Homework 4

Jaden Wang

Problem (1)**.** To show that $(M_*(f), \partial_f)$ is a complex, we need $\partial_f^2 = 0$. Given $a \in A_{n-1}, b \in B_n$,

$$\begin{aligned}
\partial_f(a,b) &= \Big(\partial_A^2(a), \partial_B(\partial_B(b) + f_{n-1}(a)) - f_{n-2}\partial_A(a)\Big) \\
&= (0, 0 + (\partial_B f_{n-1} - f_{n-2}\partial_A(a)) && A, B \text{ complex} \\
&= (0,0) && f_n \text{ chain map}
\end{aligned}$$

Let $\mathsf{S} : \mathsf{Chain} \to \mathsf{Chain}$ be the shift functor that increases the chain index by 1 and negate the morphism. That is, $A_*^+ := \mathsf{S}(A_*) = A_{*-1}$ with $\partial_{A^+} := S(\partial_A) = -\partial_A$. Consider the short exact sequence

$$0 \to B_* \xrightarrow{i} M_*(f) \xrightarrow{j} A_*^+ \to 0$$

where i is the obvious inclusion map and $j : A_{*-1} \oplus B_* \to A_{*-1}$ is the obvious projection map whose kernel is exactly B_*. Note that i, j are chain maps. Take $b \in B_n$, then $i \circ \partial_B(b) = (0, \partial_B(b)) = \partial_f(0,b) = \partial_f \circ i(b)$. Take (a,b) with $a \in A_n^+ = A_{n-1}$ and $b \in B_n$, we see that $j \circ \partial_f(a,b) = j(\partial_A^+(a), *) = \partial_A^+(a) = \partial_A^+ \circ j(a,b)$. Then the snake lemma yields a long exact sequence as stated. It remains to check that the connecting homomorphism $\partial_* = f_*$. Given $[a] \in H_n(A_*^+) = H_{n-1}(A_*)$, we have $j_*([(a,0)]) = a$ and that $[(a,0)]$. By the definition of $\partial_* : H_n(A_*^+) \to H_{n-1}(B_*)$, we have $\partial_*([a]) = [i^{-1}\partial_f(a,0)] = [i^{-1}(\partial_A(a), 0 + f_{n-1}(a))] = [f_{n-1}(a)] = f_*([a])$ since i^{-1} only picks out the second component. If $H_n(M_*(f)) = 0$ for all n, then we have $0 \to H_{n+1}(A_*^+) = H_n(A_*) \to H_n(B_*) \to 0$ for all n, which is an isomorphism on homology.

Problem (2)**.** Let $X = S^1 \vee S^1$ and Y be the 2-fold cover as figure.

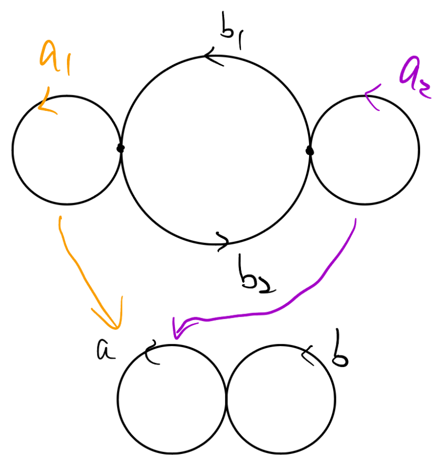


Figure 1

Consider the 1-simplices $a_1 : \Delta_1 \to Y$ and $a_2 : \Delta_1 \to Y$. Since Y is 1-dimensional, the image of any 2-simplex must be at most 1-dimensional, and the boundary must be at most 0-dimensional. Thus $a_1 - a_2$ is 1-dimensional and not a boundary of a 2-simplex, *i.e.* they are distinct elements in the homology. However, since $p \circ a_1 = a = p \circ a_2$, $p_*([a_1]) = [a] = p_*([a_2])$, showing that p_* is not injective.

