Online Appendices to: "Robust Real Rate Rules"

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Appendix A Setting nominal rates out of equilibrium

Real rate rules work as combining the rule with the Fisher equation leads the real rate terms to cancel out. Could this cancellation mask a singularity that would prevent the central bank from setting rates according to a real rate rule? To see the apparent problem, suppose that the economy is currently in period 1, and that all in future periods, the central bank's behaviour will be given by the simple real rate rule of equation (2). Assume the Fisher equation (1) always holds. Then for $t > 1$, $r_t + \mathbb{E}_t \pi_{t+1} = i_t = r_t + \phi\pi_t$.

Our discussion up to now would naturally lead the reader to conclude that $\pi_t = 0$ for all $t > 1$, unconditional on whatever happens in period 1. Suppose this were true. Then, the period 1 Fisher equation would imply that $i_1 = r_1$. Thus, apparently, nothing the central bank could do in period 1 could ever produce $i_1 \neq r_1$. In particular, it seems that the central bank cannot apply a real rate rule in period 1 if $\pi_1 \neq 0$. This is incorrect though, as if a real rate rule applies from period 1 onwards, it is only the case that $\pi_t = 0$ for all $t > 1$ if it happens that $\pi_1 = 0$. This confusion stems from us having given an incomplete description of equilibrium up to now. A full equilibrium description specifies the outcome for every possible history, not just those on the equilibrium path.

A full description of the standard equilibrium of the Fisher equation (1) and real rate rule (2) is as follows. Suppose the rule was introduced in period 1. Then, for all $t \geq 1$, if $\pi_s = 0$ for all $s \in \{1, \dots, t-1\}$, then $\pi_t = 0$. Otherwise, $\pi_t = \phi\pi_{t-1}$. This implies that on the equilibrium path, $\pi_1 = 0$ (as with $t = 1$, the set $\{1, \dots, t-1\}$ is empty), and hence $\pi_t = 0$ for all $t \geq 1$. However, suppose that off the equilibrium path, $\pi_1 \neq 0$. Then $\pi_2 = \phi\pi_1$, and hence the period 1 Fisher equation states that $i_1 - r_1 = \phi\pi_1$. Thus, $i_1 - r_1$ is not fixed; it is a function of

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period 1 inflation, something that the central bank can affect in period 1 via open market operations. There is no singularity.[1]

Appendix B Responding to other endogenous variables

The original Taylor rule contained a response to output. Even with a unit coefficient on the real interest rate, responding to output will change determinacy conditions, though it still preserves some robustness. To see this, consider the monetary rule, $i_t = r_t + \phi_\pi \pi_t + \phi_x x_t + \zeta_t$. Suppose the lag-augmented NK Phillips curve (9) holds, then this monetary rule is equivalent to the rule:

$$i_t = r_t + \phi_\pi \pi_t + \kappa^{-1}\phi_x\big[\pi_t - \tilde{\beta}(1-\varrho_\pi)\mathbb{E}_t\pi_{t+1} - \tilde{\beta}\varrho_\pi\pi_{t-1}\big] - \phi_x\omega_t + \zeta_t.$$

(This is produced by using the Phillips curve to substitute out the output gap.) Combined with the Fisher equation, we have that:

$$\mathbb{E}_t\pi_{t+1} = \phi_\pi \pi_t + \kappa^{-1}\phi_x\big[\pi_t - \tilde{\beta}(1-\varrho_\pi)\mathbb{E}_t\pi_{t+1} - \tilde{\beta}\varrho_\pi\pi_{t-1}\big] - \phi_x\omega_t + \zeta_t.$$

This has a determinate solution if the quadratic:

$$\big[1 + \kappa^{-1}\phi_x\tilde{\beta}(1-\varrho_\pi)\big]A^2 - (\phi_\pi + \kappa^{-1}\phi_x)A + \kappa^{-1}\phi_x\tilde{\beta}\varrho_\pi = 0$$

has a unique solution for A inside the unit circle. It is sufficient that the quadratic is positive at $A = -1$ but negative at $A = 1$, which holds if and only if $1 + \kappa^{-1}\phi_x(1+\tilde{\beta}) + \phi_\pi > 0$ and $1 - \kappa^{-1}\phi_x(1-\tilde{\beta}) - \phi_\pi < 0$. So, if $\kappa > 0$, $\phi_x \geq 0$ and $\tilde{\beta} \in [0,1]$ as expected, then it is sufficient that $\phi_\pi > 1$ as before.[2] This is still considerable robustness. Providing there is something like a Phillips curve linking inflation and the output gap, the standard $\phi_\pi > 1$ condition will be sufficient for determinacy. This would not hold with a more standard monetary rule: in that case determinacy depends on $\tilde{\delta}$ and ς, as shown by the Bilbiie (2008;

