APPM4350: Fourier PDE

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Contents

1 Review

1.1 2nd order ODE

Example. Solve

$$\frac{d^2y}{dx^2} = \lambda y.$$

The general solution is:

$$y(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}.$$

Or we can write it as:

$$y(x) = c_1 \cosh(\sqrt{\lambda}x) + c_2 \sinh(\sqrt{\lambda}x).$$

Note. Hyperbolic functions have easy derivatives and nice for initial conditions.

Definition: linear independence

The functions $y_1(x), \ldots, y_n(x)$ are **linearly independent** if

$$c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) = 0 \Rightarrow c1 = c2 = \ldots = c_n = 0.$$

1.2 The complex plane

The *modulus* of a complex number $a + bi$ is

$$|z| = \sqrt{z \cdot \overline{z}} = \sqrt{a^2 + b^2}.$$

1.3 Euler's formula

Proof

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \dots \\
&= 1 + i\theta + \frac{-\theta^2}{2!} + \frac{-i\theta^3}{3!} + \frac{\theta^4}{4!} + \dots \\
&= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots) + i(\theta - \frac{\theta^3}{3!} + \dots) \\
&= \cos(\theta) + i\sin(\theta)
\end{aligned}
$$

□

Note. $\left|e^{i\theta}\right| = 1$ so it's on the unit circle. Moreover,

$$z = a + ib = \rho\cos(\theta) + i\rho\sin(\theta) = \rho e^{i\theta}.$$

where $\rho = \sqrt{a^2 + b^2}$ and $\tan(\theta) = \frac{b}{a}$.

2 Fourier Series and Orthogonal Vectors (ch.1 + 2)

Definition: L2 inner product

Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$, we defined the $\boldsymbol{L^2}$**-inner product** on $[a, b]$ to be

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx.$$

with the corresponding L^2 norm,

$$\|f\|_2 = \sqrt{\langle f, f\rangle} = \left(\int_a^b f^2(x)dx\right)^{\frac{1}{2}}.$$

Definition: Fourier basis

Suppose $-\pi \leq z \leq \pi$, the **Fourier basis** is defined as

$$\{1, \cos(z), \sin(z), \cos(2z), \sin(2z), \ldots\}.$$

This is an infinite, mutually orthogonal basis of the vector space of continuous functions on $[-\pi, \pi]$.

Definition: projection

Suppose $\{\mathbf{e}_1, \mathbf{e}_2 \ldots\}$ are an orthogonal basis, then $v_i = \frac{\langle \mathbf{v}, \mathbf{e}_i\rangle}{\|\mathbf{e}_i\|^2}$.

Definition: Fourier series

Suppose $f(z)$ is defined on $[-\pi, \pi]$ and is in "the proper space of functions" (see Ch.5 notes) then $f(z)$ has a Fourier series is of the form

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nz) + \sum_{n=1}^{\infty} b_n \sin(nz).$$

where

$$b_n = \frac{\langle f(z), \sin(nz)\rangle}{\|\sin(nz)\|_2^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(z)\sin(nz)dz \text{ for } n = 1, 2, \ldots.$$

$$a_n = \frac{\langle f(z), \cos(nz)\rangle}{\|\cos(nz)\|_2^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(z)\cos(nz)dz \text{ for } n = 1, 2, \ldots.$$

$$a_0 = \frac{\langle f(z), 1\rangle}{\|1\|_2^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z)dz.$$

Example (Fourier Series)**.**

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nz) + \sum_{n=1}^{\infty} b_n \sin(nz)$$

To find a_8,

$$f(z)\cos(8t) = a_0\cos(8z) + \sum_{n=1}^{\infty} a_n \cos(nz)\cos(8z) + \sum_{n=1}^{\infty} \sin(nz)\cos(8z)$$

$$\int_{-\pi}^{\pi} f(z)\cos(8z)dz = a_0\int_{-\pi}^{\pi}\cos(8z)dz + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos(nz)\cos(8z)dz + \sum_{n=1}^{\infty} b_n\int_{-\pi}^{\pi}\sin(nz)\cos(8z)dz$$

$$\int_{-\pi}^{\pi} f(z)\cos(8z)dz = 0 + a_8\int_{-\pi}^{\pi}\cos^2(8z)dz + 0$$

So we can solve for a_8.

2.1 Taylor vs Fourier

Taylor:

1) T.S.$[f](z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ where $c_n = \frac{f^{(n)}(a)}{n!}$
2) $f(z)$ is analytic at the point z iff $f(z) =$T.S. $[f](z)$
3) RoC might be small
4) need to be differentiable
5) the basis is not orthogonal

Fourier:

1) F.S.$[f](z) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nz) + \sum_{n=1}^{\infty} b_n\sin(nz), -\pi \le z \le \pi$.

2) If it converges for $z \in [-\pi, \pi]$ then it converges for all $z \in (-\infty, \infty)$.

3) coefficients are found by integration which is a much weaker assumption.

4) basis is orthogonal.

5) $\tilde{f}_M(z)$ is the truncated Fourier series after M terms.

Example. $f(z) = |z|, \pi \leq z \leq \pi$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(z)\sin(nz)dz = \frac{1}{\pi}\int_{-\pi}^{\pi} |z|\sin(nz)dz = \frac{1}{\pi}\cdot 0 = 0.$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |z|dz = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -zdz + \int_{0}^{\pi} zdz\right] = \frac{\pi^2}{2\pi} = \frac{\pi}{2}.$$

Using integration by parts, we can show for even and odd values of $n \geq 1$ that

$$a_n = a_{2m} = 0 \text{ and } a_n = a_{2m-1} = -\frac{4}{\pi(2m-1)^2}.$$

$$\text{F.S.}[f](z) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos[(2m-1)z] = \frac{\pi}{2} - \frac{4}{\pi}\cos(z) - \frac{4}{9\pi}\cos(3z) - \ldots.$$

Convergence:

$$\begin{aligned}\left|\frac{\pi}{2}\right| + \frac{4}{9\pi}|\cos(3z)| + \ldots &\leq \frac{\pi}{2} + \frac{4}{\pi} + \frac{4}{9\pi} + \ldots \\ &= \frac{\pi}{2} + \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\end{aligned}$$

So the series is absolutely convergent.

We can use the integral test to bound the error term.

$$\begin{aligned}|\text{F.S.}[f](z) - \tilde{f}_M(z)| &= \left|-\frac{4}{\pi}\sum_{n=M}^{\infty}\frac{1}{(2m-1)^2}\cos([2n-1]z)\right| \\ &\leq \frac{4}{\pi}\sum_{n=M}^{\infty}\frac{1}{(2n-1)^2} \\ &\leq \frac{4}{\pi}\int_{M}^{\infty}\frac{1}{(2x-1)^2}dx\end{aligned}$$

3 Chapter 3

Definition: Periodicity

Suppose $g(z)$ is defined for all real numbers, if there exists a number $p > 0$ such that

$$g(z) = g(z + p), \quad \forall z \in \mathbb{R}.$$

then $g(z)$ is said to be a **periodic function**.

Note.

1) If $g(z)$ is periodic then it has many periods $p, 2p, \ldots$ we use the shortest period $p > 0$.
2) if $g_1(z)$ and $g_2(z)$ have period p then so does $h(z) = ag_1(z) \pm bg_2(z)$ for any $a, b \in \mathbb{R}$
3) constant function is trivially periodic for any $p > 0$

All Fourier basis vectors are at least 2π-periodic.

What does a Fourier series of a function represent?

Definition: periodic extension

Let $f(z)$ be a function defined on $[-L, L]$ such that $f(-L) = f(L)$. Define the **periodic extension** of $f(z)$ to be the unique periodic function $\tilde{f}(z)$ of period $2L$ such that $\tilde{f}(z) = f(z)$, for $-L \leq z \leq L$.

Note. Periodic extension requires the value at the end points be equal. Otherwise, the function would have two different outputs at the end points, which makes it not well-defined. A simple fix is to restrict the domain (remove one end point).

Definition: generalized Fourier series

Given $y = F(x)$, where $-L \leq x \leq L$ for some positive real number $L > 0$, define the inner product

$$\langle f(x), g(x)\rangle = \int_{-L}^{L} f(x)g(x)dx \text{ with norm } \|f\|_2.$$

We can show that the countably infinite set

$$\left\{1, \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right), \dots\right\}.$$

is a set of orthogonal functions with respect to the inner product given above. If we assume that

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n + \cos\left(\frac{n\pi x}{L}\right) + \sin\left(\frac{n\pi x}{L}\right).$$

then, by the projection formula for finding coordinates, the corresponding Fourier coefficients are

$$\begin{aligned}
a_0 &= \frac{\langle F(x), 1\rangle}{\|1\|^2} = \frac{1}{2L}\int_{-L}^{L} F(x)dx \\
a_n &= \frac{\langle F(x), \cos\left(\frac{n\pi x}{L}\right)\rangle}{\left\|\cos\left(\frac{n\pi x}{L}\right)\right\|^2} = \frac{1}{L}\int_{-L}^{L} F(x)\cos\left(\frac{n\pi x}{L}\right)dx \\
b_n &= \frac{\langle F(x), \sin\left(\frac{n\pi x}{L}\right)\rangle}{\left\|\sin\left(\frac{n\pi x}{L}\right)\right\|^2} = \frac{1}{L}\int_{-L}^{L} F(x)\sin\left(\frac{n\pi x}{L}\right)dx
\end{aligned}$$

Note. The generalization was achieved using a change of variables: let $z = \frac{\pi x}{L}$. The L^2-inner product on $[-L, L]$ and on $-\pi, \pi$ only differ by $\frac{L}{\pi}$.

- restrict domain
- redefine the value at end point so that $F(L) = F(-L)$.

Example. $F(x) = x, -L \leq x \leq L$. Redefine:

$$F'(-L) = F'(L) = \frac{F(-L) + F(L)}{2} = \frac{-L + L}{2} = 0.$$

3.1 Convergence

Definition

Suppose $f(x)$ and $g(x)$ are defined on $[-L, L]$, we say $f(x)$ and $g(x)$ are **equivalent** or **equal almost everywhere** (denoted as $f(x) \sim g(x)$). If

$f(x) = g(x)$ for all $x \in (-L, L)$ except possibly at finite set of points $\{x_1, x_2, \ldots\}$ (in fact measure-zero sets) at which $f(x_i) \neq g(x_i)$ where $|f(x_i)| < \infty$ and $|g(x_i)| < \infty$ for $i \leq i \leq k$.

Note. If $f(x) \sim g(x)$ then F.S. $[f](x) =$ F.S.$[g](x)$ but $f(x) \neq g(x)$. Hence the need for restriction.

Definition: dense

Let A be a non-empty set and suppose B is a subset of A. We say set B is **dense** in A if any point of a of A can be written as a limit of points from B, that is, if for any $a \in A$ there exists a sequence points (b_n) from B s.t. $\lim_{n\to\infty} b_n = a$.

Definition: trigonometric polynomial

A **trigonometric polynomial** is a finite sum of the form

$$S_N = a_0 + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Note. The Nth partial sum of a Fourier Series is a trigonometric polynomial.

Proposition

The set of trigonometric polynomials is dense in the set of continuous functions.

Intuition. The analogy is that rational numbers are dense in real numbers.

3.2 Periodicity

It "smooths" the bad end points or removable discontinuity. see lecture notes.

4 Convergence Part 1

Definition

1) $f(x)$ is **continuous** at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.
2) $f(x)$ is **piecewise continuous** on the interval $[a, b]$ if $f(x)$ is defined and continuous at all but a finite number of points $x_i \in [a, b], 1 \leq i \leq N$, where $f(x)$ has either a jump discontinuity or a removable discontinuity and $f(a^+) = \lim_{x \to a^+} f(x)$ and $f(b^-) = \lim_{x \to b^-} f(x)$ exist.
3) A differentiable function $f(x)$ is **piecewise smooth** on $[a, b]$ if both $f(x)$ and $f'(x)$ are piecewise continuous on $[a, b]$, *i.e.* $f \in$ piecewise $\mathcal{C}^1$.
4) The function $f(x)$ is **continuous piecewise smooth** on $[a, b]$ if $f(x)$ is piecewise smooth on $[a, b]$ and has no discontinuities in (a, b).

Note.

a) $f'(x)$ will not be continuous at the discontinuities of $f(x)$ (if any) and may have its own additional discontinuities.
b) If $f'(x)$ is piecewise continuous on (a, b), then $f(x)$ is also piecewise continuous on (a, b). Why? (Jaden: f' is piecewise continuous on $(a, b) \Rightarrow f'$ exists on $(a, b) \Rightarrow f$ is piecewise differentiable on $(a, b) \Rightarrow f$ is piecewise continuous on (a, b).)
c) "finite number of points" can be replaced with "a collection of points that is measure-zero" for full generality.

Example. $f(x) = \frac{1}{x}$ where $x \neq 0$ and $x \in [-1, 1]$ is not piecewise continuous because the left and right limits do not exist at $x = 0$.

Example. Consider $x \in [-1, 1]$, if

$$f(x) = |x|^{\frac{1}{4}} = \begin{cases} x^{\frac{1}{4}} & \text{if } x \in [0, 1] \\ (-x)^{\frac{1}{4}} & \text{if } x \in [-1, 0) \end{cases}$$

$$f'(x) = \begin{cases} \frac{1}{4}x^{-\frac{3}{4}} & \text{if } x \in (0,1) \\ -\frac{1}{4}(-x)^{-\frac{3}{4}} & \text{if } x \in (-1,0) \end{cases}$$

$f(x)$ is piecewise continuous on $[-1, 1]$ but the derivative is not piecewise continuous.

Example. $f(x) = |x|$ is piecewise smooth.

To prove convergence, we want:

1) $|a_0| < \infty$
2) $a_n \to 0, b_n \to 0$ as $n \to \infty$. Otherwise the series will diverge.

   Note. As $n \to \infty$, the periodicity goes to zero. So $\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)$ is the cumulative rapidly oscillating positive and negative area bounded by $|f(x)|$. Intuitively they cancel each other out as oscillation increases to infinity.

Restrictions

1 If $f(-L) = f(L)$, then we require that $\tilde{f}(x)$, the periodic extension of $f(x)$, be piecewise smooth on $[-L, L]$.

2 If $f(-L) \neq f(L)$, then redefine $f_{new}(-L) = f_{new}(L) = \frac{f(-L)+f(L)}{2}$. We denote the adjusted function $\overline{f}(x)$. Now we require that the periodic extension of the adjusted function, $\tilde{\overline{f}}(x)$, be piecewise smooth on $[-L, L]$.

Notation. We will use $\tilde{f}$ instead of $\tilde{\overline{f}}$ for brevity.

Lemma: 1

The set of real functions defined on $[-L, L]$ that satisfy Restriction 1 and 2 form a vector space (satisfies linearity).

Lemma: 2

If $f(x)$ satisfies Restriction 1 and 2 then there exists a number $0 < M < \infty$ such that

$$|\tilde{f}(x)| < M \quad \forall x \in \mathbb{R}.$$

Proof

Any closed interval on $\mathbb{R}$ is compact. Since f is piecewise continuous, the image of a compact set is also compact. And compact set in $\mathbb{R}$ means it's closed and bounded by Heine-Borel Theorem. $\square$

Lemma: 3

If $f(x)$ satisfies Restriction 1 and 2 and if M is the positive constant from Lemma 2 then the Fourier coefficients satisfy:

$$|a_0| < M, |a_n| < 2M, \text{ and } |b_n| < 2M.$$

Proof

Given $a \in \mathbb{R}$, $-|a| \leq a \leq |a|$. Thus given $f(x)$ defined on $[-L, L]$,

$$-|f(x)| \leq f(x) \leq |f(x)| \Rightarrow -\int_{-L}^{L} |f(x)|dx \leq \int_{-L}^{L} f(x)dx \leq \int_{-L}^{L} |f(x)|dx$$
$$\Rightarrow \left|\int_{-L}^{L} f(x)dx\right| \leq \int_{-L}^{L} f(x)dx$$

$$\begin{aligned}
|a_n| &= \left|\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx\right| \\
&\leq \frac{1}{L}\int_{-L}^{L} |f(x)|\left|\cos\left(\frac{n\pi x}{L}\right)\right|dx \\
&\leq \frac{1}{L}\int_{-L}^{L} |f(x)|dx \\
&\leq \frac{1}{L}\int_{-L}^{L} M dx = \frac{M}{L}2L = 2M
\end{aligned}$$

That is, $|a_n| < 2M$. Repeat for $b_n < 2M$ and $a_0 < M$. $\square$

Lemma: Riemann-Lebesgue (special case)

If $f(x)$ satisfies Restriction 1 and 2 then $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$.

Proof

Suppose $\tilde{\tilde{f}}(x)$, denote any discontinuities as $\{x_1, x_2, \ldots x_N\}$ with $x_0 = -L$ and $x_{N+1} = L$. Then we can write

$$a_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)dx = \sum_{p=1}^{N+1}\frac{1}{L}\int_{x_{p-1}}^{x_p}\tilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

Now let

$$\alpha_{n,p} = \frac{1}{L}\int_{x_{p-1}}^{x_p}\tilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

where $\tilde{f}(x) = u$ and $\cos\left(\frac{n\pi x}{L}\right)dx = dv$. Then using integration by parts,

$$\begin{aligned}
\alpha_{n,p} &= \frac{1}{L}\left(uv\Big|_{x_{p-1}}^{x_p} - \int_{x_{p-1}}^{x_p} vdu\right)\\
|a_{n,p}| &= \frac{1}{n\pi}\left|\tilde{f}(x)\sin\left(\frac{n\pi x_p}{L}\right) - \tilde{f}(x)\sin\left(\frac{n\pi x_{p-1}}{L}\right) - \int_{x_{p-1}}^{x_p}\tilde{f}'(x)\sin\left(\frac{n\pi x}{L}\right)dx\right|\\
&\leq \frac{1}{n\pi}\Bigg[\left|\tilde{f}(x)\sin\left(\frac{n\pi x_p}{L}\right)\right| + \left|\tilde{f}(x)\sin\left(\frac{n\pi x_{p-1}}{L}\right)\right|\\
&\quad + \int_{x_{p-1}}^{x_p}\left|\tilde{f}'(x)\sin\left(\frac{n\pi x}{L}\right)\right|dx\Bigg] \qquad \text{using triangle inequality}\\
&\leq \frac{1}{n\pi}\left|\tilde{f}(x)\right| + \left|\tilde{f}(x)\right| + \int_{x_{p-1}}^{x_p}|\tilde{f}(x)|dx\\
&\leq \frac{1}{n\pi}\cdot C
\end{aligned}$$

where C represents the constant that bounds $\tilde{f}(x)$ and $\tilde{f}'(x)$ on $[-L, L]$. As $n \to \infty$,

$$|\alpha_{n,p}| \to 0 \Rightarrow a_n = \sum_{p=1}^{N+1}\alpha_{n,p} \to 0.$$

$\square$

5 Convergence Part 2

Definition: Dirichlet Kernel

We define **Dirichlet Kernel** to be:

$$D_N\left(\frac{\pi u}{L}\right) = \frac{1}{2} + \sum_{n=1}^{N} \cos\left(\frac{n\pi u}{L}\right).$$

Property.

1) we have $\frac{1}{L}\int_{-L-\delta}^{L-\delta} D_N\left(\frac{\pi u}{L}\right) du = 1$ for any $\delta \in \mathbb{R}$.

2)
$$D_N\left(\frac{\pi u}{L}\right) = \frac{\sin[(N+\frac{1}{2})\frac{\pi}{L}u]}{2\sin\left(\frac{\pi}{2L}u\right)}.$$

Proof

1) Prove by direct integration.

2) use $2\sin(\alpha)\cos(\beta) = \sin(\beta+\alpha) - \sin(\beta-\alpha)$ to show

$$\sin\left(\frac{u}{2}\right) + \sum_{n=1}^{N} 2\sin(\frac{u}{2})\cos(nu) = \sin[(N+\frac{1}{2})u].$$

The sum will telescope away.

□

We will prove pointwise convergence first.

Recall that the adjusted function $\tilde{\tilde{f}}$ just average the discontinuities.

