Homework 5

Jaden Wang

Problem (1)**.**

$$J(u) = \frac{1}{2}cx^2(t_f) + \frac{1}{2}\int_0^{t_f} u^2(t)dt$$

Let $\phi(x_f) = \frac{1}{2}cx_f^2$. The Hamiltonian is

$$H = \frac{1}{2}u^2 + pu$$

To compute the Ricatti equation, we need the following: $R_2 = H_{uu} = 1$ (problem is regular), $R_{12} = H_{xu} = 0$, and $R_1 = H_{xx} = 0$. Moreover, from $\dot{x} = u$, we have $A(t) = 0$ and $B(t) = 1$. Now we have

$$\begin{aligned}
\widetilde{A} &= A - BR_2^{-1}R_{12}^T = A = 0 \\
\Sigma &= BR_2^{-1}B^T = 1 \\
\widetilde{R} &= R_1 - R_{12}R_2^{-1}R_{12}^T = 0
\end{aligned}$$

Then the Ricatti equation is

$$\begin{aligned}
-\dot{S} &= \widetilde{A}^T S + S^T \widetilde{A} - PP + \widetilde{R} = -S^2 \\
\frac{dS}{S^2} &= dt \\
s(t) &= -\frac{1}{t + C_2}
\end{aligned}$$

with $S(t_f) = c$ since $Q_f = c$. Thus we finally obtain

$$S(t) = \frac{1}{\left(t_f + \frac{1}{c}\right) - t}$$

Therefore, the optimal control is given by

$$u^* = -R_2^{-1}B^T Sx = -\frac{x}{\left(t_f + \frac{1}{c}\right) - t}$$

And we have

$$\dot{x} = u = -\frac{x}{\left(t_f + \frac{1}{c}\right) - t}$$

$$\frac{dx}{x} = \frac{dt}{t - \left(t_f + \frac{1}{c}\right)}$$
$$x(t) = C_3\left(t - \left(t_f + \frac{1}{c}\right)\right)$$
$$x(t) = \frac{x_0}{t_f + \frac{1}{c}}\left(\left(t_f + \frac{1}{c}\right) - t\right) \qquad x(0) = x_0$$
$$x(t_f) = \frac{x_0}{ct_f + 1}$$

Note that x_0 and t_f are fixed. Thus we see that as $c \to \infty$, $x(t_f) \to 0$.

Problem (2). (a) The Hamiltonian is

$$H = 1 + p_1(\cos\theta + u(y)) + p_2 \sin\theta$$

with first-order condition

$$H_\theta = -p_1 \sin\theta + p_2 \cos\theta = 0$$
$$\tan\theta = \frac{p_2}{p_1}$$

Moreover, adjoint equations yield

$$\begin{pmatrix} \dot{p_1} \\ \dot{p_2} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha p_1(3 - 3y^2) \end{pmatrix}$$

which implies that p_1 is constant. Now consider

$$\frac{d}{dt}H_\theta = -p_1 \cos\theta\dot{\theta} + \dot{p_2}\cos\theta - p_2 \sin\theta\dot{\theta} = 0$$
$$\dot{p_2}\cos\theta = \dot{\theta}(p_1\cos\theta + p_2\sin\theta)$$
$$\alpha p_1(3 - 3y^2)\cos\theta = \dot{\theta}(p_1\cos\theta + p_2\sin\theta)$$
$$3\alpha(1 - y^2)\cos\theta = \dot{\theta}(\cos\theta + \tan\theta\sin\theta)$$
$$3\alpha(1 - y^2)\cos\theta = \dot{\theta}(\frac{\cos^2\theta + \sin^2\theta}{\cos\theta})$$
$$\dot{\theta} = 3\alpha(1 - y^2)\cos^2\theta$$

Here α is a parameter that scales the rate of change for the control θ.

