Homework 5

Jaden Wang

Problem (4.22)**.** We wish to find a sequence of stopping time tending to ∞ such that it reduces N_t. Define

$$T_n(\omega) = \{\inf t \geq 0 : |U(\omega)M_t(\omega)| \geq n\}.$$

We first show that $T_n(\omega) \uparrow \infty \ \forall \ \omega$. Since M is continuous in t and U does not depend on time, N_t is continuous in t. By a fact from class, we obtain the result.

Second, since M is a CLM, there exists a sequence of stopping time (S_n) that reduces it. Since $T_n \uparrow \infty$, from class we know that $R_n := T_n \wedge S_n$ reduces M as well. we have $M_t^{R_n} \in L^1$

It remains to show that (R_n) reduces N_t. Since $|N_t^{R_n}| = |UM_t^{R_n}| \leq n$ by definition of R_n, we have $N_t^{R_n} \in L^1$ so we can consider conditional expectation on it:

$$\begin{aligned}
\mathbb{E}\left[N_t^{R_n}|\Sigma_s\right] &= \mathbb{E}\left[UM_t^{R_n}|\Sigma_s\right] \\
&= U\mathbb{E}\left[M_t^{R_n}|\Sigma_s\right] && U \in \Sigma_0 \subset \Sigma_s \\
&= UM_s^{R_n} \\
&= N_s^{R_n}.
\end{aligned}$$

Problem (4.24)**.** (1) Since M is CLM and $\mathbb{E}[M_0] = 0 < \infty$, from class we know that (T_n) reduces M.

$(\subseteq)$: if $\lim\limits_{t\to\infty} M_t(\omega) = C < \infty$, we want to show that $T_n(\omega) = \infty$ for some n. Since M is continuous, for any $\varepsilon > 0$, there exists a T s.t. $|C - M_T(\omega)| < \varepsilon$ and we know that $\sup_{t\geq 0} M_t(\omega)$ is bounded by the max of $\max_{[0,T]} M_t(\omega)$ and $|C| + \varepsilon$. Therefore, the supremum is bounded and we can take any n greater than this supremum to get $T_n(\omega) = \infty$.

$(\supseteq)$: if $T_n(\omega) = \infty$ for some n, this implies $|M_t(\omega)| \leq n$ for all $t \geq 0$. Since $M_t^{T_n} = M_t$ is a martingale with continuous path, by the Martingale Convergence it has an a.s. limit $M_\infty(\omega)$ and is finite since $M_t(\omega)$ is bounded.

$(\subseteq)$: suppose $T(\omega) = \infty$, *i.e.* $|M_t(\omega)| \leq n$ for all $t \geq 0$. Since $M_t^{T_n}$ is a martingale,

$M_0 = 0 \in L^2$, and $\mathbb{E}\left[(M_t^{T_n})^2\right] \le \sup_{t\ge0}(M_t^{T_n})^2 \le n^2$, we apply the TFAE to obtain $\mathbb{E}\left[\left\langle M^{T_n}, M^{T_n}\right\rangle_\infty\right] < \infty$. It follows that $\left\langle M^{T_n}, M^{T_n}\right\rangle_\infty < \infty$ a.s. Since $M_\infty(\omega) = M_\infty^{T_n}(\omega)$, we finally obtain $\langle M, M\rangle_\infty(\omega) < \infty$ a.s.

(2) $(\subseteq)$: let $\langle M, M\rangle_\infty(\omega) < \infty$. Since $\langle M, M\rangle_t$ is an increasing process, picking any $n > \langle M, M\rangle_\infty(\omega)$ yields $S_n(\omega) = \infty$.

$(\supseteq)$: suppose there exists an n s.t. $\langle M, M\rangle_t(\omega) \le n \ \forall\ t$, then by increasing process and Monotone Convergence Theorem, $\langle M, M\rangle_\infty(\omega) \le n$.

$(\subseteq)$: under the same assumption, first notice that S_n is a stopping time because $\langle M, M\rangle_t$ is continuous and $\{n\}$ is a closed set. Thus we have $\mathbb{E}\left[\left\langle M^{S_n}, M^{S_n}\right\rangle_\infty\right] \le n < \infty$. Since M is a CLM, M^{S_n} is also a CLM. Since $M_0 = 0 \in L^2$, we apply the TFAE to obtain that M^{S_n} is a martingale and bounded in L^2. Since $M_t^{S_n}(\omega) = M_t(\omega)$, by Martingale Convergence we obtain that $\lim_{t\to\infty} M_t^{S_n}(\omega) = \lim_{t\to\infty} M_t(\omega)$ converges a.s. and is a.s. finite since it is bounded in L^2.

Putting (1) and (2) together, we obtain the final a.s. set equality.

Problem (4.25). (1) First, T_ε^n is a stopping time because $\langle M^n, M^n\rangle_t$ is continuous and $[\varepsilon, \infty)$ is a closed set. Second, since M^n is a CLM, $M^{n,\varepsilon}$ is also a CLM. By definition we have $\langle M^{n,\varepsilon}, M^{n,\varepsilon}\rangle_t \le \varepsilon$, so by increasing process and MCT $\langle M^{n,\varepsilon}, M^{n,\varepsilon}\rangle_\infty \le \varepsilon$ exists. Since $M_0 = 0 \in L^2$, TFAE yields that $M_t^{n,\varepsilon}$ is a true martingale and is bounded in L^2.

