Theorem: Kolmogoro's 0-1 Theorem

Suppose $A_1, \ldots$ are independent events. For any A in the generated tail σ-field, we have either $P(A) = 0$ or $P(A) = 1$.

1 Simple Random Variables

Definition: simple random variable

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is a **simple random variable** if

(i) it assumes finitely many values (finite range).

(ii) For any $x \in \mathbb{R}$,

$$\{X = x\} := \{\omega : X(\omega) = x\} \in \mathcal{F}.$$

"The inverse image of x under X is in $\mathcal{F}$". We say that "X is measurable wrt $\mathcal{F}$", or X is $\mathcal{F}$-measurable.

Definition: indicator random variables

Let A be a set. The **indicator random variable**

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

Theorem

Suppose $A_1, A_2, \ldots, A_n \in \mathcal{F}$ form a finite partition of Ω. Define

$$X(\omega) = \sum_{i=1}^{n} a_i I_{A_i}(\omega).$$

for some fixed $a_1, \ldots, a_n \in \mathbb{R}$. Then X is a simple r.v.

Note. Every simple r.v. can be written in this way.

Theorem

An indicator r.v. $X(\omega) = I_A(\omega)$ is measurable wrt $\mathcal{F} \Rightarrow A \in \mathcal{F}$.

Definition

$(\Omega, \mathcal{F}, P)$, X a simple r.v. on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{G} \subseteq \mathcal{F}$ be another σ-field. We say that ***X* is measurable wrt $\boldsymbol{\mathcal{G}}$** if for any $x \in \mathbb{R}$, $\{w : X(\omega) = x\} \in \mathcal{G}$.

Note. $\mathcal{G} \subseteq \mathcal{F}$. If X is measurable wrt $\mathcal{G}$. Then for any $H \subseteq \mathbb{R}$,

$$\{\omega : X(\omega) \in H\} = \bigcup_{x \in H} \{\omega : X(\omega) = x\} \in \mathcal{G}.$$

which might be an uncountable union unless X is simple so it has finite range.

Definition: the sigma-field generated by X

The **$\boldsymbol{\sigma}$-field generated by $\boldsymbol{X}$**, denoted $\sigma(X)$, is the smallest σ-field wrt which X is measurable.

Note. It is the intersection of a σ-field wrt which X is measurable.

Definition: sigma-field generated by a sequence of r.v.

For a finite or infinite sequence $X_1, X_2, \ldots$ of simple r.v., $\sigma(X_1, X_2, \ldots)$ is the smallest σ-field wrt which each X_i is measurable.

Theorem: 5.1

Let $X_1, \ldots, X_n$ be a finite sequence of simple r.v.s.

(i) The σ-field, $\sigma(X_1, \ldots, X_n)$, consists of all subsets of Ω of the form

$$\{\omega : (X_1(\omega), \ldots, X_n(\omega)) \in H\} \text{ for any } H \subseteq \mathbb{R}^n.$$

(ii) A simple r.v. Y is measurable wrt $\sigma(X_1, \ldots, X_n)$ if and only if $Y = f(X_1, \ldots, X_n)$ for some $f : \mathbb{R}^n \to \mathbb{R}$.