Heat Equation on $\mathbb{R}$

$$\begin{cases} \text{PDE: } \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} & x \in \mathbb{R}, t > 0 \\ \text{"BCs": } u(x,t) < \infty & \forall\, x \in \mathbb{R}, t > 0 \\ \text{ICs: } u(x,0) = U(x) & \text{where } \int_{-\infty}^{\infty} |U(x)|dx = M < \infty \end{cases}$$

Again PDE and BCs form a vector space so we can use separation of variables. Then we get

$$\begin{cases} F''(x) = \lambda F(x) \\ F(x) < \infty \;\forall\, x \in \mathbb{R}, t > 0 \end{cases} \quad \text{and } G'(t) = \lambda k G(t)$$

Note. This λ has opposite sign as the one we used before.

If $\lambda > 0$, the solutions are not bounded. If $\lambda = 0$, we have $F_0(x) = B$. If $\lambda < 0$, we get $\lambda = -m^2$ for nonzero $m \in \mathbb{R}$. And

$$F_n = A_m e^{imx} + B_m e^{-imx} \text{ for } m \in \mathbb{R}.$$

For the time domain problem,

$$G'(t) = \lambda k G(t) \Rightarrow G_m(t) = G(0)e^{-m^2 kt} \text{ for } m \in \mathbb{R}.$$

Thus, summing over all product solutions with real numbers $m \in \mathbb{R}$ yields

$$u(x,t) = \int_{-\infty}^{\infty} a_m e^{imx} e^{-m^2 kt} dm + \int_{-\infty}^{\infty} b_m e^{-imx} e^{-m^2 kt} dm.$$

Note. The complex exponential is oscillating whereas the real exponential is a time-decaying term.

Notice that the second integral is redundant, since we can just use a change of variable $m = -p$ to get the same form. So

$$u(x,t) = \int_{-\infty}^{\infty} a_m e^{imx} e^{-m^2 kt} dm.$$

To find a_m, we use the IC:

$$U(x) = u(x,0) = \int_{-\infty}^{\infty} a_m e^{imx} dm \text{ and } U(x) = \int_{-\infty}^{\infty} \hat{U}(m) e^{imx} dm \Rightarrow a_m = \hat{U}(m)$$

by form matching according to our theory. So we can use the Fourier transform as the coefficient. So the solution becomes

$$u(x,t) = \int_{-\infty}^{\infty} \hat{U}(m) e^{imx} e^{-m^2 kt} dm$$

where

$$\hat{U}(m) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(x)e^{-imx}dx.$$

Note that $\hat{U}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(x)dx$ which is the area under the IC curve divided by 2π. Intuitively this is like area divided by something like a length.

$\hat{U}(x)$ is bounded because by boundedness of the integral of $U(x)$

$$|\hat{U}(m)| \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} |U(x)| \cdot |e^{-imx}|dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |U(x)|dx \leq \frac{M}{2\pi}.$$

Remark. This is part of the *Riemann-Lebesgue Lemma* for Fourier transform.

Lemma: Riemann-Lebesgue for Fourier transform

Suppose $U(x)$ is defined on $-\infty < x < \infty$ and let

$$\hat{U}(m) = \mathcal{F}[U(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(x)e^{-imx}dx, \ \forall\ m \in \mathbb{R}.$$

If $\int_{-\infty}^{\infty} |U(x)|dx = M < \infty$, then

1) $\hat{U}(m)$ is bounded and $\hat{U}(m) \leq \frac{M}{2\pi}$.
2) $\hat{U}(m)$ is continuous for all real m.
3) $\hat{U}(m) \to 0$ as $m \to \infty$.

Note. This is a special case. Intuitively if the IC is "well-behaved". then the Fourier transform is also well-behaved. Then we can truncate $\hat{U}(m)$ without much loss of information.

Proof

1) See previous.
2) Given real number m_n, define $U_n(x) = U(x)e^{-im_nx}$ then $\ \forall\ x \in r$, we have

$$\lim_{n\to\infty} U_n(x) = \lim_{n\to\infty} U(x)e^{-im_nx} = U(x)e^{-imx}, |U_n(x)| \leq |U(x)|\ \forall\ n.$$

Thus by *Lebesgue Dominated Convergence Theorem*, this result yields

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} U_n(x)dx = \int_{-\infty}^{\infty} U(x)e^{-imx}dx \Rightarrow \lim_{m_n\to m} \hat{U}(m_n) = \hat{U}(m).$$

Thus $\hat{U}(m)$ is continuous.

3) We will only prove 3 for the simple case that $U(x)$ is the indicator function over the set $[-L, L]$:

$$U(x) = \begin{cases} 1, & \text{if } x \in [-L, L] \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} \left| \int_{-\infty}^{\infty} U(x) e^{-imx} \right| &= \left| \int_{-L}^{L} e^{-imx} \right| \\ &= \left| \frac{e^{-imL} - e^{imL}}{-im} \right| \\ &\leq \frac{|i^{e^{-imL}}| + |ie^{imL}|}{|m|} \\ &= \frac{2}{|m|} \end{aligned}$$

By Squeeze Theorem, taking $m \to \pm\infty$, this bound gives us

$$\lim_{m \to \pm\infty} \int_{-\infty}^{\infty} U(x) e^{-imx} dx = 0 \Rightarrow \hat{U}(m) \to 0 \text{ as } m \to \pm\infty.$$

Note that this result can be extended to any function $U(x)$ such that $\int_{-\infty}^{\infty} |U(x)| dx < \infty$. Such result is the *Riemann-Lebesgue Theorem.*

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