Definition: Laplace Transform

Let $f(t)$ be given for $t \geq 0$ and suppose that f satisfies:

1) f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A.
2) $|f(t)| \leq Ke^{at}$ when $t \geq M$. In this inequality $K > 0, a, M > 0 \in \mathbb{R}$.

Then define the **Laplace Transform**, $\mathcal{L}\{f(t)\} = F(s)$ as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt, s > a.$$

This uses the kernel $K(s,t) = e^{-st}$ and the parameter s may be complex but we assume $s \in \mathbb{R}$.

Claim. Laplace transform is a special case of Fourier transform.

Proof

Suppose f is such a function that $f(t) = 0$ for $t < 0$ then

$$2\pi \widehat{f}(m) = 2\pi \mathcal{F}[f(t)] - \int_{-\infty}^\infty f(t) e^{-imt} dt = \int_0^\infty f(t) e^{-imt} dt.$$

If we let $m = -is, s \in \mathbb{R}$, then we have

$$2\pi \widehat{f}(m) = \int_0^\infty f(t) e^{-i(-is)t} dt = \int_0^\infty f(t) e^{-st} dt = F(s)$$
$$\Rightarrow 2\pi \mathcal{F}[f(t)] = \mathcal{L}\{f(t)\}$$

$\square$

1 Dispersive Waves

Recall the wave equation

$$\begin{cases} \text{PDE: } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} & 0 < x < L, t > 0 \\ \text{BCs: } u(0,t) = 0 = u(L,t), & t > 0 \\ \text{ICs: } u(x,0) = U(x), \frac{\partial u}{\partial t}(x,0) = V(x). & 0 \leq x \leq L \end{cases}$$

has the solution

$$u(x,t) = \sum_{n=1}^\infty \sin\left(\frac{n\pi x}{L}\right) \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right].$$

So it's a product of a wave in x and a wave in t. Therefore, suppose the solution to *wave propagation* problems have the form

$$u(x,t) = F(x)G(t) = e^{ikx} \cdot e^{-i\omega t} = e^{i(kx-\omega t)}, k, \omega \in \mathbb{R}.$$

Notation. k, ω are newly defined, not related to previous materials.

Definition: phase velocity

1) The wavelength of the space wave e^{ikx} is $L = \frac{2\pi}{k}$ so we see that k represents the number of wavelengths per 2π unit of distance (spatial frequency). For the time wave $e^{-i\omega t}$, the term ω is the number of waves per 2π unit time (time frequency).
2) the **phase velocity**, c_p, is defined as

$$c_p = \frac{\omega}{k} \text{ with units distance/time.}$$

3) The wave $e^{ik(x-\frac{\omega}{k}t)}$ represents a "traveling wave" with wave number k and wave velocity $c_p = \frac{\omega}{k}$.
4) If ω is a function of $k \in \mathbb{R}$, *i.e.* if $\omega = \omega(k)$ then $e^{-i(kx-\omega t)}$ is a **linear dispersive wave**.
5) For linear dispersive waves, we have $c_p = c_p(k)$ and if $\frac{d}{dk}c_p \neq 0$, *i.e.* if the wave velocity is not constant, then we have a wave propagation problem that is said to be **dispersive**.
6) The term $\omega(k)$ relates space and time in the expression $e^{ik(x-\omega\frac{t}{k}}$.
7) The information about the PDE (but not the BCs or ICs) is encoded in $\omega(k)$.

Example. Consider wave equation with stiffness term and ICs:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} - \alpha^2\frac{\partial^4 u}{\partial x^4}, u(x,0) = U(x), \frac{\partial}{\partial t}u(x,0) = V(x).$$

Assume $U(x,t) = e^{ikx-i\omega t}, k \in \mathbb{R}$ and it yields

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 u, c^2\frac{\partial^2 u}{\partial x^2} = c^2(ik)^2 u, \alpha^2\frac{\partial^4 u}{\partial x^4} = \alpha^2 k^4 u.$$

Substituting into the PDE and we obtain:

$$w(k) = \pm k\sqrt{c^2 + \alpha^2 k^2}.$$

Thus,

$$u_{1,k} = e^{ik(x-\sqrt{2+\alpha^2k^2t})}, u_{2,k}(x,t) = e^{ik(x+\sqrt{c^2+\alpha^2k^2})}.$$

