Homework 6

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Problem (1)**.** From the example, we know that optimal trajectories are clockwise-oriented circles centered at $(1, 0)$ for $u = 1$ and centered at $(-1, 0)$ for $u = -1$. To reach the origin, we must eventually get on the switching curves Γ_+^1 and Γ_-^1 since they are the only circles centered at $(\pm 1, 0)$ that go through the origin. Moreover, the optimal control $u^* = -\text{sign}(\Lambda \cos(\omega t + \phi))$ flips signs and thus must switch every $\frac{\pi}{\omega} = \pi$ except that it might switch sooner at the beginning or in the end.

(a) When $x_1(0) = x_2(0) = 2$, we have the following optimal trajectory:

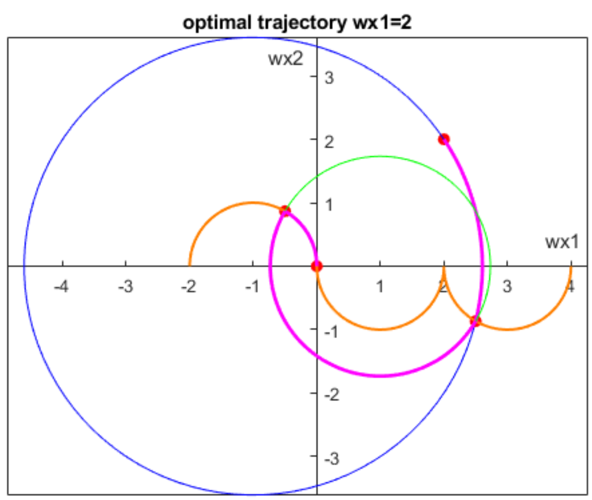


Figure 1: Magenta denotes the optimal trajectory and orange denotes the optimal switching curve. We first find clockwise-oriented circles centered at $(\pm 1, 0)$ that go through $(2, 2)$ and pick the one that reaches the switching surface the fastest. In this case it is the blue circle centered at $(-1, 0)$. Then we switch to the green circle centered at $(1, 0)$ and continue for π unit of time to reach the next switching curve which happens to be Γ_-^1 so we simply follow the singular curve to reach the origin.

Using $\widehat{\Gamma}$, we have the following trajectory:

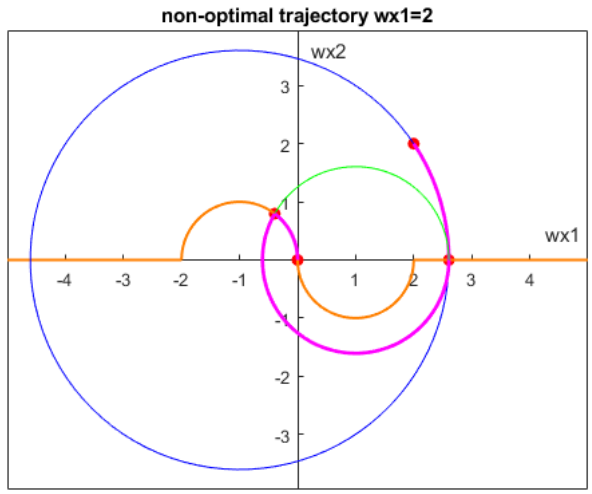
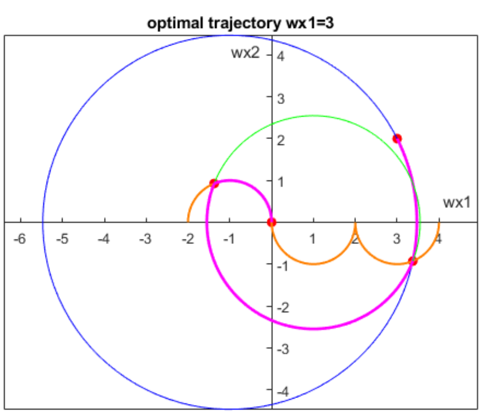
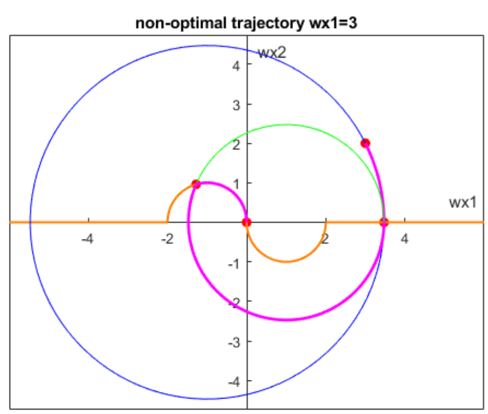


Figure 2: Magenta denotes the non-optimal trajectory and orange denotes the non-optimal switching curve. We repeat the previous first step, reach a new switching curve at $x_2 = 0$, and switch to the other center. This time we don't have to switch every π so we simply reach the next switching curve to switch.

Since $\omega = 1$, the elapsed time is the same as the angle the trajectory traced out. By adding the angles together, we obtain that $t^* = 5.0194$ and $\hat{t} = 5.1685$.

(b) We repeat the procedure for $\omega x_1 = 3, 4, 5$.





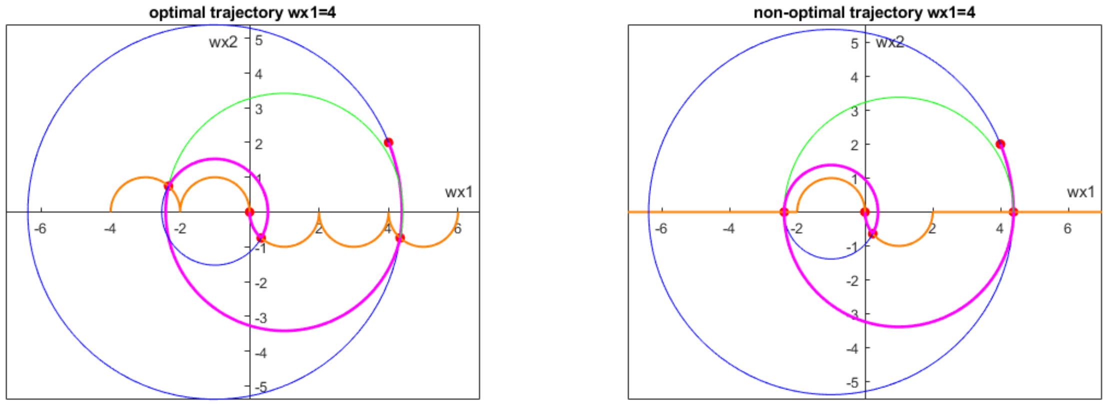
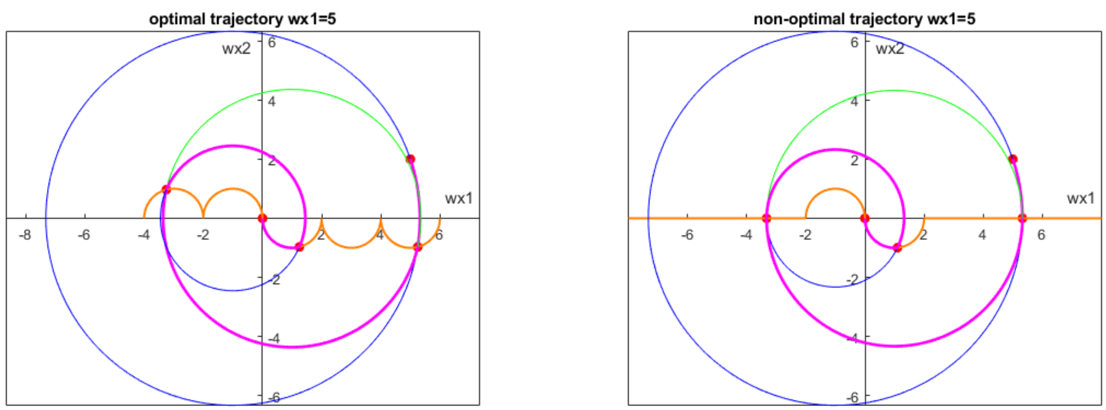


Figure 3: Both trajectories becomes more complicated. We have four arcs corresponding to three switchings.



From the figures, we can roughly see that the optimal trajectories and their non-optimal counterparts largely resembles each other. Moreover, it is not hard to imagine that as ωx_1 gets larger, the total angles traced out by both trajectories increase but their differences don't increase much. Thus the ratio $\hat{t}/t^*$ trends down to 1. There is not enough sample size in the following plot to fully illustrate that but it is a start:

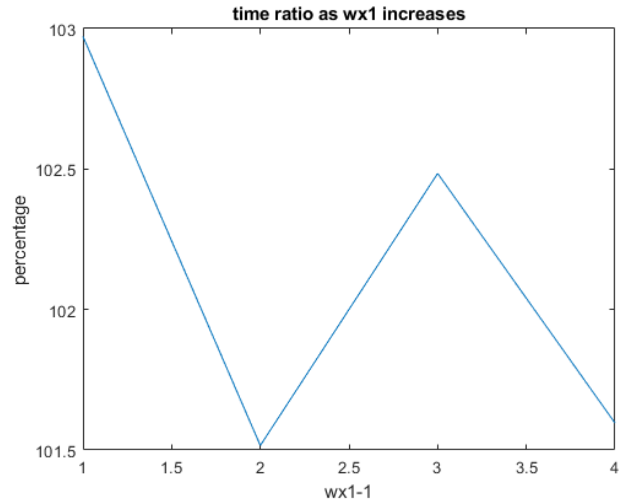


Figure 4: The ratio fluctuates but trends down.

Problem (2)**.** The Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2}x_1^2 + p_1(x_2+u) + p_2(-u) \\ &= \frac{1}{2}x_1^2 + p_1 x_2 + (p_1 - p_2)u \end{aligned}$$

The adjoint equations are

$$\dot{p_1} = -H_{x_1} = -x_1$$

$$\dot{p_2} = -H_{x_2} = -p_1$$

Since t_f is free, and H is time-independent, transversality yields $H \equiv 0$.

