1 Laplace"s Equation and Solution

We wish to solve the 2D equilibrium heat equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0.$$

Recall the Laplace Operator from prior lecture:

$$L(u) = \nabla^2 u = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Example (Thought Experiment in 1D)**.**

1) Suppose $f''(x) = 0 \Rightarrow f(x) = kx + c$, whose rate of change is constant.
2) for all possible k, the extrema only occur at the boundaries.
3) For any $x_0 \in (a, b)$, and for any $\varepsilon > 0$ such that $[x_0 - \varepsilon, x_0 + \varepsilon] \subseteq (a, b)$, we have

   $$f(x_0) = \frac{f(x_0 - \varepsilon) + f(x_0 + \varepsilon)}{2} = \frac{k(x_0 - \varepsilon) + c - k(x_0 + \varepsilon) - c}{2} = f(x_0).$$

   This is a characterization of equilibrium.

1.1 Rectangular Domain

$$\begin{cases} \text{PDE:} & \Delta u = 0, \qquad (x, y) \in (0, L) \times (0, H) \\ \text{BCs:} & u(x, 0) = f_1(x) \\ & u(0, H) = f_2(x) \\ & u(0, y) = g_1(y) \\ & u(L, y) = g_2(y) \end{cases}$$

Note that we don't have ICs because this is a steady state problem. Since Laplace Operator is linear, we can use the same trick by decomposing it into two problems. Let $u(x, y) = u_1(x, y) + u_2(x, y)$.

$$\begin{cases} \text{PDE:} & \Delta u_1 = 0, (x, y) \in (0, L) \times (0, H) \\ \text{BCs:} & u_1(x, 0) = f_1(x) \\ & u_1(0, H) = f_2(x) \\ & u_1(0, y) = 0 \\ & u_1(L, y) = 0 \end{cases} \quad \begin{cases} \text{PDE:} & \Delta u = 0, (x, y) \in (0, L) \times (0, H) \\ \text{BCs:} & u_2(x, 0) = 0 \\ & u_2(0, H) = 0 \\ & u_2(0, y) = g_1(y) \\ & u_2(L, y) = g_2(y) \end{cases}$$

Note the homogeneous BCs and PDE form a vector space. Then we can use the nonhomogeneous BCs to get the coefficients (like ICs).

1.1.1 separation of variables

Let $u_1(x,y) = F(x)G(y) \neq 0$. Then $\Delta u_1 = 0$ implies

$$F''(x)G(y) = F(x)G''(y) \Rightarrow -\frac{G''(y)}{G(y)} = \frac{F''(x)}{F(x)} = -\lambda.$$

1.1.2 F-equation

We can solve $F(x)$ under Dirichlet conditions exactly like before. Note that F is nontrivial only if $\lambda > 0$. Then we obtain

$$F_n(x) = C_n \sin\left(\frac{n\pi x}{L}\right), n = 1, 2, \ldots.$$

1.1.3 G-equation

$$G''(y) = \lambda_n G(y) \Rightarrow r^2 e^{ry} = \lambda_n e^{ry} \Rightarrow r^2 = \left(\frac{n\pi}{L}\right)^2 \Rightarrow r = \pm\frac{n\pi}{L} \Rightarrow e^{ry} = e^{\pm n\pi y/L}.$$

So $\{e^{n\pi y/L}, e^{-n\pi y/L}\}$ is a basis solutions for the G-equation. For convenience, we wish to have basis functions that vanish at the endpoints $y = 0$ and $y = H$ because it makes fining the coefficients really easy.

Recall that cosh is even and sinh is odd. Then

$$\frac{e^{n/L} + e^{-n\pi y/L}}{2} = \cosh\left(\frac{n\pi y}{L}\right) \text{ and } \frac{e^{n\pi y/L} - e^{-n\pi y/L}}{2} = \sinh\left(\frac{n\pi y}{L}\right).$$

However, using $\{\sinh\left(\frac{n\pi y}{L}\right), \cosh\left(\frac{n\pi y}{L}\right)\}$ is not the basis we seek because cosh doesn't go to zero. Let's use the identity:

$$\sinh\left(\frac{n\pi[y-H]}{L}\right) = \sinh\left(\frac{n\pi y}{L} - \frac{n\pi H}{L}\right) = \sinh\left(\frac{n\pi y}{L}\right)\cosh\left(\frac{n\pi H}{L}\right) - \cosh\left(\frac{n\pi y}{L}\right)\sinh\left(\frac{n\pi H}{L}\right).$$

Therefore, $\left\{\sinh\left(\frac{n\pi y}{L}\right), \sinh\left(\frac{n\pi[y-H]}{L}\right)\right\}$ is also a basis of solutions for the G-equation. Note that this is not an orthogonal basis. Thus, the general solution of G_n is:

$$G_n(y) = A_n \sinh\left(\frac{n\pi y}{L}\right) + B_n \sinh\left(\frac{n\pi[y-H]}{L}\right), n = 1, 2, \ldots.$$

Therefore, the general solution of $u(x,y)$ is

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + b_n \sinh\left(\frac{n\pi[y-H]}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

When $y = 0$, only the second term remains, so the nonhomogeneous BC yields

$$f_1(x) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{-n\pi H}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} \tilde{b}_n \sin\left(\frac{n\pi x}{L}\right).$$

So by projection formula,

$$b_n = \frac{\frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)}{\sinh\left(\frac{-n\pi H}{L}\right)}.$$

And similarly,

$$f_2(x) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi H}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} \tilde{a}_n \sin\left(\frac{n\pi x}{L}\right).$$

And

$$a_n = \frac{\frac{2}{L}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right)}{\sinh\left(\frac{n\pi H}{L}\right)}.$$