**Definition: order**

Let $G$ be a group and let $a \in G$. The **order** of $a$ is the number of elements in $\langle a \rangle$. Alternatively, the **order** of $a$ is the smallest $n > 0$ such that $a^n = e$ or $\infty$ if no such $n$ exists. We denote the order of $a$ by $|a|$.

# 1 Cosets and Lagrange's Theorem

See iPad screenshots for what cosets look like. We cut $G$ into pieces and each is the same size as $H$. This forms a partition of $G$. None of them are empty, intersection is empty, and union is the whole group. Only one can be a subgroup since identity can only exist in one of them.

**Definition**

$G$ is a group, $H \leq G$. Define a relation, $\sim_L$, on $G$, such that $a \sim_L b \Rightarrow a^{-1}b \in H$ (where the inverse has to be on the left).

WLOG everything below are similar for right cosets.

**Theorem**

$\sim_L$ is an equivalence relationship.

**Proof**

(i) Reflexive: need $a \sim_L a$ for $a \in G$. $a^{-1}a = e \in H$ since $H$ is a subgroup.

(ii) Symmetric: if $a \sim_L b$, then $b \sim_L a$. Since $a^{-1}b$ and $b^{-1}a$ are inverses. Since $H$ is closed under inverses, hence $b^{-1}a \in H$.

(iii) transitive: If $a \sim_L b$ and $b \sim_L c$, then $a \sim_L c$. If $a^{-1}b \in H$ and $b^{-1}c \in H$, then $a^{-1}bb^{-1}c = a^{-1}c \in H$ since $H$ is closed under operation.

□

*Note.* We used all necessary conditions of a subgroup for the above proof.

What is $[a]$, the equivalence class of $a$?

$$[a] = \{g \in G : a \sim_L g\}.$$

$a \sim_L g$ means $a^{-1}g \in H \Leftrightarrow a^{-1}g = h \in H \Leftrightarrow g = ah, h \in H$. Then

$$[a] = \{ah : h \in H\} := aH.$$

**Definition: cosets**

Group $G$ and subgroup $H \leq G$. The **left cosets of $H$ in $G$** is $aH = \{ah : h \in H\}$. The **right cosets of $H$ in $G$** is $Ha = \{ha : h \in H\}$ .

*Property.*

- Left cosets of $H$ partition $G$. (Any two left cosets are equal or disjoint).
- $xH = yH$ does **NOT** mean that $x = y$!!!!
- Consider $xH = \{xh : h \in H\}$. Then $xH$ contains $x$ because $e \in H$. So $xH$ is the left coset containing $x$. No other coset of $H$ does because they form a partition of $G$.
- $eH\{eh : h \in H\} = H$. So $H$ is one of the left cosets.
- there is a bijection between $H$ and $xH$. Can always multiply by $x^{-1}$ to undo it. This means that any two left cosets have the same size, and any two right cosets have the same size.
- there is a bijection between left cosets of $H$ and right cosets of $H$. We can just take inverses of each element in the left coset:

$$\begin{aligned} xh &= \{xh : h \in H\} \\ &= \{h^{-1}x^{-1} : h \in H\} \\ &= \{hx^{-1} : h \in H\} \\ &= Hx^{-1} \end{aligned}$$

  This is a bijection. So there are same number of left cosets as right cosets.

Question: What is the condition for $x$ and $y$ to be in the same cosets. Answer: $x \sim y$.

**Theorem**

$xH = yH \Leftrightarrow x^{-1}y \in H$ and $Hx = Hy \Leftrightarrow xy^{-1} \in H$.

**Theorem: Lagrange's Theorem**

Let $G$ be a finite group and let $H \leq G$. Then $|G|$ is equal to $|H|$ times the number of cosets of $H$ in $G$.

See iPad for picture. Each set has the same size and together they form a partition.

**Theorem**

$|H|$ divides $|G|$.

$H \leq V_4$: $H$ cannot have size 3 because 3 cannot divide 4.

*Note.* The converse of Lagrange's Theorem is false.

**Example.** $A_4$ is a group of order $4!/2 = 12$. But $A_4$ has no subgroup of order 6, although Lagrange's Theorem allows it. This is the smallest example for this.

**Example.** $D_6$ has order 12 and does have a subgroup of order 6 which are the rotations.

Question: Is $D_6$ isomorphic to $A_4$? No. Because they have a structural difference described above.

**Definition: index**

Let $G$ be a group and $H \leq G$. The **index of $H$ in $G$** is the number of cosets of $H$ in $G$. We denote this by $|G : H|$ or $\{G : H\}$ or $(G : H)$.

So Lagrange's Theorem implies $|G| = |H| \times \{G : H\}$.

**Example** (infinite group)**.** $G = \mathbb{Z}$ and $H = 3\mathbb{Z}$.