[1] It is worth noting that there are other equilibria of equations (1) and (2) that imply an identical equilibrium path but generate more plausible behaviour off this path. Suppose that in period 1 when the rule is introduced, the economy starts in state A. Suppose also that each period a biased coin is tossed which comes up heads with probability $q \in (0,1]$. If the economy is in state A in period t, then $\pi_t = 0$, whereas if the economy is in state B in period t, then $\pi_t = \frac{\phi}{q}\pi_{t-1}$. For $t > 1$, the economy is in state A at t if and only if either (i) the economy was in state A at $t-1$ and $\pi_{t-1} = 0$, or (ii) the coin comes up tails. Otherwise, the economy is in state B at t. Thus, in state B, $\mathbb{E}_t\pi_{t+1} = q\frac{\phi}{q}\pi_t + (1-q)0 = \phi\pi_t$, as required. Hence, explosions need not last for ever following a deviation.

[2] This is stronger than necessary. The second condition states that $\phi_\pi + \kappa^{-1}\phi_x(1-\tilde{\beta}) > 1$ so a response to the output gap can substitute for a response to inflation. This condition is identical to that for the standard (purely forward looking) three equation NK model with Taylor type rule found in Woodford (2001).

2019) results discussed in Subsection 3.2 of the main text.

Responding to real rates provides additional robustness even with a response to output as it disconnects the Euler equation from the rest of the model. The only remaining role of the Euler equation is to give a path for real rates, given the already determined paths of output and inflation.[3] The Fisher equation, not the Euler equation is central to monetary policy transmission under real rate rules.

For greater robustness, the central bank can replace the response to the output gap with a response to the cost push shock ω_t. With an appropriate response to ω_t, this is observationally equivalent to responding to the output gap, but ensures determinacy under the standard Taylor principle.

However, it may be hard for the central bank to observe the cost push shock. To get round this, suppose that the central bank knows that a Phillips curve in the form of equation (9) holds. (Our results would generalize to other links between real and nominal variables.) For now, suppose the central bank also knows the coefficients in equation (9). Then the central bank could use a rule of the form:

$$i_t = r_t + \phi_\pi \pi_t + \phi_x \left[x_t - \kappa^{-1} \left[\pi_t - \tilde{\beta}(1 - \varrho_\pi) \mathbb{E}_t \pi_{t+1} - \tilde{\beta} \varrho_\pi \pi_{t-1} \right] \right] + \zeta_t.$$

By equation (9), this implies that, $i_t = r_t + \phi_\pi \pi_t - \phi_x \omega_t + \zeta_t$, as desired. Of course, the central bank is also unlikely to know the exact coefficients in the Phillips curve. However, we show in Supplemental Appendix J.5 that the central bank may learn these coefficients in real time, without changing the determinacy conditions, at least under reasonable parameter restrictions.[4]

If the central bank wishes to respond to other endogenous variables, a similar approach should be possible if they are aware of the broad form of the model's structural equations. However, the central bank may legitimately worry about having fundamental misconceptions about how the economy

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[3] This is analogous to how the Euler equation is slack when solving for optimal monetary policy. In that case, the combined Euler equation and Fisher equation give the level of nominal rates required to hit the optimal output gap and inflation. The author thanks Florin Bilbiie for this observation.

[4] It is sufficient (but not necessary) that $\phi_x \geq 0$, $\phi_\pi \geq 0$, $\kappa \geq 0$, $\tilde{\beta} \in [0,1]$, $\varrho_\pi \in [0,1)$, $\rho \in [0,1)$ and $\phi_\pi > \max \left\{ \frac{1}{\tilde{\beta}(1-\varrho_\pi)}, 2(1-\varrho_\pi), \frac{\phi_x(1+\tilde{\beta})}{\kappa} \right\}$.

works. They can be reassured though that the Taylor principle is sufficient for determinacy if the response to other endogenous variables is small enough, no matter the form of the model's other equations. We prove this in Supplemental Appendix J.1. This also implies that a precise unit response to real rates is not needed for determinacy. Real rates are just another endogenous variable, so determinacy only requires a response sufficiently close to one.

