Homework 1

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Problem (1)**.** Suppose not, $\bigcap \mathcal{F}$ is not the smallest σ-field containing $\mathcal{A}$. It follows that there exists a subset $B \subseteq \Omega$ such that $B \in \bigcap \mathcal{F}$ but $B \notin \sigma(\mathcal{A})$. In other words, B is an element that makes $\bigcap \mathcal{F}$ bigger than the smallest σ-field. Notice that $B \in \bigcap \mathcal{F}$ implies that B is in every σ-field containing $\mathcal{A}$, including the smallest σ-field containing $\mathcal{A}$. Therefore, $B \in \sigma(\mathcal{A})$, which is a contradiction. Hence, $\bigcap \mathcal{F}$ must be the smallest σ-field containing $\mathcal{A}$, $\sigma(\mathcal{A})$.

Problem (2)**.** To show that $\mathcal{F}$ is a field, let's go over the definition:

(i) Since $\mathbb{N} \subseteq \mathbb{N}$, $\mathbb{N} \setminus \mathbb{N} = \emptyset$ which is finite, we see that $\mathbb{N} \in \mathcal{F}$.

(ii) Given $A \in \mathcal{F}$, we know that $A \subseteq \mathbb{N}$ and either A is finite or $\mathbb{N} \setminus A = A^c$ is finite. By symmetry, either A^c is finite or A is finite. And since $A^c \in \mathbb{N}$, we show that $A^c \in \mathcal{F}$.

(iii) To show that $\mathcal{F}$ is closed under finite unions, it suffices to show that given $A, B \in \mathcal{F}$, $A \cup B \subseteq \mathcal{F}$.

Case (1). A, B are finite, then it's easy to see that $A \cup B$ is also finite. Thus, $A \cup B \in \mathcal{F}$.

Case (2). WLOG, if A is either finite or cofinite and B is cofinite. Consider $(A \cup B)^c = A^c \cap B^c$. Since B^c is finite, this intersection must be finite. Thus, $A \cup B$ is cofinite, and $A \cup B \in \mathcal{F}$.

Hence for all possible cases, $A \cup B \in \mathcal{F}$.

To show that $\mathcal{F}$ is not a σ-field, consider the following counterexample:

Given the sequence of sets $(A_i) = \{2i\}$, $\bigcup_{i=1}^{\infty} A_i = \{2, 4, \ldots\}$. However, $\mathbb{N} \setminus \bigcup_{i=1}^{\infty} A_i = \{1, 3, 5, \ldots\}$ is also countably infinite. Thus $\bigcup_{i=1}^{\infty} A_i \notin \mathcal{F}$, and $\mathcal{F}$ is not closed under countably unions, making it not a σ-field.

Problem (3)**.**

a)

1) Since $\mathcal{F}_1$ is a field, $\Omega \in \mathcal{F}_1$. And because the sequence of sets are nested, *i.e.* $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$, it is easy to see that $\mathcal{F}_1 \subseteq \bigcup_{n=1}^{\infty} \mathcal{F}_n$. Therefore, $\Omega \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$.

2) Given a set $A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$, we know that there exists an $i \in \mathbb{N}$ such that $A \in \mathcal{F}_i$. Since $\mathcal{F}_i$ is a field, $A^c \in \mathcal{F}_i$. Therefore, $A^c \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$.

3) Given finite sets $A_1, A_2, \ldots, A_k \in \bigcup_{n=1}^{\infty} \mathcal{F}_n, k \in \mathbb{N}$, we know that each A_i must be in at least one of the $\mathcal{F}_n$. Since $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$, choose $\mathcal{F}^*$ to be the largest $\mathcal{F}_n$ that contains all the A_i's. That is, $\{A_1, A_2, \ldots, A_k\} \subseteq \mathcal{F}^*$. Since $\mathcal{F}^*$ is a field, it follows that $\bigcup_{i=1}^{k} A_i \in \mathcal{F}^* \subseteq \bigcup_{n=1}^{\infty} \mathcal{F}_n$.

By proving all three parts of the definition, we show that $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a field.

b) Let $\Omega = \mathbb{N}$, $\mathcal{F}_n$ be the collection of all subsets of $\{1, 2, \ldots, n\}$ and their complements, and $A_k = \{2k\}$. Then $\bigcup_{k=1}^{\infty} A_k = \{2, 4, \ldots\}$ which is the set of all positive even numbers. However, notice that since each set in $\mathcal{F}_n$ is either finite or cofinite, and $\mathcal{F}_n$ is nested, then $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is the collection of all finite and cofinite subsets in $\mathbb{N}$. As we have shown in Problem 2, the set of all positive even numbers is not in this set, so $\bigcup_{k=1}^{\infty} A_k \notin \bigcup_{n=1}^{\infty} \mathcal{F}_n$ and thus $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is not a σ-field.

