Homework 6

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Problem (1)**.** From the example, we know that optimal trajectories are clockwise-oriented circles centered at $(1,0)$ for $u = 1$ and centered at $(-1,0)$ for $u = -1$. To reach the origin, we must eventually get on the switching curves Γ^1_+ and Γ^1_- since they are the only circles centered at $(\pm 1, 0)$ that go through the origin. Moreover, the optimal control $u^* = -\text{sign}(\Lambda \cos(\omega t + \phi))$ flips signs and thus must switch every $\frac{\pi}{\omega} = \pi$ except that it might switch sooner at the beginning or in the end.

(a) When $x_1(0) = x_2(0) = 2$, we have the following optimal trajectory:

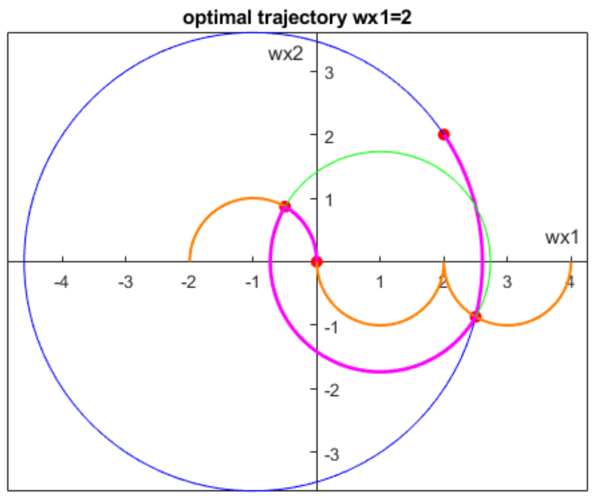


Figure 1: Magenta denotes the optimal trajectory and orange denotes the optimal switching curve. We first find clockwise-oriented circles centered at $(\pm 1, 0)$ that go through $(2,2)$ and pick the one that reaches the switching surface the fastest. In this case it is the blue circle centered at $(-1,0)$. Then we switch to the green circle centered at $(1,0)$ and continue for π unit of time to reach the next switching curve which happens to be Γ^1_- so we simply follow the singular curve to reach the origin.

Using $\widehat{\Gamma}$, we have the following trajectory:

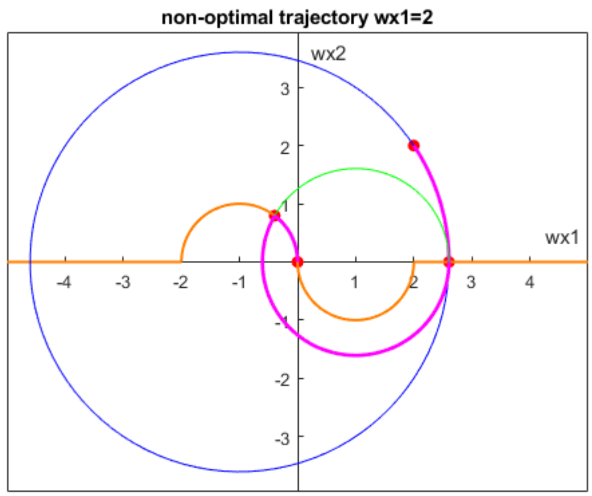
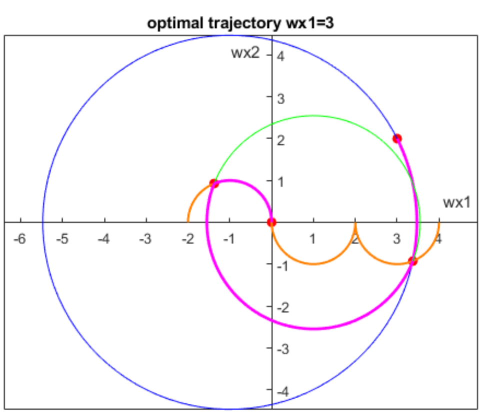
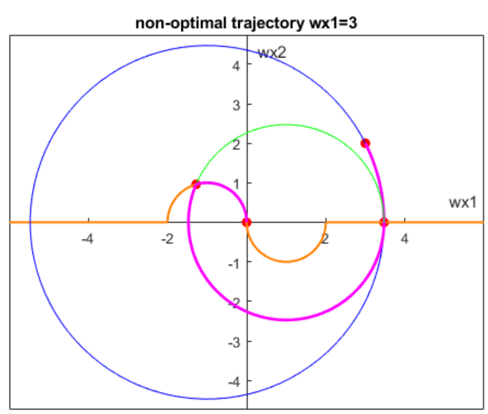