Classic results on determinacy in monetary models can be reinterpreted through this lens. Even if the central bank is not responding to real rates, it is still likely to be responding to variables that are correlated with them. Our results imply that rules sufficiently close to a real rate rule must be determinate.

For example, many models contain an Euler equation of the form:

$$1 = \beta(\exp r_t)\mathbb{E}_t\left(\frac{C_t}{C_{t+1}}\right)^{\frac{1}{\varsigma}},$$

where C_t is real consumption per capita and ς is the elasticity of intertemporal substitution. Additionally, in many models, in equilibrium, consumption growth roughly follows an ARMA(1,1) process:

$$g_t := \log\left(\frac{C_t}{C_{t-1}}\right) = (1-\rho_g)g + \rho_g g_{t-1} + \varepsilon_{g,t} + \theta_g \varepsilon_{g,t-1}, \qquad \varepsilon_{g,t} \sim \mathrm{WN}(0, \sigma_g^2).$$

(This is a good approximation to US post-war data.[5]) Combining these two equations gives that:

$$r_t = -\log\beta + \frac{1-\rho_g}{\varsigma}g - \frac{1}{2}\left(\frac{\sigma_g}{\varsigma}\right)^2 + \frac{\rho_g}{\varsigma}g_t + \frac{\theta_g}{\varsigma}\varepsilon_{g,t},$$

implying that a (roughly) $\frac{\rho_g}{\varsigma}$ response to consumption growth can substitute for a (roughly) unit response to real rates.

Of course, output (growth, level or gap) is in turn highly correlated with consumption growth, so output (growth, level or gap) may also substitute for real rates. For example, in the Smets & Wouters (2007) model of the US economy, the monetary rule is of the form $i_t = \phi_\pi \pi_t + z_t + \zeta_t$, where z_t is a linear combination of other endogenous variables and ζ_t is the monetary shock. At the estimated posterior mode, the correlation between z_t and the real interest

[5] Estimating on US data from 1947Q1 to 2021Q4 (BEA series: A794RX) with T-distributed shocks gives $\rho_g = 0.69$, $\theta_g = -0.50$ (p-values both below 10^{-5}). Using Gaussian shocks on less volatile sub-periods gives similar results.

rate is 0.63, with both variables having standard deviation of 0.46%. Thus, the Smets & Wouters (2007) estimates imply that (in a sense) the Fed is already about two thirds of the way to using a simple robust real rate rule.

There is one final way of allowing an interest rate response to other endogenous variables that is both simple and robust. Rather than placing the endogenous variables directly within the rule, the central bank can follow a time-varying inflation target which is a function of these endogenous variables. We propose this approach in Section 2 of the main text.

Appendix C Learning and bounded rationality

Our results on Fisher wedges in the main paper (Subsection 3.3) suggested that as long as θ is large enough, real rate rules should continue to work in the presence of departures from perfect rationality. Here, we verify this for several popular models of learning and bounded rationality.

C.1 Adaptive, naïve, and extrapolative expectations

Branch & McGough (2009) suppose that aggregate inflation expectations are a linear combination of rational expectations and an additional term capturing adaptive, naïve or extrapolative expectations. In particular, agents' period t expectation of period $t+1$ inflation is given by $\alpha\mathbb{E}_t\pi_{t+1} + (1-\alpha)\theta\pi_{t-1}$. Here, $\alpha \in [0,1]$ gives the weight on rational expectations, and $\theta \geq 0$ controls whether the non-rational part is adaptive ($\theta < 1$), naïve ($\theta = 1$) or extrapolative ($\theta > 1$). This leads to the behavioural Fisher equation $i_t = r_t + \alpha\mathbb{E}_t\pi_{t+1} + (1-\alpha)\theta\pi_{t-1}$.

