Corollary: 9.12

Any permutation of a finite set of at least two elements can be written as a product of its transpositions.

Example.

$$(1\ 2)(1\ 3)(1\ 4)(1\ 5) = (1\ 5\ 4\ 3\ 2) \in S_5.$$

We start from right to left. Then for (1 2 3 4 5) we can just use (1 5)(1 4)(1 3)(1 2)

Example.

$$\begin{aligned} \sigma = &\ (1\ 3\ 6)(2\ 8)(4\ 7\ 5) \\ = &\ (1\ 6)(1\ 3)(2\ 8)(4\ 5)(4\ 7) \end{aligned}$$

Example.

$$\begin{aligned} (1\ 2)(2\ 3)(1\ 2)(2\ 3)(1\ 2) = &\ (1)(2\ 3) \\ = &\ (2\ 3) \end{aligned}$$

A product of 5 transpositions = product of 1 transposition.

Example. Use the same σ as above, it is a product of 5 transpositions. We can write it as 7 transposition.

$$\sigma = (1\ 6)(1\ 3)(2\ 8)(4\ 5)(4\ 7)(1\ 2)(1\ 2)\ .$$

Could σ be the product of 10 transpositions? No!

Definition: even permutation

An **even permutation** is a product of even number of transpositions.

Likewise for odd permutation.

Note. So far any permutation is even or odd. Also, k-cycle are odd if k is even, and are even if k is odd.

Example. Consider (1 2) and (1 2 3) in S_3

Claim. No permutation is both even and odd.

Let's prove this using permutation matrix from linear algebra. Note that ith column of the permutation matrix tells you where the e_i basis goes.

Proof

Claim. A permutation is even or odd if its permutation matrix has determinant 1 or -1, respectively.

Since every time swapping rows flips the sign of the determinant. Since it cannot have determinant to be both 1 and -1 at the same time, the permutation cannot be both even and odd. $\square$

Claim. There exists an isomorphism between S_n and $n \times n$ permutation matrices under matrix multiplication.

Note. The identity is even (12)(12) and has determinant 1.

Example. If σ is a product of 5, τ is a product of 4, so $\sigma\tau$ is a product of 9 transposition.

*	even	odd
even	even	odd
odd	odd	even

So it adds like even and odd numbers.

Example. If $\alpha =$ (1 2)(2 4)(3 4), then $\alpha^{-1} = (3\ 4)(2\ 4)(1\ 2)$.

Then clearly an even permutation's inverse is also even.

Theorem

The even permutations in S_n form a subgroup.

Proof

(i) the identity is even.

(ii) the product of two even permutations is even.

(iii) the inverse of an even permutation is even.

$\square$

Definition

This is called the **alternating group on $\boldsymbol{n}$ letters**, denoted A_n.

Claim. If $n \geq 2$, then exactly half the elements in S_n are even,

Proof

The map $x \mapsto x\ (1\ 2)$ from S_n to S_n sends even to odd and vice versa. And if $n \geq 2$, then $|A_n| = \frac{n!}{2}$. $\square$

Note. If $n = 1$, S_n is trivial, and so is A_n, so $A_n = S_n$.

If $n = 2$, $S_n = \{id,\ (1\ 2)\}$, and A_n has an order 1.

If $n = 3$, S_n has order 6, and A_n has order 3. So S_n is nonabelian but A_n is abelian! This is the only time it happens.

If $n = 4$, A_n has order 12, is $\{1,\ (1\ 2\ 3),\ (1\ 3\ 2),\ (1\ 2\ 4),\ (1\ 4\ 2),\ (1\ 3\ 4),\ (1\ 4\ 3),\ (2\ 3\ 4),\ (2\ 4\ 3),\ (1\ 2)(3\ 4),$

This is nonabelian because

$$\begin{aligned}(1\ 2\ 3)(1\ 2\ 4) &= (1\ 3)(2\ 4)\\ (1\ 2\ 4)(1\ 2\ 3) &= (1\ 4)(2\ 3)\end{aligned}$$

The same counterexample can be used for A_n showing that A_n is nonabelian for $n \geq 4$.

Note. $\mathbb{Z}_n$ is abelian. D_n is nonabelian for $n \geq 3$ of order 6, A_n is nonabelian for $n \geq 4$ of order 12, S_n is nonabelian for $n \geq 3$ of order 6.