Homework 3

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Problem (1).
($\Rightarrow$): Given $\varepsilon > 0$, since almost surely convergence implies convergence in probability, $Z_n \xrightarrow{a.s} Z \Rightarrow \lim\limits_{n\to\infty} P(|Z_n - Z| \geq \varepsilon) = 0$. That is, there exists $n \in \mathbb{N}$ such that

$$P(|Z_k - Z| \geq \varepsilon, k \geq n) < \varepsilon.$$

Taking the complement yields

$$\begin{gathered} P((|Z_k - Z| \geq \varepsilon, k \geq n)^c) > 1 - \varepsilon \\ P(|Z_k - Z| < \varepsilon, n \leq k) > 1 - \varepsilon \\ P(|Z_k - Z| < \varepsilon, n \leq k \leq m) > 1 - \varepsilon \ \forall\ m \geq n \end{gathered}$$

The last step follows from that if the statement is true for all $k \geq n$, then it's true for a subset of such k where $n \leq k \leq m$ for all $m \geq n$.

($\Leftarrow$): Suppose for every $\varepsilon > 0$ there exists an n such that

$$P(|Z_k - Z| < \varepsilon, n \leq k \leq m) > 1 - \varepsilon$$

for all $m \geq n$. By taking $n \to \infty$, it will cover the n required by any arbitrarily small ε, so we can let $\varepsilon \to 0$ and obtain

$$P(\lim_{n\to\infty} Z_n = Z) \geq 1.$$

Since P is a probability measure, $P(\lim\limits_{n\to\infty} Z_n = Z) \leq 1$. Hence,

$$P(\lim_{n\to\infty} Z_n = Z) = 1$$

which is the definition of almost sure convergence.

Problem (2). ($\subseteq$): We wish to show that

$$\mathcal{G} = \{(H \cap A) \cup (H^c \cap B), A, B \in \mathcal{F}\}$$

is a σ-field containing $\mathcal{F} \cup \{H\}$.

First, notice that given $F \in \mathcal{F}$,

$$F = \Omega \cap F = (H \cup H^c) \cap F = (H \cap F) \cup (H^c \cap F) \in \mathcal{G}.$$

Moreover, since $\emptyset, \Omega \in \mathcal{F}$,

$$H = (H \cap \Omega) \cup (H^c \cap \emptyset) \in \mathcal{G}.$$

Thus, $\mathcal{F} \cup \{H\} \subseteq \mathcal{G}$.

Now let's show that $\mathcal{G}$ is a σ-field.

(i) Take $A = B = \Omega \in \mathcal{F}$, we have

$$(H \cap \Omega) \cup (H^c \cap \Omega) = H \cup H^c = \Omega \in \mathcal{G}.$$

(ii) Given $S \in \mathcal{G}$, we know there exist $A, B \in \mathcal{F}$ such that $S = (H \cap A) \cup (H^c \cap B)$. Then the complement is

$$\begin{aligned}
S^c &= ((H \cap A) \cup (H^c \cap B))^c \\
&= (H \cap A)^c \cap (H^c \cap B)^c \\
&= (H^c \cup A^c) \cap (H \cup B^c) \\
&= (H^c \cap H) \cup (H^c \cap B^c) \cup (A^c \cap H) \cup (A^c \cap B^c) \\
&= (H \cap A^c) \cup (H^c \cap B^c) \cup ((H \cup H^c) \cap (A^c \cap B^c)) \\
&= (H \cap A^c) \cup (H^c \cap B^c) \cup (H \cap (A^c \cap B^c)) \cup (H^c \cap (A^c \cap B^c)) \\
&= (H \cap (A^c \cup (A^c \cap B^c)) \cup (H^c \cap (B^c \cup (A^c \cap B^c)) \\
&= (H \cap A^c) \cup (H^c \cap B^c) \in \mathcal{G}
\end{aligned}$$

(iii) Given a sequence $G_1, G_2, \ldots \in \mathcal{G}$, we can express G_n as $(H \cap A_n) \cup (H^c \cap B_n)$ for some sequence of $(A_n), (B_n) \subseteq \mathcal{F}$. Then we have

$$\begin{aligned}
\bigcup_{n=1}^{\infty} G_n &= \bigcup_{n=1}^{\infty} (H \cap A_n) \cup (H^c \cap B_n) \\
&= \bigcup_{n=1}^{\infty} (H \cap A_n) \cup \bigcup_{n=1}^{\infty} (H^c \cap B_n) \\
&= \left(H \cap \bigcup_{n=1}^{\infty} A_n \right) \cup \left(H^c \cap \bigcup_{n=1}^{\infty} B_n \right)
\end{aligned}$$

Hence, we show that $\mathcal{G}$ is a σ-field containing $\mathcal{F} \cup \{H\}$. Since $\sigma(\mathcal{F} \cap \{H\})$ is the smallest σ-field containing $\mathcal{F} \cup \{H\}$, we prove that $\sigma(\mathcal{F} \cup \{H\}) \subseteq \mathcal{G}$, hence its elements must have such form.

