Homework 2

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Problem (LN 5.8)**.** Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ and we identify $T_p\mathbb{R}^n$ and $T_{f(p)}\mathbb{R}^m$ with $\mathbb{R}^n$ and $\mathbb{R}^m$ the standard way ($[\alpha] \mapsto \alpha'(0)$), then df_p may be identified with the linear transformation determined by the Jacobian matrix $(\partial f^i / \partial x_j)$.

Proof. Define $\alpha_i : (-\varepsilon, \varepsilon) \to \mathbb{R}^n, t \mapsto (0, \ldots, t, \ldots, 0)$ where t is at the ith entry. Then $[\alpha_i] \mapsto \alpha_i'(0) = e_i$ under the identification. The identification yields

$$\begin{aligned} df_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m, [\alpha] \mapsto [f \circ \alpha] \Rightarrow \\ df_p : \mathbb{R}^n \to \mathbb{R}^m, \alpha'(0) \mapsto (f \circ \alpha)'(0). \end{aligned}$$

Then,

$$\begin{aligned} df_p(e_i) &= df_p(\alpha'(0)) \\ &= (f \circ \alpha_i)'(0) \\ &= Df(0) \circ \alpha_i'(0) && \text{Euclidean chain rule} \\ &= Df(0)(e_i) \end{aligned}$$

Thus, we see that df_p (after identification) and the Jacobian matrix $Df(0)$ agree on the standard basis. Therefore they represent the same linear map. $\square$

The following two exercises show the functoriality of the differential operator.

Problem (LN 5.9)**.** Show that if $f : M \to N$ and $g : N \to L$ are smooth maps, then, for any $p \in M$, we have the chain rule

$$d(g \circ f)_p = dg_{f(p)} \circ df(p).$$

Proof. Let $[\gamma]$ be a tangent vector in T_pM. Thus $\gamma(0) = p$. Recall the definition (with identification):

$$df_p([\gamma]) = (f \circ \gamma)'(0).$$

By repeatedly applying this definition, we have

$$\begin{aligned} d(g \circ f)_p([\gamma]) &= ((g \circ f) \circ \gamma)'(0) \\ &= ((g \circ (f \circ \gamma))'(0) \\ &= dg_{f \circ \alpha(0)}(f \circ \gamma)'(0) \\ &= dg_{f(p)} df_p([\gamma]). \end{aligned}$$

□

Problem (LN 5.10)**.** Show that if $f : M \to N$ is a diffeomorphism, then df_p is a linear isomorphism for all $p \in M$. In particular, conclude that if M and N are diffeomorphic, then $\dim(M) = \dim(N)$.

Proof. Since f is a diffeomorphism, it admits a smooth inverse f^{-1}. Then given $p \in M$, we have

$$\begin{aligned} f^{-1} \circ f &= \mathrm{id}_M \\ d(f^{-1} \circ f)_p &= \mathrm{id}_{T_pM} && \text{differential of identity is identity} \\ d(f^{-1})_{f(p)} \circ df_p &= \mathrm{id}_{T_pM}. && \text{chain rule} \end{aligned}$$

The other direction follows similarly. Thus df_p has a two-sided linear inverse — it is a linear isomorphism. By linear algebra, the tangent spaces of M and N must have the same dimension for all points. Since the tangent space and the manifold must have the same dimension, $\dim M = \dim N$. □

Problem (do Carmo 0.2)**.** Prove that the tangent bundle of a smooth manifold M is orientable.

