From previous lecture, we obtain the **Lorentz series.**

The coefficients are

$$
\begin{aligned}
c_n = \frac{1}{2}(a_n - ib_n) &= \frac{1}{2L}\int_{-L}^{L} f(x)\left[\cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right)\right]dx \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)\cdot e^{-in\pi x/L}dx
\end{aligned}
$$

Thus the complex form of the Fourier series of $f(x)$ is

$$
\text{F.S.}[f](x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}
$$

where

$$
c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\cdot e^{-in\pi x/L}dx, n = 0, \pm 1, \pm 2, \ldots.
$$

Notice that the positive exponential term is used in the series and the negative exponential term is used to find the coefficients.

0.0.1 Orthogonality

The inner product of two complex-valued functions $f(x)$ and $g(x)$, piecewise continuous on $[-L, L]$ is defined as

$$
\langle f(x), g(x)\rangle = \int_{-L}^{L} f(x)\overline{g(x)}dx = \int_{-L}^{L}[f_1(x) + if_2(x)]\overline{[g_1(x) + ig_2(x)]}.
$$

with norm defined by

$$
\|f\| = \sqrt{\langle f(x), f(x)\rangle} = \int_{-L}^{L} |f(x)|^2 dx \in \mathbb{R}.
$$

Note that

$$
\begin{aligned}
\langle e^{im\pi x/L}, e^{in\pi x/L}\rangle &= \int_{-L}^{L} e^{im\pi x/L}\cdot\overline{e^{inx/L}}dx \\
&= \int_{-L}^{L}\left[\cos\left(\frac{(m-n)\pi x}{L}\right) + i\sin\left(\frac{(m-n)\pi x}{L}\right)\right]dx \\
&= \begin{cases} 0 & \text{if } m \neq n \\ 2L & \text{if } m = n \end{cases}
\end{aligned}
$$

Example. Compute the complex F.S. of $f(x) = e^{ax}, x \in [-L, L]$, where $a \in \mathbb{R}$.

$$\begin{aligned}
c_n &= \frac{1}{2L}\int_{-L}^{L} e^{ax} \cdot e^{-in\pi x/L} dx \\
&= \frac{1}{2L[a-(in\pi/L]} e^{[a-(in\pi/L]x}\Big|_{-L}^{L} \\
&= \frac{1}{2[aL - in\pi]}\left[e^{aL}e^{-in\pi} - e^{-aL}e^{-in\pi}\right] \\
&= \frac{1}{2[aL - in\pi]}\left[e^{aL}(\cos(n\pi) - i\sin(n\pi)) - e^{aL}(\cos(n\pi) + i\sin(n\pi))\right] \\
&= \frac{aL + in\pi}{[(aL)^2 + (n\pi)^2]} \cdot (-1)^n \cdot \frac{e^{aL} - e^{-aL}}{2} \\
&= \frac{(-1)^n(aL + in\pi)}{(aL)^2 + (n\pi)^2}\sinh(aL)
\end{aligned}$$

Therefore, the complex F.S. is

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n(aL + in\pi)}{(aL)^2 + (n\pi)^2}\sinh(aL)e^{in\pi x/L}.$$

Note that we can use $c_n = \frac{1}{2}(a_n - ib_n)$ to find the real F.S. coefficients a_n and b_n which is much easier than finding them directly!

Fun Facts:

- The coefficients c_n are usually complex even if $f(x)$ is real.
- If $f(x)$ is real then $c_{-n} = \overline{c_n}$.
- If $f(x)$ is an even function then $c_{-n} = c_n$ and if $f(x)$ is an odd function then $c_{-n} = -c_n$.
- Note that

$$c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \text{ average value of } f(x) \text{ on } [-L, L].$$

- If $f(x)$ is piecewise smooth then the complex F.S. of $f(x)$ converges to the periodic extension of the adjusted version of $f(x)$.
- Parseval's Identity states that

$$\frac{1}{2L}\int_{-L}^{L} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

0.1 Integral Transform

The Fourier transform is a *continuous analog* of the F.S. In theory, a Fourier integral would lead to more manageable and understandable solutions in closed form.

Definition

Given any "reasonable" function $K(x,z)$, we can define the **integral transform**, $T[f](z)$ of a function $f(x)$, $a \le x \le b$, by

$$T[f](z) = \int_a^b K(x,z)f(x)dx.$$

where the function $f(x)$ is transformed into a new function $T[f](z)$. Such transforms are linear. The function $K(x,z)$ is known as the **kernel** of the transform.

Remark. The Fourier transform is helpful in solving PDEs, primarily because it converts differentiation into algebraic multiplication:

$$T[f'](z) = izT[f](z).$$

0.2 Fourier transform

Given $f(x), x \in \mathbb{R}$, we wish to represent $f(x)$ as Fourier integral:

1) Suppose $\int_{-\infty}^{\infty} |f(x)|dx = M < \infty$ and $f(x)$ is piecewise smooth on every finite interval.
2) Let $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$ and let $m_n = \frac{n\pi}{L}$ then this is a partition of $(-\infty, \infty)$ for $n \in \mathbb{Z}$.
3) Note that $\Delta m_n = m_{n+1} - m_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$. Thus $\frac{L}{\pi}\Delta m_n = 1$.
4) Using this fact we can write the complex F.S. as a Riemann Sum:

   $$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L} \cdot \frac{L}{\pi}\Delta m_n = \sum_{n=-\infty}^{\infty} \left(\frac{L}{\pi}c_n\right) e^{in\pi x/L}\Delta m_n = \sum_{n=-\infty}^{\infty} \hat{f}(m_n)e^{im_n x}\Delta m_n.$$

   where we let $\hat{f}(m_n) = Lc_n/\pi$.
5) Taking the limit $L \to \infty$ on both sides $\Delta m_n \to 0$ yields:

   $$f(x) = \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} \hat{f}(m_n)e^{im_n x}\Delta m_n = \int_{-\infty}^{\infty} \hat{f}(m)e^{imx}dm.$$

   which is the Fourier integral representation of $f(x)$.
6) Note that $\hat{f}(m)$ is the *Fourier transform of* $f(x)$ and $f(x)$ is the *inverse Fourier transform of* $\hat{f}(m)$.