For the convergence of the partials, take the time partial of the solution and use the heat equation to get the 2nd space partial:

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} B_n \cdot -k\left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right) \cdot e^{-(\frac{n\pi}{L})^2 kt}$$
$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} B_n \cdot -\left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt}$$

They have the general form

$$\sum_{n=1}^{\infty} \tilde{B}_n n^2 \sin\left(\frac{n\pi x}{L}\right) e^{-Ct(n)^2}.$$

For some constant $C > 0$ and $\tilde{B}_n$. Since $t > 0$

$$n \geq 1 \Rightarrow n^2 \geq n \Rightarrow Ctn^2 \geq Ctn \Rightarrow e^{Ctn^2} \geq e^{Ctn} \Rightarrow e^{-CTn^2} \leq e^{-Ctn}.$$

So by triangle inequality we have

$$\sum_{n=1}^{\infty} \left| \tilde{B}_n n^2 \sin\left(\frac{n\pi x}{L}\right) e^{-Ctn^2} \right| \leq \sum_{n=1}^{\infty} |\tilde{B}_n| \cdot n^2 \cdot 1 \cdot e^{-Ctn}.$$

And if $\tilde{B}_n$ is bounded by some $M > 0$, then we have

$$\sum_{n=1}^{\infty} \left| \tilde{B}_n n^2 \sin\left(\frac{n\pi x}{L}\right) e^{-Ctn^2} \right| \leq \sum_{n=1}^{\infty} M n^2 e^{-Ctn}.$$

So it suffices to show that the RHS converges to show the convergence of the partials. This has been done in the homework using the ratio test. Hence the partials converge!

1 Interpret Solution

Theorem: convergence of a series solution of the heat equation

For $t > 0$, if there exists a constant $M > 0$ such that $|B_n| \leq M \ \forall\ n$, then

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt}$$

converges absolutely for each $x \in [0, L]$.

Proof

Note that given any n,

$$\left| B_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt} \right| \leq |B_n| \cdot 1 \cdot e^{-(\frac{n\pi}{L})^2 kt} \leq M e^{-(\frac{n\pi}{L})^2 kt}.$$

so given any $N > 0$, we have

$$0 < \sum_{n=1}^{N} \left| B_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt} \right| \leq \sum_{n=1}^{N} M e^{-(\frac{n\pi}{L})^2 kt}.$$

and taking the limit $N \to \infty$ yields

$$0 < \sum_{n=1}^{\infty} \left| B_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2 kt} \right| \leq \sum_{n=1}^{\infty} M e^{-(\frac{n\pi}{L})^2 kt}.$$

by Order Limit Theorem.

Now again $n \geq 1 \Rightarrow n^2 \geq n$ and since $\left(\frac{\pi}{L}\right)^2 kt > 0$,

$$e^{-(\frac{n\pi}{L})^2 kt} \leq e^{-(\frac{\pi}{L})^2 ktn}.$$

and since this holds for any n,

$$0 < \sum_{n=1}^{\infty} M e^{-(\frac{n\pi}{L})^2 kt} \leq \sum_{n=1}^{\infty} M e^{-(\frac{\pi}{L}) ktn} = \sum_{n=1}^{\infty} M \left[e^{-(\frac{\pi}{L}) kt} \right]^n < \infty.$$

The last step comes from convergence of Geometric series: Note $e^{(\frac{\pi}{L})kt} > e^0 = 1$, so the inverse is < 1. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} M e^{-(\frac{\pi}{L})^2 ktn} &= \sum_{n=1}^{\infty} M e^{-(\frac{\pi}{L})kt} e^{-(\frac{\pi}{L})ktn-1} \\
&= \sum_{n=1}^{\infty} a \cdot r^{n-1} \\
&= \frac{a}{1-r} \\
&= \frac{M e^{-(\frac{\pi}{L})kt}}{1 - e^{-(\frac{\pi}{L})kt}} < \infty
\end{aligned}$$

Therefore, by direct comparison test, the Fourier sine series converges absolutely on $[0, L]$. $\square$

Note. For the heat equation if we start with reasonable data then the solution is almost guaranteed to converge. The assumption of $|B_n| < M$ needs to hold. And since B_n is a definite integral, and its boundedness only depends on $f(x)$. As long as $f(x)$ is "nice", *i.e.* piecewise continuous with no crazy spikes, then it converges.

Example.

$$\begin{cases} \text{PDE: } \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, & 0 < x < L, t > 0 \\ \text{BC: } u(0,t) = 0 = u(L,t), & t > 0 \\ \text{IC:} u(x,0) = 100, & 0 \le x \le L \end{cases}$$

$$\begin{aligned} \frac{2}{L}\int_0^L 100\sin\left(\frac{n\pi x}{L}\right)dx &= \frac{200}{L}\cdot -\cos\left(\frac{n\pi x}{L}\right)\frac{L}{n\pi}\Bigg|_{x=0}^{x=L} \\ &= -\frac{200}{n\pi}\left[\cos(n\pi) - 1\right] \\ &= -\frac{200}{n\pi}\left[(-1)^n - 1\right] \\ &= \frac{400}{n\pi} \text{ if n is odd} \end{aligned}$$

So all even terms vanish, then B_n is a decreasing sequence. So $|B_n| \le M = \frac{400}{n\pi}$ for all $n \ge 1$. Thus for $t > 0$, $u(x,t)$ is absolutely convergent for each x. The series solution has the form:

$$u(x,t) = \frac{400}{\pi}\sum_{p=1}^{\infty}\frac{1}{2p-1}e^{-(\frac{(2p-1)\pi}{L})^2 kt}\sin\left(\frac{(2p-1)\pi x}{L}\right).$$