0.1 Binary Operations

Definition: Identity

A binary structure $(S, *)$ has an **identity** $e \in S$ if

$$e * x = x = x * e \quad \forall x \in S.$$

Definition: Inverse

If e is the identity for $(S, *)$, then y is the **inverse** for x if

$$x * y = e = y * x.$$

Theorem

An identity is unique if it exists.

Proof

Suppose not, e and e' are both identities for $(S, *)$.

$$e * e' = e'$$
$$e * e' = e$$

Hence $e = e'$ $\square$

Theorem

Inverses if it exists it is unique if $(S, *)$ is associative.

Proof

Suppose not, y and y' are both inverses for x.

$$\begin{aligned}(y * x) * y' &= y * (x * y')\\ e * y' &= y * e\\ y' &= y\end{aligned}$$

$\square$

Example. $\{1,2,3,4,5,6\}$ under $\times_7$, inverse of 3? 5.

Definition: Binary operation

A **binary operation** of a set S is a function $* : S \times S \to S$

Example. $S = \mathbb{Z}$, $* =$ subtraction.

Three things that could go wrong with binary operations (due to definition of function):

- not in S
- no ambiguity
- no gaps

How many binary operations are there on the $S = \{a, b, c\}$? 3^9. In general, for an n-element set it is n^{n^2}.

Definition: Binary structure

A **binary structure** $(S, *)$ is a set with a binary operations, $*$, on S.

Theorem

Composition of functions is associative.

Consider $(\mathbb{C}, \emptyset)$. Let's reduce this operation to $\mathbb{Q}$. If $a * b$ is still in $\mathbb{Q}$, then $*$ induces an operation on $\mathbb{Q}$.

Example. $(\mathbb{Z}, +)$ integers under addition. It's closed for even numbers but not for odd numbers.