Chapter 3: Integration

$(\Omega, \mathcal{F}, \mu)$, $f : \Omega \to \mathbb{R}$ measurable.

Notation.

$$\int f d\mu = \int_\Omega f(\omega) d\mu(\omega) = \int_\Omega f(\omega)\mu(d\omega).$$

Definition: integration

For a simple function $f(\omega) = \sum_{i=1}^n a_i I_{A_i}(\omega)$, $A_i \in \mathcal{F}$, we define

$$\int_\Omega f d\mu = \sum_{i=1}^n a_i \mu(A_i).$$

For any $B \in \mathcal{F}$, define

$$\int_B f d\mu = \sum_{i=1}^n a_i \mu(A_i \cap B).$$

Example.

1) $\int_\Omega I_A d\mu = \mu(A)$.
2) $f = \sum_{i=1}^n a_i I_{A_i}$, then $\int f \ d\lambda$ is the Riemann integral.
3) Recall Theorem 12.4 (F non-decreasing, right continuous, real-valued, there exists a unique measure μ on $\mathcal{B}(\mathbb{R})$ satisfying $\mu((a, b]) = F(b) - F(a)$. And $\mu((a, b)) = \mu((a, b^-)) = F(b^-) - F(a)$). μ is called the **Lebesgue-Stietjes meausre** given by F. Suppose f is a non-negative Riemann integrable function, and suppose F is defined by $F(x) = \int_{-\infty}^x f(y) \ dy$. Then for $a < b$,

$$\int_\mathbb{R} I_{(a,b]} d\mu = \mu((a, b]) = F(b) - F(a) = \int_a^b f(x) \ dx.$$

Moreover,

$$\int_\mathbb{R} I_{([a,b])} = \mu((-\infty, b]) - \mu((-\infty, a^-]) = F(b) - F(a^-).$$

Definition

$(\Omega, \mathcal{F}, \mu), f : \Omega \to \mathbb{R}$ measurable. If f is non-negative, define, for any

$A \in \mathcal{F}$,

$$\int_A f \; d\mu = \sup \int_A s \; d\mu$$

where the supremum is taken over all simple functions s where $0 \leq s(\omega) \leq f(\omega) \; \forall \; \omega \in A$.

Note. This is well-defined since $s(\omega) = 0 \; \forall \; \omega \in \Omega$ is one element in the set. If the supremum is infinite, we say either "f is not integrable over A" or "f has infinite integral over A".

Facts:

1) $0 \leq f \leq g \Rightarrow \int_A f \; d\mu \leq \int_A g \; d\mu$.
2) $A \subseteq B \Rightarrow \int_A f \; d\mu \leq \int_B f \; d\mu$.

Proof

Take any simple $s : \Omega \to \mathbb{R}$ such that $0 < s < f$. Then s can be written as $s(\omega) = \sum_{i=1}^n a_i I_{A_i}(\omega)$ for some partition $A_1, \ldots, A_n$ of A, assuming a_i are distinct. Then

$$A \subseteq B \Rightarrow A_i \cap A \subseteq A_i \cap B \Rightarrow \mu(A_i \cap A) \leq \mu(A_i \cap B)$$

so

$$\begin{aligned} \int_A s \; d\mu &= \sum_{i=1}^n a_i \mu(A_i \cap A) \\ &\leq \sum_{i=1}^n a_i \mu(A_i \cap B) \\ &= \int_B s \; d\mu \end{aligned}$$

Since s is arbitrary, this relationship should hold for the suprema:

$$\int_A f \; d\mu = \sup_{0 \leq s \leq f} \int_A s \; d\mu \leq \sup_{0 \leq s \leq f} \int_B s \; d\mu = \int_B f \; d\mu.$$

$\square$

3) For $c > 0$ constant, then $\int_A cf d\mu = c \int_A f d\mu$.

Proof

$0 \le s_1 \le cf$. Define $s_2 = \frac{s_1}{c}$ is still simple. And constants can be taken out of supremum. $\square$

4) $\mu(A) = 0 \Rightarrow \int_A f d\mu = 0$.

5) $\int_A f d\mu = \int_\Omega I_A \cdot f d\mu$.

More: Let S_1, S_2 be simple, then

1) $\int_A (s_1 + s_2) d\mu = \int_A s_1 d\mu + \int_A s_2 d\mu$.

2) Define $\nu(A) = \int_A s \, d\mu$. Then can show that ν is another measure on $\Omega, \mathcal{F}$.

Theorem: Lebesgue's Monotone Convergence Theorem

Let (f_n) be a sequence of measurable functions on $(\Omega, \mathcal{F}, \mu)$. Suppose $0 \le f_1(\omega) \le f_2(\omega) \le \ldots \ \forall \ \omega \in \Omega$ and that $\lim_{n\to\infty} f_n(\omega) = f(\omega) \ \forall \ \omega \in \Omega$, then

$$\lim_{n\to\infty} \int f_n \ d\mu = \int f \ d\mu.$$

Proof

We have three cases.

1) Some f_n are not integrable.

   That is, $\int f_n \, d\mu = \infty$. Then given $M > 0$, there exists a simple s with $0 \leq s \leq f_n$ and $\int s d\mu > M$. Since $f_n \nearrow f$, then $0 \leq s \leq f_n \leq f$. Hence, $\int f d\mu = \infty = \lim\limits_{n\to\infty} \int f_n \, d\mu$.

2) All f_n are integrable but $(\int f_n \, d\mu)$ diverges.

   If we assume divergence, then for any constant $M > 0$, there exists N such that $\int f_n d\mu > M + 1 \; \forall \; n \geq M$. So $\lim\limits_{n\to\infty} \int f_n \, d\mu = \infty$. By the definition of $\int f_n d\mu$, there exists a simple $s, 0 \leq s \leq f_n$ such that $\int s d\mu > M \; \forall \; n \geq N$. Since $0 \leq s \leq f_n \leq f$, this s can make $\int f d\mu$ as large as we want. Hence $\int f \, d\mu = \lim\limits_{n\to\infty} \int f_n d\mu = \infty$.

3) All f_n are integrable and $(\int f_n \, d\mu)$ converges.

   $f_n \leq f_{n+1} \Rightarrow \int f_n d\mu \leq \int f_{n+1} d\mu \Rightarrow \lim\limits_{n\to\infty} \int f_n d\mu = \sup_n \left\{ \int f_n d\mu \right\} \equiv c$. We need to show that f is integrable and the integral equals c.

Let s be simple with $0 \le s \le f$. Let b be any constant $(0,1)$. Define

$$A_n = \{\omega : f_n(\omega) \ge b \cdot s(\omega)\}.$$

Note that $A_n \in \mathcal{F}$ because both f_n, bs are both measurable and by the last theorem in lecture 17.

$\square$