Homework 2

Jaden Wang

Problem (1)**.** Let $B_n = \bigcup_{k=n}^{\infty} A_k$. Notice that $B_1 \supseteq B_2 \supseteq \ldots$. Define $B = \bigcap_{n=1}^{\infty} B_n$, then it follows that $B_n \downarrow B$. This allows us to use the continuity of probability later. Now consider:

$$
\begin{aligned}
P\left(\limsup_n A_n\right) &= P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\right) \\
&= P(B) \\
&= \lim_{n\to\infty} P(B_n) \qquad \text{by continuity of probabilities} \\
&= \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k\right) \\
&= \limsup_n P\left(\bigcup_{k=n}^{\infty} A_k\right) \\
&= \lim_{n\to\infty}\left(\sup_{m\geq n}\left\{P\left(\bigcup_{k=m}^{\infty} A_k\right)\right\}\right) \\
&= \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k\right) \qquad \text{by monotonicity of } P \\
&\geq \lim_{n\to\infty} P(A_n) \qquad \text{by monotonicity of } P \\
&= \lim_{n\to\infty}\left(\sup_{m\geq n}\{P(A_m)\}\right) \\
&= \limsup_n P(A_n)
\end{aligned}
$$

Problem (2)**.**

a) Given $a \in \limsup_n (A_n \cap B_n)$, then a is in every tail sequence $\{(A_n \cap B_n), (A_{n+1} \cap B_{n+1}), \ldots\}$. That is, given $k \in \mathbb{N}$, $\exists\ N_k \geq k$ such that $a \in (A_{N_k} \cap B_{N_k})$. Notice that $A_{N_k} \subseteq \bigcup_{n=k}^{\infty} A_n$ and $B_{N_k} \subseteq \bigcup_{n=k}^{\infty} B_n$. Since k is arbitrary, this means that no matter what the value of k is, we are guaranteed to find a N_k such that $a \in A_{N_k} \subseteq \bigcup_{n=k}^{\infty} A_n$ and

$a \in B_{N_k} \subseteq \bigcup_{n=k}^{\infty} B_n$. In the language of set theory, this translates to $a \in A_{N_k} \cap B_{N_k} \subseteq \limsup_n A_n \cap \limsup_n B_n$. Thus by containment, we show that

$$(\limsup_n A_n) \cap (\limsup_n B_n) \supseteq \limsup_n (A_n \cap B_n).$$

b) Let's show equality by double containment. Given $a \in \limsup_n (A_n \cup B_n)$,using similar logic as part (a), we can show that given $k \in \mathbb{N}$, $\exists\, N_k \geq k$ such that $a \in (A_{N_k} \cup B_{N_k})$, and $A_{N_k} \subseteq \bigcup_{n=k}^{\infty} A_n$ and $B_{N_k} \subseteq \bigcup_{n=k}^{\infty} B_n$, and therefore $a \in (A_{N_k} \cup B_{N_k}) \subseteq (\limsup_n A_n \cup \limsup_n B_n)$. This yields:

$$(\limsup_n A_n) \cup (\limsup_n B_n) \supseteq \limsup_n (A_n \cup A_n).$$

To show the other direction, given $b \in (\limsup_n A_n) \cup (\limsup_n B_n)$. This means b is either in $\limsup_n A_n$ or $\limsup_n B_n$. WLOG assume that $b \in \limsup_n A_n$. Since $A_n \subseteq A_n \cup B_n \Rightarrow \bigcup_{n=k}^{\infty} A_n \subseteq \bigcup_{n=k}^{\infty} (A_n \cup B_n) \Rightarrow \limsup_n A_n \subseteq \limsup_n (A_n \cup B_n)$. It follows that $a \in \limsup_n (A_n \cup B_n)$. By containment,

$$(\limsup_n A_n) \cup (\limsup_n B_n) \subseteq \limsup_n (A_n \cup A_n).$$

Thus by double containment, we prove the equality.

c) This follows directly by taking the complement of (b) on both sides:

$$\left(\limsup_n A_n \cup \limsup_n B_n\right)^c = \left(\limsup_n (A_n \cup B_n)\right)^c$$
$$\left(\limsup_n A_n\right)^c \cap \left(\limsup_n B_n\right)^c = \liminf_n (A_n \cup B_n)^c$$
$$\liminf_n A_n^c \cap \liminf_n B_n^c = \liminf_n \left(A_n^c \cap B_n^c\right)$$