By superposition principle, summing for all $k \in \mathbb{R}$ yields

$$u(x,t) = \int_{-\infty}^{\infty} \left[A(k) e^{ik(x-\sqrt{c^2+\alpha^2k^2})} + B(k) e^{ik(x+\sqrt{c^2+\alpha^2k^2})} \right] dk.$$

Now we can use the ICs to find $A(k), B(k)$.

$$U(x) = u(x,0) = \int_{-\infty}^{\infty} [A(k) + B(k)] e^{ikx} dk = \mathcal{F}^{-1}[A(k) + B(k)].$$

Thus, taking Fourier transform on both sides gives us

$$A(k) + B(k) = \mathcal{F}[U(x)] = \widehat{U}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(x) e^{ikx} dx.$$

And similarly,

$$-A(k) + B(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{V(x)}{ik\sqrt{c^2+\alpha^2k^2}} e^{ikx} dx.$$

These two expressions allow us to solve

$$A(k) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left[U(x) - \frac{V(x)}{ik\sqrt{c^2+\alpha^2k^2}} \right] e^{ikx} dx.$$

and

$$B(k) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left[U(x) + \frac{V(x)}{ik\sqrt{c^2+\alpha^2k^2}} \right] e^{ikx} dx.$$

1.1 Group Velocity

We wish to analyze the physical interaction of waves with proximal spatial frequency. Assume two close waves of the form $\cos[kx - \omega(k)t]$ with wave numbers k and $k + \Delta k$. That is,

$$u(x,t) = A\cos[kx - \omega(k)t] + A\cos[(k+\Delta k)x - \omega(k+\Delta k)t].$$

Using trig identity and cos is even,

$$u(x,t) = 2A \cdot \cos\left[\left(k + \frac{\Delta k}{2} \right) x - \frac{\omega(k) + \omega(k+\Delta k)}{2} t \right] \cdot \cos\left[\frac{\Delta k}{2} x - \frac{\omega(k+\Delta k) - \omega(k)}{2} t \right].$$

The wavelength of the first term is $2\pi/(k + \Delta k/2)$, which is shorter than that of the second term, $2\pi/\Delta k/2$.

The long wave acts as a wave envelop of the short waves and it travels with what is called as **group velocity** given by

$$c_g = \lim_{\Delta k \to 0} \frac{\omega(k+\Delta k) - \omega(k)}{\Delta k} = \frac{d}{dk}\omega(k).$$

The short wave moves at almost the speed $c_p = \frac{\omega(k)}{k}$ which is the phase velocity of an individual dispersive wave. The long wave acts as a wave envelope of the short waves and travels at group velocity. We claim that the **wave energy** moves with group velocity c_g.

1.2 Deep Water Waves

Gravity and surface tension affects the propagation of water waves. For surface water the equation is

$$\omega(k) = \sqrt{gk\tanh(kh)}$$

where $g = 9.8$ m/s and $h > 0$ is the depth of the water. In deep water where the depth of water h is large, notice $\lim_{h\to\infty} \tanh(kh) = 1 \Rightarrow \omega(k) = \sqrt{gk\tanh(kh)} \approx \sqrt{gk}$ thus the phase velocity of a short wave in deep water is approximately

$$c_p = \frac{\omega(k)}{k} = \frac{\sqrt{gk\tanh(kh)}}{k} = \sqrt{\frac{g\tanh(kh)}{k}} \approx \sqrt{\frac{g}{k}}.$$

Since $L = \frac{2\pi}{k}$, we finally have

$$c_p \approx \sqrt{\frac{gL}{2\pi}}.$$

Thus, $c_p = c_p(k)$ and short waves in deep water are dispersive.

For long wave

$$c_g = w'(k) = \frac{d}{dk}\sqrt{gk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\sqrt{\frac{gL}{2\pi}} \Rightarrow c_g = \frac{1}{2}c_p.$$

This shows that longer waves will have larger group velocities and arrive at a distance shoreline sooner.

For shallow water, short waves are non-dispersive and the phase velocity mainly depends on gravity and depth of water since

$$c_p = \frac{\omega(k)}{k} = \sqrt{\frac{g\tanh(kh)}{k}} = \sqrt{gh \cdot \frac{\tanh(kh)}{kh}} \approx \sqrt{gh}.$$

where we use the fact that $\tanh(kh)/kh \approx 1$ for small h (by L'Hopital Rule).