Example. Subgroups of S_3: $\{e,(1\ 2)\},\{e,(1\ 3)\},\{e,(2\ 3)\},\{e\},S_3,A_3$. The first three are not normal. The rest are normal. A_3 is 1. index is two. 2. kernel of the sign homomorphism.

Let $g \in G$, we have seen that if $H \leq G$, then so is gHg^{-1}. Furthermore, the map $\iota_g(x) = gxg^{-1}$ (injective map) is called conjugating by g.

Proposition

ι_g is a group homomorphism (in fact an isomorphism) from G to itself.

Proof

$$\iota_g(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = \iota_g(x)\iota_g(y)$$

□

Example. Isomorphism of $V_4 \to V_4$. Identity needs to go to itself, but there are 3! different isomorphism. Then the group of isomorphisms are just S_3.

But since it's abelian, conjugation is trivial.

Claim. The inverse of $\iota_g = \iota_{g^{-1}}$ (conjugation by g^{-1}).

Definition: automorphism

An isomorphism from G to itself is called an **automorphism**.

Definition: inner automorphism

Automorphisms that come from conjugation are called **inner automorphism**.

Note. Inner automorphism is a subgroup of the group of automorphisms

Note. $\iota_g(H)$ is a subgroup of G because it is the image of ι_g.

Definition

$$H \simeq gHg^{-1}.$$

is conjugate subgroup.

Example. $G = S_3, H = \{e, (1\ 2)\}, g = (1\ 3)$. Then

$$\begin{aligned} gHg^{-1} &= \{geg^{-1}, g(1\ 2)g^{-1}\} \\ &= \{e, (2\ 3)\} \\ gHg^{-1} &\simeq H \text{ but } H \neq gHg^{-1} \end{aligned}$$

Which proves that H is not normal.

Theorem

If G is abelian, then G/N is abelian.

Proof

Let $xN, yN \in G/N$. Then

$$xN * yN = (xy)N = (yx)N = yN * xN$$

Since $x = y \Rightarrow xN = yN$. □

Note. The converse is false. Example is S_3/A_3. Or the trivial N.

Note. G/N is cyclic doesn't imply G is cyclic.

What is the order of $xN \in G/N$? It is the smallest $n > 0$ such that $(xN)^n = N \Rightarrow x^n N = eN \Rightarrow e^{-1}x^n \in N \Rightarrow x^n \in N$.

Definition

The order of a coset $xN \in G/N$ is the smallest positive integer n such that $x^n \in N$.

Example (15.7)**.** $G = \mathbb{Z}_4 \times \mathbb{Z}_6$. Order is 24. If G_1 and G_2 are abelian so is $G_1 \times G_2$. If one group is not abelian, then the product isn't abelian. So G is abelian. G isn't cyclic since it isn't isomorphism to $\mathbb{Z}_{24}$.

$H = \langle (0, 1) \rangle = \{(0, 0), (0, 1), \ldots, (0, 5)\}$. The order is 6.

Is H normal in $\mathbb{Z}_4 \times \mathbb{Z}_6$? Yes because G is abelian.

Then G/H is abelian with order 4. So it's either $\mathbb{Z}_4$ or V_4. We can show $\mathbb{Z}_4$ if we find an element of order 4 (a generator). Then a coset looks like $(1, 0) + \langle (0, 1) \rangle$.

What is the order of that?

First find all elements of N. Then repeat operation on the representative of coset until it's in N. It takes 4 steps to get $(0,0) \in N$. Thus it has order 4.

Example. $\mathbb{Q}/\mathbb{Z}$ is an infinite group where every element has finite order.

$\mathbb{R}/\mathbb{Q}$ is an infinite group that has no element of finite order apart from the identity.

$\mathbb{R}/\mathbb{Z} \simeq U$. Since $\phi : \mathbb{R} \to C^*, r \mapsto e^{2\pi i r}$.