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Lemma: 1

$\mathcal{M}$ is a field.

Proof

(i) Since $\quad \forall E \subset \Omega$,

$$\begin{aligned} P^*(\emptyset \cap E) + P^*(\emptyset^c \cap E) &= P^*(\emptyset) + P^*(E) \\ &= P^*(E) \end{aligned}$$

So $\emptyset \in \mathcal{M}$.

(ii) $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$ by the symmetry of the definition of $\mathcal{M}$.

(iii) Given $A_1, A_2 \in \mathcal{M}$. Want $A_1 \cup A_2 \in \mathcal{M}$. We can show that $A_1 \cap A_2 \in \mathcal{M}$.

$$\begin{aligned} &P^*((A_1 \cap A_2) \cap E) + P^*((A_1 \cap A_2)^c \cap E) \\ &= P^*(A_1 \cap A_2 \cap E) + P^*((A_1 \cap E \cap A_2^c) \\ &\quad \cup (A_2 \cap E \cap A_1^c) \cup (A_1^c \cap A_2^c \cap E)) \qquad (1) \\ &\leq P^*(A_1 \cap A_2 \cap E) + P^*(A_1 \cap E \cap A_2^c) \\ &\quad + P^*(A_2 \cap E \cap A_1^c) + P^*(A_1^c \cap A_2^c \cap E) \\ &= A + B + C + D \end{aligned}$$

$$\begin{aligned} A + B &= P^*(A_2 \cap (A_1 \cap E)) + P^*(A_2^c \cap (A_1 \cap E)) \\ &\leq P^*(A_1 \cap E) \text{ since } A_2 \in \mathcal{M} \\ C + D &= P^*(A_2 \cap (E \cap A_1^c)) + P^*(A_2^c \cap (E \cap A_1^c)) \\ &\leq P^*(A_1^c \cap E) \end{aligned}$$

So $(1) \leq (A + B) + (C + D) \leq P^*(A_1 \cap E) + P^*(A_1^c \cap E) \leq P^*(E)$ since $A_1 \in \mathcal{M}$.

$\square$

Lemma: 2

Countable additivity holds for sets in $\mathcal{M}$.

Note. $A_1, A_2, \ldots \in \mathcal{M}$, disjoint then $P^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P^*(A_n)$

Proof

1) Prove finite additivity: given $A_1, \ldots \in \mathcal{M}$, disjoint, and $E \in \Omega$. Since $(E \cap A_1) \cup (E \cap A_2)$ are disjoint, $A_1 \cap [E \cap (A_1 \cup A_2)] = A_1 \cap E$. Also $A_2 \subset A^c$.

$$\begin{aligned} P^*(E \cap (A_1 \cup A_2)) &= P^*(A_1 \cap [E \cap (A_1 \cap A_2)]) + P^*(A_1^c \cap [E \cap (A_1 \cup A_2)]) \\ &= P^*(A_1 \cap E) + P^*(A_2 \cap E) \end{aligned}$$

2) Countable additivity: $A_1 A_2 \ldots \in \mathcal{M}$, disjoint

$$\begin{aligned} P^*(E \cap \bigcup_{n=1}^{\infty} A_n) &= P^*(\bigcup_{n=1}^{\infty} (E \cap A_n)) \\ &\leq \sum_{n=1}^{\infty} P^*(E \cap A_n) \end{aligned}$$

On the other hand:

$$\begin{aligned} P^*(E \cap \left(\bigcup_{n=1}^{\infty} A_n\right)) &\geq P^*\left(E \cap \bigcup_{n=1}^{\infty} A_n\right) \\ &= \sum_{n=1}^{\infty} P^*(E \cap A_n) \text{ by 1) above} \end{aligned}$$

Let $n \to \infty$ $P^*(E \cap \bigcup_{n=1}^{\infty} A_n) \geq \sum_{n=1}^{\infty} P^*(E \cap A_n)$

$\square$

Lemma: 3

$\mathcal{M}$ is a σ-field.

Proof

We only need to show that it is closed under countable unions. Given $A_1, A_2, \ldots \in \mathcal{M}$. Let $A = \bigcup_{n=1}^{\infty} A_n$. We want to show that $A \in \mathcal{M}$.

Assume A_1, A_2.. are disjoint. If not write $A = A_1 \cup (A_2 \setminus A_1) \cap \ldots$ which is a disjoint union.

Let $B_m := \bigcup_{n=1}^{m} A_n$. Note that $B_m \in \mathcal{M}$ since $\mathcal{M}$ is a field. Notice

$B_m \subset \bigcup_{n=1}^{\infty} A_n = A \Rightarrow A^c \subset B_m^c$. So $\quad \forall \quad E \subset \Omega$,

$$
\begin{aligned}
P^*(E) &= P^*(B_M \cap E) + P^*(B_m^c \cap E) \\
&= P^*\left(\bigcup_{n=1}^{m} (A_n \cap E)\right) + P^*(B_m^c \cap E) \\
&\geq \sum_{n=1}^{\infty} P^*(A_n \cap E) + P^*(A^c \cap E)
\end{aligned}
$$

Let $m \to \infty$,

$$
\begin{aligned}
P^*(E) &\geq \sum_{n=1}^{\infty} P^*(A_n \cap E) + P^*(A^c \cap E) \\
&= P^*\left(\bigcup_{n=1}^{\infty} (A_n \cap E)\right) + P^*(A^c \cap E) \\
&= P^*\left(\bigcup_{n=1}^{\infty} A_n \cap E\right) + P^*(A^c \cap E) \\
&= P^*(A \cap E) + P^*(A^c \cap E)
\end{aligned}
$$

Hence $A \in \mathcal{M}$. □

Lemma: 4

$\mathcal{F}_0 \subset \mathcal{M}$.

Lemma: 5

$A \in \mathcal{F}_0 \Rightarrow P^*(A) = P(A)$.

Since $P(\Omega) = 1$, by Lemma 5, $P^*(\Omega) = 1$. Now we have $P^*(\emptyset) = 0, P^*(\Omega) = 1, \emptyset \subset A \subset \Omega \Rightarrow 0 \leq P^*(A) \leq 1$, and $P*$ is countably additive on $\mathcal{M}$. It follows that $P*$ is a probability measure on $\mathcal{M}$ and $P* = P$ for set on $\mathcal{F}_0 \subset \mathcal{M}$.