Proof: Fourier Series

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nz) + \sum_{n=1}^{\infty} b_n \sin(nz)$$

To find a_8,

$$f(z)\cos(8t) = a_0 \cos(8z) + \sum_{n=1}^{\infty} a_n \cos(nz)\cos(8z) + \sum_{n=1}^{\infty} \sin(nz)\cos(8z)$$

$$\int_{-\pi}^{\pi} f(z)\cos(8z)dz = a_0 \int_{-\pi}^{\pi} \cos(8z)dz + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nz)\cos(8z)dz + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nz)\cos(8z)dz$$

$$\int_{-\pi}^{\pi} f(z)\cos(8z)dz = 0 + a_8 \int_{-\pi}^{\pi} \cos^2(8z)dz + 0$$

So we can solve for a_8. □

0.1 Taylor vs Fourier

Taylor:

1) T.S.$[f](z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ where $c_n = \frac{f^{(n)(a)}}{n!}$
2) $f(z)$ is analytic at the point z iff $f(z)$ =T.S. $[f](z)$
3) RoC might be small
4) need to be differentiable
5) the basis is not orthogonal

Fourier:

1) F.S.$[f](z) = a0 + \sum_{n=1}^{\infty} a_n \cos(nz) + \sum_{n=1}^{\infty} nn \sin(nz), -\pi \leq z \leq \pi$.
2) If it converges for $z \in [-\pi, \pi]$ then it converges for all $z \in (-\infty, \infty)$.
3) coefficients are found by integration which is a much weaker assumption.
4) basis is orthogonal.
5) $\tilde{f_M}(z)$ is the truncated Fourier series after M terms.

Example. $f(z) = |z|, \pi \leq z \leq \pi$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(z)\sin(nz)dz = \frac{1}{\pi}\int_{-\pi}^{\pi} |z|\sin(nz)dz = \frac{1}{\pi}\cdot 0 = 0.$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |z|dz = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -zdz + \int_{0}^{\pi} zdz\right] = \frac{\pi^2}{2\pi} = \frac{\pi}{2}.$$

Using integration by parts, we can show for even and odd values of $n \geq 1$ that

$$a_n = a_{2m} = 0 \text{ and } a_n = a_{2m-1} = -\frac{4}{\pi(2m-1)^2}.$$

$$\text{F.S.}[f](z) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos[(2m-1)z] = \frac{\pi}{2} - \frac{4}{\pi}\cos(z) - \frac{4}{9\pi}\cos(3z) - \ldots.$$

Convergence:

$$\begin{aligned}\left|\frac{\pi}{2}\right| + \frac{4}{9\pi}|\cos(3z)| + \ldots &\leq \frac{\pi}{2} + \frac{4}{\pi} + \frac{4}{9\pi} + \ldots \\ &= \frac{\pi}{2} + \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\end{aligned}$$

So the series is absolutely convergent.

We can use the integral test to bound the error term.

$$\begin{aligned}|\text{F.S.}[f](z) - \tilde{f_M}(z)| &= \left|-\frac{4}{\pi}\sum_{n=M}^{\infty}\frac{1}{(2m-1)^2}\cos([2n-1]z)\right| \\ &\leq \frac{4}{\pi}\sum_{n=M}^{\infty}\frac{1}{(2n-1)^2} \\ &\leq \frac{4}{\pi}\int_{M}^{\infty}\frac{1}{(2x-1)^2}dx\end{aligned}$$

1 Chapter 3

Definition: Periodicity

Suppose $g(z)$ is defined for all real numbers, if there exists a number $p > 0$ such that

$$g(z) = g(z+p), \quad \forall z \in \mathbb{R}.$$

then $g(z)$ is said to be a **periodic function**.

Note. 1) If $g(z)$ is periodic then it has many periods $p, 2p, \ldots$ we use the shortest period $p > 0$.

2) if $g_1(z)$ and $g_2(z)$ have period p then so does $h(z) = ag_1(z) \pm bg_2(z)$ for any $a, b \in \mathbb{R}$

3) constant function is trivially periodic for any $p > 0$

All Fourier basis vectors are at least $2 - \pi$-periodic.
What does a Fourier series of a function represent?

Definition: periodic extension

Let $f(z)$ be a function defined on $[-L, L]$ such that $f(-L) = f(L)$. Define the **periodic extension** of $f(z)$ to be the unique periodic function $\tilde{f}(z)$ of period $2L$ such that $\tilde{f}(z) = f(z)$, for $-L \leq z \leq L$.