Example 0.1

Recall a vector bundle $(E, B, \mathbb{R}^n)$ has a k-frame iff the structure group reduces to $\mathrm{GL}_{n-k}(\mathbb{R})$. If we choose a metric on E, then the structure group of E is $O(n)$. E has an orthonormal k-frame iff structure group reduces to $O(n-k)$. In terms of principal bundles. Let $\mathcal{F}(E)$ be the orthonormal frame bundle associated to E. Now E has an orthonormal k-frame iff $\mathcal{F}(E)/O(n-k)$ has a section. The fibers of $F(E)/O(n-k)$ are $O(n)/O(n-k) \cong V_{n,k}$. In Cor II.11, we have $\pi_i(V_{n,k})$. Unfortunately, $\pi_1(M)$ does not necessarily act trivially on $\pi_{n-k}(V_{n,k})$ if it is $\mathbb{Z}$. But if we take $\pi_{n-k}(V_{n,k})$ mod 2 then action is trivial (since any action on $\mathbb{Z}/2$ is trivial. So we have a primary obstruction to a k-frame over the $(n-k+1)$-skeleton of M. So $\gamma^{n-k+1}(E) \in H^{n-k+1}(M; \pi_{n-k}(V_{n,k}) \bmod 2)$. We set $w_\ell(E) := \gamma^\ell(E) \in H^\ell(M; \mathbb{Z}/2)$. This is called the **$\boldsymbol{\ell}$th- Steifel-Witney class of $\boldsymbol{E}$**. When ℓ is even, this is the primary obstruction to existence of an $n-\ell+1$-frame on $M^{(\ell-1)}$ that extends to $M^{(\ell)}$. If ℓ odd then $w_\ell(E)$ is the mod2 reduction of the primary obstruction.

Fact: w_ℓ determines primary obstruction for all ℓ.

Exercise: given $(E, M, \mathbb{R}^n)$,

(1) $w_\ell(E)0 \Leftrightarrow$ there exists n-frame over $M^{(0)}$ that extends over $M^{(1)}$ iff there is an orientation on E, *i.e.* structure group reduces to $\mathrm{SO}(n)$.

(2) If E is orientable, then $w_2(E) = 0 \Leftrightarrow$ there exists an $(n-1)$-frame over $M^{(1)}$ that extends to $M^{(2)}$ iff there exists an n-frame on $M^{(1)}$ that extends to $M^{(2)}$ (since there is a canonical unit vector with positive orientation orthogonal to $(n-1)$-frame). This is called a **spin** structure on E.

Example 0.2

If $(E, M, \mathbb{R}^n)$ is oriented, then $\pi_1(M)$ acts trivially on $\pi_{n-1}(V_{n,1}) \cong \mathbb{Z}$. Exercise: check this. So we get a primary obstruction $e(E) \in H^n(M; \mathbb{Z})$ to the existence of a non-zero section of E over $M^{(n)}$. $e(E)$ is called the **Euler class**.

Exercise:

(1) If $s : M \to E$ is a section and M a manifold. Then we can isotop s so it is transverse to zero section $Z = \{0 \in E_x : x \in M\}$. Then $e(E) = P.D.[s^{-1}(Z)]$ (Poincare duality of homology class of $s^{-1}(Z)$).

(2) $e(TM)([M]) = (M)$.

Example 0.3

Let $(E, M, \mathbb{C}^n)$ be a vector bundle. Structure group is $\mathrm{GL}_n(\mathbb{C})$ so this is a complex vector bundle. As discussed above, we can take structure group to be $U(n)$. Let $\mathcal{F}(E)$ be the frame bundle which is a $U(n)$ bundle over M.

Then E has a complex k-frame iff structure group reduces to $U(n)/U(n-k) \cong V_{n,k}(\mathbb{C}) \Leftrightarrow \mathcal{F}(E)/U(n-k)$ has a section. We have $\pi_i(V_{n,k}(\mathbb{C}))$ by Cor II.11.

Exercise: $\pi_1(M)$ acts trivially on $\pi_{2(n-k)+1}(V_{n,k}(\mathbb{C})$ as the fiber of $\mathcal{E}/U(n-k)$. Thus we get a primary obstruction to a $M^{2(n-k)+2}$:

$$\gamma^{2(n-k)+2}(E) \in H^{2(n-k)+2}(M; \mathbb{Z} = \pi_{2(n-k)+1}(V_{n,k}(\mathbb{C}))).$$

Define $c_k(E) = \gamma^{2k}(E) \in H^{2k}(M; \mathbb{Z})$. This is called the ***k*th Chern class of *E***. Then $c_k(E)$ is the obstruction to a complex $(n-k+1)$ frame on $M^{(2k-1)}$ that extends to $M^{(2k)}$.

Exercise:

(1) $c_n(E) = e(E)$.

(2) $w_{2i+1}(E) = 0$, which implies complex bundles are orientable.

(3) $w_{2i}(E) = c_i(E) \bmod 2$.

(4) $c_1(E) = 0 \Leftrightarrow$ structure group of E reduces to $\mathrm{SU}((n))$, “complex orientation".

(5) if $\overline{E}$ is E with the conjugate complex structure, *i.e.* $z \in \mathbb{C}$ multiply by $\overline{z}$, then $c_i(\overline{E}) = (-1)^i c_i(E)$. Hint: easy for $c_n(E)$, reduce to this case (see Milnor-Stasheff).

Example 0.4

$(E, M, \mathbb{R}^n)$, then $E \otimes_{\mathbb{R}} \mathbb{C}$ is a complex vector bundle. Then the ***i*th Pontrjagin class**

of $\boldsymbol{E}$ is

$$p_1(E) = (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(M; \mathbb{Z})$$

Exercise:

(1) show $E \otimes \mathbb{C}$ and $\overline{E \otimes \mathbb{C}}$ are isomorphic. Use this to show $c_{2i+1}(E \otimes \mathbb{C})$ is 2-torsion.

(2) If E is an oriented $\mathbb{R}^{2n}$-bundle, then

$$p_n(E) = e(E) \smile e(E)$$

(3) If E is a complex bundle and and $E^{\mathbb{R}}$ denotes the underlying real bundle. Then

$$E^{\mathbb{R}} \otimes \mathbb{C} \cong E \oplus \overline{E}.$$

(4) use 3) to show for complex $\mathbb{C}^n$-bundle E we have

$$1 - P_1(E) + p_2(E) - \ldots \pm p_n(E) = (1 + c_1(E) + \ldots c_n(E)) \smile (1 - c_1(E) + c_2(E) - \ldots \pm c_n(E))p_1$$

Characteristic class, in general, do not determine a bundle. But we have

(1) complex line bundles are determined by c_1, and any $\alpha \in H^2(M)$ is c_1 of some $\mathbb{C}$-bundle.

(2) c^2-bundles are determined by c_1 and c_2, and any $(\alpha, \beta) \in H^2(M) \times H^4(M)$ is (c_1, c_2) of some $\mathbb{C}^2$-bundle.

(3) SO(3)-bundles are isomorphic iff w_2, p_1 agree.

(4) SO(4)-bundles are isomorphic $\Leftrightarrow$ w_2, p_1, e agree.

Exercise: prove them.