Corollary: 1 of LMCT

Let $f, g \geq 0$ integrable, then $f + g$ is also integrable and

$$\int (f+g)\ d\mu = \int f\ d\mu + \int g\ d\mu.$$

Proof

Choose simple $0 \leq s_1(\omega) \leq s_2(\omega) \leq \dots$ converging up to f, and simple $0 \leq t_1(\omega) \leq t_2(\omega) \leq \dots$ converging to g. Then $(s_n + t_n)$ is a nonnegative sequence. □

Lemma

A finite "measure" μ with *only* finite additivity ($i.e.$ $\mu : \mathcal{F} \to [0, \infty), \mu(\emptyset) = 0$, finite additivity) is countably additive iff $\mu(A_n) \searrow 0$ for every $\mathcal{F}$-sequence (A_n) satisfying $A_1 \supseteq A_2 \supseteq \dots$ and $\bigcap_{i=1}^{n} A_n = \emptyset$.

Proof

($\Rightarrow$): suppose μ is countably additive and a $\mathcal{F}$-sequence $(A_n) \searrow \emptyset$. By continuity from above and the fact that μ is finite, $\mu(A_n) \searrow \mu(\emptyset) = 0$, which only makes sense if μ is countably additive. ($\Leftarrow$): □

Corollary: 2 of LMCT

Suppose $f \geq 0$ integrable, the set function $\nu(\cdot)$ defined by $\nu(A) = \int_A f\ d\mu$ is a finite measure on $(\Omega, \mathcal{F})$.

Proof

f integrable implies $\nu(\Omega) = \int_\Omega f\ d\mu < \infty$. Thus ν is a finite measure. Recall that $\mu(A) = 0 \Rightarrow \int_A f\ d\mu = 0$. So

$$\nu(\emptyset) = \int_\emptyset f\ d\mu = 0.$$

Take A_1, A_2 disjoint sets in $\mathcal{F}$.

$$\begin{aligned}
\nu(A_1 \cup A_2) &= \int_{A_1 \cup A_2} f \; d\mu \\
&= \int_\Omega I_{A_1 \cup A_2} \cdot f \; d\mu \\
&= \int_\Omega (I_{A_1} + I_{A_2}) \cdot f \; d\mu \text{ by disjoint} \\
&= \int_{A_1} f \; d\mu + \int_{A_2} f \; d\mu \\
&= \nu(A_1) + \nu(A_2)
\end{aligned}$$

Take any sequence $A_n \searrow \emptyset \Rightarrow 1 - I_{A_n} \nearrow 1 \Rightarrow (1 - I_{A_n})f \nearrow$. By LMCT,

$$\begin{gathered}
\int (1 - I_{A_n}) \cdot d\mu \nearrow \int f \; d\mu \\
\int f \; d\mu - \int_{A_n} f \; d\mu \nearrow \int f \; d\mu \\
\nu(\Omega) - \nu(A_n) \nearrow \nu(\Omega)
\end{gathered}$$

Thus by the lemma, ν is countably additive! $\square$