Proof. Let $\{U_\alpha, \phi_\alpha\}_{\alpha \in J}$ be an atlas of M. Since ϕ_α is a diffeomorphism, the differential $d\phi_\alpha : TU_\alpha \to \mathbb{R}^n \times \mathbb{R}^n, (p, v) \mapsto (\phi_\alpha(p), d\phi_p(v))$ is a diffeomorphism. Thus $\{TU_\alpha, (\phi_\alpha, d\phi_\alpha)\}_{\alpha \in J}$ is an atlas of TM. For any $p \in U_\alpha \cap U_\beta$, let $q = \phi_\alpha(p)$. The transition map is

$$\begin{aligned} d\phi_\beta \circ d\phi_\alpha^{-1}(q, v) &= (\phi_\beta \circ \phi_\alpha^{-1}(q), d\phi_{\beta p} \circ d\phi_\alpha^{-1}{}_q(v)) \\ d(d\phi_\beta \circ d\phi_\alpha^{-1})_{(q,v)} &= (d(\phi_\beta \circ \phi_\alpha^{-1})_q, d(d\phi_{\beta p} \circ d\phi_\alpha^{-1}{}_q)_v) \end{aligned}$$

$$= (d(\phi_\beta \circ \phi_\alpha^{-1})_q, d(d(\phi_\beta \circ \phi_\alpha^{-1})_q)_v).$$

Notice that $\phi_{\alpha\beta q} := d(\phi_\beta \circ \phi_\alpha^{-1})_q$ is a linear operator, so its derivative is itself for any v. It is also the derivative of a diffeomorphism so it has nonzero determinant. Thus we obtain

$$\begin{aligned}\det d(d\phi_\beta \circ d\phi_\alpha^{-1})_{(q,v)} &= \det(\phi_{\alpha,\beta q}, \phi_{\alpha,\beta q}) \\ &= \det(\phi_{\alpha,\beta q})^2 > 0,\end{aligned}$$

where the last equality comes from determinant of the product linear operator. This proves that the transition maps are orientation-preserving and thus TM is orientable. □

Problem (do Carmo 0.8)**.** Let M_1, M_2 be smooth manifolds. Let $f : M_1 \to M_2$ be a local diffeomorphism. Prove that if M_2 is orientable, then so is M_1.

Proof. Since M_2 is orientable, it admits an atlas $\{(V_\alpha, \psi_\alpha)\}_{\alpha \in J}$ where all transition maps are orientation-preserving, *i.e.* the determinant is positive. For any point $x \in M_1$, there exists an open set O_x such that $f|_{O_x}$ is a diffeomorphism onto its image. For any V_α that contain $f(x)$, we set $U_{\alpha,x} = f^{-1}(V_\alpha \cap f(O_x))$, $\phi_{\alpha,x} = \psi_\alpha \circ f|_{U_{\alpha,x}}$. Then if there is another $U_{\beta,x}$ that contains x, for any $p \in U_{\alpha,x} \cap U_{\beta,x}$, let $q = \phi_{\alpha,x}(p)$. Notice $f(p) = \psi_\alpha^{-1}(q)$. We abuse the notation f for $f|_{U_{\alpha,x} \cap U_{\beta,x}}$ below, the transition map is

$$\begin{aligned}\phi_{\beta,x} \circ \phi_{\alpha,x}^{-1}(q) &= \psi_\beta \circ f \circ f^{-1} \circ \psi_\alpha^{-1}(q) \\ d(\phi_{\beta,x} \circ \phi_{\alpha,x}^{-1})_q &= d\psi_{\beta\, f(p)} \circ df_p \circ df^{-1}_{f(p)} \circ d(\psi_\alpha^{-1})_q \\ \det d(\phi_{\beta,x} \circ \phi_{\alpha,x}^{-1})_q &= \left(\det d\psi_{\beta\, f(p)} \det d(\psi_\alpha^{-1})_q\right) \left(\det df_p \det df^{-1}_{f(p)}\right) \\ \det d(\phi_{\beta,x} \circ \phi_{\alpha,x}^{-1})_q &= \det\left(d\psi_{\beta\, f(p)} d(\psi_\alpha^{-1})_q\right) > 0.\end{aligned}$$

Therefore, $\{(U_{\alpha,x}, \phi_{\alpha,x})\}_{\alpha \in J, x \in M_1}$ is an atlas of M_1 with orientation-preserving maps. □