Since A_n, B_n are arbitrary, their complements are also arbitrary, so we can write

$$(\liminf_n A_n) \cap (\liminf_n B_n) = \liminf_n (A_n \cap B_n).$$

d) Again this is achieved by taking complements of part (a) on both sides, after similar steps, we obtain

$$\liminf_n A_n^c \cup \liminf_n B_n^c \subseteq \liminf_n \left(A_n^c \cup B_n^c\right).$$

Again due to arbitrary complements, we have

$$(\liminf_n A_n) \cup (\liminf_n B_n) \subseteq \liminf_n (A_n \cup B_n).$$

e) Let $A_n = \{1\}$ and $B_n = \{1, \ldots, n\}$. Notice $A_n \cap B_n = \{1\}\ \forall\ n \in \mathbb{N}$. And it's easy to see that $\limsup_n B_n = \mathbb{N}$. Now consider

$$\begin{aligned}
\limsup_n A_n \cap \limsup_n B_n &= \limsup_n 1 \cap \limsup_n \{1, \ldots, n\} \\
&= 1 \cap \mathbb{N} \\
&= 1 \\
&= \limsup_n 1 \\
&= \limsup_n (A_n \cap B_n)
\end{aligned}$$

Problem (3)**.**

a) Let's first show that $\limsup_n \liminf_k (A_n \cap A_k^c) = \emptyset$. Note that the dummy indices n and k do not affect each other.

$$\begin{aligned}
\limsup_n \liminf_k A_n \cap A_k^c &= \limsup_n (A_n \cap \liminf_k A_k^c) && \text{by 2(c)} \\
&\subseteq (\limsup_n A_n) \cap (\liminf_k A_k^c) && \text{by 2(a)} \\
&= (\limsup_n A_n) \cap (\limsup_k A_k)^c && \text{by De Morgan's Law} \\
&= \emptyset
\end{aligned}$$

The last step comes from that the intersection of complements is empty. Since $\emptyset$ is a subset of any set, we then must have $\limsup_n \liminf_k (A_n \cap A_k^c) = \emptyset$.

Now we are ready to show the claim:

$$\begin{aligned}
\lim_n P(\liminf_k (A_n \cap A_k^c)) &\leq \limsup_n P(\liminf_k A_n \cap A_k^c) \\
&\leq P\left(\limsup_n \liminf_k (A_n \cap A_k^c)\right) \\
&= 0
\end{aligned}$$

Since any probability $P(A) \geq 0$, this must equal to 0.

b)

$$\begin{aligned}
\lim_{n\to\infty} P(A_n \setminus A^*) &= \lim_{n\to\infty} P\left(A_n \cap A^{*^c}\right) \\
&= \lim_{n\to\infty} P\left(A_n \cap \left(\limsup_k A_k\right)^c\right) \\
&= \lim_{n\to\infty} P\left(A_n \cap \liminf_k A_k^c\right) \\
&= \lim_{n\to\infty} P\left(\liminf_k \left(A_n \cap A_k^c\right)\right) \\
&= 0 \qquad \text{by 3(a)}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} P(A_* \setminus A_n) &= \lim_{n\to\infty} P\left(\liminf_k A_k \cap A_n^c\right) \\
&= \lim_{n\to\infty} P\left(\liminf_k \left(A_k \cap A_n^c\right)\right) \\
&= \lim_{n\to\infty} P\left(\liminf_k \left(B_n \cap B_k^c\right)\right) \text{ since } A_n \text{ is arbitrary} \\
&= 0 \qquad \text{by 3(a)}
\end{aligned}$$

c)

$$\begin{aligned}
\lim_{n\to\infty} P\left(A \Delta An\right) &= \lim_{n\to\infty} P(A \setminus A_n \cup A_n \setminus A) \\
&= \lim_{n\to\infty} P(A_* \setminus A_n \cup A_n \setminus A^*) \\
&= \lim_{n\to\infty} \left(P(A_* \setminus A_n) + P(A_n \setminus A^*)\right) \text{ since they are disjoint} \\
&= 0 + 0 = 0
\end{aligned}$$

d) Let's first obtain the following results:

$$\begin{aligned}
P(A\Delta A^*) &= P\left((A \cap A^{*^c}) \cup (A^* \cap A^c)\right) \\
0 &= P(A \cap A^{*^c}) + P(A^* \cap A^c) \\
0 &= 0 + P(A^* \cap A^c) \\
P(A\Delta A_*) &= P((A \cap A_*^c) \cup (A_* \cap A^c)) \\
0 &= P(A \cap A_*^c) + P(A_* \cap A^c) \\
0 &= P(A \cap A_*^c) + 0
\end{aligned}$$

