Midterm 2 covers HW 5-8.

1 Rings

Definition: ring

Let R be a set and define two binary operations, $+_R, \times_R$ as following:

$+_R$: R is an abelian group under $+_R$.

1) $+_R$ is a binary operation.
2) $+_R$ is associative.
3) $+_R$ has an identity, O_R.
4) Everything element r has an additive inverse, $-r$.
5) $+_R$ is commutative.

$\times_R$: R is closed under an associative multiplication.

6) $\times_R$ is a binary operation.
7) $\times_R$ is associative.

Axioms governing how the two operations interact.

8) left distributivity law:

$$a \times_R (b +_R c) = a \times_R b + a \times_R c.$$

9) right distributive law:

$$(a +_R b) \times_R c = a \times_R c +_R b \times_R c.$$

Example. $(\mathbb{Z}, +_R, \times_R)$ the ring of integers. This is a ring because we assume the axioms are true for addition and multiplication of complex numbers. The same goes for rational, real, and complex numbers.

Additional properties they have:

- commutative multiplication: "commutative ring".
- multiplicative identity, I_R, exists: "ring with identity".

Example. $2\mathbb{Z}$, even integers under the usual operations. It is commutative but doesn't have multiplicative identity. Suppose $I_R = 2k$, then $2k \times 2 \Rightarrow 4k = 2 \Rightarrow k = \frac{1}{2} \notin 2\mathbb{Z}$.

Example. $(\mathbb{Z}_6, +_6, \times_6)$ is a ring (prove by quotient ring). A subset $\{0, 3\}$.

$+_6$	0	3
0	0	3
3	3	0

$\times_6$	0	3
0	0	0
3	0	3

The subring is commutative and has an identity of 3, even though the identity from the ring is not here.

Example. $M_n(\mathbb{R})$ $n \times n$ matrices with real entries under matrix addition and matrix multiplication. It is in fact a vector space of order n^2. This is also a ring. So are $M_n(\mathbb{Z}), M_n(\mathbb{Q}), M_n(\mathbb{C})$.

Are they commutative? No for $n \geq 2$. Use the usual counterexample.

Do they have an identity? Yes, I_n.

Note. $M_n(2\mathbb{Z})$: not commutative and no identity.