1 Convergence Part 1

Definition

1) $f(x)$ is **continuous** at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.
2) $f(x)$ is **piecewise continuous** on the interval $[a, b]$ if $f(x)$ is defined and continuous at all but a finite number of points $x_i \in [a, b], 1 \leq i \leq N$, where $f(x)$ has either a jump discontinuity or a removable discontinuity and $f(a^+) = \lim_{x \to a^+} f(x)$ and $f(b^-) = \lim_{x \to b^-} f(x)$ exist.
3) A differentiable function $f(x)$ is **piecewise smooth** on $[a, b]$ if both $f(x)$ and $f'(x)$ are piecewise continuous on $[a, b]$, *i.e.* $f \in$ piecewise $\mathcal{C}^1$.
4) The function $f(x)$ is **continuous piecewise smooth** on $[a, b]$ if $f(x)$ is piecewise smooth on $[a, b]$ and has no discontinuities in (a, b).

Note.

a) $f'(x)$ will not be continuous at the discontinuities of $f(x)$ (if any) and may have its own additional discontinuities.
b) If $f'(x)$ is piecewise continuous on (a, b), then $f(x)$ is also piecewise continuous on (a, b). Why? (Jaden: f' is piecewise continuous on $(a, b) \Rightarrow f'$ exists on $(a, b) \Rightarrow f$ is piecewise differentiable on $(a, b) \Rightarrow f$ is piecewise continuous on (a, b).)

Example. $f(x) = \frac{1}{x}$ where $x \neq 0$ and $x \in [-1, 1]$ is not piecewise continuous because the left and right limits do not exist at $x = 0$.

Example. Consider $x \in [-1, 1]$, if

$$f(x) = |x|^{\frac{1}{4}} = \begin{cases} x^{\frac{1}{4}} & \text{if } x \in [0, 1] \\ (-x)^{\frac{1}{4}} & \text{if } x \in [-1, 0) \end{cases}$$

$$f'(x) = \begin{cases} \frac{1}{4} x^{-\frac{3}{4}} & \text{if } x \in (0, 1) \\ -\frac{1}{4}(-x)^{-\frac{3}{4}} & \text{if } x \in (-1, 0) \end{cases}$$

$f(x)$ is piecewise continuous on $[-1, 1]$ but the derivative is not piecewise continuous.

Example. $f(x) = |x|$ is piecewise smooth.

To prove convergence, we want:

1) $|a_0| < \infty$

2) $a_n \to 0, b_n \to 0$ as $n \to \infty$. Otherwise the series will diverge.

Note. As $n \to \infty$, the periodicity goes to zero. So $\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)$ is the cumulative rapidly oscillating positive and negative area bounded by $|f(x)|$. Intuitively they cancel each other out as oscillation increases to infinity.

Restrictions

1 If $f(-L) = f(L)$, then we require that $\tilde{f}(x)$, the periodic extension of $f(x)$, be piecewise smooth on $[-L, L]$.

2 If $f(-L) \neq f(L)$, then redefine $f_{new}(-L) = f_{new}(L) = \frac{f(-L)+f(L)}{2}$. We denote the adjusted function $\overline{f}(x)$. Now we require that the periodic extension of the adjusted function, $\tilde{\overline{f}}(x)$, be piecewise smooth on $[-L, L]$.

Notation. We will use $\tilde{f}$ instead of $\tilde{\overline{f}}$ for brevity.

Lemma: 1

The set of real functions defined on $[-L, L]$ that satisfy Restriction 1 and 2 form a vector space (satisfies linearity).

