1 Complex Numbers

1.1 Cartesian Coordinates

$a+bi, a, b \in \mathbb{R}, i^2 = -1$ It is convenient to visualize it on a plane. For equivalence, $c + di = e + fi \Rightarrow c = e, d = f$.

$$(a + bi)(c + di) = ac + bic + adi + bidi = (ac - bd) + (ad + bc)i.$$

Complex number multiplication is distributive, commutative, and associative. In division, simplify denominator using complex conjugate.

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.$$

1.2 Polar Coordinates

$$\begin{aligned} e^{i\theta} &= \cos\theta + \sin\theta \\ re^{i\theta} &= r\cos\theta + r\sin\theta \end{aligned}$$

$$(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = r_1r_2e^{i(\theta_1+\theta_2)}$$

$$\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

Multiplication is easier in exponential form.

$$r_1e^{i\theta_1} = r_2e^{i\theta_2} \Rightarrow r_1 = r_2, \theta_1 - \theta_2 = 2k\pi, k \in \mathbb{Z}.$$

Does $x^5 = 12817$ have a real root? Use the horizontal line test. Yes, every real number has a unique odd number root.

Every nonzero complex number has precisely n complex roots. The root is evenly spaced around the circle centered at the origin with such radius.

Example: Find all solutions of $z^2 = i$.
Let's use the exponential form:

$$\begin{aligned} \left(re^{i\theta}\right)^2 &= i \\ \left(r^2e^{2i\theta}\right) &= 1e^{i\frac{\pi}{2}} \end{aligned}$$

So we require: $r^2 = 1$ and $2\theta - \frac{\pi}{2} = 2k\pi$. Since there are only two roots, we increment k once. So $k = 0, 1$.