In[1]:=
```
(*Problem 1*)
(*Part a*)
sol1 =
 DSolve[{x1'[t] == x2[t], x2'[t] == -x2[t] - p2[t], p1'[t] == 0, p2'[t] == p2[t] - p1[t],
   x1[0] == 0, x2[0] == 0, x1[3] == 1, x2[3] == 2}, {x1[t], x2[t], p1[t], p2[t]}, t]
xSol1 = x1[t] /. sol1[[1, 3]]
ySol1 = x2[t] /. sol1[[1, 4]]
cost := Module[{sol, x1s, x2s, u, x13, x23, x1sModified, x2sModified},
  sol = sol1;
  x1s[t_] := x1[t] /. sol[[1, 3]];
  x13 := x1s[t] /. t -> 3;
  x1sModified = x1s[t] /. x1[3] → x13;
  x2s[t_] := x2[t] /. sol[[1, 4]];
  x23 := x2s[t]  /. t -> 3;
  x2sModified = x2s[t] /. x2[3] → x23;
  u[t_] := -(p2[t] /. sol[[1, 2]]);
  1/2 ((x13 - 1)^2 + (x23 - 2)^2) + Integrate[1/2 u[t]^2, {t, 0, 3}]]
cost1 = N[cost]
u[t_] = -Simplify[N[p2[t] /. sol1[[1, 2]]]]
```

Out[1]=

$$\left\{\left\{p1[t] \to \frac{-3+e^3}{5+e^3},\ p2[t] \to \frac{3-4e^3+e^6-6e^t-6e^{3+t}}{(-1+e^3)(5+e^3)},\right.\right.$$

$$x1[t] \to \frac{e^{-t}\left(7e^3-e^6-3e^t+3e^{2t}-10e^{3+t}+e^{6+t}+3e^{3+2t}-3e^t t+4e^{3+t}t-e^{6+t}t\right)}{(-1+e^3)(5+e^3)},$$

$$\left.\left. x2[t] \to \frac{e^{-t}\left(-7e^3+e^6-3e^t+3e^{2t}+4e^{3+t}-e^{6+t}+3e^{3+2t}\right)}{(-1+e^3)(5+e^3)}\right\}\right\}$$

Out[2]=

$$\frac{e^{-t}\left(7e^3-e^6-3e^t+3e^{2t}-10e^{3+t}+e^{6+t}+3e^{3+2t}-3e^t t+4e^{3+t}t-e^{6+t}t\right)}{(-1+e^3)(5+e^3)}$$

Out[3]=

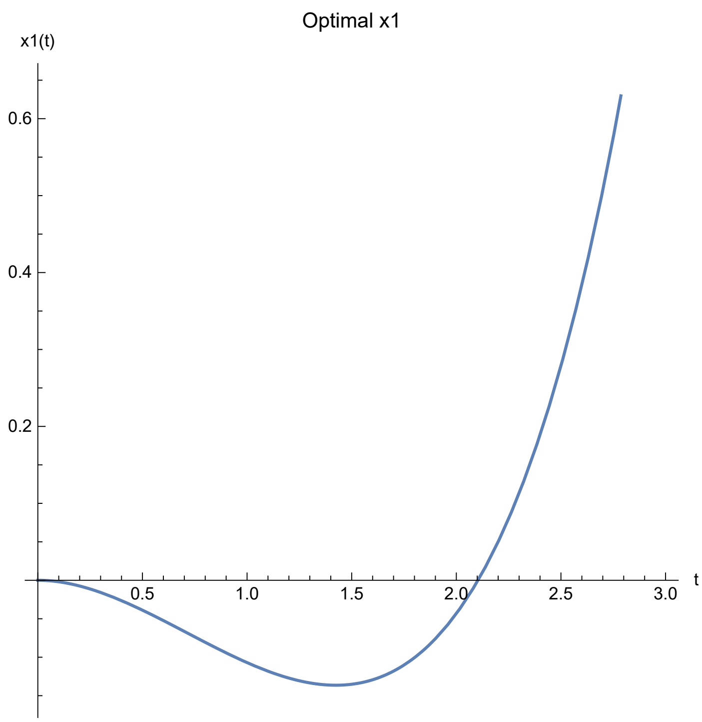
$$\frac{e^{-t}\left(-7e^3+e^6-3e^t+3e^{2t}+4e^{3+t}-e^{6+t}+3e^{3+2t}\right)}{(-1+e^3)(5+e^3)}$$

Out[5]= 4.28588

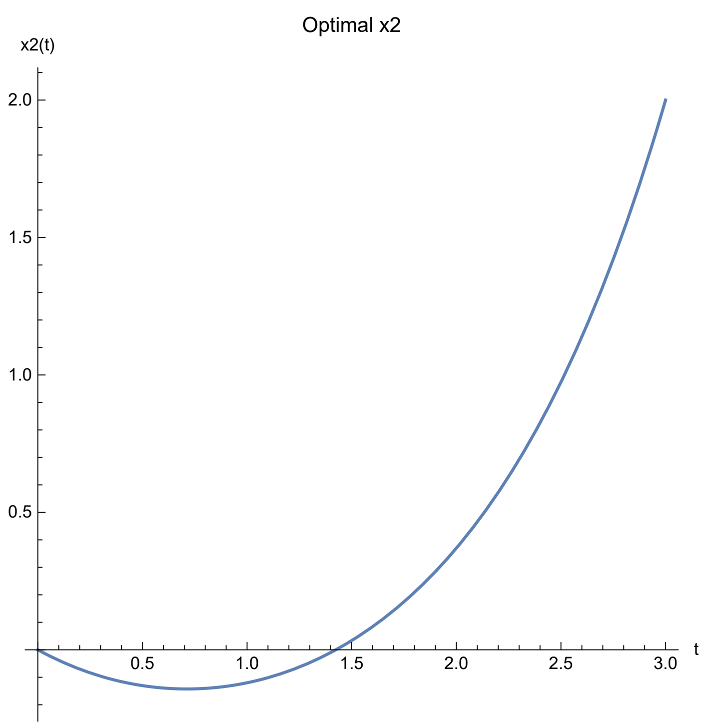
Out[6]= $-0.681091 + 0.264246 \times 2.71828^t$