Figure 2: Magenta denotes the non-optimal trajectory and orange denotes the non-optimal switching curve. We repeat the previous first step, reach a new switching curve at $x_2 = 0$, and switch to the other center. This time we don't have to switch every π so we simply reach the next switching curve to switch.

Since $\omega = 1$, the elapsed time is the same as the angle the trajectory traced out. By adding the angles together, we obtain that $t^* = 5.0194$ and $\hat{t} = 5.1685$.

(b) We repeat the procedure for $\omega x_1 = 3, 4, 5$.





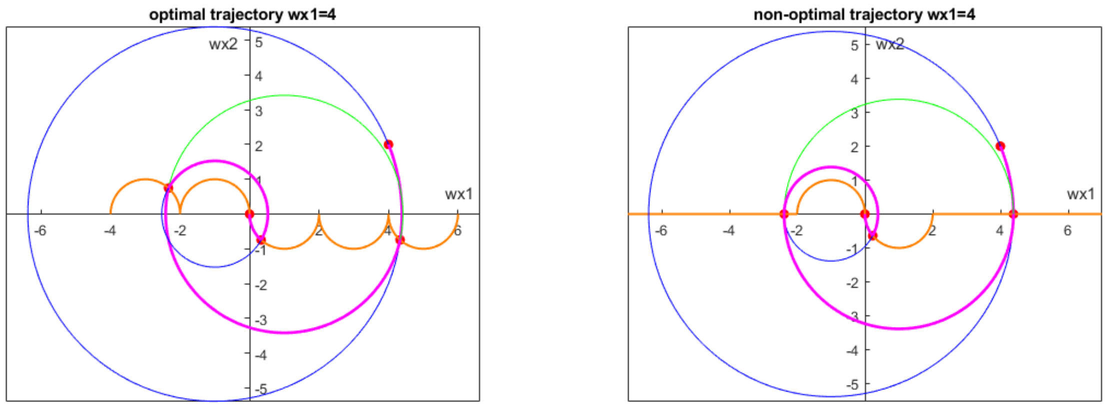
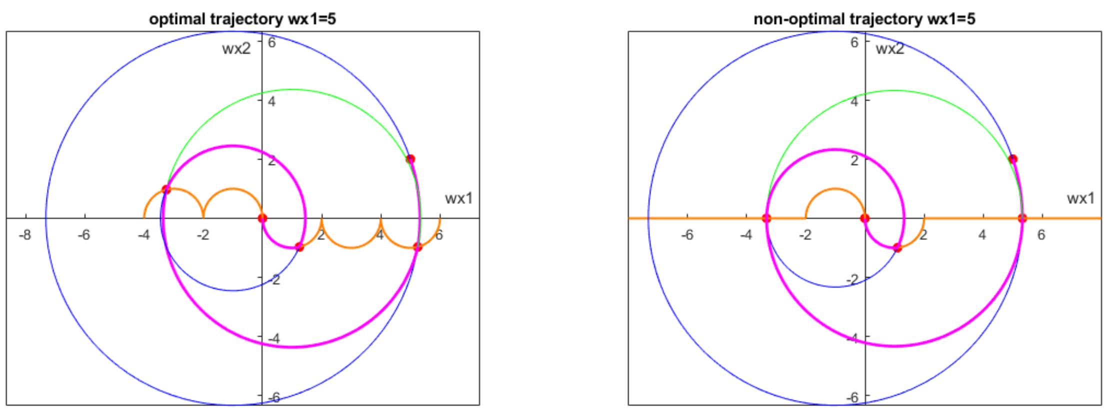


Figure 3: Both trajectories becomes more complicated. We have four arcs corresponding to three switchings.



