Homework 2

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Problem (4.4)**.** There is a canonical homeomorphism $\phi : \mathbb{R}^{m\times n} \to \mathbb{R}^{mn}$ by concatenating the columns together. This is a linear operation as it is clearly closed under addition and scalar multiplication and therefore smooth. Hence $(\phi, \mathbb{R}^{m\times n})$ is a global chart. Hausdorff and second-countable follows from homeomorphism and therefore $\mathbb{R}^{m\times n}$ is a smooth manifold.

Notice that the general linear group $\mathrm{GL}_n(R) = \{M \in \mathbb{R}^{m\times n} : \det(M) \neq 0\} = \det^{-1}(\mathbb{R} \setminus \{0\})$. Since $\{0\}$ is closed ($\mathbb{R}$ is Hausdorff and hence T1), $\mathbb{R} \setminus \{0\}$ is open. The determinant function is a polynomial of entries and therefore continuous, so $\mathrm{GL}_n(R)$ as the preimage of an open set of $\mathbb{R}$ via a continuous function is open in $\mathbb{R}^{m\times n}$. By theorem any open subset of a smooth manifold is a smooth manifold.

Problem (4.6)**.** Given $(U_1, \phi_1), (V_1, \psi_1)$ and $(U_2, \phi_2), (V_2, \phi_2)$ be two sets of local charts that satisfy the assumptions. Notice that since $\mathbb{R}^m$ and $\mathbb{R}^n$ are smooth manifolds, the transition maps $\phi_1 \circ \phi_2^{-1} : U_1 \cap U_2 \to \mathbb{R}^m$ and $\psi_2 \circ \psi_1^{-1} : V_1 \cap V_2 \to \mathbb{R}^n$ are also smooth. Thus if $\psi \circ f \circ \phi^{-1}$ is smooth, then

$$\psi_2 \circ f \circ \phi_2^{-1} = (\psi_2 \circ \psi_1^{-1}) \circ (\psi_1 \circ f \circ \phi_1^{-1}) \circ (\phi_1 \circ \phi_2^{-1})$$

is a composition of smooth functions and therefore smooth. Thus f is smooth regardless of the choice of local charts.

Problem (5.8)**.** Recall that for any smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$, $d_p f : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m, [\alpha] \mapsto [f \circ \alpha]$. By the identification, we can also think $d_p f : \mathbb{R}^n \to \mathbb{R}^m, \alpha'(0) \mapsto (f \circ \alpha)'(0)$. But by the chain rule, $(f \circ \alpha)'(0) = Df(p) \circ \alpha'(0)$. That is, $d_p f = Df(p)$ which is exactly the Jacobian of f at p.

Problem (5.9)**.** Let $f : M \to N$ and $g; N \to L$ be smooth maps. Then $d_p(g \circ f) : T_p\mathbb{R}^m \to T_p\mathbb{R}^\ell, [\alpha] \mapsto [g \circ f \circ \alpha]$. Choose any chart $(U, \phi), (V, \psi), (W, \theta)$ containing $p, f(p), g \circ f(p)$. Then

$$d_p(g \circ f)([\alpha]) = (\theta \circ g \circ f \circ \alpha)'(0)$$

$$
\begin{aligned}
&= (\theta \circ g \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \alpha))'(0) \\
&= D(\theta \circ g \circ \psi^{-1}) \circ D(\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \alpha)'(0) \\
&= d_{f(p)} g \circ d_p f([\alpha])
\end{aligned}
$$

Problem (5.10)**.** Let $f : M \to N$ be a diffeomorphism. That is, given $p \in M$ and any charts (U, ϕ) of $\mathbb{R}^m$ containing p and (V, ψ) of $\mathbb{R}^n$ containing $f(p)$, the transition map $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ (restricted to well-defined domain) is smooth and has smooth inverse. This forces the Jacobian of the transition map to be a linear isomorphism by the inverse function theorem. Then

$$
\begin{aligned}
d_p f([\alpha]) &= (\psi \circ f \circ \circ \alpha)'(0) \\
&= ((\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \alpha))'(0) \\
&= D(\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \alpha)'(0) \\
&= D(\psi \circ f \circ \phi^{-1})[\alpha].
\end{aligned}
$$

So we see that $d_p f$ can be identified with the Jacobian of the transition map which must be a linear isomorphism. Since the domain and codomain of $d_p f$ can be identified with $\mathbb{R}^m$ and $\mathbb{R}^n$, it follows that $\mathbb{R}^m = \mathbb{R}^n$ so $m = n$. That is, $\dim M = \dim N$.