Theorem: 4.2

extends to an infinite number of $\mathcal{A}_i$ and even an uncountable collection.

Corollary

Suppose that $A_{11}, A_{12}, \ldots, A_{21}, A_{22}$ are independent events. Let $\mathcal{F}_i$ be the σ-field generated by the ith row. Then $\mathcal{F}_1, \mathcal{F}_2, \ldots$ are independent.

Proof

Define $\mathcal{A}_i$ as the collection of all finite intersections of $A_{i1}, A_{i2}, \ldots$. Note that $\mathcal{A}_i$ is a π-system and $\sigma(\mathcal{A}_i) = \mathcal{F}_i$. $\square$

Lemma: The Borel-Cantelli Lemmas

1) Let $A_1, A_2, \ldots \in \mathcal{F}$. If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\limsup_n A_n) = 0$.
2) Let $A_1, A_2, \ldots \in \mathcal{F}$. If $\sum_{n=1}^{\infty} = \infty$ and if the A_ns are independent, then $P\left(\limsup_n A_n\right) = 1$.

Proof: 1

Suppose the sum is finite. Note that the "tail sums" $infsum : P(A_n) \to 0$ as $n \to \infty$.

$$\begin{aligned} P(\limsup_n A_n) &= P\left(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k\right) \\ &\leq P\left(\bigcup_{k=m}^{\infty} A_k\right) && \text{monotonicity} \\ &\leq infsum : kmP(A_k) \to 0 \text{ as } m \to \infty && \text{countable subadditivity} \end{aligned}$$

$$P(\limsup_n A_n) = \lim_{m \to \infty} P(\limsup_n A_n) \leq \lim_{m \to \infty} infsum : kmP(A_k) = 0.$$

$\square$

Proof: 2

Let $A = \limsup_n A_n = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k$. Want to show $P(A) = 1$ and will show $P(A^c) = 0$.

Note. A_1, A_2 independent $\Rightarrow A_1^c and A_2^c$ independent. Also $\Rightarrow A_1$ and A_2^c are independent.

Also, $A^c = \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} A_k^c$.

Also. $e^{-x} \geq 1 - x$ for $x \geq 0$.

For fixed m,

$$\begin{aligned} P\left(\bigcap_{k=m}^{\infty} A_k^c\right) &\leq P\left(\bigcap_{k=m}^{m+l} A_k^c\right) \\ &= \Pi_{k=m}^{m+1} P(A_k^c) \qquad \text{by independence} \\ &= \Pi_{k=m}^{m+l}(1 - P(A_k)) \\ &\leq \Pi_{k=m}^{m+l} e^{-P(A_k)} \\ &= e^{-\sum_{k=m}^{m+l} P(A_k)} \to 0 \text{ as } l \to \infty \end{aligned}$$

Since $\sum_{n=1}^{\infty} P(A_n) = \infty$. So

$$\begin{aligned} P(A^c) &= P(\bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} A_k^c) \\ &\leq \sum_{m=1}^{\infty} P(\bigcap_{m=1}^{\infty} P\left(\bigcap_{k=m}^{\infty} A_k^c\right)) \qquad \leq \sum_{m=1}^{\infty} e^{-\sum_{k=m}^{m+l} P(A_k)} \end{aligned}$$

So

$$P(A^c) = \lim_{l \to \infty} P(A^c) \leq \lim_{l \to \infty} \sum e^{\sum}.$$

□

Example (of why we need independence)**.** Given $(\Omega, \mathcal{F}, P)$. Take any $A \in \mathcal{F}$ with $0 < P(A) < 1$. Define $A_n = A$ for all n so it's not independent. Then

$$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} P(A) = \infty$$

But

$$P(A_n i.o.) = .$$