Definition

If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ and $A = \bigcup_{n=1}^{\infty} A_n$, we write $A_n \uparrow A$.

Definition

If $A_1 \supseteq A_2 \supseteq \dots$ and $A = \bigcap_{n=1}^{\infty} A_n$, we write $A_n \downarrow A$.

Example. Given $A_1, A_2, \dots \in \mathcal{F}$, consider $\bigcup_{k=1}^{\infty} A_k, \bigcup_{k=2}^{\infty} A_k \dots$ as $n \to \infty$,

$$\bigcup_{k=n}^{\infty} A_k \downarrow \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_m = \limsup_n A_n.$$

Also,

$$\bigcap_{k=n}^{\infty} A_k \uparrow \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} A_m = \liminf_n A_n.$$

Note.

$$\liminf_n A_n \subseteq \limsup_n A_n.$$

"("all but finitely many A_n") $\subseteq$ ("infinitely many of the A_n")" (this is not a proof).

Proof

Take $\omega \in \liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \Rightarrow \omega \in \bigcap_{k=n}^{\infty} A_k$ for at least one n. Then there exists a $N \geq 1$ such that $\omega \in A_N, A_{N+1}, \dots \Rightarrow \omega \in \bigcup_{k=1}^{\infty} A_k, \omega \in \bigcup_{k=2}^{\infty} A_k, \dots$. Hence, it's in all of them (the intersection) so it's in limsup. $\square$

Definition: common value

If $\liminf_n A_n = \limsup_n A_n$, define $\lim_n A_n$ to be the **common value** .

Lemma

$$\left(\limsup_n A_n\right)^c = \liminf_n A_n^c.$$

by De Morgan's law.

In probability, "sets" represent "events". liminf and limsup are also "events".

1) $\limsup_n A_n$ = "the event that infinitely many of the events A_n occur" / "A_n occurs infinitely often" / "A_n i.o.".

$$P(A_n \text{i.o.}) = P(\limsup_n A_n).$$

2) $\liminf_n A_n$ = "A_n occurs almost always"/"A_n a.a.".

Example. Let $A_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n^2}{n}\}$. So for example $A_2 \not\subseteq A_3$.

Claim: $\limsup_n A_n = \mathbb{Q}_+$.

Proof

Clearly $\limsup_n A_n \subseteq \mathbb{Q}_+$. On the other hand, take any positive rational $\frac{a}{b}$. Assume $b \neq 0$, a, b are non-negative integers, and are coprime (have no common factors).

Case (1). $a = 0 \Rightarrow \frac{a}{b} = 0 \Rightarrow \frac{a}{b} \in A_n \quad \forall n \geq 1$.

Case (2). $b = 1 \Rightarrow \frac{a}{b} = a \Rightarrow \frac{a}{b} = a \in A_n \quad \forall n \geq a$.

Case (3). Otherwise, in order for $\frac{a}{b}$ to be in A_n, choose n large enough, so n has to be a multiple of b, *i.e.* $n = kb$. Thus, $\frac{a}{b} = \frac{ka}{kb} = \frac{ka}{n}$. To get $\frac{a}{b} \in A_n$, we need $ka \in \{0, 1, 2, \ldots, n^2\}$. That is, need $a \in \{0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{n^2}{k}\}$. This will happen if $\frac{n^2}{k} \geq a \Rightarrow n \geq \sqrt{ka} \Rightarrow \frac{a}{b} \in A_n$ for infinitely many $n > \sqrt{ka}$. So, any positive rational $\frac{a}{b}$ is in all A_n for n large enough.

$$\frac{a}{b} \in \limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

$\square$

Example. Claim: $\liminf_n A_n = nn$.

Theorem: continuity of probabilities

(i) if $A_n \uparrow A$, then $\lim_{n\to\infty} P(A_n) = P(A)$.

(ii) if $A_n \downarrow A$, then $\lim_{n\to\infty} P(A_n) = P(A)$.

Proof

For (i), rewrite A as a disjoint union, $A = \bigcup_{n=1}^{\infty} B_n$ where $B_1 = A_1, B_2 = A_2 \setminus A_1 = A_2 \cap A_1^c$. Then

$$\begin{aligned}
P(A) &= P(\bigcup_{n=1}^{\infty} B_n) \\
&= \sum_{n=1}^{\infty} P(B_n) \qquad \text{by countable additivity} \\
&= \lim_{m\to\infty} \sum_{n=1}^{m} P(B_n) \\
&= \lim_{m\to\infty} P\left(\bigcup_{n=1}^{m} B_n\right) \\
&= \lim_{m\to\infty} P\left(\bigcup_{n=1}^{m} A_n\right) \\
&= \lim_{m\to\infty} P(A_m)
\end{aligned}$$

Since $A_1 \subseteq A_2 \subseteq \ldots$. □

Theorem: 4.1

(i) For any sequence $(A_n) \subseteq \mathcal{F}$.

$$P\left(\liminf_n A_n\right) \leq \liminf_n P(A_n) \leq \limsup_n P(A_n) \leq P(\limsup_n A_n).$$

(ii) If $\lim_{n\to\infty} A_n = A$, then

$$\lim_{n\to\infty} P(A_n) = P(A).$$

Definition

If (x_n) is a sequence in $\mathbb{R}$,

$$\liminf_n x_n = \lim_{n\to\infty} \left(\inf_{k\geq n} x_k\right).$$

$$\limsup_n x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right).$$

Proof

(ii) follows from (i) by the "squeeze theorem".

(i): let $B_n = \bigcap_{k=n}^{\infty} A_k$, $B = \bigcup_{n=1}^{\infty} B_n$. So $B_1 \subseteq B_2 \subseteq \ldots$ and $B_n \uparrow B$.

$$\begin{aligned}
P(\liminf_n A_n) &= P\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_n\right) \\
&= P(B) \\
&= \lim_{n\to\infty} P(B_n) \text{ by continuity of probabilities} \\
&= \lim_{m\to\infty} P\left(\bigcap_{k=n}^{\infty} A_k\right) \\
&= \liminf_n P\left(\bigcap_{k=n}^{\infty} A_k\right) \\
&\leq \liminf_n P(A_n) \text{ by monotonicity of } P
\end{aligned}$$

$\square$

Definition: independent events

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $(A_n) \subseteq \mathcal{F}$. Then $A_1, A_2, \ldots$ are mutually independent if for any $j \in \{2, 3, \ldots, n\}$ and any indices $1 \leq k_1 < \ldots < k_j \leq n$,

$$P(A_{k_1} \cap A_{k_2} \cap \ldots \cap A_{k_j}) = P(A_{k_1})P(A_{k_2})\ldots P(A_{k_j}).$$