1 Laplace's Equation and Solution Continued

$$\begin{cases} G''(y) = \lambda G(y) \\ G(0) = 0, G(H) = 0 \end{cases}$$

and $F''(x) = -\lambda F(x)$. We do the same process again to solve $u_2(x,t)$. Note that since only $\lambda < 0$ works to solve to F-equation, we require $|\lambda| = \left(\frac{n\pi}{H}\right)^2 \Rightarrow \lambda = -\left(\frac{n\pi}{H}\right)^2$. The solution is

$$u_2(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{H}\right) \sin\left(\frac{n\pi y}{H}\right) + d_n \sinh\left(\frac{n\pi[x-L]}{H}\right) \sin\left(\frac{n\pi y}{H}\right).$$

with coefficients

$$c_n = \frac{\frac{2}{H}\int_0^H g_2(y)\sin\left(\frac{n\pi y}{H}\right) dy}{\sinh\left(\frac{n\pi L}{H}\right)} \text{ and } d_n = \frac{\frac{2}{H}\int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right) dy}{\sinh\left(\frac{-n\pi L}{H}\right)}.$$

Then the **formal solution** is

$$\begin{aligned} u(x,y) &= u_1(x,y) + u_2(x,y) \\ &= \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi x}{L}\right) + b_n \sinh\left(\frac{n\pi[y-H]}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \\ &\quad + \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{H}\right) \sin\left(\frac{n\pi y}{H}\right) + d_n \sinh\left(\frac{n\pi[x-L]}{H}\right) \sin\left(\frac{n\pi y}{H}\right) \end{aligned}$$

with the coefficients:

$$a_n = \frac{\frac{2}{L}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right) dx}{\sinh\left(\frac{n\pi H}{L}\right)} \text{ and } b_n = \frac{\frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right) dx}{\sinh\left(\frac{-n\pi H}{L}\right)}$$

$$c_n = \frac{\frac{2}{H}\int_0^H g_2(y)\sin\left(\frac{n\pi y}{H}\right) dy}{\sinh\left(\frac{n\pi L}{H}\right)} \text{ and } d_n = \frac{\frac{2}{H}\int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right) dy}{\sinh\left(\frac{-n\pi L}{H}\right)}$$

Note. At the corners we have $u(0,0) = u(L,0) = u(0,H) = u(L,H) = 0$.

Intuition. Recall the formal solution $u(x,y)$ from above. We can show if $0 < a < b \Rightarrow \frac{\sinh(na)}{\sinh(nb)} < e^{n(a-b)}$ and $e^{n(a-b)} < 1$. Since $x < L$ and $y < H$, the coefficients are finite integral of quotient of two hyperbolic terms (less than 1) and thus are bounded. We can show convergence along this line of logic.

2 Laplace in Circular Geometry

We now wish to determine the formula for Laplace's equation in terms of the polar coordinates. Let's denote $u(x, y) = v(r, \theta)$. Suppose $x = r\cos(\theta), y = r\sin(\theta)$. Then

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}, r > 0.$$

Using the Chain rule,

$$\frac{\partial v}{\partial r} = \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} = u_x\cos(\theta) + u_y\sin(\theta).$$

and

$$\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial \theta} = u_x(-r\sin\theta) + u_y(r\cos\theta).$$

Now for the 2nd partials, we use the rules we discovered above:

$$\begin{aligned}
\frac{\partial^2 v}{\partial r^2} &= \frac{\partial}{\partial r}[u_x\cos\theta + u_y\sin\theta] \\
&= \frac{\partial}{\partial r}[u_x\cos\theta] + \frac{\partial}{\partial r}[u_y\sin\theta] \\
&= [u_{xx}\cos^2\theta + u_{xy}\cos\theta\sin\theta] + [u_{yx}\cos\theta\sin\theta + u_{yy}\sin^2\theta] \\
&= u_{xx}\cos^2\theta + 2u_{xy}\cos\theta\sin\theta + u_{yy}\sin^2\theta
\end{aligned}$$

Again doing this for θ,

$$\frac{\partial^2 v}{\partial \theta^2} = \frac{\partial}{\partial t}[u_x(-r\sin\theta)] + \frac{\partial}{\partial t}[u_y(r\cos\theta)].$$

Thus by product rule, we eventually obtain

$$\frac{\partial^2 v}{\partial \theta^2} = u_{xx}r^2\sin^2\theta - 2u_{yx}r^2\cos\theta\sin\theta + u_{yy}r^2\cos^2\theta - r(u_x\cos\theta + u_y\sin\theta).$$

dividing both sides by r^2 yields:

$$\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = u_{xx}\sin^2\theta - 2u_{yx}\cos\theta\sin\theta + u_{yy}\cos^2\theta - \frac{1}{r}(u_x\cos\theta + u_y\sin\theta).$$

Therefore, if we add these two second partials together:

$$v_{rr} + \frac{1}{r^2} \cdot v_{\theta\theta} = u_{xx} + u_{yy} - \frac{1}{r}v_r.$$

This gives us

$$\Delta u = u_{xx} + u_{yy} = v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta}.$$