1 Convergence Part 2

Definition: Dirichlet Kernel

We define **Dirichlet Kernel** to be:

$$D_N\left(\frac{\pi u}{L}\right) = \frac{1}{2} + \sum_{n=1}^{N} \cos\left(\frac{n\pi u}{L}\right).$$

Property.

1) we have $\frac{1}{L}\int_{-L-\delta}^{L-\delta} D_N\left(\frac{\pi u}{L}\right) du = 1$ for any $\delta \in \mathbb{R}$.

2)
$$D_N\left(\frac{\pi u}{L}\right) = \frac{\sin[(N+\frac{1}{2})\frac{\pi}{L}u]}{2\sin\left(\frac{\pi}{2L}u\right)}.$$

Proof

1) Prove by direction integration.

2) use $2\sin(\alpha)\cos(\beta) = \sin(\beta+\alpha) - \sin(\beta-\alpha)$ to show

$$\sin\left(\frac{u}{2}\right) + \sum_{n=1}^{N} 2\sin(\frac{u}{2})\cos(nu) = \sin[(N+\frac{1}{2})u].$$

The sum will telescope away.

□

We will prove pointwise convergence first.

Recall that the adjusted function $\tilde{\tilde{f}}$ just average the discontinuities.

Theorem: Dirichlet

Suppose $f(x)$ is a piecewise smooth function on $[-L, L]$ and let $\tilde{f}(x)$ denote the periodic extension of the adjusted function. For any fixed integer $N > 0$ and at each point x, we can define the Nth partial sum of the Fourier Series representing $f(x)$ as

$$S_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi x}{L}\right).$$

then

$$\text{F.S.}[f](x) = \tilde{f}(x) \quad \forall\ x \in [-L, L],$$

and

1) If $\tilde{f}(x)$ is continuous at any x_0 then

$$\lim_{N\to\infty} S_N(x_0) = \tilde{f}(x_0).$$

2) If $\tilde{f}(x)$ is discontinuous at any real x_0 then

$$\lim_{N\to\infty} S_N(x_0) = \frac{\tilde{f}(x_0^-) + \tilde{f}(x_0^+)}{2}.$$

That is, $\text{F.S.}[f](x) = \tilde{\tilde{f}}(x) \quad \forall\ x.$

Proof: Pointwise Convergence

Given $x_0 \in \mathbb{R}$, we will use the Riemann-Lebesgue Lemma, so we need to first write $S_N(x_0)$ in integral form. Note that:

$$a_n \cos\left(\frac{n\pi x}{L}\right) = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right) dt.$$

and in general, for each $n \geq 1$, we can write the nth element in the integral form:

$$\begin{aligned}
a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right) \\
&\quad + \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dt \\
&= \frac{1}{L}\int_{-L}^{L} f(t)\left[\cos\left(\frac{n\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + \sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right] dt \\
&= \frac{1}{L}\int_{-L}^{L} f(t)\left[\cos\left(\frac{n\pi(t-x)}{L}\right)\right] dt
\end{aligned}$$

Thus, we have

$$\begin{aligned}
S_N(x_0) &= a_0 + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x_0}{L}\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi x_0}{L}\right) \\
&= \frac{1}{2L}\int_{-L}^{L} f(t)dt + \sum_{n=1}^{N}\frac{1}{L} f(t)\left[\cos\left(\frac{n\pi(t-x_0)}{L}\right)\right] dt \\
&= \frac{1}{L}\int_{-L}^{L} f(t)\left[\frac{1}{2} + \sum_{n=1}^{N}\cos\left(\frac{n\pi(t-x_0)}{L}\right)\right] dt
\end{aligned}$$

we now apply the Dirichlet Kernel result to $S_N(x_0) - \tilde{f}(x_0)$.

$$S_N(x_0) = \frac{1}{L}\int_{-L}^{L} f(t) D_N\left(\frac{\pi(t-x_0)}{L}\right) dt$$

Now, using substitution $u = t - x_0$, $t = u + x_0$, we have

$$\frac{1}{L}\int_{-L-x_0}^{L-x_0} D_N\left(\frac{\pi u}{L}\right) du = 1 \Rightarrow \frac{1}{L}\int_{-L}^{L} D_N\left(\frac{\pi(t-x_0)}{L}\right) dt = 1.$$

so multiplying both sides of the equation by $\tilde{f}(x_0)$ yields

$$1 = \frac{1}{L}\int_{-L}^{L} D_N\left(\frac{\pi u}{L}\right) dt \Rightarrow \tilde{f}(x_0) = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x_0) D_N\left(\frac{\pi(t-x_0)}{L}\right) dt.$$

then applying Property 2 of Dirichlet Kernel yields

$$\begin{aligned}
S_N(x_0) - \tilde{f}(x_0) &= \frac{1}{L}\int_{-L}^{L} (f(t) - \tilde{f}(x_0) D_N\left(\frac{\pi(t-x_0)}{L}\right) dt \\
&= \frac{1}{L}\int_{-L}^{L} (f(t) - \tilde{f}(x_0)) \frac{\sin\left((N+\frac{1}{2})\frac{\pi}{L}(t-x_0)\right)}{2\sin(\frac{\pi}{2L}(t-x_0))}
\end{aligned}$$

