0.1 Asymptotic Behavior

Recall $I = \int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}$. Then for all x and $t > 0$,

$$\begin{aligned}
|u(x,t)| &\leq \int_{-\infty}^{\infty} |\hat{U}(m)| \cdot e^{-m^2kt} \cdot |e^{imx}|\ dm \\
&\leq \frac{M}{2\pi} \int_{-\infty}^{\infty} e^{-m^2kt}\ dm \\
&= \frac{M}{2\pi} \int_{-\infty}^{\infty} e^{-(m\sqrt{kt})^2}\ dm \\
&= \frac{M}{2\pi} \cdot \frac{\sqrt{\pi}}{\sqrt{kt}}
\end{aligned}$$

Therefore, $u(x,t) \to 0$ as $t \to \infty$.

0.2 Large time estimate

Note that since $-\infty < m < \infty$, there is no smallest Fourier mode, instead we expand $\hat{U}(m)$ as a Maclaurin Series (assuming it can be done) then

$$\hat{U}(m) = \hat{U}(0) + \hat{U}'(0)m + \frac{\hat{U}''(0)}{2!}m^2 + \dots.$$

By completing the square, we can show that

$$e^{-m^2kt}e^{imx} = e^{-kt(m-ix/2kt)^2}e^{-x^2/4kt}.$$

So we can approximate

$$\begin{aligned}
u(x,t) &\approx \int_{-\infty}^{\infty} \hat{U}(0)e^{imx}e^{-m^2kt}\ dm \\
&= \hat{U}(0)e^{-x^2/4kt} \int_{-\infty}^{\infty} e^{-kt(m-ix/2kt)^2}\ dm \\
&= \hat{U}(0)e^{-x^2/4kt} \cdot \frac{\sqrt{\pi}}{\sqrt{kt}}
\end{aligned}$$

So for large but finite t, we have

$$u(x,t) \approx \hat{U}(0)e^{-x^2/4kt} \cdot \frac{\sqrt{\pi}}{\sqrt{kt}}.$$

Now we can motivate the fundamental source solution:

$$\sqrt{kt} \cdot u(x,t) \approx \hat{U}(0)\sqrt{\pi} \cdot e^{-x^2/4kt}.$$

So for each fixed t, $\sqrt{kt} \cdot u(x,t)$ has a bell shape and will have a bell shape for almost initial data as $kt \to \infty$. Using the Maclaurin Series approximation of $u(x,t)$, we can show $u(x,t) \to 0$ as $|x| \to \infty$.

Definition: Dirac delta function

Define the **Dirac delta function** to be the concentrated pulse "function" $\delta(x)$ with the property that

$$\int_{-\infty}^{\infty} \delta(x-c)f(x)dx = f(c) = \int_{-\infty}^{\infty} \delta(c-x)f(x)dx, c \in \mathbb{R}.$$

That is, formally define

$$\delta(x-c) = \begin{cases} 0, & \text{if } x \neq c \\ \infty, & \text{if } x = c \end{cases}$$

Note. If we let $f(x) = 1$ and $c = 0$, then $\int_{-\infty}^{\infty} \delta(x)dx = 1$. The integrals above cannot be defined as limits of Riemann sums because $\delta(x)$ is no ordinary function. The integral statement above is true by definition.

Theorem: fundamental solution of the heat equation

Given the problem

$$\begin{cases} \text{PDE: } \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} & -\infty < x < \infty, t > 0 \\ \text{ICs: } u(x,0) = \delta(x) \end{cases}$$

Then we can show that the fundamental solution is

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \hat{\delta}(m)e^{imx}e^{-m^2kt}dm \\ &= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\overline{x})e^{-im\overline{x}}d\overline{x}\right] e^{imx}e^{-m^2kt}dm \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi}e^{-im\cdot 0}e^{imx}e^{-m^2kt}dm \\ &= \frac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt} \text{ by above approximation} \end{aligned}$$

Remark. This solution represents the evolution of the temperature due to an initial heat source at $x = 0, t = 0$ for an infinite rod and the temperature has a Gaussian distribution.