In[7]:=
```
Plot[xSol1, {t, 0, 3}, PlotLabel → "Optimal x1",
 AxesLabel → {"t", "x1(t)"}, AspectRatio → 1]
Plot[ySol1, {t, 0, 3}, PlotLabel → "Optimal x2",
 AxesLabel → {"t", "x2(t)"}, AspectRatio → 1]
ParametricPlot[{xSol1, ySol1}, {t, 0, 3}, PlotRange → {{-.5, 1.2}, {-.5, 2.2}}
, PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"}, AspectRatio → 1]
```

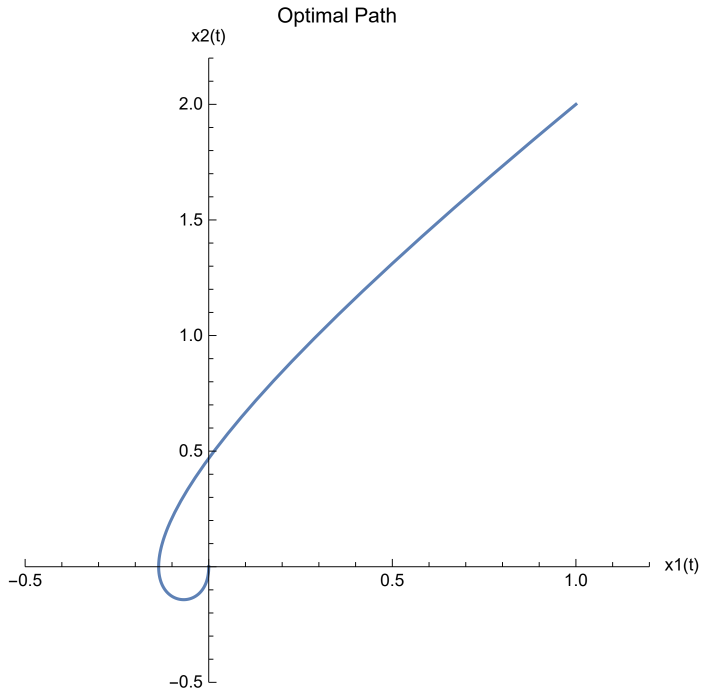
Out[7]=


Optimal x1
x1(t)
0.6
0.4
0.2
0.5
1.0
1.5
2.0
2.5
3.0
t

Out[8]=


Optimal x2
x2(t)
2.0
1.5
1.0
0.5
0.5
1.0
1.5
2.0
2.5
3.0
t

Out[9]=




```
(*Part b*)
```

In[10]:=
```
sol2 = DSolve[{x1'[t] == x2[t], x2'[t] == -x2[t] - p2[t], p1'[t] == 0, p2'[t] == p2[t] - p1[t],
    x1[0] == 0, x2[0] == 0, x1[3] == 1, p2[3] == 0}, {x1[t], x2[t], p1[t], p2[t]}, t]
xSol2 = x1[t] /. sol2[[1, 3]]
ySol2 = x2[t] /. sol2[[1, 4]]
u[t_] = -N[p2[t] /. sol2[[1, 2]]]
Plot[xSol2, {t, 0, 3}, PlotLabel → "Optimal x1",
 AxesLabel → {"t", "x1(t)"}, AspectRatio → 1]
Plot[ySol2, {t, 0, 3}, PlotLabel → "Optimal x2",
 AxesLabel → {"t", "x2(t)"}, AspectRatio → 1]
ParametricPlot[{xSol2, ySol2}, {t, 0, 3}, PlotRange → {{-.5, 1.2}, {-.5, 2.2}}
, PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"}, AspectRatio → 1]
```

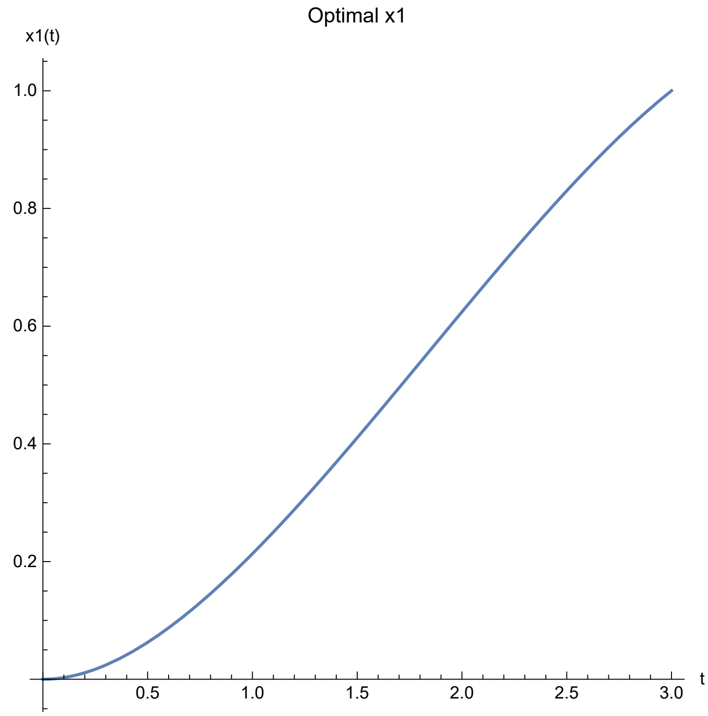
Out[10]= $\left\{\left\{p1[t] \to -\frac{2\, e^6}{-1 + 4\, e^3 + 3\, e^6},\ p2[t] \to \frac{2\, e^3\left(-e^3 + e^t\right)}{-1 + 4\, e^3 + 3\, e^6},\right.\right.$
$\left.\left. x1[t] \to -\frac{e^{3-t}\left(1 - 2\, e^3 - 2\, e^t + e^{2t} + 2\, e^{3+t} - 2\, e^{3+t}\, t\right)}{-1 + 4\, e^3 + 3\, e^6},\ x2[t] \to -\frac{e^{3-t}\left(-1 + 2\, e^3 + e^{2t} - 2\, e^{3+t}\right)}{-1 + 4\, e^3 + 3\, e^6}\right\}\right\}$

Out[11]= $-\frac{e^{3-t}\left(1 - 2\, e^3 - 2\, e^t + e^{2t} + 2\, e^{3+t} - 2\, e^{3+t}\, t\right)}{-1 + 4\, e^3 + 3\, e^6}$

