Homework 6

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Problem (10.1)**.** $4\mathbb{Z} = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$. The left cosets of $4\mathbb{Z}$ in $\mathbb{Z}$ are:

1) itself.
2) Notice 1 is missing. Let $a = 1$, then $1 + 4\mathbb{Z} = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$.
3) 2 is missing. Let $a = 2$, then $2 + 4\mathbb{Z} = \{\ldots - 6, -2, 2, 6, 10, \ldots\}$.
4) 3 is missing. Let $a = 3$, then $3 + 4\mathbb{Z} = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

This exhausts all elements in $\mathbb{Z}$. And since $\mathbb{Z}$ is abelian, it follows that the left and right cosets are the same. That's all the cosets.

Problem (10.2)**.**

1) $4\mathbb{Z}$ itself.
2) $2 + 4\mathbb{Z} = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$.

Clearly this exhausts all elements of $2\mathbb{Z} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$.

Problem (10.6)**.** By Lagrange, it should have $\frac{8}{2} = 4$ left cosets.

1) itself $H = \{\rho_0, \mu_2\}$
2) notice ρ_1 is missing. $\rho_1 H = \{\rho_1, \delta_2\}$
3) notice ρ_2 is missing. $\rho_2 H = \{\rho_2, \mu_1\}$
4) notice ρ_3 is missing. $\rho_3 H = \{\rho_3, \delta_1\}$

Problem (10.7)**.** It should have 4 right cosets. We can apply the inverse of the elements we multiplied above on the left on the right.

1) itself $H = \{\rho_0, \mu_2\}$
2) $H\rho_3^{-1} = H\rho_1 = \{\rho_1, \delta_1\}$
3) $H\rho_2^{-1} = H\rho_2 = \{\rho_2, \mu_1\}$
4) $H\rho_1^{-1} = H\rho_3 = \{\rho_3, \delta_2\}$

No they are not the same, because D_4 is not abelian.

Problem (10.12)**.** $|\langle 3\rangle| = \frac{n}{\gcd(3,24)} = 24 \div 3 = 8$.
$\{\mathbb{Z}_{24} : |\langle 3\rangle|\} = |\mathbb{Z}_{24}| \div |\langle 3\rangle| = 24 \div 8 = 3$.

Problem (10.15)**.**

$$\sigma = \ (1\ 2\ 5\ 4)(2\ 3) = (1\ 2\ 3\ 5\ 4)\ .$$

Since σ is a 5-cycle, its order $|\langle\sigma\rangle| = 5$.
$\{S_5 : |\langle\sigma\rangle|\} = \frac{|S_5|}{|\langle\sigma\rangle|} = 5!/5 = 24$ by Lagrange.

Problem (10.19)**.**

a) True. Itself.

b) True. Lagrange.

c) True. Cyclic implies abelian by Corollary 10.11.

d) False. $\{0\}$ is a subgroup and thus a left coset of $\mathbb{Z}$, yet it is finite.

e) True. $eH = H$.

f) False. $3\mathbb{Z}$ in $\mathbb{Z}$ is infinite.

g) True. $|A_n| = n!/2, |S_n| = n!$.

h) True!

i) False. $|A_4| = 12$, clearly 6 divides 12 but we know that A_4 has no subgroup of order 6.

j) True. $|\langle a\rangle| = \frac{n}{\gcd(n,a)}$.

Problem (11.1)**.**

$$
\begin{aligned}
1(0,0) = (0,0) \Rightarrow |\langle(0,0)\rangle| &= 1\\
|\langle(0,1)\rangle| = \frac{4}{\gcd(1,4)} &= 4\\
|\langle(1,0)\rangle| = \frac{2}{\gcd(1,2)} &= 2\\
|\langle(1,1)\rangle| = \operatorname{lcm}\left(\frac{2}{\gcd(1,2)}, \frac{4}{\gcd(1,4)}\right) &= 4\\
|\langle(0,2)\rangle| = \frac{4}{\gcd(2,4)} &= 2\\
|\langle(1,2)\rangle| = \operatorname{lcm}\left(\frac{2}{\gcd(1,2)}, \frac{4}{\gcd(2,4)}\right) &= 2\\
|\langle(0,3)\rangle| = \frac{4}{\gcd(3,4)} &= 4\\
|\langle(1,3)\rangle| = \operatorname{lcm}\left(\frac{2}{\gcd(1,2)}, \frac{4}{\gcd(3,4)}\right) &= 4
\end{aligned}
$$

No, by Theorem 11.5 since 2 and 4 are not relatively prime.

