1 Direct method

Consider
$$\int_{-1}^{1} f(x)dx = af(-1) + bf(0) + cf(1).$$
Now pretend $f(x) = 1, x, x^2, \ldots$, keep plugging in the next order until we get inconsistency. Then we obtain the same coefficient as Simpson's rule.

But we don't have to stick with $x = -1, 0, 1$. If we let $x = -\frac{2}{3,0,\frac{2}{3}}$, then we get something different.

We can generalize even further. Consider
$$\int_{-1}^{1} f(x)\sin\frac{\pi}{2}xdx = af(-1) + bf(0) + cf(1).$$
Repeat the same procedure and we obtain the weighted values.

Transforming integrals: Let $t = \frac{2x-a-b}{b-a}$, hence
$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{t(b-a)+a+b}{2}\right)\frac{b-a}{2}dt.$$

2 Gaussian Quadrature

We want to find:
$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} c_i f(x_i).$$
$c_i and x_i$ give us $2n$ parameters to choose, so the polynomial is at most $2n-1$ degree.

2.1 Legendre Polynomials

They are orthogonal with respect to the inner product $\int_{-1}^{1} P(x)P_n(x)dx$, where $P_n(x)$ is the nth Legendre polynomial.

Example:
$$\int_0^1 e^{(-x^2)}dx = \frac{1}{2}\int_{-1}^{1} e^{(-\frac{t+1}{2})^2}dt.$$
This is a lot less work than Simpson's.

Advantage: good accuracy

Disadvantage: uneven spacing, so if we don't know $f(x)$ there might be too much interpolation.

3 Improper Integrals

Consider the integration of functions with a singularity at $x = a$ (left endpoint) of the form:

$$f(x) = \frac{g(x)}{(x-a)^p}.$$

where $g(x)$ is continuous on $[a, b]$. And we want $\int_a^b f(x)$. Note this converges iff $0 < p < 1$.