Abstract Algebra

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0.1 Complex Numbers

0.1.1 Cartesian Coordinates

$a+bi, a, b \in \mathbb{R}, i^2 = -1$ It is convenient to visualize it on a plane. For equivalence, $c + di = e + fi \Rightarrow c = e, d = f$.

$$(a + bi)(c + di) = ac + bic + adi + bidi = (ac - bd) + (ad + bc)i.$$

Complex number multiplication is distributive, commutative, and associative. In division, simplify denominator using complex conjugate.

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.$$

0.1.2 Polar Coordinates

$$e^{i\theta} = \cos\theta + \sin\theta$$
$$re^{i\theta} = r\cos\theta + r\sin\theta$$

$$(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = r_1r_2e^{i(\theta_1+\theta_2)}$$

$$\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

Multiplication is easier in exponential form.

$$r_1e^{i\theta_1} = r_2e^{i\theta_2} \Rightarrow r_1 = r_2, \theta_1 - \theta_2 = 2k\pi, k \in \mathbb{Z}.$$

Does $x^5 = 12817$ have a real root? Use the horizontal line test. Yes, every real number has a unique odd number root.

Every nonzero complex number has precisely n complex roots. The root is evenly spaced around the circle centered at the origin with such radius.

Example: Find all solutions of $z^2 = i$.
Let's use the exponential form:

$$\left(re^{i\theta}\right)^2 = i$$
$$\left(r^2e^{2i\theta}\right) = 1e^{i\frac{\pi}{2}}$$

So we require: $r^2 = 1$ and $2\theta - \frac{\pi}{2} = 2k\pi$. Since there are only two roots, we increment k once. So $k = 0, 1$.

0.1.3 Roots of Unity

Complex number 1: $1 + 0i$

Example. $z^5 = 1$. $z = 1$ is a solution. $\zeta = 1e^{\frac{2\pi}{5}}, \ldots$.

0.1.4 Modular Arithmetic

Example. For clocks, $11 + 3 =_{12} 2$. $a =_{12} b$ means $a = b \mod 12$.

Example. $6 + 8 =_{8.5} 5.5$.

$=_k$ is an equivalence relation, where $k \in \mathbb{Z}$.

Isomorphism between multiplying argument and modulo adding.

Example. Find all solutions of $x +_{2\pi} +_{2\pi} x +_{2\pi} x +_{2\pi} x = 0$, where $x \in [0, 2\pi)$.