We suppose that the central bank follows the monetary rule of equation (6), $i_t = r_t + \phi\pi_t + \zeta_t$, where ζ_t is an AR(1) process with persistence $\rho \in (-1,1)$, and where $\phi > 0$ at least. Combining this monetary rule with the behavioural Fisher equation then gives that $\alpha\mathbb{E}_t\pi_{t+1} - \phi\pi_t + (1-\alpha)\theta\pi_{t-1} = \zeta_t$. If $\alpha = 0$ (meaning there are no rational agents), then this is purely backwards looking and hence has a unique solution, given by $\pi_t = \phi^{-1}(1-\alpha)\theta\pi_{t-1} - \phi^{-1}\zeta_t$. For stability, we need $\phi > \theta$, which is stronger than $\phi > 1$ if $\theta > 1$. When $\alpha > 0$, the system is determinate if and only if $\phi > \alpha + (1-\alpha)\theta$, which again may be stronger than $\phi > 1$ if $\theta > 1$.[6] At least for sufficiently large ϕ though, the solution is unique and stable, even in the extrapolative case of $\theta > 1$. Furthermore, as $\phi \to \infty$, $\operatorname{var}\pi_t \to 0$. This means that sufficiently aggressive monetary policy is capable of squashing the variance of inflation, even in the presence of adaptive, naïve or extrapolative expectations.

C.2 Diagnostic expectations

Under diagnostic expectations (Bordalo, Gennaioli & Shleifer 2018; L'Huillier, Singh & Yoo 2021; Bianchi, Ilut & Saijo 2023), agents use the non-rational expectation operator $\mathbb{E}_t^\theta$ defined by:

$$\mathbb{E}_t^\theta v_{t+1} := \mathbb{E}_t v_{t+1}^{\mathrm{RE}:t+1} + \theta\left[\mathbb{E}_t v_{t+1}^{\mathrm{RE}:t+1} - \frac{1}{\sum_{j=1}^J \tilde{\alpha}_j}\sum_{j=1}^J \tilde{\alpha}_j \mathbb{E}_{t-j} v_{t+1}^{\mathrm{RE}:t-j+1}\right],$$

where v_t is any endogenous variable, $\theta > 0$ governs the overreaction to new information, $\tilde{\alpha}_1, \dots, \tilde{\alpha}_J$ govern the relative importance of memory at different horizons, and where $v_t^{\mathrm{RE}:s}$ is the value v_t would take if all agents had rational expectations from period s onwards. (This definition follows Bianchi, Ilut & Saijo (2023) in assuming agents take a naïve approach to dealing with their own time inconsistency.) In the following, we will take $J = \infty$, meaning that memory matters at all horizons. We set $\tilde{\alpha}_j = (1-\alpha)\alpha^{j-1}$ giving geometric discounting to distant memories, governed by the parameter $\alpha \in (0,1)$. The $J = 1$ case of L'Huillier, Singh & Yoo (2021) is nested here as the limit $\alpha \to 0$.

As before, we are interested in the solution to the model governed by the monetary rule of equation (6) with the diagnostic Fisher equation:

$$i_t = r_t + \mathbb{E}_t^\theta p_{t+1} - p_t,$$

where p_t is the logarithm of the price level, so $\pi_t = p_t - p_{t-1}$.[7] We assume $\phi > 1$. Note that the Fisher equation is given in terms of expectations of the price level, not of inflation. The two are not equivalent under diagnostic expectations, and it is the expectation of the price level that emerges from the Euler equation (see L'Huillier, Singh & Yoo (2021) and Bianchi, Ilut & Saijo (2023)).

Now, by the results of Subsection 3.1, for $t \geq s$, $p_t^{\mathrm{RE}:s} = p_{t-1}^{\mathrm{RE}:s} - \frac{1}{\phi-\rho}\zeta_t = p_{s-1} -$

[6] For determinacy the quadratic $q(\lambda) := \alpha\lambda^2 - \phi\lambda + (1-\alpha)\theta$ must have one root for λ inside the unit circle, and another outside. Note $q(0) = (1-\alpha)\theta \geq 0$, $q''(0) > 0$ and $q'(\lambda) = 0$ if and only if $\lambda = \frac{\phi}{2\alpha} > 0$. Thus, there is determinacy if and only if $0 > q(1) = \alpha - \phi + (1-\alpha)\theta$.

[7] See Supplemental Appendix G.2 for justification for introducing the price level in this way.