(a) Let us examine the optimality condition. By the PMP,

$$u^* = \underset{u}{\operatorname{argmin}}\, H = \begin{cases} -1 & p_1 - p_2 > 0 \\ ? & p_1 - p_2 = 0 \\ 1 & p_1 - p_2 < 0 \end{cases}$$

Notice on the singular surface we have

$$H = \frac{1}{2}x_1^2 + x_1 x_2 = 0$$

$$x_1(x_1 + x_2) = 0$$

$$\Rightarrow x_1 = 0 \text{ or } x_1 = -2x_2$$

Denote the surfaces by S_1 and S_2 respectively. Now we use GLC to obtain the optimal control on the singular surface

$$\begin{aligned}
\dot{H}_u &= \dot{p_1} - \dot{p_2} = -x_1 + p_1 \Rightarrow x_1 = p_1 = p_2 \\
\ddot{H}_u &= -\dot{x_1} + \dot{p_1} = -x_2 - u - x_1 = 0 \\
u^* &= -(x_1 + x_2)
\end{aligned}$$

Moreover, since $k = 1$, $(-1)^1 \frac{\partial}{\partial u}\ddot{H}_u = 1 > 0$ which passes GLC. Thus by definition of S_1 and S_2,

$$u^* = \begin{cases} -x_2 & \text{on } S_1 \\ x_2 & \text{on } S_2 \end{cases}$$

Notice that on S_1, $\dot{x_2} = x_2$ so x_2 moves away from origin. On S_2, we have

$$\begin{cases} \dot{x_1} = 2x_2 \\ \dot{x_2} = -x_2 \end{cases}$$

which indicates that trajectory on S_2 move toward the origin as time increases.

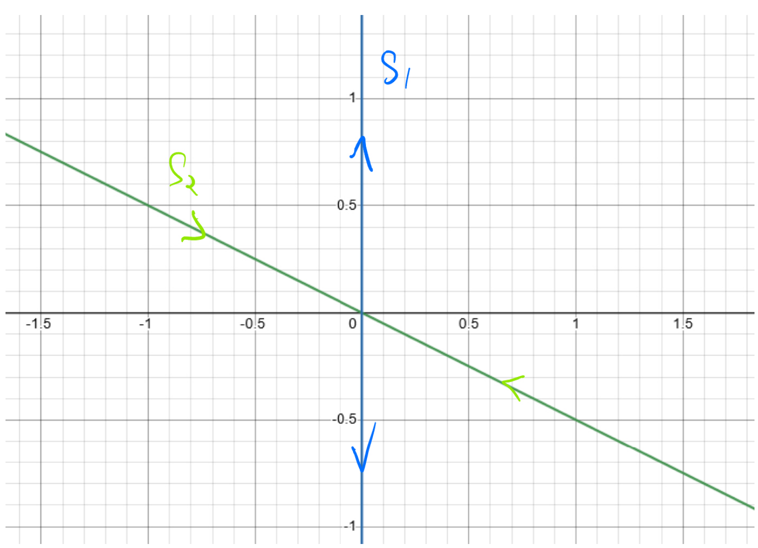


Figure 5: Switching surfaces: S_1 is blue and S_2 is green.

Now the initial point $(0, -0.5)$ is on S_1, we cannot continue on S_1 as it moves away from the target. Hence we resort to bang-bang control. If $u = 1$, dynamics are

$$\begin{cases} \dot{x_1} = x_2 + 1 & x_1(0) = 0 \\ \dot{x_2} = -1 & x_2(0) = -\frac{1}{2} \end{cases}$$

which yields $x_1 = -\frac{1}{2}\left(x_2 + \frac{1}{2}\right)^2 - \frac{1}{2}\left(x_2 + \frac{1}{2}\right)$. However, both its intersections with $x_1 = -2x_2$ are not achieved in the positive time direction.

If $u = -1$, the dynamics are

$$\begin{cases} \dot{x_1} = x_2 - 1 & x_1(0) = 0 \\ \dot{x_2} = 1 & x_2(0) = -\frac{1}{2} \end{cases}$$

which yields

$$\begin{cases} x_1(t) = \frac{1}{2}t^2 - \frac{3}{2}t \\ x_2(t) = t - \frac{1}{2} \Rightarrow t = x_2 + \frac{1}{2} \end{cases}$$

so we have the trajectory $x_1 = \frac{1}{2}\left(x + \frac{1}{2}\right)^2 - \frac{3}{2}\left(x_2 + \frac{1}{2}\right)$. This intersections with S_2 at $(-1, 0.5)$ which is achieved in positive time $t = 1$. Thus at this time we switch to singular control.

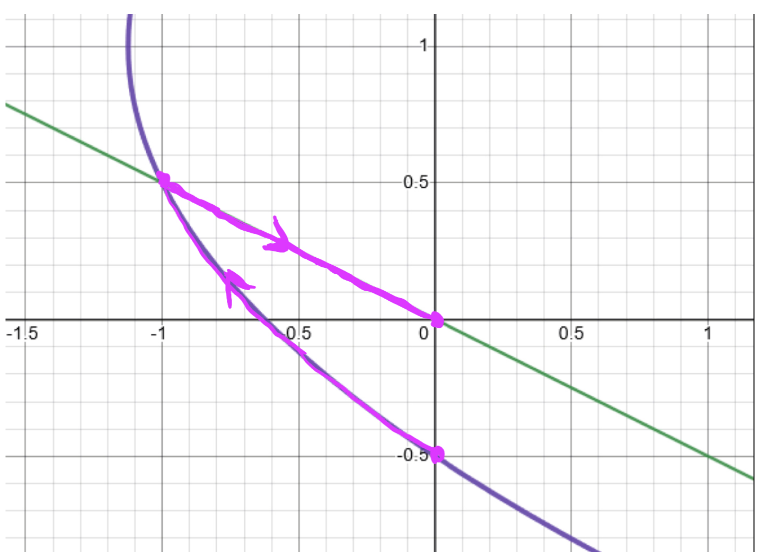


Figure 6: Optimal trajectory from initial point $(0, -0.5)$.

(b) Yes we can use bang-bang control only to reach the origin. We simply use the bang-bang control $u = -1$ above and then switch to $u = 1$ which it intersects a bang-bang trajectory using $u = 1$ that reaches the origin. We find the latter trajectory with reversed time so that it goes through the origin at $t = 0$ for simplicity:

$$\begin{cases} \dot{x_1} = x_2 + 1 & x_1(0) = 0 \\ \dot{x_2} = -1 & x_2(0) = 0 \end{cases}$$

which yields

$$\begin{cases} x_1(t) = -\frac{1}{2}t^2 + t \\ x_2(t) = -t \Rightarrow t = -x_2 \end{cases}$$

This gives the trajectory $x_1 = -\frac{1}{2}x_2^2 - x_2$. The two bang-bang arcs intersect at $(-1.1031, 0.7906)$, with $t = -0.7906$. Thus the trajectory is

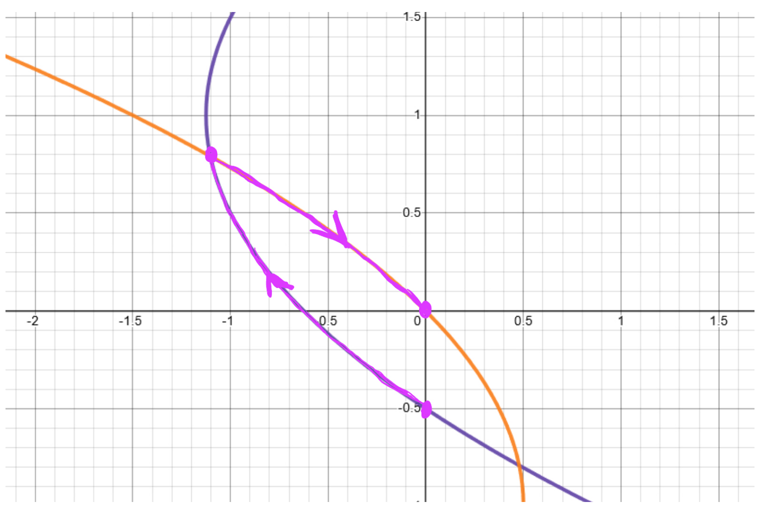


Figure 7: Bang-bang only trajectory.

(c) For the optimal control, we see that the first intersection occurs at $t_1 = x_2 + \frac{1}{2} = 1$. Moreover, since $x_1 = p_1$, the adjoint equation becomes $x_1 = -\dot{x_1}$. Thus

$$\begin{aligned} \frac{1}{2}\int_1^T x_1^2 dt &= -\frac{1}{2}\int_1^T x_1\dot{x_1} dt \\ &= -\frac{1}{2}\int_1^T \frac{d}{dt} x_1\dot{x_1} dt \\ &= -\frac{1}{4}\left(x_1(T)^2 - x_1(1)^2\right) \end{aligned}$$

$$= \frac{1}{4}x_1(1)^2$$
$$= \frac{1}{4}$$

and

$$\begin{aligned} J^* = \frac{1}{2}\int_0^T x_1^2 dt &= \frac{1}{2}\int_0^1 x_1^2 dt + \frac{1}{2}\int_1^T x_1^2 dt \\ &= \frac{1}{2}\int_0^1 \left(\frac{1}{2}t^2 - \frac{3}{2}t\right)^2 dt + \frac{1}{4} \\ &= 0.4625 \end{aligned}$$

From (d) we obtain that for bang-bang trajectory, $t_1 = 1.2906$. Thus

$$\begin{aligned} J_{BB} &= \frac{1}{2}\int_0^T x_1^2 dt \\ &= \frac{1}{2}\int_0^{1.2906} \left(\frac{1}{2}t^2 - \frac{3}{2}t\right)^2 dt + \frac{1}{2}\int_{-0.7906}^0 \left(\frac{1}{2}t^2 + t\right)^2 \\ &= 0.3755 + 0.1389 = 0.5144 \end{aligned}$$

(d) Solving the dynamics on S_2 with boundary condition $x_1(1) = -1, x_2(1) = \frac{1}{2}$ yields

$$\begin{cases} x_1(t) = -2.7183e^{-t} \\ x_2(t) = 1.3591e^{-t} \end{cases}$$

which is never going to reach $(0,0)$ in finite time.

In the bang-bang case, we have $t_1 = x_2 + \frac{1}{2} = 0.7906 + 0.5 = 1.2906$ so $T = t_1 + 0.7906 = 2.0812$.

Problem (3)**.** The Hamiltonian is

$$H = 1 + p_1 x_2 + p_2 u.$$

The adjoint equations are

$$\dot{p_1} = -H_{x_1} = 0$$

$$\dot{p_2} = -H_{x_2} = -p_1$$

We see that p_1 is constant and p_2 is linear.

