# 1 Permutation Groups

**Definition**

Let $A$ be a set (e.g. $A = \{1, 2, 3, 4, 5\}$). A **permutation** of $A$ is a bijective function $\sigma : A \to A$.

**Theorem**

A function is bijective if and only if it has a two-sided inverse (i.e. the function is invertible).

**Theorem**

A function between two finite sets with the same number of elements is injective if and only if the function is surjective.

**Example.** $A = \{1, 2, 3\}$. A permutation of $A$ might be $\dots$. There are $3!$ permutations. There are $n!$ permutations of an $n$-element set.

**Theorem: permutation group**

Let $A$ be a set. The set permutations of $A$, denoted by $S_A$ is a group under composition of functions.

**Claim.** $\tau \circ \sigma$ is a permutation of $A$.

**Proof**

Let $\sigma, \tau$ be permutations of $A$. So $\sigma$ has a two-sided inverse, $\sigma^{-1}$, and so does $\tau$, $\tau^{-1}$. The inverse of $\tau \circ \sigma$ is $\sigma^{-1} \circ \tau^{-1}$. This gives bijectivity. $\square$

**Proof**

To show it's a group,

1) the identity of $A$ is $\mathrm{id}_A$ where $\mathrm{id}_A(a) = a$.

2) the inverse is $\sigma^{-1}$.

3) composition of functions is associative.

4) the previous claim shows that it is closed under operation.

$\square$

**Definition: symmetric group**

$S_A$ is the **symmetric group** on $A$. If $A = \{1, \ldots, n\}$, we write $S_n$ for $S_A$. It has the order $n!$.

**Example.** $S_3$ permutations of $\{1, 2, 3\}$ with 6 elements. See iPad. $S_3$ is a nonabelian group with order 6, which is the smallest nonabelian group. $V_4$ is the smallest noncyclic group.