Problem (11.2)**.**

$$1(0,0) = (0,0) \Rightarrow |\langle(0,0)\rangle| = 1$$
$$|\langle(0,1)\rangle| = \frac{4}{\gcd(1,4)} = 4$$
$$|\langle(1,0)\rangle| = \frac{3}{\gcd(1,3)} = 3$$
$$|\langle(1,1)\rangle| = \operatorname{lcm}\left(\frac{3}{\gcd(1,3)}, \frac{4}{\gcd(1,4)}\right) = 12$$
$$|\langle(0,2)\rangle| = \frac{4}{\gcd(2,4)} = 2$$
$$|\langle(1,2)\rangle| = \operatorname{lcm}\left(\frac{3}{\gcd(1,3)}, \frac{4}{\gcd(2,4)}\right) = 6$$
$$|\langle(0,3)\rangle| = \frac{4}{\gcd(3,4)} = 4$$
$$|\langle(1,3)\rangle| = \operatorname{lcm}\left(\frac{3}{\gcd(1,3)}, \frac{4}{\gcd(3,4)}\right) = 12$$
$$|\langle(2,2)\rangle| = \operatorname{lcm}\left(\frac{3}{\gcd(2,3)}, \frac{4}{\gcd(2,4)}\right) = 6$$
$$|\langle(2,3)\rangle| = \operatorname{lcm}\left(\frac{3}{\gcd(2,3)}, \frac{4}{\gcd(3,4)}\right) = 12$$

Yes, because 3 and 4 are relatively prime, so by Theorem 11.5 it is cyclic.

Problem (11.14)**.**

a) $|\langle 18\rangle| = \frac{24}{\gcd(18,24)} = \frac{24}{6} = 4$.

b) $\operatorname{lcm}(3,4) = 12$.

c) $\operatorname{lcm}\left(\frac{12}{\gcd(4,12)}, \frac{8}{\gcd(8,2)}\right) = \operatorname{lcm}(3,4) = 12$.

d) $V_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

e) Choose 1 element from each group, we can have $2 \times 1 \times 4 = 8$ elements.

Problem (11.20)**.**

$$\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15} \simeq \mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$$
$$\mathbb{Z}_3 \times \mathbb{Z}_{36} \times \mathbb{Z}_{10} \simeq \mathbb{Z}_3 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_5$$

Notice that the RHSs are just rearrangement of each other, hence by the fundamental theorem of FGG, they are isomorphic to each other.

Problem (11.23)**.** Since $32 = 2^5$, let's consider the partitions of 5:

$$5 : \mathbb{Z}_{32}$$
$$4+1 : \mathbb{Z}_{16} \times \mathbb{Z}_2$$
$$3+2 : \mathbb{Z}_8 \times \mathbb{Z}_4$$
$$3+1+1 : \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$
$$2+2+1 : \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$$
$$2+1+1+1 : \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$
$$1+1+1+1+1 : \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

That's all the abelian groups of order 32, up to isomorphism.

Problem (11.24)**.** Since $720 = 2^4 3^2 5$, let's first consider the partition of $2^4 = 16$:

$$4 : \mathbb{Z}_{16}$$
$$3+1 : \mathbb{Z}_8 \times \mathbb{Z}_2$$
$$2+2 : \mathbb{Z}_4 \times \mathbb{Z}_4$$
$$2+1+1 : \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$
$$1+1+1+1 : \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

Then let's consider the partition of $3^2 = 9$:

$$2 : \mathbb{Z}_9$$
$$1+1 : \mathbb{Z}_3 \times \mathbb{Z}_3$$

Finally for$5^1 = 5$, the partition of 1 is just 1, hence it only yields $\mathbb{Z}_5$. Now choosing one from each group, we list all $5 \times 2 \times 1 = 10$ possible abelian groups of order 720 below:

$$\begin{aligned}
&\mathbb{Z}_{16} \times \mathbb{Z}_9 \times \mathbb{Z}_5 \\
&\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \\
&\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \\
&\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \\
&\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \\
&\mathbb{Z}_{16} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \\
&\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \\
&\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \\
&\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \\
&\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5
\end{aligned}$$

Problem (11.46)**.** WLOG, let's consider the direct product of two abelian groups $(G, *)$ and $(H, *)$, $G \times H$. Given two elements from this product, (g_1, h_1) and (g_2, h_2), we want to show that their product under componentwise $*$ commutes. Since G, H are abelian, $*$ certainly commutes. Then

$$\begin{aligned}
(g_1, h_1) * (g_2, h_2) &= (g_1 * g_2, h_1 * h_2) \\
&= (g_2 * g_1, h_2 * h_1) \\
&= (g_2, h_2) * (g_1, h_1)
\end{aligned}$$

This shows that componentwise $*$ commutes, and it follows that $G \times H$ is abelian.