mu(E) = \int_{\Omega\times[0,\infty)} \mathbb{1}_E \mathrm{d}\mu = \int_0^\infty \int_\Omega \mathbb{1}_{B_t=1}\mathrm{d}\lambda\mathrm{d}t = \int_0^\infty \mathbb{P}(B_t = 1)\,\mathrm{d}t = \int_0^\infty 0\mathrm{d}t = 0.
$$

Thus, E has Lebesgue measure zero.

Finally, for any $0 \le a < b$ such that $(a, b) \subset E$, by the Markov property we have BM $B_t^{(a)} \equiv 0 \; \forall \; t \in (0, b-a)$. But as a BM, it must satisfies a.s. $\sup_{t\in(0,b-a)} > 0$ and $\inf_{t\in(0,b-a)} < 0$, so it must be that $\mathbb{P}\left((a, b) \subset E : 0 \le a < b\right) = 0$.

Problem (5)**.** For any $t_0 \in \mathbb{R}$, if $B_s = t_0$, then $B_s^{(t_0)}$ again has the supremum > 0 infimum < 0 property for any neighborhood, so t_0 is not a local maximum. To show a.s. the set S of local maximum of BM is countably infinite, we need to establish a bijection between that and a countable set, $\mathbb{Q}_+$.

First, we establish that $\mathbb{Q}_+$ injects into S. WLOG take any $p < r \in \mathbb{Q}_+$, pick any $q, s \in \mathbb{Q}_+$ such that $p < q < r < s$. By continuous path and EVT, there exist local maximum$a = \sup_{[p,q]} B_t$ and $b = \sup_{[r,s]} B_t$. Then by HW2, we know a.s. $a \neq b$, which yields a.s. injectivity. Given $a \in S$, since it is a local maximum, there exists a neighborhood with rational boundary such that a is a maximum there. Taking its rational lower boundary yields surjectivity. Thus, there exists a.s. a bijection between $\mathbb{Q}_+$ and S by taking the rational lower boundary of the local maximum.

Finally, density again follows from continuity of paths and EVT. For any $0 \le s < t$, there exists an $r \in (s, t)$ such that B_r is a local maximum of $[s + \varepsilon, t - \varepsilon]$ for sufficiently small ε.