(a) By PMP, we know that the optimal control is

$$u^* = \underset{u}{\operatorname{argmin}}\, H = \begin{cases} u_{\max} & p_2 < 0 \\ ? & p_2 = 0 \\ -u_{\max} & p_2 > 0 \end{cases}$$

Since p_2 is linear, it cannot be zero for more than one point. Otherwise, $p_2 = p_1 \equiv 0$ which implies that $H \equiv 1$. However, since t_f is free, we have $H \equiv 0$, a contradiction. It follows that

$$u^* = -\operatorname{sign}(p_2) \cdot u_{\max} = \pm u_{\max},$$

i.e. we only have bang-bang control.

(b) Let $b := \pm u_{\max}$. Thus the dynamics of optimal trajectory are

$$\begin{cases} \dot{x_2} = b & \Rightarrow x_2(t) = bt + x_{20} \\ \dot{x_1} = x_2 & \Rightarrow x_1(t) = bt^2 + x_{20}t + x_{10} \end{cases}$$

By substitution we have the optimal trajectory

$$x_1 - \left(x_{10} - \frac{1}{2b}x_{20}^2\right) = \frac{1}{2b}x_2^2,$$

for initial point (x_{10}, x_{20}). To have end points at $(0, \pm 1)$, the optimal trajectory must have its last arc on one of the orange parabolas below. These are the switching curves.

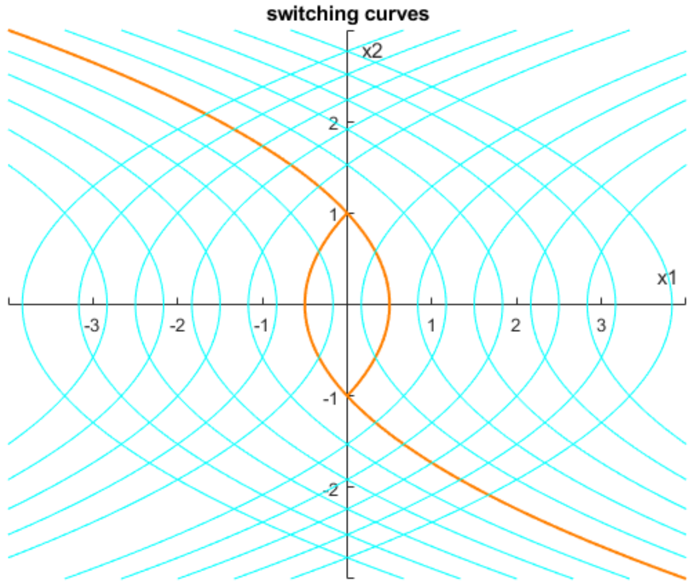


Figure 8: The orange parabolas are the only optimal trajectories that go through $(0, \pm 1)$. The cyan parabolas are other potential optimal trajectories. Given an initial point (x_{10}, x_{20}), there are always exactly two parabolas facing opposite directions that go through the point based on the equations above, and they always intersect at least one of the orange parabolas once. Note that the parabolas with mouth opening to the left goes down whereas the parabolas with mouth opening right goes up as time elapses. Note that the switching curves terminate after reaching $(0, \pm 1)$. That is, the orange curve that goes up stops at $(0, 1)$. The orange curve that goes down stops at $(0, -1)$. This is because once they reach those points, they will never be able to return.

(c) By the reasons stated in the caption above, yes solution exists for all initial points. They might not be unique, as the $(0, 0)$ case illustrates below.

(d)

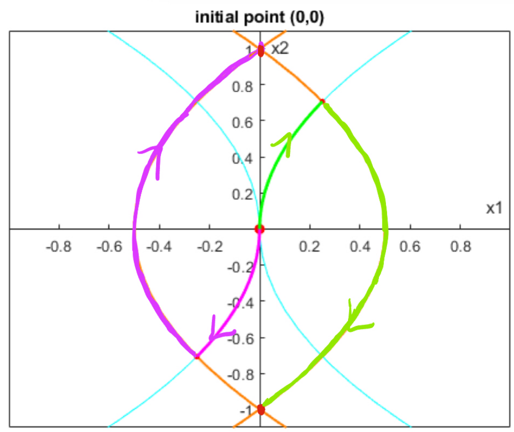
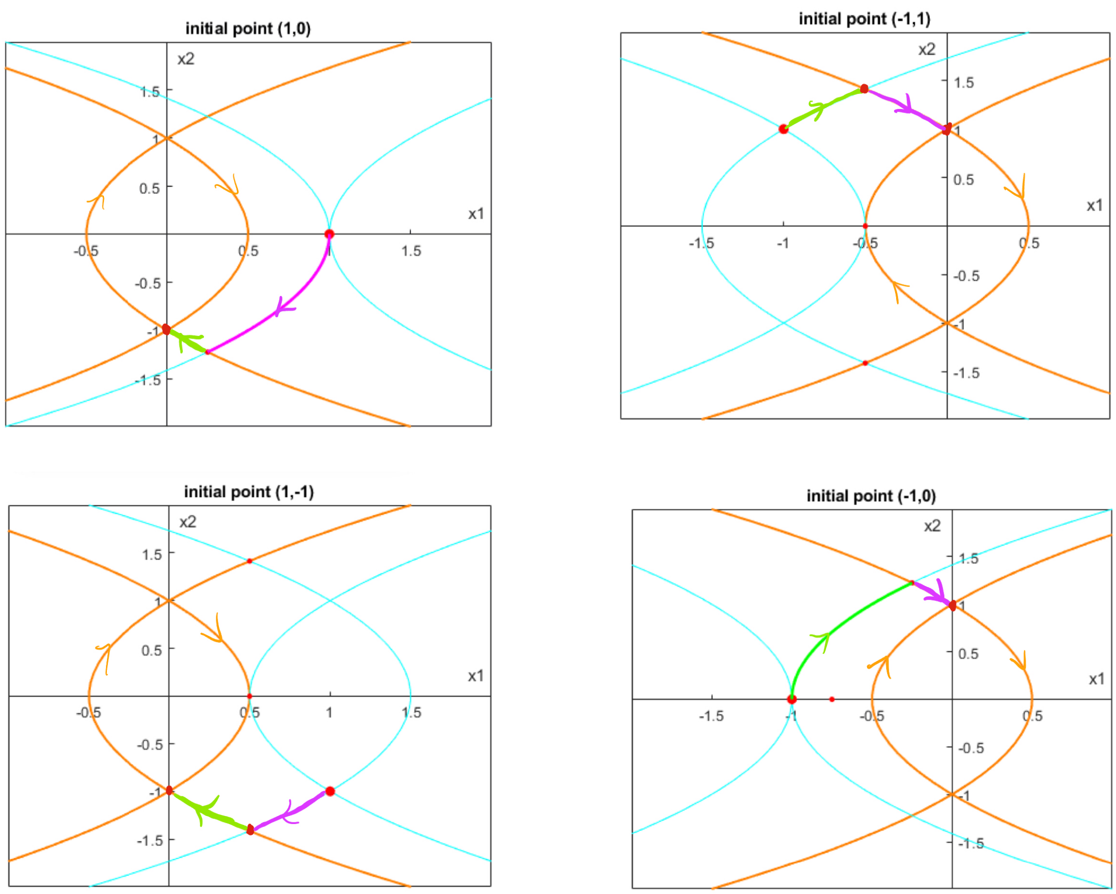


Figure 9: Due to symmetry, we see that there are two optimal trajectories from $(0, 0)$ to $(0, \pm 1)$.



Problem (4)**.** The Hamiltonian is

$$\begin{aligned}
H &= t^2 + x^2 + u^2 + pu \\
H_u &= 2u + p \\
H_{uu} &= 2 = R_2 > 0 \\
H_{ux} &= 0 = R_{12} \\
H_{xx} &= 2 = R_1
\end{aligned}$$

Moreover, $A = \frac{\partial f}{\partial x} = 0$ and $B = \frac{\partial f}{\partial u} = 1$. Recall that for a regular problem, the existence of a conjugate point is equivalent to solution of associated Ricatti equation blowing up. We have

$$\begin{aligned}
\widetilde{A} &= A - BR_2^{-1}R_{12}^T = 0 - \frac{1}{2} \cdot 0 = 0 \\
\Sigma &= BR_2^{-1}B = \frac{1}{2} \\
\widetilde{R} &= R_1 - R_{12}R_2^{-1}R_{12}^T = 2 - 0 = 2
\end{aligned}$$

Thus the Ricatti equation is

$$\begin{aligned}
-\dot{s} &= \widetilde{A}^T s + s\widetilde{A} - s\Sigma s + \widetilde{R} \\
-\dot{s} &= -\frac{1}{2}s^2 + 2
\end{aligned}$$

with $s(t_f) = 0$. This yields

$$s(t) = 2\tanh(t_f - t),$$

which is bounded! Thus we have no conjugate point.

Problem (5)**.** (a) The Hamiltonian is

$$H = 1 + pu$$

with adjoint equation

$$\dot{p} = 0$$

which implies p is constant. We know $p \neq 0$ since otherwise $H \equiv 1$ but transversality gives $H \equiv 0$, a contradiction. Thus PMP gives

$$\begin{aligned} u^* &= \begin{cases} -1 & p > 0 \\ 1 & p < 0 \end{cases} \\ &= -\operatorname{sign}(p) \end{aligned}$$

(b) By letting $b = -\operatorname{sign}(p)$, we have $\dot{x} = b$ which yields $x(t) = bt + x_0$ with $0 = x(t_f) = bt_f + x_0$ so $x_0 = -bt_f$. So the optimal value $t_f = -bx_0$.

(c) For closed-loop control, we want u^* in terms of x. Since we know that u is a constant, as above from the dynamics we get $x(t) = ut + x_0$ so boundary condition yields $u = -\frac{x_0}{t_f}$. If $x_0 > 0$, then since $t_f > 0$, we have $\dot{x} = u = -1$, so x decreases as time elapses but remains > 0 until t_f. If $x_0 < 0$, x increases with time but remains < 0 until t_f. In either case, we see that $u^* = -\operatorname{sign}(x)$.

