1 Immersions and Embeddings

How to put one manifold inside another?

Definition 1.1 — A **immersion** $f : M \to N$ is a mapping with $\operatorname{rank}(d_p f = \dim M)$ for all $p \in M$.

Remark 1.2 f is always locally 1-1 by the rank theorem.

Definition 1.3 — An **embedding** is an immersion which is globally 1-1 and a diffeomorphism onto its image (wrt subspace topology).

Example 1.4 (immersion)

Given a curve $f : (a, b) \to \mathbb{R}^n$, $f' \neq 0$.

Example 1.5

A 1-1 immersion is not always an embedding. Let $f : (a, b) \to \mathbb{R}^2$ be a figure-8 curve without intersecting. But it is not an embedding due to open neighborhood around origin. We have 4 connected components if we remove origin in the image.

Proposition 1.6

If M is compact, then a 1-1 immersion is always an embedding. More generally, let X be a compact topological space, Y be Hausdorff, then any 1-1 map $f : X \to Y$ is a homeomorphism onto its image.

Proof. It suffices to show that f^{-1} is continuous or show that f is open. Given an open set $U \subseteq X$, so $X \setminus U$ is closed so it is compact. So $f(X \setminus U)$ is compact and therefore closed since Y is Hausdorff. Hence by f 1-1, $f(U) = f(X) \setminus f(X \setminus U)$ is open. $\square$

Definition 1.7 — A topological **immersion** is a locally 1-1 continuous map. A topological **embedding** is a 1-1 immersion which is a homeomorphism onto the image.

Theorem 1.8 (Whitney Embedding)

Any (smooth) manifold M^n maybe (smoothly) embedded in $\mathbb{R}^{2n}$ and immersed in $\mathbb{R}^{2n-1}$.

Remark 1.9 $2n$ is in general a tight bound.

Example 1.10

$\mathbb{R}P^2$: Exercise 4 of lecture notes 3. $f : \mathbb{R}^3 \to \mathbb{R}^4, (x, y, z) \mapsto (xy, yz, xz, x^2 + 2y^2 + 3z^2)$. When you restricted f to S^2, then antipodal points yield the same thing.

For immersion of $\mathbb{R}P^2$ into $\mathbb{R}^3$, google Boy's surface.

Proof. Any compact manifold M^n may be topologically embedded in $\mathbb{R}^N$ for N sufficiently large). Theorem 5 LN3.

Idea: glue m finite charts together to construct an embedding into $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ $m + 1$ times. There exists $V_i \subseteq U_i$ s.t. $\{V_i\}$ still covers M. Define $\lambda_i : U_i \to \mathbb{R}$ s.t. $\lambda_i = 1$ on V_i and 0 elsewhere. Define $f_i : M \to \mathbb{R}^n, p \mapsto \lambda_i(p)\phi_i(p)$. Then $f(p) := (\lambda_1, \dots, \lambda_m, f_1, \dots, f_m)$.

Claim 1.11. f is 1-1.

Suppose $f(p) = f(q)$, that is, $\lambda_i(p) = \lambda_i(q), f_i(p) = f_i(q)$. Since $p \in V_j$ for some j, we have $\lambda_j(p) = 1 = \lambda_j(q)$ which implies that $q \in V_j$. But since f is 1-1 on each V_i, so

$$\begin{aligned}
\lambda_j(p)\phi_j(p) = f_j(p) &= f_j(q) = \lambda_j(q)\phi_j(q) \\
\phi_j(p) &= \phi_j(q) \\
p &= q
\end{aligned}$$

More details: $V_i := \phi_{-1(\int B^n)}$ unit ball in $\mathbb{R}^n$. Let $\lambda : \mathbb{R}^n \to \mathbb{R}$, $\lambda \neq 0$ on $B^n(1)$ and $\lambda = 0$ on $\mathbb{R}^n - B^n(2)$. Then $\lambda_i : M \to \mathbb{R}, \lambda_i(p) := \begin{cases} \lambda(\phi_i(p)), & p \in U_i \\ 0 & \text{else} \end{cases}$

Claim 1.12. λ_i is continuous.

$K_i = \phi_i^{-1}(B^n(2))$ so K_i is compact and therefore closed by Hausdorff. So $M - K_i$ is open. So $\{U_i, M - K_i\}$ is an open cover for M, λ_i is continuous on U_i, is continuous on $M - K_i$ so

λ_i is continuous.

Claim 1.13. $\operatorname{rank} d_p f = n \ \forall \ p \in M$.

$p \in V_i$ for some i, $f_i(p) = \lambda_i \phi_i(p) = \phi_i(p)$ which has rank n so is the derivative. The rank of f cannot be less than f_i, but the dimension of the codomain in n so the rank has to be n.

Now for the proof, there exists an embedding $f : M \to \mathbb{R}^N$, N large,

Claim 1.14. If $N > 2n+1$, then there exists a unit vector $u \in S^{n-1}$ s.t. $\pi_u \circ f : M \to \mathbb{R}^{n-1}$ is an embedding, where $\pi_k : \mathbb{R}^N \to H_n$ (hyperplane) be the orthogonal projection.

Note that u needs to be chosen s.t.

(1) For all $(p, q) \in M$, $\frac{f(p)-f(q)}{|f(p)-f(q)|}$ is not parallel to u.

(2) For all $p \in M$, $u \notin T_p M$.

Let Δ_M be the diagonal of M, by Hausdorff Δ_M is closed, so $M \times M - \Delta_M$ is open so it is a submanifold. $\dim(M \times M - \Delta_M) = 2n < N - 1 = \dim S^{N-1}$ by assumption. Define $\sigma : M \times M - \Delta_M \to S^{N-1}, (p, q) \mapsto \frac{f(p)-f(q)}{\|f(p)-f(q)\|}$. Then $\mu(\sigma(M \times M - \Delta_M)) = 0$ by lemma. Therefore, there exists a $u \in S^{N-1}$ s.t. $u \notin \sigma(M \times M - \Delta_M)$. Hence $\pi_u|_M$ is 1-1. $\square$

Lemma 1.15

If $f : M^m \to N^n$ is a C^1 map for $n > m$, then $f(M)$ is not surjective. In particular, $f(M)$ has measure zero in N.