1 Heat conduction problem

Consider a rod with length L.

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}.$$

Where ρ is the density, c is the heat capacity, and k is the heat conductivity.
Initial condition: $T(x,t)|_{t=0} = f(x)$
Boundary conditions: $T(x,t)|_{x=0} = T_0$, $T(x,t)|_{x=L} = T_0$

We want to nondimensionalize the problem to reduce complexity: Let $\alpha = \frac{k}{\rho c}$ be the thermal diffusivity, $\overline{x} = \frac{x}{L}$, $U = \frac{T - T_0}{T_r}$, so $x = \overline{x}L$, $T = UT_r + T_0$.

$$\begin{aligned}
\frac{\partial UT_r + T_0}{\partial t} &= \alpha \frac{\partial^2 UTr + T_0}{\partial \overline{x} L^2} \\
T_r \frac{\partial U}{\partial t} &= \frac{T_r \alpha}{L^2} \frac{\partial^2 U}{\partial x^2} \\
\frac{\partial U}{\partial \frac{t\alpha}{L^2}} &= \frac{\partial^2 U}{\partial \overline{x}^2}
\end{aligned}$$

Let $\tau = \frac{t\alpha}{L^2} = \frac{t}{t_{ref}}$, so $t_{ref} = \frac{L^2}{\alpha}$, and we obtain:

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial \overline{x}^2}.$$

$$\begin{aligned}
&U(\overline{x}, \tau)|_{\tau=0} = \frac{f(x) - T_0}{T_r} F(x) \\
&U(\overline{x}, \tau)|_{\overline{x}=0} = 0 \\
&U(\overline{x}, \tau)|_{\overline{x}=1} = 0
\end{aligned}$$

$$U(\overline{x}, \tau) = \sin(\pi \overline{x}) e^{-t\pi^2}$$

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Make a grid in the $\overline{x} - \tau$ plane. Let $k = \Delta\tau$, $h = \Delta\overline{x}$. Using finite difference approximation:

$$\frac{U_{i,j+1} - U_{ij}}{k} = \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{h^2}.$$

the LHS is the forward difference in time, and the RHS is the central difference in space.

$$\begin{aligned} U_{i,j+1} &= U_{ij} + \frac{k}{h^2}(U_{i+1,j} - 2U_{ij} + U_{i-1,j}) \\ &= (1-2r)U_{ij} + r(U_{i+1,j} + U_{i-1,j}) \end{aligned}$$

where $r = \frac{k}{h^2} = \frac{\Delta\tau}{\Delta x^2}$.

Set $r = \frac{1}{4}$, $\Delta x = 0.2$, then $k = 0.01$. Set $r = 1$, $\Delta x = 0.2$, then $k = 0.04$. Starts to fall apart a little bit. Set $r = 2.5$,$\Delta x = 0.2$, then $k = 0.1$. Results violate physics.

1.1 Crank-Nicolson Method

We can fix the problem above by using an implicit method. We want to average $\frac{\partial^2 U}{\partial x^2}$ at time j and $j+1$. Three points in the past and three points in the future.

$$\begin{gathered} \frac{U_{i,j+1} - U_{ij}}{k} = \frac{1}{2}\left(\frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{h^2} + \frac{U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}}{h^2}\right) \\ -rU_{i+1,j+1} + (2+2r)U_{i,j+1} - rU_{i-1,j+1} = rU_{i+1,j} + (2-2r)U_{ij} + rU_{i-1,j} \end{gathered}$$

This is much more stable. Symmetry gives us two equations for two unknowns.