1 Motion of a guitar string

For a plucked guitar string at $t = 0$, the initial position, $U(x)$, looks like either a sharp isosceles triangle or a hill-like smooth curve.

In either case $U(x)$ is continuous and $U'(x)$ is piecewise continuous or smooth. Note that if the initial velocity is $V(x) = 0$ then $B_n = 0$. If the initial position is given by

$$U(x) = \begin{cases} \frac{ax}{h}, & 0 < x \leq h \\ \frac{a(L-x)}{L-h}, & h < x < L \end{cases}$$

Then using integration by parts, we get

$$A_n = \frac{2a \sin(n\pi h/L)}{(n\pi)^2 \frac{h}{L} \frac{L-h}{L}}.$$

We can guess this converges because of n^2 term on the denominator.

So the position of the string is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2a}{\pi^2} \frac{L}{h} \frac{L}{L-h} \frac{\sin(n\pi h/L))}{n^2} \cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right).$$

I claim that this converges absolutely:

$$0 \leq \sum_{n=1}^{\infty} |u(x,t)| \leq \sum_{n=1}^{\infty} \frac{M}{n^2}$$

for some constant $M > 0$, so it converges absolutely. Hence it converges uniformly to a continuous function. But note

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} \frac{2a}{\pi^2} \frac{L}{h} \frac{L}{L-h} \frac{\sin(n\pi h/L))}{n^2} \cdot - \left(\frac{n\pi x}{L}\right)^2 \cdot \cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right).$$

Notice n^2 that helped achieving convergence got canceled! By divergence test (take n to infinity but the term doesn't go to zero), this doesn't converge. This is okay because in practice we always use a finite sum, so divergence is not a big concern.

Thus, for each fixed $N > 0$, the partial sum of the formal solution, $S_N(x)$, will satisfy the PDE and BC and will approximate the initial condition as accurately as we wish.

2 Supplement: The Sound of Music

Example (space and time waves)**.** For the general solution, if we fix a t, then we get a space wave, which represents the physical position of the string at a specific time. If we fix a x, we get a time wave. which represents the evolution of each point of the string through time.

2.1 Standing Waves

The vertical displacement is a linear combination of the simple product solutions.

Each is called the "normal nodes of vibration" for $n = 1, 2, \ldots$. And each mode has amplitude $\sqrt{A_n^2 + B_n^2}$.

The nth normal node is called the "nth harmonic".

For each fixed t, each node looks like a simple oscillation called a *standing wave*.

Note that the period of each mode (in time) is $\tau_n = 2\pi \cdot \frac{L}{n\pi c} = \frac{2L}{nc}$.

So one cycle is completed every τ_n thus the frequency is $f = \frac{1}{\tau_n} = \frac{nc}{2L} = \frac{n}{\tau_1}$ for $n = 1, 2, \ldots$

The sound produced consists of the superposition of these infinite frequencies.

2.2 Music Theory

See slides. This is common sense. E_2 is 2 octaves above E_0.

When we pluck guitar, at $n = 1$ we have the **first harmonic** or the *fundaemental mode:* $A_1 \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi ct}{L}\right)$.

Assume $\omega_1 = 82.41$Hz. We cannot mute fundamental harmonic because it has no inner stationary points.

At $n = 2$, the shape of the standing wave is determined by $\sin\left(\frac{2\pi x}{L}\right)$, which has 1 inner stationary point (that never moves). Then the frequency is twice the fundamental harmonic. The amplitude is a quarter of the fundamental harmonic.

Similarly, $n = 3$, it has 2 inner stationary points. So we can mute the third harmonic at $h = \frac{L}{3}$ or $\frac{2L}{3}$.

2.3 Fun Facts

- Each octave increase corresponds to a doubling of the frequency.
- For every octave the frequency doubles per 12 notes then the frequency per note increases by a factor of 1.06.