Problem (6)**.** The Hamiltonian is

$$H(x, u, V_x, t) = (xu)^2 + V_x xu.$$

The HJB equation is

$$V_t + \min_u \{(xu)^2 + V_x xu\} = 0.$$

To find the minimizer, we can set the derivative with respect to u to zero and obtain

$$\begin{aligned} 2x^2u + V_x x = 0 \\ u^* = -\frac{V_x}{2x}, \quad x \neq 0 \end{aligned}$$

Then the PDE becomes

$$4V_t - V_x^2 = 0.$$

Since $V(1) = x(1)^2$, we make the guess that the solution has the form $V(x,t) = p(t)x(t)^2$. Then $V_t = \dot{p}x^2$ and $V_x = 2xp(t)$, and we solve the ODE

$$4\dot{p}x^2 - p(t)^2 4x^2 = 0$$

$$\dot{p} = p^2$$

with the boundary condition $p(1) = 1$. The solution is

$$p = \frac{1}{2-t},$$

and the solution to the PDE is

$$V(x,t) = \frac{x^2}{2-t}.$$

Thus we have $V_t = \frac{x^2}{(2-t)^2}$ and $V_x = \frac{2x}{2-t}$, so the optimal control is

$$u^* = \frac{1}{t-2}.$$

Solving the dynamics $\dot{x} = \frac{x}{t-2}$ with $x(0) = 1$ gives

$$x(t) = \frac{2-t}{2}$$

which indeed doesn't equal 0 in $[0, 1]$. Also $x(1) = \frac{1}{2}$. The optimal cost is

$$J^* = x(1)^2 + \int_0^1 \left(\frac{2-t}{2}\frac{1}{t-2}\right)^2 dt = x(1)^2 + \frac{1}{4} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Problem (7)**.** The Hamiltonian is

$$H = \frac{1}{2}x^T R_1 x + x^T R_{12} u + p^T(Ax + Bu).$$

Recall the following matrix calculus rules $\frac{d(x^T Ax)}{dx} = 2(Ax)^T$, or more generally $\frac{d\langle f(x), g(x)\rangle}{dx} = g(x)^T f'(x) + f(x)^T g'(x)$. Note this is the derivative/Jacobian, which is the transpose of gradient. The adjoint equation is

$$\nabla_t p = \dot{p} = -H_x^T = -(R_1 x + R_{12} u + A^T p)$$

The singular control satisfies

$$H_u = x^T R_{12} + p^T B = 0.$$

Then

$$\dot{H}_u = \nabla_t H_u = \frac{d}{dx}(x^T R_{12})\dot{x} + \frac{d}{dp}(p^T B)\dot{p}$$

$$0 = R_{12}^T(Ax + Bu) - B^T(R_1x + R_{12}u + A^Tp)$$
$$0 = R_{12}^TAx + (R_{12}^TB - B^TR_{12})u - B^TR_1x - B^TA^Tp$$

By GLC, u can only appear in even degree of differentiation, so for singular control to be optimal we must have $R_{12}^TB - B^TR_{12} = 0$. Then

$$\ddot{H}_u = \nabla_{tt}H_u = (R_{12}^TA - B^TR_1)\dot{x} - B^TA^T\dot{p}$$
$$0 = (R_{12}^TA - B^TR_1)Bu + B^TA^TR_{12}u + f(x) + g(p)$$

Since u shows up at degree 2, $k = 1$. Thus GLC demands

$$(-1)^1\frac{\partial}{\partial u}\ddot{H}_u \geq 0$$
$$(-1)(R_{12}^TAB - B^TR_1B + B^TA^TR_{12}) \geq 0$$
$$B^TR_1B - R_{12}^TAB - B^TA^TR_{12} \geq 0$$

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Problem 1a %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
t = zeros(4,2); %store time

%%%%%%%%% initial point at (2,2) %%%%%%%%%
x0 = 2;
y0 = 2;
%time
t1 = 0; %optimal time
t2 = 0; %non-optimal time
%%%%%%%%%%%%%%%%%%%%%%%%% optimal trajectory %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%blue circle centered at (-1,0)
th = linspace(0, 2*pi, 100);
R = norm([x0+1,y0]); %radius
%circle
cx1 = R*cos(th) - 1;
cy1 = R*sin(th);
%find the first intersection of blue circle with switching curve
```

```
syms x y
%eyeball the switching curve it intersects with
eqns=[(x-3)^2+y^2==1,(x+1)^2+y^2==R^2];
sol = solve(eqns);
pt1 = [sol.x(1);sol.y(1)]
pt1 = double(pt1);
%first arc of optimal trajectory
th0 = atan(y0/(x0+1)); % initial angle for blue circle
th1 = atan(pt1(2)/(pt1(1)+1)); %first switch angle for blue circle
t1 = t1 + th0-th1; %time is the same as angle
th = linspace( th0, th1, 100);
%trajectory arc 1
trajx1 = R*cos(th) - 1;
trajy1 = R*sin(th);

%green circle centered at (1,0)
th = linspace(0, 2*pi, 100);
R = norm(pt1-[1;0]); %radius
%circle
cx2 = R*cos(th) + 1;
cy2 = R*sin(th);
%find the next intersection of green circle with switching curve
syms x y
eqns=[(x+1)^2+y^2==1,(x-1)^2+y^2==R^2];
sol = solve(eqns);
pt2 = [sol.x(2);sol.y(2)]
pt2 = double(pt2);
%second arc of optimal trajectory
th0 = atan(pt1(2)/(pt1(1)-1)); % initial angle for green circle
th1 = acos((pt2(1)-1)/R); %first switch angle for green circle
t1 = t1 + th0+2*pi - th1; %time is the same as angle
th = linspace( th0+2*pi, th1, 100);
```

```
%trajectory arc 2
trajx2 = R*cos(th) + 1;
trajy2 = R*sin(th);

%semi-circle switching curves
th = linspace( 0, -pi, 100);
R = 1; %radius
sx1 = R*cos(th) + 3;
sy1 = R*sin(th) ;
sx2 = R*cos(th) + 1;
sy2 = R*sin(th) ;
th = linspace( 0, pi, 100);
sx3 = R*cos(th) - 1;
sy3 = R*sin(th) ;
%third arc of optimal trajectory
th0 = asin(pt2(2)); % initial angle for semicircle
th1 = 0; %terminal angle for for semicircle
t1 = t1+ th0-th1;
th = linspace( th0,th1, 100);
trajx3 = R*cos(th) - 1;
trajy3 = R*sin(th);

figure;
plot(x0,y0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 3
hold on
plot(pt1(1),pt1(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(pt2(1),pt2(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(0,0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 39)
plot(sx1,sy1,Color=[1 .5 0],LineWidth=1.5); axis equal;
plot(sx2,sy2,Color=[1 .5 0],LineWidth=1.5);
plot(sx3,sy3,Color=[1 .5 0],LineWidth=1.5);
plot(cx1,cy1,Color= char 39blue char 39);
```

```
plot(trajx1,trajy1,Color= char 39magenta char 39,LineWidth=2);
plot(cx2,cy2,Color= char 39green char 39);
plot(trajx2,trajy2,Color= char 39magenta char 39,LineWidth=2);
plot(trajx3,trajy3,Color= char 39magenta char 39,LineWidth=2);
set(gca,  char 39XAxisLocation char 39,  char 39origin char 39)
set(gca,  char 39YAxisLocation char 39,  char 39origin char 39)
xlabel( char 39wx1 char 39);
ylabel( char 39wx2 char 39);
title( char 39optimal trajectory wx1=2 char 39);
hold off

%%%%%%%%%%%%%%%%%%%%%%%% non-optimal trajectory %%%%%%%%%%%%%%%%%%%%%%%%%%
%blue circle centered at (-1,0)
th = linspace(0, 2*pi, 100);
R = norm([x0+1,y0]); %radius
cx1 = R*cos(th) - 1;
cy1 = R*sin(th);
%find the first intersection of blue circle with switching curve
syms x y
eqns=[y==0,(x+1)^2+y^2==R^2];
sol = solve(eqns);
pt1 = [sol.x(2);sol.y(2)]
pt1 = double(pt1);
%first arc of optimal trajectory
th0 = atan(y0/(x0+1)); % initial angle for blue circle
th1 = atan(pt1(2)/(pt1(1)+1)); %first switch angle for blue circle
t2 = t2 + th0-th1; %time is the same as angle
th = linspace( th0, th1, 100);
trajx1 = R*cos(th) - 1;
trajy1 = R*sin(th);

%green circle centered at (1,0)
```

```
th = linspace(0, 2*pi, 100);
R = norm(pt1-[1;0]); %radius
cx2 = R*cos(th) + 1;
cy2 = R*sin(th);
%find the next intersection of green circle with switching curve
syms x y
eqns=[(x+1)^2+y^2==1,(x-1)^2+y^2==R^2];
sol = solve(eqns);
pt2 = [sol.x(2);sol.y(2)]
pt2 = double(pt2);
%second arc of non-optimal trajectory
th0 = atan(pt1(2)/(pt1(1)-1)); % initial angle for green circle
th1 = acos((pt2(1)-1)/R); %first switch angle for green circle
t2 = t2 + th0+2*pi - th1; %time is the same as angle
th = linspace( th0+2*pi, th1, 100);
trajx2 = R*cos(th) + 1;
trajy2 = R*sin(th);
%Gamma hat switching curves
th = linspace( 0, -pi, 100);
R = 1; %radius
sx1 = linspace(-5,-2);
sx11 = linspace(2,5);
sy1 = 0*sx1;
sy11 = 0*sx11;
sx2 = R*cos(th) + 1;
sy2 = R*sin(th) ;
th = linspace( 0, pi, 100);
sx3 = R*cos(th) - 1;
sy3 = R*sin(th) ;
%third arc of optimal trajectory
th0 = acos(pt2(1)+1); % initial angle for semicircle
th1 = 0; %terminal angle for for semicircle
```

