0.1 Mean Value Property

Note that

$$v(r,\theta) = a_0 + \sum_{n=1}^{\infty} a_n \left(\frac{r}{R}\right)^n \cos(n\theta) + b_n \left(\frac{r}{R}\right)^n \sin(n\theta).$$

implies that the temperature at the center is an average of the temperature at the boundary:

$$v(0,\theta) = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta.$$

For any fixed $0 < R' < R$, suppose $w(r,\theta)$ satisfies

$$\begin{cases} \text{PDE: } \Delta w = 0, & 0 < r < R', \theta \in (-\pi, \pi) \\ \text{BCs: } w(r,\theta) = g(\theta) \end{cases}$$

Then the solution is almost identical:

$$w(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n \left(\frac{r}{R'}\right)^n \cos(n\theta) + B_n \left(\frac{r}{R'}\right)^n \sin(n\theta).$$

where for $n \geq 1$, we have

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\cos(n\theta)d\theta \text{ and } B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)d\theta.$$

And note that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)d\theta = A_0 = w(0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta.$$

So every circle of radius $0 < R' < R$ centered at $r = 0$ has the same average temperature as the circle of radius $r = R$.

Theorem: Mean Value Theorem for Laplace's Equation

The temperature at any point x_0 is the average of the temperature along any circle of radius $R' > 0$ lying inside the domain and centered at x_0.

Proof

Sketch: Suppose $\Delta u = 0$ in domain D and suppose x_0 is in domain D, then $\Delta u = 0$ on any circle centered at x_0 and within D, thus solving this Laplacian "subproblem" shows that x_0 is the average of the temperature along any circle lying inside D and centered at x_0 (like we showed above). □

Theorem: Maximum Principle for Laplace's Equation

In steady state, assuming no sources, the temperature cannot attain its maximum in the interior unless the temperature is constant everywhere (In other words, the extrema values are achieved on the boundary).

Proof

Sketch: Suppose the temperature is not constant and that the maximum occurs at a point P in the domain. Now since $u(P)$ is the average temperature of all the points on any circle centered at P, it cannot be larger than any temperature on the circle (since it's an average) thus we have a contradiction. Thus the maximum temperature occurs on the boundary. $\square$

Claim. If $\Delta u = 0$ on an open region then u is $\mathcal{C}^\infty$ in the open region.

Proof

Assume the standard solution with $v(R, \theta) = f(\theta)$. Suppose $f(\theta)$ is piecewise smooth $-\pi < \theta \leq \pi$ then there exists a finite number $M > 0$ such that

$$|a_0| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)| d\theta \leq \frac{M}{2}, |a_n| \leq M, |b_n| \leq M.$$

Notice:

- if $r = R$, then the F.S. converges to $\frac{1}{2}(f(\theta^+) + f(\theta^-))$.
- if $r = 0$, then $v = 0$.
- If $0 < r < R$, then we want to show it converges absolutely.

Since $0 < \frac{r}{R} < 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \left| a_n \left(\frac{r}{R}\right)^n \cos(n\theta) \right| + \left| b_n \left(\frac{r}{R}\right)^n \sin(n\theta) \right| &\leq \sum_{n=1}^{\infty} \left[M \left(\frac{r}{R}\right)^n + M \left(\frac{r}{R}\right)^n \right] \\
&= \sum_{n=1}^{\infty} 2M \left(\frac{r}{R}\right)^n \\
&= \frac{2M(r/R)}{1-(r/R)} \\
&= \frac{2Mr}{R-r}
\end{aligned}$$

Therefore, $v(r,\theta)$ converges absolutely for each $r \in (0, R)$. $\square$

0.1.1 Derivative

Differentiation gives:

$$\frac{\partial v}{\partial \theta} = -\sum_{n=1}^{\infty} a_n \left(\frac{r}{R}\right)^n \cdot n \cdot \sin(n\theta) + b_n \left(\frac{r}{R}\right)^n \cdot n \cdot \cos(n\theta).$$

Since $\left(\frac{r}{R}\right)^n = e^{n \ln(r/R)} = e^{-n|\ln(r/R)|}$ so

$$\sum_{n=1}^{\infty} \left| a_n \left(\frac{r}{R}\right)^n n \sin(n\theta) + \_n \left(\frac{r}{R}\right)^n n \cos(n\theta) \right| \leq \sum_{n=1}^{\infty} 2M \frac{n}{e^{n|\ln(r/R)|}}.$$

By Ratio test, the series converges absolutely. This means that the derivative converges uniformly and thus we can swap differentiation and infinite sum by term-by-term differentiation theorem. This result applies to all orders of derivatives.

