Definition: Cartesian product

Given groups $(G_1, *), \ldots, (G_n, *)$. Then $G_1 \times \ldots \times G_n$ are the Cartesian product. The element in this product is the n-tuple $(g_1, \ldots, g_n) : g_i \in G_i$. The product of the elements is componentwise.

Note.

- a finite group is finitely generated. It's generated by itself.
- $\mathbb{Z}$ is finitely generated. It's generated by 1.

WARNING: $\mathbb{Z}_2 \times \mathbb{Z}_2 \not\cong \mathbb{Z}_4$. Because the former is isomorphic to V_4.

Recall the fundamental theorem of arithmetic claims that any number can be written as a product of primes. The factorization is unique up to permutation. We will generalize this to groups.

Note. Rational numbers are not finitely generated.

Theorem: The Fundamental Theorem of Finitely Generated Abelian Groups

Any finitely generated abelian group G is isomorphic to

$$G_1 \times G_2 \times \ldots \times G_n$$

where each G_i is either isomorphic to

- a cyclic group of prime power order, $\mathbb{Z}_{p^r}$, where p is prime and $r \in \mathbb{N}$.
- or $\mathbb{Z}$.

Two such groups $G_1 \times \ldots \times G_n$ and $H_1 \times \ldots \times H_m$ are isomorphic if and only if $m = n$ and the factors are rearrangements of each other.

Note. Rearrangement is the only way to have isomorphism. Unlike general groups. This theorem is powerful to tell when two finitely generated abelian groups are not isomorphic.

Example (abelian group of order 8)**.**

1) $\mathbb{Z}_8$. The building blocks are $\mathbb{Z}_8, \mathbb{Z}_4, \mathbb{Z}_2$.
2) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ not isomorphic to $\mathbb{Z}_8$ by the fundamental theorem, as they are all power of prime numbers but they are not rearrangements.
3) $\mathbb{Z}_4 \times \mathbb{Z}_2$.
4) $\mathbb{Z}_2 \times \mathbb{Z}_4$ is isomorphic to 3) by theorem.

Claim. Direct products of abelian groups are abelian.

1) $\mathbb{Z}_8 \mathbb{Z}_{2^3}$ 3

2) $\mathbb{Z}_4 \times \mathbb{Z}_2 \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^1}$ 2+1

3) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \mathbb{Z}_{2^1} \times \mathbb{Z}_{2^1} \times \mathbb{Z}_{2^1}$ 1+1+1

These are "partitions of 3": Sequences of positive integers that sums to 3 and are decreasing. Partitions of n control abelian groups of order 2^n.

Example (partition of 4)**.** $4, 3+1, 2+2, 2+1+1, 1+1+1+1$. Use this to investigate abelian groups of order 81= 3^4.

1) $\mathbb{Z}_{3^4}$

2) $\mathbb{Z}_{3^3} \times \mathbb{Z}_3$

3) $\mathbb{Z}_{3^2} \times \mathbb{Z}_{3^2}$

4) $\mathbb{Z}_{3^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3$

5) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

Example. Classify all abelian group of order 360.

$360 = 2^3 \times 3^2 \times 5^1$. Consider each power of prime term individually.

1) abelian groups of order 8.

2) order 9: $\mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$

3) order 5: $\mathbb{Z}_5$.

Then we just pick one out of each and do direct product. Note that $\mathbb{Z}_{360} \simeq \mathbb{Z}_{72} \times \mathbb{Z}_5 \simeq \mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_5$.

Example. Does $\mathbb{Z} \simeq \mathbb{Z} \times \mathbb{Z}_2$? No use theorem and observe these are not rearrangement.

Example (11.18)**.** Is $\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24}$ isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40}$?

Former $\simeq \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_8$.
Latter $\simeq \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \mathbb{Z}_5 \times \mathbb{Z}_8$. Not isomorphic. This is the repeated application of $\mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{mn}$ if $\gcd(m, n) = 1$.

Note. Abelian group has subgroups of every allowed orders by Lagrange.

Example (11.10)**.** $(8, 4, 10) \in \mathbb{Z}_{12} \times \mathbb{Z}_{60} \times \mathbb{Z}_{24}$. What is the order? Break it down to each. Order of 8 in $\mathbb{Z}_{12}$? 3. 4:15. 10:12. The answer $\operatorname{lcm}(3, 15, 12) = \frac{15 \times 12}{\gcd(15,12)} = 60$.