Definition

An infinite (even uncountable) collection of sets is independent if every finite subcollection satisfies the above definition of independence.

Note. Pairwise independence $\not\Rightarrow$ independence.

Example. Roll a fair 6-sided die twice. Let $A =$ the sum of outcomes is 7. $B =$ the first roll is 2. $C =$ the second roll is 5. $P(A) = \frac{6}{36}, P(B) = \frac{1}{6}, P(C) = \frac{1}{6}, P(A \cap B) = \frac{1}{36}$. Same for the other two pairs. However, $P(A \cap B \cap C) = P(B \cap C) = \frac{1}{36} \neq \frac{1}{6^3}$.

Definition

Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be classes/collections of sets from $\mathcal{F}$. These classes are independent if $A_1, A_2, \ldots, A_n$ are independent for all choices of $A_i \in \mathcal{A}_i$.

Or

$\mathcal{A}_1, \ldots, \mathcal{A}_n$ are independent if $P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2)\cdots P(A_n)$ for $A_i \in \mathcal{A}_i$ OR $A_i = \Omega$.

Does the independence of classes imply the independence of their sigma fields?

No! They need to be π-systems.

Let Ω be a non-empty set.

Definition

A π-system

Definition: lambda-system

A λ-system is a collection of subsets of Ω such that

(i) contains Ω.

(ii) closed under complements.

(iii) closed under countable disjoint unions.

Notation.

Example. $\Omega = \{a, b, c, d\}$. Define $\mathcal{L} = \{\{a, b\}, \{a, c\}, \{c, d\}, \{b, d\}, \Omega, \emptyset\}$. Note

$\{a, b\} \cup \{a, c\} = \{a, b, c\} \notin \mathcal{L} \Rightarrow \mathcal{L}$ is not a field.

In the definition of a λ-system, (ii) can be replaced by
(ii') $A, B \in \mathcal{L}$ with $A \subseteq B$ then $B \setminus A \in \mathcal{L}$.

Proof

(i) and (ii') $\Rightarrow$ (ii). Take $B = \Omega$. Then $B \setminus A = A^c \in \mathcal{L}$.

(ii) and (iii) $\Rightarrow A \subseteq B \Rightarrow B \setminus A = B \cap A^c = (A \cup B^c)^c \in \mathcal{L}$ since disjoint union and its complement is in $\mathcal{L}$. $\square$

Lemma: 1

If $\mathcal{A}$ is a π-system and a λ-system, then $\mathcal{A}$ is a σ-field.

Proof

(i) and (ii) for σ-field are satisfied from (i) and (ii) from the λ-system. We want to show (iii) from σ-field definition.

Take $A_1, A_2, \ldots \in \mathcal{A}$. Let $B_1 = A_1, B_2 = A_2 \setminus A_1 = A_2 \cap A_1^c, B_3 = A_3 \cap A_2^c \cap A_1^c, \ldots$ The B_n are disjoint. $\mathcal{A}$ is a π-system and λ-system implies that the B_n are in $\mathcal{A}$. So

$$\bigcup_n A_n = \bigcup_n B_n \in \mathcal{A}.$$

$\square$

Lemma: 2

Suppose that $\mathcal{L}_0$ is a λ-system. Define, for any $C \subseteq \Omega$, $\mathcal{L}_C :- \{D \subseteq \Omega : D \cap C \in \mathcal{L}_0\}$. If $C \in \mathcal{L}_0$ then $\mathcal{L}_C$ is a λ-system.

Proof

(i) $C \in \mathcal{L}_0 \Rightarrow \Omega \cap C = C \in \mathcal{L}_0 \Rightarrow \Omega \in \mathcal{L}_C$

(ii) Suppose $A \in \mathcal{L}_C$, want to show $A^c \in \mathcal{L}_C$. Write $C = (A \cap C) \cup (A^c \cap C)$. Then $A^c \cap C = C \setminus (A \cap C) \in \mathcal{L}_0$ since $\mathcal{L}_0$ is a λ-system, and $A \cap C \subseteq C \Rightarrow A^c \in \mathcal{L}_C$.

(iii) Take $A_1, A_2, \dots$ disjoint in $\mathcal{L}_C$. Then $A_1 \cap C, A_2 \cap C, \dots$ are in $\mathcal{L}_0$. Since $\mathcal{L}_0$ is a λ-system, we have

$$\bigcup_{n=1}^{\infty} (A_n \cap C) \in \mathcal{L}_0$$

$$\left(\bigcup_{n=1}^{\infty} A_n \right) \cap C \in \mathcal{L}_0$$

$\square$

Theorem: Dynkin's pi-lambda Theorem

Let $\mathcal{P}$ be a π-system and $\mathcal{L}$ be a λ-system. If $\mathcal{P} \subseteq \mathcal{L}$ then $\sigma(\mathcal{P}) \subseteq \mathcal{L}$.