Theorem: Dirichlet

Suppose $f(x)$ is a piecewise smooth function on $[-L, L]$ and let $\tilde{f}(x)$ denote the periodic extension of the adjusted function. For any fixed integer $N > 0$ and at each point x, we can define the Nth partial sum of the Fourier Series representing $f(x)$ as

$$S_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi x}{L}\right).$$

then

$$\text{F.S.}[f](x) = \tilde{f}(x) \quad \forall\, x \in [-L, L],$$

and

1) If $\tilde{f}(x)$ is continuous at any x_0 then

$$\lim_{N\to\infty} S_N(x_0) = \tilde{f}(x_0).$$

2) If $\tilde{f}(x)$ is discontinuous at any real x_0 then

$$\lim_{N\to\infty} S_N(x_0) = \frac{\tilde{f}(x_0^-) + \tilde{f}(x_0^+)}{2}.$$

That is, $\text{F.S.}[f](x) = \tilde{\tilde{f}}(x) \quad \forall\, x.$

Proof: Pointwise Convergence

Given $x_0 \in \mathbb{R}$, we will use the Riemann-Lebesgue Lemma, so we need to first write $S_N(x_0)$ in integral form. Note that:

$$a_n \cos\left(\frac{n\pi x}{L}\right) = \frac{1}{L}\int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dt.$$

and in general, for each $n \geq 1$, we can write the nth element in the integral form:

$$\begin{aligned}
a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right) \\
&\quad + \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dt \\
&= \frac{1}{L}\int_{-L}^{L} f(t)\left[\cos\left(\frac{n\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + \sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right] dt \\
&= \frac{1}{L}\int_{-L}^{L} f(t)\left[\cos\left(\frac{n\pi(t-x)}{L}\right)\right] dt
\end{aligned}$$

Thus, we have

$$\begin{aligned}
S_N(x_0) &= a_0 + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x_0}{L}\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi x_0}{L}\right) \\
&= \frac{1}{2L}\int_{-L}^{L} f(t)dt + \sum_{n=1}^{N}\frac{1}{L} f(t)\left[\cos\left(\frac{n\pi(t-x_0)}{L}\right)\right] dt \\
&= \frac{1}{L}\int_{-L}^{L} f(t)\left[\frac{1}{2} + \sum_{n=1}^{N}\cos\left(\frac{n\pi(t-x_0)}{L}\right)\right] dt
\end{aligned}$$

we now apply the Dirichlet Kernel result to $S_N(x_0) - \tilde{f}(x_0)$.

$$S_N(x_0) = \frac{1}{L}\int_{-L}^{L} f(t) D_N\left(\frac{\pi(t-x_0)}{L}\right) dt$$

Now, using substitution $u = t - x_0$, $t = u + x_0$, we have

$$\frac{1}{L}\int_{-L-x_0}^{L-x_0} D_N\left(\frac{\pi u}{L}\right) du = 1 \Rightarrow \frac{1}{L}\int_{-L}^{L} D_N\left(\frac{\pi(t-x_0)}{L}\right) dt = 1.$$

so multiplying both sides of the equation by $\tilde{f}(x_0)$ yields

$$1 = \frac{1}{L}\int_{-L}^{L} D_N\left(\frac{\pi u}{L}\right) dt \Rightarrow \tilde{f}(x_0) = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x_0) D_N\left(\frac{\pi(t-x_0)}{L}\right) dt.$$

then applying Property 2 of Dirichlet Kernel yields

$$\begin{aligned}
S_N(x_0) - \tilde{f}(x_0) &= \frac{1}{L}\int_{-L}^{L} (f(t) - \tilde{f}(x_0) D_N\left(\frac{\pi(t-x_0)}{L}\right) dt \\
&= \frac{1}{L}\int_{-L}^{L} (f(t) - \tilde{f}(x_0))\frac{\sin\left((N+\frac{1}{2})\frac{\pi}{L}(t-x_0)\right)}{2\sin(\frac{\pi}{2L}(t-x_0))}
\end{aligned}$$

Finally we will apply the Riemann-Lebesgue Lemma. Note that if we let $u = t - x_0$ and $M = (N + \frac{1}{2})\frac{\pi}{L}$, we have

$$\frac{1}{L}\int_{-L-x_0}^{L-x_0} \frac{f(u+x_0) - \tilde{f}(x_0)}{2\sin\left(\frac{\pi}{2L}u\right)} \sin\left(Mu\right).$$

Denote the quotient as $Q(u)$. Then the lemma implies that

$$\int_{-L-x_0}^{L-x_0} Q(u)\sin(Mu)du \to 0 \text{ as } M \to \infty.$$

provided that $Q(u)$ is piecewise smooth. We proceed to show this. First, we show $Q(u)$ is piecewise continuous.

Since the quotient of two continuous functions is continuous where defined we claim that $Q(u)$, being the quotient of two piecewise continuous functions is piecewise continuous on its domain. Consider the denominator of $Q(u)$ and its roots. WLOG (due to periodicity), assume that $x_0 \in [-L, L]$.

Case (1). If $x_0 \in (-L, L)$, then since $u \in [-L - x_0, L - x_0]$, we can show that $\sin\left(\frac{\pi}{2L}u\right) = 0 \Leftrightarrow u = 0$, so we need to examine the limit $\lim_{u\to 0} Q(u)$. Note that since $u + x_0 \in [-L, L]$ we have $f(u + x_0) = \tilde{f}(u + x_0)$ so

$$Q(u) = \frac{f(u+x_0) - \tilde{f}(x_0)}{2\sin\left(\frac{\pi}{2L}u\right)} = \frac{\tilde{f}(u+x_0) - \tilde{f}(x_0)}{2\sin\left(\frac{\pi}{2L}u\right)}.$$

Using L'Hopital's Rule,

$$\lim_{u\to 0} Q(u) = \lim_{u\to 0} \frac{\tilde{f}'(u+x_0)}{\frac{\pi}{L}\cos\left(\frac{\pi}{2L}u\right)} = \tilde{f}'(x_0)\frac{L}{\pi} < \infty.$$

Since $x_0 \in (-L, L)$, $f'(x_0)$ is well-defined. This implies there is a removable discontinuity at $u = 0$ so $Q(u)$ is piecewise continuous if $|x_0| < L$.

Case (2). Suppose $|x_0| = L$, and WLOG assume $x_0 = -L$ so $u \in [0, 2L]$. Notice

$$\sin\left(\frac{\pi}{2L}u\right) = 0 \Rightarrow u = 0 \text{ or } u = 2L.$$

So we only need to check these two cases. As before, since $u+x_0 \in [-L, L]$ we can interchange f and $\tilde{f}$. By the continuity and differentiability of $\tilde{f}(x)$ (at 0), we can show $\lim_{u\to 0^+} Q(u) = \lim_{u\to 0} Q(u) = \tilde{f}'(x_0)\frac{L}{\pi} < \infty$ so there is a removable discontinuity at $u = 0$.

At $u = 2L$, by periodicity, we have $\tilde{f}(x_0 + 2L) = \tilde{f}(x_0)$ so again using L'Hopital's Rule:

$$\lim_{u \to 2L^-} Q(u) = \lim_{u \to 2L^-} \frac{\tilde{f}'(u + x_0)}{\frac{\pi}{L}\cos\left(\frac{\pi}{2L}u\right)} = \tilde{f}'(2L + x_0)\frac{-L}{\pi} = \tilde{f}'(x_0)\frac{-L}{\pi} < \infty.$$

(Jaden: note $f'(x_0)$ for both cases is only defined if $\tilde{f}(x)$ is differentiable for all $x \in [-L, L]$. Additional argument is required to prove it for all piecewise smooth functions. Here we seemed to skip the proof for the cases when x_0 happens to be jumped discontinuities.) Now we have established that $Q(u)$ is piecewise continuous. To show $Q'(u)$ is also piecewise continuous we leave it to the readers. So $Q(u)$ is piecewise smooth. By the lemma, $S_N(x_0) - \tilde{f}(x_0) \to 0$ as $N \to \infty$. That is

$$\lim_{N \to \infty} S_N(x_0) = \tilde{f}(x_0) \Leftrightarrow \text{ F.S.}[f](x_0) = \tilde{f}(x_0).$$

Thus we have established the pointwise convergence.

$\square$

6 Uniform Convergence

Definition: Pointwise Convergence

For every $\varepsilon > 0$ and each $x_0 \in [-L, L]$, there exists a positive, finite integer $N_\varepsilon(x_0)$ such that if $N \geq N_\varepsilon(x_0)$, then

$$|S_N(x_0) - T(x_0)| < \varepsilon.$$

where $S_N(x_0)$ is the Nth partial sum of the Fourier series with $x = x_0$.

Definition: Uniform Convergence

For every $\varepsilon > 0$, there exists a $N_\varepsilon \in \mathbb{N}$ such that

$$|S_N(x) - T(x)| < \varepsilon.$$

for all $x \in [-L, L]$.

Note.

a) Pointwise convergence implies $\lim\limits_{N\to\infty} |S_N(x) - T(x)| = 0$ for all $x \in [-L, L]$.

b) Uniform convergence implies $\lim\limits_{N\to\infty} \max_{-L\leq x\leq L} |S_N(x) - T(x)| = 0$.

Uniform convergence is stronger and implies pointwise convergence.

Example. Suppose $f_n(x) = \frac{x+2}{4n}$ for $n \in \mathbb{N}$ and $x \in [-2, 2]$. Then $(f_n(x))$ converges uniformly to $h(x) = 0$. Note that $(f_n(x))$ is a sequence of constants for each fixed $x_0 \in [-2, 2]$.

To show that it is pointwise convergence, given $x_0 \in [-2, 2]$, we have

$$\lim_{n\to\infty} f_n(x_0) = \lim_{n\to\infty} \frac{x_0 + 2}{n} = 0 \Rightarrow \text{ pointwise convergence.}$$

For uniform convergence, we observe that for any $x \in [-2, 2]$ the maximum vertical separation of $f_n(x)$ from $h(x)$ is $\frac{1}{n}$ for each n (because the maximum difference is achieved at $x = 2$), thus

$$\lim_{n\to\infty} \max_{-2\leq x\leq 2} |f_n(x) - h(x)| = \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \text{ uniform convergence.}$$

Example.

$$g_n(x) \begin{cases} nx & 0 < x \leq \frac{1}{n} \\ 2 - nx & \frac{1}{n} < x \leq \frac{2}{n} \\ 0 & \text{for all other } x \in [-2, 2] \end{cases}$$

then $g_n(x)$ converges pointwise but not uniformly.

Pointwise: Clearly if $x_0 \in [-2, 0]$ then $\lim_{n\to\infty} g_n(x_0) = 0$. If $x_0 \in (0, 2]$ then for any $N > \frac{2}{x_0}$, if $n \geq N$, then $x_0 > \frac{2}{n}$ so $x_0 \in (\frac{2}{n}, 2]$ so $g_n(x_0) = 0$ for all $n \geq N$ so

$$\lim_{n\to\infty} g_n(x_0) = 0.$$

Note that $N > \frac{2}{x_0}$ is obtained by reverse engineering on the scratch paper.

Uniform: the maximum vertical separation of $g_n(x)$ from $h(x)$ is a fixed distance of 1 (at $x = \frac{1}{n}$) for any choice of $n \geq 1$, thus

$$\lim_{n\to\infty} \max_{x\in[-2,2]} |g_n(x) - h(x)| = \lim_{n\to\infty} 1 \neq 0.$$

Hence it doesn't converge uniformly.

Definition: Absolute Convergence

The F.S.$[f](x)$ is **absolutely convergent** if, for every $\varepsilon > 0$, there exists an integer $0 < M_\varepsilon < \infty$ such that

$$0 \leq \sum_{n=M_\varepsilon+1}^{\infty} |a_n| + \sum_{n=M_\varepsilon+1}^{\infty} |b_n| < \varepsilon \qquad \textit{i.e.} \text{ the tail converges absolutely.}$$

Note.

1) if F.S.$[f](x)$ is absolutely convergent then

$$\begin{aligned} 0 &\leq |a_0| + \sum_{n=1}^{\infty} \left|a_n \cos\left(\frac{n\pi x}{L}\right)\right| + \sum_{n=1}^{\infty} \left|b_n \sin\left(\frac{n\pi x}{L}\right)\right| \\ &\leq |a_0| + \sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n| \\ &< \infty \end{aligned}$$

2) if F.S.$[f](x)$ is absolutely convergent then it is uniformly convergent.

3) if F.S.$[f](x)$ is uniformly convergent then it is pointwise convergent.
4) there exist series of functions which are uniformly convergent but not absolutely convergent.

Theorem: Weierstrass M-test

If $(f_n(x))$ is a sequence of functions defined on a set E and (M_n) is a sequence of non-negative numbers such that $|f_n(x)| < M_n$ for all $x \in E$ and $n \geq 0$. Then $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly if $\sum_{n=0}^{\infty} M_n$ converges.

Definition: Gibbs Phenomenon

a) "Gibbs phenomenon" is a persistent overestimation or underestimation of the value of any piece wise smooth function with a jump discontinuity.
b) It occurs in truncated Fourier series of functions with jump discontinuities and does NOT go away as the number of terms is increased.
c) As the number of terms used in increased, the location of the overshoot moves closer and closer to jump discontinuity without ever reaching it.
d) As the number of terms increases, the size of the overshoot approaches a limiting value, proportional to the magnitude of the jump discontinuity with a constant of proportionality that is universal.

Example.

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi, & 0 \leq x \leq \pi \end{cases}$$

$$\text{F.S.}[f](x) = \frac{\pi}{2} + 2\sum_{n=0}^{\infty} \frac{\sin[(2n+1)x]}{2n+1}.$$

The truncated form of the Fourier series has the form

$$\tilde{f_M}(x) = \frac{\pi}{2} + 2\left(\sin(x) + \frac{\sin(3x)}{3} + \ldots + \frac{\sin[(2(M-2)+1)x]}{2(M-2)+1}\right).$$

Intuition. Gibbs phenomenon is the result of the fact that points in the middle of the interval are converging faster than points at the endpoints/discontinuities, due to pointwise convergence.

Note. Gibbs only occur if FS is truncated. Gibbs has about 9% over/undershoot.

Intuition. A sequence of continuous function cannot converge uniformly to a discontinuous function.

- if the adjusted periodic extension $\tilde{f}(x)$ is piecewise smooth on every finite interval but has a jump discontinuity then the Fourier Series of $f(x)$
  a) converges pointwise by Dirichlet's Theorem for pointwise convergence.
  b) will converge at different rates of convergence at each point.
  c) is not uniformly convergent and therefore not absolutely convergent.
  d) exhibits Gibbs Phenomenon in every open interval around a jump discontinuity and does not converge to a continuous function (but does converge).
- A series that converges uniformly will not exhibit Gibbs phenomenon.
- if $\tilde{f}(x)$ is continuous everywhere then we expect absolute convergence.

6.1 Integration and Differentiation of Fourier Series

Theorem: term-by-term integration

Let $\sum_{n=0}^{\infty} f_n(x)$ be defined on $[a,b]$. If each $f_n(x)$ is continuous on $[a,b]$ and if the series $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly to $f(x)$ on $[a,b]$ then

(i) $f(x) = \sum_{n=0}^{\infty} f_n(x)$ is continuous on $[a,b]$

(ii)
$$\int_a^b \sum_{n=0}^{\infty} f_n(x)dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)dx.$$

Theorem: term-by-term differentiation

Let f_n be differentiable functions defined on $[a,b]$ and suppose the series $\sum_{n=0}^{\infty} f_n'(x)$ converges uniformly to a limit $g(x)$ on $[a,b]$. If there exists a point $x_0 \in [a,b]$ where $\sum_{n=0}^{\infty} f_n(x_0)$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly to a differentiable function $f(x)$ satisfying $f'(x) = g(x)$ on $[a,b]$. That is,

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

and

$$f'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} f_n(x)\right] = \sum_{n=0}^{\infty} f_n'(x).$$

Theorem: univerform convergence theorem

Let $f(x)$ be a continuous piecewise smooth function $[-L,L]$ such that $f(-L) = f(L)$. Then F.S.$[f](x)$ converges uniformly to $f(x)$ on $[-L,L]$. That is,

$$\lim_{n\to\infty} \max_{-L\leq x\leq L} |S_N(x) - f(x)| = 0.$$

Pinsky book has much more analysis.

7 Derivation of the Heat Equation 1

7.1 Insulated Road

We model the transfer of thermal energy in a one dimensional rod with ends at $x = 0$ and $x = L$ and where the lateral surface of the rod is insulated perfectly.

Definition: thermal energy density

1) The **thermal energy density** $e(x,t)$ is the amount of thermal energy per unit volume.
2) Consider a thin slice of the rod with cross sectional area A between x and $x + \Delta x$. The heat energy changes in time due only to heat flowing across the edges (x and $x + \Delta x$). If Δx is small then $e(x,t)$ may be approximated as constant throughout the slice so:

$$\text{heat energy in slice } [x, x + \Delta x] = e(x,t) \cdot A \cdot \Delta x.$$

Integrating it yields:

$$\text{Total heat energy in the rod} = \int_0^L e(x,t) A dx.$$

Definition: heat flux

The **heat flux**, $\Phi(x,t)$, is the amount of thermal energy flowing to the right per unit time per unit surface area. If $\Phi(x,t) < 0$ then energy flows to the left.

The heat energy flow per unit time across the boundaries of slice $[x, x + \Delta x]$ with cross sectional surface area A is:

$$\Phi(x,t) \cdot A \text{ (heat gain) } + (-\Phi(x+\Delta x,t) \cdot A \text{ (heat loss) } = -[\Phi(x+\Delta x,t) - \Phi(x,t)] \cdot A.$$

Definition

In the model we allow for **internal sources of energy**. Let $Q(x,t)$ be the heat energy generated per unit volume per unit time within the rod then

$$\text{heat energy per unit time } = Q(x,t) \cdot A \cdot \Delta x.$$

Theorem: Heat Flow Process

The fundamental heat flow process in the rod is conceptually described as:

rate of change of heat energy wrt time = heat energy flowing across boundaries per unit time + heat energy generated inside the rod per unit time.

Now consider any finite sement of the rod (from a to b), then the conservation of heat energy principle given above implies:

$$\frac{d}{dt}\int_a^b e(x,t)Adx = -[\Phi(b,t) - \Phi(a,t)] \cdot A + \int_a^b Q(x,t)Adx.$$

which after canceling $A > 0$ can be rewritten as (by fundamental theorem of calculus):

$$\int_a^b \frac{\partial}{\partial t}e(x,t)dx = -\int_a^b \frac{\partial}{\partial x}\Phi(x,t)dx + \int_a^b Q(x,t)dx.$$

which yields the "Integral Conservation Law"

$$\int_a^b \left[\frac{\partial}{\partial t}e(x,t) + \frac{\partial}{\partial x}\Phi(x,t) - Q(x,t)\right] dx = 0.$$

which holds for any a and b within the rod, and since the integrand is assumed to be continuous, this implies (proof by contradiction):

$$\frac{\partial}{\partial t}e(x,t) + \frac{\partial}{\partial x} - Q(x,t) = 0 \Rightarrow \frac{\partial}{\partial t}e(x,t) = -\frac{\partial}{\partial x}\Phi(x,t) + Q(x,t).$$

If $\partial_x \Phi > 0$ then Φ is an increasing function in x so the heat flowing to the right at $x = b$ is greater than the heat flowing to the right at $x = a$ thus the heat energy decreases between $x = a$ and $x = b$ (hence the minus sign).

Definition: heat capacity

1) Let $u(x,t)$ be the temperature of the rod at point x and at time t. Note that it may take different amounts of thermal energy to raise two different materials from one temperature to another.

2) Define the heat capacity, $c(x,u)$, to be the heat energy required to raise its temperature one unit. We will either assume $c = c(x)$ or c is a constant.

3) An alternate description of thermal energy is that it is the amount of energy needed to raise the rod's temperature from 0 to the actual temperature $u(x,t)$. Thus if $\rho(x)$ is the mass density of the road then

$$\text{heat energy } = c(x) \cdot u(x,t) \cdot \rho(x) \cdot A \cdot \Delta x.$$

now equating the expression for heat energy derived earlier with this expression yields

$$e(x,t) \cdot A\Delta x = c(x)u(x,t)\rho(x) \cdot A\Delta x \Rightarrow e(x,t) = c(x)\rho(x)u(x,t).$$

Theorem: Heat Flow Rules

We now assume the "heat flow rules":

1) Constant temperature in a region implies that there is no heat flow.

2) heat energy flows from hotter regions to colder regions.

3) the greater the temperature difference, the greater is the flow of heat energy.

4) flow of heat energy will vary for differential materials

Theorem: Fourier's Law of Heat Conduction

$$\Phi = -K_0 \frac{\partial u}{\partial x}.$$

where K_0 is known as the **thermal conductivity constant**. This equation can be read as "the heat flux is proportional to the temperature difference (per unit length).

Theorem: the Heat Equation

If we combine the equations:

$$e(x,t) = c(x)\rho(x)u(x,t) \text{ and } \Phi = -K_0 \frac{\partial u}{\partial x}.$$

with the partial differential equation

$$\frac{\partial e}{\partial t} = -\frac{\partial \Phi}{\partial x} + Q.$$

this implies:

$$\frac{\partial}{\partial t}[c(x)\rho(x)u(x,t)] = -\frac{\partial}{\partial x}\left(-K_0 \frac{\partial u}{\partial x}\right) + Q \Rightarrow c(x)\rho(x)\frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q.$$

and if we assume $c(x)$ and $\rho(x)$ are also constants and that $Q(x,t) = 0$ then

$$\begin{aligned} c\rho \frac{\partial u}{\partial t} &= K_0 \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial u}{\partial t} &= k \frac{\partial^2 u}{\partial x^2} \\ u_t &= k u_{xx} \end{aligned}$$

where $k = \frac{K_0}{c\rho} > 0$ is the **thermal diffusivity constant**. The solution we seek is $u(x,t)$, the temperature at any time and point along the rod.

