Homework 8

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Problem (14.5)**.** It suffices to find the index of the subgroup. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, $N = \langle(1,1)\rangle$. Then the index is just $|G|/|N|$. That is,

$$|G/N| = \frac{8}{\operatorname{lcm}\left(\frac{2}{\gcd(1,2)}, \frac{4}{\gcd(1,4)}\right)} = \frac{8}{4} = 2.$$

Problem (14.6)**.** Let $G = \mathbb{Z}_{12} \times \mathbb{Z}_{18}$, $N = \langle(4,3)\rangle$. Then

$$|G/N| = \frac{12 \times 18}{\operatorname{lcm}\left(\frac{12}{\gcd(4,12)}, \frac{18}{\gcd(3,18)}\right)} = \frac{12 \times 18}{\operatorname{lcm}(3,6)} = 36.$$

Problem (14.11)**.** Let $a = (2,1)$, $G = \mathbb{Z}_3 \times \mathbb{Z}_6$, $N = \langle(1,1)\rangle$. First the elements of N are

$$N = \{(0,0),(1,1),(2,2),(0,3),(1,4)\}.$$

By theorem, the order of the coset $a + N$ is the smallest positive integer n such that $na \in N$ or infinity. Let's check:

$$2(2,1) = (1,2) \notin N$$
$$3(2,1) = (0,3) \in N$$

Hence by theorem, $|G/N| = 3$.

Problem (14.12)**.** Let $a = (3,1)$, $G = \mathbb{Z}_4 \times \mathbb{Z}_4$, $N = \langle(1,1)\rangle$. The elements of N are

$$N = \{(0,0),(1,1),(2,2),(3,3)\}.$$

Again we check:

$$2(3,1) = (2,2) \in N$$

Hence by theorem, $|G/N| = 2$.

Problem (4.23)**.** a) True. By theorem 14.4.

b) True. By a theorem from class.

c) True. $\iota_g(x) = gxg^{-1} = gg^{-1}x = ex = x$ by commutativity.

d) True. Because if G is finite, its normal subgroups are also finite, so the index is finite.

e) True.

f) False. $\mathbb{Z}$ is torsion-free, but $\mathbb{Z}/3\mathbb{Z} = \mathbb{Z}_3$, where each element of $\mathbb{Z}_3$ has finite order and is thus a torsion group.

g) True. By theorem from class.

h) False. Counterexamples are $GL_n(\mathbb{R})/SL_n(\mathbb{R}) \simeq R^*$ and $D_4/\{\rho_0, \rho_2\} \simeq V_4$. Both are abelian.

i) Since $\mathbb{Z}_n \coloneqq \mathbb{Z}/n\mathbb{Z}$ has order n.

j) False. $\mathbb{R}/n\mathbb{R} \simeq \mathbb{R}$, which has infinite order. It is also not isomorphic to $\mathbb{Z}$ and thus not cyclic.

Problem (15.1)**.** Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4, N = \langle (0, 1)\rangle$. By Theorem 15.8, N is a normal subgroup of G. Then

$$|G/N| = |G|/|N| = \frac{2 \times 4}{\frac{4}{\gcd(1,4)}} = \frac{8}{4} = 2.$$

Since there is only one group of order 2, $G/N \simeq \mathbb{Z}_2$.

Problem (15.2)**.** Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, $N = \langle (0, 2)\rangle$. Since G is abelian and $N \leq G$, N is a normal subgroup of G. Then

$$|G/N| = |G|/|N| = \frac{8}{\frac{4}{\gcd(2,4)}} = \frac{8}{2} = 4.$$

It remains to determine if this is $\mathbb{Z}_4$ or V_4. Let's show that it is not cyclic. Notice for $n = 0, 1$ and $m = 0, 1, 2, 3$, we can represent an arbitrary coset in the quotient group as $(n, m) + N$. Then let's find the order of $(n, m) + N$

(ignoring the trivial case when $(n, m) \in N$):

$$\begin{aligned}2(n, m) &= (n +_2 n, m +_4 m) \\ &= (2n \mod 2, 2m \mod 4) \\ &= \begin{cases}(0,0) \text{ if } m \text{ is even} \\ (0,2) \text{ if } m \text{ is odd}\end{cases} \in N\end{aligned}$$

Thus, $|(n, m) + N|$ is at most 2. But to be a generator of G/N it requires order 4. Therefore, there is no generator in the quotient group, proving it is not cyclic. Then it must be V_4.

Problem (15.3)**.** Let $G = \mathbb{Z}_2 \times \mathbb{Z}_4, N = \langle(1, 2)\rangle$. Then

$$N = \{(0, 0), (1, 2)\}.$$

G is abelian and $N \leq G$, therefore $N \trianglelefteq G$.

$$|G/N| = |G|/|N| = \frac{8}{\text{lcm}\left(\frac{2}{\gcd(1,2)}, \frac{4}{\gcd(2,4)}\right)} = \frac{8}{2} = 4.$$

Let $g = (1, 1) \notin N, \in G$. Notice

$$\begin{aligned}2(1, 1) &= (0, 2) \notin N \\ 3(1, 1) &= (1, 3) \notin N \\ 4(1, 1) &= (0, 0) \in N\end{aligned}$$

Hence $|g + N| = 4$ and g is a generator of G/N. Thus G/N is cyclic and isomorphic to $\mathbb{Z}_4$.

Problem (15.6)**.** Let $G = H \times K = \mathbb{Z} \times \mathbb{Z}$ and $N = \langle(0, 1)\rangle$. Notice $N = \{(0, k) : k \in K\}$. By theorem 15.8, $N \trianglelefteq G$, and $G/N \simeq H = \mathbb{Z}$.