From the figures, we can roughly see that the optimal trajectories and their non-optimal counterparts largely resembles each other. Moreover, it is not hard to imagine that as ωx_1 gets larger, the total angles traced out by both trajectories increase but their differences don't increase much. Thus the ratio $\hat{t}/t^*$ trends down to 1. There is not enough sample size in the following plot to fully illustrate that but it is a start:

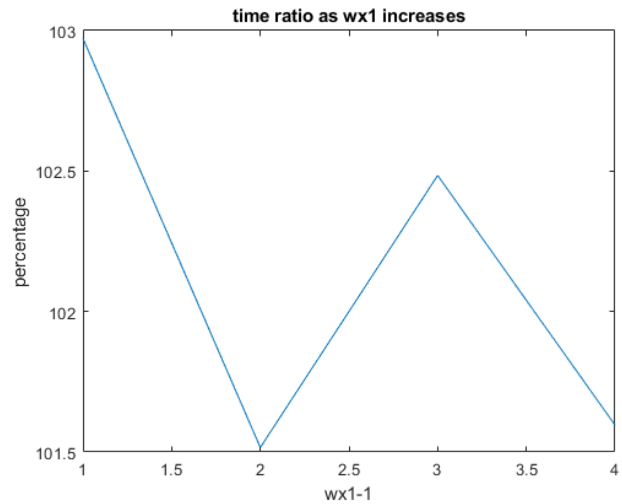


Figure 4: The ratio fluctuates but trends down.

Problem (2)**.** The Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2}x_1^2 + p_1(x_2 + u) + p_2(-u) \\ &= \frac{1}{2}x_1^2 + p_1 x_2 + (p_1 - p_2)u \end{aligned}$$

The adjoint equations are

$$\dot{p_1} = -H_{x_1} = -x_1$$

$$\dot{p_2} = -H_{x_2} = -p_1$$

Since t_f is free, and H is time-independent, transversality yields $H \equiv 0$.

(a) Let us examine the optimality condition. By the PMP,

$$u^* = \underset{u}{\operatorname{argmin}}\, H = \begin{cases} -1 & p_1 - p_2 > 0 \\ ? & p_1 - p_2 = 0 \\ 1 & p_1 - p_2 < 0 \end{cases}$$

Notice on the singular surface we have

$$H = \frac{1}{2}x_1^2 + x_1 x_2 = 0$$

$$x_1(x_1 + x_2) = 0$$

$$\Rightarrow x_1 = 0 \text{ or } x_1 = -2x_2$$

Denote the surfaces by S_1 and S_2 respectively. Now we use GLC to obtain the optimal control on the singular surface

$$\begin{aligned}
\dot{H}_u &= \dot{p_1} - \dot{p_2} = -x_1 + p_1 \Rightarrow x_1 = p_1 = p_2 \\
\ddot{H}_u &= -\dot{x_1} + \dot{p_1} = -x_2 - u - x_1 = 0 \\
u^* &= -(x_1 + x_2)
\end{aligned}$$

Moreover, since $k = 1$, $(-1)^1 \frac{\partial}{\partial u} \ddot{H}_u = 1 > 0$ which passes GLC. Thus by definition of S_1 and S_2,

$$u^* = \begin{cases} -x_2 & \text{on } S_1 \\ x_2 & \text{on } S_2 \end{cases}$$

Notice that on S_1, $\dot{x_2} = x_2$ so x_2 moves away from origin. On S_2, we have

$$\begin{cases} \dot{x_1} = 2x_2 \\ \dot{x_2} = -x_2 \end{cases}$$

which indicates that trajectory on S_2 move toward the origin as time increases.

