Example (continued)**.** According to Fourier's Law of Heat Conduction, the steady state heat flux is

$$\overline{\Phi}(x) = -k\frac{d}{dx}\overline{u}(x) = -\left(\frac{T_2 - T_1}{L}\right).$$

Example. Given a heat source $Q(x,t)$, consider the equation

$$k\frac{\partial^2 u}{\partial x^2} + Q(x,t) = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{1}{k}Q(x,t) \Rightarrow u \text{ may be a function of } t.$$

So if $\frac{\partial Q}{\partial t} \neq 0$ we will have no steady state solution.

Example. Suppose $Q(x,t) = M$ and consider $\frac{\partial^2 u}{\partial x^2} + M = 0$ with $u(0) = T_1$ and $u(L) = T_2$ then

$$u''(x) = -M \Rightarrow u(x) = -\frac{Mx^2}{2} + C_1 x + C_2.$$

and the boundary conditions imply the equilibrium solution is

$$\overline{u}(x) = T_1 + \left(\frac{T_2 - T_1}{L} + \frac{ML}{2}\right)x - \frac{Mx^2}{2}.$$

0.1 Insulated Boundaries

Consider the PDE with the BCs and IC:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, u(x,0) = f(x), \frac{\partial u}{\partial x}(0,t) = 0, \frac{\partial u}{\partial x}(L,t) = 0.$$

where IC is when $t = 0$, and BCs are $x = 0$ and $x = L$, which have zero values and means these are insulated boundaries.

With regards to the steady state solution, if we assume $u(x,t) = u(x)$ then $u(x) = C_1 x + C_2$ and using the BCs we have

$$u'(x) = C_1 \Rightarrow C_1 = u'(0) = \frac{\partial u}{\partial x}(0,t) = 0 \Rightarrow u(x) = C_2.$$

and we expect that $\overline{u}(x) = \lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} u(x) = C_2$. Rewriting the original heat equation and integrating both sides yields:

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} \Rightarrow \int_0^L c\rho\frac{\partial u}{\partial x}dx = \int_0^L K_0\frac{\partial^2 u}{\partial x^2}dx = K_0\left[\frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t)\right] = 0.$$

and multiplying by the constant area A and interchanging the derivative and integral yields

$$\int_0^L c\rho\frac{\partial u}{\partial x}Adx = 0 \Rightarrow \frac{d}{dt}\left[\int_0^L c\rho u(x,t)Adx\right] = 0.$$

This implies that the total thermal energy is constant wrt time.

Using IC,

$$\int_0^L c\rho u(x,0)A dx = \int_0^L c\rho f(x) A dx.$$

And the equilibrium thermal energy is

$$\lim_{t\to\infty}\int_0^L c\rho u(x,t)A dx = \int_0^L c\rho \left[\lim_{t\to\infty} u(x,t)\right] A dx = \int_0^L c\rho C_2 A dx = C_2 c\rho A L.$$

setting the initial and equilibrium thermal energy equal to each other and solving yields

$$c\rho A\int_0^L f(x)dx = c\rho A C_2 L \Rightarrow C_2 = \frac{1}{L}\int_0^L f(x)dx.$$

So the equilibrium solution to the heat equation with insulated boundaries is the *average value of the initial temperature* f(x) over the interval $[0, L]$.

1 Heat Equation in 3D

When can we switch integration and differentiation for partial differential equations?

Theorem

Suppose

1) $u(x,t)$ is defined for $a \le x \le b, c \le t \le d$.
2) $u(x,t)$ is Riemann integrable for every $t \in [c,d]$
3) $\partial_t u(u,t)$ is continuous for $(x,t) \in [a,b]\times[c,d]$

then $\partial_t u(x,t)$ is Riemann integrable for every $t \in [c,d]$, and

$$\frac{d}{dt}\int_a^b u(x,t)dx = \int_a^b \frac{\partial u}{\partial x} dx.$$

Note. We can replace the closed intervals above with $\mathbb{R}$.

1.1 Boundary Heat Flux

Let's generalize our result from 1D to 3D:

Let $\vec{\phi}(x)$ be the heat flux vector which specifies the direction of heat flow at the point **x**$= (x,y,z)$. Then the magnitude is the flux, and direction is the normal to the surface area. So the heat energy flowing across boundaries per unit time:

$$-\iint_{\partial R} \vec{\phi}(x) \cdot \mathbf{n} \; dS.$$

if the dot product is positive then heat is flowing out of the object and the total energy would decrease.

Then the heat flow process is:

$$\frac{d}{dt} \iiint_R c(\mathbf{x})\rho(\mathbf{x})u(\mathbf{x},t)dV = -\iint_{\partial R} \vec{\phi}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})dS + \iiint_R Q(\mathbf{x},t)dV.$$

Recall the **Divergence Theorem**, we have

$$\iint_{\partial R} \vec{\phi}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})dS = \iiint_R \nabla \cdot \vec{\phi}(\mathbf{x})dV.$$

where $\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$. Now we bring the derivative inside the integral:

$$\iiint_R c(\mathbf{x})\rho(\mathbf{x})\frac{\partial}{\partial t}u(\mathbf{x},t)dV = -\iiint_R \nabla \cdot \vec{\phi}(\mathbf{x}) + \iiint_R Q(\mathbf{x},t)dV.$$

and combining all the triple integrals on the left hand side yields

$$c(\mathbf{x})\rho(\mathbf{x})\frac{\partial}{\partial t}u(\mathbf{x},t) + \nabla \cdot \vec{\phi}(\mathbf{x}) - Q(\mathbf{x},t) = 0.$$

where the last equality follows from that the integral equation is true for any region R and by continuity.