13. Measurable Functions

Consider two measurable spaces $(\Omega, \mathcal{F}), (\Omega', \mathcal{F}')$, where $\mathcal{F}, \mathcal{F}'$ are σ-fields. Let $T : \Omega \to \Omega'$ be a mapping.

Definition: measurable

T is **measurable** if $\forall\ A' \in \mathcal{F}'$,

$$T^{-1}(A') = \{\omega \in \Omega : T(\omega) \in A'\} \in \mathcal{F}.$$

We also say T is F/F' measurable or measurable F/F'.

Example. A r.v. X is measurable $F/\mathcal{B}(\mathbb{R})$.

Theorem: 13.1

$(\Omega, \mathcal{F}), (\Omega', \mathcal{F}'), T : \Omega \to \Omega'$.

(i) If $\mathcal{F}' = \sigma(\mathcal{A}')$ and if $T^{-1}(A') \in \mathcal{F}\ \forall\ A' \in \mathcal{A}'$, then T is measurable F/F'.

(ii) Composition: If T is F/F' measurable, and $T' : \Omega' \to \Omega''$ is measurable, given $(\Omega'', \mathcal{F}''), T'(T(x)$ is measurable $\mathcal{F}'/F''$, then $T' \circ T$ is measurable F/F''.

Remark. The theorem from r.v. comes directly from (i), where we have $X^{-1}(-\infty, x]$ which is a closed interval. Instead of checking every inverse image of element in the generated σ-field is in $\mathcal{F}$, we check if the inverse image of every element in the set that generated the σ-field is in $\mathcal{F}$.

Notation. $T^{-1}A = T^{-1}(A)$.

Proof

(i) Consider a class of sets $\mathcal{G}' = \{A' \subseteq \Omega' : T^{-1}(A') \in \mathcal{F}\}$. Clearly $\mathcal{A}' \subseteq \mathcal{G}'$ by assumption. We claim that $\mathcal{G}'$ is a σ-field on Ω'. Thus, $\sigma(\mathcal{A}') = \mathcal{F}' \subseteq \mathcal{G}'$. Therefore, any sets $A' \in \mathcal{F}'$ will map back (under T) to $\mathcal{F}$.

(ii) proof omitted.

$\square$

Example. Space: $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ where μ assigns finite measure to bounded sets in $\mathbb{R}$. Define a function $F : \mathbb{R} \to \mathbb{R}$,

$$F(x) = \begin{cases} \mu((0, x]) & \text{if } x \geq 0 \\ -\mu((x, 0])) & \text{if } x < 0 \end{cases}$$

Property.

1) $F(x) < \infty \ \forall \ x \in \mathbb{R}$.
2) Non-decreasing, *i.e.* $x_1 \leq x_2 \Rightarrow F(x_1) \leq F(x_2)$.

Proof

Case. $0 \leq x_1 \leq x_2 \Rightarrow (0, x_1) \subseteq (0, x_2] \Rightarrow \mu((0, x]) \leq \mu((0, x_2]) \Rightarrow F(x_1) \leq F(x_2)$.

$\square$

3) We can write $\mu((a, b])$ in terms of F: $\mu((a, b]) = F(b) - F(a)$.

Case. $0 \leq a \leq b$. Then

$$\begin{aligned} \mu((a, b]) &= \mu((0, b] \setminus (0, a]) \\ &= \mu((0, b]) - \mu((0, a]) \\ &= F(b) - F(a) \end{aligned}$$

Case. $a < 0, b \geq 0$. Then

$$\begin{aligned} \mu((a, b]) &= \mu((a, 0]) + \mu((0, b]) \\ &= -F(a) + F(b) \end{aligned}$$

4) F is right-continuous.

Proof

Suppose (x_n) is such that $x_n \searrow x$. We want to show $F(x_n) \searrow F(x)$.

Case. $x \geq 0$, define $A_n = (0, x_n], A = (0, x]$. Then $A_n \searrow A$. By continuity from above (need $\mu(A_1) < \infty$) $\Rightarrow \mu(A_n) \searrow \mu(A) \Leftrightarrow F(x_n) \searrow F(x)$.

