Note. Periodic extension requires the value at the end points be equal. Otherwise, the function would have two different outputs at the end points, which makes it not well-defined. A simple fix is to restrict the domain (remove one end point).

Definition: generalized Fourier series

Given $y = F(x)$, where $-L \leq x \leq L$ for some positive real number $L > 0$, define the inner product

$$\langle f(x), g(x) \rangle = \int_{-L}^{L} f(x)g(x)dx \text{ with norm } \|f\|_2.$$

We can show that the countably infinite set

$$\left\{1, \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right), \ldots\right\}.$$

is a set of orthogonal functions with respect to the inner product given above. If we assume that

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n + \cos\left(\frac{n\pi x}{L}\right) + \sin\left(\frac{n\pi x}{L}\right).$$

then, by the projection formula for finding coordinates, the corresponding Fourier coefficients are

$$\begin{aligned}
a_0 &= \frac{\langle F(x), 1 \rangle}{\|1\|^2} = \frac{1}{2L}\int_{-L}^{L} F(x)dx \\
a_n &= \frac{\langle F(x), \cos\left(\frac{n\pi x}{L}\right)\rangle}{\left\|\cos\left(\frac{n\pi x}{L}\right)\right\|^2} = \frac{1}{L}\int_{-L}^{L} F(x)\cos\left(\frac{n\pi x}{L}\right)dx \\
b_n &= \frac{\langle F(x), \sin\left(\frac{n\pi x}{L}\right)\rangle}{\left\|\sin\left(\frac{n\pi x}{L}\right)\right\|^2} = \frac{1}{L}\int_{-L}^{L} F(x)\sin\left(\frac{n\pi x}{L}\right)dx
\end{aligned}$$

Note. The generalization was achieved using a change of variables: let $z = \frac{\pi x}{L}$. The L^2-inner product on $[-L, L]$ and on $-\pi, \pi$ only differ by $\frac{L}{\pi}$.

- restrict domain
- redefine the value at end point so that $F(L) = F(-L)$.

Example. $F(x) = x, -L \leq x \leq L$. Redefine:

$$F'(-L) = F'(L) = \frac{F(-L) + F(L)}{2} = \frac{-L + L}{2} = 0.$$

0.1 Convergence

Definition

Suppose $f(x)$ and $g(x)$ are defined on $[-L, L]$, we say $f(x)$ and $g(x)$ are **equivalent** or **equal almost everywhere** (denoted as $f(x) \sim g(x)$). If $f(x) = g(x)$ for all $x \in (-L, L)$ except possibly at finite set of points $\{x_1, x_2, \ldots\}$ (in fact measure-zero sets) at which $f(x_i) \neq g(x_i)$ where $|f(x_i)| < \infty$ and $|g(x_i) < \infty|$ for $i \leq i \leq k$.

Note. If $f(x) \sim g(x)$ then F.S. $[f](x) =$ F.S.$[g](x)$ but $f(x) \neq g(x)$. Hence the need for restriction.

Definition: dense

Let A be a non-empty set and suppose B is a subset of A. We say set B is **dense** in A if any point of a of A can be written as a limit of points from B, that is, if for any $a \in A$ there exists a sequence points (nn) from B s.t. $\lim_{n\to\infty} b_n = a$.

Definition: trigonometric polynomial

A **trigonometric polynomial** is a finite sum of the form

$$S_N = a_0 + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Note. The Nth partial sum of a Fourier Series is a trigonometric polynomial.

Lemma

The set of trigonometric polynomials is dense in the set of continuous functions.

Intuition. The analogy is that rational numbers are dense in real numbers.

0.2 Periodicity

It "smooths" the bad end points or removable discontinuity. see lecture notes.