Lemma: 4

$\mathcal{F}_0 \subset \mathcal{M}$

Proof

Take any $A \in \mathcal{F}_0$ and any $E \subset \Omega$, we want to show that

$$P^*(A \cap E) + P^*(A^c \cap E) \leq P^*(E).$$

Let $\varepsilon > 0$. Cover E by $\mathcal{F}_0$ sets and specifically choose a cover $\{A_n\}$ s.t. $\sum_{n=1}^{\infty} P(A_n) < P^*(E) + \varepsilon$ by the definition of infimum. Now since $A \cap E \subset \bigcup_{n=1}^{\infty}(A \cap A_n)$, $A^c \cap E \subset \bigcup_{n=1}^{\infty}(A^c \cap A_n)$

$$\begin{aligned}
P^*(A \cap E) + P^*(A^c \cap E) &\leq \sum_{n=1}^{\infty} P^*(A \cap A_n) + \sum_{n=1}^{\infty} P^*(A^c \cap A_n) \text{ by countable additivity} \\
&= \sum_{n=1}^{\infty} P^*(A_n) \\
&\leq \sum_{n=1}^{\infty} P(A_n) \\
&< P(E) + \varepsilon
\end{aligned}$$

Let $\varepsilon \to 0$, then $P^*(A \cap E) + P^*(A^c \cap E) \leq P(E) \Rightarrow A \in \mathcal{M} \Rightarrow \mathcal{F}_0 \subset \mathcal{M}$. $\square$

Lemma: 5

$A \in \mathcal{F}_0 \Rightarrow P^*(A) \geq P(A)$

Proof

We already know that $P^*(A) \leq P(A)$. We just need to show the other way around.

Cover A with $(A_n) \in \mathcal{F}_0 (A \subset \bigcup_{n=1}^\infty A_n)$. Then $A = \bigcup_{n=1}^\infty (A \cap A_n)$. So

$$\begin{aligned} P(A) &= P\left(\bigcup_{n=1}^\infty (A \cap A_n)\right) \\ &\leq \sum_{n=1}^\infty P(A \cap A_n) \text{ by c.s.} \\ &\leq \sum_{n=1}^\infty P(A_n) \end{aligned}$$

Note that $P^*(A)$ is the infimum for terms like $\sum_{n=1}^\infty P(A_n)$. But here $P(A) \leq \sum_{n=1}^\infty P(A_n)$. Hence $P(A) \leq P^*(A)$ □

Remark. $P^*(\Omega) = P(\Omega) = 1$ by Lemma 5, since $\Omega \in \mathcal{F}_0$.

So P^*, restricted to $\mathcal{F}$, is a probability measure on $\mathcal{F}$ and it agrees with P for sets in $\mathcal{F}_0$.

Lemma

The extension is unique.

Proof

Let Q be another extension of P to $\mathcal{F}$. Let $A \in \mathcal{F}$,

$$\begin{aligned} P^*(A) &= \inf\left\{\sum_{n=1}^\infty P(A_n) : A_n \in \mathcal{F}_0, A \subset \bigcup_{n=1}^\infty A_n\right\} \\ &= \inf\left\{\sum_{n=1}^\infty Q(A_n) : A_n \in \mathcal{F}_0, A \subset \bigcup_{n=1}^\infty A_n\right\} \\ &\geq \inf\left\{Q\left(\bigcup_{n=1}^\infty A_n\right) : A_n \in \mathcal{F}_0, A \subset \bigcup_{n=1}^\infty A_n\right\} \\ &\geq \inf\{Q(A) : A_n \in \mathcal{F}_0, A_n \subset \cup A_n\} \\ &= Q(A) \end{aligned}$$

$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F} \Rightarrow P^*(A^c) \geq Q(A^c) \Rightarrow 1 - p^*(A) \geq 1 - Q(A) \Rightarrow Q(A) \geq P^*(A)$

□

Definition: pi system

A **π-system** is a collection of subsets of Ω that is closed under finite intersections.

Note. This is a weak definition, and includes fields and σ-fields. Commonly denoted by $\mathcal{P}$.

Theorem: 3.3

Let $\mathcal{F}_0$ be a π-system. Suppose that P_1 and P_2 are two probability measures on $\mathcal{F} = \sigma(\mathcal{F}_0)$. If P_1 and P_2 agree on $\mathcal{F}_0$, they agree on $\mathcal{F}$.

(See Billingsley for proof.)

Note. This is a stronger theorem.

1 "Denumerable" Probabilities

Note. If we write $P(A)$, the assumption is that there is an underlying $(\Omega, \mathcal{F}, P)$.

1.1 Limit Sets

Definition: limsup

For a sequence $A_1, A_2, \ldots \in \mathcal{F}$, we define the set called "limsup over n of A_n" as

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

Example. A_1, A_2 and $A_n = \emptyset$ for $n \geq 3$. Then $\limsup_n A_n = \emptyset$.

Note.

1) Since $\mathcal{F}$ is closed under countable unions or intersections, $\limsup_n A_n, \liminf_n A_n \in \mathcal{F}$.
2) $\omega \in \limsup_n A_n \Rightarrow \omega \in \bigcup_{k=n}^{\infty} A_k$ for all $n \Rightarrow \omega$ is in infinitely many of the A_n.
3) $\omega \in \liminf_n A_n \Rightarrow \omega$ is in at least one $\bigcup_{k=n}^{\infty} A_k \Rightarrow w$ is in all but a finitely many of the A_n.