Theorem

$X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{p} X$.

Note. Almost surely is the strongest convergence.

Proof

Define $A_n = \{|X_n - X| \geq \varepsilon \ i.o.\}$.Using the inequality of the limsup and liminfs to show $\lim_{n\to\infty} P(A_n) = 0$ which is convergence in probability. $\square$

Example (The converse is false)**.** Let $\Omega = [0,1], \mathcal{F} = \mathcal{B}[0,1], P$ =Lebesgue measure. Define:

$$
\begin{aligned}
X_1(\omega) &= \omega + I_{[0,1]}(\omega) \\
X_2(\omega) &= \omega + I_{[0,\frac{1}{2}]}(\omega) \\
X_3(\omega) &= \omega + I_{[\frac{1}{2},1]}(\omega) \\
X_5(\omega) &= \omega + I_{[\frac{1}{3},\frac{2}{3}]}(\omega) \\
X_6(\omega) &= \omega + I_{[\frac{2}{3},1]}(\omega) \\
&\dots
\end{aligned}
$$

Clearly for all $\varepsilon > 0$, $P(|X_n - X| > \varepsilon) \to 0$. On the other hand, there is no $\omega \in [0,1]$ such that $X_n(\omega) \to X(\omega) = \omega$. For example, $\omega = \frac{1}{4}$, $X_1(\frac{1}{4}) = \frac{1}{4} + 1 = \frac{5}{4}$. $X_2(\frac{1}{4}) = \frac{1}{4} + 1 = \frac{5}{4}$. $X_3(\frac{1}{4}) = \frac{1}{4} + 0 = \frac{1}{4}$... So it alternates between $\frac{1}{4}$ and $\frac{5}{4}$.

Definition: expected value

For a simple r.v. $X(\omega) = \sum_{i=1}^n x_i I_{A_i}(\omega)$ (assume x_i are distinct and $A_1, \dots, A_n$ are disjoint and partition Ω). Define the **expected value** of X as

$$E[X] = \sum_{i=1}^n x_i P(A_i).$$

Property.

1) $X = 0$ a.s. $P(X = 0) = 1$. Then $E[X] = 0$. Prove using indicators.
2) For nonnegative X, *i.e.* $X \geq 0$ *a.s.* Then $E[X] \geq 0$.
3) For nonnegative X and $E[X] = 0$, then $X = 0$ a.s.
4) It is a linear operator: $E[aX + bY] = aE[X] + bE[Y]$.

5) nonnegative $X \Rightarrow E[X \cdot I_{\{X>0\}}] = E[X]$

6) $X = Y$ a.s. $(P(\{\omega : X(\omega) = Y(\omega)\}) = 1) \Rightarrow E[X] = E[Y]$.

7) $X \leq Y \ a.s. \Rightarrow E[X] \leq E[Y]$.

8) $|E[X]| \leq E[|X|]$ (special case of Jensen's inequality).

Definition: uniformly bounded

The sequence (X_n) is **uniformly bounded** if there exists a finite constant M such that $|X_n(\omega)| \leq M \ \forall \ \omega \in \Omega$ and $\forall \ n$.

Theorem

$X_n \xrightarrow{p} X$ and (X_n) uniformly bounded implies $\lim\limits_{n\to\infty} E[X_n] = E[X]$.