(b) We can use the fact that H is time-independent and thus constant to relate $H(0) = H(t_f)$ and produce the relation in the hint. But I didn't use the relation for the numerical solution. We solve the following system of equations:

$$\begin{cases} \dot{x} = \cos\theta - 0.02(3y - y^3) \\ \dot{y} = \sin\theta \\ \dot{\theta} = 3 \cdot 0.02(1 - y^2)\cos^2\theta \end{cases}$$

with the conditions $x(0) = y(0) = 1, x(t_f) = y(t_f) = 0$. Solution is $t_f = 1.2168$ and

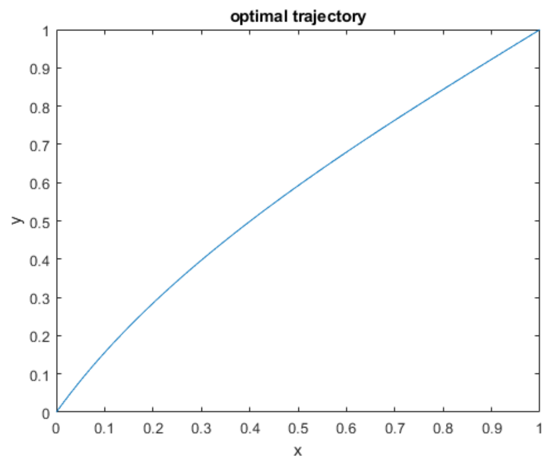

Figure 1: See the end of homework for code.

Problem (3)**.** The Hamiltonian is

$$H = u + p(-\alpha x + u) = (p+1)u - \alpha p x$$

So the switching function is $p + 1$. Since the Hamiltonian is linear in u, we want to use the Pontryagin maximum principle. To minimize $H(x^*, u, p^*, t)$, we want to make $(p+1)u$ as negative as possible. Thus

$$u^* = \begin{cases} m & p^* < -1 \\ 0 & p^* > -1 \end{cases}$$

I claim that $p^* \neq -1$. Suppose $p^* = 1$. Since T is free, transversality yields $H(T) = 0$. But since H is time-independent, H is constant so $H \equiv 0$. However, we see that

$$H(0) = 0 \cdot u - \alpha(-1)x(0) = \alpha a > 0,$$

a contradiciton.

Now solving $\dot{p} = -H_x = \alpha p$, we get $p(t) = Ke^{\alpha p}$. Notice that this is a monotone function and therefore can at most cross the line $p = -1$ once. That is, the control will switch at most once.

Case (1). If $K \geq 0$, $p(t) > -1$ for all t so we have no switching and $u^* = 0$. Then solving $\dot{x} = -\alpha x$ yields $x(t) = K_1 e^{-\alpha t}$ and $x(0) = K_1 = a$. Moreover, we have

$$\begin{aligned} x(T) = ae^{-\alpha T} &= c \\ e^{-\alpha T} &= \frac{c}{a} > 0 \\ -\alpha T &= \ln \frac{c}{a} \\ T &= \frac{1}{\alpha} \ln \frac{a}{c} \end{aligned}$$

If $a > c$, then $\frac{a}{c} > 1$ and $T > 0$. Thus, $J^* = 0$. But if $a \leq c$, then $T \leq 0$ which is unphysical. Thus in the case when $K \geq 0$, $a > c$, we have $u^* \equiv 0$.

Case (2). If $K < 0$, p might cross the line $p = -1$ once from above. Since $p(0) = K$ starts in the $p > -1$ regime, we have $u^*(0) = 0$. But

$$\begin{aligned} H(0) = (p(0)+1)u(0) - \alpha p(0)x(0) &= 0 \\ -\alpha a p(0) &= 0 \\ K = p(0) &= 0 \qquad\qquad\qquad \alpha, a > 0, \end{aligned}$$

a contradiction. Thus in this case we also have no switching and $u^* \equiv m$. Then we solve

$$\begin{aligned} \dot{x} &= -\alpha x + m \\ \frac{dx}{m - \alpha x} &= dt \\ x(t) &= \frac{m}{\alpha} - K_2 e^{-\alpha t} \\ x(0) &= \frac{m}{\alpha} - K_2 = a \\ x(t) &= \frac{m}{\alpha} - \left(\frac{m}{\alpha} - a\right) e^{-\alpha t} \end{aligned}$$

We have

$$x(T) = \frac{m}{\alpha} - \left(\frac{m}{\alpha} - a\right) e^{-\alpha T} = c$$

$$T = \frac{1}{\alpha} \ln \left(\frac{m - \alpha a}{m - \alpha c} \right)$$

If $a < c$, we see that $T > 0$, and $J^* = mT = \frac{m}{\alpha} \ln \left(\frac{m-\alpha a}{m-\alpha c} \right)$. If $a \geq c$, we get $T \leq 0$ which is unphysical. Thus in the case when $K < 0$, $a < c$, we have $u^* \equiv m$.

