0.1 2nd order ODE

Example. Solve

$$\frac{d^2y}{dx^2} = \lambda y.$$

The general solution is:

$$y(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}.$$

Or we can write it as:

$$y(x) = c_1 \cosh(\sqrt{\lambda}x) + c_2 \sinh(\sqrt{\lambda}x).$$

Note. Hyperbolic functions have easy derivatives and nice for initial conditions.

Definition: linear independence

The functions $y_1(x), \ldots, y_n(x)$ are **linearly independent** if

$$c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) = 0 \Rightarrow c1 = c2 = \ldots = c_n = 0.$$

0.2 The complex plane

The *modulus* of a complex number $a + bi$ is

$$|z| = \sqrt{z \cdot \overline{z}} = \sqrt{a^2 + b^2}.$$

0.3 Euler's formula

Proof

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \ldots \\
&= 1 + i\theta + \frac{-\theta^2}{2!} + \frac{-i\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots \\
&= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots) + i(\theta - \frac{\theta^3}{3!} + \ldots) \\
&= \cos(\theta) + i\sin(\theta)
\end{aligned}$$

$\square$

Note. $\left|e^{i\theta}\right| = 1$ so it's on the unit circle. Moreover,

$$z = a + ib = \rho\cos(\theta) + i\rho\sin(\theta) = \rho e^{i\theta}.$$

where $\rho = \sqrt{a^2 + b^2}$ and $\tan(\theta) = \frac{b}{a}$.

1 Fourier Series and Orthogonal Vectors (ch.1 + 2)

Definition: L2 inner product

Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$, we defined the $\boldsymbol{L^2}$**-inner product** on $[a, b]$ to be

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx.$$

with the corresponding L^2 norm,

$$\|f\|_2 = \sqrt{, f\rangle} = \left(\int_a^b f^2(x)dx\right)^{\frac{1}{2}}.$$

Definition: Fourier basis

Suppose $-\pi \leq z \leq \pi$, the **Fourier basis** is defined as

$$\{1, \cos(z)\sin(z), \cos(2z), \sin(2z), \ldots\}.$$

This is an infinite, mutually orthogonal basis of the vector space of continuous functions on $[-\pi, \pi]$.

Definition: projection

Suppose $\{\mathbf{e}_1, \mathbf{e}_2 \ldots\}$ are an orthogonal basis, then $v_i = \frac{\langle \mathbf{v}, \mathbf{e}_i\rangle}{\|v\|}$.

Definition: Fourier series

Suppose $f(z)$ is defined on $[-\pi, \pi]$ and is in "the proper space of functions" (see Ch.5 notes) then $f(z)$ has a Fourier series is of the form

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nz) + \sum_{n=1}^{\infty} b_n \sin(nz).$$

where

$$b_n = \frac{(z), \sin(nz)\rangle}{\|\sin(nz)\|_2^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(z)\sin(nz)dz \text{ for}$$

n=1,2,. . . .

$$a_n = \frac{(z), \cos(nz)\rangle}{\|\cos(nz)\|_2^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(z)\cos(nz)dz \text{ for}$$

n=1,2,. . . .

$$a_0 = \frac{\langle f(z), 1\rangle}{\|1\|_2^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z)dz.$$