*Intuition.* The Fourier transform is basically the "Fourier coefficient" of the basis function for the integral.

## 0.1 Notation

1)

$$\hat{f}(m) = \lim_{L\to\infty} \hat{f}(m_n) = \lim_{L\to\infty} \frac{L}{\pi} c_n = \lim_{L\to\infty} \frac{L}{\pi} \left( \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} \right) dx$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{imx} \, dx$$

*Intuition.* This is sort of the projection formula, since $2\pi$ is the circumference of the unit circle.

2)

**Definition**

Define the **Fourier transform** of $f(x)$ to be $\hat{f}(m) = \mathcal{F}[f](m)$ where

$$\hat{f}(m) = \mathcal{F}[f](m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-imx} \; \forall \; m \in \mathbb{R}.$$

where the kernel is $K(x,m) = e^{-imx}$.

3)

**Definition**

Define the **inverse Fourier transform** of $\hat{f}(m)$ to be $f(x) = \mathcal{F}^{-1}[\hat{f}](x)$ where

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \int_{-\infty}^{\infty} \hat{f}(m) e^{imx} dm \; \forall \; x \in \mathbb{R}.$$

where the kernel is $\hat{K}(m,x) = e^{imx}$.

4) Given $\hat{f}(m)$ then $f(x) = \mathcal{F}^{-1}[\hat{f}]$ and given $f(x)$ then $\hat{f}(m) = \mathcal{F}[f]$.

*Intuition.* The Fourier transform represents a function $f(x)$ in a new "coordinate system" using different eigenfunction basis.

Fun Facts:

1) If $\int_{-\infty}^{\infty} |f(x)| dx = M < \infty$ then

$$|\hat{f}(m)| \le \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)| \cdot |e^{-imx}| dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)| dx = \frac{M}{2\pi}.$$

2) Note that

$$\hat{f}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) dx = \frac{1}{2\pi} \cdot [\text{ area under the curve } f(x)].$$

3) If $f(x)$ is real then $\hat{f}(m) = \overline{\hat{f}(m)}$.

4) If $f(x)$ is even then $\hat{f}(m)$ is even, likewise for odd.

5) The data $f(x)$ is transformed to a representation $\hat{f}(m)$ in the frequency domain.

**Example.** Suppose

$$f(x) = \begin{cases} A, & -L < x < L \\ \frac{A}{2}, & x = L \text{ or } x = -L \\ 0, & \text{else} \end{cases}$$

Find the Fourier transform of $f(x)$.

$$\begin{aligned} \hat{f}(m) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-imx} dx \\ &= \frac{1}{2\pi} \int_{-L}^{L} A e^{-imx} dx \\ &= \frac{A}{2\pi i m} (e^{imL} - e^{-imL}) \\ &= \frac{A}{\pi m} \left( \frac{e^{imL} - e^{-imL}}{2i} \right) \\ &= \frac{A}{\pi m} \sin(mL) \\ &= \frac{AL}{\pi} \cdot \frac{\sin(mL)}{mL} \end{aligned}$$

Define the "sinc function" as sinc $= \frac{\sin z}{z}$, then

$$\hat{f}(m) = \frac{AL}{\pi} \operatorname{sinc}(mL) = \mathcal{F}[f](m).$$

Moreover, by applying inverse transform on both sides,

$$f(x) = \int_{-\infty}^{\infty} \frac{AL}{\pi} \operatorname{sinc}(mL) e^{imx} dm = \mathcal{F}^{-1}[\hat{f}](x).$$

*Notation.* Haberman textbook uses the negative exponential term for the transform, but they are equivalent by a change of variable.