Homework 13

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Problem (4.4.3)**.**

$$\begin{aligned}
\int_a^b c^*\omega &= \int_a^b c^* df \\
&= \int_a^b d(c^* f) \\
&= \int_a^b d(f \circ c) \\
&= \int_a^b \frac{d(f \circ c)}{dx} dx \\
&= f \circ c(b) - f \circ c(a) && \text{FTC} \\
&= f(q) - f(p)
\end{aligned}$$

Problem (4.4.5)**.** Let $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ where we view $S^1 \cong I/\partial I$. Since ω is a 1-form, we have $\omega = \sum_{i=1}^k f_i dx_i$, where $f_i : X \to \mathbb{R}$. Therefore,

$$\begin{aligned}
\gamma^*\omega &= \sum_{i=1}^k (f_i \circ \gamma)\gamma^* dx_i \\
&= \sum_{i=1}^k (f_i \circ \gamma) d(x_i \circ \gamma) \\
&= \sum_{i=1}^k (f_i \circ \gamma) d\gamma_i \\
&= \sum_{i=1}^k (f_i \circ \gamma) \frac{d\gamma_i}{dt} dt
\end{aligned}$$

Therefore,

$$\begin{aligned}
\oint_\gamma \omega &= \int_{S^1} \sum_{i=1}^k (f_i \circ \gamma(t)) \frac{d\gamma_i}{dt} dt \\
&= \sum_{i=1}^k \int_{S^1} (f_i \circ \gamma(t)) \frac{d\gamma_i}{dt} dt
\end{aligned}$$

Problem (4.4.7)**.** By Problem 3, since the endpoints p of a closed curve are the same, we immediately obtain

$$\oint_\gamma \omega = f(p) - f(p) = 0$$

Problem (4.4.8)**.**

(a) First note that under the polar coordinates, $x = r\cos\theta, y = r\sin\theta$, and $r^2 = x^2 + y^2$. Thus $dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta = \cos\theta dr - r\sin\theta d\theta$ and $dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta = \sin\theta dr + r\cos\theta d\theta$. Thus we have

$$\begin{aligned}
\int_{C_a} \omega &= \int_{C_a} \frac{-y}{x^2+y^2}dx + \int_{C_a} \frac{x}{x^2+y^2}dy \\
&= \int_a^a \frac{-\sin\theta\cos\theta}{r}dr + \int_0^{2\pi} (\sin\theta)^2 d\theta + \int_a^a \frac{\cos\theta\sin\theta}{r}dr + \int_0^{2\pi} (\cos\theta)^2 d\theta \\
&= 0 + \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)d\theta + 0 \\
&= \theta|_0^{2\pi} \\
&= 2\pi
\end{aligned}$$

(b) Note that when $x > 0$, $\frac{y}{x}$ is well-defined. Thus we have

$$\begin{aligned}
d\left(\arctan\frac{y}{x}\right) &= \frac{\partial \arctan\frac{y}{x}}{\partial x}dx + \frac{\partial \arctan\frac{y}{x}}{\partial y}dy \\
&= \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)dx + \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x}dy \\
&= \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy
\end{aligned}$$

(c) Suppose $w = \partial f$ for some $f : \mathbb{R}^2 - \{0\} \to \mathbb{R}$, then by 7 its contour integral would be 0, but that contradicts with part a.

Problem (4.4.13)**.** Suppose S is the graph of $G : \mathbb{R}^2 \to \mathbb{R}$, then S can be parameterized by $h(x_1, x_2) = (x_1, x_2, G(x_1, x_2))$. Therefore,

$$dx_3 = dG = \frac{\partial G}{\partial x_1}dx_1 + \frac{\partial G}{\partial x_2}dx_2.$$

Then the 2-form becomes

$$\begin{aligned}
dA &= -n_1\frac{\partial G}{\partial x_1}dx_1 \wedge dx_2 - n_2\frac{\partial G}{\partial x_2}dx_1 \wedge dx_2 + n_3 dx_1 \wedge dx_2 \\
&= \left(-\frac{\partial G}{\partial x_1}n_1, -\frac{\partial G}{\partial x_2}n_2, n_3\right) dx_1 \wedge dx_2 =: \mathbf{v}dx_1 \wedge dx_2 \qquad n_i \text{ are linearly independent} \\
&= \begin{pmatrix} -\frac{\partial G}{\partial x_1} & 0 & 0 \\ 0 & -\frac{\partial G}{\partial x_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} dx_1 \wedge dx_2 =: D\mathbf{n}dx_1 \wedge dx_2
\end{aligned}$$

$$= \|D\|\mathbf{n}dx_1 \wedge dx_2 \qquad \mathbf{n} \text{ unit vec}$$

$$= \sqrt{\left(\frac{\partial G}{\partial x_1}\right)^2 + \left(\frac{\partial G}{\partial x_2}\right)^2 + 1} \quad \mathbf{n}dx_1 \wedge dx_2$$

$$= \|\mathbf{v}\|\mathbf{n}dx_1 \wedge dx_2$$

which coincides with the other definition of dA after doting with $\mathbf{n}$.

Problem (4.5.1)**.** Denote the given form ω.

(a)

$$\begin{aligned} d(\omega) &= d(z^2dx \wedge dy) + d(z^2dx \wedge dz) + d(2ydx \wedge dz) \\ &= d(z^2) \wedge dx \wedge dy + d(z^2)dx \wedge dz + d(2y) \wedge dx \wedge dz \\ &= 2zdz \wedge dx \wedge dy + 2zdz \wedge dx \wedge dz + 2dy \wedge dx \wedge dz \\ &= (2z - 2)dx \wedge dy \wedge dz \end{aligned}$$

(b)

$$\begin{aligned} d\omega &= 13dx \wedge dx + 2ydy \wedge dy + d(xyz)dz \\ &= 0 + 0 + (yzdx + xzdy + xydz) \wedge dz \\ &= yzdx \wedge dz + xzdy \wedge dz \end{aligned}$$

(c)

$$\begin{aligned} d(fdg) &= df \wedge dg + 0 \qquad \text{product rule} \\ &= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right) \wedge \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz\right) \\ &= \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}dx \wedge dy + \frac{\partial f}{\partial x}\frac{\partial g}{\partial z}dx \wedge dz + \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}dy \wedge dx \\ &\quad + \frac{\partial f}{\partial y}\frac{\partial g}{\partial z}dy \wedge dz + \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}dz \wedge dx + \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}dz \wedge dy \\ &= \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) dx \wedge dy + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\right) dx \wedge dz + \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right) dy \wedge dz \end{aligned}$$

(d)

$$d\omega = 0 + 0 + 6y^2dy \wedge dz \wedge dx + 0 = 6y^2dy \wedge dz \wedge dx$$

Problem (4.5.2)**.**

$$\begin{aligned}\nabla \times F &= \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \\ &= \frac{x^2 + y^2 - 2x^2}{(x^2+y^2)^2} + \frac{x^2 + y^2 - 2y^2}{(x^2+y^2)^2} \\ &= 0\end{aligned}$$

By Problem 4.4.8, we know that F is not the differential of any function, thus it cannot be written as the gradient of any function. Otherwise, the gradient expression immediately leads to the 1-form aka differential.