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Note. Every subgroup of an abelian group is abelian.

Theorem

Let G be a group and $H_1, \ldots, H_k$ be subgroups of G. Then $H_1 \cap \ldots \cap H_k \cap \ldots$ is a subgroup of G.

Proof

(i) $e \in H_i$ for all i, so $e \in \bigcap H_i$

(ii) Let $x, y \in \bigcap H_i$, then for each i, $x \in H_i$ and $y \in H_i \Rightarrow$ for each $i, x * y \in H_i \Rightarrow x * y \in \bigcap H_i$.

(iii) for each i, $x \in H_i \Rightarrow$ for each $i, x^{-1} \in H_i \Rightarrow x^{-1} \in \bigcap H_i$.

□

Note. Unions of subgroups are usually not subgroups, since $x * y$ might not be in the set.

Theorem

Let G be a group and let $a_1, a_2, \ldots, a_k$ be elements of G. Then there is a smallest subgroup H of G that contains $a_1, \ldots, a_k$.

Note. Smallest means that every other subgroups contains this subgroup.

Proof

Take all such subgroups and intersect them. Then

$$H = \bigcap H_i.$$

is the smallest subgroup. H_i is well-defined because at least G is such subgroup. □

Theorem

Let $(G, *)$ be a group. Let $a, b, c \in G$. Let H be the smallest subgroup containing a, b, c. Then, $H = \langle a, b, c \rangle$.

Example. A word in the generators and their inverses: $a^{-2}b^3ca^{-1}c^{-1}b^2a^{-3}$. This lies in H.

Theorem

The set of "words in the generators and their inverses" forms a subgroup.

Proof

(i) $a_1a_1^{-1}$ the identity is a word.

(ii) Closure: juxtaposition gives another word.

(iii) Inverses: just another word.

$\square$

Theorem

The words $a_1, \ldots, a_k$ and their inverses is the smallest subgroup containing $a_1, \ldots, a_k$.

Proof

We want to show that $H = \langle a, b, c \rangle = \bigcap H_i$. $\square$

Intuition. The intersection starts big and the generator starts small. They both give the same result.

Definition: finitely generated

If there exists a finite set $\{a_1, \ldots, a_k\}$ of elements in G with $G = \langle a_1, \ldots, a_k \rangle$, we call G **finitely generated**.

Example. V_4. $H = \langle a, b \rangle$. Then

$$H = \{e, a, b, c\}.$$

So V_4 can be generated by 2 elements but not 1 element.

Note. Every finite group is finitely generated. Just take all elements as generators.

Example. $\mathbb{Z}$ is an infinite group that is finitely generated. $\mathbb{Z} = \langle 1 \rangle$.

Example (Challenge)**.** Show that $\mathbb{Q}$ is not finitely generated.

Example (Figure 7.11(b))**.** See iPad.