Homework 8: To check whether a group is isomorphic to V_4, we can check whether every element squares to the identity. To tell if two cosets are different, check if their difference is in the subgroup.

Example. Not assuming multiplication is commutative.

$$\begin{aligned}(1+1)(a+b) &= (1+1)a+(1+1)b = a+a+b+b \\ &= 1(a+b)+1(a+b) = a+b+a+b.\end{aligned}$$

So commutativity of addition is forced by multiplicative identity and distributive laws.

What if we had $1_R = 0_R$?

Theorem: 18.8

(i) $0a = a0 = 0$.

(ii) $a(-b) = (-a)b = -(ab)$.

(iii) $(-a)(-b) = ab$.

Proof

(i)
$$0a = (0=0)a = 0a + 0a \Rightarrow y = y + y$$
by an abelian group. Then $0 = y = a0$. Likewise for the other.

(ii)
$$0 = a0 = a(b+(-b)) = ab + a(-b) \Rightarrow a(-b) = -(ab).$$
Likewise for the other.

(iii)
$$(-a)(-b) = -((-a)b) = -(-(ab)) = ab.$$

$\square$

If $1_R = 0_R \Rightarrow 1_R r = 0_R r \Rightarrow r = 0_R$. Then $R = \{0_R\}$. This is why we restrict them to be different.

Note. If A is an abelian group, we can make A into a ring in a dull way.

Set: A. Addition: addition in A. Multiplication: $a \times b = 0$. We can check this is a ring.

Example. $R = \{f : \mathbb{R} \to \mathbb{R}\}$.

Addition: pointwise. $(f+g)(c) = f(c) + g(c)$.

Multiplication: pointwise. $(fg)(c) = f(c)g(c)$.

We can check this is a ring (vector space). Let check a distributivity law. Two functions are the same if they always give the same output for the same input.

$$\begin{aligned}((f+g)(h))(c) &= (f+g)(c)h(c) \\ &= (f(c)+g(c))h(c) \qquad \text{just real numbers} \\ &= f(c)h(c) + g(c)h(c) \\ &= fh(c) + gh(c) \\ &= (fh+gh)(c)\end{aligned}$$

They are the same because real numbers are distributive.

Is R commutative? Yes.

Does R have identity? 1_R.

Remark. Composition of functions doesn't distribute over addition.

$f(x) = x^2, g(x) = \sin(x), h(x) = e^x$. Then

$$f(g+h) = (\sin x e^x)^2 \neq \sin^2 x + e^{2x} = fg + fh.$$

Definition: homomorphism of rings

A map $\phi : R \to S$ is a **homomorphism of rings** if

$$\phi(a+b) = \phi(a) + \phi(b)$$

$$\phi(a \times b) = \phi(a) \times \phi(b)$$

for all $a, b \in \mathbb{R}$.

Example. $\phi : \mathbb{Z} \to \mathbb{Z}, a \mapsto 2a$. We can find a counterexample for multiplication: $\phi(1 \times 1) = 2 \neq 4 = \phi(1) \times \phi(1)$.