Lemma: 2

If $f(x)$ satisfies Restriction 1 and 2 then there exists a number $0 < M < \infty$ such that

$$|\tilde{f}(x)| < M \quad \forall x \in \mathbb{R}.$$

Proof

Any closed interval on $\mathbb{R}$ is compact. Since f is piecewise continuous, the image of a compact set is also compact. And compact set in $\mathbb{R}$ means it's closed and bounded by Heine-Borel Theorem. $\square$

Lemma: 3

If $f(x)$ satisfies Restriction 1 and 2 and if M is the positive constant from Lemma 2 then the Fourier coefficients satisfy:

$$|a_0| < M, |a_n| < 2M, \text{ and } |b_n| < 2M.$$

Proof

Given $a \in \mathbb{R}$, $-|a| \leq a \leq |a|$. Thus given $f(x)$ defined on $[-L, L]$,

$$-|f(x)| \leq f(x) \leq |f(x)| \Rightarrow -\int_{-L}^{L} |f(x)|dx \leq \int_{-L}^{L} f(x)dx \leq \int_{-L}^{L} |f(x)|dx$$
$$\Rightarrow \left|\int_{-L}^{L} f(x)dx\right| \leq \int_{-L}^{L} f(x)dx$$

$$\begin{aligned}
|a_n| &= \left|\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx\right| \\
&\leq \frac{1}{L}\int_{-L}^{L} |f(x)|\left|\cos\left(\frac{n\pi x}{L}\right)\right|dx \\
&\leq \frac{1}{L}\int_{-L}^{L} |f(x)|dx \\
&\leq \frac{1}{L}\int_{-L}^{L} M dx = \frac{M}{L}2L = 2M
\end{aligned}$$

That is, $|a_n| < 2M$. Repeat for $b_n < 2M$ and $a_0 < M$. $\square$

Lemma: 4, Riemann-Lebesgue Lemma (special case)

If $f(x)$ satisfies Restriction 1 and 2 then $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$.

Proof

Suppose $\tilde{\tilde{f}}(x)$, denote any discontinuities as $\{x_1, x_2, \ldots x_N\}$ with $x_0 = -L$ and $x_{N+1} = L$. Then we can write

$$a_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)dx = \sum_{p=1}^{N+1}\frac{1}{L}\int_{x_{p-1}}^{x_p} \tilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

Now let
$$\alpha_{n,p} = \frac{1}{L}\int_{x_{p-1}}^{x_p} \tilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

where $\tilde{f}(x) = u$ and $\cos\left(\frac{n\pi x}{L}\right) dx = dv$. Then using integration by parts,

$$\begin{aligned}
\alpha_{n,p} &= \frac{1}{L}\left(uv\Big|_{x_{p-1}}^{x_p} - \int_{x_{p-1}}^{x_p} vdu \right) \\
|a_{n,p}| &= \frac{1}{n\pi}\left| \tilde{f}(x)\sin\left(\frac{n\pi x_p}{L}\right) - \tilde{f}(x)\sin\left(\frac{n\pi x_{p-1}}{L}\right) - \int_{x_{p-1}}^{x_p} \tilde{f}'(x)\sin\left(\frac{n\pi x}{L}\right)dx \right| \\
&\leq \frac{1}{n\pi}\Bigg[\left|\tilde{f}(x)\sin\left(\frac{n\pi x_p}{L}\right)\right| + \left|\tilde{f}(x)\sin\left(\frac{n\pi x_{p-1}}{L}\right)\right| \\
&\quad + \int_{x_{p-1}}^{x_p} \left|\tilde{f}'(x)\sin\left(\frac{n\pi x}{L}\right)\right| dx \Bigg] \qquad \text{using triangle inequality} \\
&\leq \frac{1}{n\pi}\left|\tilde{f}(x)\right| + \left|\tilde{f}(x)\right| + \int_{x_{p-1}}^{x_p} |\tilde{f}(x)| dx \\
&\leq \frac{1}{n\pi} \cdot C
\end{aligned}$$

where C represents the constant that bounds $\tilde{f}(x)$ and $\tilde{f}'(x)$ on $[-L, L]$.
As $n \to \infty$,

$$|\alpha_{n,p}| \to 0 \Rightarrow a_n = \sum_{p=1}^{N+1} \alpha_{n,p} \to 0.$$

□