Problem (15.13)**.** By table, we see that $Z(D_4) = \{e, (1\ 3)(2\ 4)\}$ or $\{\rho_0, \rho_2\}$.

For $C(D_4)$, we want to find the smallest normal subgroup with abelian quotient. D_4 has the following normal subgroups:

$$\begin{aligned}D_4 \text{ of order } 8 :&D_4/D_4 \simeq \{e\}, \text{ abelian} \\ \{e\} \text{ of order } 1 :&D_4/\{e\} \simeq D_4, \text{ nonabelian} \\ Z(D_4) \text{ of order } 2 :&D_4/Z(D_4) \simeq V_4, \text{ abelian} \\ \{\text{ rotations }\} \text{ of order } 4 :&D_4/\{\text{ rotations }\} \simeq \mathbb{Z}_2, \text{ abelian}\end{aligned}$$

Clearly, the smallest normal subgroup with abelian quotient is $Z(D_4)$. So $C(G) = Z(D_4)$.

Problem (15.19)**.**

a) True. By theorem 15.9, if a is a generator of G, then aN is a generator of G/N.

b) False. $S_n/A_n \simeq \mathbb{Z}_2$, but S_n is non-cyclic.

c) False. Let $x = \frac{1}{2}$, then $2x = 1 \in \mathbb{Z}$, so $|x + \mathbb{Z}| = 2$.

d) True. Given $n \in \mathbb{Z}^+$, choose $x = \frac{1}{n}$ which is well-defined. Then $nx = n\frac{1}{n} = 1 \in \mathbb{Z}$, so $|x + \mathbb{Z}| = n$.

e) False. To find all elements in $\mathbb{R}/\mathbb{Z}$ of order 4, let x be a representative of any element, then $|x + \mathbb{Z}| = 4 \Leftrightarrow 4x = a \in \mathbb{Z}$, where $\gcd(a, 4) = 1$. Thus x needs to satisfy $x = \frac{a}{4}$ where $\gcd(a, 4) = 1$ to be a representative of an element of order 4. To see how many such elements exist, let $y+\mathbb{Z}$ be a different element where $y = \frac{b}{4}, \gcd(b, 4) = 1$. Since $x+\mathbb{Z} \neq y+\mathbb{Z} \Leftrightarrow x - y \notin \mathbb{Z}$, we have

$$\begin{aligned} x - y &\notin \mathbb{Z} \\ a - b &\notin 4\mathbb{Z} \\ a - b \mod 4 &\neq 0 \\ a \mod 4 +_4 (-b \mod 4) &\neq 0 \\ a \mod 4 &\neq b \mod 4 \\ p &\neq q \in \mathbb{Z}_4 \end{aligned}$$

Where $p = a \mod 4, q = b \mod 4$. Since $p, q \in \mathbb{Z}_4$, and $\gcd(a, 4) = \gcd(b, 4) = 1 \Rightarrow \gcd(p, 4) = \gcd(q, 4) = 1$, we see that there are only 2 distinct elements we can choose from $\mathbb{Z}_4$, namely 1 and 3, to form distinct cosets of order 4. So it is not infinite.

f) True. $C(G) = \{e\} \Rightarrow aba^{-1}b^{-1} = e \Rightarrow ab(ba)^{-1} = e \Rightarrow ab = ba \ \forall \ a, b \in G$.

g) False. $\{e, (1\ 3)(2\ 4)\} \not\supseteq D_4$.

h) False. Let G be an abelian simple group. Then it only has two subgroups, $\{e\}$ and itself. $G/\{e\} = G$ is abelian, and $G/G = \{e\}$ is abelian. Clearly $\{e\}$ is the smallest normal subgroup that has abelian quotient, thus $C(G) = \{e\}$.

i) True. Let G be an nonabelian simple group, then by argument above, $G/\{e\} = G$ is nonabelian, so the only candidate left is G. Thus, $C(G) = G$.

j) False. $|A_5| = 5!$ is not prime order, yet A_5 is finite and simple.

Problem (15.22)**.** Let $g : \mathbb{R} \to \mathbb{R}$ such that $g \notin K$. For $|g + K| = 2$ we need $2g \in K$. This implies that $2g$ is continuous, so it follows that g is continuous, contradicting our assumption that $g \notin K$. Hence, no such element exists.

Problem (15.23)**.** Let $g : \mathbb{R} \to \mathbb{R}$ such that $g \notin K^*$. For $|g + K| = 2$ we need $g^2 \in K^*$. Consider

$$g(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases} \notin K^*$$

However,

$$g^2(x) = \begin{cases} 1, & x \geq 0 \\ 1, & x < 0 \end{cases} = 1 \ \forall \ x \in \mathbb{R} \Rightarrow g^2 \in K^*$$

Problem (15.30)**.**

a) By definition of abelian group, every element commutes, hence $Z(G) = G$.

b) Since $Z(G) \trianglelefteq G$, and G is simple, the center is either $\{e\}$ or G. Since G is nonabelian, it must be that $Z(G) = \{e\}$.

Problem (15.31)**.**

a) If G is abelian, then we know $G/\{e\} = G$ is abelian, and clearly $\{e\}$ is the smallest normal subgroup with abelian quotient. $C(G) = \{e\}$.

b) If G is nonabelian, since G is simple, the only candidate left is G, where $G/G = \{e\}$ is abelian. So $C(G) = G$.