Theorem

Let (X_n) be a sequence of r.v. on $(\Omega, \mathcal{F}, P)$. Then

$$\left\{\omega : \lim_{n\to\infty} X_n(\omega) \text{ exists}\right\} := \left\{\lim_{n\to\infty} X_n \text{ exists}\right\}$$

is a tail event. That is, it is in $\mathcal{F}_T = \bigcap_{m=1}^{\infty} \sigma(\{X_m, X_{m+1}, \dots\})$ (let's call each individual term σ_m, and denote σ_∞ as the whole thing).

Proof

For $\omega \in \{\lim_{n\to\infty} X_n \text{ exists}\}$, $\lim_{n\to\infty} X_n(\omega)$ exists, meaning that $(X_n(\omega))$ is Cauchy sequence. That is, for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n > m \geq N$, then $|X_n - X_m| < \varepsilon$. *i.e.*:

$$\begin{aligned}\{\lim_{n\to\infty} X_n \text{ exists}\} &= \{\ \forall\, \varepsilon > 0,\ \exists\, N \text{ s.t. }\ \forall\, n > m \geq N, |X_n - X_m| < \varepsilon\} \\ &= \bigcap_{\varepsilon>0} \bigcup_{N} \bigcap_{n>m\geq N} \{|X_n - X_m| < \varepsilon\}\end{aligned}$$

since rationals are dense, we can restrict ε to be countable. Since X_n and X_m are σ_1-measurable, $|X_n - X_m|$ is also σ_1-measurable by previous proofs. Then by definition of measurable, $\{|X_n - X_m| < \varepsilon\} \in \sigma_1$. (Recall $\{Y < \varepsilon\} = \{\omega : Y(\omega) < \varepsilon\} = Y^{-1}((-\infty, \varepsilon))$, where $(-\infty, \varepsilon) \in \mathcal{B}(\mathbb{R})$.)

Then we can repeat this argument on the shifted sequence $\{X_m, X_{m+1}, \dots\}$, and by induction we can establish that $\{\lim_{n\to\infty} X_n \text{ exists}\} \in \sigma_m\ \forall\, m \geq 1$. Hence,

$$\{\lim_{n\to\infty} X_n \text{ exists}\} \in \sigma_\infty = \bigcap_{m=1}^{\infty} \sigma_m.$$

$\square$

Example (valid r.v. formed from a sequence of r.v.).

1) $\sup_n \{X_n\}$ and $\inf_n \{X_n\}$.

Proof

$$\{\omega : \sup_n X_n(\omega) \leq x\} = \bigcap_{n=1}^{\infty} \{\omega : X_n(\omega) \leq x\} \in \mathcal{F}\ \forall\, x \in \mathbb{R}$$

$$\{\omega : \inf_n X_n(\omega) \leq x\} = \bigcup_{n=1}^{\infty} \{X_n(\omega) \leq x\} \in \mathcal{F} \ \forall \ x \in \mathbb{R}.$$

$\square$

2) $\limsup_n X_n$ and $\liminf_n X_n$.

Proof

$$\begin{aligned}\{\omega : \limsup_n X_n(\omega) \leq x\} &= \{\omega : \inf_n \sup_{m \geq n} X_m(\omega) \leq x\} \\ &= \bigcup_{n=1}^{\infty} \{\omega : \sup_{m \geq n} X_m(\omega) \leq x\} \\ &= \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{\omega : X_m(\omega) \leq x\} \\ &\in \mathcal{F} \ \forall \ x \in \mathbb{R}\end{aligned}$$

$\square$

3) If $(X_n(\omega))$ converges $\ \forall \ \omega \in \Omega$, then $\lim\limits_{n\to\infty} X_n$ is a r.v.

Proof

$$\lim_{n\to\infty} X_n = \limsup_n X_n = \liminf_n X_n \in \mathcal{F}.$$

$\square$

4) If $(X_n(\omega))$ converges for "almost all" $\omega \in \Omega$. (It means that $P(\{\omega : \lim\limits_{n\to\infty} X_n(\omega)$ does not exists$\}) = 0$. Define

$$X(\omega) := \begin{cases} \lim\limits_{n\to\infty} X_n(\omega), & \text{if } (X_n(\omega)) \text{ converges} \\ 0, & \text{otherwise} \end{cases}$$

Definition

Given a sequence of r.v. (X_n) and a r.v. X on $(\Omega, \mathcal{F}, P)$. We say X_n **converges** to X w.p. 1 ("almost surely"), $X_n \xrightarrow{a.s.} X$, if

$$P\left(\lim_{n\to\infty} X_n = X\right) = P(\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 1.$$

Example. $\Omega = [0,1], \mathcal{F} = \mathcal{B}([0,1]), P =$ Lebesgue measure. Define $X_n(\omega) = \omega^n$, *i.e.* $X_1(\omega) = \omega, X_2(\omega) = \omega^2$. Then

$$\lim_{n\to\infty} X_n(\omega) = \begin{cases} 0 & \text{if } \omega \neq 1 \\ 1 & \text{if } \omega = 1 \end{cases}$$

Define

$$X(\omega) = 0$$

Then

$$P(\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = P([0,1)) = 1 - 0 = 1.$$

So

$$X_n \xrightarrow{a.s.} X.$$

An alternative characterization of almost sure convergence

Theorem

$$P\left(\lim_{n\to\infty} X_n = X\right) = 1 \Leftrightarrow P(|X_n - X| \geq \varepsilon \ i.o.) = 0 \ \forall \ \varepsilon > 0.$$

Proof

If $|X_n(\omega) - X(\omega)| \geq \varepsilon \ \forall \ \varepsilon > 0$ for finitely many n, then $X_n(\omega) \to X(\omega)$. So

$$\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\} = \{\omega : |X_n(\omega) - X(\omega)| \geq \varepsilon \ i.o.\}^c \ \forall \ \varepsilon > 0.$$

$\square$

Definition: converges in probability

X_n **converges in probability** to X if $\forall \ \varepsilon > 0$, $\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$. And we write $X_n \xrightarrow{p} X$.