Theorem: distributions of transformations

Let $X = (X_1, \ldots, X_k)$ be a r.vec. Let $g : \mathbb{R}^k \to \mathbb{R}^i$ be a measurable function. Define $Y = g(X)$ (is measurable because it's composition of measurable functions). Let P_X be the distribution of X. Then

$$P_Y(A) = P(Y \in A) = P(g(X) \in A) = P(X \in g^{-1}(A)) = P_X(g^{-1}(A)).$$

i.e. $P_Y = P_X \circ g^{-1}$.

It's also called pushforward operation.

Projection Maps

Consider the measurable spaces $(\Omega_1, \mathcal{F}_1), \ldots, (\Omega_k, \mathcal{F}_k)$. Define $\Omega = \Omega_1 \times \ldots \times \Omega_k$. Define

$$\mathcal{F} = \sigma(\mathcal{F}_1 \times \cdots \times \mathcal{F}_k).$$

Let $\pi_i : \Omega \to \Omega_i$ be the projection map

$$\pi_i(\omega_1, \ldots, \omega_k) = \omega_i.$$

Note. If $A = A_1 \times \cdots \times A_k$ then $\pi_i(A) = A_i$.

Let $\pi_i^{-1}(\mathcal{F}_i) = \{A \subseteq \Omega : \pi_i(A) \in \mathcal{F}_i\}$. Then

$$\mathcal{F} = \sigma\left(\bigcap_{i=1}^{k} \pi_i^{-1}(\mathcal{F}_i)\right).$$

Note. The thing in the parenthesis do not equal. Their σ-fields equal.

Moreover (TODO requires proof!),

$$\pi_i^{-1}(\mathcal{F}_i) \subseteq \mathcal{F} \Rightarrow \pi_i : \Omega \to \Omega_i \text{ is } \mathcal{F}/\mathcal{F}_i \text{ measurable.}$$

0.1 Marginal Distribution

Notice $\pi_i(X) = X_i$ is a composition of measurable functions hence it's measurable.

Its distribution is $P_{X_i} = P_X \circ \pi_i^{-1}$. This is called **marginal distribution**.

Question: is there a corresponding marginal density? That is, we are looking for a function f_i such that $P_{X_i}(A) = \int_A f_i(t)dt$.

Guess: If we integrate the density over all the other r.v. To do this we need to change the order of integration using Fibini's Theorem. That is, we guess

$$f_i(x_i) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_k) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_k.$$

And we would want to check that

$$\begin{aligned}
\int_A f_i(x_i)dx_i &= \int_A \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_k) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_k dx_i \\
&\overset{\text{Fibini}}{=} \int_{-\infty}^{\infty} \cdots \int_A \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_k) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_k \\
&= P_X((-\infty, \infty) \times \cdots \times A \times \cdots \times (-\infty, \infty)) \\
&= P_{X_i}(A)
\end{aligned}$$

Definition: product space

Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be measure spaces. Let's define the product σ-field $\mathcal{F} \coloneqq \sigma(\mathcal{F}_1 \times \mathcal{F}_2)$. Define $\Omega = \Omega_1 \times \Omega_2$. Then $(\Omega, \mathcal{F})$ is called the **product space**.

Proposition

Let $\mathcal{F}$ be the product σ-field. Let $A \in \mathcal{F}$. Define the "slices":

$$A\big|_{\omega_1} \coloneqq \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in A\}.$$

Then $A\big|_{\omega_1}$ and $A\big|_{\omega_2}$ are in $\mathcal{F}_2$ and $\mathcal{F}_1$, respectively.

Proof

Let $\mathcal{G} = \{B \subseteq \Omega : B\big|_{\omega_1} \in \mathcal{F}_2 \text{ and } B\big|_{\omega_2} \in \mathcal{F}_1 \ \forall\ \omega_1, \omega_2 \in \Omega\}$. We wish to show that $\mathcal{G}$ is a σ-field.

(i) Slices of Ω are $\Omega_1 \in \mathcal{F}_1, \Omega_2 \in \mathcal{F}_2$.

(ii) Take $B \in \mathcal{G}$. Notice

$$\begin{aligned}
(B\big|_{\omega_1})^c &= \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in B\}^c \\
&= \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in B^c\} \\
&= B^c\big|_{\omega_1}
\end{aligned}$$

And we know $(B\big|_{\omega_1})^c \in \mathcal{F}_2$ since $\mathcal{F}_2$ is a σ-field.

