# Homework 7

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**Problem** (13.2)**.** Not a homomorphism. Consider $1.5, 0.5 \in \mathbb{R}$, then

$$\phi(1.5 + 0.5) = \phi(2) = 2 \neq 1 = 1 + 0 = \phi(1.5) + \phi(0.5).$$

**Problem** (13.3)**.** Yes. Given $a, b \in \mathbb{R}^*$,

$$\phi(ab) = |ab| = |a||b| = \phi(a)\phi(b).$$

**Problem** (13.4)**.** Given $a, b \in \mathbb{Z}_6$,

$$\phi(a +_6 b) = (a +_6 b) \mod 2.$$

Note $\mod 2$ is a homomorphism, so we get

$$= (a \mod 2) +_2 (b \mod 2) = \phi(a) +_2 \phi(b).$$

**Problem** (13.5)**.** No. Notice

$$\phi(2 +_9 7) = \phi(0) = 0 \neq 1 = 0 +_2 1 = \phi(2) +_2 \phi(7).$$

**Problem** (13.6)**.** Yes. Given $a, b \in \mathbb{R}$,

$$\phi(a + b) = 2^{a+b} = 2^a \cdot 2^b = \phi(a)\phi(b).$$

**Problem** (13.8)**.** No. Consider $\mu_1, \rho_2 \in D_3$,

$$\phi(\mu_1\rho_2) = (\mu_1\rho_2)^{-1} = \rho_2^{-1}\mu_1^{-1} = \rho_1\mu_1 = \mu_3.$$

However,

$$\phi(\mu_1)\phi(\rho_2) = \mu_1^{-1}\rho_2^{-1} = \mu_1\rho_1 = \mu_2$$

and they clearly don't equal to each other.

**Problem** (13.12)**.** No. Consider $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_2$,

$$\phi\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

However,

$$\phi\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) + \phi\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \det \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0 + 0 = 0.$$

And they clearly don't equal.

**Problem** (13.13)**.** Yes. Given $A, B \in M_n$, consider the diagonal elements $A_{ii}$ and $B_{ii}$, where $i = 1, \ldots, n$. Since matrix addition is elementwise, we have

$$(A + B)_{ii} = A_{ii} + B_{ii}.$$

Recall

$$\begin{aligned} \operatorname{tr}(A + B) &= \sum_{i=1}^{n} (A + B)_{ii} \\ &= \sum_{i=1}^{n} (A_{ii} + B_{ii}) \\ &= \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii} \\ &= \operatorname{tr} A + \operatorname{tr} B \end{aligned}$$

Therefore,

$$\phi(A + B) = \operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B = \phi(A) + \phi(B).$$

**Problem** (13.17)**.** The identity of both groups is 0. The kernel here is

$$\ker \phi = \{x \in \mathbb{Z} : \phi(x) = 0\}.$$

Since $\mathbb{Z}_7$ is a cyclic group, and since $\gcd(4, 7) = 1$, we expect $\phi(1) = 4$ to be a generator of $\mathbb{Z}_7$ with order 7. Therefore, we expect to arrive at the identity after adding $7n$ numbers of $\phi(1)$ where $n \in \mathbb{Z}$. Since $\phi$ is a homomorphism, $7n \cdot_7 \phi(1) = \phi(7n) = 0$. Therefore,

$$\ker \phi = \{7n : n \in \mathbb{Z}\} = 7\mathbb{Z}.$$

Again by homomorphism of $\phi$ and $\mod 7$,

$$\phi(25) = 25 \cdot_7 \phi(1) = 25 \cdot \phi(1) \mod 7 = 25 \cdot 4 \mod 7 = 2.$$

**Problem** (13.18)**.** By similar argument as above, we have $\gcd(6, 10) = 2$, so we expect $\phi(1) = 6$ to have order $\frac{10}{2} = 5$. Hence $5n \cdot_{10} \phi(1) = \phi(5n) = 0$ by homomorphism. It follows that

$$\ker \phi = 5\mathbb{Z}.$$

By homomorphism of $\phi$ and $\mod 10$,

$$\begin{aligned} \phi(18) &= 18 \cdot_{10} \phi(1) \\ &= 18 \cdot 6 \mod 10 \\ &= 8 \end{aligned}$$

**Problem** (13.25)**.** There are two: $\phi(x) = x$ and $\phi(x) = -x$. The identity function is clearly a bijective homomorphism. For the latter function, given $x, y \in \mathbb{Z}$, $\phi(x+y) = -(x+y) = -x-y = \phi(x)+\phi(y)$ so it is a homomorphism. We can also find $\phi(x)^{-1} = -x$ so it is also bijective.

**Problem** (13.26)**.** There are infinitely many. By Corollary 13.18, if $\ker \phi = \{e\}$, then the homomorphism $\phi$ is injective. Let $\phi(1) = n$, where $|n| \geq 1$ and $n \in \mathbb{Z}$. Then since 1 is a generator of $\mathbb{Z}$, using homomorphism we can generate a set in $\mathbb{Z}$ using $\phi(1)$. And since $|\phi(1)| \geq 1$, we know that $\phi(1)^k \neq 0 \; \forall \, k \in \mathbb{Z}^*$. Hence, the only element that maps to 0 is 0, which gives us that $\phi$ is injective. Since we have infinitely many ways to choose $n$, we have infinitely many injective homomorphisms.

**Problem** (13.32)**.**

a) True. Every subgroup of index 2 is normal.

b) True. The trivial homomorphism.

c) False. The trivial homomorphism is clearly not one-to-one.

d) True. By Corollary 13.18.

e) False. By exercise 44.

f) False. This requires at least one element in the first group to map to more than one output, making it not a function.

g) True. Let $G = \mathbb{Z}_{12}$ and $H = \langle 2 \rangle$. Then $|\langle 2 \rangle| = \frac{12}{\gcd(12,2)} = 6$. Now let $\phi(x) = x$. In general, we can always find a $H \leq G$ where $H$ has order 6 and $G$ has order 12. This is allowed by Lagrange, since 6 is a divisor of 12.

h) False. No such subgroup described above exist here because this isn't allowed by Lagrange.

i) False. The identity is always in there, because $\phi(e_G) = \phi(a \cdot a^{-1}) = \phi(a)\phi(a^{-1}) = \phi(a)\phi(a)^{-1} = e_H$.

j) False. Let $\phi : \mathbb{Z}_n \to \mathbb{C}^*$, $\phi(x) = e^{i2\pi x/n}$.