Homework 7

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Problem (do Carmo 3.8)**.** Let M be a Riemannian manifold. Let $X \in \mathfrak{X}(M)$ and $f \in \mathcal{D}(M)$. Define the divergence of X as a function $\operatorname{div} X(p) =$ trace of the linear map $Y(p) \to \nabla_Y X(p), p \in M$, and the gradient of f as a vector field $\operatorname{grad} f$ on M defined by

$$\langle \operatorname{grad} f(p), v \rangle = df_p(v), \qquad p \in M, \quad v \in T_p\dot{M}$$

(a) Let E_i be a geodesic frame at $p \in M$. Show that

$$\operatorname{grad} f(p) = \sum_{i=1}^{n} (E_i(f))E_i(p),$$

and

$$\operatorname{div} X(p) = \sum_{i=1}^{n} E_i(f_i)(p),$$

where $X = f^i E_i$.

(b) Let $M = \mathbb{R}^n$, with coordinates $(x_1, \ldots, x_n)$ and $\frac{\partial}{\partial x_i} = (0, \ldots, 1, \ldots, 0) = e_i$. Show that

$$\operatorname{grad} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i,$$

$\operatorname{div} X = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$, where $X = f^i e_i$.

Proof. (a) Recall that a geodesic frame Write $v = v^i E_i(p)$. Then

$$\begin{aligned}
df_p(v) &= df_p\left(v^i E_i(p)\right) \\
&= v^i df_p(E_i(p)) && \text{linear map} \\
&= v^i \frac{\partial f}{\partial x_i}(p) \\
&= v^i E_i(f)(p) \\
&= \langle E_i(f)E_i(p), v^i E_i(p) \rangle \\
&= \langle \operatorname{grad} f, v \rangle.
\end{aligned}$$

Thus $\operatorname{grad} f = E^i(f)E_i(p)$.

Recall that trace of a linear map L is defined as $\operatorname{tr} L := \sum_{i=1}^{n} \langle L(E_i), E_i \rangle$.

$$\begin{aligned}
\operatorname{div} X(p) &= \operatorname{div}\left(f^i E_i\right)(p) \\
&= \sum_{j=1}^{n} \langle \nabla_{E_j}\left(f^i E_i\right)(p), E_j(p) \rangle \\
&= \sum_{j=1}^{n} \langle E_j(f^i) E_i(p) + f^i \nabla_{E_j} E_i(p), E_j(p) \rangle \\
&= \sum_{j=1}^{n} \langle E_j(f^i) E_i(p) + 0, E_j(p) \rangle && \text{geodesic frame} \\
&= E_i(f^i)(p).
\end{aligned}$$

(b) This is immediate.

$$\begin{aligned}
\operatorname{grad} f(p) &= e^i(f) e_i(p) \\
&= \frac{\partial f}{\partial x_i} e_i.
\end{aligned}$$

$$\begin{aligned}
\operatorname{div} X(p) &= E_i(f^i)(p) \\
&= \frac{\partial f^i}{\partial x_i}.
\end{aligned}$$

$\square$

Problem (do Carmo 3.9)**.** Let M be a Riemannian manifold. Define the Laplacian $\Delta : \mathcal{D}(M) \to \mathcal{D}(M)$ of M by

$$\Delta f = \operatorname{div} \operatorname{grad} f, \qquad f \in \mathcal{D}(M).$$

(a) Let E_i be a geodesic frame at $p \in M$. Prove that

$$\Delta f(p) = E_i(E^i(f))(p).$$

Conclude that if $M = \mathbb{R}^n$, Δ coincides with the usual Laplacian, namely $\Delta f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$.

(b) Show that

$$\Delta(f \cdot g) = f \Delta g + g \Delta f + 2\langle \operatorname{grad} f, \operatorname{grad} g \rangle.$$

Proof. (a) This is immediate from previous problem. Since $\operatorname{grad} f = E^i(f)E_i$,

$$\begin{aligned}\Delta f(p) &= \operatorname{div}(E^i(f)E_i)(p)\\ &= E_i(E^i(f))(p).\end{aligned}$$

If $M = \mathbb{R}^n$, then

$$\Delta f = E_i(E^i(f)) = E_i\left(\frac{\partial f}{\partial x^i}\right) = \sum_i \frac{\partial^2 f}{\partial {x_i}^2}.$$

(b) First we compute

$$\langle \operatorname{grad} f, \operatorname{grad} g\rangle = \langle E^i(f)E_i, E^i(g)E_i\rangle = E^i(f)E_i(g).$$

It follows that

$$\begin{aligned}\Delta(f\cdot g) &= E_i(E^i(f\cdot g))\\ &= E_i(E^i(f)g + fE^i(g)) && \text{Leibniz rule of derivation}\\ &= E_i(E^i(f))g + E^i(f)E_i(g) + E_i(f)E^i(g) + fE_i(E^i(g))\\ &= g\Delta f + 2\langle \operatorname{grad} f, \operatorname{grad} g\rangle + f\Delta g.\end{aligned}$$

□

Problem (do Carmo 3.10)**.** Let $f : I \times [0,a] \to M$ be a parametrized surface such that for all $t_0 \in [0,a]$, the curve $s \to f(s,t_0), s \in [0,1]$ is a geodesic parametrized by arc length, which is orthogonal to the curve $t \to f(0,t), t \in [0,a]$, at the point $f(0,t_0)$. Prove that, for all $(s_0,t_0) \in I \times [0,a]$, the curves $s \to f(s,t_0), t \to f(s_0,t)$ are orthogonal.

Proof. We want to show that $\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle \equiv 0$. Since we already know that $\langle \frac{\partial f}{\partial s}\Big|_{(0,t_0)}, \frac{\partial f}{\partial t}\Big|_{(0,t_0)}\rangle = 0$ for any $t_0 \in [0,a]$, it suffices to show that the inner product is constant as we vary s, *i.e.* it has 0 derivative wrt s. We compute

$$\begin{aligned}\frac{\partial}{\partial s}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle &= \left\langle \frac{D}{\partial s}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle + \left\langle \frac{\partial f}{\partial s}, \frac{D}{\partial s}\frac{\partial f}{\partial t}\right\rangle && \text{Leibniz rule}\\ &= \left\langle \frac{D}{\partial s}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle + \left\langle \frac{\partial f}{\partial s}, \frac{D}{\partial t}\frac{\partial f}{\partial s}\right\rangle && \text{symmetry lemma}\\ &= \left\langle 0, \frac{\partial f}{\partial t}\right\rangle + \left\langle \frac{\partial f}{\partial s}, \frac{D}{\partial t}\frac{\partial f}{\partial s}\right\rangle && \text{geodesic along } s\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{\partial}{\partial t}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right\rangle \\
&= 0.
\end{aligned}
$$

$\square$