Definition: random variable

$(\Omega, \mathcal{F}, P)$. A **random variable** X is a measurable function from $\Omega \to \mathbb{R}$. That is, for any $A \in \mathcal{B}(\mathbb{R})$,

$$X^{-1}(A) := \{\omega : X(\omega) \in A\} \in \mathcal{F}.$$

Theorem

X is a random variable if and only if $\{\omega : X(\omega) \leq x\} \in \mathcal{F} \; \forall \; x \in \mathbb{R}$.

Proof

($\Rightarrow$): the set $(-\infty, x]$ is in the Borel set.

($\Leftarrow$): show that $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{A} = \{A \subseteq \mathbb{R} : X^{-1}(A) \in \mathcal{F}\}$ $\square$

Claim. For a simple r.v. X,

$$\{\omega : X(\omega) \leq x\} \in \mathcal{F} \; \forall \; x \in \mathbb{R} \Leftrightarrow \{\omega : X(\omega) = x\} \; \forall \; x \in \mathbb{R}.$$

Proof

($\Rightarrow$): intersection of a set and a complement of another set is still in $\mathcal{F}$.
($\Leftarrow$): $\{\omega : X(\omega) \leq x\} = \bigcup_{y \leq x} \{\omega : X(\omega) = y\} \in \mathcal{F}$. Here we restrict y to be in the range of x. $\square$

Definition: measurable function

A function $f : \mathbb{R} \to \mathbb{R}$ is **(Borel) measurable** if

$$f^{-1}(A) \in \mathcal{B}(\mathbb{R}) \; \forall \; A \in \mathcal{B}(\mathbb{R}).$$

Theorem

Any composition of measurable functions is measurable. In particular, given $X : \Omega \to \mathbb{R}$ measurable and $f : \mathbb{R} \to \mathbb{R}$ measurable, then $Y = f(X)$ is measurable wrt $\mathcal{F}$, *i.e.* Y is a r.v.

Proof

Take $A \in \mathcal{B}(\mathbb{R})$, want to show that $(f \circ X)^{-1}(A) \in \mathcal{F}$. Note that $(f \circ X)^{-1}(A) = X^{-1}(f^{-1}(A)) \in \mathcal{F}$ since f^{-1} yields another Borel set by definition of measurable function and X^{-1} of a Borel set is still in $\mathcal{F}$ by definition of X being a r.v. $\square$

Note. Given a constant c, $x + c, cX, X^2, |X|$ are all measurable.

Proof: $|X|$

$f(x) = |x|$. Given $y \in \mathbb{R}$, want to show $\{x : f(x) \leq y\} \in \mathcal{B}(\mathbb{R})$. $\square$

Example. $X + Y, X - Y, XY$ are measurable.

Proof: $X + Y$

$$\begin{aligned} \{X + Y \leq x\} &= \{\omega : (X + Y)(\omega) \leq x\} \\ &= \{\omega : X(\omega) + Y(\omega) \leq x\} \\ &= \bigcup_{r \in \mathbb{R}} (\{X \leq r\} \cap \{Y \leq x - r\}) \end{aligned}$$

Claim. can restrict the union to rational number because $\mathbb{Q}$ is dense in $\mathbb{R}$.

Then the union becomes countable, hence it's in $\mathcal{F}$. $\square$

Proof: XY

$XY = \frac{1}{2}[(X + Y)^2 - X^2 - Y^2]$ $\square$