Midterm 1

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Problem (1).

a) Given $A, B \in \mathcal{G}$, by definition of $\mathcal{G}$ there exist $A_1, B_1 \in \mathcal{F}_1$ and $A_2, B_2 \in \mathcal{F}_2$ such that $A = A_1 \cap A_2$, $B = B_1 \cap B_2$. Then

$$\begin{aligned} A \cap B &= (A_1 \cap A_2) \cap (B_1 \cap B_2) \\ &= (A_1 \cap B_1) \cap (A_2 \cap B_2) \end{aligned}$$

Since σ-fields $\mathcal{F}_1, \mathcal{F}_2$ are closed under intersections, it follows that $A_1 \cap B_1 \in \mathcal{F}_1, A_2 \cap B_2 \in \mathcal{F}_2$. Hence $A \cap B \in \mathcal{G}$, and $\mathcal{G}$ is a π-system by definition.

b) We wish to show double containment. Given $a \in \mathcal{G}$, there exists $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2$ such that $a \in A_1 \cap A_2$. Since $\mathcal{F}_1, \mathcal{F}_2$ are σ-fields, the complements $A_1^c \in \mathcal{F}_1, A_2^c \in \mathcal{F}_2 \Rightarrow A_1^c \cup A_2^c \in \mathcal{F}_1 \cup \mathcal{F}_2$. Then $(A_1^c \cup A_2^c)^c = A_1 \cap A_2 \in \sigma(F_1 \cup F_2) \Rightarrow \mathcal{G} \subseteq \sigma(\mathcal{F}_1 \cup \mathcal{F}_2)$. Since a σ-field is also a λ-system, by Dynkin's π-λ theorem, $\sigma(\mathcal{G}) \subseteq \sigma(\mathcal{F}_1 \cup \mathcal{F}_2)$.

Now let's show the other direction. Given $b \in \mathcal{F}_1 \cup \mathcal{F}_2$, there exist $B_1 \in \mathcal{F}_1, B_2 \in \mathcal{F}_2$ such that $b \in B_1 \cup B_2 = (B_1^c \cap B_2^c)^c$. Clearly $B_1^c \in \mathcal{F}_1, B_2^c \in \mathcal{F}_2$, so $B_1^c \cap B_2^c \in \mathcal{G} \Rightarrow (B_1^c \cap B_2^c)^c \in \sigma(\mathcal{G}) \Rightarrow b \in \sigma(\mathcal{G}) \Rightarrow \mathcal{F}_1 \cup \mathcal{F}_2 \subseteq \sigma(\mathcal{G})$. Again by Dynkin's Theorem, $\sigma(\mathcal{F}_1 \cup \mathcal{F}_2) \subseteq \sigma(\mathcal{G})$. Therefore, we obtain $\sigma(\mathcal{G}) = \sigma(\mathcal{F}_1 \cup \mathcal{F}_2)$.

Problem (2). We would like to prove by contradiction. Suppose there exists an atom $A \in \mathcal{F}$ such that $P(A) > 0$ and $\forall\, B \in \mathcal{F}, B \subseteq A$, we have $P(B) = 0$ or $P(B) = P(A)$. Let's define

$$B_n = \begin{cases} A_n & \text{if } P(A_n \cap A) = P(A) \\ A_n^c & \text{if } P(A_n^c \cap A) = P(A) \end{cases}$$

Notice that since each B_n only depends on A_n, and A_n are independent, we know B_n are independent too. This definition intuitively means that all the

B_n always contain the mass of A. In other words,

$$\begin{aligned} P(A) &\leq P(B_1 \cap B_2 \cap \dots) \\ &= P(B_1)P(B_2)\dots \text{ by independence} \\ &= p_1 \cdot p_2 \cdot \dots \end{aligned}$$

where we let $p_n = P(B_n)$ for convenience. Recall that $1 - x \leq e^{-x} \Rightarrow x \leq e^{-(1-x)}$, applying to each p_n and we have

$$\begin{aligned} P(A) &\leq e^{-(1-p_1)}e^{-(1-p_2)}\dots \\ \ln P(A) &\leq -\sum_{n=1}^{\infty}(1-p_n) \qquad \text{by monotonicity of } \ln \\ \sum_{n=1}^{\infty}(1-p_n) &\leq -\ln P(A) = \ln \frac{1}{P(A)} \end{aligned}$$

Since $P(A) > 0$, $\ln \frac{1}{P(A)}$ is a constant so $\sum_{n=1}^{\infty}(1-p_n) < \infty$. Now consider

$$\sum_n \min\{P(A_n), 1-P(A_n)\} = \sum_n \min\{P(B_n), 1-P(B_n)\} \leq \sum_n (1-p_n) < \infty.$$

This contradicts with our assumption, therefore we prove that the probability space must be non-atomic.

Problem (3)**.** Naturally we apply Borel-Cantelli Lemma (i) and obtain $P\left(\limsup_n (A_n \cap A_{n+1}^c)\right) = 0$. Also by Theorem 4.1, $P\left(\liminf_n A_n\right) \leq \liminf_n P(A_n) = 0 \Rightarrow P\left(\liminf_n A_n\right) = 0$. By homework 2.6,

$$\begin{aligned} P\left(\limsup_n A_n\right) &= P\left(\limsup_n (A_n \cap A_{n+1}^c) \cup \liminf_n A_n\right) \\ &\leq P\left(\limsup_n (A_n \cap A_{n+1}^c\right) + P\left(\liminf_n A_n\right) \text{ by subadditivity} \\ &= 0 + 0 = 0 \end{aligned}$$

Again by Theorem 4.1, $\limsup_n P(A_n) \leq P\left(\limsup_n A_n\right) = 0 \Rightarrow \limsup_n P(A_n) = 0 = \liminf_n P(A_n) \Rightarrow \lim_{n\to\infty} P(A_n) = 0$ by definition of the limit of reals.

Problem (4)**.** We would like to show that $\{\omega : X(\omega) \leq x\} \in \mathcal{F}\ \forall\ x \in \mathbb{R}$.

Case. $x \geq 0$, then

$$\{\omega : X(\omega) \leq x\} = \{\omega : \lim_{n\to\infty} X_n(\omega) \leq x\} \cup \{\omega : \lim_{n\to\infty} X_n(\omega) \text{ diverges}\}$$

In class we have already show that the first set is in $\mathcal{F}$. Notice that $\{\omega : \lim_{n\to\infty} X_n(\omega) \text{ diverges}\} = \{\omega : \lim_{n\to\infty} X_n(\omega) \text{ exists}\}^c$, which we showed in class that it is in the tail σ-field. Since the tail σ-field is the smallest σ-field containing the tail event, and the tail event is clearly contained in $\mathcal{F}$, the tail σ-field must be a subset of $\mathcal{F}$. It follows that $\{\omega : \lim_{n\to\infty} X_n(\omega)\} \in \mathcal{F}$, which implies the complement $\{\omega : \lim_{n\to\infty} X_n(\omega) \text{ diverges}\} \in \mathcal{F}$. Therefore, the union of two sets, $\{\omega : X(\omega) \leq x\} \in \mathcal{F}$.

Case. $x < 0$, then $\{\omega : X(\omega) \leq x\} = \{\omega : \lim_{n\to\infty} X_n(\omega) \leq x\} \in \mathcal{F}$ as shown in class.

Hence for all $x \in \mathbb{R}$, we show that $\{\omega : X(\omega) \leq x\}$, proving that X is a random variable.