Homework 4

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Problem (1).

a) $T^{-1}\mathcal{F}'$: Let's check each axiom:

(i) Since $\emptyset \in \mathcal{F}'$, $T^{-1}\emptyset = \{\omega : T(\omega) \in \emptyset\} = \emptyset \in T^{-1}\mathcal{F}'$.

(ii) Given $T^{-1}A' \in T^{-1}\mathcal{F}'$, we have $A' \in \mathcal{F}'$. Since $\mathcal{F}'$ is a σ-field, $A'^c \in \mathcal{F}'$ and $T^{-1}(A'^c) \in T^{-1}\mathcal{F}'$. Then

$$\begin{aligned}
(T^{-1}A')^c &= \{\omega : \omega \in (T^{-1}A')^c\} \\
&= \{\omega : \omega \notin T^{-1}A'\} \\
&= \{\omega : T(w) \notin A'\} \\
&= \{\omega : T(\omega) \in A'^c\} \\
&= T^{-1}(A'^c) \in T^{-1}\mathcal{F}'
\end{aligned}$$

(iii) Given $T^{-1}A'_1, T^{-1}A'_2, \ldots \in T^{-1}\mathcal{F}', A'_1, A'_2, \ldots \in \mathcal{F}'$. Since $\mathcal{F}'$ is a σ-field, $\bigcup_{n=1}^{\infty} A'_n \in \mathcal{F}'$ and $T^{-1}\left(\bigcup_{n=1}^{\infty} A'_n\right) \in T^{-1}\mathcal{F}'$. Now

$$\begin{aligned}
\bigcup_{n=1}^{\infty} T^{-1}A'_n &= \bigcup_{n=1}^{\infty} \{\omega : T(\omega) \in A'_n\} \\
&= \{\omega : T(w) \in \bigcup_{n=1}^{\infty} A'_n\} \\
&= T^{-1}\left(\bigcup_{n=1}^{\infty} A'_n\right) \in T^{-1}\mathcal{F}'
\end{aligned}$$

Hence, $T^{-1}\mathcal{F}'$ is a σ-field.

$T\mathcal{F}$: Let's check each axiom:

(i) Since $\emptyset \in \mathcal{F}$, $T^{-1}\emptyset = \{\omega : T(\omega) \in \emptyset\} = \emptyset \in T^{-1}\mathcal{F}$.

(ii) Given $A' \in T\mathcal{F}$, we have $T^{-1}A' \in \mathcal{F}$. Since $\mathcal{F}$ is a σ-field, $(T^{-1}A')^c \in \mathcal{F}$. Then

$$\begin{aligned} T^{-1}(A'^c) &= \{\omega : T(\omega) \in A'^c\} \\ &= \{\omega : T(w) \notin A'\} \\ &= \{\omega : \omega \notin T^{-1}(A')\} \\ &= \{\omega : \omega \in \big(T^{-1}(A')\big)^c\} \\ &= (T^{-1}A')^c \in \mathcal{F} \end{aligned}$$

Thus $A'^c \in T\mathcal{F}$.

(iii) Given $A'_1, A'_2, \ldots \in T\mathcal{F}$, we have $T^{-1}A'_1, T^{-1}A'_2, \ldots \in \mathcal{F}$. Since $\mathcal{F}$ is a σ-field, $\bigcup_{n=1}^{\infty} T^{-1}A'_n \in \mathcal{F}$. Now

$$\begin{aligned} T^{-1}\left(\bigcup_{n=1}^{\infty} A'_n\right) &= \{\omega : T(\omega) \in \bigcup_{n=1}^{\infty} A'_n\} \\ &= \bigcup_{n=1}^{\infty} \{\omega : T(w) \in A'_n\} \\ &= \bigcup_{n=1}^{\infty} T^{-1}A'_n \in \mathcal{F} \end{aligned}$$

Thus, $(\bigcup_{n=1}^{\infty} A'_n) \in T\mathcal{F}$.

Hence, $T\mathcal{F}$ is a σ-field.

Regarding measurability, for given $A' \in \mathcal{F}'$ notice that

$$T^{-1}A' = \{\omega \in \Omega : T(\omega) \in A'\} \in \mathcal{F}$$

implies that $T^{-1}\mathcal{F}' \subseteq \mathcal{F}$ but is also the definition of T measurable F/F'. So the two statements are equivalent.

Similarly,

$$T^{-1}(A') \in \mathcal{F} \Leftrightarrow A' \in T\mathcal{F}.$$

This implies that $\mathcal{F}' \subseteq T\mathcal{F}$ but is also the definition of T measurable F/F'. Thus the two statements are equivalent.

b)

(i) We would like to prove by double containment.

($\subseteq$) : Since by part b) $T^{-1}(\sigma(\mathcal{A}'))$ is a σ-field, it suffices to show that $T^{-1}\mathcal{A}' \subseteq T^{-1}(\sigma(\mathcal{A}'))$.

Since $\mathcal{A}' \subseteq \sigma(\mathcal{A}')$,

$T^{-1}\mathcal{A}' = \{T^{-1}A' : A' \in \mathcal{A}'\} \subseteq \{T^{-1}A' : A' \in \sigma(\mathcal{A}')\} = T^{-1}(\sigma(\mathcal{A}'))$.

