# 1 Solve A Wave Equation

$$\begin{cases} \text{PDE: } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} & 0 < x < L, t > 0 \\ \text{BCs: } u(0,t) = 0 = u(L,t) & t > 0 \\ \text{ICs: } u(x,0) = U(x), \frac{\partial u}{\partial t}(x,0) = V(x) & 0 \le x \le L \end{cases}$$

We need to verify solutions that satisfy PDE and BCs form a vector space in order to use superposition principle. That is, we want to show that

$W = \{u(x,t)$ is defined on $(x,t) \in (0,L) \times (0,\infty)$ that satisfies PDE and BCs$\}$

forms a vector space.

**Proof**

Notice that $V = \{f : [0,L] \to \mathbb{R}\}$ is a vector space. We wish to show that $W$ is a subspace of $V$.

Given $u_1, u_2 \in W$, then consider $u_3(x,t) = au_1(x,t) + bu_2(x,t)$ and note that partial derivatives are linear, we have

$$\begin{aligned} \frac{\partial^2 u_3}{\partial t^2} &= \frac{\partial^2}{\partial t^2}(au_1 + bu_2) \\ a\frac{\partial^2 u_1}{\partial t^2} + b\frac{\partial^2 u_2}{\partial x^2} &= ac^2\frac{\partial^2 u_1}{\partial x^2} + bc^2\frac{\partial^2 u_2}{\partial x^2} \\ &= c^2(a\frac{\partial^2 u_1}{\partial x^2} + b\frac{\partial^2 u_2}{\partial x^2}) \end{aligned}$$

So the PDE condition is satisfied. Then $u_3(0,t) = au_1(0,t) + bu_2(0,t) = 0 + 0 = 0$ and $u_3(L,t) = au_1(L,t) + u_2(L,t) = 0 + 0 = 0$.

So the BCs condition is satisfied. Finally, since piecewise smooth functions are closed under addition and scalar multiplication, we have shown that $u_3(x,t) \in W \subseteq V$. Hence $W \le V$. $\square$

Now we assume separation of variables, $u(x,t) = F(x)G(t) \neq 0$, then the second partial derivatives are

$$F(x)G''(t) = c^2F''(x)G(t) \Rightarrow \frac{1}{c^2}\frac{G''(t)}{G(t)} = \frac{F''(x)}{F(x)} = -\lambda.$$

Notice that the eigenvalue problem is the same as the heat equation. Then the BCs become

$$u(0,t) = 0 \Rightarrow F(0)G(t) = 0 \Rightarrow F(0) = 0.$$

and

$$u(L,t) = 0 \Rightarrow F(L)G(t) = 0 \Rightarrow F(L) = 0.$$

Thus the time domain problem and eigenvalue problem become

$$G''(t) = -\lambda c^2 G(t).$$

and

$$\begin{cases} F''(x) = -\lambda F(x), \\ F(0) = 0 = F(L). \end{cases}$$

## 1.1 Eigenvalue Problem

*Case.* $\lambda < 0$, then the general solution is

$$F(x) = c_1 e^{-\sqrt{\lambda}x} + c_2 e^{\sqrt{\lambda}x}.$$

Applying the BCs,

$$F(0) = 0 \Rightarrow c_2 = -c_1 \text{ and } F(L) = 0 \Rightarrow c_1 = c_2 = 0.$$

because $e^{-\sqrt{\lambda}L} - e^{\sqrt{\lambda}L} \neq \Leftrightarrow \lambda \neq 0$. Hence this case yields the trivial solution.

*Case.* $\lambda = 0$ also yields trivial solution as before.

*Case.* $\lambda > 0$, by the exact same procedure as before, we obtain $\sqrt{\lambda} = \frac{n\pi}{L}$ and

$$F_n(x) = c_2 \sin\left(\frac{n\pi x}{L}\right) \text{ for } n = 1, 2, \ldots.$$

## 1.2 Time-Domain Problem

Similarly as above, we can obtain the general solution of $G(t)$

$$G_n(t) = d_1 \cos(\sqrt{\lambda_n} ct) + d_2 \sin(\sqrt{\lambda_n} ct) \text{ for } n = 1, 2, \ldots.$$

## 1.3 General Solution

Therefore,

$$u_n(x,t) = F_n(x)G_n(t) = \tilde{A}_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) + \tilde{B}_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) \text{ for } n = 1, 2, \ldots.$$

Now by superposition principle, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cdot u_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

Now applying the ICs to find the coefficients:

$$U(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right).$$

This is just FSS. Applying the projection formula,

$$A_n = \frac{2}{L}\int_0^L U(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

The other IC requires us to do term-by-term differentiation:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} A_n\left(-\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)+B_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right).$$

Therefore, the other IC also yields a FSS,

$$V(x) = u_t(x,0) = \sum_{n=1}^{\infty} B_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

By projection,

$$B_n = \frac{2}{n\pi c}\int_0^L V(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

## 1.4 Convergence

Note that the wave equation doesn't have an exponentially-decaying term as the heat equation. We have a spatial wave and a temporal wave. Thus, the convergence depends on how $A_n, B_n$ behave as $n \to \infty$, which in turn depends on the initial position and velocity.

While this is a drawback of the FS solution of the wave equation, it motivates d'Alembert's solution using traveling waves:

$$u(x,t) = f(x-ct) + g(x+ct)..$$