Out[12]= $-\frac{e^{3-t}\left(-1 + 2\, e^3 + e^{2t} - 2\, e^{3+t}\right)}{-1 + 4\, e^3 + 3\, e^6}$

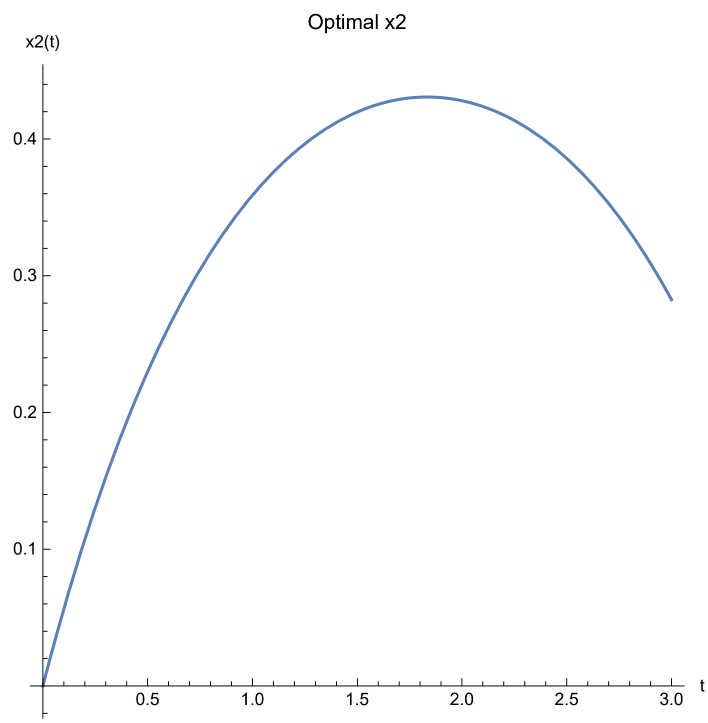
Out[13]= $-0.0311493\left(-20.0855 + 2.71828^t\right)$

Out[14]=



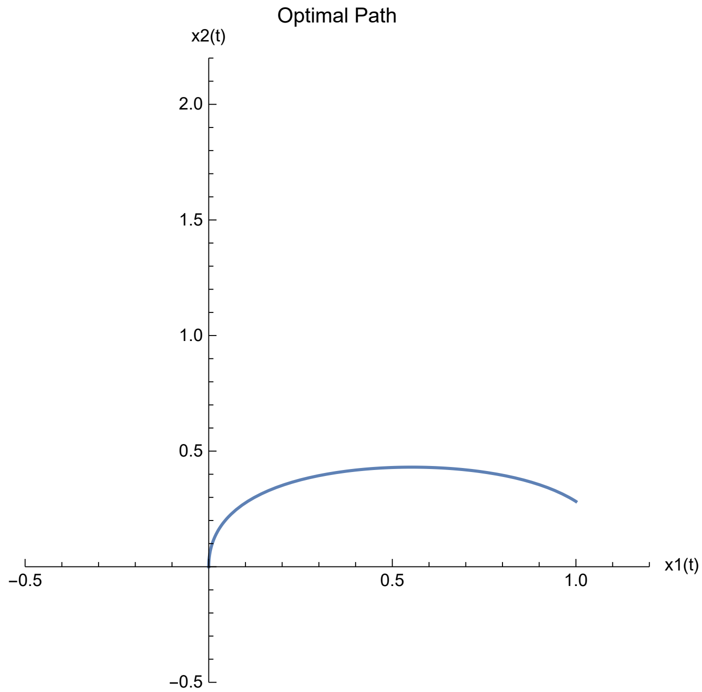
Optimal x1
x1(t)
1.0
0.8
0.6
0.4
0.2
0.5
1.0
1.5
2.0
2.5
3.0
t

Out[15]=



Optimal x2
x2(t)
0.4
0.3
0.2
0.1
0.5
1.0
1.5
2.0
2.5
3.0
t

Out[16]=




```
(*Part c*)
```

In[17]:=
```
sol3 = DSolve[
  {x1'[t] == x2[t], x2'[t] == -x2[t] - p2[t], p1'[t] == 0, p2'[t] == p2[t] - p1[t], x1[0] == 0,
   x2[0] == 0, p1[3] == x1[3] - 1, p2[3] == x2[3] - 2}, {x1[t], x2[t], p1[t], p2[t]}, t]
xSol3 = x1[t] /. sol3[[1, 3]]
ySol3 = x2[t] /. sol3[[1, 4]]
u[t_] = -Simplify[N[p2[t] /. sol3[[1, 2]]]]
Plot[xSol3, {t, 0, 3}, PlotLabel → "Optimal x1",
 AxesLabel → {"t", "x1(t)"}, AspectRatio → 1]
Plot[ySol3, {t, 0, 3}, PlotLabel → "Optimal x2",
 AxesLabel → {"t", "x2(t)"}, AspectRatio → 1]
ParametricPlot[{xSol3, ySol3}, {t, 0, 3}, PlotRange → {{-.5, 1.2}, {-.5, 2.2}}
, PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"}, AspectRatio → 1]
```