7.2 Initial and Boundary Conditions

Definition

1) Since the heat equation $u_t = k u_{xx}$ has one time derivative, we need additional information, usually in the form of an initial condition

(IC) at $t = 0$:

$$u(x, 0) = f(x).$$

where $f(x)$ specifies the initial (spatial) temperature distribution of the rod at time $t = 0$.

2) The two spatial derivatives in the term u_{xx} requires two additional boudnary conditions (BC). In theory boundary condition information an be given for any two points x_1 and x_2 in the interval $[0, L]$, however, conditions at $x = 0$ and $x = L$ are usually given. Some examples include:

   - prescribed temperature
   - insulated boundary
   - Newton's law of cooling

Example (Prescribed Temperature)**.** Let $u_B(t)$ be the temperature of a fluid bath which one end of the rod is in contact with. In this case, we can prescribe the temperature of one end of the rod with a boundary condition of the form

$$u(0, t) = u_B(t) \text{ or } u(L, t) = u_B(t).$$

Example (Insulated Boundary)**.** If we know the heat flux behavior at a single point, say $x = 0$, then the heat flux will be a function of t only (since the location is fixed) and when combined with Fourier's Law of Heat Conduction, yields a boundary condition of the form

$$-K_0 \frac{\partial u}{\partial x}(0, t) = \Phi(t) \Rightarrow \frac{\partial u}{\partial x}(0, t) = -\frac{1}{K_0}\Phi(t) \text{ where } \Phi(t) \text{ is given.}$$

The simplest case of this is when one end of the rod is **perfectly insulated** (no heat flow at the boundaryso $\Phi(t) = 0$ which yields the boundary condition:

$$\frac{\partial u}{\partial x}(0, t) = \frac{1}{-K_0} \cdot 0 \Rightarrow \frac{\partial u}{\partial x}(0, t) = 0.$$

Theorem: Newton's Law of Cooling

The heat flow leaving the rod is proportional to the temperature difference between the rod and the external temperature. In this case we can define the heat flux for the boundary condition as

$$\Phi(t) = -H[u(0,t) - u_B(t)].$$

where $H > 0$ is the **heat transfer coefficient**. Note that if for example $u(0,t) > u_B(t)$ then $\Phi(t) < 0$ and heat flows out of the rod to the left as expected. We can use Fourier's Law of Heat Conduction to specify a boundary condition:

$$-K_0 \frac{\partial u}{\partial x}(0,t) = \Phi(t) \Rightarrow -K_0 \frac{\partial u}{\partial x}(0,t) = -H[u(0,t) - u_B(t)].$$

That is the BC is

$$\frac{\partial u}{\partial x}(0,t) = \frac{H}{K_0}[u(0,t) - u_B(t)].$$

Note that for $x = L$, since the heat exits through the right we have H becoming $-H$, so

$$u_x(L,t) = -\frac{H}{K_0}[u(L,t) - u_B(t)].$$

7.3 Thermal Equilibrium

Definition

The **steady state/equilibrium solution** of the heat equation is a solution that does not depends on time, *i.e.* $u(x,t) = u(x)$.

Note.

1) No matter what the initial temperature distribution of the rod is, some systems will undergo a process that brings it into "thermal equilibrium". That is, there exists a finite time $T > 0$ such that $u_t = 0$ for $t > T$.

2) We expect that thermal equilibrium will be achieved in time, *i.e.*

$$\lim_{t\to\infty} u(x,t) = \overline{u}(x).$$

where $\overline{u}(t)$ is the steady state temperature or steady state solution.

3) We claim $u(x,t) = \overline{u}(x) + v(x,t)$ where $v(x,t) \to 0$ as $t \to \infty$. Think of

it as an interaction between short-term and long-term behaviors. We aim to solve these two solutions separately.

4) Consider blinking holiday lights, the lightbulbs are either being heated up (ON) or cooling down (OFF). Thus there is no steady state temperature for this system. Not all systems have equilibrium solutions.

Example (1)**.** Suppose that $u(x,t) = u(x)$ then this implies $\frac{\partial u}{\partial t} = 0$ and so the heat equation becomes

$$0 = k\frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{d^2 u}{dx^2} = 0 \Rightarrow \frac{du}{dx} = C_1 \Rightarrow u(x) = C_1 x + C_2.$$

and suppose the boundary conditions are steady, suppose $u(0) = T_1$ and $u(L) = T_2$ then the steady state solution is the line

$$\overline{u}(x) = T1 + \frac{T_2 - T_1}{L}x.$$

Example (continued)**.** According to Fourier's Law of Heat Conduction, the steady state heat flux is

$$\overline{\Phi}(x) = -k\frac{d}{dx}\overline{u}(x) = -\left(\frac{T_2 - T_1}{L}\right).$$

Example. Given a heat source $Q(x,t)$, consider the equation

$$k\frac{\partial^2 u}{\partial x^2} + Q(x,t) = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{1}{k}Q(x,t) \Rightarrow u \text{ may be a function of } t.$$

So if $\frac{\partial Q}{\partial t} \neq 0$ we will have no steady state solution.

Example. Suppose $Q(x,t) = M$ and consider $\frac{\partial^2 u}{\partial x^2} + M = 0$ with $u(0) = T_1$ and $u(L) = T_2$ then

$$u''(x) = -M \Rightarrow u(x) = -\frac{Mx^2}{2} + C_1 x + C_2.$$

and the boundary conditions imply the equilibrium solution is

$$\overline{u}(x) = T_1 + \left(\frac{T_2 - T_1}{L} + \frac{ML}{2}\right)x - \frac{Mx^2}{2}.$$

7.4 Insulated Boundaries

Consider the PDE with the BCs and IC:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, u(x,0) = f(x), \frac{\partial u}{\partial x}(0,t) = 0, \frac{\partial u}{\partial x}(L,t) = 0.$$

where IC is when $t = 0$, and BCs are $x = 0$ and $x = L$, which have zero values and means these are insulated boundaries.

With regards to the steady state solution, if we assume $u(x,t) = u(x)$ then $u(x) = C_1 x + C_2$ and using the BCs we have

$$u'(x) = C_1 \Rightarrow C_1 = u'(0) = \frac{\partial u}{\partial x}(0,t) = 0 \Rightarrow u(x) = C_2.$$

and we expect that $\overline{u}(x) = \lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} u(x) = C_2$. Rewriting the original heat equation and integrating both sides yields:

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} \Rightarrow \int_0^L c\rho\frac{\partial u}{\partial x}dx = \int_0^L K_0\frac{\partial^2 u}{\partial x^2}dx = K_0\left[\frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t)\right] = 0.$$

and multiplying by the constant area A and interchanging the derivative and integral yields

$$\int_0^L c\rho\frac{\partial u}{\partial x}Adx = 0 \Rightarrow \frac{d}{dt}\left[\int_0^L c\rho u(x,t)Adx\right] = 0.$$

This implies that the total thermal energy is constant wrt time.

Using IC,

$$\int_0^L c\rho u(x,0)Adx = \int_0^L c\rho f(x)Adx.$$

And the equilibrium thermal energy is

$$\lim_{t\to\infty}\int_0^L c\rho u(x,t)Adx = \int_0^L c\rho\left[\lim_{t\to\infty} u(x,t)\right]Adx = \int_0^L c\rho C_2 Adx = C_2 c\rho AL.$$

setting the initial and equilibrium thermal energy equal to each other and solving yields

$$c\rho A\int_0^L f(x)dx = c\rho AC_2 L \Rightarrow C_2 = \frac{1}{L}\int_0^L f(x)dx.$$

So the equilibrium solution to the heat equation with insulated boundaries is the *average value of the initial temperature* f(x) over the interval $[0, L]$.

7.5 Heat Equation in 3D

When can we switch integration and differentiation for partial differential equations?

Theorem

Suppose

1) $u(x,t)$ is defined for $a \le x \le b, c \le t \le d$.

2) $u(x,t)$ is Riemann integrable for every $t \in [c,d]$

3) $\partial_t u(x,t)$ is continuous for $(x,t) \in [a,b] \times [c,d]$

then $\partial_t u(x,t)$ is Riemann integrable for every $t \in [c,d]$, and

$$\frac{d}{dt}\int_a^b u(x,t)dx = \int_a^b \frac{\partial u}{\partial x}dx.$$

Note. We can replace the closed intervals above with $\mathbb{R}$.

7.6 Boundary Heat Flux

Let's generalize our result from 1D to 3D:

Theorem: Fourier's law of heat conduction 3D

$$\vec{\phi}(x)(\mathbf{x},t) = -K_0 \nabla u(\mathbf{x},t)$$

where ∇u is the spatial gradient of u and K_0 is the thermal conductivity constant.

Consider a region R with closed boundary ∂R and outward unit normal vector $\mathbf{n}$. Let $\vec{\phi}(x)$ be the heat flux vector which specifies the direction of heat flow at the point $\mathbf{x}= (x,y,z)$. Then the magnitude is the flux, and direction is the normal to the surface area. So the heat energy flowing across boundaries per unit time is:

$$-\iint_{\partial R} \vec{\phi}(x) \cdot \mathbf{n}\, dS.$$

if the dot product is positive then heat is flowing out of the object and the total energy would decrease.

Then the heat flow process is:

$$\frac{d}{dt}\iiint_R c(\mathbf{x})\rho(\mathbf{x})u(\mathbf{x},t)dV = -\iint_{\partial R} \vec{\phi}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})dS + \iiint_R Q(\mathbf{x},t)dV.$$

Recall the **Divergence Theorem**, we have

$$\iint_{\partial R} \vec{\phi}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) dS = \iiint_R \nabla \cdot \vec{\phi}(\mathbf{x}) dV.$$

where $\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$. Now we bring the derivative inside the integral:

$$\iiint_R c(\mathbf{x})\rho(\mathbf{x})\frac{\partial}{\partial t}u(\mathbf{x},t)dV = -\iiint_R \nabla \cdot \vec{\phi}(\mathbf{x}) + \iiint_R Q(\mathbf{x},t)dV.$$

and combining all the triple integrals on the left hand side yields

$$c(\mathbf{x})\rho(\mathbf{x})\frac{\partial}{\partial t}u(\mathbf{x},t) + \nabla \cdot \vec{\phi}(\mathbf{x}) - Q(\mathbf{x},t) = 0.$$

where the last equality follows from that the integral equation is true for any region R and by continuity.

Intuition. We transform the double integral to triple integral using divergence theorem so we can combine them under the same domain of $\mathbf{x}$. Now we want to replace flux with temperature function.

Definition: Laplacian

For 3D, the **Laplacian** is defined as

$$\nabla^2 u = \Delta u = u_{xx} + u_{yy} + u_{zz}.$$

Recall Fourier's Law says heat flows from hot to cold in the direction where the temperature differences are the greatest and ∇u represents the direction of greatest temperature increases, so

$$\vec{\phi} = -K_0 \cdot \nabla u \Rightarrow \nabla \cdot \vec{\phi}(\mathbf{x}) = \nabla \cdot (-K_0 \nabla u) = -K_0 \cdot \Delta u.$$

Then

$$c(\mathbf{x})\rho(\mathbf{x})\frac{\partial}{\partial t}u(\mathbf{x},t) - K_0 \Delta u - Q(\mathbf{x},t) = 0.$$

Thus **the heat equation with internal source of energy** is

$$c(\mathbf{x})\rho(\mathbf{x})\frac{\partial}{\partial t}u(\mathbf{x},t) = K_0 \Delta u + Q(\mathbf{x},t).$$

Assuming $Q = 0$ and the thermal coefficients are constant, we get

Theorem: 3D heat equation

$$\frac{\partial u}{\partial t} = k\Delta u.$$

where $k = \frac{K_0}{c\rho}$ = "thermal diffusivity", with initial condition $u(\mathbf{x}, 0) = f(\mathbf{x})$ and boundary condition $u(\mathbf{x}, t) = T(\mathbf{x}, t)$ for $\mathbf{x} \in \partial R$.

7.7 Steady State

Theorem: Laplace's Equation

Consider the heat equation with internal source of energy defined above, then if $u_t = 0$ this gives **Poisson's Equation**, $\Delta u = -\frac{Q}{K_0}$, and if $Q = 0$ this yields **Laplace's Equation**:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Theorem: Laplace's Equation in Cylindrical Coordinates

Let $x = r\cos(\theta), y = r\sin(\theta), z = z$ then using the Chain Rule,

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

Theorem: Spherical

$$\Delta u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2\sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial u}{\partial \phi}\right) + \frac{1}{\rho^2\sin^2(\phi)}\frac{\partial^2 u}{\partial \phi^2}.$$

8 Solving the Heat Equation

Definition: the Heat Operator

1) Define the **heat operator** as

$$L(u) = \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2}.$$

for any $u(x,t)$ in the appropriate function space (once differentiable in t and twice differentiable in x). Then $L(u)$ is a linear operator.

2) The set of functions that satisfy the boundary conditions $u(0,t) = 0 = u(L,t)$ form a vector space. That is, if u_i satisfy these boundary condition for $i = 1, 2$ and if $u_3(x,t) = c_1 u_1(x,t) + c_2 u_2(x,t)$ then $u_3(0,t) = 0 = u_3(L,t)$ for any $c_1, c_2 \in \mathbb{R}$.

Note. The set of function that satisfy the initial condition $u(x,0) = f(x) \neq 0$ does NOT form a vector space.

8.1 Separation of Variables

Consider the following boundary value problem, there are three pieces of the full story:

$$\begin{cases} \text{PDE:} & \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, & 0 < x < L, t > 0 \\ \text{BC:} & u(0,t) = 0 = u(L,t), & t > 0 \\ \text{IC:} & u(x,0) = f(x), & 0 \leq x \leq L \end{cases}$$

Intuition. We will take the non-zero part of the boundary conditions into the steady-state ODE, so that the PDE forms a vector space and becomes easier to solve.

Typically we would assume $u(x,t) = \overline{u}(x) + v(x,t)$ but in this case $\overline{u}(x) = 0$, so we apply separation of variables directly to $u(x,t)$. Assume (separable functions wrt t and x): $u(x,t) = F(x) \cdot G(t) \neq 0$ then

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \Rightarrow F(x)\frac{dG}{dt} = k\frac{d^2 F}{dx^2}G(t) \Rightarrow \frac{1}{k} \cdot \frac{G'(t)}{G(t)} = \frac{F''(x)}{F(x)}.$$

For this to satisfy, the ratio must be a constant.

Proof

We aim to show that the derivative of LHS wrt t is zero for all t. Note that

$$\frac{d}{dt}\left(\frac{1}{k}\cdot\frac{G'(t)}{G(t)}\right)=\frac{d}{dt}\left(\frac{F''(x)}{F(x)}\right)=0.$$

and likewise for the RHS

$$\frac{d}{dx}\left(\frac{F''(x)}{F(x)}\right)=0.$$

Together 0 derivative everywhere implies a constant function:

$$\frac{1}{k}\cdot\frac{G'(t)}{G(t)}=\frac{F''(x)}{F(x)}=-\lambda.$$

where λ is some constant. □

Now consider the differential equations

$$G'(t)=-\lambda kG(t) \text{ and } F''(x)=-\lambda F(x).$$

Intuition. The PDE and BCs allow us to form a vector space of solutions to the homogeneous equation.

The **time domain problem** is

$$G'(t)=-\lambda kG(t)\Rightarrow\int\frac{1}{G(t)}G'(t)dt=\int-\lambda k\ dt\Rightarrow\ln|G(t)|=-\lambda kt+C_1.$$

which finally yields the general solution

$$G(t)=Ce^{-\lambda kt},C\in\mathbb{R}.$$

Physically we expect that $\lambda>0$ (because since boundary condition is 0 we expect temperature to decay as time goes on). Now the boundary conditions give:

$$u(0,t)=0\Rightarrow F(0)G(t)=0\Rightarrow F(0)=0.$$

Because otherwise , if $G(t)=0\Rightarrow u(x,t)=F(x)G(t)=0$ which is a trivial solution that violates our separation assumption. Similarly $u(L,t)=0\Rightarrow F(L)=0$. Thus the **boundary value problem** (or *eigenvalue problem)* can be formulated as

$$\begin{cases} \frac{d^2F}{dx^2} = -\lambda F(x) \\ F(0) = 0 = F(L) \end{cases}$$

To solve this ODE, let $F(x) = e^{rx}$, so

$$F''(x) = -\lambda F(x) \Rightarrow r^2 e^{rx} = -\lambda e^{rx} \Rightarrow r^2 = -\lambda.$$

and the last equation is the characteristic equation. We claim that $\lambda \leq 0$ gives the trivial solution $F(x) = 0$.

Proof

Case (1). If $\lambda = 0$, then $r = 0$ with repeated roots. Let $F(x) = C_1 e^0 + C_2 x e^0 \Rightarrow F(x) = C_1 + C_2 x$. Since $0 = F(0) = C_1$, $0 = F(L) = C_2 L \Rightarrow C_2 = 0$. Hence $F(x) = 0$ which is the trivial solution.

Case (2). Similarly when $\lambda < 0$ also gives us the trivial solution. (See homework).

□

The only interesting case is $\lambda > 0$. Let's solve the BVP: $r^2 = -\lambda$ then we have purely imaginary roots $r_{1,2} = \pm i\sqrt{\lambda}$ and by Euler's Formula, we have

$$e^{r_1 x} = e^{i\sqrt{\lambda}x} = \cos(\sqrt{\lambda}x) + i\sin(\sqrt{\lambda}x).$$

and

$$e^{r_2 x} = e^{-i\sqrt{\lambda}x} = \cos(\sqrt{\lambda}x) - i\sin(\sqrt{\lambda}x).$$

Then the general solution can be any linear combination of these functions. We can now convert them to the wave form:

$$\frac{e^{i\sqrt{\lambda}x} + e^{-i\sqrt{\lambda}x}}{2} = \cos(\sqrt{\lambda}x).$$

and

$$\frac{e^{i\sqrt{\lambda}x} - e^{-i\sqrt{\lambda}x}}{2i} = \sin(\sqrt{\lambda}x).$$

So a general solution of the ODE is

$$F(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x).$$

The boundary conditions give

$$F(0) = 0 \Rightarrow C_1 = 0 \text{ and } F(L) = 0 = C_2 \sin(\sqrt{\lambda}L).$$

Since $C_2 \neq 0$ (or it would be trivial solution), this implies

$$\sin(\sqrt{\lambda}L) = 0 \Rightarrow \sqrt{\lambda}L = n\pi \Rightarrow \sqrt{\lambda} = \frac{n\pi}{L} \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 = \lambda_n, \text{ for } n = \pm 1, \pm 2, \ldots.$$

Note n should not be 0 since it would give $\lambda = 0$. Since sine is an odd function, for $n = 1, 2, 3, \ldots$ we can write $F_n(x) = C_2 \sin(\frac{n\pi x}{L})$. Thus we have the *product solution*

$$u_n(x,t) = F_n(x)G_n(t) = B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} \text{ for } n = 1, 2, 3, \ldots.$$

and some constants B_n.

Now by superposition principle, the solution of the homogeneous PDE (if it converges) is the linear combination of the product solutions,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} \text{ for some constants } B_n.$$

and now using the initial condition to determine B_n. Note that we have an orthogonal basis of sines. This allows us to use the projection formula to find B_n. When $t = 0$, we get a **Fourier Sine Series** (FSS), thus by projection formula (f(x) inner product with a sine basis vector):

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \Rightarrow B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Note that the domain is only from 0 to L, and the odd extension of $f(x)$ makes the whole integrand odd, allowing us to simply multiply by 2 using symmetry. That is, the solution of the heat equation assuming convergence is

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx\right) \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}.$$

Note that the exponentially decaying term guarantees the convergence. More on Fourier Sine Series:

Definition

1) Define the **odd extension** of $f(x), 0 < x < L$, to be

$$f_{odd}(x) = \begin{cases} f(x), & \text{if } 0 < x < L \\ -f(-x), & \text{if } -L < x < 0 \end{cases}$$

2) If $f(x)$ is piecewise smooth then $f(x)$ has a Fourier series representation and if

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), \text{ for } 0 < x < L.$$

then note that the RHS is continuous, odd, and $2L$-periodic thus the Fourier sine series of $f(x)$ represents the periodic extension of the adjusted odd extension of $f(x)$. That is,

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = \tilde{\bar{f}}_{odd}(x).$$

3) In general, note that $\text{FSS}[f](x) = \text{ F.S.}[f_{odd}](x)$.

See lecture slides for an pictorial example. Fourier cosine series is defined similarly using even extension.

What if we don't have homogeneous boundary conditions?

Theorem: General Solution

If:

1) the set of functions that satisfy the PDE and BC form a vector space.

2) there is a non-trivial function $v(x,t)$ that satisfies the PDE and BC (the transient solution).

3) if the PDE has a steady state solution, $\overline{u}(x)$

then the function $u(x,t) = \overline{u}(x) + v(x,t)$ will be a solution to the PDE that satisfies the BCs and IC.

So since our steady state solution in the previous example is trivial, we only obtain the transient solution.

