1 Heat Equation conitnues

Theorem: Heat Flow Rules

We now assume the "heat flow rules":

1) Constant temperature in a region implies that there is no heat flow.
2) heat energy flows from hotter regions to colder regions.
3) the greater the temperature difference, the greater is the flow of heat energy.
4) flow of heat energy will vary for differential materials

Theorem: Fourier's Law of Heat Conduction

$$\Phi = -K_0 \frac{\partial u}{\partial x}.$$

where K_0 is known as the **thermal conductivity constant**. This equation can be read as "the heat flux is proportional to the temperature difference (per unit length).

Theorem: the Heat Equation

If we combine the equations:

$$e(x,t) = c(x)\rho(x)u(x,t) \text{ and } \Phi = -K_0 \frac{\partial u}{\partial x}.$$

with the partial differential equation

$$\frac{\partial e}{\partial t} = -\frac{\partial \Phi}{\partial x} + Q.$$

this implies:

$$\frac{\partial}{\partial t}[c(x)\rho(x)u(x,t)] = -\frac{\partial}{\partial x}\left(-K_0 \frac{\partial u}{\partial x}\right) + Q \Rightarrow c(x)\rho(x)\frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q.$$

and if we assume $c(x)$ and $\rho(x)$ are also constants and that $Q(x,t) = 0$

then

$$
\begin{aligned}
c\rho\frac{\partial u}{\partial t} &= K_0\frac{\partial^2 u}{\partial x^2} \\
\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} \\
u_t &= ku_{xx}
\end{aligned}
$$

where $k = \frac{K_0}{c\rho} > 0$ is the **thermal diffusivity constant**. The solution we seek is $u(x,t)$, the temperature at any time and point along the rod.

1.1 Initial and Boundary Conditions

Definition

1) Since the heat equation $u_t = ku_{xx}$ has one time derivative, we need additional information, usually in the form of an initial condition (IC) at $t = 0$:

$$u(x,0) = f(x).$$

where $f(x)$ specifies the initial (spatial) temperature distribution of the rod at time $t = 0$.

2) The two spatial derivatives in the term u_{xx} requires two additional boudnary conditions (BC). In theory boundary condition information an be given for any two points x_1 and x_2 in the interval $[0, L]$, however, conditions at $x = 0$ and $x = L$ are usually given. Some examples include:

   - prescribed temperature
   - insulated boundary
   - Newton's law of cooling

Example (Prescribed Temperature)**.** Let $u_B(t)$ be the temperature of a fluid bath which one end of the rod is in contact with. In this case, we can prescribe the temperature of one end of the rod with a boundary condition of the form

$$u(0,t) = u_B(t) \text{ or } u(L,t) = u_B(t).$$

Example (Insulated Boundary)**.** If we know the heat flux behavior at a single point, say $x = 0$, then the heat flux will be a function of t only (since the location is fixed) and when combined with Fourier's Law of Heat Conduction, yields a boundary condition of the form

$$-K_0\frac{\partial u}{\partial x}(0,t) = \Phi(t) \Rightarrow \frac{\partial u}{\partial x}(0,t) = -\frac{1}{K_0}\Phi(t) \text{ where } \Phi(t) \text{ is given.}$$

The simplest case of this is when one end of the rod is **perfectly insulated** (no heat flow at the boundaryso $\Phi(t) = 0$ which yields the boundary condition:

$$\frac{\partial u}{\partial x}(0,t) = \frac{1}{-K_0}\cdot 0 \Rightarrow \frac{\partial u}{\partial x}(0,t) = 0.$$

Theorem: Newton's Law of Cooling

The heat flow leaving the rod is proportional to the temperature difference between the rod and the external temperature. In this case we can define the heat flux for the boundary condition as

$$\Phi(t) = -H[u(0,t) - u_B(t)].$$

where $H > 0$ is the **heat transfer coefficient**. Note that if for example $u(0,t) > u_B(t)$ then $\Phi(t) < 0$ and heat flows out of the rod to the left as expected. We can use Fourier's Law of Heat Conduction to specify a boundary condition:

$$-K_0\frac{\partial u}{\partial x}(0,t) = \Phi(t) \Rightarrow -K_0\frac{\partial u}{\partial x}(0,t) = -H[u(0,t) - u_B(t)].$$

That is the BC is

$$\frac{\partial u}{\partial x}(0,t) = \frac{H}{K_0}[u(0,t) - u_B(t)].$$

Note that for $x = L$, since the heat exits through the right we have H becoming $-H$, so

$$u_x(L,t) = -\frac{H}{K_0}[u(L,t) - u_B(t)].$$

2 Thermal Equilibrium

Definition

The **steady state/equilibrium solution** of the heat equation is a solution that does not depends on time, *i.e.* $u(x,t) = u(x)$.

Note.

1) No matter what the initial temperature distribution of the rod is, some systems will undergo a process that brings it into "thermal equilibrium". That is, there exists a finite time $T > 0$ such that $u_t = 0$ for $t > T$.

2) We expect that thermal equilibrium will be achieved in time, *i.e.*

$$\lim_{t\to\infty} u(x,t) = \overline{u}(x).$$

where $\overline{u}(t)$ is the steady state temperature or steady state solution.

3) We claim $u(x,t) = \overline{u}(x) + v(x,t)$ where $v(x,t) \to 0$ as $t \to \infty$. Think of it as an interaction between short-term and long-term behaviors. We aim to solve these two solutions separately.

4) Consider blinking holiday lights, the lightbulbs are either being heated up (ON) or cooling down (OFF). Thus there is no steady state temperature for this system. Not all systems have equilibrium solutions.

Example (1)**.** Suppose that $u(x,t) = u(x)$ then this implies $\frac{\partial u}{\partial t} = 0$ and so the heat equation becomes

$$0 = k\frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{d^2u}{dx^2} = 0 \Rightarrow \frac{du}{dx} = C_1 \Rightarrow u(x) = C_1x + C_2.$$

and suppose the boundary conditions are steady, suppose $u(0) = T_1$ and $u(L) = T_2$ then the steady state solution is the line

$$\overline{u}(x) = T1 + \frac{T_2 - T_1}{L}x.$$