Problem (4)**.** First, by Property 3 of P^*,

$$P^*(A) \leq P(A) \tag{1}$$

Since $A \subseteq \bigcup_{n=1}^{\infty} B_n$, by the monotonicity of probability measure, we have

$$P(A) \leq P\left(\bigcup_{n=1}^{\infty} B_n\right) \tag{2}$$

The final missing link can be found by the countable subadditivity of probability measure:

$$P\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} P(B_n) \tag{3}$$

The three inequalities above together yields:

$$P^*(A) \leq P(A) \leq P\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} P(B_n).$$

Problem (5)**.**

a) We would like to show that $P_2^*(A)$ is also the infimum of the set $S = \{\sum_{n=1}^{\infty} P(A_n) : A_n \in \mathcal{F}_0, A \subseteq \bigcup_{n=1}^{\infty} A_n\}$. That is, we will show that $P^*(A)$ is a lower bound of S and is no less than any lower bound of S.

   1) Given a sequence $(A_n) \subseteq \mathcal{F}_0$ such that $A \subseteq \bigcup_{n=1}^{\infty} A_n$, since $\mathcal{F} = \sigma(\mathcal{F}_0)$, $(A_n) \subseteq \mathcal{F}$ and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$. And since probability measure P is defined on $\mathcal{F}$, by subadditivity we immediately have $P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n)$. Since $P_2^*(A)$ is defined as the infimum of the set $\{P(B) : A \subseteq B, B \in \mathcal{F}\}$, we have $P_2^*(A) \leq P\left(\bigcup_{n=1}^{\infty} A_n\right)$. It follows that $P_2^*(A) \leq \sum_{n=1}^{\infty} P(A_n)$, which makes $P_2^*(A)$ a lower bound of S.

   2) To show that $P_2^*(A)$ is the greatest lower bound, it suffices to show that $P_2^*(A) \geq P_1^*(A)$ since $P_1^*(A)$ is already no less than all lower bounds of S. Using the ε definition of infimum, we know that given $\varepsilon > 0$, there exists a $B \in \mathcal{F}$ such that $A \subseteq B$ and $P_2^*(A) + \varepsilon > P(B)$. We can use this existence machine to construct a sequence as the following: let $\varepsilon_n = \frac{1}{n}$, there exists a $B_n \in \mathcal{F}$ such that $A \subseteq B_n$ and $P_2^*(A) + \varepsilon_n > P(B_n)$. Let $B = \bigcap_{n=1}^{\infty} B_n$. Clearly $B \in \mathcal{F}$ as $\mathcal{F}$ is closed under countable intersection and B is a subset of any B_n. By monotonicity of probability measure, $P(B) \leq P(B_n)$ for all $n \in \mathbb{N}$. Since A is a subset of all B_n's, it must be a subset of B as well. By monotonicity of outer measures, $P_1^*(A) \leq P_1^*(B)$. By the theorem from class, $P_1^*(B) = P(B)$. Putting these together and letting $n \to \infty$, we obtain

   $$P_1^*(A) \leq P_1^*(B) = P(B) \leq P(B_n) \leq P_2^*(A)$$

   as required.

Hence, $P_2 * (A)$ is also an infimum of S. Since infimum is unique, it must be that $P_1^*(A) = P_2^*(A)$.

b) Consider

$$\begin{aligned}\sup\{P(B) : B \subseteq A, B \in \mathcal{F}\} &= \sup\{(1-(1-P(B)) : B \subseteq A, B \in \mathcal{F}\} \\ &= 1 - \inf\{P(B^c) : A^c \subseteq B^c, B^c \in \mathcal{F}\} \\ &= 1 - P_2^*(A^c)\end{aligned}$$

To show that the infimum and supremum are always achieved, we need to show that there exist $B_1, B_2 \in \mathcal{F}$ such that $P(B_1) = P_2^*(A^c)$ and $P(B_2) = P_{2*}(A)$. Notice that in a), we have already proven that for the specific B we construct, $P_2^*(A) \geq P(B)$. And since $P_2^*(A)$ is the infimum, we also have $P_2^*(A) \leq P(B)$. Hence, this specific B is our candidate such that $P(B) = P_2^*(A)$. Then by symmetry of the complement, there must exist a B_1 such that $P(B_1) = P_2^*(A^c)$. For the supremum, choose $B_2 = \Omega \setminus B_1$. Then by additivity of probability measure, $P(B_2) = 1 - P(B_1) = 1 - P_2^*(A^c) = P_{2*}(A)$, as required.

c) ($\Rightarrow$) Suppose $P^*(A \cap E) + P^*(A^c \cap E) = P^*(E), \quad \forall E \subseteq \Omega$. Choose $E = \Omega$, and we have $A \cap E = A$, $A^c \cap E = A^c$. It follows that

$$\begin{aligned}P^*(A) + P^*(A^c) = P^*(\Omega) &= 1 \\ P^*(A) &= 1 - P^*(A^c) \\ \inf\{P(B) : A \subseteq B, B \in \mathcal{F}\} &= \sup\{P(B) : B \subseteq A, B \in \mathcal{F}\}\end{aligned}$$

