1 Transversality

Motivation: we want to prove the following theorem but have trouble at tangency.

Theorem 1.1 (Whitney)

If $X \subseteq \mathbb{R}^n$ is any closed set, there exists a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with $X = f^{-1}(0)$. Then the graph $M = \{(x, f(x)) : x \in \mathbb{R}^n\}$ is a submanifold of $\mathbb{R}^{n+1}$.

Definition 1.2 — Let $X, Y \subseteq Z$ be submanifolds of Z. We say X, Y are **transversal**, denoted $X \pitchfork Y$ if

$$T_pX + T_pY = T_pZ \ \forall \ p \in X \cap Y.$$

Remark 1.3 If $X \cap Y = \emptyset$, then X, Y are trivially transversal.

If $\dim X + \dim Y < \dim Z$ and $X \pitchfork Y$, then $X \cap Y = \emptyset$.

The main results are

(1) $X \pitchfork Y$ and $X \cap Y = \emptyset$, then $X \cap Y$ is a manifold. Moreover, $\dim(X \cap Y) = \dim X + \dim Y - \dim Z$.

(2) Any pair of submanifold becomes transversal after a perturbation (we say they assume general position).

(3) If $\dim X + \dim Y = \dim Z$, and $X \pitchfork Y$, then $X \cap Y$ is a discrete set. If X, Y are compact, then $\#(X \cap Y)$ is finite, called the **intersection number**. This number mod 2 is invariant under homotopy.

Application: general Jordan curve theorem.

Definition 1.4 — Let $Y \subseteq Z$, $f : X \to Z$. Then we say f is **transversal** to Y, $f \pitchfork Y$, if

$$df_p(T_pX) + T_{f(p)}Y = T_{f(p)}Z \ \forall \ p \in f^{-1}(Y).$$

Note that X is a submanifold of Y if there exists an embedding $f : X'tpY$ s.t. $f(X') = X$, this is an immersion and a homemorphism. Rewrite $X = X'$ and consider $i : X \to Z$ the inclusion map. The rank theorem says that

Locally any submanifold is the inverse image of 0 via a submersion.

To prove (1), since $X \cap Y = i^{-1}(Y)$, it is equivalent to prove the following:

Theorem 1.5

If $f : X \to Z$ is transversal to Y, then $f^{-1}(Y)$ is a manifold.

Proof. Locally $Y = g^{-1}(0)$ where g is a submersion. Then

$$\begin{aligned} f^{-1}(Y) &= f^{-1}(g^{-1}(0)) \\ &= (g \circ f)^{-1}(0) \end{aligned}$$

It suffices to show that 0 is a regular value of $g \circ f$. Exercise. □