Definition: simplied irreducible

An element $f(x) \in F[x]$ is **irreducible** if

(i) It is not constant;

(ii) It cannot be factorized into two polynomials both of strictly smaller degree.

Example. In $\mathbb{Q}[x]$, $3x + 3$ is irreducible because 3 is a unit. Also the degree wouldn't work if $3x + 3 = g(x)h(x)$ where $g(x), h(x)$ are nonconstant.

Corollary

In $F[x]$, a polynomial of degree 1 is irreducible.

Remark. There is always a question about degree 4 factorization on the final.

Let's now look at degree 2 or 3.

Suppose $f(x) \in F[x]$ has degree 2, *i.e.* $f(x) = ax^2 + bx + c, a, b, c \in F, a \neq 0$. Suppose $f(x) = g(x)h(x)$. Then we have three cases. When $g(x)$ or $h(x)$ has degree 0, they are units and we ignore these cases. So if f is not irreducible, then $\deg g = \deg h = 1$. $g(x) = ax + b, a \neq 0$, since $a^{-1} \in F$, $g(x)$ has a root in F, which is $-\frac{b}{a}$. Then $f(x)$ has a root in F (the same one).

Suppose $f(x)$ has degree 3, *i.e.* $f(x) = ax^3 + bx^2 + cx + d, a, b, c, d \in F, a \neq 0$. Then $f(x) = g(x)h(x)$, ignoring degree 0 cases, implies two cases. When $\deg g = 1$, then g has a root in F, so $f(x)$ does as well by evaluation homomorphism. When $\deg h = 1$, then h has a root in F, so $f(x)$ does as well.

Example. $x^2 + 1 \in \mathbb{R}[x]$ has a zero i but it's not in $\mathbb{R}$.

Theorem: 23.10

If $f(x) \in F[x]$ has degree 2 or 3, then either $f(x)$ is irreducible or $f(x)$ has a zero in F.

Corollary

If $f(x) \in F[x]$ is a polynomial of degree 2 or 3 with no roots in F, then $f(x)$ is irreducible.

Example. $x^2 + 1 \in R[x]$ is irreducible.

Note. This is not true in degree 4 or higher.

Example (counterexample in degree 4)**.** Consider $x^4 + 1 \in \mathbb{Q}[x]$. Does $f(x)$ have any roots in $\mathbb{Q}$? No. What about $\mathbb{R}[x]$? No. Suppose $f(x) = g(x)h(x)$. Ignoring degree 0 cases, for degree 1 cases there is a root. But for degree 2 case we might have 2 irreducible quadratics (no root). If f is not irreducible we must have 2 irreducible quadratics case. WLOG, we can pretend the first term is monic because we can convert to monic anyway:

$$x^4 + 1 = (x^2 + ax + b)(x^2 + cx + d)$$

Now we can equate coefficients:

Constant terms: $1 = bd$.

Coefficient of x^3: $0 = a + c$.

x: $0 = ad + bc$.

x^2: $0 = ac + b + d$.

Let's write everything in terms of a, b: $d = \frac{1}{b}, c = -a$. So we have

$$\begin{cases} 0 &= \frac{a}{b} - ab \\ 0 &= b + \frac{1}{b} - a^2 \end{cases}$$

First gives $a(1 - b^2) = 0 \Rightarrow a = 0$ or $b = \pm 1$. If $a = 0$, then $b + \frac{1}{b} = 0$ (no solution). If $b = -1$, $a^2 = -2$. If $b = 1$, then $a^2 = 2 \Rightarrow a = \sqrt{2}, b = 1, c = -\sqrt{2}, d = 1$. So

$$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1).$$

These two factors are not units and irreducible because they are degree 2 and have no roots by determinant < 0 of quadratic.