$\frac{1}{\phi-\rho}\sum_{k=s}^{t}\zeta_k$. Hence for $j \geq 0$:

$$\mathbb{E}_{t-j}p_{t+1}^{\mathrm{RE}:t-j+1} = p_{t-j} - \frac{1}{\phi-\rho}\sum_{k=t-j+1}^{t+1}\mathbb{E}_{t-j}\zeta_k = p_{t-j} - \rho\frac{\zeta_{t-j}}{\phi-\rho}\frac{1-\rho^{j+1}}{1-\rho}.$$

So, from the Fisher equation and the definition of diagnostic expectations:

$$\begin{aligned} i_t &= r_t + \mathbb{E}_t p_{t+1}^{\mathrm{RE}:t+1} + \theta\left[\mathbb{E}_t p_{t+1}^{\mathrm{RE}:t+1} - (1-\alpha)\sum_{j=1}^{\infty}\alpha^{j-1}\mathbb{E}_{t-j}p_{t+1}^{\mathrm{RE}:t-j+1}\right] - p_t \\ &= r_t - \rho\frac{\zeta_t}{\phi-\rho} + \theta\left[\pi_t - \rho\frac{\zeta_t}{\phi-\rho} - (1-\alpha)s_{t-1}\right], \end{aligned}$$

where the auxiliary state s_t evolves according to $s_t = \alpha s_{t-1} - \frac{\alpha}{1-\alpha}\pi_t - \rho(1+\rho)\frac{\zeta_t}{\phi-\rho} - z_{t-1}$, and where the auxiliary state z_t evolves according to $z_t = \alpha\rho z_{t-1} + \alpha\rho^3\frac{\zeta_t}{\phi-\rho}$. Thus, by the monetary rule:

$$\pi_t = -\left(1-\frac{\theta}{\phi}\right)^{-1}\left[\left(1+\frac{\theta}{\phi}\rho\right)\frac{\zeta_t}{\phi-\rho} + \frac{\theta}{\phi}(1-\alpha)s_{t-1}\right],$$

so:

$$s_t = \alpha\frac{\phi}{\phi-\theta}s_{t-1} + \left[\frac{\alpha}{1-\alpha}\left(1-\frac{\theta}{\phi}\right)^{-1}\left(1+\frac{\theta}{\phi}\rho\right) - \rho(1+\rho)\right]\frac{\zeta_t}{\phi-\rho} - z_{t-1}.$$

Inflation π_t is stationary if and only if s_t is stationary. Since z_t is clearly stationary, for s_t to be stationary, we need that $\alpha\frac{\phi}{\phi-\theta} \in (-1,1)$. Given our assumptions, this requires that $\phi > \max\{1, \frac{\theta}{1-\alpha}\}$. Bianchi, Ilut & Saijo (2023) estimate $\theta = 1.97$ and a mean memory horizon of around 5.44, corresponding to $\alpha = 0.18$, so we would need $\phi > 2.40$. If ϕ were below this value, then inflation would explode. This comes from compounding overreactions to past overreactions.

Still, for ϕ large enough, inflation is stationary. Furthermore, as $\phi \to \infty$, $\operatorname{var}\pi_t \to 0$, and $\operatorname{var}\big((\phi-\rho)\pi_t + \zeta_t\big) \to 0$ as well. The latter fact means that with even moderately high ϕ, inflation's dynamics are very close to its dynamics under rational expectations. Hence, as long as the central bank is moderately aggressive, a real rate rule gets inflation to target even in the presence of diagnostic expectations.

C.3 Finite horizon planning

Woodford (2019) gives a model of limited planning horizons. Agents are assumed to optimize over decisions in finitely many future periods, using a learned value function to evaluate outcomes at their planning horizon. We will focus on the simple case in which planning horizons are heterogeneous across agents, with a fraction $(1-\alpha)\alpha^j$ of households having planning horizon $j \in \mathbb{N}$, where $\alpha \in (0,1)$. With a learned terminal value function, this leads to the Fisher equation:[8]

$$i_t - \bar{\imath}_t = r_t - \bar{r}_t + \alpha \mathbb{E}_t(\pi_{t+1} - \bar{\pi}_{t+1}),$$

