What are the cosets of kernel?

Theorem

Let $\phi : G \to H$ be a homomorphism, and $K = \ker \phi = \{x \in G : \phi(x) = e_H\}$. The left and right cosets of K are the same. That is,

$$xK = yK \Leftrightarrow Kx = Ky.$$

Proof

Note. $xK = yK \Leftrightarrow x^{-1}y \in K$.

Thus $x^{-1}y \in \ker \phi \Leftrightarrow \phi(x^{-1}y) = e_H \Leftrightarrow \phi(x)^{-1} *_H \phi(y) = e_H$. Therefore, $xK = yK \Leftrightarrow \phi(x) = \phi(y)$. Similarly, $Kx = Ky \Leftrightarrow \phi(x) = \phi(y)$. $\square$

Definition: normal subgroup

Let $H \leq G$. We say H is a **normal subgroup** of G if the left and right cosets of H in G agree. We denote this as $H \trianglelefteq G$.

Note. There is no such thing as a "normal group". It only applies to subgroups.

Claim.

- If G is a group then $G \trianglelefteq G$.
- The trivial subgroup $\{e\}$ is a normal subgroup.
- If $G = S_3$ and $H = \{e, (1\ 2)\}$, then H is not normal in G. This is the smallest example of non-normal subgroup. (even though H is abelian!)
- If G is abelian and $H \leq G$ then $H \trianglelefteq G$. "Any subgroup of an abelian group is normal."
- Any group G has normal subgroups G and $\{e\}$.
- **EVERY SUBGROUP OF INDEX 2 IS NORMAL**.

  Proof

  Suppose $H \leq G$ and H has index 2 (number of cosets). Then the left cosets are: H and everything else. The right cosets are again H and everything else. Therefore they must agree. $\square$

- The kernel of a homomorphism is a normal subgroup.

Example (determinant map)**.** $\phi : GL_N(\mathbb{R}) \to \mathbb{R}^*, \phi(A) = \det A$. $\ker \phi = \{A \in GL_n(\mathbb{R}) : \phi(A) = 1\} = \{A \in GL_n(\mathbb{R}) : \det A = 1\} = SL_n(\mathbb{R})$, the special linear group.

Is $SL_n(\mathbb{R}) \trianglelefteq Gl_n(\mathbb{R})$? Yes, because it's the kernel of a homomorphism.

Example (sign map)**.** $\ker \varepsilon = A_n$ so $A_n \trianglelefteq S_n$ because 1. it's the kernel of a known homomorphism 2. index of A_n in S_n is 2.