Intuition. If the rope is tight enough, *i.e.* T_0 is large, then there will be barely any oscillation. It's some sort of restorative force.

Recall the slope of the string may be represented as $\frac{\partial u}{\partial x}$ or $\tan\theta$.

$$\tan\theta = \frac{\partial u}{\partial x}.$$

Thus

$$\frac{\partial}{\partial x}\tan\theta = \frac{\partial}{\partial x}\cdot\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}.$$

Then the PDE becomes

$$\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho_0}\cdot\frac{\partial^2 u}{\partial x^2} - g.$$

Assuming $g = 0$, let $c^2 = \frac{T_0}{\rho_0} = \frac{T_0}{\delta\cdot A} > 0$, then we obtain the 1D wave equation

Theorem: 1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

The more general version is

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} - g + f(x,t).$$

The LHS is the vertical acceleration of string, and the RHS is the restoring force due to tension (- gravitational acceleration + acceleration from known external forces).

0.1 initial conditions

Second order time derivative needs two ICs!

Initial position: $u(x,0) = U(x)$.

Initial velocity: $\frac{\partial}{\partial t}u(x,0) = V(x)$.

The boundary conditions are fixed: $u(x,0) = u(x,L) = 0$.

So we have the following:

$$\begin{cases} \text{PDE: } \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} & 0 < x < L, t > 0 \\ \text{BCs: } u(0,t) = 0 = u(L,t) & t > 0 \\ \text{ICs: } u(x,0) = U(x), \frac{\partial u}{\partial t}(x,0) = V(x) & 0 \le x \le L \end{cases}$$

In general, we assume that the solution has the form $u(x,t) = \overline{u}(x) + w(x,t)$, where $\overline{u}(x)$ is the steady state solution and $w(x,t)$ is in the vector space of functions that satisfy the PDE and BCs.

Example. Suppose
$$\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho_0} \cdot \frac{\partial^2 u}{\partial x^2} - g.$$
with $u(0,t) = 0 = u(L,t)$. Find the steady state position of the string.

For steady state, $\frac{\partial u}{\partial t} = 0$, so $\frac{\partial^2 u}{\partial t^2} = 0$, and
$$\frac{T_0}{\rho_0}\overline{u}''(x) - g = 0 \Rightarrow \overline{u}(x) = \frac{\rho_0 g}{2T_0}x^2 + Ax + B.$$
Then BC $\overline{u}(0) = 0$ implies $B = 0$. And the BC $\overline{u}(L) = 0$ implies
$$0 = \overline{u}(L) = \frac{\rho_0 g}{2T_0}L^2 + A \cdot L \Rightarrow A = \frac{\rho_0 g}{2T_0}L.$$
therefore the steady state solution becomes
$$\overline{u}(x) = \frac{\rho_0 g}{2T_0} \cdot x \cdot (x - L).$$
Therefore, the graph is a parabola with minimum occurring when $x = \frac{L}{2}$, then the minimum $\overline{u}_{\min} = \overline{u}\left(\frac{L}{2}\right) = \frac{\rho_0 g}{2T_0}L^2$. This is the sag in the center of the string due to gravity.

Intuition. Let's take a look at the units of $c = \sqrt{\frac{T_0}{\rho_0}}$.
$$c = \sqrt{\text{ mass} \cdot \text{ length/time}^2 \cdot \text{ length/ mass}} = \text{ speed.}$$
So c should be some sort of speed. The only likely candidate seems to be the speed of propagation. If that's true, then we can also calculate propagation time,
$$\tau = \frac{L}{c} + \frac{L}{c} = \frac{2L}{c}.$$
Then frequency (cycles divided by time) follows:
$$f = \text{ constant} \cdot \frac{c}{L} = \text{ constant}\frac{1}{2} \cdot \frac{1}{L} \cdot \sqrt{\frac{T_0}{\delta \cdot A}}.$$
How can we increase the frequency?

1) Decrease length L.
2) Increase tension T_0.
3) Decrease density δ.

4) Increase area A.

They all match our intuition!

Note. Gravity doesn't change frequency according to this model!

0.2 Variations on the Wave Equation (optional)

Transverse Motion: stiffness.

Damped Motion: eventually return to stationary.

Transverse Vibrations: 2D membrane (drum).

Longitudinal Vibration: does not assume totally vertical motion.