Homework 2

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Problem (1)**.** Since $Dg(x,y) = \begin{bmatrix} 4x^3 - 3x^2 \\ 2y \end{bmatrix}$, it equals zero exactly when $y = 0$ and $x = 0$ or $\frac{4}{3}$. Strong normality is satisfied for all other points. This this case, the Lagrangian is

$$\mathscr{L}((x,y),1,\lambda) = x + \lambda(y^2 + x^4 - x^3).$$

The first-order conditions require that

$$\begin{aligned} \mathscr{L}_x &= 1 + 4\lambda x^3 - 3\lambda x^2 = 0 \\ \mathscr{L}_y &= 2\lambda y = 0 \\ \mathscr{L}_\lambda &= y^2 + x^4 - x^3 = 0 \end{aligned}$$

which forces $y = 0$, $x = 1$, and thus $\lambda = -1$. Thus $(1,0) \in \operatorname{int} \mathbb{R}^2$ is a candidate local minimizer. Since $g'(1,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the null space is $N = t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Since

$$\mathscr{L}_{(x,y)(x,y)} = \begin{pmatrix} -12x^2 + 6x & 0 \\ 0 & -2 \end{pmatrix}$$

we see that for the basis vector of N,

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -6 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -2 < 0$$

Thus $(1,0)$ is not a local minimum. The only remaining possibilities are the abnormal cases $(0,0)$ or $(\frac{4}{3}, 0)$. The Lagrangian is

$$\begin{aligned} \mathscr{L}((x,y),0,\lambda) &= \lambda(y^2 + x^4 - x^3) \\ \mathscr{L}_x &= 4\lambda x^3 - 3\lambda x^2 = 0 \\ \mathscr{L}_y &= 0 \\ \mathscr{L}_\lambda &= 0 \end{aligned}$$

which yields $x = 0$, $y = 0$, and λ can be anything nonzero. I claim that $(0,0)$ is the global minimum. Suppose $x < 0$, then $x^4 > 0$ and $-x^3 > 0$, thus $y^2 + x^4 - x^3 > 0$, not satisfying the constraint. Thus $f(x)$ is lower bounded by $x = 0$, which is achieved by $(0,0)$.

Problem (2)**.** For $\mu = 1$:

$$\mathscr{L}(x, 1, \lambda, \nu) = x_1^2 - x_2 + \lambda_1(x_1^2 + x_2^2 - 1) + \lambda_2(x_2 - 2) + \nu(x_1^3 + x_2 - 1)$$
$$\mathscr{L}_x = \begin{pmatrix} x_1(2 + 2\lambda_1 + 3\nu x_1) \\ 2\lambda_1 x_2 + \lambda_2 + \nu - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\lambda_1(x_1^2 + x_2^2 - 1) = 0$$
$$\lambda_2(x_2 - 2) = 0$$
$$\lambda_1, \lambda_2 \geq 0$$
$$x_1^3 + x_2 - 1 = 0$$
$$x_1^2 + x_2^2 - 1 \leq 0$$
$$x_2 - 2 \leq 0$$

Since if $x_2^* = 2$, $x_1^2 + (x_2^*)^2 - 1 = x_1^2 + 3 > 0$ not primal feasible, so it must be that $\lambda_2^* = 0$. The solutions that satisfy the system of equations and inequality constraints above are $x_1 = 0$, $x_2 = 1$, $\lambda_2 = 0$, and $\nu = 1 - 2\lambda_1$; $x_1 = 0.544, x_2 = 0.839, \lambda_1 = 4.92, \lambda_2 = 0$, and $\nu = -7.26$.

We see that only λ_1 is active, so the Jacobian to test normality is

$$\nabla \begin{pmatrix} x_1^2 + x_2^2 \\ x_1^3 + x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 & 2x_2 \\ 3x_1^2 & 1 \end{pmatrix}$$

Since $\ell = 2$, the only time this matrix is not rank 2 is when $x_1 = 0$. This means that our solution $(0, 1)$ is abnormal, but $(0.544, 0.839)$ is strongly normal.

$$\mathscr{L}_{xx} = \begin{pmatrix} 6\nu x_1 + 2\lambda_1 + 2 & 0 \\ 0 & 2\lambda_1 \end{pmatrix} = \begin{pmatrix} -11.86 & 0 \\ 0 & 9.84 \end{pmatrix}$$

is a local saddle point.