Since $\sigma(T^{-1}A')$ is the smallest σ-field containing $T^{-1}\mathcal{A}'$, we obtain $\sigma(T^{-1}\mathcal{A}') \subseteq T^{-1}(\sigma(\mathcal{A}'))$ as required.

($\supseteq$) : Give $A' \in \mathcal{A}'$, clearly $T^{-1}A' \in T^{-1}\mathcal{A}'$ and thus $T^{-1}A' \in \sigma(T^{-1}A') \coloneqq \mathcal{F}$. Let $\mathcal{F}' \coloneqq \sigma(\mathcal{A}')$. Then by Theorem 13.1, T is measurable F/F'. Then by 1a, this is equivalent to

$$T^{-1}\mathcal{F}' = T^{-1}(\sigma(\mathcal{A}')) \subseteq \sigma(T^{-1}\mathcal{A}') = \mathcal{F}.$$

As we obtain both directions,

$$\sigma(T^{-1}\mathcal{A}') = T^{-1}(\sigma(\mathcal{A}')).$$

(ii) Suppose $\Omega_0 \subseteq \Omega, T : \Omega_0 \to \Omega$ be the identity map. Theorem 10.1 states:

1) If $\mathcal{F}$ is a σ-field on Ω, then $\mathcal{F}_0 = \mathcal{F} \cap \Omega_0$ is a σ-field on Ω_0.

Proof

Given $A \in \mathcal{F}$,

$$\begin{aligned} T^{-1}A &= \{\omega \in \Omega_0 : T(\omega) \in A\} \\ &= \{\omega \in \Omega_0 : T(\omega) \in A \cap \Omega_0\} \text{ by identity map} \\ &= A \cap \Omega_0 \end{aligned}$$

Therefore, $T^{-1}\mathcal{F} = \{T^{-1}A : A \in F\} = \{A \cap \Omega_0 : A \in \mathcal{F}\} = \mathcal{F} \cap \Omega_0$. Recall from part b) that $T^{-1}\mathcal{F}$ is the smallest σ-field such that T is measurable $(\mathcal{F} \cap \Omega_0)/F$.

Thus, $\mathcal{F}_0 = \mathcal{F} \cap \Omega_0$ is a σ-field. ☐

2) If $\mathcal{F} = \sigma(\mathcal{A})$ on Ω, then $\mathcal{F}_0 = \mathcal{F} \cap \Omega_0 = \sigma(\mathcal{A} \cap \Omega_0)$.

Proof

By part c (i), we have

$$\begin{aligned}
\sigma(T^{-1}\mathcal{A}) &= T^{-1}(\sigma(A)) \\
\sigma(\{T^{-1}A : A \in \mathcal{A}\}) &= T^{-1}\mathcal{F} \\
\sigma(A \cap \Omega_0 : A \in \mathcal{A}) &= \mathcal{F} \cap \Omega_0 \text{ by identity map} \\
\sigma(\mathcal{A} \cap \Omega_0) &= \mathcal{F}_0
\end{aligned}$$

as required. ☐

Problem (2)**.** Suppose $s = \sum_{i=1}^{n} a_i I_{A_i}, A_i \in \mathcal{F}$.

(i) $\nu(\emptyset) = 0$:

$$\begin{aligned}
\nu(\emptyset) &= \int_{\emptyset} s \; d\mu \\
&= \sum_{i=1}^{n} a_i \mu(A_i \cap \emptyset) \\
&= \sum_{i=1}^{n} a_i \mu(\emptyset) \\
&= \sum_{i=1}^{n} a_i \cdot 0 = 0
\end{aligned}$$

(ii) $\nu : \Omega \to [0, \infty)$: Since $s \geq 0$, we have $a_i \geq 0 \; \forall \; i$. Since μ is a measure on $(\Omega, \mathcal{F})$, $\mu(A) \geq 0 \; \forall \; A \in \mathcal{F}$. Given $B \in \mathcal{F}$, since $\mathcal{F}$ is a σ-field,

$A_i \cap B \in \mathcal{F}$. So $a_i\mu(A_i \cap B) \geq 0 \ \forall\ i$.

$$\begin{aligned}
\nu(B) &= \int_B s\ d\mu \\
&= \sum_{i=1}^{n} a_i\mu(A_i \cap B) \\
&\geq 0
\end{aligned}$$

(iii) countable additivity: Given disjoint $B_1, B_2, \ldots \in \mathcal{F}$,

$$\begin{aligned}
\nu\left(\bigcup_{n=1}^{\infty} B_n\right) &= \sum_{i=1}^{m} a_i\mu\left(A_i \cap \bigcup_{n=1}^{\infty} B_n\right) \\
&= \sum_{i=1}^{m} a_i\mu\left(\bigcup_{n=1}^{\infty}(A_i \cap B_n)\right) \\
&= \sum_{i=1}^{m} a_i \sum_{n=1}^{\infty} \mu(A_i \cap B_n) \text{ countable add. of } \mu \\
&= \sum_{n=1}^{\infty} \sum_{i=1}^{m} a_i\mu(A_i \cap B_n) \qquad \text{linearity} \\
&= \sum_{n=1}^{\infty} \nu(B_n)
\end{aligned}$$

Hence, ν is a measure on $(\Omega, \mathcal{F})$.