Intuition. The PDE and BCs allow us to form a vector space of solutions to the homogeneous equation.

The **time domain problem** is

$$G'(t) = -\lambda k G(t) \Rightarrow \int \frac{1}{G(t)} G'(t) dt = \int -\lambda k \; dt \Rightarrow \ln |G(t)| = -\lambda k t + C_1.$$

which finally yields the general solution

$$G(t) = Ce^{-\lambda k t}, C \in \mathbb{R}.$$

Physically we expect that $\lambda > 0$ (because since boundary condition is 0 we expect temperature to decay as time goes on). Now the boundary conditions give:

$$u(0,t) = 0 \Rightarrow F(0)G(t) = 0 \Rightarrow F(0) = 0.$$

Because otherwise , if $G(t) = 0 \Rightarrow u(x,t) = F(x)G(x) = 0$ which is a trivial solution that violates our separation assumption. Similarly $u(L,t) = 0 \Rightarrow F(L) = 0$. Thus the **boundary value problem** (or *eigenvalue problem)* can be formulated as

$$\begin{cases} \frac{d^2F}{dx^2} = -\lambda F(x) \\ F(0) = 0 = F(L) \end{cases}$$

To solve this ODE, let $F(x) = e^{rx}$, so

$$F''(x) = -\lambda F(x) \Rightarrow r^2 e^{rx} = -\lambda e^{rx} \Rightarrow r^2 = -\lambda.$$

and the last equation is the characteristic equation. We claim that $\lambda \leq 0$ gives the trivial solution $F(x) = 0$.

Proof

Case (1). If $\lambda = 0$, then $r = 0$ with repeated roots. Let $F(x) = C_1 e^0 + C_2 x e^0 \Rightarrow F(x) = C_1 + C_2 x$. Since $0 = F(0) = C_1$, $0 = F(L) = C_2 L \Rightarrow C_2 = 0$. Hence $F(x) = 0$ which is the trivial solution.

Case (2). Similarly when $\lambda < 0$ also gives us the trivial solution. (See homework).

$\square$

The only interesting case is $\lambda > 0$. Let's solve the BVP: $r^2 = -\lambda$ then we have purely imaginary roots $r_{1,2} = \pm i\sqrt{\lambda}$ and by Euler's Formula, we have

$$e^{r_1 x} = e^{i\sqrt{\lambda}x} = \cos(\sqrt{\lambda}x) + i\sin(\sqrt{\lambda}x).$$

and

$$e^{r_2 x} = e^{-i\sqrt{\lambda}x} = \cos(\sqrt{\lambda}x) - i\sin(\sqrt{\lambda}x).$$

Then the general solution can be any linear combination of these functions. We can now convert them to the wave form:

$$\frac{e^{i\sqrt{\lambda}x}+e^{-i\sqrt{\lambda}x}}{2}=\cos(\sqrt{\lambda}x).$$

and

$$\frac{e^{i\sqrt{\lambda}x}-e^{-i\sqrt{\lambda}x}}{2i}=\sin(\sqrt{\lambda}x).$$

So a general solution of the ODE is

$$F(x)=C_1\cos(\sqrt{\lambda}x)+C_2\sin(\sqrt{\lambda}x).$$

The boundary conditions give

$$F(0)=0\Rightarrow C_1=0 \text{ and } F(L)=0=C_2\sin(\sqrt{\lambda}L).$$

Since $C_2\neq 0$ (or it would be trivial solution), this implies

$$\sin(\sqrt{\lambda}L)=0\Rightarrow\sqrt{\lambda}L=n\pi\Rightarrow\sqrt{\lambda}=\frac{n\pi}{L}\Rightarrow\lambda=\left(\frac{n\pi}{L}\right)^2=\lambda_n,\ \text{for } n=\pm1,\pm2,\ldots.$$

Note n should not be 0 since it would give $\lambda=0$. Since sine is an odd function, for $n=1,2,3,\ldots$ we can write $F_n(x)=C_2\sin(\frac{n\pi x}{L})$. Thus we have the *product solution*

$$u_n(x,t)=F_n(x)G_n(t)=B_n\sin\left(\frac{n\pi x}{L}\right)e^{-\left(\frac{n\pi}{L}\right)^2kt} \text{ for } n=1,2,3,\ldots.$$

and some constants B_n.

Now by superposition principle, the solution of the homogeneous PDE (if it converges) is the linear combination of the product solutions,

$$u(x,t)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{L}\right)e^{-\left(\frac{n\pi}{L}\right)^2kt} \text{ for some constants } B_n.$$

and now using the initial condition to determine B_n. Note that we have an orthogonal basis of sines. This allows us to use the projection formula to find B_n. When $t=0$, we get a **Fourier Sine Series** (FSS), thus by projection formula (f(x) inner product with a sine basis vector):

$$f(x)=u(x,0)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{L}\right)\Rightarrow B_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Note that the domain is only from 0 to L. That is, the solution of the heat equation assuming convergence is

$$u(x,t)=\sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)\sin\left(\frac{n\pi x}{L}\right)e^{-\left(\frac{n\pi}{L}\right)^2kt}.$$

Note that the exponentially decaying term guarantees the convergence.