When $\mu = 0$,

$$\mathscr{L}(x, 0, \lambda, \nu) = \lambda_1(x_1^2 + x_2^2 - 1) + \lambda_2(x_2 - 2) + \nu(x_1^3 + x_2 - 1)$$
$$\mathscr{L}_x = \begin{pmatrix} x_1(2\lambda_1 + 3\nu x_1) \\ 2\lambda_1 x_2 + \lambda_2 + \nu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\lambda_1(x_1^2+x_2^2-1)=0$$
$$\lambda_2(x_2-2)=0$$
$$\lambda_1,\lambda_2\geq 0$$
$$x_1^3+x_2-1=0$$
$$x_1^2+x_2^2-1\leq 0$$
$$x_2-2\leq 0$$

and we get the same solution as the case when $\mu=1$.

Since this is the only candidate, $(0,1)$ is the minimizer we seek, the minimum is $f(0,1)=-1$.

Problem (3)**.** Since maximizing x_1 is the same as minimizing $-x_1$, we have $f(x_1,x_2)=-x_1$ For all other points, assume $\mu=1$:

$$\mathscr{L}(x_1,x_2,1,\lambda_1,\lambda_2)=-x_1+\lambda_1(x_2-(1-x_1)^3)-\lambda_2x_2$$
$$\mathscr{L}_x=\begin{pmatrix}-1+3\lambda_1(1-x_1)^2\\ \lambda_1-\lambda_2\end{pmatrix}=0$$
$$\lambda_1(x_2-(1-x_1)^3)=0$$
$$-\lambda_2x_2=0$$
$$\lambda_1,\lambda_2\geq 0$$
$$x_2-(1-x_1)^3\leq 0$$
$$-x_2\leq 0$$

We see that $\lambda_2\neq 0$ since otherwise $\lambda_1=0$ and we have $-1=0$. It must be that $x_2=0$ and thus $-(1-x_1)^3=0$ which forces $x_1=1$, but this violates the first equation, so there is no solution.

Now for $\mu=0$,

$$\mathscr{L}(x_1,x_2,0,\lambda_1,\lambda_2)=\lambda_1(x_2-(1-x_1)^3)-\lambda_2x_2$$
$$\mathscr{L}_x=\begin{pmatrix}3\lambda_1(1-x_1)^2\\ \lambda_1-\lambda_2\end{pmatrix}=0$$
$$\lambda_1(x_2-(1-x_1)^3)=0$$

$$-\lambda_2 x_2 = 0$$
$$\lambda_1, \lambda_2 \geq 0$$
$$x_2 - (1-x_1)^3 \leq 0$$
$$-x_2 \leq 0$$

The solution is $(1,0)$ with both constraints active. We have $g(x_1,x_2) = \begin{pmatrix} x_2 - (1-x_1)^3 \\ -x_2 \end{pmatrix}$.

$$g'(x_1,x_2) = \begin{pmatrix} 3(1-x_1)^2 & 1 \\ 0 & -1 \end{pmatrix}$$

Thus the point is abnormal iff $x_1 = 1$. So $(1,0)$ is abnormal! Thus the maximum of x_1 is achieved at 1.

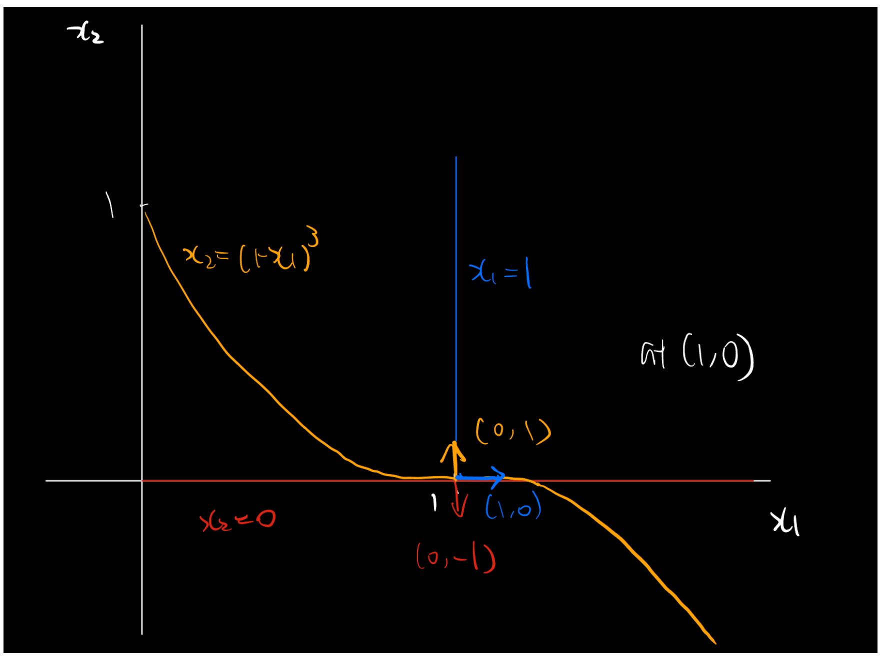

Figure 1: We see that at $(1,0)$ the gradients of two active constraints are parallel.