where the (learned) trend levels of nominal rates, real rates and inflation, $\bar{\imath}_t$, $\bar{r}_t$ and $\bar{\pi}_t$, respectively satisfy $\phi\bar{\pi}_t = \bar{\imath}_t - \bar{r}_t = \alpha\bar{\pi}_t + (1-\alpha)\mu_{t-1}$, assuming the monetary rule is again given by equation (6), where μ_t (a measure of the relative marginal value of real over nominal bonds) evolves according to:

$$\mu_t = (1-\gamma)\mu_{t-1} + \gamma(\pi_t - \bar{\pi}_t),$$

where $\gamma \in (0,1)$ controls the speed of household learning.[9] Thus, if we define $m := \frac{1-\alpha}{\phi-\alpha}$, then $\bar{\pi}_t = m\mu_{t-1}$, so:

$$\mathbb{E}_t \begin{bmatrix} \pi_{t+1} \\ \mu_t \end{bmatrix} = \begin{bmatrix} \alpha^{-1}[\phi + \alpha\gamma m] & -\alpha^{-1}m\big[\phi - \alpha[1-\gamma(1+m)]\big] \\ \gamma & 1-\gamma(1+m) \end{bmatrix} \begin{bmatrix} \pi_t \\ \mu_{t-1} \end{bmatrix} + \begin{bmatrix} \alpha^{-1} \\ 0 \end{bmatrix} \zeta_t.$$

The large matrix here has eigenvalues $1-\gamma \in (0,1)$ and $\frac{\phi}{\alpha}$. Thus, for determinacy, we just need that $\phi > \alpha$, which is strictly weaker than $\phi > 1$. The solution has:

$$\pi_t = m\mu_{t-1} - \frac{\zeta_t}{\phi - \alpha\rho}, \qquad \mu_t = (1-\gamma)\mu_{t-1} - \gamma\frac{\zeta_t}{\phi - \alpha\rho}.$$

As with diagnostic expectations, as $\phi \to \infty$, $\operatorname{var}\pi_t \to 0$, and $\operatorname{var}\big((\phi-\rho)\pi_t + \zeta_t\big) \to 0$ as well, so again a large ϕ brings dynamics towards those under rational expectations. It is particularly reassuring that with finite horizon planning, determinacy conditions are weaker than under rational expectations. Given a mix of finite horizon expectations and diagnostic or extrapolative ones, it is likely that ϕ not much larger than one would be sufficient.

C.4 Least squares learning

Under least squares learning (Marcet & Sargent 1989; Evans & Honkapohja 2001), agents update their beliefs about the laws of motion of endogenous

[8] We derive this by combining the Euler equation for nominal bonds in equation (61) of Woodford (2019) with an Euler equation for real bonds produced by setting $\pi_{t+1} = \bar{\pi}_{t+1} = 0$ in the same equation, (61).

[9] The right hand side of the equation for $\bar{\imath}_t - \bar{r}_t$ and the law of motion for μ_t are derived from subtracting versions of equations (46), (59), (65) of Woodford (2019) for real bonds from the corresponding equations for nominal bonds.

variables via recursive least squares. We suppose the real rate rule of equation (6) is introduced in period 1, and we allow agents to begin with prior beliefs that may not be centred on the rational expectations solution. For simplicity, we assume agents can directly observe the monetary shock ζ_t. (Without this assumption, we can still prove local convergence of beliefs to rational expectations. With it, we will have global convergence.) We also assume that the shock to ζ_t is normally distributed.

Since agents observe the exogenous process ζ_t, by the strong law of large numbers, their estimates of the parameters of ζ_t's law of motion converge almost surely. Thus, without loss of generality, we can assume that they already know these coefficients. Then, let v be the known variance of ζ_t. We suppose that in period t, agents believe that for all s, $\pi_s = a_t + b_t\zeta_s + \varepsilon_s$, where they believe $\mathbb{E}_{s-1}\varepsilon_s = 0$. Allowing for a constant seems natural, as they may not know the inflation target (here zero), or the size of the static Fisher equation wedge (here also zero). They estimate the coefficients a_t and b_t by recursive least squares, given some initial prior beliefs a_0 and b_0 with weight $w \geq 0$, and given the known value of v. Hence:

$$\begin{bmatrix} a_t \\ b_t \end{bmatrix} = \begin{bmatrix} a_{t-1} \\ b_{t-1} \end{bmatrix} + \frac{1}{t+w}\frac{1}{v}\begin{bmatrix} v \\ \zeta_t \end{bmatrix}(\pi_t - a_{t-1} - b_{t-1}\zeta_t).$$

