1 Taylor expansion for multivariable functions

$$f(x + dx, y + dy, z + dz) = [e^{(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z})}]f(x, y, z)$$

2 1st order ODE IVP

2.1 Euler's Method

$y' = f(x, y), y(a) = y_a, a \le x \le b$. Divide $[a, b]$ into equal parts, $h = \frac{b-a}{n}$, so for each step, we have x_i and x_{i+1}.

Then

$$\int_{x_i}^{x_{i+1}} \frac{dy}{dx} dx = \int_{x_i}^{x_{i+1}} f(x, y(x)) dx.$$

Then approximately we get

$$y_{i+1} = y_i + f(x_i, y_i) \cdot h.$$

Using Taylor's expansion, we can get

$$y(x + h) = y(x) + \frac{h}{1!}y'(x) + \frac{h^2}{2!}y''(x) + \ldots \approx y(x) + hy'(x) = y(x) + hf(x, y).$$

which gives us the same thing. The error is $\mathcal{O}(h)$.

2.2 Higher order Taylor method

We know that $y^{(n)} = f^{(n-1)}(x_i, y_i)$. Let $T^{(n)}(x_i, y_i) = \sum_{k=1}^{n} \frac{h^{k-1}}{k!} f^{(k-1)}(x_i, y_i)$. Then we can write $w_{i+1} = w_i + hT^{(n)}(x_i, y_i)$.

2.3 Runge-Kutta Methods

Taylor needs to compute the derivative of the function. We can do something similar to Gaussian quadrature, to find the ideal points so it is equivalent as evaluating the slope. Consider

$$w_{i+1} = w_i + h[a \cdot f(x_i + \alpha, w_i + \beta)].$$

When $n = 2$,

$$T^{(2)}(x, y) = f(x, y) + \frac{h}{2} f'(x, y)$$
$$= f(x, y) + \frac{h}{2} \left[\frac{\partial f}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x} \right]$$
$$= f(x, y) + \frac{h}{2} [f_x(x, y) \frac{\partial x}{\partial x} + f_y(x, y) f(x, y)]$$

and

$$af(x + \alpha, y + \beta) = a[f(x, y) + \alpha f_x + \beta f_y] + \dots.$$

Now we compare and get $a = 1, a \cdot \alpha = \frac{h}{2}, a \cdot \beta = \frac{h}{2} f(x, y)$. Therefore,

$$w_{i+1} = w_i + hf(x_i + \frac{h}{2}, w_i + \frac{h}{2} f(x_i, w_i)).$$

Now the local truncation error is $\mathcal{O}(h^2)$. This is called 2nd order R-K method, or midpoint method. It uses the slope at the midpoint.

In general, there is no one magic location that gives you the equivalence of 4th order Taylor. We need four.

When $n = 4$

$$k_1 = hf(x_i, w_i)$$
$$k_2 = hf(x_i + \frac{h}{2}, w_i + \frac{k_1}{2})$$
$$k_3 = hf(x_1 + \frac{h}{2}, w_i + \frac{k_2}{2})$$
$$k_4 = hf(x_i + h, w_i + k_3)$$
$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

The local truncation error is $\mathcal{O}(h^4)$.

3 Extrapolation

4 Higher order