1 Insulated Rod continued

Note. The BCs in this problem is the **Von Neumann condition**, which gives rise to FCS. The BCs from a previous problem with no derivatives is the **Dirichlet condition**, which gives rise to FSS.

Note. In the case when PDE and BCs already form a vector space, we don't need to solve for steady-state and transient solutions separately because the eigenvalue problem at $\lambda = 0$ case gives the steady-state solution. Let's directly use $u(x,t) = F(x)G(t) \neq 0$ and apply separation of variables.

Then the time domain problem is

$$G'(t) = -\lambda k G(t).$$

And the solution is again $G(t) = Ce^{-\lambda kt}, C \in \mathbb{R}$. The boundary value problem is:

$$\begin{cases} \frac{d^2F}{dx^2} = -\lambda F(x) \\ F'(0) = 0 = F'(L) \end{cases}$$

This is equivalent to the eigenvalue problem:

Case. $\lambda < 0$ we get trivial solution.

Case. $\lambda = 0$, then

$$F''(x) = 0 \Rightarrow F(x) = Ax + B.$$

and the BCs yields $A = 0$ thus $F(x) = B, B \in \mathbb{R}$.

Note. We didn't get trivial solution here because it is the Von Neumann condition, as opposed to the Dirichlet condition from before.

Case. $\lambda > 0$, this is the same as before, we have

$$F(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

And apply BCs:

$$0 = F'(0) \Rightarrow c_2 = 0 \Rightarrow F(x) = c_1 \cos(\sqrt{\lambda}x).$$

$$0 = F'(L) \Rightarrow \sin(\sqrt{\lambda}L) = 0 \Rightarrow \sqrt{\lambda} = \frac{n\pi}{L} \text{ for } n = \pm 1, \pm 2, \ldots.$$

which implies $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, \ldots$.

Now we have $u_n(x,t) = a_n \cos\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt}$ for $n = 1, 2, \ldots$. The superposition principle asserts that the solution of the homogeneous PDE (if it converges) is the linear combination of all $u_n(x,t)$. That is,

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt}$$

for some constants A_n. Now we apply the IC to find these constants. Since we have orthogonal cosine basis, it is in fact a Fourier Consine Series (FCS):

$$f(x) = u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right).$$

we use the projection formula for the case $t = 0$ to find the coefficients of this basis:

$$A_0 = \frac{1}{L}\int_0^L f(x)dx.$$

and

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

Theorem: convergence

For $t > 0$, if there exists a constant $0 < M < \infty$ such that $|A_n| \leq M$, for all n, then

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt}$$

converges absolutely for each $x \in [0, L]$

Therefore, our final solution is

$$u(x,t) = \frac{1}{L}\int_0^L f(x)dx + \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx\right)\cos\left(\frac{n\pi x}{L}\right)e^{-(\frac{n\pi}{L})^2 kt}.$$

and note that when $t \to \infty$, we obtain the steady state solution.

For large but finite time, we can use the slowest decaying term to approximate the solution

$$u(x,t) = A_0 + A_1\cos\left(\frac{\pi x}{L}\right)e^{-(\frac{\pi}{L})^2 kt}.$$

1.1 Fourier Cosine Series

What does it represent?

Definition: even extension

1) Define the **even extension** of $f(x)$ to be

$$f_{even}(x) = \begin{cases} f(x), & \text{if } 0 < x < L \\ f(-x), & \text{if } -L < x < 0 \end{cases}$$

Then $f_{even}(-x) = f_{even}(x)$ for any $x \in (-L, L)$.

2) If $f(x)$ is piecewise smooth then $f(x)$ has a Fourier series representation and if

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), \text{ for } 0 < x < L.$$

Then note that the RHS is continuous, even, and $2L$-periodic. Thus the Fourier cosine series of $f(x)$ represents the periodoc extension of the (adjusted) even extension of $f(x)$ that is

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = \tilde{\bar{f}}_{even}(x).$$

3) in general, $\text{FCS}[f](x) = \text{ F.S.}[f_{even}](x)$.

Example. See lecture notes for figures of FCS.

MIDTERM 1 MATERIALS STOP HERE.