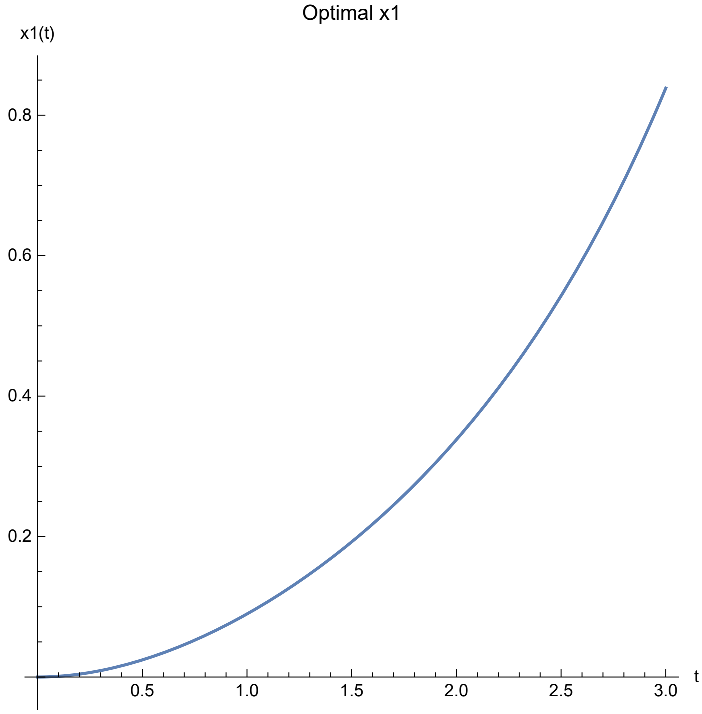
Out[17]=
$$\left\{\left\{p1[t] \to \frac{3-4\,e^3-e^6}{-7+8\,e^3+7\,e^6},\ p2[t] \to -\frac{-3+4\,e^3+e^6+6\,e^t+8\,e^{3+t}}{-7+8\,e^3+7\,e^6},\right.\right.$$
$$x1[t] \to \frac{e^{-t}\left(8\,e^3+e^6-3\,e^t+3\,e^{2t}-12\,e^{3+t}-e^{6+t}+4\,e^{3+2t}-3\,e^t\,t+4\,e^{3+t}\,t+e^{6+t}\,t\right)}{-7+8\,e^3+7\,e^6},$$
$$\left.\left. x2[t] \to \frac{e^{-t}\left(-8\,e^3-e^6-3\,e^t+3\,e^{2t}+4\,e^{3+t}+e^{6+t}+4\,e^{3+2t}\right)}{-7+8\,e^3+7\,e^6}\right\}\right\}$$

Out[18]=
$$\frac{e^{-t}\left(8\,e^3+e^6-3\,e^t+3\,e^{2t}-12\,e^{3+t}-e^{6+t}+4\,e^{3+2t}-3\,e^t\,t+4\,e^{3+t}\,t+e^{6+t}\,t\right)}{-7+8\,e^3+7\,e^6}$$

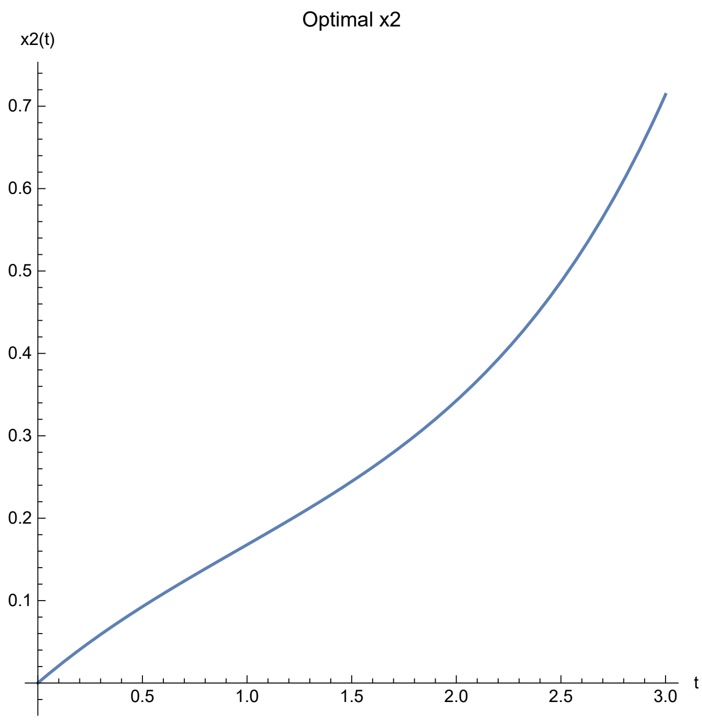
Out[19]=
$$\frac{e^{-t}\left(-8\,e^3-e^6-3\,e^t+3\,e^{2t}+4\,e^{3+t}+e^{6+t}+4\,e^{3+2t}\right)}{-7+8\,e^3+7\,e^6}$$

