1 Probability Measures

Let Ω be a non-empty set. (Think of it as a sample space = set of all possible outcomes of an experiment involving randomness.) Ex: Flip a coin twice.

$$\Omega = \{HH, HT, TH, TT\}.$$

Let $A \subset \Omega$. Ex: A = an event, $\{HT, TH, TT\}$. With fair coin $P_r(A) = \frac{3}{4}$

To measure a 2D blob: usual length, area, volume ..

We want to assign probability to subsets of Ω
P : subsets of $\Omega \to [0, 1]$.

Definition: field

Let Ω be a non-empty set. Let $\mathcal{F}$ a collection of subsets of Ω. $\mathcal{F}$ is called a **field (or algebra)** if

1. $\Omega \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ ("closed under complements")
3. Given $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{F}$ ("closed under finite unions").

Remark. We can also use "closed under intersection" to define because of De Morgan's law.

$$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c.$$

Definition: sigma-field

If (iii) is replaced by a *countable* union, e.g. $\bigcup_{n=1}^{\infty} A_n$, then $\mathcal{F}$ is called a $\boldsymbol{\sigma}$**-field**.

Remark. σ field is stronger than field.

Example (sigma-fields)**.** TODO

Ω.

- $\mathcal{F} = \{\emptyset, \Omega\}$
- $\mathcal{F}$ = the power sets (all possible subsets of Ω). (Billingsley:"power class").
- Take any $A \subset \Omega$. Define $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$

A counterexample of a field that is not a σ-field. Let $\Omega = \mathbb{R}$, $\mathcal{F}$ = the empty set and all finite disjoint unions of things like $(a, b]$ and/or (a, ∞) for $-\infty \le a < b < \infty$.
(i) $\Omega = (-\infty, \infty) \in \mathcal{F}$.
(ii) Complements: Ex: $(a, b]^c = (-\infty, a] \cup (b, \infty)$. More generally a set in $\mathcal{F}$ has the form a bunch of disjointed (] or (∞).
(iii)

Case (1). A_1, A_2 have no overlaps. Then $A_1 \cup A_2$ is still a finite disjointed union.

Case 2: Some overlap, then $A_1 \cup A_2$ is still the same type of interval.
However, it is not a σ-field. We want to find a countable collection of sets that isn't in here. Let $A_n = (0, 1 - \frac{1}{n}]$. Then $\bigcup_{n=1}^{\infty} A_n = (0, 1) \notin \mathcal{F}$.

Definition: generated sigma-field

Let $\mathcal{A}$ be a collection of subsets of Ω. The **$\boldsymbol{\sigma}$-field generated by $\boldsymbol{\mathcal{A}}$** is the smallest σ-field containing all the sets in $\mathcal{A}$. We write it as $\sigma(\mathcal{A})$.

i.e. Any σ-field containing $\mathcal{A}$ must contain $\sigma(\mathcal{A})$. Note:

- If $\mathcal{F}$ is a σ-field, $\sigma(\mathcal{F}) = \mathcal{F}$.
- If $\mathcal{F}$ is a σ-field and $\mathcal{A} \subset \mathcal{F}$, then $\mathcal{A} \subset \mathcal{F}$.
- $\sigma(\mathcal{A}) = \bigcap \mathcal{F}$ the intersection over all σ-field that contain $\mathcal{A}$.
- $\mathcal{A} \subset \mathcal{A}' \Rightarrow \sigma(\mathcal{A}) \subset \sigma(\mathcal{A}')$

Example:

- $A \subset \Omega, \mathcal{A} = \{A\} \Rightarrow \sigma(\mathcal{A}) = \sigma(A) = \{\emptyset, A, A^c, \Omega\}$
- **Borel Sets** in $\mathbb{R}$. Let $\Omega = \mathbb{R}$, $\mathcal{A}$ = all open finite intervals in $\mathbb{R}$. $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{A})$. Note: include all half open intervals. It also contains single points: $\{a\} = \bigcap_{n=1}^{\infty} (a, a + \frac{1}{n}]$. Cantor sets are not in here.

1.1 Unions and Intersections of sigma-fields

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- The union of two σ-Fields is not a necessarily a σ-Field. Take $A, B \subset \Omega, A \neq B, \sigma(A) = \{\emptyset, A, A^c, \Omega\}, \sigma(B) = \emptyset, B, B^c, \Omega$. So $\sigma(A) \cup \sigma(B)$
- The intersection (including uncountable intersection) of two σ-Fields is a σ-Field. Let $\mathcal{F}_1, \mathcal{F}_2$ be two σ-Fields. $\mathcal{F}_1 \cap \mathcal{F}_2$ (i) $\Omega \in \mathcal{F}_1 \cap \mathcal{F}_2$ since $\emptyset \in \mathcal{F}_1$ and $\emptyset \in \mathcal{F}_2$. (ii) Let $A \in \mathcal{F}_1 \cap \mathcal{F}_2$ $A \in \bigcap_{n=1}^{\infty} \mathcal{F}_n \Rightarrow A \in \mathcal{F}_n \forall n \Rightarrow A^c \in \mathcal{F}_n \forall n \Rightarrow A^c \in \bigcap_{n=1}^{\infty} \mathcal{F}_n$