```
t2 = t2+ th0-th1;
th = linspace( th0,th1, 100);
trajx3 = R*cos(th) - 1;
trajy3 = R*sin(th);

figure;
plot(x0,y0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 3
hold on
plot(pt1(1),pt1(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(pt2(1),pt2(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(0,0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 39)
plot(sx1,sy1,Color=[1 .5 0],LineWidth=1.5); axis equal;
plot(sx11,sy11,Color=[1 .5 0],LineWidth=1.5);
plot(sx2,sy2,Color=[1 .5 0],LineWidth=1.5);
plot(sx3,sy3,Color=[1 .5 0],LineWidth=1.5);
plot(cx1,cy1,Color= char 39blue char 39);
plot(trajx1,trajy1,Color= char 39magenta char 39,LineWidth=2);
plot(cx2,cy2,Color= char 39green char 39);
plot(trajx2,trajy2,Color= char 39magenta char 39,LineWidth=2);
plot(trajx3,trajy3,Color= char 39magenta char 39,LineWidth=2);
set(gca,  char 39XAxisLocation char 39,  char 39origin char 39)
set(gca,  char 39YAxisLocation char 39,  char 39origin char 39)
xlabel( char 39wx1 char 39);
ylabel( char 39wx2 char 39);
title( char 39non-optimal trajectory wx1=2 char 39);
hold off

fprintf( char 39The optimal time is %f, the non-optimal time is
    %f. char 92n char 39,t1,t2)

t(1,1) = t1;
t(1,2) = t2;
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%% Problem 1b %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%% initial point at (3,2) %%%%%%%%
x0 = 3;
y0 = 2;
%time
t1 = 0; %optimal time
t2 = 0; %non-optimal time
%blue circle centered at (-1,0)
th = linspace(0, 2*pi, 100);
R = norm([x0+1,y0]); %radius
cx1 = R*cos(th) - 1;
cy1 = R*sin(th);
%find the first intersection of blue circle with switching curve
syms x y
eqns=[(x-3)^2+y^2==1,(x+1)^2+y^2==R^2];
sol = solve(eqns);
pt1 = [sol.x(1);sol.y(1)]
pt1 = double(pt1);
%first arc of optimal trajectory
th0 = atan(y0/(x0+1)); % initial angle for blue circle
th1 = atan(pt1(2)/(pt1(1)+1)); %first switch angle for blue circle
t1 = t1 + th0-th1; %time is the same as angle
th = linspace( th0, th1, 100);
trajx1 = R*cos(th) - 1;
trajy1 = R*sin(th);

%green circle centered at (1,0)
th = linspace(0, 2*pi, 100);
R = norm(pt1-[1;0]); %radius
cx2 = R*cos(th) + 1;
```

```
cy2 = R*sin(th);
%find the next intersection of green circle with switching curve
syms x y
eqns=[(x+1)^2+y^2==1,(x-1)^2+y^2==R^2];
sol = solve(eqns);
pt2 = [sol.x(2);sol.y(2)]
pt2 = double(pt2);
%%%%%%second arc of optimal trajectory
th0 = atan(pt1(2)/(pt1(1)-1)); % initial angle for green circle
th1 = acos((pt2(1)-1)/R); %first switch angle for green circle
t1 = t1 + th0+2*pi - th1; %time is the same as angle
th = linspace( th0+2*pi, th1, 100);
trajx2 = R*cos(th) + 1;
trajy2 = R*sin(th);
%semi-circle switching curves
th = linspace( 0, -pi, 100);
R = 1; %radius
sx1 = R*cos(th) + 3;
sy1 = R*sin(th) ;
sx2 = R*cos(th) + 1;
sy2 = R*sin(th) ;
th = linspace( 0, pi, 100);
sx3 = R*cos(th) - 1;
sy3 = R*sin(th) ;
%%%%third arc of optimal trajectory
th0 = acos(pt2(1)+1); % initial angle for semicircle
th1 = 0; %terminal angle for for semicircle
t1 = t1+ th0-th1;
th = linspace( th0,th1, 100);
trajx3 = R*cos(th) - 1;
trajy3 = R*sin(th);
```

```
figure;
plot(x0,y0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 3
hold on
plot(pt1(1),pt1(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(pt2(1),pt2(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(0,0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 39)
plot(sx1,sy1,Color=[1 .5 0],LineWidth=1.5); axis equal;
plot(sx2,sy2,Color=[1 .5 0],LineWidth=1.5);
plot(sx3,sy3,Color=[1 .5 0],LineWidth=1.5);
plot(cx1,cy1,Color= char 39blue char 39);
plot(trajx1,trajy1,Color= char 39magenta char 39,LineWidth=2);
plot(cx2,cy2,Color= char 39green char 39);
plot(trajx2,trajy2,Color= char 39magenta char 39,LineWidth=2);
plot(trajx3,trajy3,Color= char 39magenta char 39,LineWidth=2);
set(gca,  char 39XAxisLocation char 39,  char 39origin char 39)
set(gca,  char 39YAxisLocation char 39,  char 39origin char 39)
xlabel( char 39wx1 char 39);
ylabel( char 39wx2 char 39);
title( char 39optimal trajectory wx1=3 char 39);
hold off

%%%%%%%%%%%%%%%%%%%%%%%% non-optimal trajectory %%%%%%%%%%%%%%%%%%%%%%%%%%%
%blue circle centered at (-1,0)
th = linspace(0, 2*pi, 100);
R = norm([x0+1,y0]); %radius
cx1 = R*cos(th) - 1;
cy1 = R*sin(th);
%find the first intersection of blue circle with switching curve
syms x y
eqns=[y==0,(x+1)^2+y^2==R^2];
sol = solve(eqns);
pt1 = [sol.x(2);sol.y(2)]
```

```
pt1 = double(pt1);
%first arc of optimal trajectory
th0 = atan(y0/(x0+1)); % initial angle for blue circle
th1 = atan(pt1(2)/(pt1(1)+1)); %first switch angle for blue circle
t2 = t2 + th0-th1; %time is the same as angle
th = linspace( th0, th1, 100);
trajx1 = R*cos(th) - 1;
trajy1 = R*sin(th);

%green circle centered at (1,0)
th = linspace(0, 2*pi, 100);
R = norm(pt1-[1;0]); %radius
cx2 = R*cos(th) + 1;
cy2 = R*sin(th);
%find the next intersection of green circle with switching curve
syms x y
eqns=[(x+1)^2+y^2==1,(x-1)^2+y^2==R^2];
sol = solve(eqns);
pt2 = [sol.x(2);sol.y(2)]
pt2 = double(pt2);
%second arc of non-optimal trajectory
th0 = atan(pt1(2)/(pt1(1)-1)); % initial angle for green circle
th1 = acos((pt2(1)-1)/R); %first switch angle for green circle
t2 = t2 + th0+2*pi - th1; %time is the same as angle
th = linspace( th0+2*pi, th1, 100);
trajx2 = R*cos(th) + 1;
trajy2 = R*sin(th);
%Gamma hat switching curves
th = linspace( 0, -pi, 100);
R = 1; %radius
sx1 = linspace(-6,-2);
sx11 = linspace(2,6);
```

```
sy1 = 0*sx1;
sy11 = 0*sx11;
sy1 = 0*sx1;
sx2 = R*cos(th) + 1;
sy2 = R*sin(th) ;
th = linspace( 0, pi, 100);
sx3 = R*cos(th) - 1;
sy3 = R*sin(th) ;
%third arc of optimal trajectory
th0 = acos(pt2(1)+1); % initial angle for semicircle
th1 = 0; %terminal angle for for semicircle
t2 = t2+ th0-th1;
th = linspace( th0,th1, 100);
trajx3 = R*cos(th) - 1;
trajy3 = R*sin(th);

figure;
plot(x0,y0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 3
hold on
plot(pt1(1),pt1(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(pt2(1),pt2(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(0,0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 39)
plot(sx1,sy1,Color=[1 .5 0],LineWidth=1.5); axis equal;
plot(sx11,sy11,Color=[1 .5 0],LineWidth=1.5);
plot(sx2,sy2,Color=[1 .5 0],LineWidth=1.5);
plot(sx3,sy3,Color=[1 .5 0],LineWidth=1.5);
plot(cx1,cy1,Color= char 39blue char 39);
plot(trajx1,trajy1,Color= char 39magenta char 39,LineWidth=2);
plot(cx2,cy2,Color= char 39green char 39);
plot(trajx2,trajy2,Color= char 39magenta char 39,LineWidth=2);
plot(trajx3,trajy3,Color= char 39magenta char 39,LineWidth=2);
set(gca,  char 39XAxisLocation char 39,  char 39origin char 39)
```

```
set(gca,  char 39YAxisLocation char 39,  char 39origin char 39)
xlabel( char 39wx1 char 39);
ylabel( char 39wx2 char 39);
title( char 39non-optimal trajectory wx1=3 char 39);
hold off


t(2,1) = t1;
t(2,2) = t2;

%%%%%%%%%%%%%%%%%%%%%%%% initial point at (4,2) %%%%%%%%%%%%%%%%%%%%%%%%%%
x0 = 4;
y0 = 2;

%time
t1 = 0; %optimal time
t2 = 0; %non-optimal time
%blue circle centered at (-1,0)
th = linspace(0, 2*pi, 100);
R = norm([x0+1,y0]); %radius
cx1 = R*cos(th) - 1;
cy1 = R*sin(th);
%find the first intersection of blue circle with switching curve
syms x y
eqns=[(x-5)^2+y^2==1,(x+1)^2+y^2==R^2];
sol = solve(eqns);
pt1 = [sol.x(1);sol.y(1)]
pt1 = double(pt1);
%first arc of optimal trajectory
th0 = atan(y0/(x0+1)); % initial angle for blue circle
th1 = atan(pt1(2)/(pt1(1)+1)); %first switch angle for blue circle
th0-th1
```

```
t1 = t1 + th0-th1; %time is the same as angle
th = linspace( th0, th1, 100);
trajx1 = R*cos(th) - 1;
trajy1 = R*sin(th);

%green circle centered at (1,0)
th = linspace(0, 2*pi, 100);
R = norm(pt1-[1;0]); %radius
cx2 = R*cos(th) + 1;
cy2 = R*sin(th);
%find the next intersection of green circle with switching curve
syms x y
eqns=[(x+3)^2+y^2==1,(x-1)^2+y^2==R^2];
sol = solve(eqns);
pt2 = [sol.x(2);sol.y(2)]
pt2 = double(pt2);
%%%%%%second arc of optimal trajectory
th0 = atan(pt1(2)/(pt1(1)-1)); % initial angle for green circle
th1 = acos((pt2(1)-1)/R); %first switch angle for green circle
th0+2*pi - th1
t1 = t1 + th0+2*pi - th1; %time is the same as angle
th = linspace( th0+2*pi, th1, 100);
trajx2 = R*cos(th) + 1;
trajy2 = R*sin(th);

%blue circle centered at (-1,0)
th = linspace(0, 2*pi, 100);
R = norm(pt2+[1;0]); %radius
cx3 = R*cos(th) - 1;
cy3 = R*sin(th);
%find the next intersection of blue circle with switching curve
syms x y
```