Problem (4)**.** Note that any possible pair is a good pair in this problem, since contractible implies that we can just treat the subspace as a point, and the subspace deformation retracts to the point via contractibility. Denote $X_{12} := X_1 \cup X_2$. First I claim that $H_n(X_{12}) = 0$ for $n \geq 2$. If X_2 is empty then it is trivially true. If X_2 is contractible, we have

$$H_n(X_{12}) \cong \widetilde{H}_n(X_{12}/X_2) \cong \widetilde{H}_n(X_1/(X_1 \cap X_2)) = \widetilde{H}_n(*) = 0$$

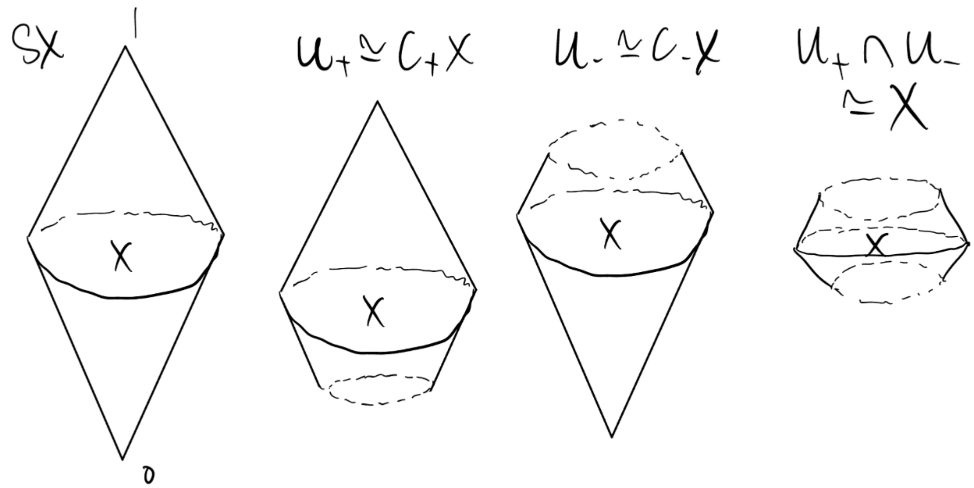
Next, I claim that $H_n(X_{12} \cap X_3) = 0$ for $n \geq 1$. Suppose $(X_1 \cap X_3) \cap (X_2 \cap X_3) = X_1 \cap X_2 \cap X_3 = \emptyset$. Then by additivity, $H_n(X_{12} \cap X_3) = H_n(X_1 \cap X_3) \oplus H_n(X_2 \cap X_3) = 0 \oplus 0 = 0$. Suppose the three-way intersection is not empty, WLOG we can assume both two-way intersections are contractible (if they are empty we can use additivity again to get trivial homology), by Mayer-Vietoris on the two-way intersections in the obvious way we obtain $H_n(X_{12} \cap X_3) = 0$.

Finally, by Mayer-Vietoris, for $n \geq 2$ we have

$$H_n(X_{12} \cap X_3) = 0 \to H_n(X_{12}) \oplus H_n(X_3) = 0 \to H_n(X) \to H_{n-1}(X_{12} \cap X_3) = 0$$

So $H_n(X) = 0$.

Problem (8)**.** Recall that a cone is contractible via the straight-line homotopy to the cone tip. Denote the top cone $C_+(X)$ and bottom cone $C_-(X)$. Let U_+ be the top cone with some "open skirt" and U_- be the bottom cone with open skirt. They union to SX and deformation retract to their respective cone.



