x commutes with everything. Multiplication:

$$\left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{j=0}^{n} b_j x^j\right) = \sum_{k=0}^{n} c_k x^k$$

where

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}.$$

Example. The coefficient of x^3 is contributed by terms with degree up to 3. So $c_3 = a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0$.

Proposition

The zero element in $R[x]$ is the zero polynomial 0_R. If R has identity, then so does $R[x]$: constant polynomial 1_R. If R has zero divisors, then so does $R[x]$.

Example. $R = \mathbb{Z}_6, \mathbb{Z}_6[x]$. $2 \times 3 = 0$, then think of these as constant polynomials, so they are zero divisors of $R[x]$.

Theorem

If R is a domain, then $R[x]$ is a domain.

Proof

(i) $R[x]$ is commutative, we can show by doing multiplication the other way around.

(ii) $R[x]$ has identity: we can check 1_R is the identity.

(iii) $R[x]$ has no zero divisors: Suppose $f(x), g(x) \in R[x]$ and $f(x) \neq 0, g(x) \neq 0$. We need to show that $f(x)g(x) \neq 0$.

Let $f(x)$ have degree n and $g(x)$ have degree m. So $f(x) = a_nx^n + \ldots, a_n \neq 0$, $g(x) = b_mx^m + \ldots, b_n \neq 0$. So

$$f(x)g(x) = a_nb_mx^{m+n} + \ldots.$$

Then $a_nb_m \neq 0$ because R is a domain and $a_n, b_m \neq 0$.

$\square$

Corollary

If R is a domain and $f(x), g(x) \in R[x]$ are nonzero, then $\deg(f(x)g(x)) = \deg f(x) + \deg g(x)$.

Example (counterexample when R is not a domain)**.** $\mathbb{Z}_6[x]$. Consider

$$\begin{aligned}(1+2x)(1-x+3x^2) &= 1-x+3x^2+2x-2x^2+6x^3 \\ &= 1+x+x^2\end{aligned}$$

Is $x^6 = x^0$? No because coefficient of x^6 is 1 but that of x^0 is 0.

Note. The exponents are not ring elements. They are natural numbers.

What are the units in $R[x]$?

If R is not a domain, there is no easy answer.

If R is a domain, the units in $R[x]$ are the constant units in R.

Example. $In \mathbb{Q}[x]$, the units are the nonzero constants because units in $\mathbb{Q}$ are $\mathbb{Q} \setminus \{0\}$.

Example. The units in $\mathbb{Z}_5[x]$ are $\{1, 2, 3, 4\}$. The units in $\mathbb{Z}[x]$ are ± 1.

Theorem

If R is a domain, then $U(R[x]) = U(R)$.

Proof

Suppose $f(x)g(x) = 1$ (they are units). The degree 0 polynomials are the nonzero constant polynomials. The degree of 1 is 0, and equals to the sum of degrees of f, g, thus f, g has degree 0 too. This means that $f(x), g(x)$ are nonzero constants, so they are units in R. $\square$