Definition: simple group

A group is **simple** if it is nontrivial and it has no **NORMAL** subgroups other than itself and the trivial subgroup.

Claim. $\mathbb{Z}_p$ is a simple group if p is prime. By Lagrange, there aren't any subgroups other than the trivial group and itself. These are the only abelian simple groups!

Theorem: 15.15

The alternating groups A_n for $n \geq 5$ are simple and (nonabelian).

Note. A_4 has a normal subgroup of order 4: $\{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$

Note. $(1\ 2\ 3\ 4\ 5) \in A_5$, 5-cycles are even, $(1\ 5)(1\ 4)(1\ 3)(1\ 2)$. $\langle(1\ 2\ 3\ 4\ 5)\rangle$ is a subgroup of A_5 of order 5. A_5 has many subgroups but only A_5, e are normal.

Remark. Finite simple groups were classified in 1981. The largest "sporadic group" is the Monster. It is NOT the largest nonabelian group because A_n can as large as we like.

Definition: center

Let G be a group. Then **center** (zentrum) of G, $Z(G)$, is the set of elements of G that commute with everything. That is,

$$Z(G) = \{x \in G : xg = gx \ \forall \ g \in G\}.$$

Example. $Z(D_4) = \{\text{ identity, rotation by } 180\}$

$$Z(D_n) = \begin{cases} \text{trivial group if n is odd} \\ \{e, \text{ identity, rotation by } 180\} \end{cases}$$

Note. If G is abelian then $Z(G) = G$.

Note. $Z(S_n)$ if $n \leq 3$ is trivial.

Theorem

$$Z(G) \leq G.$$

Proof

$e \in Z(G)$. If $x \in Z(G)$ then $x^{-1} \in Z(G)$.

$$xg = gx$$
$$x^{-1}xgx^{-1} = x^{-1}gxx^{-1}$$
$$gx^{-1} = x^{-1}g$$

If $x, y \in Z(G)$, then show it's closed:

$$xyg = xgy = gxy.$$

To show it's normal, let $g \in G$ and $x \in Z(G)$. Show $gx^{-1}g^{-1} \in Z(G)$. Need to know it is a subgroup which we just proved.

$$gxg^{-1} = xgg^{-1} = x \in Z(G).$$

Thus, $Z(G) \trianglelefteq G$. $\square$

Example. What is $Z(A_5)$? It is $\{e\}$ or A_5 because A_5 is simple and $Z(A_5) \trianglelefteq A_5$. Since A_5 is nonabelian, so $Z(A_5) \neq A_5$, so $Z(A_5) = \{e\}$.

Note. If a nonabelian group, the center is definitely not itself.

Definition: commutator

A **commutator** is an element of for $aba^{-1}b^{-1}$, $[a, b]$.

Note. Inverse can either be on the left or right. It doesn't matter.

Definition: commutator subgroup

The **commutator subgroup** (derived subgroup) $C(G)$, (or G') of G, is the subgroup generated by the commutators.

Note. There is no guarantee that products of commutators are commutators.

Theorem

$$C(G) \trianglelefteq G..$$

Proof

$$gaba^{-1}b^{-1}g^{-1} = gag^{-1}gbg^{-1}ga^{-1}g^{-1}gb^{-1}g^{-1} = [gag^{-1}, gbg^{-1}].$$

$\square$

Theorem: IMPORTANT

The commutator subgroup, $C(G)$ of G, is the smallest normal subgroup with abelian quotient. That is, $G/C(G)$ is abelian, and if $N \trianglelefteq G$ with G/N abelian, then $C(G) \leq N$.

Example. $G = D_4$, normal subgroups of G: $\{$ rotations$\}$ because index is 2, $\{e\}, D_4, \{e,$ rotation by 180$\}$ because it's the center .

- D_4/D_4 is trivial group. Abelian
- $D_4/\{e\}$ is isomorphic to D_4. Nonabelian.
- $D_4/Z(G)$ has order 4, it's in fact V_4. So abelian.
- $D_4/\{$ rotations$\}$ has order 2, so isomorphic to $\mathbb{Z}_2$ abelian.

Now among the three abelian ones, which is the smallest? Guess: $C(D_4) = \{e,$ rotations by 180$\}$.

Let's check. Normal? Yes because it's the center.

Abelian quotient? Yes because has order 4.

Smallest? If not, the smallest would have to be one of its subgroups. The subgroup of this is just the identity. But identity doesn't work as it yields nonabelian group.

Intuition. What if $[a, b] = e$?

$$aba^{-1}b^{-1} = e \Rightarrow aba^{-1} = b \Rightarrow ab = ba.$$

Note. If G is abelian, $C(G) = \{e\}$ and vice versa. A group is abelian iff it's center is the whole thing.

Example. What is $C(A_5)$? It's A_5 since $\{e\}$ yields A_5 which is nonabelian.