1 Matrices

1.1 Round-off error

Ill-conditioned: if the columns are almost dependent. A small change in the RHS can yield drastically different solutions as an almost parallel line shifted.

If γ is the angle between two linear equations, then

$$\tan\gamma = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}a_{12} + a_{21}a_{22}}.$$

As this quantity $\to 0$, $\gamma \to 0$.

Symptoms of ill-conditioned:

- if $|detA| << \max|a_{ij}| or \max|b_i|$
- poor approximation solutions with small residuals
- elements of A^{-1} are large compared to elements of

Signs of well-conditioned: —diagonal elements—¿¿—off-diagonal elements—

To tackle this problem:

- want the largest coefficient in all rows to be comparable, rescale the rows
- rearrange the rows to place the largest elements on diagonal

1.2 Gauss-Seidel Method

$$\begin{aligned}
x_i &= \frac{b_i}{a_{ii}} \\
&\dots \\
x_i^* &= \frac{b_i - \sum_{j=1,j\neq i}^{n} a_{ij}x_j}{a_{ii}} \\
x_i* &= \frac{b_i - \sum_{j=1}^{i-1} a_{ij}x_j^* - \sum_{j=i+1}^{n} a_{ij}x_j^*}{a_{ii}}
\end{aligned}$$

1.3 Residuals

$$\begin{aligned}
r &= b - Ax \\
r_i &= b_i - \sum_{j=1}^{n} a_{ij}x_j \\
&= b_i - \sum_{j=1}^{i-1} a_{ij}x_j - a_{ii}x_i - \sum_{j=i+1}^{n} a_{ij}x_j \\
&= b_i - a_{ii}x_i - (\sum_{j=1}^{i-1} a_{ij}x_j * + \sum_{j=i+1}^{n} a_{ij}x_j) \\
&= a_{ii}x_i^* - a_{ii}x_i \\
x_i^* &= x_i + \frac{r_i}{a_{ii}}
\end{aligned}$$

1.4 Relaxation Methods

$$x_i^* = x_i + \omega\frac{r_i}{a_{ii}}.$$

where ω is the relaxation constant. If $0 < \omega < 1$, it is under-relaxation; if $\omega > 1$, it is over-relaxation. Some systems do not converge unless we use $0 < \omega < 1$. When using systems to solve PDEs can use over-relaxation to speed up convergence.

1.5 Matrix Inversion

1.5.1 Iterative Method

For the problem $x \cdot a = 1$, we can solve it using Newton's method on $f(x) = \frac{1}{x} - a = 0$. Then we have

$$x_{i+1} = x_i(2 - ax_i).$$

Can we extend this finding to matrices? Yes but it's only guaranteed to converge if all eigenvalues of $I - Ax$ $|\lambda_i| < 1$.

$$X_{i+1} = X_i(2I - AX_i).$$

This X_i will eventually give us A^{-1}. The error is also $\mathcal{O}(h^2)$, for each entry.

1.6 Eigenvalues and Eigenvectors

The set of all eigenvalues are called spectrum. And $|\lambda_{\max}|$ is called the spectral radius.

1.6.1 Gershgorin Theorem

Let λ be an eigenvalue of an arbitrary matrix $A = (a_{ij})$. Then

$$|a_{ii} - \lambda| \leq \sum_{j=1, j\neq i}^{n} |a_{ij}|.$$

The eigenvalues lie in the union of the Gershgorin disks (Gershgorin domain) centered at the diagonal entries with radius of the sum of the off-diagonal entries of that row.

1.6.2 Collatz Theorem

Let $A = (a_{ij})$ be a real square matrix with positive elements, and x be any real vector with positive components, y be the components of $y = Ax$. Then the closed interval bounded by $\left|\frac{y_i}{x_i}\right|_{\min}$ and $\left|\frac{y_i}{x_i}\right|_{\max}$ contains at least an eigenvalue of A.

1.6.3 Rayleigh Quotient

Given a real symmetric matrix A, a real and non-zero vector x, compute

$$\begin{aligned} x_i &= Ax_{i-1} \\ m_i &= x_i^T x_i TODO \end{aligned}$$

Then $q = \frac{m_i}{m_{i-1}} = \frac{x_{i-1}^T x_i}{x_{i-1}^T x_{i-1}} TODO$ is an approximate to an eigenvalue of A. And the error $\epsilon = q - \lambda$ is

$$|\epsilon| \leq \sqrt{\frac{m_2}{m_0} - q^2} = \sqrt{\frac{x_i^T x_i}{x_{i-1}^T x_{i-1}} - \frac{x_{i-1}^T x_i}{x_{i-1}^T x_{i-1}}} TODO.$$

1.6.4 Positive-Definite Matrix