Homework 4

Jaden Wang

Problem (7.1)**.** Recall that 1 is a generator of $\mathbb{Z}_{12}$. If 1 can be represented by a word of $\{2,3\}$, then we can just let the $\{2,3\}$ representation of 1 generate $\mathbb{Z}_{12}$. This is indeed possible here:

$$1 = 3 +_{12} 3 +_{12} 3 +_{12} 2 +_{12} 2.$$

Therefore, $\{2,3\}$ should generate the whole $\mathbb{Z}_{12}$.

Problem (7.2)**.** Since even numbers under $+_{12}$ can only output even numbers, we know that the odd numbers $\{1,3,5,7,9,11\}$ cannot be in the subgroup. Let's check the rest:

$$\begin{aligned} 0 &= 6 +_{12} 6 \\ 2 &= 6 +_{12} 4 +_{12} 4 \\ 4 &= 4 \\ 6 &= 6 \\ 8 &= 4 +_{12} 4 \\ 10 &= 4 +_{12} 6 \end{aligned}$$

Hence, the elements of the subgroup are $\{0,2,4,6,8,10\}$.

Problem (7.7)**.**

a) $(a^2b)a^3 = eaabaaa = a^3b$

b) $(ab)(a^3b) = eabaaab = a^2$

c) $b(a^2b) = ebaab = a^2$

Problem (7.8)**.** To fill out the table, we simply need to represent c in terms of a, b. The graph tells us $c = ab = ba$. Hence we have the following: $aa = e, ab = c, ac = aab = eb = b, ba = c, bb = e, bc = bba = ea = a, ca = b, cb = a, cc = c(ab) = ce = c$. That is,

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Problem (7.10)**.** Let's represent other elements using a, b: $c = aad = ba, f = ab$. Thus we have the following: $aa = c, ab = f, ac = aaa = e, ad = aba = b, af = aab = d, ba = d, bb = e, bc = baa = f, bd = bba = a, bf = bab = c, ca = e, cb = d, cc = caa = a, cd = cba = f, cf = cab = b, da = f, db = c, dc = daa = b, dd = dba = e, df = dab = a, fa = b, fb = a, fc = faa = d, fd = fba = c, ff = fab = e$.

*	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	c	f	e	b	d
b	b	d	e	f	a	c
c	c	e	d	a	f	b
d	d	f	c	b	e	a
f	f	b	a	d	c	e

Problem (8,1)**.**

$$\begin{aligned}\tau\sigma &= \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 2 & 4 & 1 & 3 & 6 & 5\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 2 & 4 & 1 & 3 & 6 & 5\end{pmatrix}\\ &= \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 1 & 2 & 3 & 6 & 5 & 4\end{pmatrix}\end{aligned}$$

Problem (8.2)**.**

$$\begin{aligned}\tau^2\sigma = \tau(\tau\sigma) &= \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 2 & 4 & 1 & 3 & 6 & 5\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 1 & 2 & 3 & 6 & 5 & 4\end{pmatrix}\\ &= \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 2 & 4 & 1 & 5 & 6 & 3\end{pmatrix}\end{aligned}$$

Problem (8.6)**.**

$$\begin{aligned}
\sigma^2 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 5 & 6 & 2 & 1 \end{pmatrix} \\
\sigma^3 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 5 & 6 & 2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 2 & 1 & 3 \end{pmatrix} \\
\sigma^4 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 2 & 1 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix} \\
\sigma^5 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 3 & 4 & 5 \end{pmatrix} \\
\sigma^6 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 3 & 4 & 5 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}
\end{aligned}$$

Since σ^6 is clearly the identity, we know that $\sigma^6 = \sigma^0$. Therefore, $|\langle\sigma\rangle| = 6$.

Problem (8.9)**.** Notice

$$\begin{aligned}
\mu^2 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 4 & 3 & 1 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 4 & 3 & 1 & 6 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}
\end{aligned}$$

Hence $\mu^2 = \mu^0$. It is easy to see that $\mu^{100} = \mu^0$ which is the identity above.

Problem (8.20)**.**

	ρ^0	ρ	ρ^2	ρ^3	ρ^4	ρ^5
ρ^0	ρ^0	ρ	ρ^2	ρ^3	ρ^4	ρ^5
ρ	ρ	ρ^2	ρ^3	ρ^4	ρ^5	ρ^0
ρ^2	ρ^2	ρ^3	ρ^4	ρ^5	ρ^0	ρ
ρ^3	ρ^3	ρ^4	ρ^5	ρ^0	ρ	ρ^2
ρ^4	ρ^4	ρ^5	ρ^0	ρ	ρ^2	ρ^3
ρ^5	ρ^5	ρ^0	ρ	ρ^2	ρ^3	ρ^4

This cannot be isomorphic to S_3, since it is symmetric about the diagonal and thus abelian yet S_3 is nonabelian.

Problem (8.32)**.** It suffices to show that f_3 has a two-sided inverse to determine it is a permutation. Let $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = -x^{\frac{1}{3}}$. It is clear that given $x \in \mathbb{R}$, $f_3 \circ g(x) = -\left(-x^{\frac{1}{3}}\right)^3 = x = -\left(-x^3\right)^{\frac{1}{3}} = g \circ f_3(x)$. Therefore, $g(x) = f^{-1}(x)$ thus f_3 is bijective and hence is a permutation.

Problem (8.33)**.** f_4 is not a permutation because it is not bijective on $\mathbb{R} \to \mathbb{R}$. Choose $y = -1 \in \mathbb{R}$, notice that since $e^x > 0$ for all $x \in \mathbb{R}$, there is no $x \in \mathbb{R}$ such that $f_4(x) = y < 0$. This violates the surjectivity requirement.

Problem (8.34)**.** f_5 is not a permutation. $f_5(x) = x^3 - x^2 - 2x = x(x - 2)(x + 1)$. We can immediately see that it is not bijective because although $f_4(0) = f_4(-1) = f_4(2) = 0$, we have $-1 \neq 0 \neq 2$, violating the injective requirement.