Thus $H_n(C_\pm X) \cong H_n(U_\pm) = 0$ for $n \geq 1$ and $\mathbb{Z}$ for $n = 0$. Moreover, $U_+ \cap U_- \simeq X$ so $H_n(U_+ \cap U_-) \cong H_n(X)$. Now apply Mayer-Vietoris for $n \geq 1$:

$$\to \underbrace{H_{n+1}(U_+) \oplus H_{n+1}(U_-)}_{0} \to H_{n+1}(SX) \to \underbrace{H_n(U_+ \cap U_-)}_{\cong H_n(X)} \to \underbrace{H_n(U_+) \oplus H_n(U_-)}_{0} \to$$

So $H_{n+1}(SX) \cong H_n(X)$ for $n \geq 1$, and the reduced homology coincide with homology for this range. Outside this range, we have

$$\to \underbrace{H_1(U_+) \oplus H_1(U_-)}_{0} \to H_1(SX) \xrightarrow{\partial} \underbrace{H_0(U_+ \cap U_-)}_{\cong H_0(X)} \xrightarrow{\phi} \underbrace{H_0(U_+) \oplus H_0(U_-)}_{\mathbb{Z} \oplus \mathbb{Z}} \xrightarrow{\psi} \underbrace{H_0(SX)}_{\cong \mathbb{Z}} \to 0$$

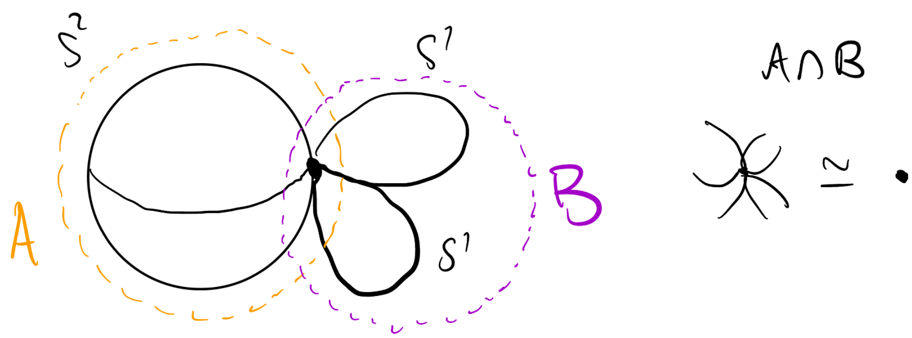
Since ψ is surjective, exactness and 1st isomorphism theorem yield $\mathbb{Z}^2 / \operatorname{im} \phi \cong \mathbb{Z}$. Since $\mathbb{Z}$ is free, we have a split short exact sequence $0 \to \operatorname{im} \phi \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$ and hence $\operatorname{im} \phi \oplus \mathbb{Z} \cong \mathbb{Z}^2$. Thus $H_0(X)/\ker \phi \cong \operatorname{im} \phi \cong \mathbb{Z}$. By the same argument, $H_0(X) \cong \ker \phi \oplus \mathbb{Z}$ so $\widetilde{H}_0(X) \cong \ker \phi = \operatorname{im} \partial \cong H_1(SX) = \widetilde{H}_1(SX)$ since ∂ is injective.

Finally, since SX is path-connected (can always connect two points via the cone tip), $\widetilde{H}_0(SX) = 0 = \widetilde{H}_{-1}(X)$.

Problem (9)**.** We already know that $T^2 := S^1 \times S^1$ have

$$H_n(T^2) = \begin{cases} \mathbb{Z}^2 & n = 2 \\ \mathbb{Z} & n = 0, 1 \end{cases}$$

Denote $X = S^1 \vee S^1 \vee S^2$. Let A and B be as shown in the figure. Clearly $A \simeq S^2$, $B \simeq S^1 \vee S^1$, $A \cap B$ is contractible.



