Problem (22.3)**.** In $\mathbb{Z}_6[x]$ (note $+_6, \times_6$ are implied):

$$\begin{aligned} f(x)g(x) &= (2+3)x^2 + (3+2)x + (4+3) \\ &= 5x^2 + 5x + 1 \end{aligned}$$

$$\begin{aligned} f(x)g(x) &= (2\times 3)x^4 + (2\times 2 + 3\times 3)x^3 + (2\times 3 + 3\times 2 + 4\times 3)x^2 + (3\times 3 + 4\times 2)x + 4\times 3 \\ &= x^3 + 5x \end{aligned}$$

Problem (22.4)**.** In $\mathbb{Z}_5[x]$:

$$\begin{aligned} f(x) + g(x) &= 3x^4 + 2x^3 + 4x^2 + (3+2)x + (2+4) \\ &= 3x^4 + 2x^3 + 4x^2 + 1 \end{aligned}$$

$$\begin{aligned} f(x)g(x) &= (2\times 3)x^7 + (4\times 3)x^6 + (3\times 3)x^5 + (2\times 3 + 2\times 2)x^4 + (2\times 4 + 4\times 2)x^3 \\ &\quad + (4\times 4 + 3\times 2)x^2 + (3\times 4 + 2\times 2)x + (2\times 4) \\ &= x^7 + 2x^6 + 4x^5 + x^3 + 2x^2 + x + 3 \end{aligned}$$

Problem (22.8)**.** In $\mathbb{C}$:

$$\begin{aligned} \phi_i(2x^3 - x^2 + 3x + 2) &= 2i^3 - i^2 + 3i + 2 \\ &= -2i + 1 + 3i + 2 \\ &= 3 + i \end{aligned}$$

Problem (22.9)**.** In $\mathbb{Z}_7[x]$:

$$\begin{aligned} \phi_3[(x^4 + 2x)(x^3 - 3x^2 + 3)] &= \phi_3(x^4 + 2x) \times \phi_3(x^3 - 3x^2 + 3) \\ &= (3^4 \bmod 7 + 2\times 3)(3^3 \bmod 7 - 3^3 \bmod 7 + 3) \\ &= (4+6)3 \\ &= 2 \end{aligned}$$

Problem (22.12)**.** In $\mathbb{Z}_2[x]$, trying exhaustively:

$$\begin{aligned} \phi_0(x^2+1) &= 0^2 + 1 = 1 \\ \phi_1(x^2+1) &= 1^2 + 1 = 0 \end{aligned}$$

Thus 1 is the zero of $x^2 + 1$ in $\mathbb{Z}_2[x]$.

Problem (22.13)**.** In $\mathbb{Z}_7[x]$, trying exhaustively:

$$\begin{aligned}
\phi_0 &= 0^3 + 2 \times 0 + 2 = 2 \\
\phi_1 &= 1^3 + 2 \times 1 + 2 = 5 \\
\phi_2 &= 2^3 + 2 \times 2 + 2 = 0 \\
\phi_3 &= 3^3 + 2 \times 3 + 2 = 0 \\
\phi_4 &= 4^3 + 2 \times 4 + 2 = 4 \\
\phi_5 &= 5^3 + 2 \times 5 + 2 = 4 \\
\phi_6 &= 6^3 + 2 \times 6 + 2 = 6
\end{aligned}$$

Thus $2, 3$ are the zeros of $x^3 + 2x + 2 \in \mathbb{Z}_7[x]$.

Problem (22.16)**.** In $\mathbb{Z}_5[x]$, since $\gcd(3, 5) = 1$, by Fermat $3^4 = 1 \bmod 5$.

$$\begin{aligned}
\phi_3(x^{231} + 3x^{117} - 2x^{117} - 2x^{53} + 1) &= 3^{231} + 3^{118} - 2 \times 3^{53} + 1 \\
&= (3^4)^{57} \times 3^3 + (3^4)^{29} \times 3^2 - 2 \times (3^4)^{13} \times 3 \\
&= 3^3 + 3^2 - 2 \times 3 \\
&= 2 + 4 - 1 \\
&= 0
\end{aligned}$$

Problem (22.18)**.** A polynomial with coefficients in a ring R is an infinite formal sum

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots.$$

where $a_i \in \mathbb{R}$ and only finitely many $a_i \neq 0$.

Problem (22.20)**.**

$$\begin{aligned}
f(x, y) &= 3x^3y^3 + 2xy^3 + x^2y^2 - 6xy^2 + y^2 + x^4y - 2xy + x^4 - 3x^2 + 2 \\
&= (1 + y)x^4 + (3y^3)x^3 + (y^2 - 3)x^2 + (2y^3 - 6y^2 - 2y)x + (y^2 + 2)
\end{aligned}$$

Problem (22.22)**.** In $\mathbb{Z}_4[x]$, consider $2x + 1$:

$$(2x + 1)(1 - 2x) = 1^2 - (2x)^2 = 1 - 0x^2 = 1.$$

Since $2x + 1 \neq 0$, it is a unit in $\mathbb{Z}_4[x]$.

