Remark.
Wave Equation: we have the same eigenvalue problem as heat equation. The time domain problem has a negative sign which leads to oscillating terms in time.

Laplace Equation: Same eigenvalue problem. The time domain problem has positive sign so we obtain hyperbolic functions in order to easily solve the coefficients.

0.1 different PDE domain

1) Inside the disc of radius R. Then $0 < r < R, \theta_0 < \eta \leq \theta_0 + 2\pi$. Physical boundary is $r = R$.
2) Outside the disc of radius R. Then $R < r < \infty, \theta_0 < \theta \leq \theta_0 + 2\pi$. Physical boundary: $r = R$.
3) Annulus: $R_i < r < R_o, \theta_0 < \eta \leq \theta_0 + 2\pi$. Physical boundaries: $r = R_i, r = R_0$.
4) Pie shaped sector: $0 < r < R, \theta_1 < \theta \leq \theta_2$. Physical boundary: $r = R, \theta = \theta_1, \theta = \theta_2$.

Example (circular disc)**.** Suppose $\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}$ on the domain $D = \{(r,\theta) | 0 \leq r \leq R, -\pi < \theta \leq \pi\}$.

Note. $r = 0, r = \infty$ are **singular points** of the coordinate system for $u = 0$ but *not* of the physical system. For physical reasons it is reasonable to assume boundedness at the origin: $|v(0,\theta)| < \infty$.

By periodicity we can assume continuity on the derivatives at the boundaries $\theta = \pm\pi$.

Claim. The set of functions that satisfy the *boundedness condition* or the *periodicity conditions* form a vector space.

0.2 separation of variables

Consider

$$\begin{cases} \text{PDE: } \Delta u = 0 & 0 < r < R, \theta \in (-\pi, \pi) \\ \text{BCs: } v(R,\theta) = f(\theta) \end{cases}$$

Assume $v(r,\theta) = F(\theta)G(r) \neq 0$. Then we obtain

$$\begin{cases} F''(\theta) & = -\lambda F(\theta) \\ F(-\pi) & = F(\pi) \\ F'(-\pi) & = F'(\pi) \end{cases} \qquad r^2 G''(r) + rG'(r) - \lambda G(r) = 0$$

0.3 F-equation

The same eigenvalue problem. Only $\lambda > 0$ is nontrivial, so

$$F_n(\theta) = A_n \sin(\sqrt{\lambda_n}\theta) + B_n \cos(\sqrt{\lambda_n}\theta)$$

where $\lambda_n = n^2, n = 1, 2, \ldots$, since $L = \pi$.

0.4 G-equation

Plug in $\lambda = n^2$, we have

$$r^2 G''(r) + rG'(r) - n^2 G(r) = 0.$$

Let $G(r) = r^p$. Then we get

$$r^2 p(p-1)r^{p-2} + rpr^{p-1} - n^2 r^2 = 0.$$

After cancellation, we obtain

$$p = \pm n, n = 1, 2, \ldots.$$

If $n \neq 0$, then $r_1 = r^n, r_2 = r^{-n}$. So by superposition, we get

$$G(r) = c_1 r^n + c_2 r^{-n}.$$

The boundedness condition yields,

$$|G(0)| < \infty \Rightarrow c_2 = 0 \Rightarrow G_n(r) = c_1 r^n, n = 1, 2, \ldots$$

Because $r = 0$ makes the second term undefined.

If $n = 0$, then

$$r^2 G''(r) + rG'(r) = 0 \Rightarrow rG''(r) + G'(r) = 0 \Rightarrow \frac{d}{dr}(rG'(r)) = 0 \Rightarrow rG'(r) = C_1.$$

Finally note that

$$rG'(r) = C_1 \Rightarrow G'(r) = \frac{C_1}{r} \Rightarrow G(r) = C_1 \ln(r) + C_2.$$

Boundedness again forces $C_1 = 0 \Rightarrow G_0(r) = C_2$.

Note. If the domain is Annulus we would keep both terms. Also if the BCs aren't so nice we might have to use periodicity condition.

Now the general solution is

$$\begin{aligned} v(r,\theta) &= a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) \\ &= a_0 \sum_{n=1}^{\infty} A_n \left(\frac{r}{R}\right)^n \cos(n\theta) + B_n \left(\frac{r}{R}\right)^n \sin(n\theta) \end{aligned}$$

where $A_n = a_n R^n, B_n = b_n R^n$. Now using the BCs:

$$f(\theta) = v(R, \theta) = a_0 + \sum_{n=1}^{\infty} A_n \cdot 1 \cdot \cos(n\theta) + B_n \cdot 1 \cdot \sin(n\theta).$$

which is a F.S.! So,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) d\theta.$$