Now let's consider the main problem:

$$\begin{aligned}
\lim_{n\to\infty} P(A\Delta A_n) &= \lim_{n\to\infty} P\left((A\cap A_n^c)\cup(A_n\cap A^c)\right) \\
&= \limsup_n P\left((A\cap A_n^c)\cup(A_n\cap A^c)\right) \\
&\leq P\left(\limsup_n \left((A\cap A_n^c)\cup(A_n\cap A^c)\right)\right) \\
&= P\left(\left(\limsup_n A_n^c \cap A\right)\cup\left(\limsup_n A_n\cap A^c\right)\right) \text{ by 2(b)} \\
&= P\left(\left(\left(\liminf_n A_n\right)^c \cap A\right)\cup\left(\left(\limsup_n A_n\right)\cap A^c\right)\right) \\
&\leq P(A\cap A_*^c) + P(A^*\cap A^c) \text{ by subadditivity} \\
&= 0+0 = 0
\end{aligned}$$

Since $\lim_{n\to\infty} P(A\Delta A_n) \geq 0$, we must have $\lim_{n\to\infty} P(A\Delta A_n) = 0$.

Problem (4)**.** Let's show that $\mathscr{L}_C$ satisfies the three conditions of a λ-system.

(i) $\Omega \in \mathscr{L}_C$: Let $D=\Omega$, then clearly $\Omega\subseteq\Omega$ and $\Omega\cap C = C\in\mathscr{L}_0$, hence $\Omega\in\mathscr{L}_C$.

(ii) closed under complements: Given $A\in\mathscr{L}_C$, we want to show that $A^c\cap C\in\mathscr{L}_0$. Since $A\in\mathscr{L}_C$ implies $A\cap C\in\mathscr{L}_0$, and $\mathscr{L}_0$ is a λ-system, it follows that $A^c\cup C^c\in\mathscr{L}_0$. Since $C\in\mathscr{L}_0$ and $\mathscr{L}_0$ is also closed under countable intersections,

$$\begin{aligned}
(A^c\cup C^c)\cap C &\in \mathscr{L}_0 \\
(A^c\cap C)\cup(C^c\cap C) &\in \mathscr{L}_0 \\
A^c\cap C &\in \mathscr{L}_0
\end{aligned}$$

as required. Hence $A^c\in\mathscr{L}_C$ and $\mathscr{L}_C$ is closed under complements.

(iii) closed under disjoint unions: Given disjoint $A_1, A_2, \ldots \in\mathscr{L}_C$, we want to show that $\bigcup_{n=1}^\infty A_n\cap C\in\mathscr{L}_0$. Since $A_n\cap C\in\mathscr{L}_0$ and are disjoint for all $n \in \mathbb{N}$ and $\mathscr{L}_0$ is closed under countable disjoint unions, we have

$$\bigcup_{n=1}^{\infty} (A_n \cap C) \in \mathscr{L}_0$$
$$\left(\bigcup_{n=1}^{\infty} A_n\right) \cap C \in \mathscr{L}_0$$

Hence $\bigcup_{n=1}^{\infty} A_n \in \mathscr{L}_C$ and $\mathscr{L}_C$ is closed under countable disjoint unions. Hence, $\mathscr{L}_C$ is a λ-system.

Problem (5)**.**

a) Let's show that $\mathscr{L}$ satisfies the three conditions of a λ-system:

   (i) Clearly $\Omega \in F$. Since P, Q are two probability measure on $\mathcal{F}$, by definition we have $P(\Omega) = Q(\Omega) = 1$. Hence $\Omega \in \mathscr{L}$.