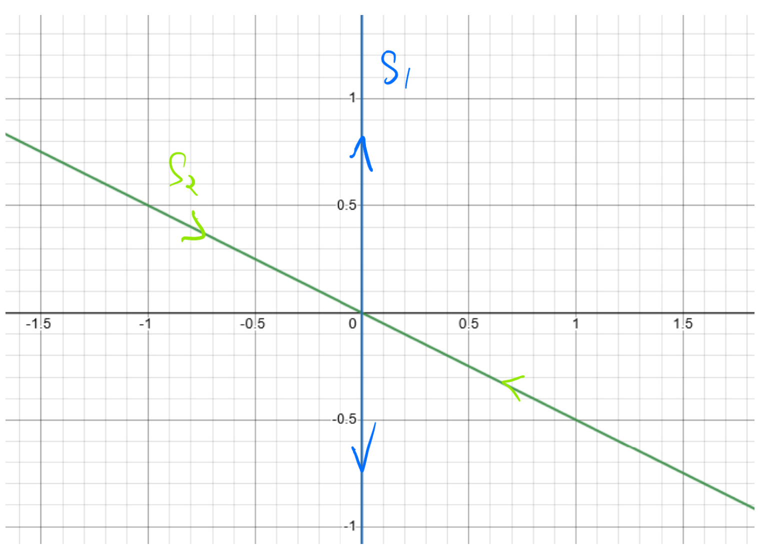


Figure 5: Switching surfaces: S_1 is blue and S_2 is green.

Now the initial point $(0, -0.5)$ is on S_1, we cannot continue on S_1 as it moves away from the target. Hence we resort to bang-bang control. If $u = 1$, dynamics are

$$\begin{cases} \dot{x_1} = x_2 + 1 & x_1(0) = 0 \\ \dot{x_2} = -1 & x_2(0) = -\frac{1}{2} \end{cases}$$

which yields $x_1 = -\frac{1}{2}\left(x_2 + \frac{1}{2}\right)^2 - \frac{1}{2}\left(x_2 + \frac{1}{2}\right)$. However, both its intersections with $x_1 = -2x_2$ are not achieved in the positive time direction.

If $u = -1$, the dynamics are

$$\begin{cases} \dot{x_1} = x_2 - 1 & x_1(0) = 0 \\ \dot{x_2} = 1 & x_2(0) = -\frac{1}{2} \end{cases}$$

which yields

$$\begin{cases} x_1(t) = \frac{1}{2}t^2 - \frac{3}{2}t \\ x_2(t) = t - \frac{1}{2} \Rightarrow t = x_2 + \frac{1}{2} \end{cases}$$

so we have the trajectory $x_1 = \frac{1}{2}\left(x + \frac{1}{2}\right)^2 - \frac{3}{2}\left(x_2 + \frac{1}{2}\right)$. This intersections with S_2 at $(-1, 0.5)$ which is achieved in positive time $t = 1$. Thus at this time we switch to singular control.

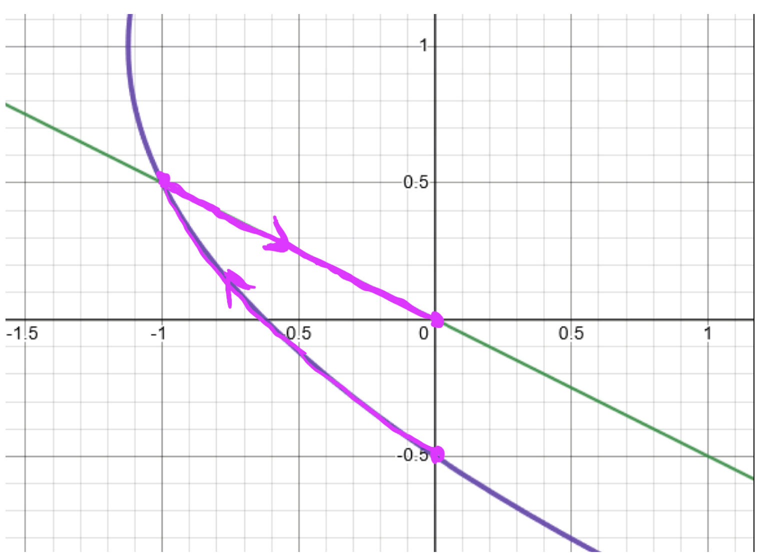


Figure 6: Optimal trajectory from initial point $(0, -0.5)$.