Example. Consider

$$\begin{cases} \text{PDE: } \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, & 0 < x < L, t > 0 \\ \text{BCs: } u(0,t) = T_1, u(L,t) = T_2, & t > 0 \\ \text{IC: } u(x,0) = f(x), & 0 \leq x \leq L \end{cases}$$

We need to make sure the solutions of PDE form a vector space, let $u(x,t) = \overline{u}(x) + v(x,t)$ then

$$\frac{\partial u}{\partial t} = 0 + \frac{\partial v}{\partial t}.$$

and

$$k\frac{\partial^2 u}{\partial x^2} = k\overline{u}''(x) + k\frac{\partial^2 v}{\partial x^2}.$$

The heat equation equates the two equations above. If we assume that steady-state (plus nonhomogeneous part if any) and transient solutions behave independently, then by matching terms we obtain the following ODE and PDE:

$$\overline{u}''(x) = 0 \text{ and } \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}.$$

and by assigning the nonhomogeneous part of the BCs to ODE,

$$T_1 = u(0,t) = \overline{u}(0) + v(0,t) \Rightarrow \overline{u}(0) = T_1 \text{ and } v(0,t) = 0.$$

This way, the PDE solutions still form a vector space.

Similarly,

$$T_2 = u(L,t) = \overline{u}(L) + v(L,t) \Rightarrow \overline{u}(L) = T_2 \text{ and } v(L,t) = 0.$$

Then finally for IC:

$$f(x) = u(x,0) = \overline{u}(x) + v(x,0) \Rightarrow v(x,0) = f(x) - \overline{u}(x).$$

Now we are able to solve for the solution of both the steady state and transient problems.

Steady state problem:

$$\overline{u}''(x) = 0, \overline{u}(0) = T_1, \overline{u}(L) = T_2.$$

As we have seen earlier, the solution is

$$\overline{u}(x) = T_1 + \frac{T_2 - T_1}{L}x.$$

Transient problem:

$$\begin{cases} \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}, & 0 < x < L, t > 0 \\ v(0,t) = 0 = v(L,t), & t > 0 \\ v(x,0) = f(x) - \overline{u}(x), & 0 \le x \le L \end{cases}$$

To check that we set up the problems correctly, we see that $\frac{\partial u}{\partial t} = 0 + v_t(x,t) = k\overline{u}''(x) + kv_{xx} = k\frac{\partial^2 u}{\partial x^2}$. Hence heat equation is satisfied. Also,

$$\begin{aligned} u(0,t) &= \overline{u}(0) + v(0,t) = T_1 + 0 = T_1 \\ u(L,t) &= \overline{u}(L) + v(L,t) = T_2 + 0 = T_2 \\ u(x,0) &= \overline{u}(x) + v(x,0) = \overline{u}(x) + [f(x) - \overline{u}(x)] = f(x) \end{aligned}$$

Hence as before, we obtain the transient solution:

$$v(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{L} \int_0^L [f(x) - \overline{u}(x)] \sin\left(\frac{n\pi x}{L}\right)\, dx \right) \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}.$$

Again note that the whole integral is just a constant, B_n. Then the full solution is

$$u(x,t) = T_1 + \frac{T_2 - T_1}{L}x + \sum_{n=1}^{\infty} \left(\frac{2}{L} \int_0^L [f(x) - \overline{u}(x)] \sin\left(\frac{n\pi x}{L}\right)\, dx \right) \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}.$$

where the first part comes from BCs, the integral constant comes from IC, and the sin and exponential terms come from the homogeneous PDE plus BCs. To verify our solution is correct, we plug in $x = 0, L$ or $t = 0$, we can verify that it satisfies the BCs and IC.

It remains to show that the full solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

converges.

For the convergence of the partials, take the time partial of the solution and use the heat equation to get the 2nd space partial:

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \sum_{n=1}^{\infty} B_n \cdot -k\left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right) \cdot e^{-\left(\frac{n\pi}{L}\right)^2 kt} \\ \frac{\partial^2 u}{\partial x^2} &= \sum_{n=1}^{\infty} B_n \cdot -\left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} \end{aligned}$$

They have the general form

$$\sum_{n=1}^{\infty} \tilde{B}_n n^2 \sin\left(\frac{n\pi x}{L}\right) e^{-Ct(n)^2}.$$

For some constant $C > 0$ and $\tilde{B}_n$. Since $t > 0$

$$n \geq 1 \Rightarrow n^2 \geq n \Rightarrow Ctn^2 \geq Ctn \Rightarrow e^{Ctn^2} \geq e^{Ctn} \Rightarrow e^{-CTn^2} \leq e^{-Ctn}.$$

So by triangle inequality we have

$$\sum_{n=1}^{\infty} \left| \tilde{B}_n n^2 \sin\left(\frac{n\pi x}{L}\right) e^{-Ctn^2} \right| \leq \sum_{n=1}^{\infty} |\tilde{B}_n| \cdot n^2 \cdot 1 \cdot e^{-Ctn}.$$

And if $\tilde{B}_n$ is bounded by some $M > 0$, then we have

$$\sum_{n=1}^{\infty} \left| \tilde{B}_n n^2 \sin\left(\frac{n\pi x}{L}\right) e^{-Ctn^2} \right| \leq \sum_{n=1}^{\infty} M n^2 e^{-Ctn}.$$

So it suffices to show that the RHS converges to show the convergence of the partials. This has been done in the homework using the ratio test. Hence the partials converge!

8.2 Interpret Solution

Theorem: convergence of a series solution of the heat equation

For $t > 0$, if there exists a constant $M > 0$ such that $|B_n| \leq M \;\forall\, n$, then

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt}$$

converges absolutely for each $x \in [0, L]$.

Proof

Note that given any n,

$$\left| B_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt} \right| \leq |B_n| \cdot 1 \cdot e^{-(\frac{n\pi}{L})^2 kt} \leq M e^{-(\frac{n\pi}{L})^2 kt}.$$

so given any $N > 0$, we have

$$0 < \sum_{n=1}^{N} \left| B_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt} \right| \leq \sum_{n=1}^{N} M e^{-(\frac{n\pi}{L})^2 kt}.$$

and taking the limit $N \to \infty$ yields

$$0 < \sum_{n=1}^{\infty} \left| B_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt} \right| \leq \sum_{n=1}^{\infty} M e^{-(\frac{n\pi}{L})^2 kt}.$$

by Order Limit Theorem.

Now again $n \geq 1 \Rightarrow n^2 \geq n$ and since $\left(\frac{\pi}{L}\right)^2 kt > 0$,

$$e^{-(\frac{n\pi}{L})^2 kt} \leq e^{-(\frac{\pi}{L})^2 ktn}.$$

and since this holds for any n,

$$0 < \sum_{n=1}^{\infty} M e^{-(\frac{n\pi}{L})^2 kt} \leq \sum_{n=1}^{\infty} M e^{-(\frac{\pi}{L}) ktn} = \sum_{n=1}^{\infty} M \left[e^{-(\frac{\pi}{L}) kt} \right]^n < \infty.$$

The last step comes from convergence of Geometric series: Note $e^{(\frac{\pi}{L})kt} > e^0 = 1$, so the inverse is < 1. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} M e^{-(\frac{\pi}{L})^2 ktn} &= \sum_{n=1}^{\infty} M e^{-(\frac{\pi}{L}) kt} \left[e^{-(\frac{\pi}{L}) kt} \right]^{n-1} \\
&= \sum_{n=1}^{\infty} a \cdot r^{n-1} \\
&= \frac{a}{1-r} \\
&= \frac{M e^{-(\frac{\pi}{L}) kt}}{1 - e^{-(\frac{\pi}{L}) kt}} < \infty
\end{aligned}$$

Therefore, by direct comparison test, the Fourier sine series converges absolutely on $[0, L]$. $\square$

Note. For the heat equation if we start with reasonable data then the solution is almost guaranteed to converge. The assumption of $|B_n| < M$ needs to hold. And since B_n is a definite integral, and its boundedness only depends on $f(x)$. As long as $f(x)$ is "nice", *i.e.* piecewise continuous with no crazy spikes, then it converges.

Example.

$$\begin{cases}
\text{PDE: } \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, & 0 < x < L, t > 0 \\
\text{BC: } u(0,t) = 0 = u(L,t), & t > 0 \\
\text{IC:} u(x,0) = 100, & 0 \leq x \leq L
\end{cases}$$

$$
\begin{aligned}
\frac{2}{L}\int_0^L 100\sin\left(\frac{n\pi x}{L}\right)dx &= \frac{200}{L}\cdot -\cos\left(\frac{n\pi x}{L}\right)\frac{L}{n\pi}\bigg|_{x=0}^{x=L} \\
&= -\frac{200}{n\pi}\left[\cos(n\pi)-1\right] \\
&= -\frac{200}{n\pi}\left[(-1)^n-1\right] \\
&= \frac{400}{n\pi} \text{ if n is odd}
\end{aligned}
$$

So all even terms vanish, then B_n is a decreasing sequence. So $|B_n| \leq M = \frac{400}{n\pi}$ for all $n \geq 1$. Thus for $t > 0$, $u(x,t)$ is absolutely convergent for each x. The series solution has the form:

$$
u(x,t) = \frac{400}{\pi}\sum_{p=1}^{\infty}\frac{1}{2p-1}e^{-(\frac{(2p-1)\pi}{L})^2 kt}\sin\left(\frac{(2p-1)\pi x}{L}\right).
$$

Example. Let $L = 10$cm, $k = 1$ cm^2/sec (copper) and $t = 0.35$sec and use the first 8 terms.

$$
u(x,0.35) \approx \frac{400}{\pi}\sum_{p=1}^{8}\frac{1}{2p-1}e^{-[(2p-1)\frac{\pi}{10}]^2 0.35}\sin\left(\frac{(2p-1)\pi x}{10}\right).
$$

See lecture slide for graph.

In this case, $\overline{u}(x) = 0$ and for each x,

$$
u(x,t) \to 0 \text{ as } t \to \infty.
$$

Note. Recall $\Phi = -K_0\frac{\partial u}{\partial x}$. Consider the x-term in $u_n(x,t)$

$$
\left[\sin\left(\frac{n\pi x}{L}\right)\right]' = n\cdot\frac{\pi}{L}\cos\left(\frac{n\pi x}{L}\right)
$$

Thus the derivative wrt x is proportional to n and Φ. That is, as n increases, the derivative increases, and the heat flux (loss) increases. So the slowest decaying term is when n is smallest, *i.e.* $n = 1$.

Establishing that the slowest decaying term (dominant term) is at $n = 1$ and for large t, we can use the $n = 1$ term (called the "first Fourier mode") as an approximation

$$
u(x,t) \approx B_1\sin\left(\frac{\pi x}{L}\right)e^{-(\frac{\pi}{L})^2 kt}.
$$

and we can use this for long term temperature prediction. So for this problem we use the approximation:

$$
u(x,t) \approx \frac{400}{\pi}\sin\left(\frac{\pi x}{L}\right)e^{-(\frac{\pi}{L})^2 kt}.
$$

to analyze the dynamics of the temperature when t grows. We expect $u(x,t) \to 0$ as $t \to \infty$. See lecture slides for graph.

Example (estimating cooling time)**.** How long will it take for the maximum absolute temperature of the rod to be less than $\frac{1}{10}$ the initial maximum absolute temperature?

We wish to find t such that

$$\max_{0<x<L} |u(x,t)| \leq \frac{1}{10} \max_{0<x<L} |f(x)|.$$

Using the first Fourier mode approximation obtained above, recall there is an upper bound (in fact it's the least upper bound/supremum)

$$|u(x,t)| \leq \frac{400}{\pi} e^{-(\frac{\pi}{L})^2 kt}.$$

Hence this upper bound is greater or equal to the maximum (in this case they are in fact equal). Thus it suffices to find t such that

$$\frac{400}{\pi} e^{-(\frac{\pi}{L})^2 kt} \leq \frac{1}{10} \cdot 100 = 10.$$

Solving this inequality yields

$$t \geq \frac{L^2}{k} \frac{1}{\pi^2} \ln\left(\frac{40}{\pi}\right).$$

9 Insulated Rods

Consider the following BVP:

$$\begin{cases} \text{PDE:} & \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, t > 0 \\ \text{BC:} & \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t), \qquad t > 0 \\ \text{IC:} & u(x,0) = f(x), \qquad 0 \leq x \leq L \end{cases}$$

Recall Fourier's Law of Heat Conduction regarding the heat flux

$$\Phi = -K_0\frac{\partial u}{\partial x}.$$

So here the BCs imply that there is no heat flow at the ends of the rod, *i.e.* the rod is insulated on all sides.

Since there is no external source of heat, we expect this BVP to have a steady state solution.

Hence we can assume the solution has the form

$$u(x,t) = \overline{u}(x) + v(x,t).$$

In fact, we can skip this decomposition step because the BVP already gives us a nice vector space due to the homogeneous BCs. And $\lambda = 0$ case will give us the steady-state solution anyway.

Note. The BCs in this problem is the **Von Neumann condition**, which gives rise to FCS. The BCs from a previous problem with no derivatives is the **Dirichlet condition**, which gives rise to FSS.

Note. In the case when PDE and BCs already form a vector space, we don't need to solve for steady-state and transient solutions separately because the eigenvalue problem at $\lambda = 0$ case gives the steady-state solution. Let's directly use $u(x,t) = F(x)G(t) \neq 0$ and apply separation of variables.

Then the time domain problem is

$$G'(t) = -\lambda k G(t).$$

And the solution is again $G(t) = Ce^{-\lambda kt}, C \in \mathbb{R}$. The boundary value problem is:

$$\begin{cases} \frac{d^2F}{dx^2} = -\lambda F(x) \\ F'(0) = 0 = F'(L) \end{cases}$$

This is equivalent to the eigenvalue problem:

Case. $\lambda < 0$ we get trivial solution.

Case. $\lambda = 0$, then

$$F''(x) = 0 \Rightarrow F(x) = Ax + B.$$

and the BCs yields $A = 0$ thus $F(x) = B, B \in \mathbb{R}$.

Note. We didn't get trivial solution here because it is the Von Neumann condition, as opposed to the Dirichlet condition from before.

Case. $\lambda > 0$, this is the same as before, we have

$$F(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

And apply BCs:

$$0 = F'(0) \Rightarrow c_2 = 0 \Rightarrow F(x) = c_1 \cos(\sqrt{\lambda}x).$$

$$0 = F'(L) \Rightarrow \sin(\sqrt{\lambda}L) = 0 \Rightarrow \sqrt{\lambda} = \frac{n\pi}{L} \text{ for } n = \pm 1, \pm 2, \ldots.$$

which implies $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, \ldots$.

Now we have $u_n(x,t) = a_n \cos\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt}$ for $n = 1, 2, \ldots$. The superposition principle asserts that the solution of the homogeneous PDE (if it converges) is the linear combination of all $u_n(x,t)$. That is,

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt}$$

for some constants A_n. Now we apply the IC to find these constants. Since we have orthogonal cosine basis, it is in fact a Fourier Consine Series (FCS):

$$f(x) = u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right).$$

we use the projection formula for the case $t = 0$ to find the coefficients of this basis:

$$A_0 = \frac{1}{L} \int_0^L f(x) dx.$$

and

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx.$$

Theorem: convergence

For $t > 0$, if there exists a constant $0 < M < \infty$ such that $|A_n| \leq M$, for all n, then

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt}$$

converges absolutely for each $x \in [0, L]$

Therefore, our final solution is

$$u(x,t) = \frac{1}{L}\int_0^L f(x)dx + \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx\right)\cos\left(\frac{n\pi x}{L}\right)e^{-(\frac{n\pi}{L})^2 kt}.$$

and note that when $t \to \infty$, we obtain the steady state solution.

For large but finite time, we can use the slowest decaying term to approximate the solution

$$u(x,t) = A_0 + A_1 \cos\left(\frac{\pi x}{L}\right) e^{-(\frac{\pi}{L})^2 kt}.$$

9.1 Fourier Cosine Series

What does it represent?

Definition: even extension

1) Define the **even extension** of $f(x)$ to be

$$f_{even}(x) = \begin{cases} f(x), & \text{if } 0 < x < L \\ f(-x), & \text{if } -L < x < 0 \end{cases}$$

Then $f_{even}(-x) = f_{even}(x)$ for any $x \in (-L, L)$.

2) If $f(x)$ is piecewise smooth then $f(x)$ has a Fourier series representation and if

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), \text{ for } 0 < x < L.$$

Then note that the RHS is continuous, even, and $2L$-periodic. Thus the Fourier cosine series of $f(x)$ represents the periodoc extension of the (adjusted) even extension of $f(x)$ that is

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = \tilde{\bar{f}}_{even}(x).$$

3) in general, $\mathrm{FCS}[f](x) =$ F.S.$[f_{even}](x)$.

Example. See lecture notes for figures of FCS.

10 Thin Circular Ring

The wire is circular with circumference $2L$ and insulated. The radius is therefore $r = \frac{L}{\pi}$. If the wire is thin enough then we assume the temperature is constant along the cross sections of the wire and satisfies the following BVP:

$$\begin{cases} \text{PDE:} & \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, & -L < x < L, t > 0 \\ \text{BCs:} & u(-L,t) = u(L,t), \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t), & t > 0 \\ \text{IC:} & u(x,0) = f(x), & -L \leq x \leq L \end{cases}$$

The BCs here assume that at the ends, the temperature is continuous and the flux is also continuous.

Due to the circular nature, $u(x_0,t) = u(x_0 + 2L, t)\ \forall\ x_0 \in [-L, L]$. Then we can define $u(x,t)\ \forall\ x \in \mathbb{R}$.

Do the PDE and BCs form a vector space? See homework, where we check linearity. Yes, so we can try separable of variables $u(x,t) = F(x) \cdot G(t) \neq 0$. We turn this into a time domain problem and an eigenvalue problem.

Note. As before we have

$$\frac{1}{k}\frac{G'(t)}{G(t)} = \frac{F''(x)}{F(x)} = -\lambda \Rightarrow G'(t) = -\lambda k G(t) \text{ and } F''(x) = -\lambda F(x).$$

Then BCs respectively becomes

$$F(-L) = F(L) \text{ and } F'(-L) = F'(L)$$

1) *time domain problem:* $G(t) = Ce^{-\lambda kt}, C \in \mathbb{R}$.

2) *eigenvalue problem:*

Case. $\lambda < 0$, again we get the trivial solution.

Case. $\lambda = 0$, then

$$F''(x) = 0 \Rightarrow F(x) = Ax + B.$$

Then the BC $F(-L) = F(L) \Rightarrow A = 0$. Therefore, $F(x) = B, B \in \mathbb{R}$. The other BC is trivial and redundant in this case.

Case. $\lambda > 0$, we solve the characteristic equation as before and obtain

$$F(x), c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

Then the BC $F(-L) = F(L)$ yields

$c_1 \cos(-\sqrt{\lambda}L)+c_2 \sin(-\sqrt{\lambda}L) = c_1 \cos(\sqrt{\lambda}L)+c_2 \sin(\sqrt{\lambda}L) \Rightarrow 2c_2 \sin(\sqrt{\lambda}L) = 0.$

This implies either $c_2 = 0$ or $\sqrt{\lambda}L = n\pi$ for $n = \pm 1, \pm 2, \ldots$ Applying the other BC $F'(-L) = F'(L)$, as above we get

$$c_1 \sin(\sqrt{\lambda}L) = 0.$$

This implies either $c_1 = 0$ or $\sqrt{\lambda}L = n\pi$ for $n = \pm 1, \pm 2, \ldots$ Recall that we do not want $c_1 = 0$ and $c_2 = 0$, *i.e.* the trivial solution, so we require either $c_1 \neq 0$ or $c_2 \neq 0$. This means that both the cosine and sine terms survive in the general solution.

Moreover, we get $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2 \ldots$

Hence by superposition principle, the general solution is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt}.$$

The coefficients a_0, a_n, b_n are obtained just as before using projection.

Note. Suppose $f(x)$ is odd, then $a_0, a_n = 0$, and $b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)$, just like in FSS. Similar with even $f(x)$.

Now for large but finite time, we can again approximate our temperature prediction using the slowest decaying exponential term, which includes both sine and cosine terms when $n = 1$:

$$u(x,t) \approx \frac{1}{2L}\int_{-L}^{L} f(x)dx + \left[a_1 \cos\left(\frac{\pi x}{L}\right) + b_1 \sin\left(\frac{\pi x}{L}\right)\right] e^{-(\frac{\pi}{L})^2 kt}.$$

And the steady-state solution ($t \to \infty$) is just

$$\overline{u}(x) = a_0.$$

How are the FSS, FCS, and FS related?

Definition

Note that for any function $f(x)$, we have

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)].$$

1) Define the **even part** of $f(x)$ to be $f_e(x) = \frac{1}{2}[f(x) + f(-x)]$, then

$f_e(-x) = f_e(x)$.

2) Define the **odd part** of $f(x)$ to be $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$, then $f_o(-x) = -f_o(x)$.