Problem (4). (a) We obtain

$$\begin{cases} \dot{x} = 15 \cos\theta + 2 \\ \dot{y} = 15 \sin\theta - 6 \end{cases}$$

So the Hamiltonian is

$$H = 1 + p_1(15\cos\theta + 2) + p_2(15\sin\theta - 6).$$

Adjoint equations yields that p_1 and p_2 are constants. First-order condition yields

$$H_\theta = -15p_1 \sin\theta + 15p_2 \cos\theta = 0$$
$$\tan\theta = \frac{p_2}{p_1}$$

so θ is also constant. We can integrate to get

$$x(t) = (15\cos\theta + 2)t - 20$$
$$y(t) = (15\sin\theta - 6)t$$

Using the terminal conditions, we solve

$$\begin{cases} (15\cos\theta + 2)t_f - 20 = -15 \\ (15\sin\theta - 6)t_f = 35.5 \end{cases}$$

which yields $\theta^* = 1.6198$ and $t_f = 3.9524$. Since t_f is free, transversality condition yields

$$H(t_f) = 0$$

Since everything in H is a constant, H is a constant and must equal to 0.

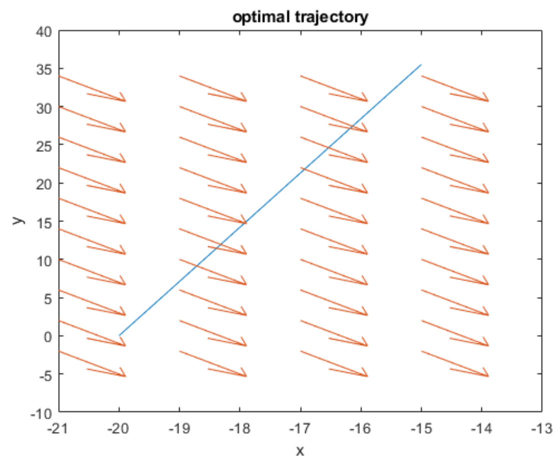

Figure 2: The optimal trajectory is a straight line.

(b) The only thing changed here is the transversality conditions:

$$H(t_f) = \psi_t \lambda = 0$$
$$-p_1(t_f) = \psi_x \lambda = \lambda(-0.25 - 0.006x_f^2)$$
$$-p_2(t_f) = \psi_y \lambda = -\lambda$$

where $\psi(x_f, y_f)$ is the terminal constraint. Thus

$$\tan\theta = \frac{p_2}{p_1} = \frac{1}{0.25 + 0.006x_f^2}$$

is still constant. This allows us to integrate the dynamics with the same initial conditions just as part (a).

$$x(t) = (15\cos\theta + 2)t - 20$$
$$y(t) = (15\sin\theta - 6)t$$

Now we can use the equations above to obtain x_f and y_f in terms of θ and t_f, and plug them into the terminal constraint and first-order condition:

$$\begin{cases} 25 - 0.25x_f - 0.002x_f^3 - y_f = 0 \\ \tan\theta = \frac{1}{0.25+0.006x_f^2} \end{cases}$$

and obtain $\theta^* = 1.3031$, $t_f = 3.0144$, which yields $x_f = -2.0114$ and $y_f = 25.5191$. Similar to part (a), the Hamiltonian consists of only constants so it is constant (zero).

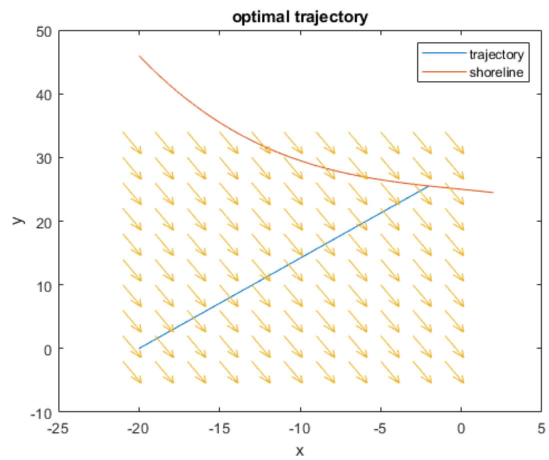

Figure 3: The optimal trajectory is still a straight line.

