*Intuition.* For linear map T, $T$ is injective iff $\ker T = \{0\}$

**Theorem**

If $\phi : G \to H$ homomorphism, then $\phi$ is injective $\Leftrightarrow \ker\phi = \{e_G\}$

**Proof**

Let's prove the contrapositives:

($\Leftarrow$):If $\ker\phi \neq \{e_G\}$, then there exists $x \in G \setminus \{e_G\}$ such that $\phi(x) = e_H$. Since $e_G \neq x$ and $\phi(x) = \phi(e_G) = e_H$, this implies $\phi$ is not injective.

($\Rightarrow$): We want to use the same idea from linear algebra. If $\phi$ is not injective, then there exist $x \neq y$ with $\phi(x) = \phi(y)$. Then

$$\begin{aligned}\phi(x^{-1}y) &= \phi(x^{-1})\phi(y)\\ &= \phi(x)^{-1}\phi(y)\\ &= e_H\end{aligned}$$

However, $x^{-1}y \neq e_G$ since $x \neq y$. Hence, $e_G$ and $x^{-1}y$ are distinct elements of $\ker\phi \Rightarrow \ker \neq \{e_G\}$. $\square$

**Example.** Read differentiation example in textbook.

**Example.** $\phi : \mathbb{C}^* \to \mathbb{R}*$, $\phi(z) = |z|$. We claim this is a homomorphism.

$$\phi(z *_{\mathbb{C}} w) = |zw| = |z||w| = \phi(z) *_{\mathbb{R}} \phi(w).$$

$\operatorname{im}\phi = \mathbb{R}^+$. This is a subgroup because it is the image of a known homomorphism!

$\ker\phi = U$, the unit circle $\{z \in \mathbb{C}^* : |z| = 1\}$. Hence $U$ is a normal subgroup of $\mathbb{C}^*$. It follows from that the subgroup of an abelian group is normal. Or it's the kernel of a known homomorphism.

*Note.* T/F: If $H \leq G$ and $H$ is abelian, is $H$ a normal subgroup? NO, think $\{e, (1\ 2)\} \leq S_3$.

*Notation.* $\pi$ stands for surjective/projection. $\iota$ stands for injective.

**Example.** $\pi_1 : G_1 \times G_2 \to G_1$, $\pi((g_1, g_2)) = g_1$. "Projection to first component".

$$\begin{aligned}\pi_1((x_1,x_2)*(y_1,y_2)) &= \pi_1((x_1*y_1,x_2*y_2))\\ &= (x_1*y_1)\\ &= \phi((x_1,x_2))\phi((y_1,y_2))\end{aligned}$$

$\operatorname{im}\pi_1 = G_1$.

$\ker\pi_1 = \{(g_1,g_2) : g_1 = e_1\}$. $K \trianglelefteq G_1 \times G_2$.

# 1 Factor/Quotient Groups

*Note.* Quotient groups are NOT a special kind of subgroup. We try to find a group of cosets. See screenshot for the motivation. For some cases, if you call the inputs by different names, we obtain the same result.

Well-defined: the operation is not confused by picking different representations for the inputs.

Goal: $H \le G$, try to make the left cosets of $H$ in $G$ into a group. That is, $(xH)*(yH) = xyH$. Here we should think of each coset as a single object.

Problem: the result seems to depend on the representatives $x, y$ that were chosen. If $x, x'$ in the same left coset, $xH = x'H \Leftrightarrow h = x^{-1}x' \in H, x' = xh$. So $xH = xhH$. Then

$$(xhH)*(yh'H) = xhyh'H.$$

We need $xhyh'H$ and $xyH$ to be the representations of the same things for all $x, y, h, h'$. Again this means

$$\begin{aligned}xyH = xhyh'H &\Leftrightarrow (xy)^{-1}xhyh'H \in H\\ &\Leftrightarrow y^{-1}x^{-1}xhyh'H \in H\\ &\Leftrightarrow y^{-1}hyh' = h_0 \in H\\ &\Leftrightarrow y^{-1}hy = h_0(h')^{-1} \in H\end{aligned}$$

Summary: this would work iff $y^{-1}hy \in H$ for all $h \in H, y \in G$.