Corollary

Let $\mathcal{P}$ be a π-system. Suppose that μ_1, μ_2 are finite measures on $\sigma(\mathcal{P})$ and that Ω is a countable union of sets in $\mathcal{P}$. If μ_1, μ_2 agree on $\mathcal{P}$, they agree on $\sigma(\mathcal{P})$.

Intuition. We sort of replace the σ-finiteness of μ with Ω as a countable union of sets.

Definition: general outer-measure

Let Ω be a non-empty set. An **outer measure** on Ω is a function

$$\mu^* : \text{ Power Set of } \Omega \to [0, \infty]$$

that satisfies

(i) $\mu^*(\emptyset) = 0$

(ii) monotone

(iii) countable subadditivity

Definition

Let μ^* be an outer measure on Ω. A set $A \subseteq \Omega$ is said to be **μ^*-measurable** if

$$\mu^*(A \cap E) + \mu^*(A^c \cap E) = \mu^*(E) \ \forall \ E \subseteq \Omega.$$

Equivalently, by finite subadditivity of μ^*:

$$\mu^*(E) = \mu^*((A \cap E) \cup (A^c \cap E)) \leq \mu^*(A \cap E) + \mu^*(A^c \cap E).$$

Theorem

A set A is μ^*-measurable if $\mu^*(A \cap E) + \mu^*(A^c \cap E) \leq \mu^*(E)$.

Let $\mathcal{M}(\mu^*) =$ the collection of μ^*-measurable sets.

Theorem: 11.1

μ^* is an outer measure implies:

(i) $\mathcal{M}(\mu^*)$ is a σ-field.

(ii) μ^*, restricted to $\mathcal{M}(u^*)$, is a measure on $\mathcal{M}(\mu^*)$

(The same as the 5 lemma proof for probability outer measure.)

Theorem: 11.2

A measure μ on a field has an extension to the generated σ-field.

Claim. If the original measure is finite or σ-finite on the field (cover Ω with a countable collection $\mathcal{F}$-sets with finite measure), then the extension is unique.

Definition: semiring

A class $\mathcal{A}$ of subsets of Ω is called a **semiring** if

(i) $\emptyset \in \mathcal{A}$.

(ii) closed under finite intersections.

(iii) For any $A, B \in \mathcal{A}$ such that $A \subseteq B$, there exists disjoint sets $A_1, A_2, \ldots \in \mathcal{A}$ such that $B \setminus A = \bigcup_{i=1}^{n} A_i$. Equivalently, $B = A \cup A_1 \cup \ldots \cup A_n$ all disjoint.

Note. For (iii), think of a picture with A in B and the "ring" formed by the boundaries of A and B can be partitioned into finite disjoint sets.

A semiring doesn't have additive inverses compared to a ring.

In the set theory context, addition becomes $A \Delta B$ (symmetric difference) and multiplication becomes $A \cap B$.

Example. Any field is a semiring.

(i) Field (i) and (ii)

(ii) Field (ii) and (iii)

(iii) Take any $A, B \in \mathcal{F}$ with $A \subseteq B$, then

$$\begin{aligned} B &= A \cap (B \setminus A) \\ &= A \cup (B \cap A^c) \in \mathcal{F} \end{aligned}$$

Example. $\mathbb{R}, \mathcal{A} =$ collection of half-open intervals of the form $\{x : a < x \leq b\}$ for any $a, b \in \mathbb{R}$. Then $\mathcal{A}$ is a semiring.

(i) Take $b < a$.

(ii) $A, B \in A \Rightarrow A = \{x : a < x \leq b\}, B = \{x : c < x \leq d\}$. Then $a \cap B = \{x : \max(a, c) < x \leq \min(d, b)\}$

(iii) Take $A, B \in \mathcal{A}$ with $A \subseteq B$, then $A = (a, b], B = (c, d]$ with $c \leq a < b \leq d$. It follows that $B \setminus A = (c, a]$
$cup(b, d]$ disjoint.

Example. $\mathbb{R}^k$ and half-open rectangles $\{(x_1, \ldots, x_k) : a_i < x_i \leq b_i \text{ for } i = 1, \ldots, k\}$ is a semiring because any Cartesian product of semiring is a semiring.

Theorem: 11.3

Suppose that μ is a measure on a semiring $\mathcal{A}$. Then μ extends to a measure on the $\sigma(\mathcal{A})$.

Note. Theorem 11.2 is a special case of this.

Only need finite additivity of μ, as opposed to countable additivity.

Proof

Claim. μ is monotone on $\mathcal{A}$.

We need to prove this because a semiring might not contain complements, which is what we used to prove monotonicity previously. Since $\mathcal{A}$ is a semiring, there exist disjoint $A_1, \ldots, A_n$ such that $B = A \cup A_1 \cup \ldots \cup A_n$, where all these sets are in $\mathcal{A}$. By finite additivity of μ,

$$\mu(B) = \mu(A) + \sum_{i=1}^{n} \mu(A_n) \geq \mu(A).$$

Define an outer measure, for any $A \subseteq \Omega$. Using infimum like probability measure might not be well-defined because there might not exists $\mathcal{A}$-covering of A, since $\mathcal{A}$ might not contain Ω, yet previously we can choose $A_1 = \Omega, A_2 = \emptyset, \ldots$ to cover A. Therefore, we modify the definition to be

$$u^*(A) = \begin{cases} \infty, & \text{if a } \mathcal{A} \text{ covering doesn't exist for } A \\ \inf\left\{\sum_n \mu(A_n) : A_n \in \mathcal{A}, A \subseteq \bigcup_n A_n\right\}, & \text{otherwise} \end{cases}$$

Claim. μ^* is an outer measure.

