0.1 Classification Theorem

It asserts that every second-order linear PDE with constant coefficients, where the unknown function has *two* independent variables, can be transformed by a change of variables into exactly one of the following forms:

1) generalized wave equation

$$-c^2 u_{xx} + u_{tt} + \alpha u = f(x,t), c > 0 \text{ (hyperbolic case)}$$

since it's 2nd derivative minus 2nd derivative.

2) generalized Poisson/Laplace equation ($t = y$)

$$a^2 u_{xx} + u_{tt} + \alpha u = g(x,t), a > 0 \text{ (elliptic case).}$$

3) generalized heat equation

$$-k^2 u_{xx} + u_t + \alpha u = h(x,t), k > 0 \text{ (parabolic case).}$$

4)

$$u_{xx} + cu = g(x,t), \text{ (degenerate case).}$$

1 Complex Fourier Series

Recall by Euler's formula,

$$\cos\left(\frac{n\pi x}{L}\right) = \frac{e^{in\pi x/L} + e^{-in\pi x/L}}{2} \text{ and } \sin\left(\frac{n\pi x}{L}\right) = \frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i}.$$

Therefore, the F.S. becomes

$$\begin{aligned} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) &= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} + \frac{b_n}{2i}\right) e^{in\pi x/L} + \left(\frac{a_n}{2} - \frac{b_n}{2i}\right) e^{-in\pi x/L} \\ &= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2}\right) e^{in\pi x/L} + \left(\frac{a_n + ib_n}{2}\right) e^{-in\pi x/L} \\ &= a_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/L} + c_{-n} e^{-in\pi x/L} \end{aligned}$$

If we let $n = -m$, then by projection,

$$\begin{aligned}
c_{-n} = \frac{1}{2}(a_n + ib_n) &= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx + \frac{i}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right) \\
&= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{-m\pi x}{L}\right)dx + \frac{i}{L}\int_{-L}^{L} f(x)\sin\left(\frac{-m\pi x}{L}\right)dx\right) \\
&= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx - \frac{i}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx\right) \\
&= \frac{1}{2}(a_m - ib_m) = c_m
\end{aligned}$$

Therefore,

$$\text{F.S.}[f](x) = a_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/L} + \sum_{m=-1}^{-\infty} c_m e^{in\pi x/L} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}.$$