Homework 3

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Problem (6.5)**.** No. Since $f:\mathbb{R}\to\mathbb{R}, x\mapsto\sqrt{x^2+1}$, notice $\sqrt{x^2+1}>x, \sqrt{y^2+1}>y$, and $x-y$ and $\sqrt{x^2+1}+\sqrt{y^2+1}$ has the same sign so we can drop the absolute value, then

$$\begin{aligned}\frac{(x-y)}{\sqrt{x^2+1}-\sqrt{y^2+1}} &= \frac{(x-y)(\sqrt{x^2+1}+\sqrt{y^2+1})}{x^2+1-y^2-1}\\ &= \frac{\sqrt{x^2+1}+\sqrt{y^2+1}}{x+y}\\ &> 1\end{aligned}$$

So we establish that $|f(x)-f(y)|<|x-y|$. However, suppose $\sqrt{x^2+1}=x$ this forces $0=1$ a contradiction so f has no fix-point. Since $\mathbb{R}$ is a complete metric space, we have a counterexample.

Problem (6.7)**.** Define $\gamma:[0,1]\to\mathbb{R}^n, t\mapsto(1-t)p+tq$. Then $f\circ\gamma:[0,1]\to\mathbb{R}$. Notice that $\gamma'(t)=q-p$. Applying the 1D mean value theorem to this function yields a $t\in(0,1)$ s.t.

$$\begin{aligned}\frac{f\circ\gamma(1)-g\circ\gamma(0)}{1-0} &= (f\circ\gamma)'(t)\\ f(q)-f(p) &= Df(s)\circ\gamma'(t) && s:=(1-t)p+tq\\ f(q)-f(p) &= Df(s)(q-p).\end{aligned}$$

Theorem 0.1

Let $f:\mathbb{R}\to\mathbb{R}$ be a smooth map. Suppose that f' is nonzero at some $p\in\mathbb{R}$. Then f^{-1} exists in some open interval around p and is also smooth. Moreover,

$$(f^{-1})'(f(p))=\frac{1}{f'(p)}.$$

Problem (6.10)**.** *Proof.* Since $f'(p)$ is nonzero, it is either positive or negative. WLOG suppose $f'(p)>0$, then since f' is continuous, f is monotone on some interval (c,d) containing p. Hence f^{-1} exists on this interval. Smoothness of f^{-1} on (c,d) is the result that

$f \circ f^{-1} = \text{id}$ and f and id are both smooth so f^{-1} must be smooth. By chain rule,

$$\begin{aligned}(f \circ f^{-1})'(f(p)) &= \text{id}'(f(p)) \\ f'(p)(f^{-1})'(f(p)) &= 1 \\ (f^{-1})'(f(p)) &= \frac{1}{f'(p)}\end{aligned}$$

□

Problem (6.12)**.** Suppose the theorem is true when $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. Then for a smooth map of general manifolds (with dim m, n) $f : M \to N$, take any local charts $(U, \phi), (V, \psi)$ of p and $f(p)$, we see that $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ by assumption yields local charts (linear isomorphisms) (U', ϕ') and (V', ψ') for $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively s.t.

$$\begin{aligned}\psi' \circ (\psi \circ f \circ \phi^{-1}) \circ \phi'^{-1}(x_1, \dots, x_n) &= \Psi \circ f \circ \Phi^{-1}(x_1, \dots, x_n) \\ &= (x_1, \dots, x_k, 0 \dots, 0)\end{aligned}$$

Then $\Phi := \phi' \circ \phi$ and $\Psi := \psi' \circ \psi$ are the local charts we seek for f.

Now assume $f : \mathbb{R}^m \to \mathbb{R}^n$. If $p \neq 0$, then we can set $\widehat{f}(x) = f(x - p)$ which has the same Jacobian as f so it doesn't change the proof. Similarly, if $f(p) \neq 0$, we can set $\widetilde{f}(x) = f(x) - f(p)$ and again it doesn't change the Jacobian. Finally, by assumption rank $Df = k$, so $Df(0)$ has k linearly-independent columns and k linearly independent rows. So by a series of permutation matrices we can swap all the linearly-independent columns to the first k-columns, and then swap all the linearly-independent rows to the first k-columns. Then the first $k \times k$ submatrix has k pivots so it is nonsingular. These permutations are nonsingular and smooth which still allow us to use the inverse function theorem.

Problem (6.13)**.** Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is C^1 and 1-1. Then $\operatorname{rank} d_p f \leq \dim \mathbb{R} = 1$ so it is either 0 or 1. Suppose $\operatorname{rank} d_p f = 0$ for all $p \in \mathbb{R}^2$, then f is clearly not 1-1 as it is constant. Suppose there exists a $p \in \mathbb{R}^2$ s.t. $\operatorname{rank} d_p f = 1$. Then we know that $f(p)$ is a regular value by definition, but since f is 1-1, the preimage $f^{-1}(f(p)) = \{p\}$ which is a submanifold. But p is clearly not a manifold, a contradiction. Hence in both cases no such function can exist.

Problem (1.3.5)**.** Recall that $f : X \to Y$ is a local diffeomorphism if for all $x \in X$, f maps a neighborhood of x diffeomorphically to a neighborhood of $f(x)$.

Since f is clearly surjective onto its image, and f is 1-1 by assumption, we have $f : X \to f(X)$ is a bijection so f^{-1} is well-defined as a set map. Denote each neighborhood of x locally diffeomorphic to open set of Y as U_x. Let $V_y = f(U_x)$ with $y = f(x)$. Then each V_y is open by local diffeomorphism so $W := \bigcup_{x \in X} V_y$ is an open subset of Y. Since $W \subseteq f(X)$ but $f(x) \in W \ \forall \ x \in X$, we have $W = f(X)$. Moreover, $(f^{-1})|_{V_y} = (f|_{U_x})^{-1}$ so f^{-1} is locally smooth as well. Smoothness of f follows from that f is locally smooth so any transition map restricted to such neighborhood is smooth. These restricted neighborhood for all points in X still form an atlas of X so any transition map from this atlas is smooth. Likewise $f^{-1} : W \to X$ is smooth. Therefore, f is a diffeomorphism from X to W.