Agents then approximate $\mathbb{E}_t\pi_{t+1}$ by $a_t + \rho b_t\zeta_t$, so from the monetary rule, $a_t + \rho b_t\zeta_t = \phi\pi_t + \zeta_t$. Thus, if we define $m_t := \frac{1}{t+w}\left(1 + \rho\frac{\zeta_t^2}{v}\right)$, then:

$$\pi_t = \frac{1}{\phi}[a_t + (\rho b_t - 1)\zeta_t] = \frac{1}{\phi}m_t\pi_t + \frac{1}{\phi}[(1-m_t)a_{t-1} + (\rho - m_t)b_{t-1}\zeta_t - \zeta_t].$$

So $\pi_t = (\phi - m_t)^{-1}[(1-m_t)a_{t-1} + (\rho - m_t)b_{t-1}\zeta_t - \zeta_t]$. Substituting this back into the law of motion for $\begin{bmatrix} a_t \\ b_t \end{bmatrix}$ gives a recurrence for these variables to which we can apply a slight generalization of Theorem 6.10 of Evans & Honkapohja (2001). We do this in Supplemental Appendix J.11 and so prove that if $\phi > 1$, then with probability one, a_t converges to 0 and b_t converges to $-\frac{1}{\phi-\rho}$. Since m_t converges in probability to zero, this implies $\pi_t + \frac{\zeta_t}{\phi-\rho}$ converges in probability to zero as well. Thus, agents succeed in learning the rational expectations solution, no matter what the initial conditions are. This guarantee of global stability under least squares learning is a large improvement over the situation with standard monetary rules, for which at best local stability can be proven (see e.g. Bullard & Mitra (2002)).

C.5 Constant gain learning

If agents believe parameters may be non-stationary, then it is no longer reasonable to perform standard least squares learning. Instead, it is natural to assume that they learn with a constant gain coefficient on new observations (Evans & Honkapohja 2001). This replaces the $\frac{1}{t+w}$ gain in the law of motion for $\begin{bmatrix} a_t \\ b_t \end{bmatrix}$ above with some constant, $\gamma > 0$. For simplicity, we start by looking at the $\rho = 0$ case. As before we assume that agents know the coefficients governing ζ_t, so they know that $\rho = 0$. Then a_t and b_t evolve according to:

$$\begin{aligned}\begin{bmatrix} a_t \\ b_t \end{bmatrix} &= \begin{bmatrix} a_{t-1} \\ b_{t-1} \end{bmatrix} + \gamma \frac{1}{v} \begin{bmatrix} v \\ \zeta_t \end{bmatrix} \left[\frac{(1-\gamma)a_{t-1} - \gamma b_{t-1}\zeta_t - \zeta_t}{\phi - \gamma} - a_{t-1} - b_{t-1}\zeta_t \right] \\ &= \begin{bmatrix} 1 - \gamma \dfrac{\phi - 1}{\phi - \gamma} & -\gamma \dfrac{\phi}{\phi - \gamma} \zeta_t \\ -\gamma \dfrac{\phi - 1}{\phi - \gamma} \dfrac{\zeta_t}{v} & 1 - \gamma \dfrac{\phi}{\phi - \gamma} \dfrac{\zeta_t^2}{v} \end{bmatrix} \begin{bmatrix} a_{t-1} \\ b_{t-1} \end{bmatrix} - \frac{\gamma}{\phi - \gamma} \begin{bmatrix} \zeta_t \\ \dfrac{\zeta_t^2}{v} \end{bmatrix}.\end{aligned}$$