Problem (4)**.** For $\mu = 1$, we have

$$\mathscr{L}(x,1,\lambda) = -5x_1 - x_2 + \lambda_1(-x_1) + \lambda_2(3x_1 + x_2 - 11) + \lambda_3(x_1 - 2x_2 - 2)$$
$$\mathscr{L}_x = \begin{pmatrix} -5 - \lambda_1 + 3\lambda_2 + \lambda_3 \\ -1 + \lambda_2 - 2\lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$
\begin{aligned}
\lambda_1(-x_1) &= 0 \\
\lambda_2(3x_1 + x_2 - 11) &= 0 \\
\lambda_3(x_1 - 2x_2 - 2) &= 0 \\
\lambda_1, \lambda_2, \lambda_3 &\geq 0 \\
-x_1 &\leq 0 \\
3x_1 + x_2 - 11 &\leq 0 \\
x_1 - 2x_2 - 2 &\leq 0
\end{aligned}
$$

The only solution is $x_1 = \frac{24}{7}$, $x_2 = \frac{5}{7}$, $\lambda_1 = 0$, $\lambda_2 = \frac{11}{7}$, and $\lambda_3 = \frac{2}{7}$.

Since $\lambda_1 = 0$, the Jacobian to test normality is We compute

$$
\nabla \begin{pmatrix} 3x_1 + x_2 - 11 \\ x_1 - 2x_2 - 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}
$$

which has rank 2, same as the dimension of its image. Thus the solution is strongly normal.

$$
\mathscr{L}_{xx} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
$$

For $\mu = 0$, we have

$$
\begin{aligned}
\mathscr{L}(x, 0, \lambda) &= \lambda_1(-x_1) + \lambda_2(3x_1 + x_2 - 11) + \lambda_3(x_1 - 2x_2 - 2) \\
\mathscr{L}_x &= \begin{pmatrix} -\lambda_1 + 3\lambda_2 + \lambda_3 \\ \lambda_2 - 2\lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\lambda_1(-x_1) &= 0 \\
\lambda_2(3x_1 + x_2 - 11) &= 0 \\
\lambda_3(x_1 - 2x_2 - 2) &= 0 \\
\lambda_1, \lambda_2, \lambda_3 &\geq 0 \\
-x_1 &\leq 0 \\
3x_1 + x_2 - 11 &\leq 0 \\
x_1 - 2x_2 - 2 &\leq 0
\end{aligned}
$$

which has no solution with one of multipliers nonzero.

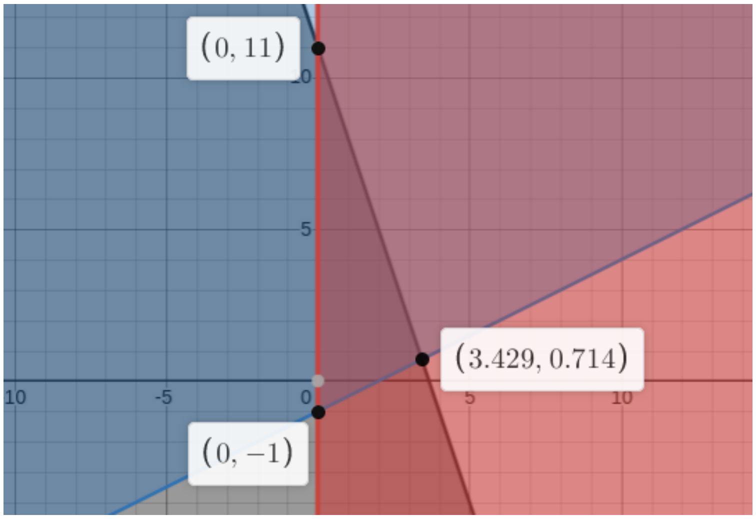

Figure 2: The feasible region is the triangle formed by the three points. We see that the minimizer is at a vertex of the triangle.

