Midterm 1: HW 1-4. Midterm 2: HW 5-9.

VERY IMPORTANT: the easiest example of nonnormal subgroup: $G = S_3$, and H has order 2, *i.e.* $H = \{e, \{1\ 2\}\}$.

True: any subgroup of an abelian group is normal. False: any abelian subgroup of a group is normal. The example above!

The group G and the trivial subgroup are normal.

Any subgroup of index 2 is normal.

The kernel of a homomorphism is normal.

False: subgroup of index 3 is normal. The example above again!

True: $3\mathbb{Z} \trianglelefteq \mathbb{Z}$ because $\mathbb{Z}$ is abelian.

Note. $H \trianglelefteq G$ is equivalent to:

- $gH = Hg$ for all $g \in G$. The left/right cosets containing g. Because e is in H.
- $gHg^{-1} = H$ for all $g \in G$. $gHg^{-1} = \{ghg^{-1} : h \in H\}$.

  Claim. gHg^{-1} is a subgroup of G (even if H is not normal).

  Proof

  It is clearly a subset.

  Identity: since $e \in H$, $geg^{-1} = gg^{-1} = e \in gHg^{-1}$.

  Closure: $gh_1g^{-1}gh_2g^{-1} = gh_1h_2g^{-1} \in gHg^{-1}$.

  Inverse: $(ghg^{-1})^{-1} = ghg^{-1} \in gHg^{-1}$.

  $\square$

  Example. $G = S_3, H = \{e, (1\ 2)\}$. Let $g = (1\ 2\ 3)$.

  $$gHg^{-1} = \{(1\ 2\ 3)e(1\ 3\ 2), (1\ 2\ 3)(1\ 2)(1\ 3\ 2)\} = \{e, (2\ 3)\} \neq H.$$

  This is a subgroup of S_3, this proves that it is not normal.

  This form is called **conjugation**. We conjugated H by g to get gHg^{-1}. This might not give us the same subgroup but it would have the same order.

  Then $gH = Hg \Leftrightarrow gHg^{-1} = Hgg^{-1} = H$.

- $ghg^{-1} \in H$ for all $g \in G$, $h \in H$. This is very useful if everything else doesn't work.

Warning: this only checks that a known subgroup is normal. It doesn't prove that something is a subgroup.

Recall last time we tried to put a group structure on the (left) cosets of H in G. That is,

$$(xH) * (yH) = xyH.$$

However, this is not well-defined unless $y^{-1}hy \in H$ for all $y \in G, h \in H$. Let $y = g^{-1}$, then $ghg^{-1} \in H \ \forall \ h \in H, g \in G$.

Example (not well-defined function). $f\left(\frac{a}{b}\right) = a$ is not well-defined because by choosing different representations we get different answers.

Definition: quotient group

Let G be a group and $N \trianglelefteq G$. We define a new group, G/N (read G mod N), where G/N is the set of cosets of N in G, and the operation is $(xN) * (yN) = xyN$.

Intuition. N is normal guarantees that if we choose different elements from the same cosets, we would get answers in another same coset.

To show that the quotient group is indeed a group,

Proof

(i) identity: $eH = N$ so that $(eN) * (xN) = exN = xN = (xN) * (eN)$.

(ii) inverses: $(xN) * (x^{-1}N) = xx^{-1}N = N = (x^{-1}N) * (xN)$.

(iii) associativity: $(xN) * (yN) * (zN) = (xyN) * (zN) = xyzN = x(yz)N = (xN) * ((yN) * (zN))$ by associativity in G.

$\square$

Example. $G = \mathbb{Z}, N = 6\mathbb{Z}$. Note N is normal because it is a subgroup of an abelian group. Then $G/N = \mathbb{Z}/6\mathbb{Z}$ is the definition of the integers mod 6, $\mathbb{Z}_6$. It follows that $\mathbb{Z}_6$ is a group and $+_n$ is associative.

$$G/N = \{0 + 6\mathbb{Z}, 1 + 6\mathbb{Z}, \ldots, 5 + 6\mathbb{Z}\}.$$

Then an example is

$$(3 + 6\mathbb{Z}) + (5 + 6\mathbb{Z}) = 8 + 6\mathbb{Z} = 2 + 6\mathbb{Z}.$$

This is because $8 - 2 \in 6\mathbb{Z}$, so $8 + 6\mathbb{Z} = 2 + 6\mathbb{Z}$.

Theorem: fundamental homomorphsim theorem (1st isomorphism theorem

Let $\phi : G \to H$ be a homomorphism. Then $\ker \phi \trianglelefteq G$ and $\operatorname{im} \phi \trianglelefteq H$, and

$$G/\ker \phi \simeq \operatorname{im} \phi.$$

Furthermore, an isomorphism is given by

$$\psi : g(\ker \phi) \to \phi(g).$$

WARNING: the input of ψ is coset. So we need to prove that it is a function first.