1 Multi-step Methods

1.1 Adams-Bashforth 3-step

$$y_{i+1} = y_i + h[af(x_i, y_i) + bf(x_{i-1}, y_{i-1}) + cf(x_{i-2}, y_{i-2})].$$

Using Taylor's series expansion on two previous points, -h and -2h away from x_i,

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2}y_i''' + \frac{h^3}{6} + \ldots.$$

$$f(x_{i-1}, y_{i-1}) = y_i' - hy_i'' + \frac{h^2}{2}y_i'' - \frac{h^3}{6}y_i''' + \ldots.$$

$$f(x_{i-2}, y_{i-2}) = y_i' - 2hy_i'' + 2h^2yi''' - \frac{4}{3}h^3y_i''''.$$

Plugging back into the first equation, we obtain

$$\begin{aligned}
h :& y_i' = ay_i' + 6y_i' + cy_i' \\
h^2 :& \frac{1}{2}y_i'' = -by_i'' - 2cy_i'' \\
h^3 :& \frac{1}{6}y_i''' = \frac{b}{2}y_i''' + 2cy_i''' \\
h^4 :& \frac{1}{24} = \ldots \text{ error}
\end{aligned}$$

And the grand accounting gives us the coefficients.

$$y_{i+1} = y_i + \frac{h}{12}[23f(x_i, y_i) - 16f(x_{i-1}, y_{i-1}) + 5f(x_{i-2}, y_{i-2})].$$

Let $\phi = \frac{h}{12}[23f(x_i, y_i) - 16f(x_{i-1}, y_{i-1}) + 5f(x_{i-2}, y_{i-2})]$. The local truncation error is $\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - \frac{\phi}{h}$ and is $\mathcal{O}(h^3)$.

1.2 Adams-Moulton Methods 2-step

We can also let

$$y_{i+1} = y_i + h[af(x_{i+1}, y_{i+1}) + bf(x_i, y_i + cf(x_{i-1}, y_{i-1})].$$

Note that this uses two old and one new points. Adams-Bashforth only uses old points. "Step" refers to old points only.

1.3 Predictor-Corrector Method/Modified Euler Method

We would like to use explicit method to find implicit solution.

$$y_{i+1}* = y_i + hf(x_i, y_i)y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_{i+1}*)].$$

Average the slopes between two points.

1.4 Generating seed values

Suppose $y' = y + x$. If x is small, then $\frac{y'}{y} \approx 1$ and $y(0) = 0$, whose solution is $y = 0$, but this solution violated our assumptions. If y is small, we get $y = \frac{x^2}{2}$. Since $x >> y = \frac{x^2}{2}$ for $|x| << 1$, so we can approximate $y \approx \frac{x^2}{2}$