For $i > 2$, we have

$$\cdots \to \underbrace{H_i(A \cap B)}_{0} \to \underbrace{H_i(A) \oplus H_i(B)}_{0 \oplus 0} \to H_i(X) \to \underbrace{H_{i-1}(A \cap B)}_{0} \to \cdots$$

which implies $H_i(X) = 0$. Else,

$$\underbrace{H_2(A \cap B)}_{0} \xrightarrow{\phi_2} \underbrace{H_2(A) \oplus H_2(B)}_{\cong \mathbb{Z} \oplus 0} \xrightarrow{\psi_2} H_2(X) \xrightarrow{\partial_2} \underbrace{H_1(A \cap B)}_{0} \xrightarrow{\phi_1} \underbrace{H_1(A) \oplus H_1(B)}_{0 \oplus \mathbb{Z}^2} \xrightarrow{\psi_2} H_1(X) \xrightarrow{\partial_1}$$

$$\underbrace{H_0(A \cap B)}_{\cong \mathbb{Z}} \xrightarrow{\phi_0} \underbrace{H_0(A) \oplus H_0(B)}_{\mathbb{Z} \oplus \mathbb{Z}} \to \cdots$$

We immediately have $H_2(X) \cong H_2(A) \oplus H_2(B) \cong \mathbb{Z}$. Moreover, ψ_1 is injective so $\mathbb{Z}^2 \cong \operatorname{im} \psi_1 = \ker \partial_1$. We see that $\phi_0 : 1 \mapsto (1, 1)$ is also injective, so $\operatorname{im} \partial_1 = \ker \phi_0 = 0$, therefore by first isomorphism theorem, $0 = \operatorname{im} \partial_1 \cong H_1(X)/\ker \partial_1 = H_1(X)/\mathbb{Z}^2$ which implies that $H_1(X) = \mathbb{Z}^2$. Finally, X is clearly path-connected so $H_0(X) \cong \mathbb{Z}$. Therefore, the homology of X coincides with T^2. However, the universal cover of T^2 is $\mathbb{R}^2$ which is contractible, yet the universal cover of X is the Caley tree of F_2 where each vertex wedges a S^2, so it is homotopy equivalent to an infinity wedge of circles $\bigvee^\infty S^2$ by quotienting out the contractible

Caley tree. Notice that we can apply Mayer-Vietoris on $\bigvee^\infty S^2$ by letting A bean open set containing exactly one sphere, and B be an open set containing the rest. Clearly $A \cap B$ is contractible, so it yields

$$0 \to H_2(A) \oplus H_2(B) \to H_2(\bigvee^{\infty} S^2) \to 0$$

That is, $H_2(\bigvee^\infty S^2) \cong \mathbb{Z} \oplus H_2(B)$ which is not trivial. Since $H_2(\mathbb{R}^2) = 0$ yet $H_2(\bigvee^\infty S^2) \neq 0$, we prove the statement.

Problem (10)**.** We think of $\mathbb{R}P^2$ as a disk with antipodal points in the boundary identified. Let A, B be as shown in the figure below.

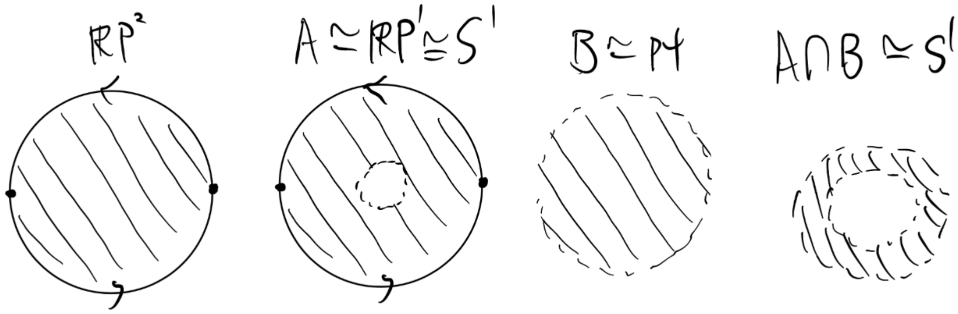


Figure 2

We see that B is an open disk which is contractible, and $A \cap B$ is an open annulus which is homotopy equivalent to S^1. Finally, we see that by deformation retracting to the boundary, A is homotopy equivalent to a circle with antipodal points identified, which is exactly $\mathbb{R}P^1$. It is easy to see that for $i > 2$, all parts involved have zero homology so $H_i(\mathbb{R}P^2) = 0$ for this range. Consider,