(2) The previous TFAE also yields $\mathbb{E}\left[|M_t^{n,\varepsilon}|^2\right] = \mathbb{E}\left[\langle M^{n,\varepsilon}, M^{n,\varepsilon}\rangle_t\right] \le \varepsilon$. Applying Doob's inequality, we obtain

$$\mathbb{E}\left[\sup_{t\ge0} |M_t^{n,\varepsilon}|^2\right] \le \left(\frac{2}{2-1}\right)^2 \mathbb{E}\left[|M_t^{n,\varepsilon}|^2\right] \le 4\varepsilon.$$

(3) Since when $T_\varepsilon^n = \infty$, we have $M_t^n = M_t^{n,\varepsilon}$, we can rewrite

$$\begin{aligned}\mathbb{P}\left(\sup_{t\ge0} |M_t^n| \ge a\right) &= \mathbb{P}\left(\{\sup_{t\ge0} |M_t^{n,\varepsilon}| \ge a\} \cap \{T_\varepsilon^n = \infty\}\right) + \mathbb{P}\left(\{\sup_{t\ge0} |M_t^n| \ge a\} \cap \{T_\varepsilon^n < \infty\}\right) \\ &\le \mathbb{P}\left(\{\sup_{t\ge0} |M_t^{n,\varepsilon}| \ge a\}\right) + \mathbb{P}\left(T_\varepsilon^n < \infty\right)\end{aligned}$$

by monotonicity. By maximal inequality of martingale, we can bound the first term as

$$\frac{2}{a}\mathbb{E}\left[\sup_{t\ge0} |M_t^{n,\varepsilon}|\right] \le \frac{4\sqrt{\varepsilon}}{a}$$

Since quadratic variation is increasing, the second term can be bounded by

$$\mathbb{P}\left(\langle M^n, M^n\rangle_\infty > \varepsilon\right)$$

Let $\varepsilon \to 0$ and $n \to \infty$, then by the assumption that $\lim_{n\to\infty} \langle M^n, M^n\rangle_\infty = 0$ in probability, we obtain that

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{t\geq 0} |M_t^n| \geq a\right) = 0.$$

Since $a > 0$ is arbitrary, this converges in probability.

Problem (2)**.** Since M is a Gaussian process, and as a martingale$\mathbb{E}\left[M_t\right] = \mathbb{E}\left[M_0\right] = 0$, M is a centered Gaussian process and $M_{t+s} - M_t$ is a centered Gaussian. To show that $M_{t+s} - M_t$ is independent of $\sigma(M_r : r \in [0,t])$, it suffices to show that $\operatorname{Cov}\left(M_{t+s} - M_t, M_r\right) = 0$. We have

$$\begin{aligned}
\mathbb{E}\left[(M_{t+s} - M_t)M_r\right] &= \mathbb{E}\left[\mathbb{E}\left[M_{t+s}M_r|\Sigma_t\right] - M_tM_r\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[M_{t+s}|\Sigma_t\right]M_r - M_tM_r\right] && M_r \in \Sigma_r \subset \Sigma_t \\
&= \mathbb{E}\left[M_tM_r - M_tM_r\right] = 0.
\end{aligned}$$

Since M_t is Gaussian, we know $M_t \in L^2$. Since M is also a continuous martingale with $M_0 = 0$, by TFAE we conclude that $M^2 - l$

We wish to show that $M_t^2 - \mathbb{E}\left[M_t^2\right]$ is a continuous martingale, so by (indist) uniqueness of quadratic variation, we would obtain that $\langle M, M\rangle_t = \mathbb{E}\left[M_t^2\right]$ which is a deterministic continuous monotone nondecreasing (by definition of quadratic variation) function. Consider

$$\begin{aligned}
\mathbb{E}\left[\left(M_t^2 - \mathbb{E}\left[M_t^2\right]\right)|\Sigma_s\right] &= \mathbb{E}\left[M_t^2|\Sigma_s\right] - \mathbb{E}\left[\mathbb{E}\left[M_t^2\right]|\Sigma_s\right] \\
&= \mathbb{E}\left[M_t^2 - M_s^2 + M_s^2|\Sigma_s\right] - \mathbb{E}\left[\mathbb{E}\left[M_t^2 - M_s^2 + M_s^2\right]|\Sigma_s\right] \\
&= \mathbb{E}\left[M_t^2 - M_s^2|\Sigma_s\right] + M_s^2 - \underbrace{\mathbb{E}\left[\mathbb{E}\left[M_t^2 - M_s^2\right]|\Sigma_s\right]}_{\text{tower}} - \mathbb{E}\left[M_s^2\right] \\
&= M_s^2 - \mathbb{E}\left[M_s^2\right].
\end{aligned}$$

This shows that it is indeed a continuous martingale.

