Theorem: 11.4 (Approximation Theorem)

Suppose $\mathcal{A}$ is a semiring, and μ is a measure on $\mathcal{F} := \sigma(\mathcal{A})$, and μ is σ-finite on $\mathcal{A}$. Take $\varepsilon > 0$ and any $B \in \mathcal{F}$. Then

(i) There exists a disjoint sequence $A_1, A_2, \ldots \in \mathcal{A}$ (maybe finite with empty sets) such that $B \subseteq \bigcup_n A_n$ and

$$\mu\left(\bigcup_n A_n \setminus B\right) < \varepsilon.$$

(ii) If $\mu(B) < \infty$, there exists a finite disjoint sequence $A_1, A_2, \ldots, A_n \in \mathcal{A}$ such that

$$\mu\left(B\Delta\left(\bigcup_{i=1}^{n} A_i\right)\right) < \varepsilon.$$

Note. Recall $A\Delta B = (A \setminus B) \cup (B \setminus A)$.

12: Measures in Euclidean Space

Example. Consider $\mathbb{R}$. Let $\mathcal{A}$ be the collection of half intervals $(a, b]$. We saw from last time that $\mathcal{A}$ is a semiring. Define a measure λ as $\lambda(\emptyset) = 0$ and $\lambda((a, b]) = b - a$. Note that λ is defined on the field $\mathbb{R}$. Then $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$ since the Borel sets can be generated by these half intervals. By Theorem 11.3, λ can be extended to a measure on $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$. Since $\mathcal{A}$ is a π-system, $\mathcal{A}, \lambda$ is σ-finite, Theorem 10.3 tells us that the extension of λ from $\mathcal{A}$ to $\sigma(\mathcal{A})$ is unique. So there is no other measure on $\mathcal{B}(\mathbb{R})$ that will assign measure $(b - a)$ to $(a, b]$. But Lebesgue measure assigns $(b-a)$ to $(a, b]$, then λ must be Lebesgue measure.

Example. In $\mathbb{R}^k$, the analogy is let

$$R = \{(x_1, \ldots, x_k) : a_i < x_i \leq b_i \text{ for } i = 1, \ldots, k\},$$

$$\lambda(\mathbb{R}) := \prod_{i=1}^{k}(b_i - a_i)$$

and assign $\lambda(\emptyset) = 0$. Then this extend to all of $\mathbb{R}^k$ so we can define Lebesgue measure on $\mathbb{R}^k$.

Property.

1) **Translation invariance**: For $A \in \mathcal{B}(\mathbb{R}^k)$ and any $\mathbf{x} \in \mathbb{R}^k$, define the set $A + \mathbf{x}$ as

$$A + \mathbf{x} = \{\mathbf{a} + \mathbf{x} : \mathbf{a} \in A\}.$$

Then

(i) $A + \mathbf{x} \in \mathcal{B}(\mathbb{R}^{k)}$.

(ii) $\lambda(A) = \lambda(A + \mathbf{x})$.

Note. $A \in \mathcal{B}(\mathbb{R}^k)$ does not have to be a rectangle.

Proof

(i) Let $\mathcal{G} = \{A \subseteq R^k : A + \mathbf{x} \in \mathcal{B}(\mathbb{R}^k)\ \forall\ \mathbf{x} \in \mathbb{R}^k\}$. We can show that $\mathcal{G}$ is a σ-field. Let $\mathcal{A}$ be the class of half open rectangles in $\mathbb{R}^k$. Then $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R}^k)$ by the definition of $\mathcal{B}(\mathbb{R}^k)$. We can show that $\mathcal{A} \subseteq \mathcal{G}$. Thus,

$$\mathcal{A} \subseteq \sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R}^k) \subseteq \mathcal{G}.$$

Therefore, by the definition of $\mathcal{G}$, $A \in \mathcal{B}(\mathbb{R}^k) \subseteq \mathcal{G} \Rightarrow A + \mathbf{x} \in \mathcal{B}(\mathbb{R}^k)$.

$\square$

2) For a linear mapping $T : \mathbb{R}^k \to \mathbb{R}^k$,

(i) $A \in \mathcal{B}(\mathbb{R}^k) \Rightarrow T(A) \in \mathcal{B}(\mathbb{R}^k)$.

(ii) $\lambda(T(A)) = |\det(T)| \cdot \lambda(A)\ \forall\ A \in \mathcal{B}(\mathbb{R}^k)$.

Note. A linear map $T : \mathbb{R}^k \to \mathbb{R}^k$ can be written as $T(\mathbf{x}) = T\mathbf{x}$ where T is a $k \times k$