(c) The dynamics becomes

$$\begin{cases} \dot{x} = 15\cos\theta - (y - 50) \\ \dot{y} = 15\sin\theta + 2(x - 15) \end{cases}$$

And the Hamiltonian becomes

$$H = 1 + p_1(15\cos\theta - (y - 50)) + p_2(15\sin\theta + 2(x - 15))$$

The adjoint equations become

$$\dot{p}_1 = -H_x = -2p_2$$
$$\dot{p}_2 = -H_y = p_1$$

that satisfying the same transversality conditions as part (b). We still have

$$\tan\theta = \frac{p_2}{p_1}$$

from first-order condition, although it is no longer constant. Moreover,

$$\frac{d}{dt}H_\theta = -15\dot{p}_1\sin\theta - 15p_1\cos\theta\dot{\theta} + 15\dot{p}_2\cos\theta - 15p_2\sin\theta\dot{\theta} = 0$$

$$2p_2\sin\theta + p_1\cos\theta = (p_1\cos\theta + p_2\sin\theta)\dot{\theta}$$

$$\dot{\theta} = 1 + \frac{p_2\sin\theta}{p_1\cos\theta + p_2\sin\theta} = 1 + \frac{1}{\cot^2\theta + 1} = 1 + \sin^2\theta$$

with boundary condition

$$\tan\theta(t_f) = \frac{p_2(t_f)}{p_1(t_f)} = \frac{1}{0.25 + 0.006x_f^2}$$

Together with state dynamics, initial conditions, and the terminal constraint, we obtain the following solution with $t_f = 0.8974$:

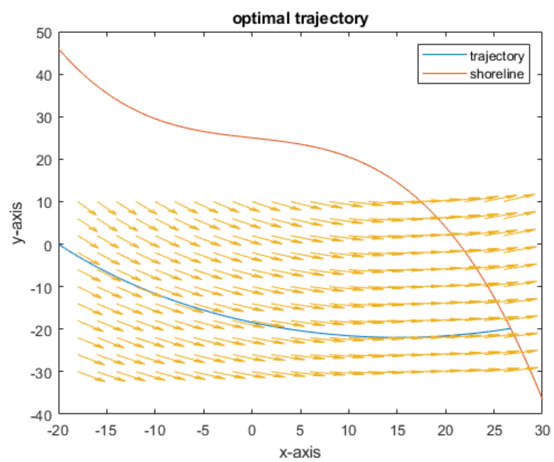

Figure 4: The trajectory aligns closely with the current this time.

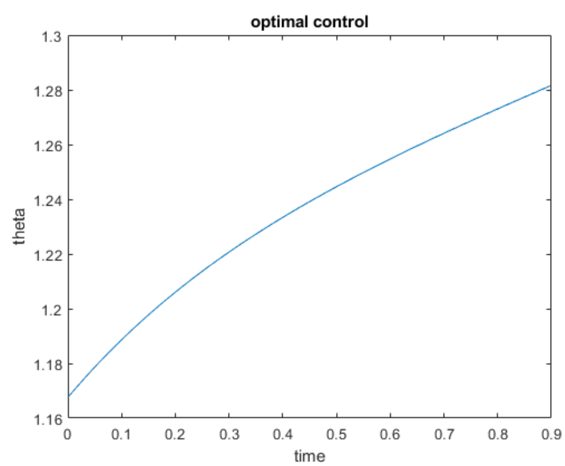

Figure 5: The optimal control θ.

We can compute the Hamiltonian by solving for p_1, p_2 and λ using three transversality conditions of $p_1(t_f), p_2(t_f)$, and $H(t_f) = 0$, but I ran out of time to do it.