3) The F.S.$[f](x)$ equals the FCS of $f_e(x)$ plus the FSS of $f_o(x)$. That is

$$\text{F.S.}[f](x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt}.$$

Note. The even and odd parts of $f(x)$ is NOT the even and odd extension of $f(x)$!

This concludes our discussion of the heat equation, for now.

11 Summary of Heat Equations

BCs whose solutions form a vector space:

1) Dirichlet BCs: temperature fixed at ends, homogeneous function BCs. Solution is FSS.
2) Neumann BCs: perfectly insulated, homogeneous derivative BCs. Solution is FCS.
3) Cauchy BCs: thin circular wire, equal function and derivative BCs. Solution is FS.
4) Variations: mixture of Dirichlet and Neumann BCs. Depending on the mixture, we get different answers. See homework and practice exam.

For those that don't form a vector space, we move the nonhomogeneous part to the steady state BCs.

Note. The initial condition changes after removing the steady state component.

12 Motion of Stretched String

Motivation. We consider a *horizontally stretched string* with ends that are tied down (something like a guitar). The string moves in time and we wish to track the position of each point on the string during vibration. The motion of a point on the string is NOT entirely vertical, but we are going to assume the motion is entirely vertical. See lecture slides for illustrations.

12.1 Assumptions

1) With **no motion**, the string has
   - $\delta(s)$: density
   - $A(s)$: cross-sectional area
   - $u(s)$: vertical displacement at arc length s.
2) The *linear mass density* of the string is $\rho_0 = (\delta \cdot A)$.
3) *Boundary Conditions*: The ends of the string with length L are fixed: $u(0,t) = u(L,t) = 0$.
4) Possible external forces: gravity, violin bow, guitar pick, etc.
5) **Trivial equilibrium** is if there are no external forces and no motion, then we assume the string lies along a straight line, so $ds = dx$. Then we assume there is a constant **tensile force** or *tension* along the string.
6) For small vibrations, $u = u(x,t)$ measures the vertical displacement from the trivial equilibrium at time t. That is, the shape of the string at time $t = t_0$ is given by $u(x, t_0)$.

12.2 Additional Assumptions

1) We assume mass is constant.
2) Assume the string is perfectly flexible and has no stiffness.
3) The forces exerted by the string on the ends act purely in the *tangential direction* and there are no transverse forces and no torque (twisting).

12.3 Derivation

Let $T(x,t) \geq 0$ represent the magnitude of the tangential force due to the tension. Then the horizontal tension balances each other out because there is no horizontal motion. That is,

$$T(x,t)\cos(x,\theta) = T(x+\Delta x,t)\cos(x+\Delta x,\theta).$$

Therefore, we conclude that T is constant. $T(x,t)\cos(x,t) = T_0$.

12.4 Vertical Forces

According to Newton's second law, $\mathbf{F} = m\mathbf{a}$, the vertical net force equals the tensile forces plus the vertical components of any external forces. Therefore,

$$\rho_0 \Delta x \cdot \frac{\partial^2 u}{\partial t^2} = T(x+\Delta x,t)\sin(\theta(x+\Delta x,t)) - T(x,t)\sin(\theta(x,t)) - \rho_0 \Delta x \cdot g.$$

This is force equals to opposing vertical tensile forces minus the gravity. Note that $T(x,t) = \frac{T_0}{\cos(x,t)} \Rightarrow T(x,t)\sin(\theta(x,t)) = T_0 \tan(\theta(x,t))$, so

$$\rho_0 \cdot \frac{\partial^2 u}{\partial t^2} = T_0 \cdot \left[\frac{\tan(\theta(x+\Delta x,t)) - \tan(\theta(x,t))}{\Delta x}\right] - \rho_0 \cdot g.$$

Taking the limit $\Delta x \to 0$,

$$\rho_0 \cdot \frac{\partial^2 u}{\partial t^2} = T_0 \cdot \frac{\partial}{\partial x} \tan(\theta(x,t)) - \rho_0 \cdot g.$$

Intuition. If the rope is tight enough, *i.e.* T_0 is large, then there will be barely any oscillation. It's some sort of restorative force.

Recall the slope of the string may be represented as $\frac{\partial u}{\partial x}$ or $\tan\theta$.

$$\tan\theta = \frac{\partial u}{\partial x}.$$

Thus

$$\frac{\partial}{\partial x}\tan\theta = \frac{\partial}{\partial x}\cdot\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}.$$

Then the PDE becomes

$$\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho_0}\cdot\frac{\partial^2 u}{\partial x^2} - g.$$

Assuming $g = 0$, let $c^2 = \frac{T_0}{\rho_0} = \frac{T_0}{\delta \cdot A} > 0$, then we obtain the 1D wave equation

Theorem: 1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

The more general version is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - g + f(x,t).$$

The LHS is the vertical acceleration of string, and the RHS is the restoring force due to tension (- gravitational acceleration + acceleration from known external forces).

12.5 initial conditions

Second order time derivative needs two ICs!

Initial position: $u(x,0) = U(x)$.

Initial velocity: $\frac{\partial}{\partial t}u(x,0) = V(x)$.

The boundary conditions are fixed: $u(x,0) = u(x,L) = 0$.

So we have the following:

$$\begin{cases} \text{PDE: } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} & 0 < x < L, t > 0 \\ \text{BCs: } u(0,t) = 0 = u(L,t) & t > 0 \\ \text{ICs: } u(x,0) = U(x), \frac{\partial u}{\partial t}(x,0) = V(x) & 0 \le x \le L \end{cases}$$

In general, we assume that the solution has the form $u(x,t) = \overline{u}(x) + w(x,t)$, where $\overline{u}(x)$ is the steady state solution and $w(x,t)$ is in the vector space of functions that satisfy the PDE and BCs.

Example. Suppose

$$\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho_0} \cdot \frac{\partial^2 u}{\partial x^2} - g.$$

with $u(0,t) = 0 = u(L,t)$. Find the steady state position of the string.

For steady state, $\frac{\partial u}{\partial t} = 0$, so $\frac{\partial^2 u}{\partial t^2} = 0$, and

$$\frac{T_0}{\rho_0}\overline{u}''(x) - g = 0 \Rightarrow \overline{u}(x) = \frac{\rho_0 g}{2T_0}x^2 + Ax + B.$$

Then BC $\overline{u}(0)=0$ implies $B=0$. And the BC $\overline{u}(L)=0$ implies

$$0=\overline{u}(L)=\frac{\rho_0 g}{2T_0}L^2+A\cdot L \Rightarrow A=\frac{\rho_0 g}{2T_0}L.$$

therefore the steady state solution becomes

$$\overline{u}(x)=\frac{\rho_0 g}{2T_0}\cdot x\cdot(x-L).$$

Therefore, the graph is a parabola with minimum occurring when $x=\frac{L}{2}$, then the minimum $\overline{u}_{\min}=\overline{u}\left(\frac{L}{2}\right)=\frac{\rho_0 g}{2T_0}L^2$. This is the sag in the center of the string due to gravity.

Intuition. Let's take a look at the units of $c=\sqrt{\frac{T_0}{\rho_0}}$.

$$c=\sqrt{\text{ mass}\cdot\text{ length/time}^2\cdot\text{ length/ mass}}=\text{ speed}.$$

So c should be some sort of speed. The only likely candidate seems to be the speed of propagation. If that's true, then we can also calculate propagation time,

$$\tau=\frac{L}{c}+\frac{L}{c}=\frac{2L}{c}.$$

Then frequency (cycles divided by time) follows:

$$f=\text{ constant}\cdot\frac{c}{L}=\text{ constant}\frac{1}{2}\cdot\frac{1}{L}\cdot\sqrt{\frac{T_0}{\delta\cdot A}}.$$

How can we increase the frequency?

1) Decrease length L.
2) Increase tension T_0.
3) Decrease density δ.
4) Increase area A.

They all match our intuition!

Note. Gravity doesn't change frequency according to this model!

12.6 Variations on the Wave Equation (optional)

Transverse Motion: stiffness.

Damped Motion: eventually return to stationary.

Transverse Vibrations: 2D membrane (drum).

Longitudinal Vibration: does not assume totally vertical motion.

13 Solve A Wave Equation

$$\begin{cases} \text{PDE: } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} & 0 < x < L, t > 0 \\ \text{BCs: } u(0,t) = 0 = u(L,t) & t > 0 \\ \text{ICs: } u(x,0) = U(x), \frac{\partial u}{\partial t}(x,0) = V(x) & 0 \le x \le L \end{cases}$$

We need to verify solutions that satisfy PDE and BCs form a vector space in order to use superposition principle. That is, we want to show that

$$W = \{u(x,t) \text{ is defined on } (x,t) \in (0,L) \times (0,\infty) \text{ that satisfies PDE and BCs}\}$$

forms a vector space.

Proof

Notice that $V = \{f : [0,L] \to \mathbb{R}\}$ is a vector space. We wish to show that W is a subspace of V.

Given $u_1, u_2 \in W$, then consider $u_3(x,t) = au_1(x,t) + bu_2(x,t)$ and note that partial derivatives are linear, we have

$$\begin{aligned} \frac{\partial^2 u_3}{\partial t^2} &= \frac{\partial^2}{\partial t^2}(au_1 + bu_2) \\ a\frac{\partial^2 u_1}{\partial t^2} + b\frac{\partial^2 u_2}{\partial x^2} &= ac^2\frac{\partial^2 u_1}{\partial x^2} + bc^2\frac{\partial^2 u_2}{\partial x^2} \\ &= c^2(a\frac{\partial^2 u_1}{\partial x^2} + b\frac{\partial^2 u_2}{\partial x^2}) \end{aligned}$$

So the PDE condition is satisfied. Then $u_3(0,t) = au_1(0,t) + bu_2(0,t) = 0 + 0 = 0$ and $u_3(L,t) = au_1(L,t) + u_2(L,t) = 0 + 0 = 0$.

So the BCs condition is satisfied. Finally, since piecewise smooth functions are closed under addition and scalar multiplication, we have shown that $u_3(x,t) \in W \subseteq V$. Hence $W \le V$. $\square$

Now we assume separation of variables, $u(x,t) = F(x)G(t) \neq 0$, then the second partial derivatives are

$$F(x)G''(t) = c^2 F''(x)G(t) \Rightarrow \frac{1}{c^2}\frac{G''(t)}{G(t)} = \frac{F''(x)}{F(x)} = -\lambda.$$

Notice that the eigenvalue problem is the same as the heat equation. Then the

BCs become

$$u(0,t) = 0 \Rightarrow F(0)G(t) = 0 \Rightarrow F(0) = 0.$$

and

$$u(L,t) = 0 \Rightarrow F(L)G(t) = 0 \Rightarrow F(L) = 0.$$

Thus the time domain problem and eigenvalue problem become

$$G''(t) = -\lambda c^2 G(t).$$

and

$$\begin{cases} F''(x) = -\lambda F(x), \\ F(0) = 0 = F(L). \end{cases}$$

13.1 Eigenvalue Problem

Case. $\lambda < 0$, then the general solution is

$$F(x) = c_1 e^{-\sqrt{\lambda}x} + c_2 e^{\sqrt{\lambda}x}.$$

Applying the BCs,

$$F(0) = 0 \Rightarrow c_2 = -c_1 \text{ and } F(L) = 0 \Rightarrow c_1 = c_2 = 0.$$

because $e^{-\sqrt{\lambda}L} - e^{\sqrt{\lambda}L} \neq \Leftrightarrow \lambda \neq 0$. Hence this case yields the trivial solution.

Case. $\lambda = 0$ also yields trivial solution as before.

Case. $\lambda > 0$, by the exact same procedure as before, we obtain $\sqrt{\lambda} = \frac{n\pi}{L}$ and

$$F_n(x) = c_2 \sin\left(\frac{n\pi x}{L}\right) \text{ for } n = 1, 2, \ldots.$$

13.2 Time-Domain Problem

Similarly as above, we can obtain the general solution of $G(t)$

$$G_n(t) = d_1 \cos(\sqrt{\lambda_n}ct) + d_2 \sin(\sqrt{\lambda_n}ct) \text{ for } n = 1, 2, \ldots.$$

13.3 General Solution

Therefore,

$$u_n(x,t) = F_n(x)G_n(t) = \tilde{A}_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) + \tilde{B}_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) \text{ for } n = 1, 2, \ldots.$$

Now by superposition principle, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cdot u_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

Now applying the ICs to find the coefficients:

$$U(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right).$$

This is just FSS. Applying the projection formula,

$$A_n = \frac{2}{L}\int_0^L U(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

The other IC requires us to do term-by-term differentiation:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} A_n \left(-\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) + B_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right).$$

Therefore, the other IC also yields a FSS,

$$V(x) = u_t(x,0) = \sum_{n=1}^{\infty} B_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

By projection,

$$B_n = \frac{2}{n\pi c}\int_0^L V(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

13.4 Convergence

Note that the wave equation doesn't have an exponentially-decaying term as the heat equation. We have a spatial wave and a temporal wave. Thus, the convergence depends on how A_n, B_n behave as $n \to \infty$, which in turn depends on the initial position and velocity.

While this is a drawback of the FS solution of the wave equation, it motivates d'Alembert's solution using traveling waves:

$$u(x,t) = f(x-ct) + g(x+ct)..$$

14 Motion of a guitar string

For a plucked guitar string at $t = 0$, the initial position, $U(x)$, looks like either a sharp isosceles triangle or a hill-like smooth curve.

In either case $U(x)$ is continuous and $U'(x)$ is piecewise continuous or smooth. Note that if the initial velocity is $V(x) = 0$ then $B_n = 0$. If the initial position is given by

$$U(x) = \begin{cases} \frac{ax}{h}, & 0 < x \le h \\ \frac{a(L-x)}{L-h}, & h < x < L \end{cases}$$

Then using integration by parts, we get

$$A_n = \frac{2a\sin(n\pi h/L)}{(n\pi)^2 \frac{h}{L}\frac{L-h}{L}}.$$

We can guess this converges because of n^2 term on the denominator.

So the position of the string is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2a}{\pi^2}\frac{L}{h}\frac{L}{L-h}\frac{\sin(n\pi h/L))}{n^2}\cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

I claim that this converges absolutely:

$$0 \le \sum_{n=1}^{\infty} |u(x,t)| \le \sum_{n=1}^{\infty} \frac{M}{n^2}$$

for some constant $M > 0$, so it converges absolutely. Hence it converges uniformly to a continuous function. But note

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} \frac{2a}{\pi^2}\frac{L}{h}\frac{L}{L-h}\frac{\sin(n\pi h/L))}{n^2} \cdot -\left(\frac{n\pi x}{L}\right)^2 \cdot \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

Notice n^2 that helped achieving convergence got canceled! By divergence test (take n to infinity but the term doesn't go to zero), this doesn't converge. This is okay because in practice we always use a finite sum, so divergence is not a big concern.

Thus, for each fixed $N > 0$, the partial sum of the formal solution, $S_N(x)$, will satisfy the PDE and BC and will approximate the initial condition as accurately as we wish.

15 Supplement: The Sound of Music

Example (space and time waves)**.** For the general solution, if we fix a t, then we get a space wave, which represents the physical position of the string at a specific time. If we fix a x, we get a time wave. which represents the evolution of each point of the string through time.

15.1 Standing Waves

The vertical displacement is a linear combination of the simple product solutions.

Each is called the "normal nodes of vibration" for $n = 1, 2, \ldots$. And each mode has amplitude $\sqrt{A_n^2 + B_n^2}$.

The nth normal node is called the "nth harmonic".

For each fixed t, each node looks like a simple oscillation called a *standing wave*.

Note that the period of each mode (in time) is $\tau_n = 2\pi \cdot \frac{L}{n\pi c} = \frac{2L}{nc}$.

So one cycle is completed every τ_n thus the frequency is $f = \frac{1}{\tau_n} = \frac{nc}{2L} = \frac{n}{\tau_1}$ for $n = 1, 2, \ldots$

The sound produced consists of the superposition of these infinite frequencies.

15.2 Music Theory

See slides. This is common sense. E_2 is 2 octaves above E_0.

When we pluck guitar, at $n = 1$ we have the **first harmonic** or the *fundaemental mode:* $A_1 \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi c t}{L}\right)$.

Assume $\omega_1 = 82.41\text{Hz}$. We cannot mute fundamental harmonic because it has no inner stationary points.

At $n = 2$, the shape of the standing wave is determined by $\sin\left(\frac{2\pi x}{L}\right)$, which has 1 inner stationary point (that never moves). Then the frequency is twice the fundamental harmonic. The amplitude is a quarter of the fundamental harmonic.

Similarly, $n = 3$, it has 2 inner stationary points. So we can mute the third harmonic at $h = \frac{L}{3}$ or $\frac{2L}{3}$.

15.3 Fun Facts

- Each octave increase corresponds to a doubling of the frequency.
- For every octave the frequency doubles per 12 notes then the frequency per note increases by a factor of 1.06.

How large is the signal propagation speed $c = \sqrt{\frac{T_0}{\delta A}}$?

The time required to complete one cycle, $T_1 = \frac{2L}{c}$. So

$$\omega_1 = \frac{1}{T_1} = \frac{c}{2L} \Rightarrow c = 2L\omega_1 = 106.9 \text{ m/sec}.$$

Then for nth harmonics, $c = n \cdot \omega_1 = n \cdot 106.9$ m/sec.

16 d'Alembert's solution to the wave equation

We want to use trigonometric identity to transform the solution.

Recall

$$\sin(a)\cos(b) = \frac{1}{2}[\sin(a+b)+\sin(a-b)] \text{ and } \sin(a)\sin(b) = \frac{1}{2}[\cos(a-b)-\cos(a+b)].$$

Applying this to the solution and we obtain

$$\begin{aligned}
&A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) \\
&= \frac{A_n}{2}\left(\sin\left(\frac{n\pi}{L}(x+ct)\right) + \sin\left(\frac{n\pi}{L}(x-ct)\right)\right) + \frac{B_n}{2}\left(\cos\left(\frac{n\pi}{L}(x-ct)\right) - \cos\left(\frac{n\pi}{L}(x+ct)\right)\right) \\
&= f(x+ct) + g(x-ct)
\end{aligned}$$

Let's introduce the natural variables

$$y = x + ct \text{ and } z = x - ct \text{ for } c > 0 \Rightarrow x = \frac{y+z}{2}, t = \frac{y-z}{2c}.$$

Note that any point $(x,t) \in \mathbb{R} \times [0,\infty)$ can be written as a point $(y,z) \in \mathbb{R}^2$.

Intuition. y, z represent two lines in the x, t plane and their intersection gives us (x,t).

We want to solve the wave equation in terms of these new variables so we need to write the wave equation in terms of (y,z). Note this is not $u(y,z)$, as we can see from the counterexample below.

Example. Let $u(x,t) = x \cdot t$. Then

$$\begin{aligned}
u(x,t) &= \left(\frac{y+z}{2}\right)\left(\frac{y-z}{2c}\right) \\
&= \frac{y^2 - z^2}{4c} \\
&\neq u(y,z) = yz
\end{aligned}$$

Rewrite $u(x,t) = v(y,z)$, then by Chain Rule,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}(y,z) = \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x} = v_y + v_z.$$

In general, for any function $f(y,z)$, we have $\frac{\partial f}{\partial x} = f_y + f_z$.

Again by Chain Rule,

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} \\ &= \frac{\partial}{\partial x}(v_y + v_z) \\ &= [v_{yy} + v_{yz}] + [v_{zy} + v_{zz}] \\ &= \frac{\partial^2 v}{\partial u^2} + 2\frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 v}{\partial z^2}\end{aligned}$$

the combining is due to Clairaut's Theorem. Similarly,

$$\frac{\partial u}{\partial t} = c(v_y - v_z).$$

and

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 v}{\partial y^2} - 2\frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 v}{\partial z^2}\right).$$

Now plug them into the wave equation and we can rewrite it after cancellation as

$$4c^2 \frac{\partial^2 v}{\partial y \partial z} = 0.$$

We keep c^2 there because it contains useful information about the problem. This is a much simpler equation to solve! Note $\frac{\partial}{\partial y} v_z = 0$ implies that v_z constant with respect to y. That is,

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial z}\right) = 0 \Rightarrow \frac{\partial v}{\partial z} = f'(z).$$

Integrating both sides wrt z, we have

$$v(y, z) = f(z) + C_1(y).$$

Note that the constant is only wrt z, so it can be a function of y.

Now doing the same thing wrt y, we obtain

$$v(y, z) = g(y) + C_2(z).$$

By term matching, we obtain

$$v(y, z) = f(z) + g(y).$$

Plugging back x, t, we obtain

$$v(y, z) = u(x, t) = f(x - ct) + g(x + ct).$$

We can use the BCs and ICs to determine $f(z), g(y)$. This is the **d'Alembert's solution to the traveling wave equation**.

Definition: traveling waves

1) The function $f(x - ct)$ is a waveform that moves to the right with velocity $c > 0$ and is called a **traveling wave**.
2) Likewise, the function $g(x + ct)$ is a traveling wave moving to the left with velocity $-c$.

Note.

