Note. Gibbs only occur if FS is truncated. Gibbs has about 9% over/undershoot.

Intuition. A sequence of continuous function cannot converge uniformly to a discontinuous function.

- if the adjusted periodic extension $\tilde{f}(x)$ is piecewise smooth on every finite interval but has a jump discontinuity then the Fourier Series of $f(x)$
  a) converges pointwise by Dirichlet's Theorem for pointwise convergence.
  b) will converge at different rates of convergence at each point.
  c) is not uniformly convergent and therefore not absolutely convergent.
  d) exhibits Gibbs Phenomenon in every open interval around a jump discontinuity and does not converge to a continuous function (but does converge).
- A series that converges uniformly will not exhibit Gibbs phenomenon.
- if $\tilde{f}(x)$ is continuous everywhere then we expect absolute convergence.

0.1 Integration and Differentiation of Fourier Series

Theorem: term-by-term integration

Let $\sum_{n=0}^{\infty} f_n(x)$ be defined on $[a, b]$. If each $f_n(x)$ is continuous on $[a, b]$ and if the series $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly to $f(x)$ on $[a, b]$ then

(i) $f(x) = \sum_{n=0}^{\infty} f_n(x)$ is continuous on $[a, b]$

(ii)
$$\int_a^b \sum_{n=0}^{\infty} f_n(x)dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)dx.$$

Theorem: term-by-term differentiation

Suppose $\sum_{n=0}^{\infty} f_n(x)$ converges pointwise to $f(x)$ in $[a, b]$.

Suppose $f_n'(x)$ exists for each n and is continuous on $[a, b]$ and suppose the series $\sum_{n=0}^{\infty} f_n'(x)$ converges uniformly on $[a, b]$ then

$$f'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} f_n(x)\right] = \sum_{n=0}^{\infty} \frac{d}{dx} f_n(x).$$

Theorem: univerform convergence theorem

Let $f(x)$ be a continuous piecewise smooth function $[-L, L]$ such that $f(-L) = f(L)$. Then F.S.$[f](x)$ converges uniformly to $f(x)$ on $[-L, L]$. That is,

$$\lim_{n\to\infty} \max_{-L\leq x\leq L} |S_N(x) - f(x)| = 0.$$

Pinsky book has much more analysis.

1 Derivation of the Heat Equation 1

1.1 Insulated Road

We model the transfer of thermal energy in a one dimensional rod with ends at $x = 0$ and $x = L$ and where the lateral surface of the rod is insulated perfectly.

Definition

1) The **thermal energy density** $e(x, t)$ is the amount of thermal energy per unit volume.
2) Consider a thin slice of the rod with cross sectional area A between x and $x + \Delta x$. The heat energy changes in time due only to heat flowing across the edges (x and $x + \Delta x$). If Δx is small then $e(x, t)$ may be approximated as constant throughout the slice so:

$$\text{heat energy in slice } [x, x + \Delta x] = e(x, t) \cdot A \cdot \Delta x.$$

Integrating it yields:

$$\text{Total heat energy in the rod} = \int_0^L e(x, t) A dx.$$

Definition: heat flux

The **heat flux**, $\Phi(x, t)$, is the amount of thermal energy flowing to the right per unit time per unit surface area. If $\Phi(x, t) < 0$ then energy flows to the left.

The heat energy flow per unit time across the boundaries of slice $[x, x + \Delta x]$

with cross sectional surface area A is:

$\Phi(x,t)\cdot A$ (heat gain) $+(-\Phi(x+\Delta x,t)\cdot A$ (heat loss) $= -[\Phi(x+\Delta x,t) - \Phi(x,t)]\cdot A$.

Definition

In the model we allow for **internal sources of energy**. Let $Q(x,t)$ be the heat energy generated per unit volume per unit time within the rod then

$$\text{heat energy per unit time } = Q(x,t)\cdot A\cdot \Delta x.$$

Theorem: Heat Flow Process

The fundamental heat flow process in the rod is conceptually described as:

rate of change of heat energy wrt time = heat energy flowing across boundaries per unit time + heat energy generated inside the rod per unit time.

Now consider any finite sement of the rod (from a to b), then the conservation of heat energy principle given above implies:

$$\frac{d}{dt}\int_a^b e(x,t)A dx = -[\Phi(b,t) - \Phi(a,t)]\cdot A + \int_a^b Q(x,t)A dx.$$

which after canceling $A > 0$ can be rewritten as (by fundamental theorem of calculus):

$$\int_a^b \frac{\partial}{\partial t}e(x,t)dx = -\int_a^b \frac{\partial}{\partial x}\Phi(x,t)dx + \int_a^b Q(x,t)dx.$$

which yields the "Integral Conservation Law"

$$\int_a^b \left[\frac{\partial}{\partial t}e(x,t) + \frac{\partial}{\partial x}\Phi(x,t) - Q(x,t)\right] dx = 0.$$

which holds for any a and b within the rod, and since the integrand is assumed to be continuous, this implies (proof by contradiction):

$$\frac{\partial}{\partial t}e(x,t) + \frac{\partial}{\partial x} - Q(x,t) = 0 \Rightarrow \frac{\partial}{\partial t}e(x,t) = -\frac{\partial}{\partial x}\Phi(x,t) + Q(x,t).$$

If $\partial_x \Phi > 0$ then Φ is an increasing function in x so the heat flowing to the right at $x = b$ is greater than the heat flowing to the right at $x = a$ thus the heat energy decreases between $x = a$ and $x = b$ (hence the minus sign).

Definition: heat capacity

1) Let $u(x,t)$ be the temperature of the rod at point x and at time t. Note that it may take different amounts of thermal energy to raise two different materials from one temperature to another.

2) Define the heat capacity, $c(x,u)$, to be the heat energy required to raise its temperature one unit. We will either assume $c = c(x)$ or c is a constant.

3) An alternate description of thermal energy is that it is the amount of energy needed to raise the rod's temperature from 0 to the actual temperature $u(x,t)$. Thus if $\rho(x)$ is the mass density of the road then

$$\text{heat energy } = c(x) \cdot u(x,t) \cdot \rho(x) \cdot A \cdot \Delta x.$$

now equating the expression for heat energy derived earlier with this expression yields

$$e(x,t) \cdot A\Delta x = c(x)u(x,t)\rho(x) \cdot A\Delta x \Rightarrow e(x,t) = c(x)\rho(x)u(x,t).$$