1 D'Alembert's Solution continued

$$\begin{cases} \text{PDE:} & x \in \mathbb{R}, t > 0 \\ \text{ICs: } u(x,0) = U(x), \frac{\partial u}{\partial t}(x,0) = V(t) & x \in \mathbb{R} \end{cases}$$

Note. Since $x \in \mathbb{R}$, we no longer have boundaries or BCs.

The solution has the form:

$$u(x,t) = f(x - ct) + g(x + ct).$$

where f, g are traveling waves and need to be twice differentiable. We find $f(z)$ and $g(y)$ using the ICs.

We know that the set of functions that satisfy the wave equation form a vector space. So we can divide and conquer by separating the PDE into two problems:

$$\begin{cases} \partial_t^2 u_1 = c^2 \partial_x^2 u_1 \\ u_1(x,0) = U(x) \\ \partial_x u_1(x,0) = 0 \end{cases} \qquad \begin{cases} \partial_t^2 u_2 = c^2 \partial_x^2 u_2 \\ u_2(x,0) = 0 \\ \partial_x u_2(x,0) = V(x) \end{cases}$$

Intuition. By having 0 on the RHS, we can express f in terms of g, and solve an ODE involving one function instead.

If there exists two solutions u_1, u_2, then their sum satisfies the PDE and ICs.

Let's assume $u_1(x,t) = f_1(x - ct) + g_1(x + ct)$. Using the Chain Rule,

$$0 = \frac{\partial u_1}{\partial t}(x,0) = f_1'(x) \cdot (-c) + g_1'(x) \cdot (c) \Rightarrow f_1'(x) = g_1'(x).$$

Therefore, $f_1(x) = g_1(x) + k$ for some constant k. Using the other initial condition,

$$U(x) = u_1(x,0) = f_1(x) + g_1(x) = [g_1(x) + k] + g_1(x) \Rightarrow g_1(x) = \frac{U(x)}{2} - \frac{k}{2}.$$

Then

$$f_1(x) = g_1(x) + k = \frac{U(x)}{2} + \frac{k}{2}.$$

Therefore, the solution is

$$u_1(x,t) = f_1(x - ct) + g_1(x + ct) = \frac{U(x - ct)}{2} + \frac{U(x + ct)}{2}.$$

Intuition. If started at rest, the initial position of the string breaks in two, half moving to the left and half moving to the right at equal speeds c, each with half the amplitude of the original. The solution is the simple sum of these traveling waves.

Similarly, we assume $u_2 = f_2(x - ct) + g2(x + ct)$, and

$$0 = u_2(x, 0) = f_2(x) + g_2(x) \Rightarrow f_2(x) = -g_2(x) \Rightarrow f_2'(x) = -g_2'(x).$$

Using the second initial condition:

$$V(x) = \frac{\partial u}{\partial t}(x, 0) = f_2'(x)\cdot(-c) + g_2'(x)\cdot c = -g_2'(x)\cdot(-c) + g_2'(x)\cdot c = 2cg_2'(x) \Rightarrow g_2'(x) = \frac{V(x)}{2c}.$$

That is, $g_2'(x)$ is an antiderivative of $\frac{V(x)}{2}$. Recall by Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x)) \cdot g'(x).$$

So by the FTC, we can define

$$g_2(x) = \int_a^x \frac{V(t)}{2c}dt \text{ and } f_2(x) = \int_x^a \frac{V(t)}{2c}dt.$$

Therefore,

$$u_2(x.t) = f_2(x - ct) + g_2(x + ct) = \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{V(t)}{2c}dt.$$

Hence, the **general solution** is

$$u(x, t) = \frac{1}{2}U(x - ct) + \frac{1}{2}U(x + ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} V(s)ds.$$

We can check it satisfies the ICs. When checking the second IC, we would need to apply the FTC using chain rule:

$$\begin{aligned}
\frac{1}{2c}\frac{d}{dt}\int_{x-ct}^{x+ct} V(s)ds &= \frac{1}{2c}\frac{d}{dt}\left(-\int_a^{x-ct} V(s)ds + \int_a^{x+ct} V(s)ds\right) \\
&= \frac{1}{2c}(-V(x - ct) \cdot (-c) + V(x + ct) \cdot c) \\
&= \frac{1}{2}(V(x - ct) + V(x + ct))
\end{aligned}$$

At $t = 0$, we obtain $\frac{\partial u}{\partial t}(x, 0) = V(x)$.

Example. Solve the PDE:

$$\begin{cases} \text{PDE: } \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} & x \in \mathbb{R}, t > 0 \\ \text{ICs: } u(x, 0) = 0, \frac{\partial u}{\partial t}(x, 0) = \frac{2}{1+x^2} & x \in \mathbb{R} \end{cases}$$

Apply the d'Alembert's formula we have

$$\begin{aligned} u(x,t) &= 0 + \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{2}{1+x^2} ds \\ &= \frac{1}{c}[\arctan(x+ct) - \arctan(x-ct)] \end{aligned}$$

Taking the limit as $t \to \infty$, for each fixed x,

$$\lim_{t\to\infty} \frac{1}{c}[\arctan(x+ct) - \arctan(x-ct)] = \frac{1}{c}\left\{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right\} = \frac{\pi}{c}.$$

See lecture slides for figures. The top of the wave just flattens out.