1 Heat Equation in 3D continued

Intuition. We transform the double integral to triple integral using divergence theorem so we can combine them under the same domain of $\mathbf{x}$. Now we want to replace flux with temperature function.

Definition: Laplacian

For 3D, the **Laplacian** is defined as

$$\nabla^2 u = \Delta u = u_{xx} + u_{yy} + u_{zz}.$$

Recall Fourier's Law says heat flows from hot to cold in the direction where the temperature differences are the greatest and ∇u represents the direction of greatest temperature increases, so

$$\vec{\phi} = -K_0 \cdot \nabla u \Rightarrow \nabla \cdot \vec{\phi}(\mathbf{x}) = \nabla \cdot (-K_0 \nabla u) = -K_0 \cdot \Delta u.$$

Then

$$c(\mathbf{x})\rho(\mathbf{x})\frac{\partial}{\partial t}u(\mathbf{x}, t) - K_0 \Delta u - Q(\mathbf{x}, t) = 0.$$

Thus **the heat equation with internal source of energy** is

$$c(\mathbf{x})\rho(\mathbf{x})\frac{\partial}{\partial t}u(\mathbf{x}, t) = K_0 \Delta u + Q(\mathbf{x}, t).$$

Assuming $Q = 0$ and the thermal coefficients are constant, we get

$$\frac{\partial u}{\partial t} = k\Delta u \text{ where } k = \frac{K_0}{c\rho} = \text{ "thermal diffusivity".}$$

with initial condition $u(\mathbf{x}, 0) = f(\mathbf{x})$ and boundary condition $u(\mathbf{x}, t) = T(\mathbf{x}, t)$ for $\mathbf{x} \in \partial R$.

1.1 Steady State

Theorem: Laplace's Equation

Consider the heat equation with internal source of energy defined above, then if $u_t = 0$ this gives **Poisson's Equation**, $\Delta u = -\frac{Q}{K_0}$, and if $Q = 0$ this yields **Laplace's Equation**:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Theorem: Laplace's Equation in Cylindrical Coordinates

Let $x = r\cos(\theta), y = r\sin(\theta), z = z$ then using the Chain Rule,

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

Theorem: Spherical

$$\Delta u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2\sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial u}{\partial \phi}\right) + \frac{1}{\rho^2\sin^2(\phi)}\frac{\partial^2 u}{\partial \phi^2}.$$

2 Solving the Heat Equation 1

Definition: the Heat Operator

1) Define the **heat operator** as

$$L(u) = \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2}.$$

for any $u(x,t)$ in the appropriate function space (once differentiable in t and twice differentiable in x). Then $L(u)$ is a linear operator.

2) The set of functions that satisfy the boundary conditions $u(0,t) = 0 = u(L,t)$ form a vector space. That is, if u_i satisfy these boundary condition for $i = 1, 2$ and if $u_3(x,t) = c_1u_1(x,t) + c_2u_2(x,t)$ then $u_3(0,t) = 0 = u_3(L,t)$ for any $c_1, c_2 \in \mathbb{R}$.

Note. The set of function that satisfy the initial condition $u(x,0) = f(x) \neq 0$ does NOT form a vector space.

2.1 Separation of Variables

Consider the following boundary value problem, there are three pieces of the full story:

$$\begin{cases} \text{PDE:} & \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, & 0 < x < L, t > 0 \\ \text{BC:} & u(0,t) = 0 = u(L,t), & t > 0 \\ \text{IC:} & u(x,0) = f(x), & 0 \leq x \leq L \end{cases}$$

Intuition. We will take the non-zero part of the boundary conditions into the steady-state ODE, so that the PDE forms a vector space and becomes easier to solve.

Typically we would assume $u(x,t) = \overline{u}(x) + v(x,t)$ but in this case $\overline{u}(x) = 0$, so we apply separation of variables directly to $u(x,t)$. Assume (separable functions wrt t and x): $u(x,t) = F(x) \cdot G(t) \neq 0$ then

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \Rightarrow F(x)\frac{dG}{dt} = k\frac{d^2F}{dx^2}G(t) \Rightarrow \frac{1}{k} \cdot \frac{G'(t)}{G(t)} = \frac{F''(x)}{F(x)}.$$

For this to satisfy, the ratio must be a constant.

Proof

We aim to show that the derivative of LHS wrt t is zero for all t. Note that

$$\frac{d}{dt}\left(\frac{1}{k} \cdot \frac{G'(t)}{G(t)}\right) = \frac{d}{dt}\left(\frac{F''(x)}{F(x)}\right) = 0.$$

and likewise for the RHS

$$\frac{d}{dx}\left(\frac{F''(x)}{F(x)}\right) = 0.$$

Together 0 derivative everywhere implies a constant function:

$$\frac{1}{k} \cdot \frac{G'(t)}{G(t)} = \frac{F''(x)}{F(x)} = -\lambda.$$

where λ is some constant. $\square$

Now consider the differential equations

$$G'(t) = -\lambda k G(t) \text{ and } F''(x) = -\lambda F(x).$$