   (ii) Given $A \in \mathscr{L}$, we know $A \in \mathcal{F}$ and $P(A) = Q(A)$. Since $\mathcal{F}$ is a σ-field closed under complements, $A^c \in \mathcal{F}$. Thus by countable additivity of probability measure we obtain:

$$P(A^c) = P(\Omega) - P(A) = Q(\Omega) - Q(A) = Q(A^c).$$

   (iii) Given disjoint $A_1, A_2, \ldots \in \mathscr{L}$, we know that $A_n \in \mathcal{F}$ and $P(A_n) = Q(A_n)$ for all $n \in \mathbb{N}$. We want to show that $\bigcup_{n=1}^{\infty} A_n \in \mathscr{L}$. Since $\mathcal{F}$ is a σ-field closed under countable unions, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$. Since the equality holds under summation,

$$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} Q(A_n)$$
$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = Q\left(\bigcup_{n=1}^{\infty} A_n\right)$$

   by countable additivity of P, Q. Hence $\mathscr{L}$ is closed under countable disjoint unions.

Therefore, $\mathscr{L}$ is a λ-system.

b) We want to show that $\mathcal{F} \subseteq \mathscr{L}$. It suffices to show that $\mathcal{P} \subseteq \mathscr{L}$ and apply Dynkin's Theorem. To prove containment, given $A \in \mathcal{P}$, we want to show $A \in \mathscr{L}$. Since $\mathcal{P} \subseteq \sigma(\mathcal{P})$ by definition of a generated σ-field, clearly $A \in \mathcal{F}$. Since $A \in \mathcal{P}$, we are given that $P(A) = Q(A)$. Thus, $A \in \mathscr{L}$ and we obtain $P \subseteq \mathscr{L}$. By Dynkin's Theorem, $\mathcal{F} = \sigma(\mathcal{P}) \subseteq \mathscr{L}$, This implies any element A in $\mathcal{F}$ satisfies $P(A) = Q(A)$ by the definition of $\mathscr{L}$.

Problem (6)**.**

a) Let's first prove a claim:

Claim.

$$\limsup_n A_n \cap \limsup_n A_{n+1}^c = \limsup_n (A_n \cap An_{n+1}^c).$$

Note that since we already have one direction of containment by 2(a), it suffices to show the other direction. Given $a \in (\limsup_n A_n \cap \limsup_n {A_{n+1}}^c)$. Then we know that given any $n \in \mathbb{N}$, there exists an $N \geq n$ such that $a \in {A_{N+1}}^c$. Choose $N \geq n$ to be the smallest index such that $a \in {A_{N+1}}^c$, then this implies that $a \notin {A_N}^c$. If a is not in the complement, then it must be in A_N. It follows that $a \in (A_N \cap {A_{N+1}}^c)$, which leads to $a \in \bigcup_{k=n}^{\infty}(A_k \cap {A_{k+1}}^c)$ for every $n \in \mathbb{N}$. This is equivalent to $a \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty}(A_k \cap {A_{k+1}}^c) = \limsup_n (A_n \cap {A_{n+1}}^c)$ and we obtain the containment as required. Together with 2(a), we prove the claim.

Now we begin the proof proper:

$$
\begin{aligned}
&\{(A_n \cap A_{n+1}^c)\ i.o.\} \cup \liminf_n A_n \\
&= \left(\limsup_n A_n \cap \limsup_n A_{n+1}^c\right) \cup \liminf_n A_n \qquad \text{by the claim above} \\
&= \left(\limsup_n A_n \cup \liminf_n A_n\right) \cap \left(\limsup_n A_{n+1}^c \cup \liminf_n A_n\right) \\
&= \limsup_n A_n \cap \left(\left(\liminf_n A_{n+1}\right)^c \cup \liminf_n A_n\right) \\
&= \limsup_n A_n \cap \left(\left(\liminf_n A_n\right)^c \cap \liminf_n A_n\right) \\
&= \limsup_n A_n \cap \Omega \\
&= \{A_n\ i.o.\}
\end{aligned}
$$

b) Since $\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty$, it follows from Borel-Cantelli Lemma (i) that $p(A_n \cap A_{n+1}^c\ i.o.) = 0$. Then the equality from part (a) yields

$$
\begin{aligned}
P(A_n\ i.o.) &= P(\{A_n \cap A_{n+1}\ i.o.\} \cup \liminf_n A_n) \\
&\leq P(A_n \cap A_{n+1}\ i.o.) + P(\liminf_n A_n) \text{ by subadditivity of } P \\
&\leq 0 + \liminf_n P(A_n) \\
&= \lim_{n\to\infty} P(A_n) \\
&= 0
\end{aligned}
$$

Then clearly $P(A_n\ i.o.) = 0$.