Out[20]= $0.161458 + 0.0559778 \times 2.71828^{t}$

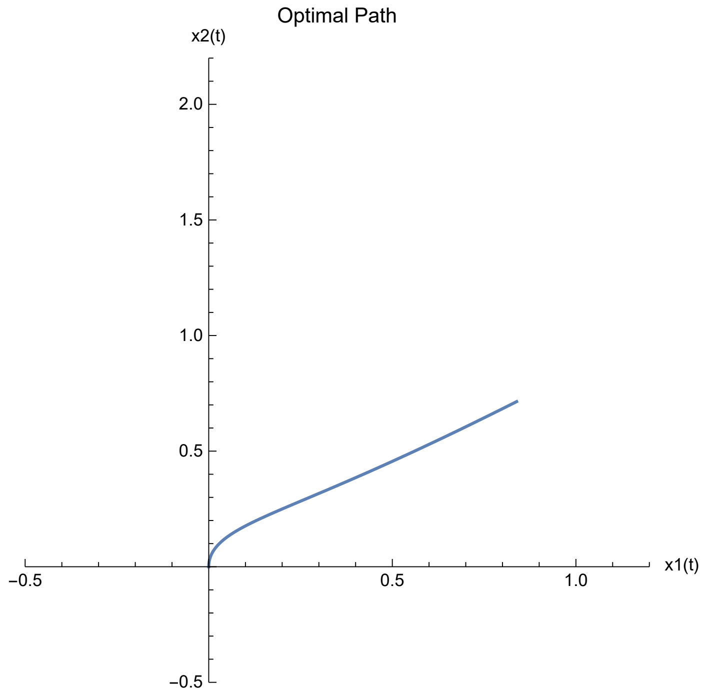
Out[21]=



Out[22]=



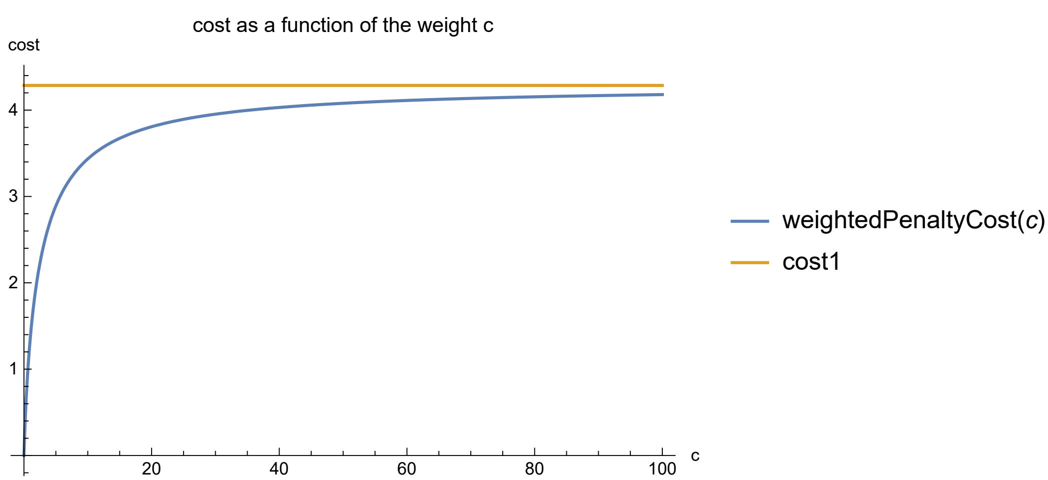
Out[23]=



In[24]:=
```
weightedPenaltyCost[c_] := Module[{sol, x1s, x2s, u, x13, x23, x1sModified, x2sModified},
  sol = DSolve[{x1'[t] == x2[t], x2'[t] == -x2[t] - p2[t], p1'[t] == 0,
     p2'[t] == p2[t] - p1[t], x1[0] == 0, x2[0] == 0, p1[3] == c (x1[3] - 1),
     p2[3] == c (x2[3] - 2)}, {x1[t], x2[t], p1[t], p2[t]}, t];
  x1s[t_] := x1[t] /. sol[[1, 3]];
  x13 := x1s[t] /. t -> 3;
  x1sModified = x1s[t] /. x1[3] → x13;
  x2s[t_] := x2[t] /. sol[[1, 4]];
  x23 := x2s[t] /. t -> 3;
  x2sModified = x2s[t] /. x2[3] → x23;
  u[t_] := -(p2[t] /. sol[[1, 2]]);
  c/2 ((x13 - 1)^2 + (x23 - 2)^2) + Integrate[1/2 u[t]^2, {t, 0, 3}]]

Plot[{weightedPenaltyCost[c], cost1}, {c, 0, 100},
 PlotLabel → "cost as a function of the weight c", AxesLabel → {"c", "cost"},
 PlotLegends → "Expressions", PlotPoints → 10, PlotRange → Full]
```

Out[25]=



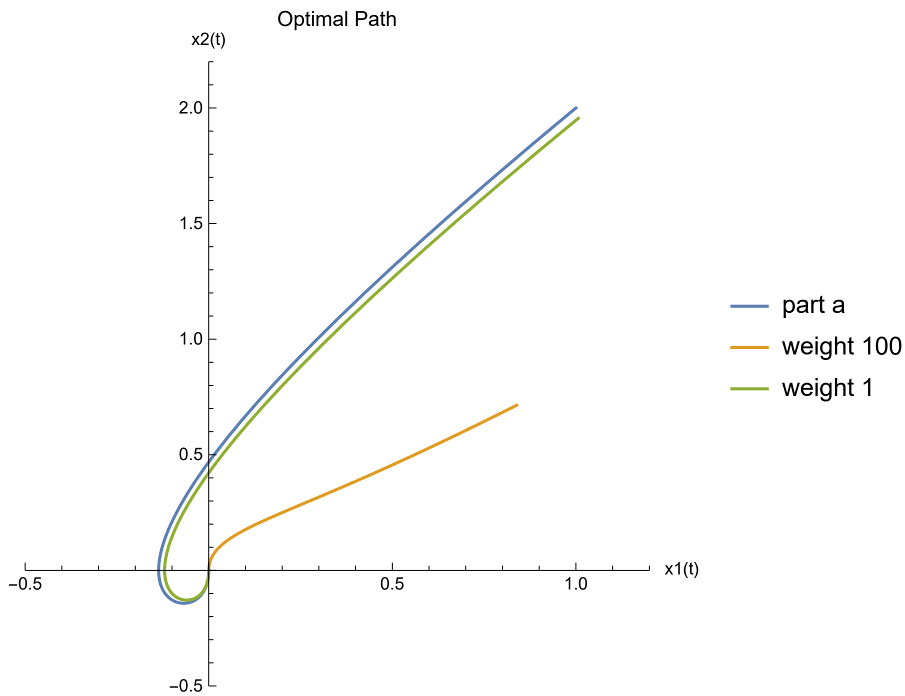
Out[•]= 4.28577

In[26]:=
```
N[weightedPenaltyCost[100000]]
(*we see that for weight 100000 the value is super close to cost from (a)*)
sol3w = DSolve[{x1'[t] == x2[t], x2'[t] == -x2[t] - p2[t], p1'[t] == 0,
    p2'[t] == p2[t] - p1[t], x1[0] == 0, x2[0] == 0, p1[3] == 100 (x1[3] - 1),
    p2[3] == 100 (x2[3] - 2)}, {x1[t], x2[t], p1[t], p2[t]}, t];
x1s[t_] := x1[t] /. sol3w[[1, 3]];
x13 := x1s[t] /. t -> 3;
x1sModified = x1s[t] /. x1[3] → x13;
x2s[t_] := x2[t] /. sol3w[[1, 4]];
x23 := x2s[t] /. t -> 3;
x2sModified = x2s[t] /. x2[3] → x23;
```

Out[26]= 4.28577

In[34]:=
```
ParametricPlot[{{xSol1, ySol1}, {xSol3, ySol3}, {x1sModified, x2sModified}},
 {t, 0, 3}, PlotRange → {{-.5, 1.2}, {-.5, 2.2}}
, PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"},
 PlotLegends → {"part a", "weight 100", "weight 1"}, AspectRatio → 1]
```

Out[34]=




```
(*Part d*)
```

In[35]:=
```
sol42 =
 Simplify[DSolve[{x2'[t] == -x2[t] - (-3 lam Exp[-3] Exp[t] - 2 lam), x2[0] == 0}, x2[t], t]]
 (*pre-solve p1,p2*)
sol4x2[t_] = x2[t] /. sol42[[1]]
sol41 = Simplify[DSolve[{x1'[t] == sol4x2[t], x1[0] == 0}, x1[t], t]]
sol4x1[t_] = x1[t] /. sol41[[1]]
lamSol = N[Solve[2 sol4x1[3] + 5 sol4x2[3] == 20, lam]] (*solve for lambda*)
lamS = lam /. lamSol[[1]]
u[t_] = 3 lamS Exp[-3] Exp[t] + 2 lamS
```