Finally we will apply the Riemann-Lebesgue Lemma. Note that if we let $u = t - x_0$ and $M = (N + \frac{1}{2})\frac{\pi}{L}$, we have

$$\frac{1}{L}\int_{-L-x_0}^{L-x_0} \frac{f(u+x_0) - \tilde{f}(x_0)}{2\sin\left(\frac{\pi}{2L}u\right)} \sin(Mu).$$

Denote the quotient as $Q(u)$. Then the lemma implies that

$$\int_{-L-x_0}^{L-x_0} Q(u)\sin(Mu)du \to 0 \text{ as } M \to \infty.$$

provided that $Q(u)$ is piecewise smooth. We proceed to show this. First, we show $Q(u)$ is piecewise continuous.

Since the quotient of two continuous functions is continuous where defined we claim that $Q(u)$, being the quotient of two piecewise continuous functions is piecewise continuous on its domain. Consider the denominator of $Q(u)$ and its roots. WLOG (due to periodicity), assume that $x_0 \in [-L, L]$.

Case (1). If $x_0 \in (-L, L)$, then since $u \in [-L - x_0, L - x_0]$, we can show that $\sin\left(\frac{\pi}{2L}u\right) = 0 \Leftrightarrow u = 0$, so we need to examine the limit $\lim_{u \to 0} Q(u)$. Note that since $u + x_0 \in [-L, L]$ we have $f(u + x_0) = \tilde{f}(u + x_0)$ so

$$Q(u) = \frac{f(u+x_0) - \tilde{f}(x_0)}{2\sin\left(\frac{\pi}{2L}u\right)} = \frac{\tilde{f}(u+x_0) - \tilde{f}(x_0)}{2\sin\left(\frac{\pi}{2L}u\right)}.$$

Using L'Hopital's Rule,

$$\lim_{u\to 0} Q(u) = \lim_{u \to 0} \frac{\tilde{f}'(u+x_0)}{\frac{\pi}{L}\cos\left(\frac{\pi}{2L}u\right)} = \tilde{f}'(x_0)\frac{L}{\pi} < \infty.$$

Since $x_0 \in (-L, L)$, $f'(x_0)$ is well-defined. This implies there is a removable discontinuity at $u = 0$ so $Q(u)$ is piecewise continuous if $|x_0| < L$.

Case (2). Suppose $|x_0| = L$, and WLOG assume $x_0 = -L$ so $u \in [0, 2L]$. Notice

$$\sin\left(\frac{\pi}{2L}u\right) = 0 \Rightarrow u = 0 \text{ or } u = 2L.$$

So we only need to check these two cases. As before, since $u + x_0 \in [-L, L]$ we can interchange f and $\tilde{f}$. By the continuity and differentiability of $\tilde{f}(x)$ (at 0), we can show $\lim\limits_{u\to 0^+} Q(u) = \lim\limits_{u\to 0} Q(u) = \tilde{f}(x_0)\frac{L}{\pi}$ so there is a removable discontinuity at $u = 0$.

At $u = 2L$, by periodicity, we have $\tilde{f}(x_0 + 2L) = \tilde{f}(x_0)$ so again using L'Hopital's Rule:

$$\lim_{u\to 2L^-} Q(u) = \lim_{u\to 2L^-} \frac{\tilde{f}'(u + x_0)}{\frac{\pi}{L}\cos\left(\frac{\pi}{2L}u\right)} = \tilde{f}'(2L + x_0)\frac{-L}{\pi} = \tilde{f}'(x_0)\frac{-L}{\pi} < \infty.$$

(Jaden: note $f'(x_0 = -L)$ is only defined if $\tilde{f}(x)$ is continuous on the entire $[-L, L]$. This doesn't work for piecewise.) Now we have established that $Q(u)$ is piecewise continuous. To show $Q'(u)$ is also piecewise continuous we leave it to the readers. So $Q(u)$ is piecewise smooth. By the lemma, $S_N(x_0) - \tilde{f}(x_0) \to 0$ as $N \to \infty$. That is

$$\lim_{N\to\infty} S_N(x_0) = \tilde{f}(x_0) \Leftrightarrow \text{ F.S.}[f](x_0) = \tilde{f}(x_0).$$

Thus we have established the pointwise convergence.

$\square$