Integrals for General Measurable Functions

Consider $f : \Omega \to [-\infty, \infty]$. Write $f = f^+ - f^-$ where $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$.

Note.

1) $f^+, f^- \geq 0$.
2) $|f| = f^+ + f^-$.

Definition: integrable

$f : \Omega \to [-\infty, \infty]$ is **integrable** if f^+, f^- are integrable. In this case,

$$\int_A f \ d\mu = \int_A f^+ \ d\mu - \int_A f^- \ d\mu.$$

Note. $f^{\pm}$ integrable $\Rightarrow$ $f^{\pm}$ measurable $\Rightarrow$ f is measurable.

Theorem

Suppose f is measurable. Then f is integrable if and only if $|f|$ is integrable.

Proof

($\Rightarrow$): Suppose f is integrable, then $\int f^+ \ d\mu < \infty$ and $\int f^- \ d\mu < \infty$ so $\int |f| \ d\mu = \int f^+ \ d\mu + \int f^- \ d\mu < \infty$.

($\Leftarrow$): Suppose $|f|$ is integrable, then $|f|$ is measurable. Write $f^+ = \frac{1}{2}(f + |f|) \Rightarrow f^+$ is measurable. Moreover,

$$f^+ \leq |f| \Rightarrow \int f^+ \ d\mu \leq \int |f| \ d\mu < \infty.$$

So f^+ is integrable. Likewise f^- is integrable. So $f = f^+ - f^-$ is integrable. $\square$

Claim. If f, g are measurable, then

1) $\min\{f, g\}, \max\{f, g\}$ are measurable. Think $\{\omega : \max(f(\omega), g(\omega)) \leq x\}$.
2) $-f$ is measurable.

Remark. Billingsley defines the integral as

$$\int f d\mu = \sup \sum_{i=1}^{\infty} \left[\inf_{A_i} f\right] \mu(A_i).$$

where the sup is taken of all finite partitions of Ω into $\mathcal{F}$-sets A_i.

Adventure in "Almost Everywhere" Properties

Definition: almost everywhere

A property holds **almost everywhere** if it holds for all sets except for possible some sets of measure zero.

1) $f = 0$ a.e. $\Rightarrow \int f d\mu = 0$ (need f measurable).
2) $f = g$ a.e. $\Rightarrow \int f \ d\mu = \int g \ d\mu$.
3) $f \leq g$ a.e., f, g measurable $\Rightarrow \int f \ d\mu \leq \int g \ d\mu$.
4) Suppose f is integrable and $\int_A f \ d\mu \geq 0$ for every $A \in \mathcal{F}$. Then $f \geq 0$ a.e.

Proof

Let $B = \{\omega : f(\omega) < 0\}$. We want to show that $\mu(B) = 0$. Then we have

$$I_B(\omega) = \begin{cases} 0 & \text{if } f(\omega) \geq 0 \\ f(\omega) & \text{if } f(\omega) < 0 \end{cases}$$

So $f(\omega)I_B(\omega) \leq f(\omega)$. Since $f(\omega)I_B$ is non-positive, we have

$$nf(\omega)I_B(\omega) \leq f(\omega) \ \forall \ n \in \mathbb{N}$$

$$\int nfI_B \ d\mu \leq \int f \ d\mu$$

$$\int_B f \ d\mu \leq \frac{1}{n} \int f \ d\mu$$

Taking $n \to \infty$, we have $\int_B fd \ \mu \leq 0$. Since by assumption, $\int_B f \ d\mu \geq 0$, we have $\int_B f \ d\mu = 0$. And since we define $f(\omega) < 0 \ \forall \ \omega \in B$, it must be that $\mu(B) = 0 \Rightarrow f \geq 0$ a.e. $\square$

20: Random Variables

$(\Omega, \mathcal{F}, P)$.

Definition: random variable

A **random variable** is a measurable function $X : \Omega \to \mathbb{R}$.

Definition: random vector

A **random vector** X is a measurable function $X : \Omega \to \mathbb{R}^k$. It necessarily has the form

$$X(\omega) = (X_1(\omega), \ldots, X_k(\omega)).$$

Claim. X is a random vector if and only if each X_i is measurable.