(iii) same technique as above.

It remains to show that $\mathcal{F}$ is a subset of $\mathcal{G}$. Given $A_i \in \mathcal{F}_i$, notice

$$\begin{aligned}(A_1 \times A_2)\big|_{w_1} &= \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in A_1 \times A_2\} \\ &= \begin{cases} A_2 & \text{if } \omega_1 \in A_1 \\ \emptyset & \text{if } \omega_1 \notin A_1 \end{cases} \in \mathcal{F}_2\end{aligned}$$

Likewise for the other. Thus $A_1 \times A_2 \in \mathcal{G}$. Since $\mathcal{F}$ is the smallest σ-field containing sets of the form $A_1 \times A_2$, it follows that $\mathcal{F} \subseteq \mathcal{G}$. □

Theorem

$$\mathcal{B}(\mathbb{R}^{n+m}) = \sigma(\mathcal{B}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^m)).$$

Definition: product measure

Given two spaces $(\Omega_1, \mathcal{F}_1, \mu_1), (\Omega_2, \mathcal{F}_2, \mu_2)$. Define a **product measure** as a measure $\mu = \mu_1 \times \mu_2$ on $(\Omega, \mathcal{F})$ such that

$$\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2) \;\forall\; A_i \in \mathcal{F}_i.$$

Theorem: Fubini's Theorem

Given two σ-finite measure spaces and their product measure space. If $f : \Omega \to \mathbb{R}$ is a measurable function such that $\int_\Omega |f| \; d\mu < \infty$, then

1) For almost all $\omega_1 \in \Omega_1$, $f(\omega_1, \omega_2)$ is an integrable function of ω_2 (vice-versa).
2) There exists an integrable $h : \Omega_1 \to \mathbb{R}$ such that

$$h(\omega_1) = \int_{\Omega_2} f(\omega_1, \omega_2) d\omega_2$$

for almost all ω_1 (vice versa).

3)

$$\begin{aligned}\int_\Omega f \; d\mu = \int_{\Omega_1 \times \Omega_2} f \; d(\mu_1 \times \mu_2) &= \int_{\Omega_1} \left[\int_{\Omega_2} f(\omega_1, \omega_2) d\omega_2 \right] d\omega_1 \\ &= \int_{\Omega_2} \left[\int_{\Omega_1} f(\omega_1, \omega_2) d\omega_1 \right] d\omega_2\end{aligned}$$

.

Lemma: 1

Suppose μ_1 and μ_2 are σ-finite. For every $A \in \mathcal{F} = \sigma(\mathcal{F}_1 \times \mathcal{F}_2)$, the function $h : \Omega_1 \to \mathbb{R}$ defined as

$$h(\omega_1) = \mu_2(A\big|_{\omega_1})$$

is $\mathcal{F}_1$-measurable.

Proof

Case (1). μ_1, μ_2 are finite. Define $\mathcal{G} = \{B \in \mathcal{F} : h_B \text{ is } \mathcal{F}_1 - \text{measurable}\}$. We wish to show that $\mathcal{G}$ is a λ-system.

(i) $\Omega \in \mathcal{G}$ because

$$h_\Omega(\omega_1) = \mu_2(\Omega\big|_{\omega_1}) = \mu_2(\Omega_2) < \infty.$$

Then h_Ω is a finite constant function which is measurable as we have shown in Lecture 9.

(ii) Take $B \in \mathcal{G}$,

$$\begin{aligned} h_{B^c}(\omega_1) &= \mu_2(B^c\big|_{\omega 1}) \\ &= \mu_2((B\big|_{\omega_1})^c) \\ &= \mu_2(\Omega_2) - \mu_2(B\big|_{\omega_1} \end{aligned}$$

Since a constant function minus a measurable function is still $\mathcal{F}_1$ measurable, $B^c \in \mathcal{G}$.

(iii) Given B_n disjoint, using the same technique, we can show

$$h_{\bigcup_n B_i}(\omega_1) = \sum_n h_{B_i}(\omega_1).$$

The sum is measurable. Since the limit of measurable functions is measurable, taking $n \to \infty$ gives us a $\mathcal{F}_1$ measurable function. closure under countable disjoint union.

□