How does $[X,Y]_0$ depend on the base point?

Definition 0.1 — Given $f_0, f_1 : X \to Y$ and a path $u : I \to Y$, if there is a homotopy $H : X \times I \to Y$ s.t. $H(x,0) = f_0(x)$, $H(x,1) = f_1(x)$, and $H(x_0,t) = u(t)$, then we say H is a **homotopy along $\boldsymbol{u}$**, denote $f_0 \simeq_u f_1$.

If f_0, f_1 are base-point preserving, then u is a loop in Y. To be able to say anything about moving base point we read (X, x_0) to be "non-degenerate" by which we mean that (X, x_0) is an **NDR-pair** (neighborhood deformation retract).

Definition 0.2 — $A \subseteq X$ is an **NDR-pair** if there are maps $u : X \to I$ and $h : X \times I \to X$ s.t.

(1) $A = u^{-1}(0)$;

(2) $h(x,0) = x \ \forall\ x \in X$;

(3) $h(a,t) = a \ \forall\ a \in A$;

(4) $h(x,1) \in A \ \forall\ x \in X$ with $u(x) < 1$.

Note: $u^{-1}([0,1))$ is an open neighborhood of $A \in X$ that retracts to A.

Example 0.3 (NDR-pairs)

We have

(1) sub CW-complex of a CW-complex (think that sphere can contract to boundary if we remove a point).

(2) submanifold of a manifold.

Lemma 0.4

If (X, A) is an NDR-pair, then $(X \times \{0\}) \cup (A \times I)$ is a retract of $X \times I$.

Proof. Define $R : X\times I \to (X\times\{0\})\cup(A\times I), (x,t) \mapsto \begin{cases} (x,t) & x \in A \text{ or } t = 0 \\ (h(x,1), t-u(x)) & t \geq u(x), t > 0 \\ (h(x, \frac{t}{u(x)}), 0) & u(x) \geq t \text{ and } u(x) > 0 \end{cases}$

Exercise: this is a retract. □

Lemma 0.5

If (X, x_0) is an NDR-pair and $f_0 : X \to Y$, $f_0(x_0) = y_0$, $\gamma : I \to Y$ path from y_0 to y, then there exists $f_1 : X \to Y$ s.t. $f_1(x_0) = y_1$ and $f_0 \simeq_\gamma f_1$. We denote f_1 by $\gamma \cdot f_0$ (well-defined once R from previous lemma is fixed).

Proof. Let R be from previous lemma for (X, x_0). Let $H : X \times I \to Y$ to be

$$H(x,t) = \begin{cases} f_0(R(x,t)) & R(x,t) \in X \times \{0\} \\ \gamma(R(x,t)) & R(x,t) \in A \times I \end{cases}$$

Let $f_1(x) = H(x,1)$. Then H yields $f_0 \simeq_\gamma f_1$. □

Lemma 0.6

Suppose $f_0, f_1, f_2 : X \to Y, (X, x_0)$ an NDR pair if $f_0 \simeq_\gamma f_1, f_0 \simeq_{\gamma'} f_2$ with $\gamma \simeq \gamma'$ rel boundary, then $f_1 \simeq f_2$ rel base point.

Proof. Since (X, x_0) is an NDR-pair, we can show that $(X \times I, (X \times \{0,1\} \cup \{x_0\} \times I))$ is also an NDR-pair. Exercise: show this (ε tubes retract, just need to be compatible on overlaps). So lemma yields a retraction $R : (X\times I)\times I \to ((X\times I)\times\{0\})\cup((X\times\{0,1\})\cup(\{x_0\}\times I))\times I$.

Let H be homotopy $f_0 \simeq_\gamma f_1$, G be homotopy $f_0 \simeq_{\gamma'} f_2$, K be homotopy $\gamma \simeq \gamma'$ rel boundary. Let $\overline{R} = () \circ R$. Check $\overline{R}|_{X\times I\times\{1\}}$ is a homotopy f_1 to f_2 rel base point. □

Lemma 0.7

Suppose $f_0, f_1, f_2 : X \to Y$, $f_0 \simeq_{\gamma_1} f_1$, $f1 \simeq_{\gamma_2} f_2$ with $\gamma_1(1) = \gamma_2(0)$, then $f_0 \simeq_{\gamma_1 \times \gamma_2} f_2$.

Proof. Concatenate homotopies, H for $f_0 \simeq_{\gamma_1} f_1$ and G for $f_1 \simeq_{\gamma_2} f_2$. $\square$

Theorem 0.8

If x_0 is a non-degenerate base point of X, then $\pi_1(Y, y_0)$ acts on $[X, Y]_0$. Moreover, $[X, Y]$ is the quotient of $[X, Y]_0$ by the $\pi_1(Y, y_0)$ action if Y is path-connected.

Proof. Take $[\gamma] \in \pi_1(Y, y_0), [f] \in [X, Y]_0$, lemma yields $\gamma \cdot f : X \to Y$ and $[\gamma \cdot f]$ clearly $[X, Y]_0$.

Claim 0.9. $[\gamma \cdot f]$ is well-defined.

If $f, g \in [f]$ so $f \simeq g$ rel base point. By lemma, we get f, g s.t. $f \simeq_\gamma f_1$ and $g \simeq_\gamma g_1$. Thus $f_1 \simeq_{\gamma^{-1}} f \simeq g \simeq_\gamma g_1$. Lemma says $f_1 \simeq \gamma^{-1} \times \text{const} \times \gamma g_1 \simeq \text{const}$. So Lemma 11 says $f_1 \simeq g_1$ rel base points.

That is, $[\gamma \cdot f]$ does not depend on choice of f. By lemma 11, $[\gamma \cdot f]$ does not depend on γ. Let $\Phi : [X, Y]_0 \to [X, Y]$ by just forgetting base point. Clearly $\Phi([\gamma] \cdot [f]) = \Phi([f])$. Hence we obtain an induced map

$$\Phi : [X, Y]_0 / \pi_1(Y, y_0) \to [X, Y].$$

$\square$

If $\Phi([f]) = \Phi([g])$, then let H be the free homotopy from f to g. Set $\gamma(t) = H(x_0, t)$. Both f, g take x_0 to y_0 so $[\gamma] \in \pi_1(Y, y_0)$ and $[\gamma \cdot f] = [g]$. So Φ is injective.

Φ is surjective by lemma 10 if Y is path-connected.

Corollary 0.10

A based map is null-homotopic iff it is based null-homotopic.

Proof. Based null-homotopy clearly implies null-homotopic. If $f \simeq e$, WLOG $e(x) = y_0$, by homotopy H, let $\gamma(t) = H(x_0, t)$ so $f \simeq_\gamma e$ so $e \simeq_{\gamma^{-1}} f$. By lemma 11, $f \simeq e$ rel base point. $\square$