Note (hw20.10). $d = 3$, so there are three evenly spaced solutions, $x = 3 \bmod 8$ or $x = 3, 11, 19 \bmod 24$. We can either multiply by 7 or think 7 as -1.

Question: is $R[x]$ a field?

No, x is never a unit.

$$x(a_0 + a_1x + \ldots + a_nx^n) = a_0x + a_1x^2 + \ldots + a_nx^{n+1} \neq 0.$$

Since $x0 \neq 1$ by theorem and definition of unit.

Example (Evaluation Homomorphism)**.** Idea: "plug in values" for x.

Evaluate a polynomial of $\mathbb{R}[x]$ at i. $f(x) = \sqrt{2} - \pi x^2 + \frac{36}{7}x^3$. Then

$$\phi_i(f(x)) = \sqrt{2} + \pi - \frac{36}{7}i \in \mathbb{C}.$$

$\phi_\alpha : F[x] \to E$. Two fields $F, E, F \leq E, \alpha \in E, \phi_\alpha(f(x)) = f(\alpha)$.

Why is ϕ_α a ring homomorphism?

Consider $\phi_i : \mathbb{R}[x] \to \mathbb{C}$,

$$\begin{aligned} \phi_i((\sqrt{2} - \pi x)(\frac{3}{7} + 4x^3)) &= \frac{3\sqrt{2}}{7} - \frac{3\pi}{7}i + 4\sqrt{2}i^2 - 4\pi i^3 \text{ if we distributive} \\ &= \phi_i(\sqrt{2} - \pi x)\phi_i(\frac{3}{7} + 4x^3) \end{aligned}$$

This is very trivial because we just substitute i for x and everything else stays the same.

Definition: kernel of ring homomorphism

The **kernel** of a ring homomorphism are elements that are mapped to zero.

What about $\ker \phi_i$? Notice that $\phi_i(0) = \phi_i(1 + x^2) = 0$.

What is $\operatorname{im} \phi_i$? $\mathbb{R}[x] \to \mathbb{C}$.

Example (22.6)**.** $\phi_0 : \mathbb{Q}[x] \to \mathbb{R}$.

$$\phi_0(a_0 + a_1x + \ldots + a_nx^n) = a_0 + \ldots = a_0.$$

ϕ_0 is a ring homomorphism and ϕ_0 takes the constant term imply constant term of a sum is the sum of the constant terms.

$$\phi_0(a(x) + b(x)) = \phi_0(a(x)) + \phi_0(b(x)).$$

Same with the products.

But consider $\phi_1 : \mathbb{Q}[x] \to \mathbb{Q}$.

$$\phi_1(a_0 + a_1 x_1 + \ldots + a_n x^n) = \sum_{i=1}^{n} a_i.$$

This is a ring homomorphism too! So does ϕ_{-1}.

Goal: we want to study the (zero) roots of polynomials.

"$\sqrt{2}$ is a root of the polynomial $x^2 - 2$". We need evaluation homomorphism $\phi_{\sqrt{2}} : \mathbb{Q}[x] \to \mathbb{R}$. Then $\phi_{\sqrt{2}}(f(x)) = 0$ means that $\sqrt{2}$ is a root of $x^2 - 2$ in $\mathbb{R}$.

Does $x^2 - 2 \in \mathbb{Q}[x]$ have any roots in $\mathbb{Q}$? No.

Theorem

If $n \in \mathbb{N}$, then a positive k-th root of n is either an integer or irrational.

23:

Consider $F[x]$ and F is field. So $F[x]$ is a domain.

Example. We want to divide $sx^4 - 3x^3 + x^2 + 4x - 1$ by $x^2 - 2x + 3$ in $\mathbb{Z}_5[x]$. The reminder term will have less degree than the divisor or 0.