1 Summary of Heat Equations

BCs whose solutions form a vector space:

1) Dirichlet BCs: temperature fixed at ends, homogeneous function BCs. Solution is FSS.
2) Neumann BCs: perfectly insulated, homogeneous derivative BCs. Solution is FCS.
3) Cauchy BCs: thin circular wire, equal function and derivative BCs. Solution is FS.
4) Variations: mixture of Dirichlet and Neumann BCs. Depending on the mixture, we get different answers. See homework and practice exam.

For those that don't form a vector space, we move the nonhomogeneous part to the steady state BCs.

Note. The initial condition changes after removing the steady state component.

2 Motion of Stretched String

Motivation. We consider a *horizontally stretched string* with ends that are tied down (something like a guitar). The string moves in time and we wish to track the position of each point on the string during vibration. The motion of a point on the string is NOT entirely vertical, but we are going to assume the motion is entirely vertical. See lecture slides for illustrations.

2.1 Assumptions

1) With **no motion**, the string has
   - $\delta(s)$: density
   - $A(s)$: cross-sectional area
   - $u(s)$: vertical displacement at arc length s.
2) The *linear mass density* of the string is $\rho_0 = (\delta \cdot A)$.
3) *Boundary Conditions*: The ends of the string with length L are fixed: $u(0,t) = u(L,t) = 0$.
4) Possible external forces: gravity, violin bow, guitar pick, etc.
5) **Trivial equilibrium** is if there are no external forces and no motion, then we assume the string lies along a straight line, so $ds = dx$. Then we assume there is a constant **tensile force** or *tension* along the string.

6) For small vibrations, $u = u(x,t)$ measures the vertical displacement from the trivial equilibrium at time t. That is, the shape of the string at time $t = t_0$ is given by $u(x, t_0)$.

2.2 Additional Assumptions

1) We assume mass is constant.
2) Assume the string is perfectly flexible and has no stiffness.
3) The forces exerted by the string on the ends act purely in the *tangential direction* and there are no transverse forces and no torque (twisting).

2.3 Derivation

Let $T(x,t) \geq 0$ represent the magnitude of the tangential force due to the tension. Then the horizontal tension balances each other out because there is no horizontal motion. That is,

$$T(x,t)\cos(x,\theta) = T(x+\Delta x, t)\cos(x+\Delta x, \theta).$$

Therefore, we conclude that T is constant. $T(x,t)\cos(x,t) = T_0$.

2.4 Vertical Forces

According to Newton's second law, $\mathbf{F} = m\mathbf{a}$, the vertical net force equals the tensile forces plus the vertical components of any external forces. Therefore,

$$\rho_0 \Delta x \cdot \frac{\partial^2 u}{\partial t^2} = T(x+\Delta x, t)\sin(\theta(x+\Delta x, t)) - T(x,t)\sin(\theta(x,t)) - \rho_0 \Delta x \cdot g.$$

This is force equals to opposing vertical tensile forces minus the gravity. Note that $T(x,t) = \frac{T_0}{\cos(x,t)} \Rightarrow T(x,t)\sin(\theta(x,t)) = T_0 \tan(\theta(x,t))$, so

$$\rho_0 \cdot \frac{\partial^2 u}{\partial t^2} = T_0 \cdot \left[\frac{\tan(\theta(x+\Delta x, t)) - \tan(\theta(x,t))}{\Delta x}\right] - \rho_0 \cdot g.$$

Taking the limit $\Delta x \to 0$,

$$\rho_0 \cdot \frac{\partial^2 u}{\partial t^2} = T_0 \cdot \frac{\partial}{\partial x}\tan(\theta(x,t)) - \rho_0 \cdot g.$$