1 Thin Circular Ring

The wire is circular with circumference $2L$ and insulated. The radius is therefore $r = \frac{L}{\pi}$. If the wire is thin enough then we assume the temperature is constant along the cross sections of the wire and satisfies the following BVP:

$$\begin{cases} \text{PDE:} & \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, & -L < x < L, t > 0 \\ \text{BCs:} & u(-L,t) = u(L,t), \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t), & t > 0 \\ \text{IC:} & u(x,0) = f(x), & -L \leq x \leq L \end{cases}$$

The BCs here assume that at the ends, the temperature is continuous and the flux is also continuous.

Due to the circular nature, $u(x_0, t) = u(x_0 + 2L, t) \; \forall \; x_0 \in [-L, L]$. Then we can define $u(x,t) \; \forall \; x \in \mathbb{R}$.

Do the PDE and BCs form a vector space? See homework, where we check linearity. Yes, so we can try separable of variables $u(x,t) = F(x) \cdot G(t) \neq 0$. We turn this into a time domain problem and an eigenvalue problem.

Note. As before we have

$$\frac{1}{k}\frac{G'(t)}{G(t)} = \frac{F''(x)}{F(x)} = -\lambda \Rightarrow G'(t) = -\lambda k G(t) \text{ and } F''(x) = -\lambda F(x).$$

Then BCs respectively becomes

$$F(-L) = F(L) \text{ and } F'(-L) = F'(L)$$

1) *time domain problem:* $G(t) = Ce^{-\lambda k t}, C \in \mathbb{R}$.

2) *eigenvalue problem:*

   Case. $\lambda < 0$, again we get the trivial solution.

   Case. $\lambda = 0$, then

   $$F''(x) = 0 \Rightarrow F(x) = Ax + B.$$

   Then the BC $F(-L) = F(L) \Rightarrow A = 0$. Therefore, $F(x) = B, B \in \mathbb{R}$. The other BC is trivial and redundant in this case.

   Case. $\lambda > 0$, we solve the characteristic equation as before and obtain

   $$F(x), c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

   Then the BC $F(-L) = F(L)$ yields

   $$c_1 \cos(-\sqrt{\lambda}L) + c_2 \sin(-\sqrt{\lambda}L) = c_1 \cos(\sqrt{\lambda}L) + c_2 \sin(\sqrt{\lambda}L) \Rightarrow 2c_2 \sin(\sqrt{\lambda}L) = 0.$$

   This implies either $c_2 = 0$ or $\sqrt{\lambda}L = n\pi$ for $n = \pm 1, \pm 2, \ldots$ Applying the other BC $F'(-L) = F'(L)$, as above we get

   $$c_1 \sin(\sqrt{\lambda}L) = 0.$$

This implies either $c_1 = 0$ or $\sqrt{\lambda}L = n\pi$ for $n = \pm 1, \pm 2, \ldots$ Recall that we do not want $c_1 = 0$ and $c_2 = 0$, *i.e.* the trivial solution, so we require either $c_1 \neq 0$ or $c_2 \neq 0$. This means that both the cosine and sine terms survive in the general solution.

Moreover, we get $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2 \ldots$

Hence by superposition principle, the general solution is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt}.$$

The coefficients a_0, a_n, b_n are obtained just as before using projection.

Note. Suppose $f(x)$ is odd, then $a_0, a_n = 0$, and $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)$, just like in FSS. Similar with even $f(x)$.

Now for large but finite time, we can again approximate our temperature prediction using the slowest decaying exponential term, which includes both sine and cosine terms when $n = 1$:

$$u(x,t) \approx \frac{1}{2L}\int_{-L}^{L} f(x)dx + \left[a_1 \cos\left(\frac{\pi x}{L}\right) + b_1 \sin\left(\frac{\pi x}{L}\right)\right] e^{-(\frac{\pi}{L})^2 kt}.$$

And the steady-state solution ($t \to \infty$) is just

$$\overline{u}(x) = a_0.$$

How are the FSS, FCS, and FS related?

Definition

Note that for any function $f(x)$, we have

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)].$$

1) Define the **even part** of $f(x)$ to be $f_e(x) = \frac{1}{2}[f(x) + f(-x)]$, then $f_e(-x) = f_e(x)$.
2) Define the **odd part** of $f(x)$ to be $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$, then $f_o(-x) = -f_o(x)$.
3) The F.S.$[f](x)$ equals the FCS of $f_e(x)$ plus the FSS of $f_o(x)$. That is

$$\text{F.S.}[f](x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt}.$$

Note. The even and odd parts of $f(x)$ is NOT the even and odd extension of $f(x)$!

This concludes our discussion of the heat equation, for now.