($\supseteq$): Given $G \in \mathcal{G}$, let's show that $G \in \sigma(\mathcal{F} \cup \{H\})$. By definition of $\mathcal{G}$, there exists $A, B \in \mathcal{F}$ such that $G = (H^c \cap A) \cup (H^c \cap B)$.

$$\begin{aligned}(H \cap A) \cup (H^c \cap B) &= ((H \cup H^c) \cap (H \cup B)) \cap ((A \cup H^c) \cap (A \cup B)) \\ &= (H \cup B) \cap (H^c \cup A) \cap (A \cup B)\end{aligned}$$

Since $H, H^c, A, B \in \sigma(\mathcal{F} \cup \{H\})$, their unions and intersections are also in it. Thus, $G \in \sigma(\mathcal{F} \cup \{H\})$ and $\mathcal{G} \subseteq \sigma(\mathcal{F} \cup \{H\})$.

By double-containment, we show that $\sigma(\mathcal{F} \cup \{H\}) = \mathcal{G}$.

Problem (3)**.** Recall that $\mu(A \cup B) = \mu((A \cup B) \setminus (A \cap B)) + \mu(A \cap B)$ by finite additivity. Moreover, $\mu(A), \mu(B) \leq \mu(A \cup B)$ and $\mu(A), \mu(B) \geq \mu(A \cap B)$ by monotonicity. Thus, $\mu(A) - \mu(B) \leq \mu(A \cup B) - \mu(A \cap B), \mu(B) - \mu(A) \leq \mu(A \cup B) - \mu(A \cap B) \Rightarrow |\mu(A) - \mu(B)| \leq \mu(A \cup B) - \mu(A \cap B)$.

$$\begin{aligned}\mu(A \Delta B) &= \mu((A \cap B^c) \cup (A^c \cap B)) \\ &= \mu((A \cup A^c) \cap (B^c \cap A^c) \cap (A \cup B) \cap (B^c \cup B)) \\ &= \mu(\Omega \cap (A^c \cup B^c) \cap (A \cup B) \cap \Omega) \\ &= \mu((A \cap B)^c \cap (A \cup B)) \\ &= \mu((A \cup B) \setminus (A \cap B)) \\ &= \mu(A \cup B) - \mu(A \cap B) \\ &\geq |\mu(A) - \mu(B)|\end{aligned}$$

Problem (4)**.**

(i) Since P is a probability measure, $0 \leq P(\{\omega : X(\omega) \in A\}) \leq 1 \ \forall \ A \subseteq \mathbb{R} \Rightarrow 0 \leq P_X(A) \leq 1 \ \forall \ A \subseteq \mathbb{R}$.

(ii) $P_X(\emptyset) = P(\{\omega : X(\omega) \in \emptyset\}) = P(\emptyset) = 0$. And $P_X(\mathbb{R}) = P(\{\omega : X(\omega) \in \mathbb{R}\}) = P(\Omega) = 1$.

(iii) Given a sequence of disjoint sets in $\mathbb{R}$, (A_n),

$$\begin{aligned} P_X\left(\bigcup_{n=1}^{\infty} A_n\right) &= P\left(\left\{\omega : X(\omega) \in \bigcup_{n=1}^{\infty} A_n\right\}\right) \\ &= P\left(\bigcup_{n=1}^{\infty}\{\omega : X(\omega) \in A_n\}\right) \\ &= \sum_{n=1}^{\infty} P(\{\omega : X(\omega) \in A_n\}) \text{ by countable additivity of } P \\ &= \sum_{n=1}^{\infty} P_X(A_n) \end{aligned}$$

Thus, P_X is indeed a probability measure.

Problem (5)**.** First let's show that μ is finitely additive.

Given disjoint $A, B \in \sigma(\Omega)$, if both A, B are finite, then disjointness gives us

$$\mu(A \cup B) = \sum_{k \in A \cup B} 2^{-k} = \sum_{k \in A} 2^{-k} + \sum_{k \in B} 2^{-k} = \mu(A) + \mu(B).$$

If at least one of them is infinite, then $A \cup B$ is infinite. Thus,

$$\mu(A \cup B) = \infty = \mu(A) + \mu(B).$$

However, we claim that μ is not countably additive. Here is an counterexample: let $A_n = \{n\}$, so they are disjoint and finite. It's easy to see that $\bigcup_{n=1}^{\infty} A_n = \Omega$ is infinite, so $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \infty$. However,

$$\begin{aligned} \sum_{n=1}^{\infty} \mu(A_n) &= \sum_{n=1}^{\infty} \sum_{k=n}^{n} 2^{-k} \\ &= \sum_{n=1}^{\infty} 2^{-n} \\ &= 1 < \infty \end{aligned}$$

Therefore, μ is not countably additive.