($\Leftarrow$) Suppose $\inf\{P(B) : A \subseteq B, B \in \mathcal{F}\} = \sup\{P(B) : B \subseteq A, B \in \mathcal{F}\}$. Based on previous problems, it's easy to see that this statement is equivalent to $P^*(A) + P^*(A^c) = 1$. We would like to prove both directions to show equality. One direction comes directly from subadditivity of the outer measure: $P^*(A \cap E) + P^*(A^c \cap E) \geq P^*((A \cap E) \cup (A^c \cap E)) = P^*(E)$. The other direction requires more work. By b), since the infimum is always achieved, there exist $B_1, B_2 \in \mathcal{F}$ such that $A \subseteq B_1, P^*(A) = P(B_1)$ and $A^c \subseteq B_2, P^*(A^c) = P(B_2)$. It follows that $P(B_1) + P(B_2) = 1 = P(\Omega)$. disjointed??? Given $E \subseteq \Omega$, there also exists $B_3 \in \mathcal{F}$ such that $E \subseteq B_3, P^*(E) = P(B_3)$. Now by monotonicity of the outer measure, we can establish

$$P^*(A \cap E) + P^*(A^c \cap E) \leq P^*(B_1 \cap E) + P^*(B_2 \cap E) \tag{4}$$

And because $B_1 \cap E \subseteq B_1 \cap B_3 \in \mathcal{F}$ and $B_2 \cap E \subseteq B_2 \cap B_3 \in \mathcal{F}$, by the definition of infimum,

$$P^*(B_1 \cap E) + P^*(B_2 \cap E) \leq P(B_1 \cap B_3) + P(B_2 \cap B_3) \tag{5}$$

By the additivity of probability measure,

$$P(B_1 \cap B_3) + P(B_2 \cap B_3) = P(\Omega \cap B_3) = P(B3) = P^*(E) \quad (6)$$

Combining (4) to (6), we obtain $P^*(A \cap E) + P^*(A^c \cap E) \leq P^*(E)$. Hence, we proved both directions so we obtain the equality as required.

d) We want to show equality by proving inequalities in both directions. Let's denote the first set as S again and the new set as S'.

   1) Since $\mathcal{F}_0 \subseteq \mathcal{F}$, by definition of infimum, $P_3^*(A) \leq \sum_{n=1}^{\infty} P(A_n) \quad \forall A_n \in \mathcal{F}_0 \subseteq \mathcal{F}$. Thus $P_3^*(A)$ is a lower bound of S. And the greatest lower bound of S $P_1^*(A) \geq P_3^*(A)$.

   2) Now we want to show that $P_3^*(A) \geq P_1^*(A)$ and thus it is the greatest lower bound of S. It suffices to show that $P_1^*(A)$ is in fact a lower bound of S'. Given a sequence of $(A_n) \subseteq \mathcal{F}$, since $\mathcal{F}$ is a σ-field, we know that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$. By subadditivity and the definition of infimum, respectively, we obtain the following inequalities:

   $$\sum_{n=1}^{\infty} P(A_n) \geq P\left(\bigcup_{n=1}^{\infty} A_n\right) \geq P_2^*(A) = P_1^*(A).$$

   Thus $P_1^*(A)$ is a lower bound of S'. It follows that the greatest lower bound of S' $P_3^*(A) \geq P_1^*(A)$.

By proving both directions, we obtain $P_3^*(A) = P_1^*(A) = \inf\{S\}$.

Problem (6)**.** To prove that the triple is a probability measure space, we need to show that $\mathcal{F}$ is a σ-field and λ is a probability measure.

1)

   (i) Since $(0,1] \in \mathcal{B}$, it is easy to see that $\Omega \in \mathcal{F}$.

   (ii) Given $S \in \mathcal{F}$,

   $$S^c = \{(x,y) : x \in A^c, 0 < y \leq 1\}.$$

   Since $\mathcal{B}$ is closed under complements, $A^c \in \mathcal{B}$. Hence $S^c \in \mathcal{F}$.

(iii) Given $(S_n) \in \mathcal{F}$,

$$\bigcup_{n=1}^{\infty} S_n = \left\{ (x, y) : x \in \bigcup_{n=1}^{\infty} A_n, 0 < y \leq 1 \right\}.$$

Since each $A_n \in \mathcal{B}$, and $\mathcal{B}$ is a σ-field, we know that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$. Hence $\bigcup_{n=1}^{\infty} S_n \in \mathcal{F}$.

2) It's easy to see that $P(\Omega) = \lambda((0,1]) = |1-0| = 1$ and that $P : \mathcal{F} \to [0,1]$. Hence P is a probability measure.

Taken together, $(\Omega, \mathcal{F}, P)$ is a probability measure space.

Given $B \in \mathcal{F}$ such that $A \subseteq B$, then this constraint yields

$$\{(x, y) : x \in (0,1], 0 < y \leq 1\} = \Omega \subseteq B.$$

But since $B \in \mathcal{F}$ is a subset of Ω, this yields $B = \Omega$. That is, the only possible B is Ω itself. Now consider

$$\begin{aligned} P^*(A) &= \inf \{P(B) : A \subseteq B, B \in \mathcal{F}\} \\ &= \inf \{\Omega\} \\ &= P(\Omega) \\ &= 1 \end{aligned}$$

By definition of the inner measure, $P_*(A) = 1 - P^*(A) = 0$.