A soliton in a conduit can be described by the following PDE:

$$A_t + (A^2)_z - \left(A^2\left(\frac{A_t}{A}\right)_z\right)_z = 0 \tag{1}$$

where the function $A(z,t)$ describes the non-dimensionalized cross-sectional area of the wave at location z and time t. Furthermore, the PDE satisfies the following boundary conditions:

$$\lim_{z\to\pm\infty} A(z,t) = 1 \tag{2}$$

$$\lim_{z\to\pm\infty} A_z(z,t) = 0 \tag{3}$$

$$\lim_{z\to\pm\infty} A_{zz}(z,t) = 0 \tag{4}$$

$$A(0,t) = a, A'(0,t) = 0 \tag{5}$$

Since we know that the soliton is a traveling wave in the $+z$ direction, we can convert this PDE into an ODE using change of variables $\zeta = z - ct$, where c is an unknown constant that represents traveling speed. Thus we let $A(z,t) = f(\zeta)$ and by using the Chain rule we obtain the following boundary value problem:

$$-cf' + (f^2)' - (f^2(-cf^{-1}f')')' = 0 \tag{6}$$

$$\lim_{\zeta\to\pm\infty} f(\zeta) = 1 \tag{7}$$

$$\lim_{\zeta\to\pm\infty} f'(\zeta) = 0 \tag{8}$$

$$\lim_{\zeta\to\pm\infty} f''(\zeta) = 0 \tag{9}$$

$$f(0) = a, f'(0) = 0 \tag{10}$$

where the prime notation is short for $\frac{d}{d\zeta}$.

We aim to reduce this ODE to first order. Notice since all terms on LHS are derivatives with respect to ζ, we can integrate both sides with respect to ζ and obtain

$$\begin{aligned} -cf + f^2 - f^2(-cf^{-1}f')' &= D \\ -cf + f^2 - cf^2f^{-2}f' + cf^2f^{-1}f'' &= D \\ -cf + f^2 - cf' + cff'' &= D \end{aligned} \tag{11}$$

We can find D by letting $\zeta \to \infty$ and applying the BCs:

$$\begin{aligned} -c + 1 - 0 + 0 &= D \\ D &= 1 - c \end{aligned} \tag{12}$$

Then the ODE becomes

$$-cf + f^2 - cf' + cff'' = 1 - c \tag{13}$$

To obtain first order ODE, we need to multiply the integrating factor $f^{-3}f'$ on both side and integrate:

$$\begin{aligned}
-cf^{-2}f' + f^{-1}f' - cf^{-3}f'^2 + cf^{-2}f'f'' &= (1-c)f^{-3}f' \\
cf^{-1} + \ln f + \frac{1}{2}cf^{-2}f'^2 &= \frac{1}{2}(c-1)f^{-2} + B
\end{aligned} \tag{14}$$

Again we take $\zeta \to \infty$ and apply the BCs to find the constant B:

$$\begin{aligned}
c + 0 + 0 &= \frac{1}{2}(c-1) + B \\
B &= \frac{1}{2}(c+1)
\end{aligned} \tag{15}$$

And the ODE becomes:

$$\begin{aligned}
cf^{-1} + \ln f + \frac{1}{2}cf^{-2}f'^2 &= \frac{1}{2}(c-1)f^{-2} + \frac{1}{2}(c+1) \\
cf + f^2 \ln f + \frac{1}{2}cf'^2 &= \frac{1}{2}(c-1) + \frac{1}{2}(c+1)f^2
\end{aligned} \tag{16}$$

It remains to find the constant c using the eq:ic. At $\zeta = 0$:

$$\begin{aligned}
cf(0) + f(0)^2 \ln f(0) + \frac{1}{2}cf(0)'^2 &= \frac{1}{2}(c-1) + \frac{1}{2}(c+1)f(0)^2 \\
ca + a^2 \ln a + 0 &= \frac{1}{2}(c-1) + \frac{1}{2}(c+1)a^2 \\
ac - \frac{1}{2}c - \frac{1}{2}a^2 c &= -\frac{1}{2} + \frac{1}{2}a^2 - a^2 \ln a \\
(2a - 1 - a^2)c &= -1 + a^2 - 2a^2 \ln a \\
c &= \frac{a^2 - 2a^2 \ln a - 1}{2a - a^2 - 1}
\end{aligned} \tag{17}$$