$$\cdots \to \underbrace{H_2(A) \oplus H_2(B)}_{0 \oplus 0} \xrightarrow{\psi_2} H_2(\mathbb{R}P^2) \xrightarrow{\partial_2} \underbrace{H_1(A \cap B)}_{\langle f_1 - f_2 \rangle \cong \mathbb{Z}} \xrightarrow{\phi_1} \underbrace{H_1(A) \oplus H_1(B)}_{\langle g_1 - g_2 \rangle \cong \mathbb{Z} \oplus 0} \xrightarrow{\psi_1} H_1(\mathbb{R}P^2) \xrightarrow{\partial_1}$$
$$\underbrace{H_0(A \cap B)}_{\cong \mathbb{Z}} \xrightarrow{\phi_0} \underbrace{H_0(A) \oplus H_0(B)}_{\cong \mathbb{Z} \oplus \mathbb{Z}} \to \cdots$$

By figure, we see that inlusion of $f_1 - f_2$ into A wraps around the generator $g_1 - g_2$ twice due to the identification. Hence ϕ_1 indices multiplication by 2 on the homology. Hence $0 = \ker \phi_1 =$

$\operatorname{im}\partial_2 = H_2(\mathbb{R}P^2)$ and $\operatorname{im}\phi_1 = 2\mathbb{Z}$. Since $\phi_0 : 1 \mapsto (1,1)$ is injective, $\operatorname{im}\partial_1 = \ker\phi_0 = 0$. Thus ψ_1 is surjective and $H_1(\mathbb{R}P^2) = \operatorname{im}\psi_1 = \mathbb{Z}/\ker\psi_1 \cong \mathbb{Z}/\operatorname{im}\phi_1 = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2$.

In summary, we have

$$H_i(\mathbb{R}P^2) = \begin{cases} 0 & i > 1 \\ \mathbb{Z}/2 & i = 1 \\ \mathbb{Z} & i = 0 \end{cases}.$$

For $\mathbb{R}P^3$, let A, B be analogous open set on D^3 with boundary antipodal points identified. Then $A \simeq \mathbb{R}P^2$, $B \simeq *$, and $A \cap B \simeq S^2$. Again for $i > 3$, $H_i(\mathbb{R}P^3) = 0$. Consider

$$\cdots \to \underbrace{H_3(A) \oplus H_3(B)}_{0\oplus 0} \xrightarrow{\psi_3} H_3(\mathbb{R}P^3) \xrightarrow{\partial_3} \underbrace{H_1(A \cap B)}_{\cong\mathbb{Z}} \xrightarrow{\phi_2} \underbrace{H_2(A) \oplus H_2(B)}_{0\oplus 0} \xrightarrow{\psi_2} H_2(\mathbb{R}P^3) \xrightarrow{\partial_2}$$

$$\underbrace{H_1(A \cap B)}_{0} \xrightarrow{\phi_1} \underbrace{H_1(A) \oplus H_1(B)}_{\mathbb{Z}/2\oplus 0} \xrightarrow{\psi_1} H_1(\mathbb{R}P^3) \xrightarrow{\partial_1} \underbrace{H_0(A \cap B)}_{\cong\mathbb{Z}} \xrightarrow{\phi_0} \underbrace{H_0(A) \oplus H_0(B)}_{\cong\mathbb{Z}\oplus\mathbb{Z}} \to \cdots$$

We immediately have $H_3(\mathbb{R}P^3) \cong \mathbb{Z}$. By the same argument as in $\mathbb{R}P^2$, ∂_1 is surjective, so $H_1(\mathbb{R}P^3) \cong \mathbb{Z}/2/\ker\psi_1 \cong \mathbb{Z}/2$.

In summary, we have

$$H_i(\mathbb{R}P^3) = \begin{cases} 0 & i > 3 \\ \mathbb{Z}/2 & i = 1 \\ \mathbb{Z} & i = 0, 3 \end{cases}.$$