1) This solution is the general solution of the 1D wave equation as long as the 2nd derivatives are continuous. So we don't need to deal with convergence issues.
2) For the Fourier Series solution of the wave equation, it can be shown that in order for the series for $\partial_x^2 u(x,t)$ to converge for $t \geq 0$, we need $U''(x)$ to have a convergent series at $t = 0$. Then if the initial data, that is $U(x)$ and $V(t)$, are smooth enough then the Fourier series solution is the solution to the BVP.

17 D'Alembert's Solution continued

$$\begin{cases} \text{PDE: } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} & x \in \mathbb{R}, t > 0 \\ \text{ICs: } u(x,0) = U(x), \frac{\partial u}{\partial t}(x,0) = V(t) & x \in \mathbb{R} \end{cases}$$

Note. Since $x \in \mathbb{R}$, we no longer have boundaries or BCs.

The solution has the form:

$$u(x,t) = f(x - ct) + g(x + ct).$$

where f, g are traveling waves and need to be twice differentiable. We find $f(z)$ and $g(y)$ using the ICs.

We know that the set of functions that satisfy the wave equation form a vector space. So we can divide and conquer by separating the PDE into two problems:

$$\begin{cases} \partial_t^2 u_1 = c^2 \partial_x^2 u_1 \\ u_1(x,0) = U(x) \\ \partial_x u_1(x,0) = 0 \end{cases} \qquad \begin{cases} \partial_t^2 u_2 = c^2 \partial_x^2 u_2 \\ u_2(x,0) = 0 \\ \partial_x u_2(x,0) = V(x) \end{cases}$$

Intuition. By having 0 on the RHS, we can express f in terms of g, and solve an ODE involving one function instead.

If there exists two solutions u_1, u_2, then their sum satisfies the PDE and ICs.

Let's assume $u_1(x,t) = f_1(x-ct) + g_1(x+ct)$. Using the Chain Rule,

$$0 = \frac{\partial u_1}{\partial t}(x,0) = f_1'(x)\cdot(-c) + g_1'(x)\cdot(c) \Rightarrow f_1'(x) = g_1'(x).$$

Therefore, $f_1(x) = g_1(x) + k$ for some constant k. Using the other initial condition,

$$U(x) = u_1(x,0) = f_1(x) + g_1(x) = [g_1(x) + k] + g_1(x) \Rightarrow g_1(x) = \frac{U(x)}{2} - \frac{k}{2}.$$

Then

$$f_1(x) = g_1(x) + k = \frac{U(x)}{2} + \frac{k}{2}.$$

Therefore, the solution is

$$u_1(x,t) = f_1(x-ct) + g_1(x+ct) = \frac{U(x-ct)}{2} + \frac{U(x+ct)}{2}.$$

Intuition. If started at rest, the initial position of the string breaks in two, half moving to the left and half moving to the right at equal speeds c, each with half the amplitude of the original. The solution is the simple sum of these traveling waves.

Similarly, we assume $u_2 = f_2(x-ct) + g_2(x+ct)$, and

$$0 = u_2(x,0) = f_2(x) + g_2(x) \Rightarrow f_2(x) = -g_2(x) \Rightarrow f_2'(x) = -g_2'(x).$$

Using the second initial condition:

$$V(x) = \frac{\partial u}{\partial t}(x,0) = f_2'(x)\cdot(-c) + g_2'(x)\cdot c = -g_2'(x)\cdot(-c) + g_2'(x)\cdot c = 2cg_2'(x) \Rightarrow g_2'(x) = \frac{V(x)}{2c}.$$

That is, $g_2'(x)$ is an antiderivative of $\frac{V(x)}{2}$. Recall by Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x))\cdot g'(x).$$

So by the FTC, we can define

$$g_2(x) = \int_a^x \frac{V(t)}{2c}dt \text{ and } f_2(x) = \int_x^a \frac{V(t)}{2c}dt.$$

Therefore,

$$u_2(x.t) = f_2(x-ct) + g_2(x+ct) = \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{V(t)}{2c} dt.$$

Hence, the **general solution** is

$$u(x,t) = \frac{1}{2}U(x-ct) + \frac{1}{2}U(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} V(s)ds.$$

We can check it satisfies the ICs. When checking the second IC, we would need to apply the FTC using chain rule:

$$\begin{aligned}
\frac{1}{2c}\frac{d}{dt}\int_{x-ct}^{x+ct} V(s)ds &= \frac{1}{2c}\frac{d}{dt}\left(-\int_a^{x-ct} V(s)ds + \int_a^{x+ct} V(s)ds\right) \\
&= \frac{1}{2c}(-V(x-ct)\cdot(-c) + V(x+ct)\cdot c) \\
&= \frac{1}{2}(V(x-ct) + V(x+ct))
\end{aligned}$$

At $t = 0$, we obtain $\frac{\partial u}{\partial t}(x,0) = V(x)$.

Example. Solve the PDE:

$$\begin{cases} \text{PDE: } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} & x \in \mathbb{R}, t > 0 \\ \text{ICs: } u(x,0) = 0, \frac{\partial u}{\partial t}(x,0) = \frac{2}{1+x^2} & x \in \mathbb{R} \end{cases}$$

Apply the d'Alembert's formula we have

$$\begin{aligned}
u(x,t) &= 0 + \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{2}{1+x^2} ds \\
&= \frac{1}{c}[\arctan(x+ct) - \arctan(x-ct)]
\end{aligned}$$

Taking the limit as $t \to \infty$, for each fixed x,

$$\lim_{t\to\infty} \frac{1}{c}[\arctan(x+ct) - \arctan(x-ct)] = \frac{1}{c}\left\{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right\} = \frac{\pi}{c}.$$

See lecture slides for figures. The top of the wave just flattens out.

18 Laplace's Equation and Solution

We wish to solve the 2D equilibrium heat equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0.$$

Recall the Laplace Operator from prior lecture:

$$L(u) = \nabla^2 u = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Example (Thought Experiment in 1D)**.**

1) Suppose $f''(x) = 0 \Rightarrow f(x) = kx + c$, whose rate of change is constant.

2) for all possible k, the extrema only occur at the boundaries.

3) For any $x_0 \in (a, b)$, and for any $\varepsilon > 0$ such that $[x_0 - \varepsilon, x_0 + \varepsilon] \subseteq (a, b)$, we have

$$f(x_0) = \frac{f(x_0 - \varepsilon) + f(x_0 + \varepsilon)}{2} = \frac{k(x_0 - \varepsilon) + c - k(x_0 + \varepsilon) - c}{2} = f(x_0).$$

This is a characterization of equilibrium.

18.1 Rectangular Domain

$$\begin{cases} \text{PDE:} & \Delta u = 0, \qquad (x,y) \in (0,L) \times (0,H) \\ \text{BCs:} & u(x,0) = f_1(x) \\ & u(0,H) = f_2(x) \\ & u(0,y) = g_1(y) \\ & u(L,y) = g_2(y) \end{cases}$$

Note that we don't have ICs because this is a steady state problem. Since Laplace Operator is linear, we can use the same trick by decomposing it into two problems. Let $u(x, y) = u_1(x, y) + u_2(x, y)$.

$$\begin{cases} \text{PDE:} & \Delta u_1 = 0, (x,y) \in (0,L) \times (0,H) \\ \text{BCs:} & u_1(x,0) = f_1(x) \\ & u_1(0,H) = f_2(x) \\ & u_1(0,y) = 0 \\ & u_1(L,y) = 0 \end{cases} \begin{cases} \text{PDE:} & \Delta u = 0, (x,y) \in (0,L) \times (0,H) \\ \text{BCs:} & u_2(x,0) = 0 \\ & u_2(0,H) = 0 \\ & u_2(0,y) = g_1(y) \\ & u_2(L,y) = g_2(y) \end{cases}$$

Note the homogeneous BCs and PDE form a vector space. Then we can use the nonhomogeneous BCs to get the coefficients (like ICs).

separation of variables

Let $u_1(x,y) = F(x)G(y) \neq 0$. Then $\Delta u_1 = 0$ implies

$$F''(x)G(y) = F(x)G''(y) \Rightarrow -\frac{G''(y)}{G(y)} = \frac{F''(x)}{F(x)} = -\lambda.$$

F-equation

We can solve $F(x)$ under Dirichlet conditions exactly like before. Note that F is nontrivial only if $\lambda > 0$. Then we obtain

$$F_n(x) = C_n \sin\left(\frac{n\pi x}{L}\right), n = 1, 2, \ldots.$$

G-equation

$G''(y) = \lambda_n G(y) \Rightarrow r^2 e^{ry} = \lambda_n e^{ry} \Rightarrow r^2 = \left(\frac{n\pi}{L}\right)^2 \Rightarrow r = \pm\frac{n\pi}{L} \Rightarrow e^{ry} = e^{\pm n\pi y/L}$.

So $\{e^{n\pi y/L}, e^{-n\pi y/L}\}$ is a basis solutions for the G-equation. For convenience, we wish to have basis functions that vanish at the endpoints $y = 0$ and $y = H$ because it makes fining the coefficients really easy.

Recall that cosh is even and sinh is odd. Then

$$\frac{e^{n\pi y/L} + e^{-n\pi y/L}}{2} = \cosh\left(\frac{n\pi y}{L}\right) \text{ and } \frac{e^{n\pi y/L} - e^{-n\pi y/L}}{2} = \sinh\left(\frac{n\pi y}{L}\right).$$

However, using $\{\sinh\left(\frac{n\pi y}{L}\right), \cosh\left(\frac{n\pi y}{L}\right)\}$ is not the basis we seek because cosh doesn't go to zero. Let's use the identity:

$$\sinh\left(\frac{n\pi[y-H]}{L}\right) = \sinh\left(\frac{n\pi y}{L} - \frac{n\pi H}{L}\right) = \sinh\left(\frac{n\pi y}{L}\right)\cosh\left(\frac{n\pi H}{L}\right) - \cosh\left(\frac{n\pi y}{L}\right)\sinh\left(\frac{n\pi H}{L}\right).$$

Therefore, $\left\{\sinh\left(\frac{n\pi y}{L}\right), \sinh\left(\frac{n\pi[y-H]}{L}\right)\right\}$ is also a basis of solutions for the G-equation. Note that this is not an orthogonal basis. Thus, the general solution of G_n is:

$$G_n(y) = A_n \sinh\left(\frac{n\pi y}{L}\right) + B_n \sinh\left(\frac{n\pi[y-H]}{L}\right), n = 1, 2, \ldots.$$

Therefore, the general solution of $u(x,y)$ is

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + b_n \sinh\left(\frac{n\pi[y-H]}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

When $y = 0$, only the second term remains, so the nonhomogeneous BC yields

$$f_1(x) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{-n\pi H}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} \tilde{b}_n \sin\left(\frac{n\pi x}{L}\right).$$

So by projection formula,

$$b_n = \frac{\frac{2}{L}\int_0^L f_1(x) \sin\left(\frac{n\pi x}{L}\right)}{\sinh\left(\frac{-n\pi H}{L}\right)}.$$

And similarly,

$$f_2(x) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi H}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} \tilde{a}_n \sin\left(\frac{n\pi x}{L}\right).$$

And

$$a_n = \frac{\frac{2}{L}\int_0^L f_2(x) \sin\left(\frac{n\pi x}{L}\right)}{\sinh\left(\frac{n\pi H}{L}\right)}.$$

$$\begin{cases} G''(y) = \lambda G(y) \\ G(0) = 0, G(H) = 0 \end{cases}$$

and $F''(x) = -\lambda F(x)$. We do the same process again to solve $u_2(x, t)$. Note that since only $\lambda < 0$ works to solve to F-equation, we require $|\lambda| = \left(\frac{n\pi}{H}\right)^2 \Rightarrow \lambda = -\left(\frac{n\pi}{H}\right)^2$. The solution is

$$u_2(x, y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{H}\right) \sin\left(\frac{n\pi y}{H}\right) + d_n \sinh\left(\frac{n\pi[x-L]}{H}\right) \sin\left(\frac{n\pi y}{H}\right).$$

with coefficients

$$c_n = \frac{\frac{2}{H}\int_0^H g_2(y) \sin\left(\frac{n\pi y}{H}\right) dy}{\sinh\left(\frac{n\pi L}{H}\right)} \text{ and } d_n = \frac{\frac{2}{H}\int_0^H g_1(y) \sin\left(\frac{n\pi y}{H}\right) dy}{\sinh\left(\frac{-n\pi L}{H}\right)}.$$

Then the **formal solution** is

$$\begin{aligned} u(x, y) &= u_1(x, y) + u_2(x, y) \\ &= \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi x}{L}\right) + b_n \sinh\left(\frac{n\pi[y-H]}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \\ &\quad + \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{H}\right) \sin\left(\frac{n\pi y}{H}\right) + d_n \sinh\left(\frac{n\pi[x-L]}{H}\right) \sin\left(\frac{n\pi y}{H}\right) \end{aligned}$$

with the coefficients:

$$a_n = \frac{\frac{2}{L}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right)dx}{\sinh\left(\frac{n\pi H}{L}\right)} \text{ and } b_n = \frac{\frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)dx}{\sinh\left(\frac{-n\pi H}{L}\right)}$$

$$c_n = \frac{\frac{2}{H}\int_0^H g_2(y)\sin\left(\frac{n\pi y}{H}\right)dy}{\sinh\left(\frac{n\pi L}{H}\right)} \text{ and } d_n = \frac{\frac{2}{H}\int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right)dy}{\sinh\left(\frac{-n\pi L}{H}\right)}$$

Note. At the corners we have $u(0,0) = u(L,0) = u(0,H) = u(L,H) = 0$.

Intuition. Recall the formal solution $u(x,y)$ from above. We can show if $0 < a < b \Rightarrow \frac{\sinh(na)}{\sinh(nb)} < e^{n(a-b)}$ and $e^{n(a-b)} < 1$. Since $x < L$ and $y < H$, the coefficients are finite integral of quotient of two hyperbolic terms (less than 1) and thus are bounded. We can show convergence along this line of logic.

19 Laplace in Circular Geometry

We now wish to determine the formula for Laplace's equation in terms of the polar coordinates. Let's denote $u(x, y) = v(r, \theta)$. Suppose $x = r\cos(\theta), y = r\sin(\theta)$. Then

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}, r > 0.$$

Using the Chain rule,

$$\frac{\partial v}{\partial r} = \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = u_x\cos(\theta) + u_y\sin(\theta).$$

and

$$\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial \theta} = u_x(-r\sin\theta) + u_y(r\cos\theta).$$

Now for the 2nd partials, we use the rules we discovered above:

$$\begin{aligned}
\frac{\partial^2 v}{\partial r^2} &= \frac{\partial}{\partial r}[u_x\cos\theta + u_y\sin\theta] \\
&= \frac{\partial}{\partial r}[u_x\cos\theta] + \frac{\partial}{\partial r}[u_y\sin\theta] \\
&= [u_{xx}\cos^2\theta + u_{xy}\cos\theta\sin\theta] + [u_{yx}\cos\theta\sin\theta + u_{yy}\sin^2\theta] \\
&= u_{xx}\cos^2\theta + 2u_{xy}\cos\theta\sin\theta + u_{yy}\sin^2\theta
\end{aligned}$$

Again doing this for θ,

$$\frac{\partial^2 v}{\partial \theta^2} = \frac{\partial}{\partial t}[u_x(-r\sin\theta)] + \frac{\partial}{\partial t}[u_y(r\cos\theta)].$$

Thus by product rule, we eventually obtain

$$\frac{\partial^2 v}{\partial \theta^2} = u_{xx}r^2\sin^2\theta - 2u_{yx}r^2\cos\theta\sin\theta + u_{yy}r^2\cos^2\theta - r(u_x\cos\theta + u_y\sin\theta).$$

dividing both sides by r^2 yields:

$$\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = u_{xx}\sin^2\theta - 2u_{yx}\cos\theta\sin\theta + u_{yy}\cos^2\theta - \frac{1}{r}(u_x\cos\theta + u_y\sin\theta).$$

Therefore, if we add these two second partials together:

$$v_{rr} + \frac{1}{r^2}\cdot v_{\theta\theta} = u_{xx} + u_{yy} - \frac{1}{r}v_r.$$

This gives us

$$\Delta u = u_{xx} + u_{yy} = v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta}.$$

Remark.

Wave Equation: we have the same eigenvalue problem as heat equation. The time domain problem has a negative sign which leads to oscillating terms in time.

Laplace Equation: Same eigenvalue problem. The time domain problem has positive sign so we obtain hyperbolic functions in order to easily solve the coefficients.

19.1 different PDE domain

1) Inside the disc of radius R. Then $0 < r < R, \theta_0 < \theta \leq \theta_0 + 2\pi$. Physical boundary is $r = R$.
2) Outside the disc of radius R. Then $R < r < \infty, \theta_0 < \theta \leq \theta_0 + 2\pi$. Physical boundary: $r = R$.
3) Annulus: $R_i < r < R_o, \theta_0 < \theta \leq \theta_0 + 2\pi$. Physical boundaries: $r = R_i, r = R_o$.
4) Pie shaped sector: $0 < r < R, \theta_1 < \theta \leq \theta_2$. Physical boundary: $r = R, \theta = \theta_1, \theta = \theta_2$.

Example (circular disc)**.** Suppose $\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}$ on the domain $D = \{(r, \theta) | 0 \leq r \leq R, -\pi < \theta \leq \pi\}$.

Note. $r = 0, r = \infty$ are **singular points** of the coordinate system for $\Delta u = 0$ but *not* of the physical system. For physical reasons it is reasonable to assume boundedness at the origin: $|v(0, \theta)| < \infty$.

By periodicity we can assume continuity on the derivatives at the boundaries $\theta = \pm\pi$.

Claim. The set of functions that satisfy the *boundedness condition* or the *periodicity conditions* form a vector space.

19.2 separation of variables

Consider

$$\begin{cases} \text{PDE: } \Delta u = 0 & 0 < r < R, \theta \in (-\pi, \pi) \\ \text{BCs: } v(R, \theta) = f(\theta) \end{cases}$$

Assume $v(r, \theta) = F(\theta)G(r) \neq 0$. Then we obtain

$$\begin{cases} F''(\theta) & = -\lambda F(\theta) \\ F(-\pi) & = F(\pi) \\ F'(-\pi) & = F'(\pi) \end{cases} \qquad r^2 G''(r) + rG'(r) - \lambda G(r) = 0$$

19.3 F-equation

The same eigenvalue problem. Only $\lambda > 0$ is nontrivial, so

$$F_n(\theta) = A_n \sin(\sqrt{\lambda_n}\theta) + B_n \cos(\sqrt{\lambda_n}\theta)$$

where $\lambda_n = n^2, n = 1, 2, \ldots$, since $L = \pi$.

19.4 G-equation

Plug in $\lambda = n^2$, we have

$$r^2 G''(r) + rG'(r) - n^2 G(r) = 0.$$

Let $G(r) = r^p$. Then we get

$$r^2 p(p-1) r^{p-2} + rpr^{p-1} - n^2 r^2 = 0.$$

After cancellation, we obtain

$$p = \pm n, n = 1, 2, \ldots.$$

If $n \neq 0$, then $r_1 = r^n, r_2 = r^{-n}$. So by superposition, we get

$$G(r) = c_1 r^n + c_2 r^{-n}.$$

The boundedness condition yields,

$$|G(0)| < \infty \Rightarrow c_2 = 0 \Rightarrow G_n(r) = c_1 r^n, n = 1, 2, \ldots$$

Because $r = 0$ makes the second term undefined.

If $n = 0$, then

$$r^2 G''(r) + rG'(r) = 0 \Rightarrow rG''(r) + G'(r) = 0 \Rightarrow \frac{d}{dr}(rG'(r)) = 0 \Rightarrow rG'(r) = C_1.$$

Finally note that

$$rG'(r) = C_1 \Rightarrow G'(r) = \frac{C_1}{r} \Rightarrow G(r) = C_1 \ln(r) + C_2.$$

Boundedness again forces $C_1 = 0 \Rightarrow G_0(r) = C_2$.

Note. If the domain is Annulus we would keep both terms. Also if the BCs aren't so nice we might have to use periodicity condition.

Now the general solution is

$$\begin{aligned} v(r,\theta) &= a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) \\ &= a_0 + \sum_{n=1}^{\infty} A_n \left(\frac{r}{R}\right)^n \cos(n\theta) + B_n \left(\frac{r}{R}\right)^n \sin(n\theta) \end{aligned}$$

where $A_n = a_n R^n, B_n = b_n R^n$. Now using the BCs:

$$f(\theta) = v(R,\theta) = a_0 + \sum_{n=1}^{\infty} A_n \cdot 1 \cdot \cos(n\theta) + B_n \cdot 1 \cdot \sin(n\theta).$$

which is a F.S.! So,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)d\theta, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)d\theta.$$

19.5 Mean Value Property

Note that

$$v(r,\theta) = a_0 + \sum_{n=1}^{\infty} a_n \left(\frac{r}{R}\right)^n \cos(n\theta) + b_n \left(\frac{r}{R}\right)^n \sin(n\theta).$$

implies that the temperature at the center is an average of the temperature at the boundary:

$$v(0,\theta) = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta.$$

For any fixed $0 < \tilde{R} < R$, suppose $w(r,\theta)$ satisfies

$$\begin{cases} \text{PDE: } \Delta w = 0, & 0 < r < \tilde{R}, \theta \in (-\pi,\pi) \\ \text{BCs: } w(\tilde{R},\theta) = g(\theta) \end{cases}$$

Then the solution is almost identical:

$$w(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n \left(\frac{r}{\tilde{R}}\right)^n \cos(n\theta) + B_n \left(\frac{r}{\tilde{R}}\right)^n \sin(n\theta).$$

where for $n \geq 1$, we have

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\cos(n\theta)d\theta \text{ and } B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)d\theta.$$

And note that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)d\theta = A_0 = w(0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta.$$

So every circle of radius $0 < \tilde{R} < R$ centered at $r = 0$ has the same average temperature as the circle of radius $r = R$.