Problem (5). (a) We have $J = \max x(t_f) = -\min -x(t_f)$ so $\phi(x_f) = -x_f$ and

$$\begin{cases} \dot{x} = \cos\theta + u_0 \sin^2 y \\ \dot{y} = \sin\theta \end{cases}$$

The Hamiltonian is

$$H = p_1(\cos\theta + u_0 \sin^2 y) + p_2 \sin\theta$$

The adjoint equations yield p_1 is constant,

$$\dot{p_2} = -p_1 u_0 \sin 2y$$

Since $x(t_f)$ is free, transversality condition yields

$$p_1 \equiv p_1(t_f) = \phi_{x_f} = -1$$

First-order condition yields

$$H_\theta = -p_1 \sin\theta + p_2 \cos\theta = \sin\theta + p_2 \cos\theta = 0$$
$$\tan\theta = -p_2$$

Hence we solve the following system:

$$\begin{cases} \dot{x} = \frac{1}{\sqrt{1+p_2^2}} + u_0 \sin^2 y \\ \dot{y} = -\frac{p_2}{\sqrt{1+p_2^2}} \\ \dot{p_2} = u_0 \sin 2y \end{cases}$$

with $x(0) = y(0) = 0$ and $y(t_f) = 0$.

(b) We solve the Ricatti equation to find the conjugate point (when P blows up). We have $R_2 = H_{\theta\theta} = \cos\theta - p_2 \sin\theta$, $R_{12} = H_{xu} = 0$, and $R_1 = \begin{pmatrix} 0 & 0 \\ 0 & -2u_0 \end{pmatrix}$. Furthermore, $A(t) = \frac{\partial f}{\partial x} = \begin{pmatrix} 0 & u_0 \sin 2y \\ 0 & 0 \end{pmatrix}$ and $B(t) = \frac{\partial f}{\partial \theta} = \begin{pmatrix} -\sin\theta \\ cos\theta \end{pmatrix}$. Staying on x-axis means that $\theta = 0$ and $y = 0$ throughout, so we have the following (with $R_2 = 1 > 0$ so regular):

$$\widetilde{A} = A - BR_2^{-1}R_{12}^T = 0 - 0 = 0$$

$$\Sigma = BR_2^{-1}B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} 1 \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\widetilde{R} = R_1 - R_{12}R_2^{-1}R_{12}^T = R_1 = \begin{pmatrix} 0 & 0 \\ 0 & -2u_0 \end{pmatrix}$$

Then Ricatti equation states:

$$-\dot{P} = 0 + 0 - \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -2u_0 \end{pmatrix}$$

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 + 2u_0 \end{pmatrix}$$

$$\dot{p_{22}} = p_{22}^2 + 2u_0$$

$$p_{22}(t) = \sqrt{2u_0}\tan\left(\sqrt{2u_0}(t - t_f)\right) \qquad P(t_f) = 0$$

We know that $\tan t$ blows up at $t = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$. Thus p_{22} blows up at

$$t_k = \frac{1}{\sqrt{2u_0}}\left(\frac{\pi}{2} + k\pi\right) + t_f$$

The conjugate point t_c is the first singularity encountered since the starting time $t = 0$. Thus t_c has the smallest k s.t. $t_c > 0$. Thus x-axis is a maximizing path only when there is no conjugate point, *i.e.* $t_f < t_c$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Problem 2(b)
solinit=bvpinit(linspace(0,1),[1,1,-2.2],1.2); %tau is in [0,1]
sol2=bvp4c(@odehw52,@bchw52,solinit);


x=sol2.y(1,:);
y=sol2.y(2,:);
theta = sol2.y(3,:);
tf= sol2.parameters;
time=tf*sol2.x;


figure
plot(x,y)
```

```
xlabel( char 39x char 39);
ylabel( char 39y char 39);
title( char 39optimal trajectory char 39);

function dydt = odehw52(t,y,tf)
dydt = tf*[cos(y(3))-0.2*(3*y(2)-y(2)^3)
          sin(y(3))
          0.6*(1-y(2)^2)*cos(y(3))^2];
end

function res = bchw52(ya,yb,tf)
res = [ya(1)-1
       yb(1)
       ya(2)-1
       yb(2)];
end