Out[35]= $\left\{\left\{x2[t] \to \frac{1}{2} e^{-3-t} \left(-3 - 4 e^{3} + 3 e^{2t} + 4 e^{3+t}\right) lam\right\}\right\}$

Out[36]= $\frac{1}{2} e^{-3-t} \left(-3 - 4 e^{3} + 3 e^{2t} + 4 e^{3+t}\right) lam$

Out[37]= $\left\{\left\{x1[t] \to \frac{1}{2} e^{-3-t} lam \left(3 + 4 e^{3} - 6 e^{t} + 3 e^{2t} + 4 e^{3+t} (-1 + t)\right)\right\}\right\}$

Out[38]= $\frac{1}{2} e^{-3-t} lam \left(3 + 4 e^{3} - 6 e^{t} + 3 e^{2t} + 4 e^{3+t} (-1 + t)\right)$

Out[39]= $\{\{lam \to 0.717067\}\}$

Out[40]= 0.717067

Out[41]= $1.43413 + 0.107102\, e^{t}$

In[42]:=
```
xSol4[t_] = sol4x1[t] /. lam → lamS
ySol4[t_] = sol4x2[t] /. lam → lamS
xSol4[3]
ySol4[3]
Plot[xSol4[t], {t, 0, 3}, PlotLabel → "Optimal x1",
 AxesLabel → {"t", "x1(t)"}, AspectRatio → 1]
Plot[ySol4[t], {t, 0, 3}, PlotLabel → "Optimal x2",
 AxesLabel → {"t", "x2(t)"}, AspectRatio → 1]
x2[x1_] = 1/5 (20 - 2 x1);
plot = Plot[x2[x1], {x1, -10, 10}, PlotStyle → Orange];
parametric = ParametricPlot[{xSol4[t], ySol4[t]},
   {t, 0, 3}, PlotRange -> {{-.1, 5}, {-.1, 3}}, PlotStyle → Blue,
   PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"}, AspectRatio → 1];
Show[parametric, plot, PlotLabel → "Optimal Path"]
```

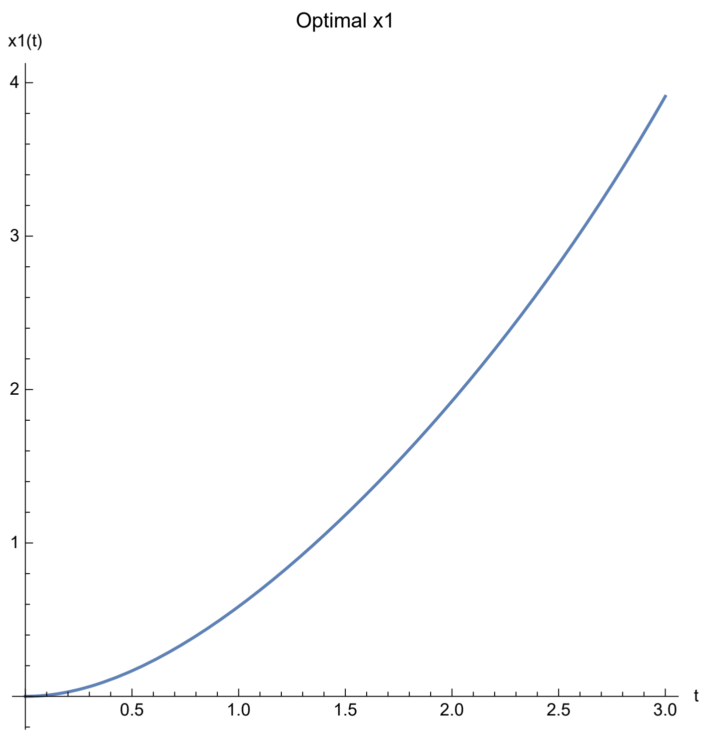
Out[42]= $0.358533\, e^{-3-t} \left(3 + 4 e^{3} - 6 e^{t} + 3 e^{2t} + 4 e^{3+t} (-1 + t)\right)$