```
eqns=[(x-1)^2+y^2==1,(x+1)^2+y^2==R^2];
sol = solve(eqns);
pt3 = [sol.x(1);sol.y(1)]
pt3 = double(pt3);
%third arc of optimal trajectory
th0 = acos((pt2(1)+1)/R); % initial angle for blue circle
th1 = atan(pt3(2)/(pt3(1)+1)); %first switch angle for blue circle
th0-th1
t1 = t1 + th0-th1; %time is the same as angle
th = linspace( th0, th1, 100);
trajx3 = R*cos(th) - 1;
trajy3 = R*sin(th);

%semi-circle switching curves
th = linspace( 0, -pi, 100);
R = 1; %radius
sx1 = R*cos(th) + 3;
sy1 = R*sin(th) ;
sx2 = R*cos(th) + 1;
sy2 = R*sin(th) ;
sx3 = R*cos(th) + 5;
sy3 = R*sin(th) ;
th = linspace( 0, pi, 100);
sx4 = R*cos(th) - 1;
sy4 = R*sin(th) ;
sx5 = R*cos(th) - 3;
sy5 = R*sin(th) ;
%%%%fourth arc of optimal trajectory
th0 = -acos(pt3(1)-1); % initial angle for semicircle
th1 = -pi; %terminal angle for for semicircle
abs(th1-th0)
t1 = t1+ abs(th1-th0);
```

```
th = linspace( th0,th1, 100);
trajx4 = R*cos(th) + 1;
trajy4 = R*sin(th);

figure;
plot(x0,y0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 3
hold on
plot(pt1(1),pt1(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(pt2(1),pt2(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(pt3(1),pt3(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(0,0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 39)
plot(sx1,sy1,Color=[1 .5 0],LineWidth=1.5); axis equal;
plot(sx2,sy2,Color=[1 .5 0],LineWidth=1.5);
plot(sx3,sy3,Color=[1 .5 0],LineWidth=1.5);
plot(sx4,sy4,Color=[1 .5 0],LineWidth=1.5);
plot(sx5,sy5,Color=[1 .5 0],LineWidth=1.5);
plot(cx1,cy1,Color= char 39blue char 39);
plot(trajx1,trajy1,Color= char 39magenta char 39,LineWidth=2);
plot(cx2,cy2,Color= char 39green char 39);
plot(trajx2,trajy2,Color= char 39magenta char 39,LineWidth=2);
plot(cx3,cy3,Color= char 39blue char 39);
plot(trajx3,trajy3,Color= char 39magenta char 39,LineWidth=2);
plot(trajx4,trajy4,Color= char 39magenta char 39,LineWidth=2);
set(gca,  char 39XAxisLocation char 39,  char 39origin char 39)
set(gca,  char 39YAxisLocation char 39,  char 39origin char 39)
xlabel( char 39wx1 char 39);
ylabel( char 39wx2 char 39);
title( char 39optimal trajectory wx1=4 char 39);
hold off

%%%%%%%%%%%%%%%%%%%%%%%% non-optimal trajectory %%%%%%%%%%%%%%%%%%%%%%%%%%%
%blue circle centered at (-1,0)
```

```
th = linspace(0, 2*pi, 100);
R = norm([x0+1,y0]); %radius
cx1 = R*cos(th) - 1;
cy1 = R*sin(th);
%find the first intersection of blue circle with switching curve
syms x y
eqns=[y==0,(x+1)^2+y^2==R^2];
sol = solve(eqns);
pt1 = [sol.x(2);sol.y(2)]
pt1 = double(pt1);
%first arc of optimal trajectory
th0 = atan(y0/(x0+1)); % initial angle for blue circle
th1 = atan(pt1(2)/(pt1(1)+1)); %first switch angle for blue circle
th0-th1
t2 = t2 + th0-th1; %time is the same as angle
th = linspace( th0, th1, 100);
trajx1 = R*cos(th) - 1;
trajy1 = R*sin(th);

%green circle centered at (1,0)
th = linspace(0, 2*pi, 100);
R = norm(pt1-[1;0]); %radius
cx2 = R*cos(th) + 1;
cy2 = R*sin(th);
%find the next intersection of green circle with switching curve
syms x y
eqns=[y==0,(x-1)^2+y^2==R^2];
sol = solve(eqns);
pt2 = [sol.x(1);sol.y(1)]
pt2 = double(pt2);
%second arc of non-optimal trajectory
th0 = atan(pt1(2)/(pt1(1)-1)); % initial angle for green circle
```

```
th1 = acos((pt2(1)-1)/R); %first switch angle for green circle
th0+2*pi - th1
t2 = t2 + th0+2*pi - th1; %time is the same as angle
th = linspace( th0+2*pi, th1, 100);
trajx2 = R*cos(th) + 1;
trajy2 = R*sin(th);

%blue circle centered at (-1,0)
th = linspace(0, 2*pi, 100);
R = norm(pt2+[1;0]); %radius
cx3 = R*cos(th) - 1;
cy3 = R*sin(th);
%find the next intersection of blue circle with switching curve
syms x y
eqns=[(x-1)^2+y^2==1,(x+1)^2+y^2==R^2];
sol = solve(eqns);
pt3 = [sol.x(1);sol.y(1)]
pt3 = double(pt3);
%third arc of non-optimal trajectory
th0 = acos((pt2(1)+1)/R); % initial angle for blue circle
th1 = atan(pt3(2)/(pt3(1)+1)); %first switch angle for blue circle
th0-th1
t2 = t2 + th0-th1; %time is the same as angle
th = linspace( th0, th1, 100);
trajx3 = R*cos(th) - 1;
trajy3 = R*sin(th);

%Gamma hat switching curves
th = linspace( 0, -pi, 100);
R = 1; %radius
sx1 = linspace(-7,-2);
sx11 = linspace(2,7);
```

```
sy1 = 0*sx1;
sy11 = 0*sx11;
sy1 = 0*sx1;
sx2 = R*cos(th) + 1;
sy2 = R*sin(th) ;
th = linspace( 0, pi, 100);
sx3 = R*cos(th) - 1;
sy3 = R*sin(th) ;

%%%%fourth arc of non-optimal trajectory
th0 = -acos(pt3(1)-1); % initial angle for semicircle
th1 = -pi; %terminal angle for for semicircle
abs(th1-th0)
t2 = t2 + abs(th1-th0);
th = linspace( th0,th1, 100);
trajx4 = R*cos(th) + 1;
trajy4 = R*sin(th);

figure;
plot(x0,y0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 3
hold on
plot(pt1(1),pt1(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(pt2(1),pt2(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(pt3(1),pt3(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(0,0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 39)
plot(sx1,sy1,Color=[1 .5 0],LineWidth=1.5); axis equal;
plot(sx11,sy11,Color=[1 .5 0],LineWidth=1.5);
plot(sx2,sy2,Color=[1 .5 0],LineWidth=1.5);
plot(sx3,sy3,Color=[1 .5 0],LineWidth=1.5);
plot(cx1,cy1,Color= char 39blue char 39);
plot(trajx1,trajy1,Color= char 39magenta char 39,LineWidth=2);
plot(cx2,cy2,Color= char 39green char 39);
```

```
plot(cx3,cy3,Color= char 39blue char 39);
plot(trajx2,trajy2,Color= char 39magenta char 39,LineWidth=2);
plot(trajx3,trajy3,Color= char 39magenta char 39,LineWidth=2);
plot(trajx4,trajy4,Color= char 39magenta char 39,LineWidth=2);
set(gca,  char 39XAxisLocation char 39,  char 39origin char 39)
set(gca,  char 39YAxisLocation char 39,  char 39origin char 39)
xlabel( char 39wx1 char 39);
ylabel( char 39wx2 char 39);
title( char 39non-optimal trajectory wx1=4 char 39);
hold off

t(3,1) = t1;
t(3,2) = t2;
%%%%%%%%%%%%%%%%%%%%%% initial point at (5,2) %%%%%%%%%%%%%%%%%%%%%%%%%%%%%
x0 = 5;
y0 = 2;

%time
t1 = 0; %optimal time
t2 = 0; %non-optimal time
%blue circle centered at (-1,0)
th = linspace(0, 2*pi, 100);
R = norm([x0+1,y0]); %radius
cx1 = R*cos(th) - 1;
cy1 = R*sin(th);
%find the first intersection of blue circle with switching curve
syms x y
eqns=[(x-5)^2+y^2==1,(x+1)^2+y^2==R^2];
sol = solve(eqns);
pt1 = [sol.x(1);sol.y(1)]
pt1 = double(pt1);
%first arc of optimal trajectory
```

```
th0 = atan(y0/(x0+1)); % initial angle for blue circle
th1 = atan(pt1(2)/(pt1(1)+1)); %first switch angle for blue circle
t1 = t1 + th0-th1; %time is the same as angle
th = linspace( th0, th1, 100);
trajx1 = R*cos(th) - 1;
trajy1 = R*sin(th);

%green circle centered at (1,0)
th = linspace(0, 2*pi, 100);
R = norm(pt1-[1;0]); %radius
cx2 = R*cos(th) + 1;
cy2 = R*sin(th);
%find the next intersection of green circle with switching curve
syms x y
eqns=[(x+3)^2+y^2==1,(x-1)^2+y^2==R^2];
sol = solve(eqns);
pt2 = [sol.x(2);sol.y(2)]
pt2 = double(pt2);
%%%%%%second arc of optimal trajectory
th0 = atan(pt1(2)/(pt1(1)-1)); % initial angle for green circle
th1 = acos((pt2(1)-1)/R); %first switch angle for green circle
t1 = t1 + th0+2*pi - th1; %time is the same as angle
th = linspace( th0+2*pi, th1, 100);
trajx2 = R*cos(th) + 1;
trajy2 = R*sin(th);

%blue circle centered at (-1,0)
th = linspace(0, 2*pi, 100);
R = norm(pt2+[1;0]); %radius
cx3 = R*cos(th) - 1;
cy3 = R*sin(th);
%find the next intersection of blue circle with switching curve
```