Theorem: Mean Value Theorem for Laplace's Equation

The temperature at any point x_0 is the average of the temperature along any circle of radius $\tilde{R} > 0$ lying inside the domain and centered at x_0.

Proof

Sketch: Suppose $\Delta u = 0$ in domain D and suppose x_0 is in domain D, then $\Delta u = 0$ on any circle centered at x_0 and within D, thus solving this Laplacian "subproblem" shows that x_0 is the average of the temperature along any circle lying inside D and centered at x_0 (like we showed above).

□

Theorem: Maximum Principle for Laplace's Equation

In steady state, assuming no sources, the temperature cannot attain its maximum in the interior unless the temperature is constant everywhere (In other words, the extrema values are achieved on the boundary).

Proof

Sketch: Suppose the temperature is not constant and that the maximum occurs at a point P in the domain. Now since $u(P)$ is the average temperature of all the points on any circle centered at P, it cannot be larger than any temperature on the circle (since it's an average) thus we have a contradiction. Thus the maximum temperature occurs on the boundary. $\square$

Claim. If $\Delta u = 0$ on an open region then u is $\mathcal{C}^\infty$ in the open region.

Proof

Assume the standard solution with $v(R, \theta) = f(\theta)$. Suppose $f(\theta)$ is piecewise smooth $-\pi < \theta \leq \pi$ then there exists a finite number $M > 0$ such that

$$|a_0| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)| d\theta \leq \frac{M}{2}, |a_n| \leq M, |b_n| \leq M.$$

Notice:

- if $r = R$, then the F.S. converges to $\frac{1}{2}(f(\theta^+) + f(\theta^-))$.
- if $r = 0$, then $v = 0$.
- If $0 < r < R$, then we want to show it converges absolutely.

Since $0 < \frac{r}{R} < 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \left|a_n \left(\frac{r}{R}\right)^n \cos(n\theta)\right| + \left|b_n \left(\frac{r}{R}\right)^n \sin(n\theta)\right| &\leq \sum_{n=1}^{\infty} \left[M \left(\frac{r}{R}\right)^n + M\left(\frac{r}{R}\right)^n\right] \\
&= \sum_{n=1}^{\infty} 2M \left(\frac{r}{R}\right)^n \\
&= \frac{2M(r/R)}{1-(r/R)} \\
&= \frac{2Mr}{R-r}
\end{aligned}$$

Therefore, $v(r, \theta)$ converges absolutely for each $r \in (0, R)$. $\square$

Derivative

Differentiation gives:

$$\frac{\partial v}{\partial \theta} = -\sum_{n=1}^{\infty} a_n \left(\frac{r}{R}\right)^n \cdot n \cdot \sin(n\theta) + b_n \left(\frac{r}{R}\right)^n \cdot n \cdot \cos(n\theta).$$

Since $\left(\frac{r}{R}\right)^n = e^{n\ln(r/R)} = e^{-n|\ln(r/R)|}$ so

$$\sum_{n=1}^{\infty} \left| a_n \left(\frac{r}{R}\right)^n n\sin(n\theta) + b_n \left(\frac{r}{R}\right)^n n\cos(n\theta) \right| \leq \sum_{n=1}^{\infty} 2M \frac{n}{e^{n|\ln(r/R)|}}.$$

By Ratio test, the series converges absolutely. This means that the derivative converges uniformly and thus we can swap differentiation and infinite sum by term-by-term differentiation theorem. This result applies to all orders of derivatives.

Theorem: general solution of annulus

If $v(R_i, \theta) = f_i(\theta), v(R_o, \theta) = f_o(\theta)$, then the general solution over an annulus is

$$v(r,\theta) = [a_0 + A_0 \ln(r)] + \sum_{n=1}^{\infty} \left(\frac{r}{R_o}\right)^n [a_n \cos(n\theta) + b_n \sin(n\theta)]$$
$$+ \sum_{p=1}^{\infty} \left(\frac{R_i}{r}\right)^p [A_p \cos(p\theta) + B_p \sin(p\theta)]$$

Note. This solution encompasses the cases of completely inside the disk and outside the disk.

19.6 Uniqueness of Solutions

Uniqueness of Laplacian's equation

Suppose $u_1(x,y), u_2(x,y)$ both satisfy the PDE and BCs. Let $v(x,y) = \tilde{v}(r,\theta) = u_1 - u_2$. Notice

$\Delta v = \Delta u_1 - \Delta u_2 = 0 - 0 = 0$ and $\tilde{v}(R,\theta) = \tilde{u}_1(R,\theta) - \tilde{u}_2(R,\theta) = f(\theta) - f(\theta) = 0$.

Thus, $\tilde{u}$ satisfies the above PDE and BCs. Note that the Maximum Principle states that the maximum and minimum of $v(x,y) = \tilde{v}(r,\theta)$ occur on the boundary but $\tilde{v}(R,\theta) = 0$ so $\tilde{v} = 0 \Rightarrow u_1 = u_2$.

Theorem: Uniqueness of Heat Equation

Suppose $u_1(x,t), u_2(x,t)$ are $\mathcal{C}^2$ solutions of the problem

$$\begin{cases} \text{PDE: } u_t = ku_{xx} & 0 < x < L, t > 0 \\ \text{BCs: } u(0,t) = a(t), u(L,t) = b(t) & t > 0 \\ \text{ICs: } u(x,0) = f(x) & 0 \le x \le L \end{cases}$$

where $a(t), b(t), f(x)$ are given $\mathcal{C}^2$ functions when $u_1(x,t) = u_2(x,t)$ for all $x \in [0,L], t \ge 0$.

Proof

Sketch: Let $v(x,t) = u_1(x,t) - u_2(x,t)$. Define

$$F(x) = \int_0^L |v(x,t)|^2 \, dt \ge 0.$$

By using integration by parts, we can show that $F'(x) \le 0$ and since $F(0) = 0, F(t) \ge 0$, this is only possible if $F(t) = 0 \Rightarrow v(x,t) = 0 \Rightarrow u_1(x,t) = u_2(x,t)$. $\square$

Theorem: Uniqueness of Wave Equation

Suppose $u_1(x,t), u_2(x,t)$ are $\mathcal{C}^2$ solutions of the problem

$$\begin{cases} \text{PDE: } u_t = c^2 u_{xx} & 0 < x < L, t > 0 \\ \text{BCs: } u(0,t) = a(t), u(L,t) = b(t) & t > 0 \\ \text{ICs: } u(x,0) = U(x), u_t(x,0) = V(x) & 0 \le x \le L \end{cases}$$

where $a(t), b(t), U(x), V(x)$ are given $\mathcal{C}^2$ functions when $u_1(x,t) = u_2(x,t)$ for all $x \in [0,L], t \ge 0$.

Proof

Sketch: Define *Lyapunov functional* as

$$H(t) = int_{x=0}^{x=L}[c^2 \cdot v_x^2(x,t) + v_t^2(x,t)] \; dx.$$

Then we can show that $H'(t) = 0$ and since $H(0) = 0$, then it must be that $H(t) = 0 \Rightarrow v_t(x,t) \Rightarrow v(x,t) = \int_0^t v_t(x,s) \; dx \Rightarrow u_1(x,t) = u_2(x,t)$. □

19.7 Classification Theorem

It asserts that every second-order linear PDE with constant coefficients, where the unknown function has *two* independent variables, can be transformed by a change of variables into exactly one of the following forms:

1) generalized wave equation

$$-c^2 u_{xx} + u_{tt} + \alpha u = f(x,t), c > 0 \text{ (hyperbolic case)}$$

since it's 2nd derivative minus 2nd derivative.

2) generalized Poisson/Laplace equation $(t = y)$

$$a^2 u_{xx} + u_{tt} + \alpha u = g(x,t), a > 0 \text{ (elliptic case)}.$$

3) generalized heat equation

$$-k^2 u_{xx} + u_t + \alpha u = h(x,t), k > 0 \text{ (parabolic case)}.$$

4)

$$u_{xx} + cu = g(x,t), \text{ (degenerate case)}.$$

20 Fourier Transform

20.1 Complex Fourier Series

Recall by Euler's formula,

$$\cos\left(\frac{n\pi x}{L}\right) = \frac{e^{in\pi x/L} + e^{-in\pi x/L}}{2} \text{ and } \sin\left(\frac{n\pi x}{L}\right) = \frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i}.$$

Therefore, the F.S. becomes

$$\begin{aligned}
a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) &= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} + \frac{b_n}{2i}\right) e^{in\pi x/L} + \left(\frac{a_n}{2} - \frac{b_n}{2i}\right) e^{-in\pi x/L} \\
&= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2}\right) e^{in\pi x/L} + \left(\frac{a_n + ib_n}{2}\right) e^{-in\pi x/L} \\
&= a_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/L} + c_{-n} e^{-in\pi x/L}
\end{aligned}$$

Notation. c_{-n} denotes the coefficient of the exponential term with $-n$.

If we let $n = -m$, then by projection,

$$\begin{aligned}
c_{-n} = \frac{1}{2}(a_n + ib_n) &= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx + \frac{i}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right) \\
&= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{-m\pi x}{L}\right)dx + \frac{i}{L}\int_{-L}^{L} f(x)\sin\left(\frac{-m\pi x}{L}\right)dx\right) \\
&= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx - \frac{i}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx\right) \\
&= \frac{1}{2}(a_m - ib_m) = c_m
\end{aligned}$$

Therefore,

$$\text{F.S.}[f](x) = a_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/L} + \sum_{m=-1}^{-\infty} c_m e^{in\pi x/L} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}.$$

From previous lecture, we obtain the **Lorentz series.**

The coefficients are

$$\begin{aligned} c_n = \frac{1}{2}(a_n - ib_n) &= \frac{1}{2L}\int_{-L}^{L} f(x)\left[\cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right)\right]dx \\ &= \frac{1}{2L}\int_{-L}^{L} f(x)\cdot e^{-in\pi x/L}dx \end{aligned}$$

Thus the complex form of the Fourier series of $f(x)$ is

$$\text{F.S.}[f](x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\cdot e^{-in\pi x/L}dx, n = 0, \pm 1, \pm 2, \ldots.$$

Notice that the positive exponential term is used in the series and the negative exponential term is used to find the coefficients.

20.2 Orthogonality

The inner product of two complex-valued functions $f(x)$ and $g(x)$, piecewise continuous on $[-L, L]$ is defined as

$$\langle f(x), g(x)\rangle = \int_{-L}^{L} f(x)\overline{g(x)}dx = \int_{-L}^{L}[f_1(x) + if_2(x)]\overline{[g_1(x) + ig_2(x)]}.$$

with norm defined by

$$\|f\| = \sqrt{\langle f(x), f(x)\rangle} = \int_{-L}^{L} |f(x)|^2 dx \in \mathbb{R}.$$

Note that

$$\begin{aligned} \langle e^{im\pi x/L}, e^{in\pi x/L}\rangle &= \int_{-L}^{L} e^{im\pi x/L}\cdot\overline{e^{inx/L}}dx \\ &= \int_{-L}^{L}\left[\cos\left(\frac{(m-n)\pi x}{L}\right) + i\sin\left(\frac{(m-n)\pi x}{L}\right)\right]dx \\ &= \begin{cases} 0 & \text{if } m \neq n \\ 2L & \text{if } m = n \end{cases} \end{aligned}$$

Example. Compute the complex F.S. of $f(x) = e^{ax}, x \in [-L, L]$, where $a \in \mathbb{R}$.

$$
\begin{aligned}
c_n &= \frac{1}{2L}\int_{-L}^{L} e^{ax}\cdot e^{-in\pi x/L}dx \\
&= \frac{1}{2L[a-(in\pi/L]} e^{[a-(in\pi/L]x}\Big|_{-L}^{L} \\
&= \frac{1}{2[aL-in\pi]}\left[e^{aL}e^{-in\pi} - e^{-aL}e^{-in\pi}\right] \\
&= \frac{1}{2[aL-in\pi]}\left[e^{aL}(\cos(n\pi) - i\sin(n\pi)) - e^{aL}(\cos(n\pi)+i\sin(n\pi))\right] \\
&= \frac{aL+in\pi}{[(aL)^2+(n\pi)^2]}\cdot(-1)^n\cdot\frac{e^{aL}-e^{-aL}}{2} \\
&= \frac{(-1)^n(aL+in\pi)}{(aL)^2+(n\pi)^2}\sinh(aL)
\end{aligned}
$$

Therefore, the complex F.S. is

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} = \sum_{n=-\infty}^{\infty}\frac{(-1)^n(aL+in\pi)}{(aL)^2+(n\pi)^2}\sinh(aL)e^{in\pi x/L}.$$

Note that we can use $c_n = \frac{1}{2}(a_n - ib_n)$ to find the real F.S. coefficients a_n and b_n which is much easier than finding them directly!

Fun Facts:

- The coefficients c_n are usually complex even if $f(x)$ is real.
- If $f(x)$ is real then $c_{-n} = \overline{c_n}$.
- If $f(x)$ is an even function then $c_{-n} = c_n$ and if $f(x)$ is an odd function then $c_{-n} = -c_n$.
- Note that

$$c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \text{ average value of } f(x) \text{ on } [-L, L].$$

- If $f(x)$ is piecewise smooth then the complex F.S. of $f(x)$ converges to the periodic extension of the adjusted version of $f(x)$.
- Parseval's Identity states that

$$\frac{1}{2L}\int_{-L}^{L}|f(x)|^2dx = \sum_{n=-\infty}^{\infty}|c_n|^2.$$

20.3 Integral Transform

The Fourier transform is a *continuous analog* of the F.S. In theory, a Fourier integral would lead to more manageable and understandable solutions in closed form.

Definition

Given any "reasonable" function $K(x, z)$, we can define the **integral transform**, $T[f](z)$ of a function $f(x)$, $a \leq x \leq b$, by

$$T[f](z) = \int_a^b K(x, z) f(x) dx.$$

where the function $f(x)$ is transformed into a new function $T[f](z)$. Such transforms are linear. The function $K(x, z)$ is known as the **kernel** of the transform.

Remark. The Fourier transform is helpful in solving PDEs, primarily because it converts differentiation into algebraic multiplication:

$$T[f'](z) = izT[f](z).$$

20.4 Fourier transform

Given $f(x), x \in \mathbb{R}$, we wish to represent $f(x)$ as Fourier integral:

1) Suppose $\int_{-\infty}^{\infty} |f(x)| dx = M < \infty$ and $f(x)$ is piecewise smooth on every finite interval.

2) Let $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$ and let $m_n = \frac{n\pi}{L}$ then this is a partition of $(-\infty, \infty)$ for $n \in \mathbb{Z}$.

3) Note that $\Delta m_n = m_{n+1} - m_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$. Thus $\frac{L}{\pi} \Delta m_n = 1$.

4) Using this fact we can write the complex F.S. as a Riemann Sum:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L} . \frac{L}{\pi} \Delta m_n = \sum_{n=-\infty}^{\infty} \left(\frac{L}{\pi} c_n \right) e^{in\pi x/L} \Delta m_n = \sum_{n=-\infty}^{\infty} \hat{f}(m_n) e^{im_n x} \Delta m_n.$$

where we let $\hat{f}(m_n) = Lc_n/\pi$.

5) Taking the limit $L \to \infty$ on both sides $\Delta m_n \to 0$ yields:

$$f(x) = \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} \hat{f}(m_n)e^{im_n x}\Delta m_n = \int_{-\infty}^{\infty} \hat{f}(m)e^{imx}dm.$$

which is the Fourier integral representation of $f(x)$.

6) Note that $\hat{f}(m)$ is the *Fourier transform of* $f(x)$ and $f(x)$ is the *inverse Fourier transform of* $\hat{f}(m)$.

Intuition. The Fourier transform is basically the "Fourier coefficient" of the basis function for the integral.

20.5 Notation

1)

$$\hat{f}(m) = \lim_{L\to\infty} \hat{f}(m_n) = \lim_{L\to\infty} \frac{L}{\pi}c_n = \lim_{L\to\infty} \frac{L}{\pi}\left(\frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\right) dx$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{imx}\, dx$$

Intuition. This is sort of the projection formula, since 2π is the circumference of the unit circle.

2)

Definition

Define the **Fourier transform** of $f(x)$ to be $\hat{f}(m) = \mathcal{F}[f](m)$ where

$$\hat{f}(m) = \mathcal{F}[f](m) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-imx} \ \forall\ m \in \mathbb{R}.$$

where the kernel is $K(x, m) = e^{-imx}$.

3)

Definition

Define the **inverse Fourier transform** of $\hat{f}(m)$: $f(x) = \mathcal{F}^{-1}[\hat{f}](x)$,

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \int_{-\infty}^{\infty} \hat{f}(m)e^{imx}dm \; \forall \; x \in \mathbb{R}.$$

where the kernel is $\hat{K}(m, x) = e^{imx}$.

4) Given $\hat{f}(m)$ then $f(x) = \mathcal{F}^{-1}[\hat{f}]$ and given $f(x)$ then $\hat{f}(m) = \mathcal{F}[f]$.

   Intuition. The Fourier transform represents a function $f(x)$ in a new "coordinate system" using different eigenfunction basis.

Fun Facts:

1) If $\int_{-\infty}^{\infty} |f(x)|dx = M < \infty$ then

$$|\hat{f}(m)| \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x)| \cdot |e^{-imx}|dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x)|dx = \frac{M}{2\pi}.$$

2) Note that

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)dx = \frac{1}{2\pi} \cdot [\text{ area under the curve } f(x)].$$

3) If $f(x)$ is real then $\hat{f}(m) = \overline{\hat{f}(m)}$.

4) If $f(x)$ is even then $\hat{f}(m)$ is even, likewise for odd.

5) The data $f(x)$ is transformed to a representation $\hat{f}(m)$ in the frequency domain.

Example. Suppose

$$f(x) = \begin{cases} A, & -L < x < L \\ \frac{A}{2}, & x = L \text{ or } x = -L \\ 0, & \text{else} \end{cases}$$

Find the Fourier transform of $f(x)$.

$$
\begin{aligned}
\hat{f}(m) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-imx}dx \\
&= \frac{1}{2\pi}\int_{-L}^{L} Ae^{-imx}dx \\
&= \frac{A}{2\pi im}(e^{imL} - e^{-imL}) \\
&= \frac{A}{\pi m}\left(\frac{e^{imL} - e^{-imL}}{2i}\right) \\
&= \frac{A}{\pi m}\sin(mL) \\
&= \frac{AL}{\pi}\cdot\frac{\sin(mL)}{mL}
\end{aligned}
$$

Define the "sinc function" as sinc $= \frac{\sin z}{z}$, then

$$\hat{f}(m) = \frac{AL}{\pi}\operatorname{sinc}(mL) = \mathcal{F}[f](m).$$

Moreover, by applying inverse transform on both sides,

$$f(x) = \int_{-\infty}^{\infty} \frac{AL}{\pi}\operatorname{sinc}(mL)e^{imx}dm = \mathcal{F}^{-1}[\hat{f}](x).$$

Notation. Haberman textbook uses the negative exponential term for the transform, but they are equivalent by a change of variable.

21 Heat Equation on $\mathbb{R}$

$$\begin{cases} \text{PDE: } \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} & x \in \mathbb{R}, t > 0 \\ \text{"BCs": } u(x,t) < \infty & \forall\ x \in \mathbb{R}, t > 0 \\ \text{ICs: } u(x,0) = U(x) & \text{where } \int_{-\infty}^{\infty} |U(x)|dx = M < \infty \end{cases}$$

Again PDE and BCs form a vector space so we can use separation of variables. Then we get

$$\begin{cases} F''(x) = \lambda F(x) \\ F(x) < \infty\ \forall\ x \in \mathbb{R}, t > 0 \end{cases} \quad \text{and } G'(t) = \lambda k G(t)$$

Note. This λ has opposite sign as the one we used before.

