Problem (1.25).
$$\frac{1}{2} +_1 \frac{7}{8} = \frac{4}{8} +_1 \frac{7}{8} = \frac{3}{8}.$$

Problem (1.26).
$$\frac{3\pi}{4} +_{2\pi} \frac{3\pi}{2} = \frac{3\pi}{4} +_{2\pi} \frac{6\pi}{4} = \frac{\pi}{4}.$$

Problem (1.31). Given the small set, we would like to find the solution exhaustively:
If $x = 0$, $0 +_7 0 = 0 \neq 3$.
If $x = 1$, $1 +_7 1 = 2 \neq 3$.
If $x = 2$, $2 +_7 2 = 4 \neq 3$.
If $x = 3$, $3 +_7 3 = 6 \neq 3$.
If $x = 4$, $4 +_7 4 = 1 \neq 3$.
If $x = 5$, $5 +_7 5 = 3$.
If $x = 6$, $6 +_7 6 = 5 \neq 3$.
We have covered all cases of $x \in \mathbb{Z}_7$, and have found that $x = 5$ is the only solution.

Problem (1.32). Again using exhaustive search:
If $x = 0$, $0 +_7 0 +_7 0 = 0 \neq 5$.
If $x = 1$, $1 +_7 1 +_7 1 = 3 \neq 5$.
If $x = 2$, $2 +_7 2 +_7 2 = 6 \neq 5$.
If $x = 3$, $3 +_7 3 +_7 3 = 2 \neq 5$.
If $x = 4$, $4 +_7 4 +_7 4 = 5$.
If $x = 5$, $5 +_7 5 +_7 5 = 1 \neq 5$.
If $x = 6$, $6 +_7 6 +_7 6 = 4 \neq 5$.
Hence, $x = 4$ is the only solution.

Problem (1.35). Due to isomorphism, we know that $\zeta \times \zeta = \zeta^2$ is isomorphic to $5 +_8 5 = 2$. Repeating this process yields:

$$\begin{aligned}
5 +_8 5 &= 2 & \zeta^2 &\leftrightarrow 2\\
2 +_8 +5 &= 7 & \zeta^3 &\leftrightarrow 7\\
7 +_8 5 &= 4 & \zeta^4 &\leftrightarrow 4\\
4 +_8 5 &= 1 & \zeta^5 &\leftrightarrow 1\\
1 +_8 5 &= 6 & \zeta^6 &\leftrightarrow 6\\
6 +_8 5 &= 3 & \zeta^7 &\leftrightarrow 3\\
3 +_8 5 &= 0 & \zeta^0 &\leftrightarrow 0\\
0 +_8 5 &= 5 & \zeta^1 &\leftrightarrow 5
\end{aligned}$$

Problem (1.37)**.** Because $\zeta \leftrightarrow 4$ implies that $\zeta \times \zeta \leftrightarrow 4 +_8 4 = 0$, and $\zeta \times \zeta \times \zeta \leftrightarrow 4 +_8 4 +_8 4 = 4$. And since isomorphism requires an one-to-one mapping between U_6 and $\mathbb{R}_6$, yet both ζ and ζ^3 map to 4, the mapping cannot be one-to-one and therefore isomorphism doesn't exist.

Problem (2.1)**.** Table 2.26 tells us that $b * d = e$, $c * c = b$, and

$$\begin{aligned}
[(a * c) * e] * a &= [c * e] * a\\
&= a * a\\
&= a
\end{aligned}$$

Problem (2.5)**.** For $*$ to be commutative, we require the table to be symmetric about the diagonal. Note that answers are bolded.

*	a	b	c	d
a	a	b	c	**d**
b	b	d	**a**	c
c	c	a	d	b
d	d	**c**	**b**	a

Problem (2.6)**.** The missing entries represents $d*a$, $d*b$, $d*c$, and $d*d$. Since we have the complete information of how the operation is defined among pairs consisting of the other three elements, we can replace d with the other elements. Notice that there is only one such pair that engendered d: $c * b = d$. Now let's consider each entry and apply associativity:

$$d * a = (c * b) * a = c * (b * a) = c * b = d.$$
$$d * b = (c * b) * b = c * (b * b) = c * a = c.$$
$$d * c = (c * b) * c = c * (b * c) = c * c = c.$$
$$d * d = (c * b) * d = c * (b * d) = c * d = d.$$

Hence, the missing entires are

d	d	c	c	d

Problem (2.7)**.**

1) Commutativity: Let $a = 1$,$b = 2$, it is easy to see that $a * b = 1 - 2 \neq 2 - 1 = b * a$. Hence $*$ is not commutative.

2) Associativity: Let $a = 1, b = 2, c = 3$. Since $(a * b) * c = (a - b) - c = (1 - 2) - 3 = -4$ and $a * (b * c) = a - (b - c) = 1 - (2 - 3) = 2$, clearly $(a * b) * c \neq a * (b * c)$ so $*$ is not associative.

Problem (2.8)**.**

1) Commutativity: Given $a, b \in \mathbb{Q}$, we have $a * b = ab + 1 = ba + 1 = b * a$ since scalar multiplication is commutative. Hence $*$ is commutative.

2) Associativity: Let $a = 1, b = \frac{1}{2}, c = \frac{1}{3}$, $(a*b)*c = \left(1 \times \frac{1}{2} + 1\right) \times \frac{1}{3} + 1 = \frac{3}{2}$ and $a * (b * c) = 1 \times \left(\frac{1}{2} \times \frac{1}{3} + 1\right) + 1 = \frac{13}{6}$. Hence $(a * b) * c \neq a * (b * c)$ and $*$ is not associative.

Problem (2.24)**.**

a) True. Consider $S_0 = \{a\}$. Since $*$ is a binary operation on any arbitrary set, it is defined on S_0 as well. It follows that $a * a \in S_0$. Since there is only one element in S_0, this forces $a * a = a$.

b) True. Since $*$ is a commutative binary operation on S, given $a, b, c \in S$, we know that $b*c \in S$, and thus $a*(b*c) = (b*c)*a$ by commutativity.

c) False. Consider S as the set of all 3×3 permutation matrices and $*$ is the matrix multiplication, which we know is associative but not commutative. Let $a = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, and $c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

LHS yields:

$$a * (b * c) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

However, the RHS yields:

$$(b * c) * a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

LHS and RHS do not equal so the statement is false.

d) False. The more abstract math gets the less important numbers are...

e) False. It should be for all $a, b \in S$.

f) True. It is easy to see that given $S_0 = \{a\}$, $a * a = a * a = a$, and $(a * a) * a = a * a = a = a * (a * a)$.

g) True. Because at least one includes exactly one.

h) True. Because at most one includes exactly one.

i) True. Because binary operation is a function and can only have one output.

j) False. As above.