1 Finite Rod

Consider the finite interval problem:

$$\begin{cases} \text{PDE: } \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} & 0 < x < L, t > 0 \\ \text{BCs: } u(0,t) = 0 = u(L,t) & t > 0 \\ \text{ICs: } u(x,0) = f(x) & 0 \leq x \leq L \end{cases}$$

where $f(x)$ is continuous and $f(0) = f(L) = 0$. Note that the DE is not required to hold when $t = 0$ (so $f(x)$ is not required to be twice differentiable). However, $u(x,t)$ is required to be continuous for $(x,t) \in [0,L] \times [0,\infty)$. That is,

$$\lim_{(x,t)\to(x_0,0^+} u(x,t) = u(x_0,0) = f(x_0), x_0 \in \mathbb{R}.$$

Theorem: heat equation by method of images

Let $\tilde{f}_{odd}(x), -\infty < x < \infty$, be the periodic extension of the odd extension of $f(x)$. Then the unique solution of problem above, which is continuous for all $(x,t) \in (-\infty,\infty) \times [0,\infty)$, is

$$u(x,t) = \begin{cases} \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-\overline{x})^2/4kt} \tilde{f}_{odd}(\overline{x}) d\overline{x}, & \text{if } t > 0 \\ f(x), & \text{if } t = 0 \end{cases}$$

Claim. The fundamental source solution and the F.S. solution to the heat equation are the same.

Proof

Since $u(x,t) \in \mathcal{C}^\infty$ odd periodic function of x, $t > 0$. Thus, fundamental source solution is equal to its F.S.S on $[0,L], t > 0$. That is,

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\left(\frac{n\pi x}{L}\right), B_n(t) = \frac{2}{L}\int_0^L u(x,t) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Using integration by parts, we move derivative of f wrt x to derivative of

g wrt x.

In general, $\displaystyle\int_a^b \frac{df}{dx} g(x)dx = g(x)f(x)\Big|_a^b - \int_a^b f(x)\cdot\frac{dg}{dx}dx$

$$\begin{aligned}
\text{In our case, } \frac{\partial u}{\partial t} &= \sum_{n=1}^{\infty} \frac{\partial}{\partial t} B_n(t)\sin\left(\frac{n\pi x}{L}\right) \\
&= \sum_{n=1}^{\infty} \frac{2}{L}\left[\int_0^L u_t(x,t)\sin\left(\frac{n\pi x}{L}\right)dx\right]\sin\left(\frac{n\pi x}{L}\right) \\
&= \sum_{n=1}^{\infty} \frac{2}{L}\left[\int_0^L ku_{xx}(x,t)\sin\left(\frac{n\pi x}{L}\right)dx\right]\sin\left(\frac{n\pi x}{L}\right) \\
&= \sum_{n=1}^{\infty} \frac{2}{L}\left[\int_0^L ku(x,t)\frac{d^2}{dx^2}\left(\sin\left(\frac{n\pi x}{L}\right)\right)dx\right]\sin\left(\frac{n\pi x}{L}\right)
\end{aligned}$$

After differentiation we can show that $B_n(t)$ satisfies the following ODE:

$$B_n(t) = -k\left(\frac{n\pi}{L}\right)^2 B_n'(t) \Rightarrow B_n(t) = c_n e^{-(n\pi/L)^2 kt}.$$

We can finally show that $c_n = b_n$ and thus complete the proof. $\square$

Theorem: transform method

Suppose $f(x) \in \mathcal{C}^1$ and suppose $|f(x)| + |f'(x)| \le K|x|^{-2}$, then

$$\widehat{\frac{df}{dx}}(m) = im\cdot\widehat{f}(m) \Leftrightarrow \mathcal{F}[f'(x)] = im\cdot\mathcal{F}[f(x)].$$

Proof

Use integration by parts on $\mathcal{F}[f'(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f'(x)e^{-imx}dx$ and the fact that $\lim\limits_{x\to\pm\infty} f(x) = 0$ to prove it. $\square$

Example. Take the Fourier transform of both sides of the heat equation wrt x, fix t, yields:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} u_t e^{-imx}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} ku_{xx}e^{-imx}dx \Rightarrow \frac{\partial}{\partial t}\hat{u}(m,t) = k\widehat{u_{xx}}(m,t).$$

Apply the property above and we have

$$\frac{\partial}{\partial t}\hat{u}(m,t) = k(im)^2\hat{u}(m,t) \Rightarrow \frac{\partial}{\partial t}\hat{u}(m,t) = -km^2\hat{u}(m,t).$$

Thus the PDE has been essentially transformed into an ODE (because x doesn't affect the frequency domain). It can be shown that the solution is $\hat{u}(m,t) = \hat{f}(m)e^{-km^2t}$ which implies that $u(x,t) = \mathcal{F}^{-1}[\hat{u}(m,t)]$ is equal to the Fourier solution.