%Problem 4(a)
syms theta t
eqn=[(15*cos(theta)+2)*t-20==-15,(15*sin(theta)-6)*t==35.5];
sol4a=solve(eqn);

tf=double(sol4a.t);
tf=tf(1); %choose positive solution
theta = double(sol4a.theta);
theta = theta(1);

time=linspace(0,tf);
x=(15*cos(theta)+2)*time-20;
y=(15*sin(theta)-6)*time;
figure
plot(x,y)
```

```
hold on

[X, Y] = meshgrid(-21:2:-14, -2:4:36);
u = 2*ones(size(X));
v = -6*ones(size(Y));

% Normalize vectors for better visualization
magnitude = sqrt(u.^2 + v.^2);
u = u ./ magnitude;
v = v ./ magnitude;

quiver(X, Y, u, v);
hold off
xlabel( char 39x char 39);
ylabel( char 39y char 39);
title( char 39optimal trajectory char 39);

%Problem 4(b)
syms tf theta
x=@(t) (15*cos(theta)+2)*t-20;
y=@(t) (15*sin(theta)-6)*t;
eqn=[theta==atan(1/(0.25+0.006*x(tf)^2)),25-0.25*x(tf) ...
    -0.002*x(tf)^3-y(tf)==0];
sol4b = solve(eqn);
theta4b = double(sol4b.theta);
tf4b = double(sol4b.tf);

x=@(t) (15*cos(theta4b)+2)*t-20;
y=@(t) (15*sin(theta4b)-6)*t;
xf4b = x(tf4b);
yf4b = y(tf4b);
```

```
f=@(x) 25-0.25*x-0.002*x.^3;
xplot = linspace(-20,2);

[X, Y] = meshgrid(-21:2:0, -2:4:36);
u = 2*ones(size(X));
v = -6*ones(size(Y));
% Normalize vectors for better visualization
magnitude = sqrt(u.^2 + v.^2);
u = u ./ magnitude;
v = v ./ magnitude;
time=linspace(0,tf4b);

figure
plot(x(time),y(time))
xlabel( char 39x char 39);
ylabel( char 39y char 39);
title( char 39optimal trajectory char 39);
hold on
plot(xplot,f(xplot))
quiver(X, Y, u, v);
hold off
legend( char 39trajectory char 39, char 39shoreline char 39)

%Problem 4(c)
tinit = linspace(0,1);
solinit=bvpinit(tinit,@guess54,3); %tau is in [0,1]
sol4c=bvp4c(@odehw54,@bchw54,solinit);

x=sol4c.y(1,:);
y=sol4c.y(2,:);
theta = atan(sol4c.y(3,:));
tf= sol4c.parameters;
```

```
time=tf*sol4c.x;
f=@(x) 25-0.25*x-0.002*x.^3;
xplot = linspace(-20,30);

[X, Y] = meshgrid(-18:2:26, -30:4:10);
u = -(Y-50);
v = 2*(X-15);
% Normalize vectors for better visualization
magnitude = sqrt(u.^2 + v.^2);
u = u ./ magnitude;
v = v ./ magnitude;

figure
plot(x,y)
xlabel( char 39x char 39);
ylabel( char 39y char 39);
title( char 39optimal trajectory char 39);
hold on
plot(xplot,f(xplot))
quiver(X, Y, u, v);
hold off
legend( char 39trajectory char 39, char 39shoreline char 39)

figure
plot(time,theta)
xlabel( char 39time char 39);
ylabel( char 39theta char 39);
title( char 39optimal control char 39);

function dydt = odehw54(t,y,tf)
dydt = tf*[15*cos(y(3))-y(2)+50 %x
          15*sin(y(3))+2*(y(1)-15) %y
```

```
        1+sin(y(3))^2]; %theta
end

function res = bchw54(ya,yb,tf)
res = [ya(1)+20 %x0
     ya(2) %y0
     25-0.25*yb(1)-0.002*yb(1)^3-yb(2) %endpoint constraint
     tan(yb(3))-1/(0.25+0.006*yb(1)^2)]; %transversality costates
end

function g=guess54(t)
xinit = @(t) -20+t;
yinit = @(t) -t;
g=[xinit(t) yinit(t) 1];
end

%Problem 5

syms p22(t) u0 tf
sol5=dsolve(diff(p22,t)==2*u0+p22^2,p22(tf)==0);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```