Out[43]= $0.358533\, e^{-3-t} \left(-3 - 4 e^{3} + 3 e^{2t} + 4 e^{3+t}\right)$

Out[44]= 3.91083

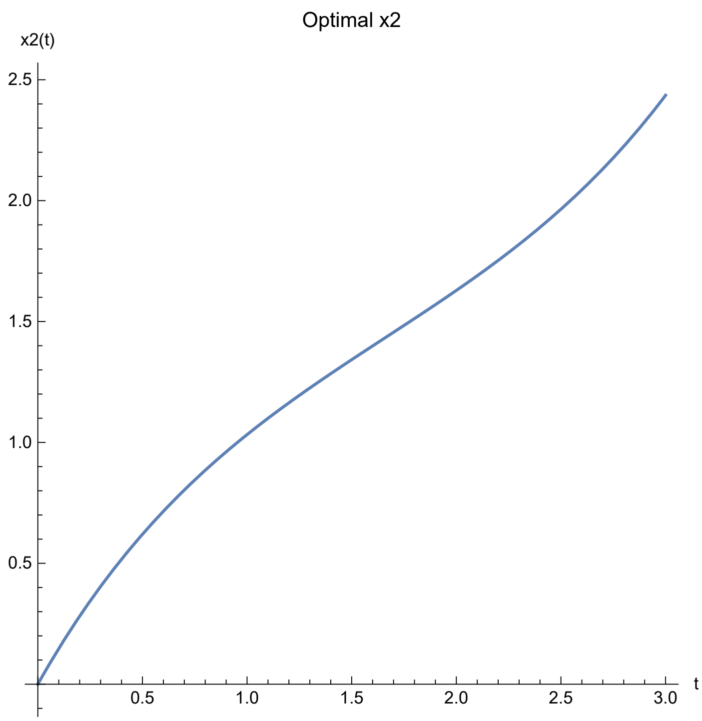
Out[45]= 2.43567

Out[46]=



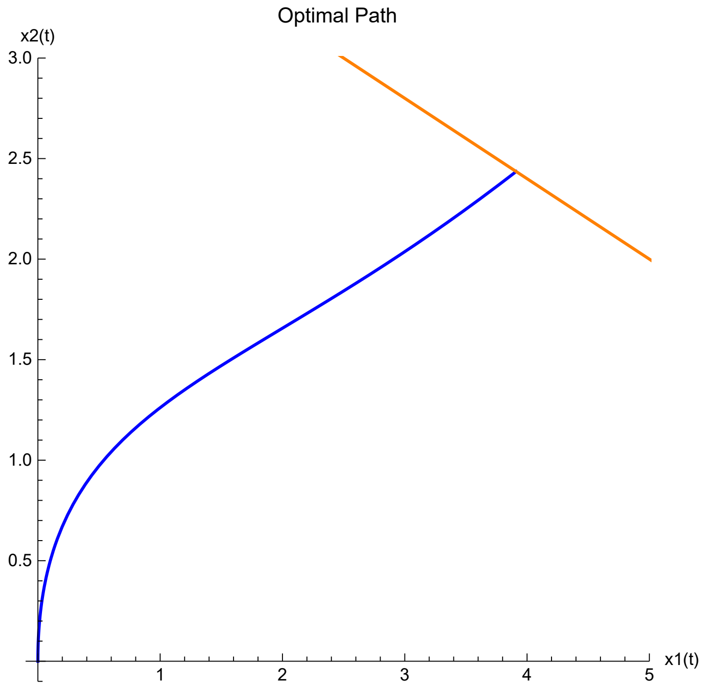
Optimal x1
x1(t)
4
3
2
1
0.5
1.0
1.5
2.0
2.5
3.0
t

Out[47]=



Optimal x2
x2(t)
2.5
2.0
1.5
1.0
0.5
0.5
1.0
1.5
2.0
2.5
3.0
t

Out[51]=



In[69]:=

```
(*Part e*)
ClearAll[lam, x2, lamS, tf, tfS, u]
sol52 = Simplify[
  DSolve[{x2'[t] == -x2[t] - (-3 lam Exp[-tf] Exp[t] - 2 lam), x2[0] == 0}, x2[t], t]]
sol5x2[t_] = x2[t] /. sol52[[1]]
sol51 = Simplify[DSolve[{x1'[t] == sol5x2[t], x1[0] == 0}, x1[t], t]]
sol5x1[t_] = x1[t] /. sol51[[1]]
sol5x2[tf]
(*sol5=Solve[
  {2*sol5x1[tf]+5 *sol5x2[tf] == 20+tf^2/2,-25/2 lam+3*sol5x2[tf]== tf},{lam,tf}]*)
(*lamS = lam /.sol5[[1,1]]*)
(*tfS= tf /.sol5[[1,2]]*)
lamS = 0.917966088109113799928811368357O6;
(*computed from MATLAB since Mathematica cannot compute this*)
tfS = 2.38067401962272398753992308724 5;
(*computed from MATLAB since Mathematica cannot compute this*)
xSol5[t_] = sol5x1[t] /. {lam → lamS, tf → tfS};
ySol5[t_] = sol5x2[t] /. {lam → lamS, tf → tfS};
Plot[xSol5[t], {t, 0, tfS}, PlotLabel → "Optimal x1",
 AxesLabel → {"t", "x1(t)"}, AspectRatio → 1]
Plot[ySol5[t], {t, 0, tfS}, PlotLabel → "Optimal x2",
 AxesLabel → {"t", "x2(t)"}, AspectRatio → 1]
x2[x1_] = 1/5 (20 + tfS^2/2 - 2 x1);
plot = Plot[x2[x1], {x1, -10, 10}, PlotStyle → Orange];
parametric = ParametricPlot[{xSol5[t], ySol5[t]},
   {t, 0, tfS}, PlotRange -> {{-.1, 5}, {-.1, 4}}, PlotStyle → Blue,
   PlotLabel → "Optimal Path", AxesLabel → {"x1(t)", "x2(t)"}, AspectRatio → 1];
Show[parametric, plot, PlotLabel → "Optimal Path"]
u[t_] = 3 lamS Exp[-tfS] Exp[t] + 2 lamS
```

Out[70]= $\left\{\left\{x2[t] \to \frac{1}{2} e^{-t-tf} \left(-1 + e^{t}\right) \left(3 + 3 e^{t} + 4 e^{tf}\right) lam\right\}\right\}$

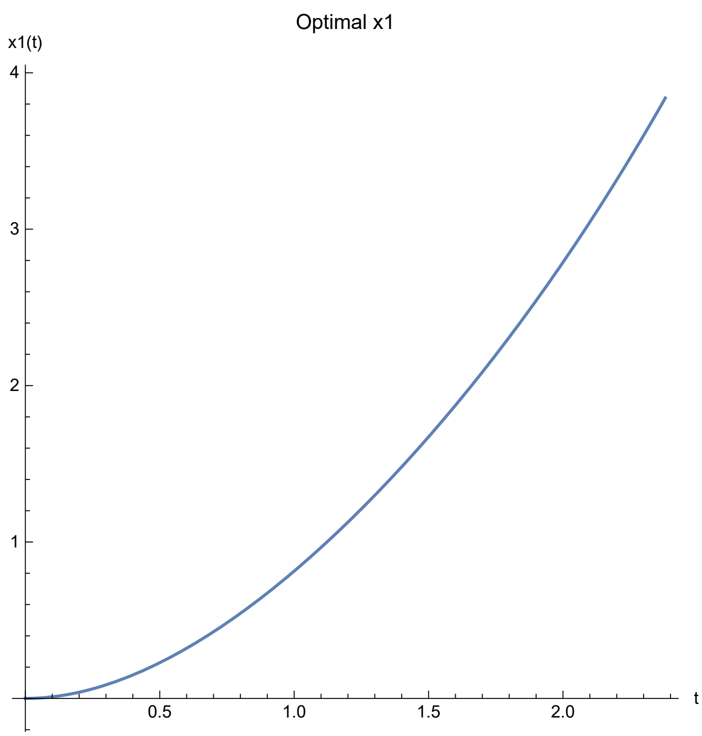
Out[71]= $\frac{1}{2} e^{-t-tf} \left(-1 + e^{t}\right) \left(3 + 3 e^{t} + 4 e^{tf}\right) lam$

