1 Homomorphism

Note. **Homomorphism** is a structure preserving map.

In linear algebra: it preserves vector addition and scalar multiplication (linear maps).

In group theory: preserves group operation.

Definition

Let $(G, *)$ and $(H, *)$ be groups. A **homomorphism** (of groups) from G to H is a function $\phi : G \to H$ such that

$$\phi(g_1 *_G g_2) = \phi(g_1) *_H \phi(g_2).$$

Note. This resembles linear maps $T(u+v) = T(u) + T(v)$. Any linear map is a group homomorphism. An isomorphism is a bijective homomorphism.

Example (uninteresting)**.** $T : V \to W$? Let $T(v) = 0$ for all $v \in V$. Similarly, let $(G, *_G), (H, *_H)$ be groups. Define $\phi : G \to H$ by $\phi(g) = e_H$ for all $g \in G$. Then ϕ is a homomorphism.

Proof

$\phi(x *_G y) = e_H = e_H *_H e_H = \phi(x) *_H \phi(y)$ □

Example. $T : V \to V$. Another trivial example of homomorphism is the identity map $T(v) = v$. Similarly, $(G, *_G)$ is a group. Let $\phi : G \to G$ be defined as $\phi(g) = g$. The proof is trivial.

Example (interesting, not isomorphism)**.** Consider $GL_n(\mathbb{R})$: $n \times n$ invertible matrices with entries from $\mathbb{R}$ under matrix multiplication.

Is it abelian? Counterexample: use 2×2 upper and lower triangle of all 1 as non-zero entries. Then we can just insert this to any $n \times n$ identity matrices to the top left.

$GL_1(\mathbb{R}) \simeq \mathbb{R}^*$. This is abelian.

So it is abelian if and only if $n = 1$.

It is an infinite group. Just change one element of an identity matrix with infinite number of choices.

$\det : GL_n(\mathbb{R}) \to \mathbb{R}^*$, $A \mapsto \det(A)$.

Claim. det is a homomorphism of groups.

It is surjective but not injective. It isn't an isomorphism because det is abelian.

$\det(AB) = \det(A)\det(B)$ by a Theorem from linear algebra. This satisfies the definition of homomorphism.

Example (sign map)**.** $\varepsilon : S_n \to \mathbb{Z}_2$.

$$\varepsilon(g) \begin{cases} 0 & \text{if } g \text{ is even} \\ 1 & \text{if } g \text{ is odd} \end{cases}$$

	even	odd
even	even	odd
odd	odd	even

$+_2$	0	1
0	0	1
1	1	0

WLOG assume x is even, y is odd. Then

$$\begin{aligned} \varepsilon(x * y) &= ?\varepsilon(x) * \varepsilon(y) \\ 1 &= 0 +_2 1 \end{aligned}$$

True by table.

Note. Given linear map $T : V \to W$. Then $T(0_V) = 0_W$. $T(-v) = -T(v)$.

Similarly, for group homomorphism $\phi : G \to H$.

Claim. $\phi(e_G) = e_H$.

Proof

$$\begin{aligned} \phi(x *_G y) &= \phi(x) *_H \phi(y) \\ \phi(e_G) &= \phi(e_G * e_G) = \phi(e_G) *_H \phi(e_G) \\ Y &= Y *_H Y \Rightarrow Y = \phi(e_G) = e_H \end{aligned}$$

□

Claim. $\phi(g^{-1}) = \phi(g)^{-1}$.

Proof

$$\begin{aligned}\phi(x *_G x^{-1}) &= \phi(x) *_H \phi(x^{-1}) \\ e_H = \phi(e_G) &= \end{aligned}$$

Thus $\phi(x^{-1}) = \phi(x)^{-1}$. $\square$

Example. Consider $\det : GL_n(\mathbb{R}) \to \mathbb{R}^*$. By the theorem above:

$$\det(A^{-1}) = \frac{1}{\det A}$$

.

Note. In linear algebra, $T : V \to W$ linear map,

$$\ker T = \{v \in V : T(v) = 0_W\}$$

and

$$\operatorname{im} T = \{w \in W : T(v) = w, v \in V\}$$

Definition: kernel and image

$\phi : G \to H$ a homomorphism of groups.

$$\ker \phi = \{g \in G : \phi(g) = e_H\}.$$

$$\operatorname{im} \phi = \{h \in H : \phi(g) = h, g \in G\}.$$

$\ker \phi \leq G$, $\operatorname{im} \phi \leq H$.

See screenshot for illustration.

Proof

$\ker \phi \subseteq G$ by definition. Then

1) $\phi(e_G) = e_H$ by previous proof.
2) closure: If $x, y \in \ker \phi$, $\phi(x) = e_H, \phi(y) = e_H$, and $\phi(x * y) = e_H * e_H = e_H$.
3) If $x \in \ker \phi$, $\phi(x^{-1}) = \phi(x)^{-1} = e_H^{-1} = e_H$.

$\square$