Example. Question: Are 19 and 6 in the same coset?

No, because $x + H = y + H \Leftrightarrow x - y \in H$. And $19 - 6 = 13 \notin H$.

Are 19 and 7 in the same coset?

Yes, because $7 - 19 = -12 \in H$.

Example (easiest one: left/right cosets are different, IMPORTANT)**.** $G = S_3$, $H = \{e, (1\ 2)\}$.

What are the left cosets of H in G?

1) H itself.
2) The left coset containing $(1\,2\,3)H = \{(1\,2\,3)e, (1\,2\,3)(1\,2)\} = \{(1\,2\,3), (1\,3)\}$.
3) by elimination, we know the last left coset is $\{(2\ 3), (1\ 3\ 2)\}$.

Question: is $(1\ 3\ 2)H = (2\ 3)H$? Yes because let $x = (1\ 3\ 2)$ and $y = (2\ 3)$, then $x^{-1} = (1\ 2\ 3)$, so $x^{-1}y = (1\ 2\ 3)(2\ 3) = (1\ 2) \in H$. QUestion: is $(1\ 2\ 3)H = (2\ 3)H$? $x^{-1}y = (1\ 3\ 2)(2\ 3) = (1\ 3) \notin H$, so no.

We can just take every element of the left cosets and individually invert it to convert to right cosets.

Show that $H(1\ 3) = H(1\ 3\ 2)$. $xy^{-1} = (1\ 3)(1\ 2\ 3) = (1\ 2) \in H$.

Example. $G = S_3, H = \{e\}$. There are six cosets. Left/right cosets are the same even if the group is nonabelian.

Example. $G = S_3, H = S_3$. There is only one coset (left/right).

Example. $G = S_3, H = A_3$. There are two cosets. Left and right again agree because there is a subgroup and everything else.

Claim. Left and right cosets always agree if there are only two cosets (because one must a subgroup).

Theorem

If G is finite, then the order of x, $o(x)$ divides $|G|$.

Proof

$\langle x \rangle$ is a subgroup of G, so $|\langle x \rangle|$ divides G by Lagrange. $\square$

Theorem: 10.11

Every group of prime order is cyclic.

Example. We know this doesn't have to be true for non-prime number: V_4.

Proof

Let G be a group of order p and let $H \geq g$. Then $|H|$ divides $|G|$. But since $|G|$ is a prime, so $|H| = 1 \Rightarrow H = \{e\}$ or $|H| = p \Rightarrow H = G$. $\square$

Claim. Let $x \in G \setminus \{e\}$. Then $\langle x \rangle = G$. Not only is G cyclic, but any nonidentity element is a generator.

Definition: direct products

Let $(G, *_G)$ and $(H, *_H)$ be two groups. The **direct product** $G \times H$ as

$$G \times H = \{(g, h) : g \in G, h \in H\}.$$

Operation: componentwise

$$(g_1, h_2) * (g_2, h_2) = (g_1 * g_2, h_1 * h_2).$$

where closure follows immediately.

Identity: $e = (e_G, e_H)$. $(e_G, e_H) * (g, h) = (e_G * g, e_H * H) = (g, h)$. Same for the other way.

Inverse: (g^{-1}, h^{-1}).

Associativity: see book.

Example. $\mathbb{Z}_2 \times \mathbb{Z}_2$ under $+_2$. See iPad screenshots for the table. $|A \times B| = |A| \times |B|$. It is isomorphic to V_4. This is the proof that V_4 is a group and is associative.

Therefore, $\mathbb{Z}_2 \times \mathbb{Z}_2$ is NOT isomorphic to $\mathbb{Z}_4$!

Example. $\mathbb{Z}_2 \times \mathbb{Z}_3$ this is an abelian group of order 6. $x = (1, 1) \in \mathbb{Z}_2 \times \mathbb{Z}_3$. $x + x = (0, 2)$. $x + x + x = (1, 0)$. $4x = (0, 1)$. $5x = (1, 2)$. $6x = (0, 0)$. Hence $\mathbb{Z}_2 \times \mathbb{Z}_3$ is generated by $(1, 1)$. So it is a cyclic group of order 6. Then it must be isomorphic to $\mathbb{Z}_6$!

Claim. Direct product of abelian groups are abelian.

Claim. $\mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{mn}$ if and only if $\gcd(m, n) = 1$. If $\gcd(m, n) = 1$, then the order of $(1, 1)$ is $\operatorname{lcm}(m, n) = mn$.

Goal: classify all abelian groups of order n.