```
syms x y
eqns=[(x-1)^2+y^2==1,(x+1)^2+y^2==R^2];
sol = solve(eqns);
pt3 = [sol.x(1);sol.y(1)]
pt3 = double(pt3);
%third arc of optimal trajectory
th0 = acos((pt2(1)+1)/R); % initial angle for blue circle
th1 = atan(pt3(2)/(pt3(1)+1)); %first switch angle for blue circle
t1 = t1 + th0-th1; %time is the same as angle
th = linspace( th0, th1, 100);
trajx3 = R*cos(th) - 1;
trajy3 = R*sin(th);

%semi-circle switching curves
th = linspace( 0, -pi, 100);
R = 1; %radius
sx1 = R*cos(th) + 3;
sy1 = R*sin(th) ;
sx2 = R*cos(th) + 1;
sy2 = R*sin(th) ;
sx3 = R*cos(th) + 5;
sy3 = R*sin(th) ;
th = linspace( 0, pi, 100);
sx4 = R*cos(th) - 1;
sy4 = R*sin(th) ;
sx5 = R*cos(th) - 3;
sy5 = R*sin(th) ;
%%%%fourth arc of optimal trajectory
th0 = -acos(pt3(1)-1); % initial angle for semicircle
th1 = -pi; %terminal angle for for semicircle
t1 = t1+ abs(th1-th0);
th = linspace( th0,th1, 100);
```

```
trajx4 = R*cos(th) + 1;
trajy4 = R*sin(th);

figure;
plot(x0,y0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 3
hold on
plot(pt1(1),pt1(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(pt2(1),pt2(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(pt3(1),pt3(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(0,0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 39)
plot(sx1,sy1,Color=[1 .5 0],LineWidth=1.5); axis equal;
plot(sx2,sy2,Color=[1 .5 0],LineWidth=1.5);
plot(sx3,sy3,Color=[1 .5 0],LineWidth=1.5);
plot(sx4,sy4,Color=[1 .5 0],LineWidth=1.5);
plot(sx5,sy5,Color=[1 .5 0],LineWidth=1.5);
plot(cx1,cy1,Color= char 39blue char 39);
plot(trajx1,trajy1,Color= char 39magenta char 39,LineWidth=2);
plot(cx2,cy2,Color= char 39green char 39);
plot(trajx2,trajy2,Color= char 39magenta char 39,LineWidth=2);
plot(cx3,cy3,Color= char 39blue char 39);
plot(trajx3,trajy3,Color= char 39magenta char 39,LineWidth=2);
plot(trajx4,trajy4,Color= char 39magenta char 39,LineWidth=2);
set(gca,  char 39XAxisLocation char 39,  char 39origin char 39)
set(gca,  char 39YAxisLocation char 39,  char 39origin char 39)
xlabel( char 39wx1 char 39);
ylabel( char 39wx2 char 39);
title( char 39optimal trajectory wx1=5 char 39);
hold off

%%%%%%%%%%%%%%%%%%%%%%%% non-optimal trajectory %%%%%%%%%%%%%%%%%%%%%%%%%
%blue circle centered at (-1,0)
th = linspace(0, 2*pi, 100);
```

```
R = norm([x0+1,y0]); %radius
cx1 = R*cos(th) - 1;
cy1 = R*sin(th);
%find the first intersection of blue circle with switching curve
syms x y
eqns=[y==0,(x+1)^2+y^2==R^2];
sol = solve(eqns);
pt1 = [sol.x(2);sol.y(2)]
pt1 = double(pt1);
%first arc of non-optimal trajectory
th0 = atan(y0/(x0+1)); % initial angle for blue circle
th1 = atan(pt1(2)/(pt1(1)+1)); %first switch angle for blue circle
t2 = t2 + th0-th1; %time is the same as angle
th = linspace( th0, th1, 100);
trajx1 = R*cos(th) - 1;
trajy1 = R*sin(th);

%green circle centered at (1,0)
th = linspace(0, 2*pi, 100);
R = norm(pt1-[1;0]); %radius
cx2 = R*cos(th) + 1;
cy2 = R*sin(th);
%find the next intersection of green circle with switching curve
syms x y
eqns=[y==0,(x-1)^2+y^2==R^2];
sol = solve(eqns);
pt2 = [sol.x(1);sol.y(1)]
pt2 = double(pt2);
%second arc of non-optimal trajectory
th0 = atan(pt1(2)/(pt1(1)-1)); % initial angle for green circle
th1 = acos((pt2(1)-1)/R); %first switch angle for green circle
t2 = t2 + th0+2*pi - th1; %time is the same as angle
```

```
th = linspace( th0+2*pi, th1, 100);
trajx2 = R*cos(th) + 1;
trajy2 = R*sin(th);

%blue circle centered at (-1,0)
th = linspace(0, 2*pi, 100);
R = norm(pt2+[1;0]); %radius
cx3 = R*cos(th) - 1;
cy3 = R*sin(th);
%find the next intersection of blue circle with switching curve
syms x y
eqns=[(x-1)^2+y^2==1,(x+1)^2+y^2==R^2];
sol = solve(eqns);
pt3 = [sol.x(1);sol.y(1)]
pt3 = double(pt3);
%third arc of non-optimal trajectory
th0 = acos((pt2(1)+1)/R); % initial angle for blue circle
th1 = atan(pt3(2)/(pt3(1)+1)); %first switch angle for blue circle
t2 = t2 + th0-th1; %time is the same as angle
th = linspace( th0, th1, 100);
trajx3 = R*cos(th) - 1;
trajy3 = R*sin(th);

%Gamma hat switching curves
th = linspace( 0, -pi, 100);
R = 1; %radius
sx1 = linspace(-8,-2);
sx11 = linspace(2,8);
sy1 = 0*sx1;
sy11 = 0*sx11;
sy1 = 0*sx1;
sx2 = R*cos(th) + 1;
```

```
sy2 = R*sin(th) ;
th = linspace( 0, pi, 100);
sx3 = R*cos(th) - 1;
sy3 = R*sin(th) ;

%%%%fourth arc of non-optimal trajectory
th0 = -acos(pt3(1)-1); % initial angle for semicircle
th1 = -pi; %terminal angle for for semicircle
t2 = t2+ abs(th1-th0);
th = linspace( th0,th1, 100);
trajx4 = R*cos(th) + 1;
trajy4 = R*sin(th);

figure;
plot(x0,y0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 3
hold on
plot(pt1(1),pt1(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(pt2(1),pt2(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(pt3(1),pt3(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(0,0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 39)
plot(sx1,sy1,Color=[1 .5 0],LineWidth=1.5); axis equal;
plot(sx11,sy11,Color=[1 .5 0],LineWidth=1.5);
plot(sx2,sy2,Color=[1 .5 0],LineWidth=1.5);
plot(sx3,sy3,Color=[1 .5 0],LineWidth=1.5);
plot(cx1,cy1,Color= char 39blue char 39);
plot(trajx1,trajy1,Color= char 39magenta char 39,LineWidth=2);
plot(cx2,cy2,Color= char 39green char 39);
plot(cx3,cy3,Color= char 39blue char 39);
plot(trajx2,trajy2,Color= char 39magenta char 39,LineWidth=2);
plot(trajx3,trajy3,Color= char 39magenta char 39,LineWidth=2);
plot(trajx4,trajy4,Color= char 39magenta char 39,LineWidth=2);
set(gca,  char 39XAxisLocation char 39,  char 39origin char 39)
```

```
set(gca,  char 39YAxisLocation char 39,  char 39origin char 39)
xlabel( char 39wx1 char 39);
ylabel( char 39wx2 char 39);
title( char 39non-optimal trajectory wx1=5 char 39);
hold off

t(4,1) = t1;
t(4,2) = t2;

figure;
plot(t(:,2)./t(:,1)*100)
ylabel( char 39percentage char 39);
xlabel( char 39wx1-1 char 39);
title( char 39time ratio as wx1 increases char 39);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%% Problem 2 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%a
syms x y
eqns = [x==-1/2*(y+1/2)^2-1/2*(y+1/2),x==-2*y];
sol2 = solve(eqns);
sol2.x
sol2.y
eqns = [x==1/2*(y+1/2)^2-3/2*(y+1/2),x==-2*y];
sol2 = solve(eqns);
sol2.x
sol2.y
%b
eqns = [x==1/2*(y+1/2)^2-3/2*(y+1/2),x==-1/2*y^2-y];
sol2 = solve(eqns);
sol2.x
sol2.y
pt = [sol2.x(2);sol2.y(2)];
```

```
pt = double(pt)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%% Problem 3 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
t = linspace(-2,2);
x0 = 0;
y0 = 1;
%up
Gamma1xf =@(t) 0.5*t.^2 + y0*t + x0;
Gamma1yf =@(t) t + y0;
%down
Gamma2xf =@(t) -0.5*t.^2 + y0*t + x0;
Gamma2yf =@(t) -t + y0;
Gamma1x = Gamma1xf(t);
Gamma1y = Gamma1yf(t);
Gamma2x = Gamma2xf(t);
Gamma2y = Gamma2yf(t);

figure;
hold on
plot(Gamma2x,Gamma2y,Color=[1 .5 0],LineWidth=1.5)
for n=1:5
    k=2*n/3;
    plot(Gamma1x-k,Gamma1y,Color= char 39cyan char 39)
    plot(Gamma1x+k,Gamma1y,Color= char 39cyan char 39)
    plot(Gamma2x-k,Gamma2y,Color= char 39cyan char 39)
    plot(Gamma2x+k,Gamma2y,Color= char 39cyan char 39)
end
set(gca,  char 39XAxisLocation char 39,  char 39origin char 39)
set(gca,  char 39YAxisLocation char 39,  char 39origin char 39)
xlabel( char 39x1 char 39);
ylabel( char 39x2 char 39);
title( char 39switching curves char 39);
```

```
axis([-4,4,-3,3]);

t = linspace(-2,2);
x0 = 0;
y0 = -1;
%up
Gamma1xf =@(t) 0.5*t.^2 + y0*t + x0;
Gamma1yf =@(t) t + y0;
%down
Gamma2xf =@(t) -0.5*t.^2 + y0*t + x0;
Gamma2yf =@(t) -t + y0;
Gamma1x = Gamma1xf(t);
Gamma1y = Gamma1yf(t);
Gamma2x = Gamma2xf(t);
Gamma2y = Gamma2yf(t);

plot(Gamma1x,Gamma1y,Color=[1 .5 0],LineWidth=1.5)
for n=1:5
    k=2*n/3;
    plot(Gamma1x-k,Gamma1y,Color= char 39cyan char 39)
    plot(Gamma1x+k,Gamma1y,Color= char 39cyan char 39)
    plot(Gamma2x-k,Gamma2y,Color= char 39cyan char 39)
    plot(Gamma2x+k,Gamma2y,Color= char 39cyan char 39)
end
hold off