Theorem: 20.1

Let $X = (X_1, \ldots, X_k)$ be a random vector.

(i) $\sigma(X) = \sigma(X_1, \ldots, X_k)$ consists precisely of the sets $\{\omega : X(\omega) \in H\}$ $\forall$ $H \in \mathcal{B}(\mathbb{R}^k)$.

(ii) A r.v. Y is measurable wrt $\sigma(X)$ if and only if there exists a measurable $f : \mathbb{R}^k \to \mathbb{R}$ such that $Y(\omega) = f(X_1(\omega), \ldots, X_k(\omega))$ $\forall$ $\omega \in \Omega$.

Note. This is Theorem 5.1 but with general random variables.

Proof

(i) same as 5.1 by defining $\mathcal{G} = \{X^{-1}(H) : H \in \mathcal{B}(\mathbb{R}^k)\}$ and show equivalence.

(ii) ($\Leftarrow$) Suppose there exists a measurable $f : \mathbb{R}^k \to \mathbb{R}$ such that $Y(\omega) = f(X(\omega))$ $\forall$ $\omega \in \Omega$. Then by Theorem 13.1 (ii), composite measurable function is measurable $\Rightarrow$ Y measurable wrt $\sigma(X)$.

($\Rightarrow$): Suppose $Y : \Omega \to \mathbb{R}$ is measurable wrt $\sigma(X)$. Consider the following cases:

Case (1). Y is simple, *i.e.* $Y(\omega) = \sum_{i=1}^n a_i I_{A_i}$ where A_is are disjoint. We want to find a f measurable $\mathcal{B}(\mathbb{R}^k)$ such that $Y(\omega) = f(X(\omega))$ $\forall$ $\omega \in \Omega$.

Y is measurable wrt $\sigma(X)$ implies that

$$A_i = Y^{-1}(\{a_i\}) \in \sigma(X).$$

By part (i) of this theorem, we know A_i has the form

$$A_i = \{\omega : X(\omega) \in H_i\} \text{ for some } H_i \in \mathcal{B}(\mathbb{R}^k).$$

Now let's define $f : \mathbb{R}^k \to \mathbb{R}$ to be

$$f(x) = \sum_{i=1}^{n} a_i I_{H_i}(x).$$

Note that f is measurable (since the inverse image would give us H_i or $\emptyset$, both in $\mathcal{B}(\mathbb{R}^k)$). Therefore,

$$\begin{aligned} f(X(\omega)) &= \sum_{i=1}^{n} a_i I_{H_i}(X(\omega)) \\ &= \sum_{i=1}^{n} a_i I_{A_i}(\omega) \\ &= Y(\omega) \end{aligned}$$

Case (2). Y is simple. Then we want to approximate Y with simple functions Y_n, *i.e.* $0 \le Y_n(\omega) \nearrow Y(\omega)$ if $Y(\omega) \ge 0$ and $0 \ge Y_n(\omega) \searrow Y(\omega)$ if $Y(\omega) < 0$ (see details in Theorem 13.5). We can use Case 1 to find measurable $f : \mathbb{R}^k \to \mathbb{R}$ such that $f_n(X(\omega)) = Y_n(\omega)\ \forall\, \omega \in \Omega$.

Now consider the set $A = \{x : f_n(x) \text{ converges}\}$. Then by the theorem right after 13.4 ($A = \{\liminf_n f_n(x) = \limsup_n f_n(x)\}$ where LHS and RHS are both measurable), we know $A \in \mathcal{B}(\mathbb{R}^k)$. Let's define $f : \mathbb{R}^k \to \mathbb{R}$ as

$$f(x) = \begin{cases} \lim\limits_{n\to\infty} f_n(x) & , x \in A \\ 0 & , x \notin A \end{cases}$$

Note that $f = \left(\lim\limits_{n\to\infty} f_n\right) \cdot I_A = \lim\limits_{n\to\infty} (f_n \cdot I_A)$. Since f_n, I_A are measurable, $f_n \cdot I_A$ is also measurable, and the limit is also measurable by 13.4. Thus we showed that f is measurable and thus $f(X(\omega)) = Y(\omega)$ is what we seek.

$\square$