Midterm 2

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Problem (1)**.** We wish to show inequality in both directions to prove equality.

($\geq$): Since $f \geq 0$ is measurable, there exists a sequence of simple measurable functions (f_n) such that $0 \leq f_n \nearrow f$. This means that given f_n, ω, $f_n(\omega) \leq f(\omega)$.

$$\begin{aligned}
\int_\Omega f \ d\mu &= \int_\Omega \lim_{n\to\infty} f_n \ d\mu \\
&= \lim_{n\to\infty} \sum_i a_{n_i} \mu(A_{n_i}) \\
&= \limsup_n \sum_i a_{n_i} \mu(A_{n_i}) \\
&\leq \limsup_n \sum_i \left[\inf_{\omega \in A_{n_i}} f(\omega) \right] \mu(A_{n_i}) \\
&\leq \sup \sum_i \left[\inf_{\omega \in A_i} f(\omega) \right] \mu(A_i)
\end{aligned}$$

($\leq$) : Since the supremum of a set is always greater or equal to the supremum of its subset,

$$\begin{aligned}
\int_\Omega f \ d\mu &= \sup_{0 \leq s \leq f} s \ d\mu \\
&= \sup \sum_i a_i \mu(A_i) \text{ where } a_i \leq f(w) \text{ for } \omega \in A_i \\
&\geq \sup \sum_i \left[\inf_{\omega \in A_i} f(\omega) \right] \mu(A_i) \text{ since } \inf \text{ is a special case of } a_i
\end{aligned}$$

By inequality in both direction we show that they are equal.

Problem (2)**.**

Case (1). f, g are simple. Then let $f = \sum_{i=1}^n a_i I_{A_i}$ and $g = \sum_{j=1}^m b_j I_{B_j}$ where

A_i, B_j are respectively disjoint partitions of Ω. Then

$$\begin{aligned} gf &= \left(\sum_{j=1}^{m} b_j I_{B_j}\right)\left(\sum_{i=1}^{n} a_i I_{A_i}\right) \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n} b_j a_i I_{A_i \cap B_j} \end{aligned}$$

Note that if a given $\omega \notin B_j$ or $\omega \notin A_i$, then the product $b_j a_i I_{B_j} I_{A_i}$ will be zero. Thus the only terms remain are the ones with $I_{A_i \cap B_j}$. Now consider

$$\begin{aligned} \int_\Omega g\ d\nu &= \sum_{j=1}^{m} b_j \nu(B_j) \\ &= \sum_{j=1}^{m} b_j \int_{B_j} fd\ \mu \\ &= \sum_{j=1}^{m} b_j \sum_{i=1}^{n} a_i \mu(A_i \cap B_j) \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n} b_j a_i \mu(A_i \cap B_j) \\ &= \int_\Omega gf\ d\mu \end{aligned}$$

Case (2). f, g are not simple. Since $f, g \geq 0$ are measurable, there exists sequences of simple measurable functions $(f_n), (g_m)$ such that $0 \leq f_n \nearrow f, 0 \leq g_m \nearrow g$. By Case 1, we know that for each pair of f_n, g_m,

$$\int_\Omega g_m\ d\nu = \int_\Omega g_m f_n\ d\mu.$$

Taking $n, m \to \infty$, by Lebesgue's Monotone Convergence Theorem,

$$\begin{aligned} \lim_{n,m\to\infty} \int_\Omega g_m d\ \nu &= \lim_{n,m\to\infty} \int_\Omega g_m f_n\ d\mu \\ \int_\Omega g\ d\nu &= \int_\Omega gf\ d\mu \end{aligned}$$

Problem (3)**.** ($\Rightarrow$): Suppose $E(X) = 0$, we want to show that $P(X = 0) = 1$ by showing $P((X = 0)^c) = 0$. Since $X \geq 0$, the complement of $X = 0$ is $X > 0$. Let $B = \{\omega : X(\omega) > 0\}$, it suffices to show that $P(B) = 0$.

Note that since $X \geq 0$, $0 \leq XI_B \leq X$. Therefore,

$$0 \leq \int XI_B \; dP \leq \int X \; dP = E(X) = 0.$$

It follows that $\int XI_B \; dP = \int_B X \; dP = 0$. Recall that $X(\omega) > 0 \; \forall \; \omega \in B$, then it must be that $P(B) = P(X > 0) = 0$ as required.

($\Leftarrow$): Suppose $P(X = 0) = 1 \Rightarrow P(X > 0) = 0$. $X = 0$ is clearly a simple function where $X(\omega) = 0 \cdot I_\Omega$.

$$\begin{aligned} E(X) &= \int X \; dP \\ &= 0 \cdot P(\Omega) \\ &= 0 \end{aligned}$$