%%%%%%%%%%% initial (0,0) %%%%%%%%%%%%
x0 = 0;
y0 = 0;
upx =@(t) 0.5*t.^2 + y0*t + x0;
upy =@(t) t + y0;
downx =@(t) -0.5*t.^2 + y0*t + x0;
```

```
downy =@(t) -t + y0;

%u=-1 goes down, intersects Gamma2 u=1 goes up
eqns = [x-(x0+1/2*y0^2)==-1/2*(y)^2,x+1/2==1/2*(y)^2];
sol31 = solve(eqns);
sol31.x
sol31.y
pt1 = [sol31.x(1);sol31.y(1)]; %pick the lowest x2 to go up
pt1 = double(pt1);
tf1 = -pt1(2);
t1 = linspace(0,tf1);

%u=1 goes up, intersects Gamma1 u=-1 goes down
eqns = [x-(x0-1/2*y0^2)==1/2*(y)^2,x-1/2==-1/2*(y)^2];
sol32 = solve(eqns);
sol32.x
sol32.y
pt2 = [sol32.x(1);sol32.y(2)]; %pick the highest x2 to go down
pt2 = double(pt2);
tf2 = pt2(2);
t2 = linspace(0,tf2);
figure;
plot(Gamma1x,Gamma1y,Color=[1 .5 0],LineWidth=1.5)
hold on
plot(Gamma2x,Gamma2y,Color=[1 .5 0],LineWidth=1.5)
plot(upx(t),upy(t),Color= char 39cyan char 39)
plot(downx(t),downy(t),Color= char 39cyan char 39)
plot(x0,y0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 3
plot(pt1(1),pt1(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(pt2(1),pt2(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(downx(t1),downy(t1),Color= char 39magenta char 39,LineWidth=2)
plot(upx(t2),upy(t2),Color= char 39green char 39,LineWidth=2)
```

```
set(gca,  char 39XAxisLocation char 39,  char 39origin char 39)
set(gca,  char 39YAxisLocation char 39,  char 39origin char 39)
xlabel( char 39x1 char 39);
ylabel( char 39x2 char 39);
title( char 39initial point (0,0) char 39);
axis([-1,1,-1.1,1.1]);
hold off


%%%%%%%%%%% initial (1,0) %%%%%%%%%%%%
x0 = 1;
y0 = 0;
upx =@(t) 0.5*t.^2 + y0*t + x0;
upy =@(t) t + y0;
downx =@(t) -0.5*t.^2 + y0*t + x0;
downy =@(t) -t + y0;

%u=-1 goes down, intersects Gamma2 u=1 goes up
eqns = [x-(x0+1/2*y0^2)==-1/2*(y)^2,x+1/2==1/2*(y)^2];
sol31 = solve(eqns);
sol31.x
sol31.y
pt1 = [sol31.x(1);sol31.y(1)]; %pick the lowest x2 to go up
pt1 = double(pt1);
tf1 = -pt1(2);
t1 = linspace(0,tf1);

%u=1 goes up, intersects Gamma1 u=-1 goes down
eqns = [x-(x0-1/2*y0^2)==1/2*(y)^2,x-1/2==-1/2*(y)^2];
%no solution!
figure;
plot(Gamma1x,Gamma1y,Color=[1 .5 0],LineWidth=1.5)
```

```
hold on
plot(Gamma2x,Gamma2y,Color=[1 .5 0],LineWidth=1.5)
plot(upx(t),upy(t),Color= char 39cyan char 39)
plot(downx(t),downy(t),Color= char 39cyan char 39)
plot(x0,y0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 
plot(pt1(1),pt1(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 3
plot(downx(t1),downy(t1),Color= char 39magenta char 39,LineWidth=2)
set(gca,  char 39XAxisLocation char 39,  char 39origin char 39)
set(gca,  char 39YAxisLocation char 39,  char 39origin char 39)
xlabel( char 39x1 char 39);
ylabel( char 39x2 char 39);
title( char 39initial point (1,0) char 39);
axis([-1,2,-2,2]);
hold off

%%%%%%%%%% initial (-1,1) %%%%%%%%%%%
x0 = -1;
y0 = 1;
upx =@(t) 0.5*t.^2 + y0*t + x0;
upy =@(t) t + y0;
downx =@(t) -0.5*t.^2 + y0*t + x0;
downy =@(t) -t + y0;

%u=-1 goes down, intersects Gamma2 u=1 goes up
eqns = [x-(x0+1/2*y0^2)==-1/2*(y)^2,x+1/2==1/2*(y)^2];
sol31 = solve(eqns);
sol31.x
sol31.y
pt1 = [sol31.x(1);sol31.y(1)]; %pick the lowest x2 to go up
pt1 = double(pt1);
tf1 = -pt1(2);
t1 = linspace(0,tf1);
```

```
%u=1 goes up, intersects Gamma1 u=-1 goes down
eqns = [x-(x0-1/2*y0^2)==1/2*(y)^2,x-1/2==-1/2*(y)^2];
sol32 = solve(eqns);
sol32.x
sol32.y
pt2 = [sol32.x(1);sol32.y(2)]; %pick the highest x2 to go down
pt2 = double(pt2);
tf2 = pt2(2);
t2 = linspace(0,tf2);
figure;
plot(Gamma1x,Gamma1y,Color=[1 .5 0],LineWidth=1.5)
hold on
plot(Gamma2x,Gamma2y,Color=[1 .5 0],LineWidth=1.5)
plot(upx(t),upy(t),Color= char 39cyan char 39)
plot(downx(t),downy(t),Color= char 39cyan char 39)
plot(x0,y0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 3
plot(pt1(1),pt1(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(pt2(1),pt2(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(downx(t1),downy(t1),Color= char 39magenta char 39,LineWidth=2)
set(gca,  char 39XAxisLocation char 39,  char 39origin char 39)
set(gca,  char 39YAxisLocation char 39,  char 39origin char 39)
xlabel( char 39x1 char 39);
ylabel( char 39x2 char 39);
title( char 39initial point (-1,1) char 39);
axis([-2,1,-2,2]);
hold off

%%%%%%%%%%% initial (1,-1) %%%%%%%%%%%%
x0 = 1;
y0 = -1;
```

```
upx =@(t) 0.5*t.^2 + y0*t + x0;
upy =@(t) t + y0;
downx =@(t) -0.5*t.^2 + y0*t + x0;
downy =@(t) -t + y0;

%u=-1 goes down, intersects Gamma2 u=1 goes up
eqns = [x-(x0+1/2*y0^2)==-1/2*(y)^2,x+1/2==1/2*(y)^2];
sol31 = solve(eqns);
sol31.x
sol31.y
pt1 = [sol31.x(1);sol31.y(1)]; %pick the lowest x2 to go up
pt1 = double(pt1);
tf1 = -pt1(2);
t1 = linspace(0,tf1);

%u=1 goes up, intersects Gamma1 u=-1 goes down
eqns = [x-(x0-1/2*y0^2)==1/2*(y)^2,x-1/2==-1/2*(y)^2];
sol32 = solve(eqns);
sol32.x
sol32.y
pt2 = [sol32.x(1);sol32.y(1)]; %pick the highest x2 to go down
pt2 = double(pt2);
tf2 = pt2(2);
t2 = linspace(0,tf2);
figure;
plot(Gamma1x,Gamma1y,Color=[1 .5 0],LineWidth=1.5)
hold on
plot(Gamma2x,Gamma2y,Color=[1 .5 0],LineWidth=1.5)
plot(upx(t),upy(t),Color= char 39cyan char 39)
plot(downx(t),downy(t),Color= char 39cyan char 39)
plot(x0,y0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 3
plot(pt1(1),pt1(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
```

```
plot(pt2(1),pt2(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
set(gca,  char 39XAxisLocation char 39,  char 39origin char 39)
set(gca,  char 39YAxisLocation char 39,  char 39origin char 39)
xlabel( char 39x1 char 39);
ylabel( char 39x2 char 39);
title( char 39initial point (1,-1) char 39);
axis([-1,2,-2,2]);
hold off

%%%%%%%%%%% initial (-1,0) %%%%%%%%%%%
x0 = -1;
y0 = 0;
upx =@(t) 0.5*t.^2 + y0*t + x0;
upy =@(t) t + y0;
downx =@(t) -0.5*t.^2 + y0*t + x0;
downy =@(t) -t + y0;

%u=-1 goes down, intersects Gamma2 u=1 goes up
eqns = [x-(x0+1/2*y0^2)==-1/2*(y)^2,x+1/2==1/2*(y)^2];
sol31 = solve(eqns);
sol31.x
sol31.y
pt1 = [sol31.x(1);sol31.y(1)]; %pick the lowest x2 to go up
pt1 = double(pt1);
tf1 = -pt1(2);
t1 = linspace(0,tf1);

%u=1 goes up, intersects Gamma1 u=-1 goes down
eqns = [x-(x0-1/2*y0^2)==1/2*(y)^2,x-1/2==-1/2*(y)^2];
sol32 = solve(eqns);
sol32.x
sol32.y
```

```
pt2 = [sol32.x(1);sol32.y(2)]; %pick the highest x2 to go down
pt2 = double(pt2);
tf2 = pt2(2);
t2 = linspace(0,tf2);
figure;
plot(Gamma1x,Gamma1y,Color=[1 .5 0],LineWidth=1.5)
hold on
plot(Gamma2x,Gamma2y,Color=[1 .5 0],LineWidth=1.5)
plot(upx(t),upy(t),Color= char 39cyan char 39)
plot(downx(t),downy(t),Color= char 39cyan char 39)
plot(x0,y0, char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39r char 3
plot(pt1(1),pt1(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(pt2(1),pt2(2), char 39o char 39,MarkerEdgeColor= char 39r char 39,MarkerFaceColor= char 39
plot(upx(t2),upy(t2),Color= char 39green char 39,LineWidth=2)
set(gca,  char 39XAxisLocation char 39,  char 39origin char 39)
set(gca,  char 39YAxisLocation char 39,  char 39origin char 39)
xlabel( char 39x1 char 39);
ylabel( char 39x2 char 39);
title( char 39initial point (-1,0) char 39);
axis([-2,1,-2,2]);
hold off
```