(b) Yes we can use bang-bang control only to reach the origin. We simply use the bang-bang control $u = -1$ above and then switch to $u = 1$ which it intersects a bang-bang trajectory using $u = 1$ that reaches the origin. We find the latter trajectory with reversed time so that it goes through the origin at $t = 0$ for simplicity:

$$\begin{cases} \dot{x_1} = x_2 + 1 & x_1(0) = 0 \\ \dot{x_2} = -1 & x_2(0) = 0 \end{cases}$$

which yields

$$\begin{cases} x_1(t) = -\frac{1}{2}t^2 + t \\ x_2(t) = -t \Rightarrow t = -x_2 \end{cases}$$

This gives the trajectory $x_1 = -\frac{1}{2}x_2^2 - x_2$. The two bang-bang arcs intersect at $(-1.1031, 0.7906)$, with $t = -0.7906$. Thus the trajectory is

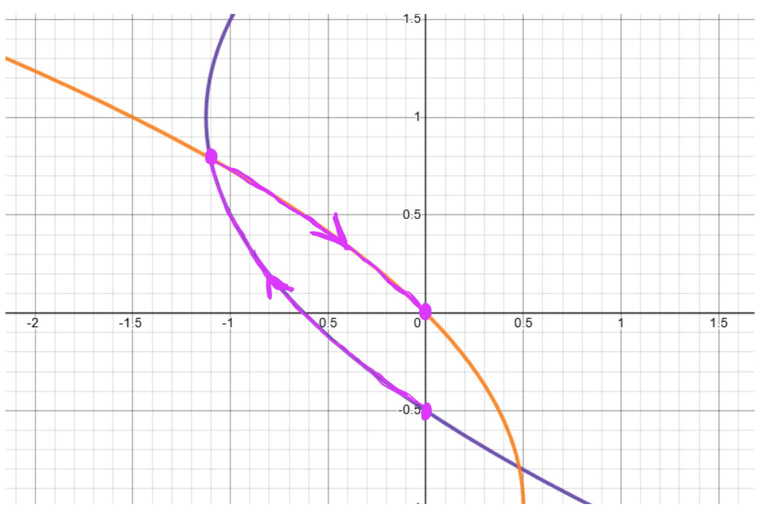


Figure 7: Bang-bang only trajectory.

(c) For the optimal control, we see that the first intersection occurs at $t_1 = x_2 + \frac{1}{2} = 1$. Moreover, since $x_1 = p_1$, the adjoint equation becomes $x_1 = -\dot{x_1}$. Thus

$$\begin{aligned} \frac{1}{2}\int_1^T x_1^2 dt &= -\frac{1}{2}\int_1^T x_1\dot{x_1}dt \\ &= -\frac{1}{2}\int_1^T \frac{d}{dt}x_1\dot{x_1}dt \\ &= -\frac{1}{4}\left(x_1(T)^2 - x_1(1)^2\right) \end{aligned}$$

$$= \frac{1}{4}x_1(1)^2$$
$$= \frac{1}{4}$$

and

$$\begin{aligned} J^* = \frac{1}{2}\int_0^T x_1^2 dt &= \frac{1}{2}\int_0^1 x_1^2 dt + \frac{1}{2}\int_1^T x_1^2 dt \\ &= \frac{1}{2}\int_0^1 \left(\frac{1}{2}t^2 - \frac{3}{2}t\right)^2 dt + \frac{1}{4} \\ &= 0.4625 \end{aligned}$$

From (d) we obtain that for bang-bang trajectory, $t_1 = 1.2906$. Thus

$$\begin{aligned} J_{BB} &= \frac{1}{2}\int_0^T x_1^2 dt \\ &= \frac{1}{2}\int_0^{1.2906} \left(\frac{1}{2}t^2 - \frac{3}{2}t\right)^2 dt + \frac{1}{2}\int_{-0.7906}^0 \left(\frac{1}{2}t^2 + t\right)^2 \\ &= 0.3755 + 0.1389 = 0.5144 \end{aligned}$$

(d) Solving the dynamics on S_2 with boundary condition $x_1(1) = -1, x_2(1) = \frac{1}{2}$ yields

$$\begin{cases} x_1(t) = -2.7183e^{-t} \\ x_2(t) = 1.3591e^{-t} \end{cases}$$

which is never going to reach $(0,0)$ in finite time.