$\square$

Theorem: 12.4

If F is a non-decreasing, right-continuous, real-valued ($\mathbb{R} \to \mathbb{R}$), then there exists a unique measure μ on $\mathcal{B}(\mathbb{R})$ such that $\mu((a,b]) = F(b) - F(a)\ \forall\ a \leq b$.

Proof

Define a measure μ on $\mathcal{A}$ which is the collection of all intervals of the form $(a,b]$ for $a < b$ such that $\mu(\emptyset) = 0$ and $\mu((a,b])) = F(b) - F(a)$. Note that $\mathcal{A}$ is a semiring and can generate the Borel sets. By Theorem 11.3, μ extends to $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$. Since $F : \mathbb{R} \to \mathbb{R}$, we cannot get infinity for the measure and can cover $\mathbb{R}$ by these half closed intervals, so it is σ-finite. By Theorem 10.3 (note μ is σ-finite because intervals are finite and cover the reals), the extension is unique. $\square$

Note. Borel sets on $\mathbb{R}$ can be generated by

1) open intervals on $\mathbb{R}$
2) closed intervals on $\mathbb{R}$.
3) half open intervals.
4) $(-\infty, a]$ or $(-\infty, a)$.
5) open sets on $\mathbb{R}$.

Claim.

1) A continuous function $f : \mathbb{R} \to \mathbb{R}$ is measurable.

   Proof

   We need to show that inverse image of a set in $\mathcal{B}(\mathbb{R})$ is in $\mathcal{B}(\mathbb{R})$. Take $\mathcal{A}$ as all the open sets in $\mathbb{R}$. Then the inverse image of open sets is open since F is continuous. Thus by Theorem, 13.1, we are done. $\square$

2) $f_i : \Omega \to \mathbb{R}$ for $i = 1, \ldots, k$, each measurable $F/\mathcal{B}(\mathbb{R})$ and $g : \mathbb{R}^k \to \mathbb{R}$ measurable $\mathcal{B}(\mathbb{R}^k)/\mathcal{B}(\mathbb{R})$. Then $g(f_1(\omega), \ldots, f_k(\omega))$ is measurable $F/\mathcal{B}(\mathbb{R})$.

   Proof

f_i measurable implies $f(\omega) \coloneqq (f_1(\omega), \ldots, f_k(\omega))$ is measurable. Then the result follows from composition of functions. $\square$

3) Suppose $f : \Omega \to [-\infty, \infty]$ (extended reals) is measurable if $A \in \mathcal{B}(\mathbb{R}) \Rightarrow f^{-1}(A) \in \mathcal{F}$ and $\{\omega : f(\omega) = \infty\}$ and $\{\omega : f(\omega) = -\infty\}$ are in $\mathcal{F}$.

Theorem: 13.4

$(\Omega, \mathcal{F})$. Consider a sequence of measurable functions, $f_1, f_2, \ldots$ from $\Omega \to \mathbb{R}$. Then, the following are true:

(i) $\sup_n f_n, \inf_n f_n, \limsup_n f_n, \liminf_n f_n$ are measurable.

(ii) $\lim\limits_{n\to\infty} f_n$ if it exists everywhere is measurable.

(iii) $\{\omega : f_n(\omega) \text{ converges}\} \in \mathcal{F}$

Theorem

$(\Omega, \mathcal{F})$, $f, g : \Omega \to \mathbb{R}$ measurable. Then

(i) $\{\omega : f(\omega) > g(\omega)\} \in \mathcal{F}$.

(ii) $\{\omega : f(\omega) = g(\omega)\} \in \mathcal{F}$.

Proof

(i) Note that for any real $a > b$, there exists a $r \in \mathbb{Q}$ such that $a > r > b$. Hence

$$\{\omega : f(\omega) > g(\omega)\} = \bigcup_{r\in\mathbb{Q}} \left[\{\omega : f(\omega) > r\} \cap \{\omega : g(\omega) < r\}\right].$$

The first set is the complement in $\mathcal{F}$ since f is measurable, The second is also in $\mathcal{F}$ since g is measurable, Their unions and intersections are in $\mathcal{F}$.

(ii)

$$\{\omega : f(\omega) = g(\omega)\} = \left(\{\omega : f(\omega) > g(\omega)\} \cup \{\omega : f(\omega) < g(\omega)\}\right)^c \in \mathcal{F}.$$

$\square$