Problem (3)**.** First, we establish the following equality:

$$\begin{aligned}
\mathbb{E}\left[M_{t_i}A_{t_{i-1}}\right] &= \mathbb{E}\left[\mathbb{E}\left[M_{t_i}A_{t_{i-1}}|\Sigma_{t_{i-1}}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[M_{t_i}|\Sigma_{t_{i-1}}\right]A_{t_{i-1}}\right] && A_{t_{i-1}} \in \Sigma_{t_{i-1}} \\
&= \mathbb{E}\left[M_{t_{i-1}}A_{t_{i-1}}\right].
\end{aligned}$$

Since both M and A are bounded, take the usual mesh, we have

$$\left|\sum_{i=1}^{p_n} M_{t_i}(A_{t_i} - A_{t_{i-1}})\right| \leq CA \leq C'.$$

Thus, by bounded convergence theorem, we have

$$\begin{aligned}
\mathbb{E}\int_0^\infty M_t \mathrm{d}A_t &= \mathbb{E}\left[\lim_{n\to\infty}\sum_{i=1}^{p_n} M_{t_i}(A_{t_i} - A_{t_{i-1}})\right] \\
&= \lim_{n\to\infty}\sum_{i=1}^{p_n}\left(\mathbb{E}\left[M_{t_i}A_{t_i}\right] - \mathbb{E}\left[M_{t_i}A_{t_{i-1}}\right]\right) \\
&= \lim_{n\to\infty}\sum_{i=1}^{p_n}\left(\mathbb{E}\left[M_{t_i}A_{t_i}\right] - \mathbb{E}\left[M_{t_{i-1}}A_{t_{i-1}}\right]\right) \\
&= \mathbb{E}\left[M_tA_t\right] - \mathbb{E}\left[M_0A_0\right] && \text{telescope} \\
&= \mathbb{E}\left[M_tA_t\right].
\end{aligned}$$

Since $\mathbb{E}\left[M_tA_t\right]$ is bounded, by bounded convergence theorem we can take $t \to \infty$ for the equality to hold.

Problem (4)**.** $(\Rightarrow)$: this follows immediately from the uniqueness of quadratic variation. Suppose $0 \leq a \leq b$ and $M_t = M_a$ a.s. for all $t \in [a,b]$. Then consider M_a as a martingale with contant sample paths and the stopped CLM $N_t = M^b_{t+a}$ with $N_0 = M_a$. Since they equal for all $t \geq 0$, by uniqueness, $\langle M_a, M_a\rangle_t = \langle N, N\rangle_t$ a.s. In particular, we have a.s.

$$\begin{aligned}
\langle M, M\rangle_b &= \left\langle M^b_{b-a+a}, M^b_{b-a+a}\right\rangle \\
&= \langle N, N\rangle_{b-a} \\
&= \langle M, M\rangle_a.
\end{aligned}$$

$(\Leftarrow)$: First, since the rationals are dense in the reals, and M_t and $\langle M, M\rangle_t$ are continuous, it suffices to prove for $a = q, b \in \mathbb{Q}$ and perform the usual countable intersection. Define $N_t := M_t - M_{t\wedge q}$ (notice that $N_0 = M_0 - M_0 = 0$) and $T_q = \inf\{t > 0 : \langle N, N\rangle_t > 0\}$. Since

$(0, \infty)$ is open, T_q is a stopping time for (Σ_{t+}). Under the usual condition of right-continuous filtration, T_q is indeed a stopping time for (Σ_t). Moreover, since M is a CLM, stopped process $M_{t\wedge q}$ is also a CLM, thus N_t and $N_t^{T_q}$ are CLMs. Suppose $\langle M, M\rangle_b = \langle M, M\rangle_q$ where $b \geq q$. If $T_q < q$, we have $N_t^{T_q} = M_{t\wedge T_q} - M_{t\wedge T_q} = 0$ so $\left\langle N^{T_q}, N^{T_q}\right\rangle_t \equiv 0$ a.s. but this contradict with the definition of T_q. So we must have $T_q \geq q$. By the definition of T_q, we have

$$\begin{aligned} 0 \equiv \left\langle N^{T_q}, N^{T_q}\right\rangle_t &= \left\langle M_{t\wedge T_q} - M_{t\wedge q\wedge T_q}, M_{t\wedge T_q} - M_{t\wedge q\wedge T_q}\right\rangle \\ &= \left\langle M_{t\wedge T_q} - M_{t\wedge q}, M_{t\wedge T_q} - M_{t\wedge q}\right\rangle. \end{aligned}$$

Since $N_0 = 0$, we conclude that $M_{t\wedge T_q} = M_{t\wedge q}$ a.s. Thus, $M_{b\wedge T_q} = M_q$. If we show that $b \leq T_q$, we are done. This can be shown from the fact that $\langle M, M\rangle_t$ is an increasing process so $\langle M, M\rangle_t = \langle M, M\rangle_q$ for all $t \in [q, b]$. But $\langle N, N\rangle_t = \langle M_t - M_q, M_t - M_q\rangle$ for $t \in [q, b]$.