Homework 9

Jaden Wang

Problem (2.3.11)**.** Move X around until it satisfies the assumption. Since x is a 1-submanifold, take a neighborhood U of 0 so that $\phi : \phi^{-1}(X) \subseteq \mathbb{R} \to X \cap U \subseteq \mathbb{R}^2$ is a diffeomorphism. Let $\pi : X \cap U \to \mathbb{R}, (x, y) \mapsto x$ be the projection to x-axis. Then define $f := \pi \circ \phi : \phi^{-1}(X) \subseteq \mathbb{R} \to \mathbb{R}$. It is easy to see that df_p has rank 1 and therefore full rank for all $p \in \phi^{-1}(X)$ by chain rule. That is, df_0 is invertible. Consider the map $F : X \cap U \to \Gamma(f), x \mapsto (x, f(x))$. The Jacobian of F at 0 is simply $\begin{pmatrix} I & 0 \\ 0 & df_0 \end{pmatrix}$ which is invertible. Hence by IFT, there exists a neighborhood V of 0 s.t. F is a local diffeomorphism with F^{-1} being injective. That is, if $x_1 = x_2$, then $f(x_1) = f(x_2)$ so f is a well-defined function on the Euclidean space, and $X = \Gamma(f)$ on $U \cap V$.

We wish to find a point y in $\mathbb{R}^2$ s.t. some $q \in h^{-1}(y)$ makes dh_q not a submersion. First, notice that for any point $\overline{x} \in X \cap U \cap V$, we can represent it by the graph $\overline{x} = (x, f(x))$ for some $x \in \mathbb{R}$. The tangent space can be identified as the tangent line to the curve, which is spanned by $(1, f'(x))$. The set of vectors in $\mathbb{R}^2$ normal to $T_{\overline{x}}X$ is the line spanned by $(-f'(x), 1)$ by analytic geometry. So an element in NX is $((x, f(x)), c(-f'(x), 1))$ for some $x, c \in \mathbb{R}$. Then the normal bundle map becomes $h : ((x, f(x)), c(-f'(x), 1)) \mapsto (x - cf'(x), f(x) + c)$. Then dh is simply the derivative wrt to x, which is $(1 - cf''(x), f'(x))$. Since we are given that $f''(0) \neq 0$ and $f'(0) = 0$, we quickly see that setting $x = 0$, $c = \frac{1}{f''(0)}$, and $q = ((0, f(0)), \frac{1}{f''(0)}(-f'(0), 1))$ makes dh_q not a submersion as it is the zero map. Hence $y = (0, f(0)) + \frac{1}{f''(0)}(-f'(0), 1) = (0, \frac{1}{\kappa(p)})$ is a focal point as desired.

Problem (2.3.14)**.** Given an equivalent class of smooth curves $[\gamma]$ based at $z \in Z$, since 0 is in any subspace, and any curve in $N(Z; Y)$ is determined by specifying a curve on each factor, we have the equivalence class of smooth curves $[(\gamma, 0)]$ based at $(z, 0)$. Then $d\sigma_{(z,0)} : [(\gamma, 0)] \mapsto [\sigma \circ (\gamma, 0)] = [\gamma]$ so σ is a submersion.

$$\sigma^{-1}(z) = \{(z', v) \in N(Z; Y) : \sigma(z', v) = z' = z\} = \{(z, v) : v \in T_zY, v \in T_zZ\}.$$

That is, it is the orthogonal complement of T_zZ.

Problem (2.4.4)**.** First we must assume Y is connected. As I said before, since manifold is locally path-connected, Y is also path-connected. Suppose $f : X \to Y$ is homotopic to a constant function $e_0 : x \mapsto y_0$ for some $y_0 \in Y$. If $\dim X > 0$, then $\dim Z < \dim Y$. Hence there exists a $y \notin Z$. Let γ be a path between y_0 and y. Then we see that $f \simeq e : x \mapsto y$ by composing the homotopy with the path. Since intersection number is invariant under homotopy, and $y \notin Z$, we have that $I_2(f, Z) = I_2(e, Z) = 0$.

Problem (2.4.5)**.** Note that if $\dim X > 0$, it suffices to show that every $f : X \to Y$ is homotopic to a constant map and then apply problem 4. Since Y is contractible, we have $1_Y \simeq e_{y_0}$. But notice

$$1_Y \circ f \simeq e_{y_0} \circ f$$
$$f \simeq \widetilde{f}$$

where $\widetilde{f} : X \to Y, x \mapsto y_0$ is a constant function, as desired.

If $\dim X = 0$, X can be covered by using local diffeomorphism neighborhoods, so f maps X to a union of open sets so $\operatorname{im} f$ is open. It is also closed because $\operatorname{im} f$ is compact and Y is Hausdorff. Hence $\operatorname{im} f$ is clopen and by connectedness and nonemptyness (since f is a function) $\operatorname{im} f = Y$. But that means that $\dim Y = \dim X = 0$, a contradiction.

Problem (2.4.6)**.** Suppose Y is a compact, contractible manifold. Then let $f : Y \to Y$ be the identity map. Let $Z = \{y\}$ be a single point with $\dim Z = 0$ which is closed. Then $I_2(f, Z) = 0$ if $\dim Y > 0$ according to Problem 5. However, since $Z \cap f(Y) = \{y\}$ so they intersect exactly once, a contradiction. This forces $\dim Y = 0$. Since Y is compact, Y must be a finite set of points. Since Y is contractible, there exists a homotopy that maps finite points to a single point, *i.e.* finite paths from points to a single point contained in the manifold. Then the paths must be the trivial path or Y would be at least dimension 1. That is, Y must be a one-point space.

Problem (2.4.10)**.** Take any two transversal 1-manifolds in S^2, then since S^2 has no boundary, by classification of 1-manifold they must be loops. Homotop them so that one is contained in the north hemisphere and one is contained in the south hemisphere. Their intersection number mod 2 is clearly 0. However, if we take a horizontal circle and a vertical circle in a torus, their intersection number mod 2 is 1. This is a structural difference so they cannot be diffeomorphic.

Problem (2.4.17)**.** We wish to use the Boundary Theorem to prove the No Retraction Theorem.

Suppose X is a compact manifold with boundary, and suppose to the contrary that there is a smooth map $g : X \to \partial X$ s.t. $\partial g : \partial X \to \partial X$ is the identity. In other words, the smooth map ∂g extends to all of X. Let $Z = \{z\}$ be a single point of ∂X. Then clearly $\partial g(\partial X) \cap Z = \{z\}$ so $I_2(\partial g, Z) = 1$. However, $I_2(\partial g, Z) = 0$ by the Boundary Theorem, a contradiction.