In the bang-bang case, we have $t_1 = x_2 + \frac{1}{2} = 0.7906 + 0.5 = 1.2906$ so $T = t_1 + 0.7906 = 2.0812$.

Problem (3)**.** The Hamiltonian is

$$H = 1 + p_1 x_2 + p_2 u.$$

The adjoint equations are

$$\dot{p_1} = -H_{x_1} = 0$$
$$\dot{p_2} = -H_{x_2} = -p_1$$

We see that p_1 is constant and p_2 is linear.

(a) By PMP, we know that the optimal control is

$$u^* = \operatorname*{argmin}_u H = \begin{cases} u_{\max} & p_2 < 0 \\ ? & p_2 = 0 \\ -u_{\max} & p_2 > 0 \end{cases}$$

Since p_2 is linear, it cannot be zero for more than one point. Otherwise, $p_2 = p_1 \equiv 0$ which implies that $H \equiv 1$. However, since t_f is free, we have $H \equiv 0$, a contradiction. It follows that

$$u^* = -\operatorname{sign}(p_2) \cdot u_{\max} = \pm u_{\max},$$

i.e. we only have bang-bang control.

(b) Let $b := \pm u_{\max}$. Thus the dynamics of optimal trajectory are

$$\begin{cases} \dot{x_2} = b & \Rightarrow x_2(t) = bt + x_{20} \\ \dot{x_1} = x_2 & \Rightarrow x_1(t) = bt^2 + x_{20}t + x_{10} \end{cases}$$

By substitution we have the optimal trajectory

$$x_1 - \left(x_{10} - \frac{1}{2b}x_{20}^2\right) = \frac{1}{2b}x_2^2,$$

for initial point (x_{10}, x_{20}). To have end points at $(0, \pm 1)$, the optimal trajectory must have its last arc on one of the orange parabolas below. These are the switching curves.

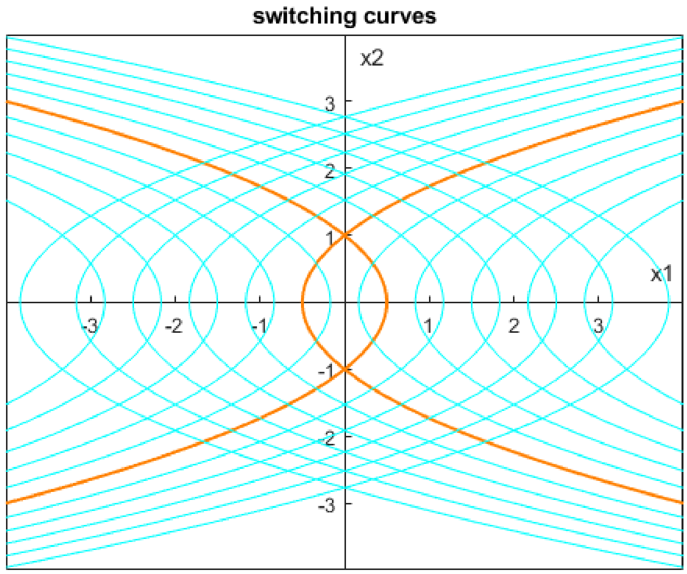


Figure 8: The orange parabolas are the only optimal trajectories that go through $(0, \pm 1)$. The cyan parabolas are other potential optimal trajectories. Given an initial point (x_{10}, x_{20}), there are always exactly two parabolas facing opposite directions that go through the point based on the equations above, and they always intersect at least one of the orange parabolas once. Note that the parabolas with mouth opening to the left goes down whereas the parabolas with mouth opening right goes up as time elapses.

(c) By the reasons stated in the caption above, yes solution exists for all initial points. They might not be unique, as the $(0, 0)$ case illustrates below.