Let $\mathcal{M}(\mu^*)$ be the class of μ^*-measurable sets.

Claim. $\mathcal{A} \subseteq \mathcal{M}(\mu^*)$.

Take any $A \in \mathcal{A}$ and $E \in \Omega$, we want to show that

$$\mu^*(A \cap E) + \mu^*(A^c \cap E) \leq \mu^*(E) \tag{1}$$

Case. $\mu^*(E) = \infty$. This makes (1) trivially true.

Case. $\mu^*(E) < \infty$. This implies that there exists a $\mathcal{A}$-covering of E. Let $\varepsilon > 0$. Choose $\mathcal{A}$-sets $A_1, A_2, \ldots$ such that $E \subseteq \bigcup_n A_n$ and

$$\sum_n \mu(A_n) < \mu^*(E) + \varepsilon$$

by the definition of infimum. Since $\mathcal{A}$ is a semiring, and $A, A_n \in \mathcal{A}$, define $B_n \coloneqq A \cap A_n \in \mathcal{A} \ \forall\ n \in \mathbb{N}$. Note that $B_n \subseteq A_n$. So by definition of a semiring, there exists $C_{n,1}, C_{n,2}, \ldots, C_{n,m_n}$ such that

$$A_n \setminus B_n = \bigcup_{i=1}^{m_n} C_{n,i} \ \forall\ n \in \mathbb{N}.$$

Notice that $A_n \cap A^c = A_n \setminus B_n = \bigcup_{i=1}^{m_n} C_{n,i}$ by how we define B_n. Then $A_n = B_n \cup \bigcup_{i=1}^{m_n} C_{n,i}$. Thus by finite additivity of μ,

$$\mu(A_n) = \mu(B_n) + \sum_{i=1}^{m_n} \mu(C_{n,i}).$$

Since E is covered by A_n, $A \cap E \subseteq A \cap \bigcup_n A_n = \bigcup_n (A \cap A_n) = \bigcup_n B_n$, and $A^c \cap E \subseteq A^c \cap \bigcup_n A_n = \bigcup_n (A^c \cap A_n) = \bigcup_n \bigcup_{i=1}^{m_n} C_{n,i}$. Notice both are $\mathcal{A}$-covering of the sets. By definition of infimum,

$$\begin{aligned}
\mu^*(A \cap E) + \mu^*(A^c \cap E) &\leq \sum_n \mu(B_n) + \sum_n \sum_{i=1}^{m_n} \mu(C_{n,i}) \\
&= \sum_n \left(\mu(B_n) + \sum_{i=1}^{m_n} \mu(C_{n,i}) \right) \\
&= \sum_n \mu(A_n) \\
&< \mu^*(E) + \varepsilon
\end{aligned}$$

Take $\varepsilon \to 0$, we have (1) as required. Thus, $\mathcal{A} \subseteq \mathcal{M}(\mu^*)$.

Claim. μ^* is a measure on $\mathcal{M}(\mu^*)$

by Theorem 11.1.

Claim. μ and μ^* agree on $\mathcal{A}$.

Take any $A \in \mathcal{A}$.

Since $A_1 = A, A_2 = \emptyset, \ldots$ is a $\mathcal{A}$-covering of A, $\mu^*(A) \le \mu(A)$ follows from $\mu^*(A)$ being an infimum.

For the other direction, we want to show that $\mu(A)$ is a lower bound. let (A_n) be an $\mathcal{A}$-covering of A. This is well-defined because we at least have the covering above. Since $\mathcal{A}$ is a semiring, $A \cap A_n \in \mathcal{A} \ \forall \ n \ge 1$. Since $A \subseteq \bigcup_n A_n$, we have

$$\bigcup_{n=1}^{\infty} (A \cap A_n) = A \cap \bigcup_{n=1}^{\infty} A_n = A.$$

By countable subadditivity of μ,

$$\mu(A) = \mu\left(\bigcup_{n=1}^{\infty} A \cap A_n\right) \le \sum_{n=1}^{\infty} \mu(A \cap A_n).$$

Since $\mu^*(A)$ is the infimum of the RHS, and $\mu(A)$ is a lower bound of the RHS, we have $\mu(A) \le \mu * (A)$.

Since $\mathcal{A} \subseteq \mathcal{M}(u^*)$ and $\mathcal{M}(\mu^*)$ is a σ-field by Theorem 11.1, $\mathcal{A} \subseteq \sigma(\mathcal{A}) \subseteq \mathcal{M}(\mu^*)$. Since μ^* is a measure on $\mathcal{M}(\mu^*)$, then μ^* is a measure on $\sigma(\mathcal{A})$ and it agrees with μ for all sets in $\mathcal{A}$.

$\square$