

Chapter 4 Numerical Derivatives

Using Taylor's expansion:

$$\begin{aligned}f'(x) &= \frac{f(x+h) - f(x) - \frac{h^2}{2}f''(x) - \dots}{h} \\&= \frac{f(x+h) - f(x)}{h} + \mathcal{O}(h)\end{aligned}$$

This is the **forward difference**.

Now try with $f(x-h)$:

$$f'(x) = \frac{f(x) - f(x-h)}{h} + \mathcal{O}(h).$$

Now subtracting forward and backward differences:

$$\begin{aligned}f(x+h) - f(x-h) &= 2hf'(x) + \frac{h^3}{3}f'''(x) \\f'(x) &= \frac{1}{2h}(f(x+h) - f(x-h)) + \mathcal{O}(h)\end{aligned}$$

This is the **central difference**.

Now consider the second derivatives by adding the forward and backward differences.

$$f''(x) = \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} + \mathcal{O}(h^2).$$

Using three-point Lagrange Polynomials: see lecture notes. Differences are just a weighted average. Errors are all $\mathcal{O}(h^3)$.

For second derivative using Lagrange polynomials, we obtain similar answer as from Taylor. But for third derivatives, Lagrange wouldn't work anymore because all three differences are zero now.

Midpoint for 2nd derivatives from book:

$$f''(x_0) = \frac{1}{h^2} [f(x_0-h) - 2f(x_0) + f(x_0+h)] - \frac{h^2}{12}f^{(4)}(\xi)$$

for some ξ , where $x_0 - h < \xi < x_0 + h$

Taylor Matching:

$$\begin{aligned}f'(x) &= af(x+h) + bf(x) \\&= a[f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \dots] \\&\quad + b[f(x)]\end{aligned}$$