(d)

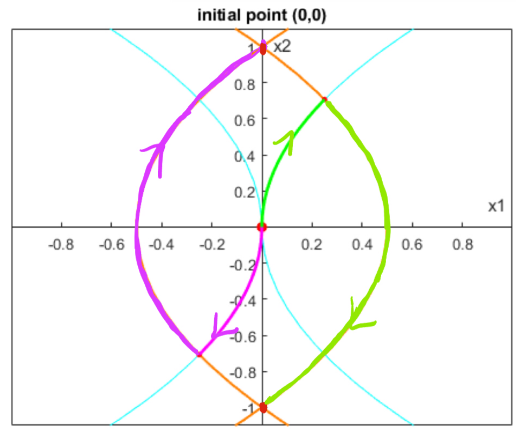
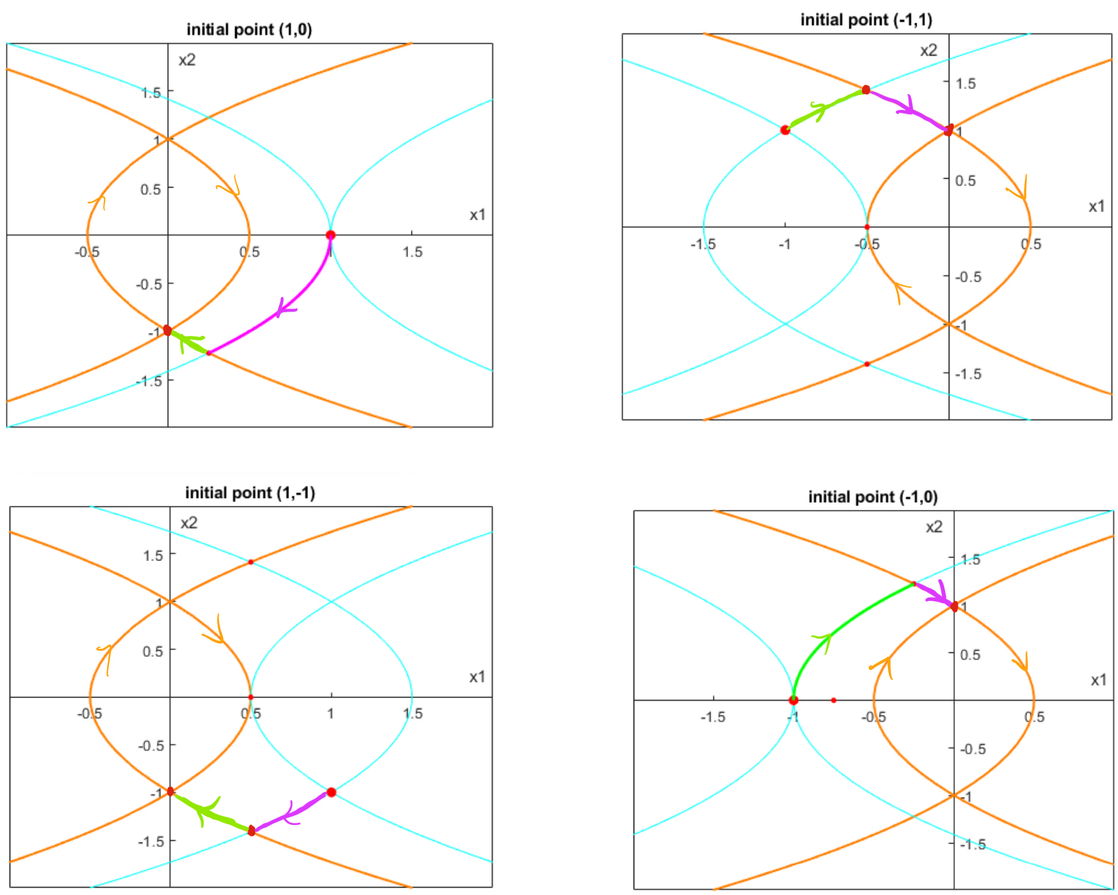


Figure 9: Due to symmetry, we see that there are two optimal trajectories from $(0,0)$ to $(0,\pm 1)$.



Problem (4)**.**