If $\lambda > 0$, the solutions are not bounded. If $\lambda = 0$, we have $F_0(x) = B$. If $\lambda < 0$, we get $\lambda = -m^2$ for nonzero $m \in \mathbb{R}$. And

$$F_n = A_m e^{imx} + B_m e^{-imx} \text{ for } m \in \mathbb{R}.$$

For the time domain problem,

$$G'(t) = \lambda k G(t) \Rightarrow G_m(t) = G(0)e^{-m^2kt} \text{ for } m \in \mathbb{R}.$$

Thus, summing over all product solutions with real numbers $m \in \mathbb{R}$ yields

$$u(x,t) = \int_{-\infty}^{\infty} a_m e^{imx} e^{-m^2kt} dm + \int_{-\infty}^{\infty} b_m e^{-imx} e^{-m^2kt} dm.$$

Note. The complex exponential is oscillating whereas the real exponential is a time-decaying term.

Notice that the second integral is redundant, since we can just use a change of variable $m = -p$ to get the same form. So

$$u(x,t) = \int_{-\infty}^{\infty} a_m e^{imx} e^{-m^2kt} dm.$$

To find a_m, we use the IC:

$$U(x) = u(x,0) = \int_{-\infty}^{\infty} a_m e^{imx} dm \text{ and } U(x) = \int_{-\infty}^{\infty} \hat{U}(m) e^{imx} dm \Rightarrow a_m = \hat{U}(m)$$

by form matching according to our theory. So we can use the Fourier transform as the coefficient. So the solution becomes

$$u(x,t) = \int_{-\infty}^{\infty} \hat{U}(m) e^{imx} e^{-m^2kt} dm$$

where

$$\hat{U}(m) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(x)e^{-imx}dx.$$

Note that $\hat{U}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(x)dx$ which is the area under the IC curve divided by 2π. Intuitively this is like area divided by something like a length.

$\hat{U}(x)$ is bounded because by boundedness of the integral of $U(x)$

$$|\hat{U}(m)| \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} |U(x)| \cdot |e^{-imx}|dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |U(x)|dx \leq \frac{M}{2\pi}.$$

Remark. This is part of the *Riemann-Lebesgue Lemma* for Fourier transform.

Lemma: Riemann-Lebesgue for Fourier transform

Suppose $U(x)$ is defined on $-\infty < x < \infty$ and let

$$\hat{U}(m) = \mathcal{F}[U(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(x)e^{-imx}dx, \ \forall\, m \in \mathbb{R}.$$

If $\int_{-\infty}^{\infty} |U(x)|dx = M < \infty$, then

1) $\hat{U}(m)$ is bounded and $\hat{U}(m) \leq \frac{M}{2\pi}$.
2) $\hat{U}(m)$ is continuous for all real m.
3) $\hat{U}(m) \to 0$ as $m \to \infty$.

Note. This is a special case. Intuitively if the IC is "well-behaved". then the Fourier transform is also well-behaved. Then we can truncate $\hat{U}(m)$ without much loss of information.

Proof

1) See previous.
2) Given real number m_n, define $U_n(x) = U(x)e^{-im_n x}$ then $\forall\ x \in r$, we have

$$\lim_{n\to\infty} U_n(x) = \lim_{n\to\infty} U(x)e^{-im_n x} = U(x)e^{-imx}, |U_n(x)| \leq |U(x)|\ \forall\ n.$$

Thus by *Lebesgue Dominated Convergence Theorem*, this result yields

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} U_n(x)dx = \int_{-\infty}^{\infty} U(x)e^{-imx}dx \Rightarrow \lim_{m_n\to m}\hat{U}(m_n) = \hat{U}(m).$$

Thus $\hat{U}(m)$ is continuous.

3) We will only prove 3 for the simple case that $U(x)$ is the indicator function over the set $[-L, L]$:

$$U(x) = \begin{cases} 1, & \text{if } x \in [-L, L] \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} \left|\int_{-\infty}^{\infty} U(x)e^{-imx}\right| &= \left|\int_{-L}^{L} e^{-imx}\right| \\ &= \left|\frac{e^{-imL} - e^{imL}}{-im}\right| \\ &\leq \frac{|i^{e^{-imL}}| + |ie^{imL}|}{|m|} \\ &= \frac{2}{|m|} \end{aligned}$$

By Squeeze Theorem, taking $m \to \pm\infty$, this bound gives us

$$\lim_{m\to\pm\infty}\int_{-\infty}^{\infty} U(x)e^{-imx}dx = 0 \Rightarrow \hat{U}(m) \to 0 \text{ as } m \to \pm\infty.$$

Note that this result can be extended to any function $U(x)$ such that $\int_{-\infty}^{\infty} |U(x)|dx < \infty$. Such result is the *Riemann-Lebesgue Theorem.*

□

21.1 Asymptotic Behavior

Recall $I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$. Then for all x and $t > 0$,

$$\begin{aligned}
|u(x,t)| &\leq \int_{-\infty}^{\infty} |\hat{U}(m)| \cdot e^{-m^2kt} \cdot |e^{imx}|\, dm \\
&\leq \frac{M}{2\pi} \int_{-\infty}^{\infty} e^{-m^2kt}\, dm \\
&= \frac{M}{2\pi} \int_{-\infty}^{\infty} e^{-(m\sqrt{kt})^2}\, dm \\
&= \frac{M}{2\pi} \cdot \frac{\sqrt{\pi}}{\sqrt{kt}}
\end{aligned}$$

Therefore, $u(x,t) \to 0$ as $t \to \infty$.

21.2 Large time estimate

Note that since $-\infty < m < \infty$, there is no smallest Fourier mode, instead we expand $\hat{U}(m)$ as a Maclaurin Series (assuming it can be done) then

$$\hat{U}(m) = \hat{U}(0) + \hat{U}'(0)m + \frac{\hat{U}''(0)}{2!}m^2 + \dots.$$

By completing the square, we can show that

$$e^{-m^2kt}e^{imx} = e^{-kt(m-ix/2kt)^2}e^{-x^2/4kt}.$$

So we can approximate

$$\begin{aligned}
u(x,t) &\approx \int_{-\infty}^{\infty} \hat{U}(0)e^{imx}e^{-m^2kt}\, dm \\
&= \hat{U}(0)e^{-x^2/4kt} \int_{-\infty}^{\infty} e^{-kt(m-ix/2kt)^2}\, dm \\
&= \hat{U}(0)e^{-x^2/4kt} \cdot \frac{\sqrt{\pi}}{\sqrt{kt}}
\end{aligned}$$

So for large but finite t, we have

$$u(x,t) \approx \hat{U}(0)e^{-x^2/4kt} \cdot \frac{\sqrt{\pi}}{\sqrt{kt}}.$$

Now we can motivate the fundamental source solution:

$$\sqrt{kt} \cdot u(x,t) \approx \hat{U}(0)\sqrt{\pi} \cdot e^{-x^2/4kt}.$$

So for each fixed t, $\sqrt{kt} \cdot u(x,t)$ has a bell shape and will have a bell shape for almost initial data as $kt \to \infty$. Using the Maclaurin Series approximation of $u(x,t)$, we can show $u(x,t) \to 0$ as $|x| \to \infty$.

Definition: Dirac delta function

Define the **Dirac delta function** to be the concentrated pulse "function" $\delta(x)$ with the property that

$$\int_{-\infty}^{\infty} \delta(x-c)f(x)dx = f(c) = \int_{-\infty}^{\infty} \delta(c-x)f(x)dx, c \in \mathbb{R}.$$

That is, formally define

$$\delta(x-c) = \begin{cases} 0, & \text{if } x \neq c \\ \infty, & \text{if } x = c \end{cases}$$

Note. If we let $f(x) = 1$ and $c = 0$, then $\int_{-\infty}^{\infty} \delta(x)dx = 1$. The integrals above cannot be defined as limits of Riemann sums because $\delta(x)$ is no ordinary function. The integral statement above is true by definition.

Theorem: fundamental solution of the heat equation

Given the problem

$$\begin{cases} \text{PDE: } \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} & -\infty < x < \infty, t > 0 \\ \text{ICs: } u(x,0) = \delta(x) \end{cases}$$

Then we can show that the fundamental solution is

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \hat{\delta}(m)e^{imx}e^{-m^2kt}dm \\ &= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\overline{x})e^{-im\overline{x}}d\overline{x}\right] e^{imx}e^{-m^2kt}dm \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi}e^{-im\cdot 0}e^{imx}e^{-m^2kt}dm \\ &= \frac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt} \text{ by above approximation} \end{aligned}$$

Remark. This solution represents the evolution of the temperature due to an initial heat source at $x = 0, t = 0$ for an infinite rod and the temperature has a Gaussian distribution.

22 Finite Rod

Consider the finite interval problem:

$$\begin{cases} \text{PDE: } \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} & 0 < x < L, t > 0 \\ \text{BCs: } u(0,t) = 0 = u(L,t) & t > 0 \\ \text{ICs: } u(x,0) = f(x) & 0 \le x \le L \end{cases}$$

where $f(x)$ is continuous and $f(0) = f(L) = 0$. Note that the DE is not required to hold when $t = 0$ (so $f(x)$ is not required to be twice differentiable). However, $u(x,t)$ is required to be continuous for $(x,t) \in [0,L] \times [0,\infty)$. That is,

$$\lim_{(x,t)\to(x_0,0^+} u(x,t) = u(x_0,0) = f(x_0), x_0 \in \mathbb{R}.$$

Theorem: heat equation by method of images

Let $\tilde{f}_{odd}(x), -\infty < x < \infty$, be the periodic extension of the odd extension of $f(x)$. Then the unique solution of problem above, which is continuous for all $(x,t) \in (-\infty,\infty) \times [0,\infty)$, is

$$u(x,t) = \begin{cases} \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-\overline{x})^2/4kt} \tilde{f}_{odd}(\overline{x}) d\overline{x}, & \text{if } t > 0 \\ f(x), & \text{if } t = 0 \end{cases}$$

Claim. The fundamental source solution and the F.S. solution to the heat equation are the same.

Proof

Since $u(x,t) \in \mathcal{C}^\infty$ odd periodic function of x, $t > 0$. Thus, fundamental source solution is equal to its F.S.S on $[0,L], t > 0$. That is,

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\left(\frac{n\pi x}{L}\right), B_n(t) = \frac{2}{L}\int_0^L u(x,t) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Using integration by parts, we move derivative of f wrt x to derivative of

g wrt x.

In general, $\displaystyle\int_a^b \frac{df}{dx} g(x)dx = g(x)f(x)\Big|_a^b - \int_a^b f(x) \cdot \frac{dg}{dx} dx$

$$\begin{aligned} \text{In our case, } \frac{\partial u}{\partial t} &= \sum_{n=1}^{\infty} \frac{\partial}{\partial t} B_n(t) \sin\left(\frac{n\pi x}{L}\right) \\ &= \sum_{n=1}^{\infty} \frac{2}{L} \left[\int_0^L u_t(x,t) \sin\left(\frac{n\pi x}{L}\right) dx\right] \sin\left(\frac{n\pi x}{L}\right) \\ &= \sum_{n=1}^{\infty} \frac{2}{L} \left[\int_0^L k u_{xx}(x,t) \sin\left(\frac{n\pi x}{L}\right) dx\right] \sin\left(\frac{n\pi x}{L}\right) \\ &= \sum_{n=1}^{\infty} \frac{2}{L} \left[\int_0^L k u(x,t) \frac{d^2}{dx^2}\left(\sin\left(\frac{n\pi x}{L}\right)\right) dx\right] \sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

After differentiation we can show that $B_n(t)$ satisfies the following ODE:

$$B_n(t) = -k\left(\frac{n\pi}{L}\right)^2 B_n'(t) \Rightarrow B_n(t) = c_n e^{-(n\pi/L)^2 kt}.$$

We can finally show that $c_n = b_n$ and thus complete the proof. □

Theorem: transform method

Suppose $f(x) \in \mathcal{C}^1$ and suppose $|f(x)| + |f'(x)| \leq K|x|^{-2}$, then

$$\widehat{\frac{df}{dx}}(m) = im \cdot \widehat{f}(m) \Leftrightarrow \mathcal{F}[f'(x)] = im \cdot \mathcal{F}[f(x)].$$

Proof

Use integration by parts on $\mathcal{F}[f'(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f'(x)e^{-imx}dx$ and the fact that $\lim_{x\to\pm\infty} f(x) = 0$ to prove it. □

Example. Take the Fourier transform of both sides of the heat equation wrt x, fix t, yields:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} u_t e^{-imx}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} k u_{xx} e^{-imx} dx \Rightarrow \frac{\partial}{\partial t}\hat{u}(m,t) = k\widehat{u_{xx}}(m,t).$$

Apply the property above and we have

$$\frac{\partial}{\partial t}\hat{u}(m,t) = k(im)^2\hat{u}(m,t) \Rightarrow \frac{\partial}{\partial t}\hat{u}(m,t) = -km^2\hat{u}(m,t).$$

Thus the PDE has been essentially transformed into an ODE (because x doesn't affect the frequency domain). It can be shown that the solution is $\hat{u}(m,t) = \hat{f}(m)e^{-km^2t}$ which implies that $u(x,t) = \mathcal{F}^{-1}[\hat{u}(m,t)]$ is equal to the Fourier solution.

Definition: Laplace Transform

Let $f(t)$ be given for $t \geq 0$ and suppose that f satisfies:

1) f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A.

2) $|f(t)| \leq Ke^{at}$ when $t \geq M$. In this inequality $K > 0, a, M > 0 \in \mathbb{R}$.

Then define the **Laplace Transform**, $\mathcal{L}\{f(t)\} = F(s)$ as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt, s > a.$$

This uses the kernel $K(s,t) = e^{-st}$ and the parameter s may be complex but we assume $s \in \mathbb{R}$.

Claim. Laplace transform is a special case of Fourier transform.

Proof

Suppose f is such a function that $f(t) = 0$ for $t < 0$ then

$$2\pi\widehat{f}(m) = 2\pi\mathcal{F}[f(t)] - \int_{-\infty}^\infty f(t)e^{-imt}dt = \int_0^\infty f(t)e^{-imt}dt.$$

If we let $m = -is, s \in \mathbb{R}$, then we have

$$2\pi\widehat{f}(m) = \int_0^\infty f(t)e^{-i(-is)t}dt = \int_0^\infty f(t)e^{-st}dt = F(s)$$

$$\Rightarrow 2\pi\mathcal{F}[f(t)] = \mathcal{L}\{f(t)\}$$

□

23 Dispersive Waves

Recall the wave equation

$$\begin{cases} \text{PDE: } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} & 0 < x < L, t > 0 \\ \text{BCs: } u(0,t) = 0 = u(L,t), & t > 0 \\ \text{ICs: } u(x,0) = U(x), \frac{\partial u}{\partial t}(x,0) = V(x). & 0 \le x \le L \end{cases}$$

has the solution

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right].$$

So it's a product of a wave in x and a wave in t. Therefore, suppose the solution to *wave propagation* problems have the form

$$u(x,t) = F(x)G(t) = e^{ikx} \cdot e^{-i\omega t} = e^{i(kx-\omega t)}, k, \omega \in \mathbb{R}.$$

Notation. k, ω are newly defined, not related to previous materials.

Definition: phase velocity

1) The wavelength of the space wave e^{ikx} is $L = \frac{2\pi}{k}$ so we see that k represents the number of wavelengths per 2π unit of distance (spatial frequency). For the time wave $e^{-i\omega t}$, the term ω is the number of waves per 2π unit time (time frequency).

2) the **phase velocity**, c_p, is defined as

$$c_p = \frac{\omega}{k} \text{ with units distance/time.}$$

3) The wave $e^{ik(x-\frac{\omega}{k}t)}$ represents a "traveling wave" with wave number k and wave velocity $c_p = \frac{\omega}{k}$.

4) If ω is a function of $k \in \mathbb{R}$, *i.e.* if $\omega = \omega(k)$ then $e^{-i(kx-\omega t)}$ is a **linear dispersive wave**.

5) For linear dispersive waves, we have $c_p = c_p(k)$ and if $\frac{d}{dk}c_p \neq 0$, *i.e.* if the wave velocity is not constant, then we have a wave propagation problem that is said to be **dispersive**.

6) The term $\omega(k)$ relates space and time in the expression $e^{ik(x-\omega\frac{t}{k}}$.

7) The information about the PDE (but not the BCs or ICs) is encoded in $\omega(k)$.

Example. Consider wave equation with stiffness term and ICs:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} - \alpha^2\frac{\partial^4 u}{\partial x^4}, u(x,0) = U(x), \frac{\partial}{\partial t}u(x,0) = V(x).$$

Assume $U(x,t) = e^{ikx-i\omega t}, k \in \mathbb{R}$ and it yields

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 u, c^2\frac{\partial^2 u}{\partial x^2} = c^2(ik)^2 u, \alpha^2\frac{\partial^4 u}{\partial x^4} = \alpha^2 k^4 u.$$

Substituting into the PDE and we obtain:

$$w(k) = \pm k\sqrt{c^2+\alpha^2k^2}.$$

Thus,

$$u_{1,k} = e^{ik(x-\sqrt{^2+\alpha^2k^2t})}, u_{2,k}(x,t) = e^{ik(x+\sqrt{c^2+\alpha^2k^2})}.$$

By superposition principle, summing for all $k \in \mathbb{R}$ yields

$$u(x,t) = \int_{-\infty}^{\infty}\left[A(k)e^{ik(x-\sqrt{c^2+\alpha^2k^2})} + B(k)e^{ik(x+\sqrt{c^2+\alpha^2k^2})}\right]dk.$$

Now we can use the ICs to find $A(k), B(k)$.

$$U(x) = u(x,0) = \int_{-\infty}^{\infty}[A(k)+B(k)]e^{ikx}dk = \mathcal{F}^{-1}[A(k)+B(k)].$$

Thus, taking Fourier transform on both sides gives us

$$A(k)+B(k) = \mathcal{F}[U(x)] = \widehat{U}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}U(x)e^{ikx}dx.$$

And similarly,

$$-A(k)+B(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{V(x)}{ik\sqrt{c^2+\alpha^2k^2}}e^{ikx}dx.$$

These two expressions allow us to solve

$$A(k) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left[U(x) - \frac{V(x)}{ik\sqrt{c^2+\alpha^2k^2}}\right]e^{ikx}dx.$$

and

$$B(k) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left[U(x) + \frac{V(x)}{ik\sqrt{c^2+\alpha^2k^2}}\right]e^{ikx}dx.$$

23.1 Group Velocity

We wish to analyze the physical interaction of waves with proximal spatial frequency. Assume two close waves of the form $\cos[kx-\omega(k)t]$ with wave numbers k and $k+\Delta k$. That is,

$$u(x,t) = A\cos[kx-\omega(k)t] + A\cos[(k+\Delta k)x - \omega(k+\Delta k)t].$$

Using trig identity and cos is even,

$$u(x,t) = 2A\cdot\cos\left[\left(k+\frac{\Delta k}{2}\right)x - \frac{\omega(k)+\omega(k+\Delta k)}{2}t\right]\cdot\cos\left[\frac{\Delta k}{2}x - \frac{\omega(k+\Delta k)-\omega(k)}{2}t\right].$$

The wavelength of the first term is $2\pi/(k+\Delta k/2)$, which is shorter than that of the second term, $2\pi/\Delta k/2$.

The long wave acts as a wave envelop of the short waves and it travels with what is called as **group velocity** given by

$$c_g = \lim_{\Delta k\to 0}\frac{\omega(k+\Delta k)-\omega(k)}{\Delta k} = \frac{d}{dk}\omega(k).$$

The short wave moves at almost the speed $c_p = \frac{\omega(k)}{k}$ which is the phase velocity of an individual dispersive wave. The long wave acts as a wave envelope of the short waves and travels at group velocity. We claim that the **wave energy** moves with group velocity c_g.

23.2 Deep Water Waves

Gravity and surface tension affects the propagation of water waves. For surface water the equation is

$$\omega(k) = \sqrt{gk\tanh(kh)}$$

where $g = 9.8$ m/s and $h > 0$ is the depth of the water. In deep water where the depth of water h is large, notice $\lim\limits_{h\to\infty}\tanh(kh) = 1 \Rightarrow \omega(k) = \sqrt{gk\tanh(kh)} \approx \sqrt{gk}$ thus the phase velocity of a short wave in deep water is approximately

$$c_p = \frac{\omega(k)}{k} = \frac{\sqrt{gk\tanh(kh)}}{k} = \sqrt{\frac{g\tanh(kh)}{k}} \approx \sqrt{\frac{g}{k}}.$$

Since $L = \frac{2\pi}{k}$, we finally have

$$c_p \approx \sqrt{\frac{gL}{2\pi}}.$$

Thus, $c_p = c_p(k)$ and short waves in deep water are dispersive.

For long wave

$$c_g = w'(k) = \frac{d}{dk}\sqrt{gk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\sqrt{\frac{gL}{2\pi}} \Rightarrow c_g = \frac{1}{2}c_p.$$

This shows that longer waves will have larger group velocities and arrive at a distance shoreline sooner.

For shallow water, short waves are non-dispersive and the phase velocity mainly depends on gravity and depth of water since

$$c_p = \frac{\omega(k)}{k} = \sqrt{\frac{g\tanh(kh)}{k}} = \sqrt{gh\cdot\frac{\tanh(kh)}{kh}} \approx \sqrt{gh}.$$

where we use the fact that $\tanh(kh)/kh \approx 1$ for small h (by L'Hopital Rule).