Out[72]= $\left\{\left\{x1[t] \to \frac{1}{2} e^{-t-tf} lam \left(3 - 6 e^{t} + 3 e^{2 t} + 4 e^{tf} + 4 e^{t+tf} (-1 + t)\right)\right\}\right\}$

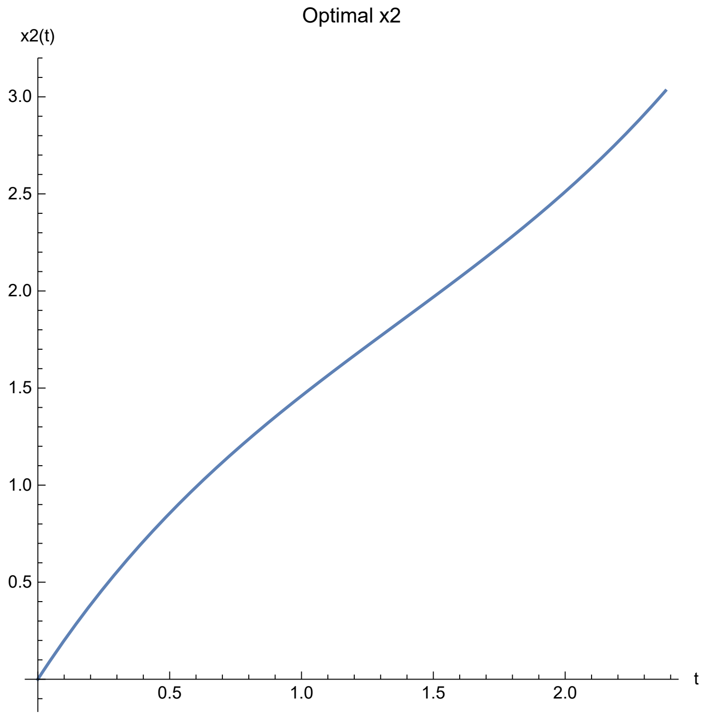
Out[73]= $\frac{1}{2} e^{-t-tf} lam \left(3 - 6 e^{t} + 3 e^{2 t} + 4 e^{tf} + 4 e^{t+tf} (-1 + t)\right)$

Out[74]= $\frac{1}{2} e^{-2 tf} \left(-1 + e^{tf}\right) \left(3 + 7 e^{tf}\right) lam$

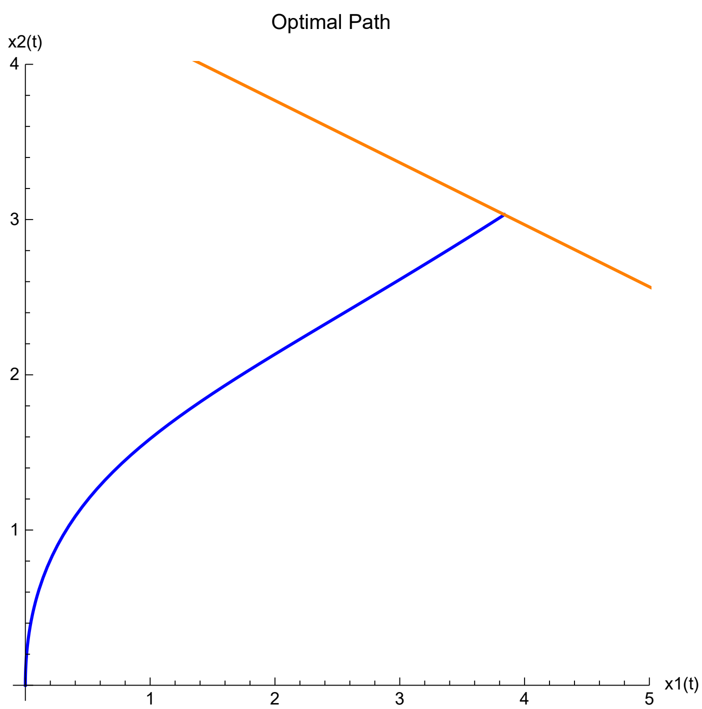
Out[79]=



Out[80]=



Out[84]=



Out[85]= $1.8359321762182275998576227367141 + 0.2547031419834292829896440029 97\, e^{t}$

(*Problem 3*)

In[136]:=
```
sol = Simplify[DSolve[
    {x'[t] == -p[t], p'[t] == -x[t], p[2] == x[2] - 1 + lam, p[0] == lam}, {x[t], p[t]}, t]]
xSol[t_] = x[t] /. sol[[1, 2]]
pSol[t_] = p[t] /. sol[[1, 1]]
xSol0[t_] = xSol[t] /.
    lam → 15164043596883160878601741802992 / 50239011507324117710738448669621;
(*lambda solved by MATLAB*)
pSol0[t_] = pSol[t] /.
    lam → 15164043596883160878601741802992 / 50239011507324117710738448669621;
u[t_] = Simplify[-N[pSol0[t]]]
cost = N[1/2 (xSol0[2] - 1)^2 + 1/2 Integrate[xSol0[t]^2 + pSol0[t]^2, {t, 0, 2}]]
hamiltonian[t_] = 1/2 (xSol0[t]^2 - pSol0[t]^2);
Plot[hamiltonian[t], {t, 0, 2}]
```

Out[136]= $\left\{\left\{p[t] \to \frac{1}{2} e^{-2-t} \left(1 + e^{2t} (-1 + lam) - lam + 2 e^2 lam\right),\; x[t] \to \frac{1}{2} e^{-2-t} \left(1 - e^{2t} (-1 + lam) - lam + 2 e^2 lam\right)\right\}\right\}$

Out[137]= $\frac{1}{2} e^{-2-t} \left(1 - e^{2t} (-1 + lam) - lam + 2 e^2 lam\right)$

Out[138]= $\frac{1}{2} e^{-2-t} \left(1 + e^{2t} (-1 + lam) - lam + 2 e^2 lam\right)$

Out[141]= $e^{-1.\, t} \left(-0.349081 + 0.047243\, e^{2.\, t}\right)$

Out[142]= 0.301838

Out[144]=