Theorem: general solution of annulus

If $v(R_i, \theta) = f_i(\theta), v(R_o, \theta) = f_o(\theta)$, then the general solution over an annulus is

$$\begin{aligned}
v(r,\theta) = [a_0 + A_0 \ln(r)] + \sum_{n=1}^{\infty} \left(\frac{r}{R_o}\right)^n [a_n \cos(n\theta) + b_n \sin(n\theta)] \\
+ \sum_{p=1}^{\infty} \left(\frac{R_i}{r}\right)^p [A_p \cos(p\theta) + B_p \sin(p\theta)]
\end{aligned}$$

Note. This solution encompasses the cases of completely inside the disk and outside the disk.

0.2 Uniqueness

Suppose $u_1(x,y), u_2(x,y)$ both satisfy the PDE and BCs. Let $u'(x,y) = u_1 - u_2$. Notice

$\Delta u' = \Delta u_1 - \Delta u_2 = 0 - 0 = 0$ and $u'(R,\theta) = u_1(R,\theta) - u_2(R,\theta) = f(\theta) - f(\theta) = 0$.

Thus, u' also satisfies the PDE and BCs. Note that the Maximum Principle states that the maximum and minimum of $u'(x,y)$ occur on the boundary but $v'(R,\theta) = 0$ so $v' = 0 \Rightarrow u_1 = u_2$.

Theorem: Uniqueness of Heat Equation

Suppose $u_1(x,t), u_2(x,t)$ are $\mathcal{C}^2$ solutions of the problem

$$\begin{cases} \text{PDE: } u_t = k u_{xx} & 0 < x < L, t > 0 \\ \text{BCs: } u(0,t) = a(t), u(L,t) = b(t) & t > 0 \\ \text{ICs: } u(x,0) = f(x) & 0 \leq x \leq L \end{cases}$$

where $a(t), b(t), f(x)$ are given $\mathcal{C}^2$ functions when $u_1(x,t) = u_2(x,t)$ for all $x \in [0,L], t \geq 0$.

Proof

Sketch: Let $u'(x,t) = u_1(x,t) - u_2(x,t)$. Define

$$F(x) = \int_0^L |u'(x,t)|^2 \, dt \geq 0.$$

By using integration by parts, we can show that $F'(x) \leq 0$ and since $F(0) = 0, F(t) \geq 0$, this is only possible if $F(t) = 0 \Rightarrow u'(x,t) = 0 \Rightarrow u_1(x,t) = u_2(x,t)$. □

Theorem: Uniqueness of Wave Equation

Suppose $u_1(x,t), u_2(x,t)$ are $\mathcal{C}^2$ solutions of the problem

$$\begin{cases} \text{PDE: } u_t = c^2 u_{xx} & 0 < x < L, t > 0 \\ \text{BCs: } u(0,t) = a(t), u(L,t) = b(t) & t > 0 \\ \text{ICs: } u(x,0) = U(x), u_t(x,0) = V(x) & 0 \leq x \leq L \end{cases}$$

where $a(t), b(t), U(x), V(x)$ are given $\mathcal{C}^2$ functions when $u_1(x,t) = u_2(x,t)$ for all $x \in [0,L], t \geq 0$.

Proof

Sketch: Define *Lyapunov functional* as

$$H(t) = int_{x=0}^{x=L}[c^2 \cdot v_x^2(x,t) + v_t^2(x,t)]\ dx.$$

Then we can show that $H'(t) = 0$ and since $H(0) = 0$, then it must be that $H(t) = 0 \Rightarrow v_t(x,t) \Rightarrow v(x,t) = \int_0^t v_t(x,s)\ dx \Rightarrow u_1(x,t) = u_2(x,t)$. $\square$