0.1 Hyperbolic functions

- $\cosh()(x) + \sinh()(x) = e^x$ and $\cosh()(x) - \sinh()(x) = e^{-x}$
- $\cosh()(x)^2 - \sinh()(x)^2 = 1$
- ...

0.2 Series

There are three ways to diverge: 1. goes to ∞ 2. goes to $-\infty$ 3. oscillates.

0.2.1 Convergence

- divergent test: if $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=0}^{\infty} a_n$ diverges.
- Geometric Series: Note that if $|r| < 1$, then $\sum_{n=0}^{\infty} ar^{n = \frac{1}{1-r}}$ and diverges otherwise.
- Direct Comparison Test: suppose $0 \leq a_n \leq b_n \forall\ n \geq 0$, then $0 \leq \sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{\infty} b_n$ and if $\sum_{n=0}^{\infty} b_n$ converges then $\sum_{n=0}^{\infty} a_n$ converges. Likewise for divergence.

Geometric series:

Proof

$$
\begin{aligned}
s_N &= a + ar + ar^2 + \ldots + ar^{N-1} \\
s_N r &= ar + ar^2 + \ldots + ar^N \\
s_N - s_N r &= a - ar^N = a(1 - r^{N)} \\
s_N &= \frac{a(1 - r^N}{1 - r} \\
\lim_{N\to\infty} s_N &= \lim_{N\to\infty} \frac{a(1 - r^{N)}}{1 - r} = \frac{a}{1 - r} \text{ iff |r|<1}
\end{aligned}
$$

□

Proof

Example (Absolutely convergent test)**.** Using the comparison test, we have $0 \leq a_n + |a_n| \leq 2|a_n| \Rightarrow 0 \leq \sum_{n=0}^{\infty}(a_n + |a_n|) \leq \sum_{n=0}^{\infty} 2|a_n|$. Since $\sum_{n=0}^{\infty} |a_n|$ converges, there exists some finite number L such that $\sum_{n=0}^{\infty} |a_n| = L$, which implies $\sum_{n=0}^{\infty} 2|a_n| = 2L$ so $\sum_{n=0}^{\infty} 2|a_n|$ converges. Thus, $\sum_{n=0}^{\infty}(a_n +$

$|a_n|$) converges by comparison test. Finally,

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty}(a_n + |a_n|) - \sum_{n=0}^{\infty} |a_n|.$$

Since both terms on the RHS are finite, their difference is finite and therefore the original series converges. □

Note. $|\sum_{n=0}^{\infty} a_n| \leq \sum_{n=0}^{\infty} |a_n|$ doesn't guarantee convergence because it can be oscillating divergence.

0.2.2 Rearrangement

- The real numbers possesses a property known as the *communitive property of addition* which states that the order in which we form a finite sum doesn't matter.
- Given a series $\sum_{n=1}^{\infty} a_n$ with partial sums (s_N), if we formulate the sum in a different order then this results in a different series and is known as a *rearrangement* of the original series.

Theorem

Let $\sum a_n$ be a series of real numbers which converges but not absolutely. Suppose $-\infty \leq \alpha \leq \beta \leq \infty$. Then there exists rearrangements of the original series, say, $\sum \hat{\alpha}_n$ and $\sum \hat{\beta}_n$, such that

$$\sum_{n=1}^{\infty} \hat{\alpha}_n \text{ and } \sum_{n=1}^{\infty} \hat{\beta}_n \text{ where } -\infty \leq \alpha \leq \beta \leq \infty.$$

0.2.3 Ratio test

Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n} = R\right|$. If $R < 1$, then the series $\sum_{n=0}^{\infty} a_n$ converges absolutely, if $R > 1$ then the series diverges and if $R = 1$ the it's inconclusive.

Use ratio test to determine the convergence of Taylor series, which is "radius of convergence".