Example. Let $L = 10$cm, $k = 1$ cm^2/sec (copper) and $t = 0.35$sec and use the first 8 terms.

$$u(x, 0.35) \approx \frac{400}{\pi} \sum_{p=1}^{8} \frac{1}{2p-1} e^{-[(2p-1)\frac{\pi}{10}]^2 0.35} \sin\left(\frac{(2p-1)\pi x}{10}\right).$$

See lecture slide for graph.

In this case, $\overline{u}(x) = 0$ and for each x,

$$u(x, t) \to 0 \text{ as } t \to \infty.$$

Note. Recall $\Phi = -K_0 \frac{\partial u}{\partial x}$. Consider the x-term in $u_n(x, t)$

$$\left[\sin\left(\frac{n\pi x}{L}\right)\right]' = n \cdot \frac{\pi}{L} \cos\left(\frac{n\pi x}{L}\right)$$

Thus the derivative wrt x is proportional to n and Φ. That is, as n increases, the derivative increases, and the heat flux (loss) increases. So the slowest decaying term is when n is smallest, *i.e.* $n = 1$.

Establishing that the slowest decaying term (dominant term) is at $n = 1$ and for large t, we can use the $n = 1$ term (called the "first Fourier mode") as an approximation

$$u(x, t) \approx B_1 \sin\left(\frac{\pi x}{L}\right) e^{-(\frac{\pi}{L})^2 kt}.$$

and we can use this for long term temperature prediction. So for this problem we use the approximation:

$$u(x, t) \approx \frac{400}{\pi} \sin\left(\frac{\pi x}{L}\right) e^{-(\frac{\pi}{L})^2 kt}.$$

to analyze the dynamics of the temperature when t grows. We expect $u(x, t) \to 0$ as $t \to \infty$. See lecture slides for graph.

Example (estimating cooling time)**.** How long will it take for the maximum absolute temperature of the rod to be less than $\frac{1}{10}$ the initial maximum absolute temperature?

We wish to find t such that

$$\max_{0<x<L} |u(x, t)| \leq \frac{1}{10} \max_{0<x<L} |f(x)|.$$

Using the first Fourier mode approximation obtained above, recall there is an upper bound (in fact it's the least upper bound/supremum)

$$|u(x, t)| \leq \frac{400}{\pi} e^{-(\frac{\pi}{L})^2 kt}.$$

Hence this upper bound is greater or equal to the maximum (in this case they are in fact equal). Thus it suffices to find t such that

$$\frac{400}{\pi} e^{-(\frac{\pi}{L})^2 kt} \leq \frac{1}{10} \cdot 100 = 10.$$

Solving this inequality yields

$$t \geq \frac{L^2}{k} \frac{1}{\pi^2} \ln\left(\frac{40}{\pi}\right).$$

1 Insulated Rods

Consider the following BVP:

$$\begin{cases} \text{PDE:} & \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, t > 0 \\ \text{BC:} & \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t), \qquad t > 0 \\ \text{IC:} & u(x,0) = f(x), \qquad 0 \leq x \leq L \end{cases}$$

Recall Fourier's Law of Heat Conduction regarding the heat flux

$$\Phi = -K_0 \frac{\partial u}{\partial x}.$$

So here the BCs imply that there is no heat flow at the ends of the rod, *i.e.* the rod is insulated on all sides.

Since there is no external source of heat, we expect this BVP to have a steady state solution.

Hence we can assume the solution has the form

$$u(x,t) = \overline{u}(x) + v(x,t).$$

In fact, we can skip this decomposition step because the BVP already gives us a nice vector space due to the homogeneous BCs. And $\lambda = 0$ case (from homework) will give us the steady-state solution anyway.