Problem (22.23)**.** a) True.

b) True.

c) True. By theorem.

d) True.

e) False. It cannot exceed 7.

f) False. $f(x) = 2x^3, g(x) = 2x^4 \in \mathbb{Z}_4[x]$, $f(x)g(x) = 0$ doesn't have degree 7.

g) True. By theorem.

h) True. By theorem.

i) True. Given $f(x) \neq 0 \in R[x]$, then $xf(x)$ keeps all coefficients the same and add 1 degree to each term. Thus $xf(x) \neq 0$.

j) False. In $\mathbb{Z}_{12}[x]$, $3x \times 4x = 0$ so they are zero divisors but they are not in $\mathbb{Z}_{12}$.

Problem (23.1)**.**

Figure 1

Problem (23.2)**.**

Figure 2

Problem (23.6)**.** By theorem since 7 is prime, $U(\mathbb{Z}_7) \simeq (\mathbb{Z}_6, +_6)$. We know the generators of $\mathbb{Z}_6$ are 1,5 since they are coprime to 6, and notice $5 = -1$ which is the inverse of 1. Since isomorphism preserves structure, we know that $U(\mathbb{Z}_7)$ must have only 2 generators that are a pair of inverses. It suffices to find one generator in $U(\mathbb{Z}_7)$ and the other is the inverse. Notice:

$$\begin{aligned}3^1 &= 3\\3^2 &= 2\\3^3 &= 6\\3^4 &= 4\\3^5 &= 5\\3^6 &= 1\end{aligned}$$

Thus 3 is a generator of $U(\mathbb{Z}_7)$. The other generator is thus its inverse 5.

Problem (23.9)**.** In $\mathbb{Z}_5[x]$:

$$\begin{aligned}x^4 + 4 &= x^4 - 1\\&= (x^2 - 1)(x^2 + 1)\\&= (x - 1)(x + 1)(x^2 - 4)\\&= (x - 1)(x + 1)(x - 2)(x + 2)\end{aligned}$$

Problem (23.12)**.** In $\mathbb{Z}_5[x]$:

$$\phi_4(x^3 + 2x + 3) = (-1)^3 + (-2) + 3 = 0.$$

Thus 4 is a zero. Since $f(x)$ has degree 3 and $\mathbb{Z}_5$ is a field, by Theorem 23.10 $f(x)$ is not irreducible. Thus by inspection,

$$x^3 + 2x + 3 = (x - 1)(x^2 - x + 3) = (x - 1)(x + 1)(x - 3)$$

since $\phi_{-1}(x^2 - x + 3) = 0$.

Problem (23.14)**.** Using the quadratic formula:

$$r_{\pm} = \frac{-8 \pm \sqrt{8^2 + 8}}{2} = -4 \pm 3\sqrt{2} \notin \mathbb{Q}.$$

So the roots of $f(x)$ are not in $\mathbb{Q}$, and since $f(x)$ has degree 2 and no zeros in $\mathbb{Q}$, it is irreducible over $\mathbb{Q}$ by Theorem 23.10.

And since $-4 \pm 3\sqrt{2} \in \mathbb{R} \subseteq \mathbb{C}$, $f(x)$ has zeros in $\mathbb{R}$ and $\mathbb{C}$, so it is not irreducible over $\mathbb{R}$ and $\mathbb{C}$.

Problem (23.15)**.** Using the quadratic formula:

$$r_{\pm} = \frac{-6 \pm \sqrt{36 - 48}}{2} = -3 \pm i\sqrt{3} \notin \mathbb{Q} \text{ or } \mathbb{R}.$$

Since $f(x)$ has degree 2, by Theorem 23.10 it is irreducible over $\mathbb{Q}$ and $\mathbb{R}$.

Since $-3 \pm i\sqrt{3} \in \mathbb{C}$, it is not irreducible over $\mathbb{C}$.

Problem (23.16)**.** Let $p = 3$ which is a prime. Notice $f(x) \in \mathbb{Z}[x], 1 \neq 0 \bmod 3, 8 \neq 0 \bmod 3^2, 3 = 0 \bmod 3$, by Eisenstein Criterion $f(x)$ is irreducible over $\mathbb{Q}$.

Problem (23.18)**.** Let $p = 5$ which is a prime. Notice $f(x) \in \mathbb{Z}[x], 1 \neq 0 \bmod 5, -12 \neq 0 \bmod 5^2, 0 = 0 \bmod 5$, by Eisenstein Criterion $f(x)$ is irreducible over $\mathbb{Q}$.

Problem (23.19)**.** Let $p = 3$ which is a prime. Notice $f(x) \in \mathbb{Z}[x], 8 \neq 0 \bmod 3, 6 = 0 \bmod 3, 9 = 0 \bmod 3, 24 \neq 0 \bmod 3^2$, by Eisenstein Criterion $f(x)$ is irreducible over $\mathbb{Q}$.

Problem (23.25)**.**

a) True. Since $\mathbb{Q}$ is a field, all degree 1 polynomials in $\mathbb{Q}[x]$ are irreducible.

b) True. By the same reasoning.

c) True. Since the roots $\pm\sqrt{3} \notin \mathbb{Q}$.

d) False. In $\mathbb{Z}_7$, $x^2 + 3 = x^2 - 4 = (x+2)(x-2)$.

e) True. By theorem.

f) True (repeat).

g) True. By Corollary 23.5.

h) True. By factor theorem.

i) True. Because its coefficient has inverse since F is a field.

j) True. Because we only have finitely many nonzero terms, and we can only at most have as many zeros as the leading degree of the polynomial.