1 Measures

Definition: Measurable Space

Let Ω be a non-empty set, $\mathcal{F}$ be a σ-field on Ω. $(\Omega, \mathcal{F})$ is a **measurable space**.

Definition: Measure

A **measure** on this space is a function $\mu : \mathcal{F} \to [0, \infty]$ with the properties:

(i) $\mu(\emptyset) = 0$

(ii) $A_1, A_2, \ldots \in \mathcal{F}$, disjoint $\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ (countable additivity)

Definition: Probability Measure

Let $(\Omega, \mathcal{F})$ be a measurable space. A **probability measure** is a function $P : \mathcal{F} \to [0, 1]$ and has $P(\Omega) = 1$.

Notation. μ = general measure, P = probability measure, λ = Lebesgue measure.

Note.

1. countable additivity $\Rightarrow$ finite additivity: if $A_1, \ldots, A_n \in \mathcal{F}$ disjoint. $\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$.
2. For any $A \in \mathcal{F}$, $1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c) \Rightarrow P(A^c) = 1 - P(A)$.
3. If $A, B \in \mathcal{F}$ and $A \subset B$, then $P(A) \le P(B)$. Since $B = A \cup (B \setminus A) = A \cup (B \cap A^c)$ which are disjoint.
4. A measure is "countably subadditive". *i.e.*: $A_1, \ldots \in \mathcal{F}$ are not necessarily disjoint,
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu(A_n).$$

Proof

$$
\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu(A_1 \cup \dots) \\
&= \mu(A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2 \setminus A_1) \dots) \\
&= \mu(A_1) + \mu(A_2 \setminus A_1) + \dots \qquad \text{(countable additivity)} \\
&\le \mu(A_1) + \mu(A_2) + \dots
\end{aligned}
$$

□

Definition: Probability space

Let $(\Omega, \mathcal{F})$ be a measurable space. Let P be a probability measure on $(\Omega, \mathcal{F})$. The triple $(\Omega, \mathcal{F}, P)$ is called a **probability space**.

Example. $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$ = all Borel sets on $[0,1]$. Let P = Lebesgue measure. Then $(\Omega, \mathcal{F}, P)$ is a probability space.

2 Existence and Uniqueness of Measures

Let Ω be a non-empty set. Let $\mathcal{F}_0$ be a field on Ω. Let $\mathcal{F} = \sigma(\mathcal{F}_0)$. Suppose P is a probability measure on $\mathcal{F}_0$. ($P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$ only holds if $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}_0$, disjointed.

Let's extend P from $\mathcal{F}_0$ to $\mathcal{F}$.

Definition: Outer measure P^*

The **outer measure** is defined as

$$
P^*(A) = \inf\left\{\sum_{n=1}^{\infty} P(A_n) : A_n \in \mathcal{F}_0, A \subset \bigcup_{n=1}^{\infty} A_n\right\} \qquad \text{for any} A \in \mathcal{F}
$$

Note. This is well-defined since there exists at least one cover $A \subset \bigcup_{n=1}^{\infty} A_n$ *i.e.*: $A_1 = \Omega, A_n = \emptyset$ for $n \geq 2$.

2.1 Properties of P^*

1. $P^*(\emptyset) = 0$. Pf: Take $A_n = \emptyset$, $P(A_n) = 0$. The infimum cannot go lower than 0.

2. For any $A \subset B$, $P^*(A) \leq P^*(B)$ (monotone). Pf: Any cover of B is a cover of A.

3. For any $A \in \mathcal{F}_0$, $P^*(A) \leq P(A)$. Pf: Take $A_1 = A, A_n = \emptyset$ for $n \geq 2$.

4. "countable subadditivity": *i.e.* for any $A_1, \ldots, A_n \in \mathcal{F}$

$$P^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P^*(A_n).$$

Proof

Let $A = \bigcup_{n=1}^{\infty} A_n$. Then we want to prove that $P^*(A) \leq \sum_{n=1}^{\infty} P^*(A_n)$. For each A_n, $P^*(A_n) = \inf\{\sum_{k=1}^{\infty} P(A_{n_k}) : A_{n_k} \in \mathcal{F}_0$ and $A_n \subset \bigcup_{k=1}^{\infty} A_{n_k}\}$. Take any particular sequence that covers A_n.

$$A_n \subset \bigcup_{k=1}^{\infty} A^*_{n_k}.$$

Given $\varepsilon > 0$, by the definition of infimum, $\sum_{k=1}^{\infty} P(A^*_{n_k}) < P^*(A_n) + \varepsilon$. Do this with $\varepsilon_n = \frac{\varepsilon}{2^n}$.

$$\sum_{k=1}^{\infty} P(A^*_{n_k}) < P^*(A_n) + \frac{\varepsilon}{2^n}.$$

Sum both sides over n:

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} P(A^*_{n_k}) < \sum_{n=1}^{\infty} P^*(A_n) + \varepsilon$$
$$\sum_{n,k} P(A^*_{n_k}) < \sum_{n=1}^{\infty} P^*(A_n) + \varepsilon$$

Note that $A_n \subset \bigcup_{k=1}^{\infty} A^*_{n_k} \Rightarrow A := \bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n,k} A^*_{n_k}$. Therefore, $P(A) \leq P\left(\bigcup_{n,k} A^*_{n_k}\right) \leq \sum_{n,k} P(A^*_{n_k})$ by monotone and countable subadditivity of P. By Property 3 pf P^*, $P^*(A) \leq P(A)$. Putting all together, we obtain

$$\begin{aligned} P^*(A) &\le P(A) \\ &\le \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} P(A_{n_k}) \\ &< \sum_{n=1}^{\infty}\left(P^*(A_n) + \frac{\varepsilon}{2^n}\right) \\ &= \sum_{n=1}^{\infty} P^*(A_n) + \varepsilon \end{aligned}$$

Let $\varepsilon \to 0$, then $P^*(A) \le \sum_{n=1}^{\infty} P^*(A_n)$. □

Definition: Inner measure

$$P_*(A) := 1 - P^*(A^c).$$

Note. The inner and outer measures agree if $P^*(A^c) = 1 - P^*(A)$. We proceed in defining P for $A \in \mathcal{F}$ as P^* whenever this holds.

Consider all sets for which $p^*(A^c) = 1 - P^*(A)$ and ..

$$\mathcal{M} := \{A \subset \Omega : P^*(A \cap E) + P^*(A^c \cap E) = p^*(E) \forall E \subset \Omega\}.$$

Fact: $P^*(\Omega) = 1$. If we take $E = \Omega$, then we obtain the agreement. $\mathcal{M}$ is the collection of P^*-measurable sets.

Note. For any $A, E \subset \Omega$,

$$P^*(E) = P^*((A \cap E) \cup (A^c \cap E)) \le P^*(A \cap E) + P^*(A^c \cap E) \qquad \text{count add.}$$

So $\mathcal{M}$ can be defined equivalently as $\mathcal{M} = \{A \subset \Omega : P^*(A \cap E) + P^*(A^c \cap E \le p*(E) \forall E \subset \Omega\}$