1 Orbits, Cycles, and the Alternating Groups

Note (equivalence relation). Reflexive: aRa for all $a \in S$. Symmetric: If aRb then bRa. Transitive: If aRb and bRc then aRc.

Example.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 7 & 4 & 1 & 5 & 2 \end{pmatrix}.$$

See iPad for orbits. Orbits of σ (sets) are: $\{1,3,6\}, \{2,8\}, \{4,5,7\}$. They are disjoint, their union is the whole set, they are a partition of $\{1,2,\ldots,8\}$.

Definition

We define an equivalence relation on $\{1,\ldots,8\}$. $a \sim b$ if $b = \sigma^n(a) for some n \in \mathbb{Z}$.

Claim. $\sim$ is an equivalence relation.

Proof

Reflexive: Take $n = 0$, $a = \sigma^0(a)$.
Symmetric: If $b = \sigma^n(a)$ then $a = \sigma^{-n}(b)$.
Transitive: If $b = \sigma^n(a)$ and $c = \sigma^m(b)$ then $c = \sigma^{n+m}(a)$. $\square$

Notation (cycles). (1,3,6) or (1 3 6) means "1 maps to 3 which maps to 6, which maps back to 1". This is the same as (3 6 1) and (6 1 3). We prefer to use (1 3 6) where the smallest number to be in the front. This is called a cycle of length 3 or "3-cycle".

Example (other cycles of σ)**.** (2 8) = (8 2) "2-cycle (transposition)" and (4 7 5) = (7 5 4) = (5 4 7) "3-cycle".

Notation. (4 7 5) means

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 7 & 4 & 6 & 5 & 8 \end{pmatrix}.$$

So there is one orbit of length 3 and others are length 1. This permutation would be called "3-cycle". If a permutation has two orbits with more than one element, then it's not a cycle.

Example.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 7 & 4 & 1 & 5 & 2 \end{pmatrix} = (1\ 3\ 6)(2\ 8)(4\ 7\ 5).$$

These are disjoint cycles (no elements are mentioned more than once). Any permutation can be written as a product of disjoint cycles.

Example (9.10)**.** S_6. Consider

$$(1\ 4\ 5\ 6)(2\ 1\ 5) = (1\ 6)(2\ 4\ 5)(3).$$

Where we start with the smallest number and we feed the number to the other cycle and if it's not mentioned then we come back to the first cycle and stop after one iteration. If the number is not appeared in any cycle we close the parenthesis.

Note. Disjoint cycles commute.

$$(2\ 1\ 5)(1\ 4\ 5\ 6) = (1\ 4\ 2)(3)(5\ 6).$$