The minimum is therefore $-5 \cdot \frac{24}{7} - \frac{5}{7} = -\frac{125}{7}$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Problem 4
function [c,ceq] = constraints4(x)
x1=x(1);
x2=x(2);
c=[-x1;3*x1+x2-11;x1-2*x2-2];
ceq = [];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
f=@(x) -5*x(1)-x(2);
A = [];
b = [];
Aeq = [];
beq = [];
lb = [];
ub = [];
nonlcon=@constraints4;
```

```
x0=[1;0];
x=fmincon(f,x0,A,b,Aeq,beq,lb,ub,nonlcon)
fx = f(x)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

The output is a match!

Problem (5). (1)

$$\min \quad \begin{pmatrix} 400 & 360 & 550 & 470 & 600 & 500 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{pmatrix}^T$$

$$\text{subject to} \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{pmatrix}^T = \begin{pmatrix} 200 \\ 360 \\ 340 \\ 500 \\ 400 \end{pmatrix}$$

(2)

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Problem 5
f=[400;360;550;470;600;500];
A = [];
b = [];
Aeq = [1 1 0 0 0 0;0 0 1 1 0 0; 0 0 0 0 1 1; 1 0 1 0 1 0; 0 1 0 1 0 1];
beq = [200;360;340;500;400];
lb = [0;0;0;0;0;0];
ub = [500;400;500;400;500;400];
x=linprog(f,A,b,Aeq,beq,lb,ub);
fx=f char 39*x
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

The minimum cost schedule is $\begin{pmatrix} 200 & 0 & 300 & 60 & 0 & 340 \end{pmatrix}^T$ and the minimum cost is 443200 dollars.

Problem (6). (1)

$$\mathscr{L} = V\sin\gamma + \lambda_1(T(V)\cos(\alpha+\varepsilon) - D(V,\alpha) - mg\sin\gamma) + \lambda_2(T(V)\sin(\alpha+\varepsilon) + L(V,\alpha) - mg\cos\gamma)$$

So the first-order necessary conditions are

$$\begin{aligned}
\mathscr{L}_V &= \sin\gamma + \lambda_1(T'(V)\cos(\alpha+\varepsilon) - D_V(V,\alpha)) + \lambda_2(T'(V)\sin(\alpha+\varepsilon) + L_V(V,\alpha)) = 0 \\
&= \sin\gamma + (\lambda_1\cos(\alpha+0.0349) + \lambda_2\sin(\alpha+0.0349))(-0.04312 + 2\cdot 0.008392V) \\
&\quad - 2\lambda_1 V(0.07351 - 0.08617\alpha + 1.996\alpha^2) + 2\lambda_2 V(0.1667 + 6.231\alpha - 21.65[\max(0,\alpha-0.2094)]^2) \\
\mathscr{L}_\alpha &= -\lambda_1(T(V)\sin(\alpha+\varepsilon) - V^2(-0.08617 + 2\cdot 1.996\alpha + 6.231 - 2\cdot 21.65[(\alpha-0.2094) \text{ or } 0])) \\
\mathscr{L}_\gamma &= (V - mg)\cos\gamma + mg\sin\gamma \\
0 &= T(V)\cos(\alpha+\varepsilon) - D(V,\alpha) - mg\sin\gamma \\
0 &= T(V)\sin(\alpha+\varepsilon) + L(V,\alpha) - mg\cos\gamma
\end{aligned}$$

(2)

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Problem 6b
function [c,ceq] = constraints6(x)
T=@(V) 0.2476-0.04312*V+0.008392*V^2;
D=@(V,a) V^2*(0.07351-0.08617*a+1.996*a^2);
L=@(V,a) V^2*(0.1667+6.231*a-21.65*(max([0 a-0.2094])^2));
e=0.0349;
w=180000;
c=[];
V=x(1);
a=x(2);
gamma=x(3);
ceq = [T(V)*cos(a+e)-D(V,a)-w*sin(gamma);T(V)*sin(a+e)+L(V,a)-w*cos(gamma)];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
g=32.17;
W=180000;
l=2*W/(0.002203*1560*g);
f=@(x) -x(1)*sin(x(3)); %maximize is the same as minimize the negative
A = [];
b = [];
Aeq = [];
```

```
beq = [];
lb = [1,0.1,0.1];
ub = [2,0.2,0.2];
nonlcon=@constraints6;
x0=[342/sqrt(g*l);6.39/180*pi;6.31/180*pi];
[x,fval,exitflag,output,lambda,grad,hessian]=fmincon(f,x0,A,b,Aeq,beq,lb,ub,nonlcon)
fval
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

I cannot get the correct values after hours of debugging.

(3) I do not have the lambdas from b so I cannot check if it is zero.