Proof

Suppose $\mathcal{P} \subseteq \mathcal{L}$. Let $\mathcal{L}_0$ be the smallest π-system containing $\mathcal{P}$. Then $\mathcal{P} \subseteq \mathcal{L}_0 \subseteq \mathcal{L}$. If we can show that $\mathcal{L}_0$ is a σ-field then $\sigma(\mathcal{P}) \subseteq \mathcal{L}_0 \subseteq \mathcal{L}$.

By Lemma 1, we need to show that $\mathcal{L}_0$ is a π-system. Take $A, B \in \mathcal{L}_0$, we want to show that $A \cap B \in \mathcal{L}_0$.

Claim: $\mathcal{P} \subseteq \mathcal{L}_B = \{C \subseteq \Omega : C \cap B \in \mathcal{L}_0\}$. Then, since $B \in \mathcal{L}_0$, by Lemma 2, $\mathcal{L}_B$ is a λ-system, assuming the claim is true. Since $\mathcal{P} \subseteq \mathcal{L}_B$ and $\mathcal{L}_B$ is a λ-system, we have $\mathcal{P} \subseteq \mathcal{L}_0 \subseteq \mathcal{L}_B$ since $\mathcal{L}_0$ is the smallest λ-system containing $\mathcal{P}$. So $A \in \mathcal{L}_B \Rightarrow A \cap B \in \mathcal{L}_0$ by the definition of $\mathcal{L}_B$. $\square$

Theorem: 4.2

Given $(\Omega, \mathcal{F}, P)$. If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ are independent π-systems, then $\sigma(\mathcal{A}_1), \sigma(\mathcal{A}_2), \dots, \sigma(\mathcal{A}_n)$ are independent.

Proof

For $i = 1, 2, \ldots, n$, define $\mathcal{B}_i = \mathcal{A}_i \cup \{\Omega\}$. Note that $\mathcal{B}_i$ is still a π-system and $\sigma(\mathcal{B}_i) = \sigma(\mathcal{A}_i)$. $\mathcal{A}_1, \ldots, \mathcal{A}_n$ independent $\Leftrightarrow$ $P(A_1 \cap \ldots \cap A_n) = P(A_1) \cdots P(A_n)$ for all $A_i \in \mathcal{A}_i$ or $A_i = \Omega \Leftrightarrow P(B_1 \cap \ldots \cap B_n) = P(B_1) \cdots P(B_n)$ for any $B_i \in \mathcal{B}_i$.

For $i = 2, \ldots, n$, fix sets $B_i \in \mathcal{B}_i$ and define $\mathcal{L} = \{A \in \mathcal{F} : P(A \cap B_2 \cap \ldots \cap$

$B_n) = P(A)P(B_2)\cdots P(B_n)\}$.

Claim: $\mathcal{L}$ is a λ-system.

(i) (i) $\Omega \in \mathcal{L}$ since $P(B_2 \cap \ldots \cap B_n) = P(B_2)\cdots P(B_n)$

(ii) Take $A \in \mathcal{L}$. Let $B = B_2 \cap \ldots \cap B_n$. Then $P(B) = P(A \cap B) + P(A^c \cap B) \Rightarrow P(B_2)\cdots P(B_n) = P(A)P(B_2)\cdots P(B_n) + P(A^c \cap B) \Rightarrow P(A^c \cap B) = (1 - P(A)P(B_2)\cdots P(B_n) \Rightarrow A^c \in \mathcal{L}$.

(iii) Take $A_1, \ldots \in \mathcal{L}$ disjoint. Let $A = \bigcup_{m=1}^{\infty} A_m$. Then

$$\begin{aligned} P(A \cap B_2 \cap \ldots \cap B_n) &= P(\bigcup_{m=1}^{\infty} A_m \cap B_2 \cap \ldots \cap B_n \\ &= P\left(\bigcup_{m=1}^{\infty} A_m \cap (B_2 \cap \ldots \cap B_n)\right) && \text{by countable additivity} \\ &= \left(\sum_{m=1}^{\infty} P(A_m)\right) P(B_2)\cdots P(B_n) \\ &= P\left(\bigcup_{m=1}^{\infty} A_m\right) P(B_2)\cdots P(B_n) && \text{by countable additivity} \end{aligned}$$

Hence $\bigcup_{m=1}^{\infty} A_m \in \mathcal{L}$.

Note that $B_1 \subseteq \mathcal{L}$, $\mathcal{B}_i$ is a π-system, $\mathcal{L}$ is a λ-system, by Dynkin we have $\sigma(\mathcal{A}_1)\sigma(\mathcal{B}_1) \subseteq \mathcal{L}$. By definition of $\mathcal{L}$, we have $\sigma(\mathcal{A}_1)$ is independent of $\mathcal{A}_2, \ldots, \mathcal{A}_n$. $\square$