The results of Conlisk (1974) imply that the mean and variance of $\begin{bmatrix} a_t \\ b_t \end{bmatrix}$ converge to finite constants if and only if the eigenvalues of the expectation of the Kronecker product of the transition matrix with itself are in the unit circle. These eigenvalues are $1 - \frac{2\phi - 1}{\phi}\gamma + \mathrm{O}(\gamma^2)$, $1 - \frac{2\phi - 1}{\phi}\gamma + \mathrm{O}(\gamma^2)$, $1 - 2\frac{\phi - 1}{\phi}\gamma + \mathrm{O}(\gamma^2)$, $1 - 2\gamma + \mathrm{O}(\gamma^2)$ as $\gamma \to 0$. So $\phi > 1$ is sufficient for the mean and variance of $\begin{bmatrix} a_t \\ b_t \end{bmatrix}$ converge to finite constants for all sufficiently low γ. In this case, $\mathbb{E}_0 a_t \to 0$ and $\mathbb{E}_0 b_t = -\frac{1}{\phi}$ as $t \to \infty$, as expected. Moreover, note that by continuity in ρ, the convergence of means and variances generalises from the $\rho = 0$ case. In particular, for all $\phi > 1$ and all ρ and γ sufficiently close to 0, the mean and variance of $\begin{bmatrix} a_t \\ b_t \end{bmatrix}$ will converge to finite values (continuous in ρ).

Finally, from the explicit formula for the variance given in Conlisk (1974), we have that with $\rho = 0$, $\phi > 1$ and γ sufficiently low, $\operatorname{var}_0 \begin{bmatrix} a_t \\ b_t \end{bmatrix} \to 0$ as $t \to \infty$, meaning that a_t and b_t converge in probability to the truth. (Note that if $a_{t-1} = 0$ and $b_{t-1} = -\frac{1}{\phi}$, then $a_t = 0$ and $b_t = -\frac{1}{\phi}$ as well.) Thus, even though agents are using a constant gain, they still manage to exactly learn the true parameters, whatever the initial conditions. It is easy for agents to learn the rational expectations equilibrium under a real rate rule!

Appendix D Non-linear expectational difference equations

D.1 General set-up

We are interested in the non-linear expectational difference equation:

$$\left(\frac{\Pi^*_{t|t-1}}{\Pi_t}\right)^{\phi} = \mathbb{E}_t \frac{\Xi_{t+1}}{\mathbb{E}_t \Xi_{t+1}} \frac{\Pi^*_{t+1|t}}{\Pi_{t+1}}.$$

If we define $X_t := \frac{\Pi^*_{t|t-1}}{\Pi_t}$ and $Z_t := \frac{\Xi_{t+1}}{\mathbb{E}_t \Xi_{t+1}}$ then this difference equation is a particular example of the more general equation:

$$X_t^{\phi} = \mathbb{E}_t Z_{t+1} X_{t+1}.$$

We show in Appendix D.2 below that if $Z_t = 1$ for all t, then this has a unique solution for $\phi > 1$, and we show in Appendix D.3 that it still has a unique solution for arbitrary Z_t under a few additional conditions, and that the solution is approximately unique under even milder conditions.

For the results of Appendix D.3 to apply, we need that Π_t is bounded above. This is true in any model with monopolistic competition in which at least some small fraction of firms do not adjust their price each period. This does not seem an unrealistic assumption, at least if the model's time periods are sufficiently short. Even under hyper-inflation, it is still unlikely that firms adjust prices many times per day.

Π_t is bounded above in such a model because the price level remains finite even if adjusting firms set an infinite price, as all demand switches to non-adjusting firms. For example, the model of Fernández-Villaverde et al. (2015) contains the equation: $1 = \theta \Pi_t^{\varepsilon-1} + (1-\theta)\widetilde{\Pi}_t^{1-\varepsilon}$, where $\widetilde{\Pi}_t$ is the relative price of adjusting firms and $\varepsilon > 1$. This equation comes from the definition of the aggregate price. As $\widetilde{\Pi}_t \to \infty$, $\Pi_t \to \theta^{-\frac{1}{\varepsilon-1}} < \infty$, thus inflation is always bounded above, as required.

D.2 Uniqueness of the solution of a simple non-linear expectational difference equation

Let $\phi > 1$. We seek to prove that the non-linear expectational difference equation:

$$X_t^{\phi} = \mathbb{E}_t X_{t+1},$$

has a unique solution that is:

a) positive (i.e., $X_t > 0$ for all $t \in \mathbb{Z}$),

b) strictly stationary (so for example $\mathbb{E}X_t = \mathbb{E}X_s$ for all $t, s, \in \mathbb{Z}$),

c) and has bounded unconditional mean and log mean (i.e., $\mathbb{E}X_t < \infty$ and $|\mathbb{E}\log X_t| < \infty$ for all $t \in \mathbb{Z}$).

Clearly $X_t = 1$ is one such solution.

Let X_t be a solution to $X_t^{\phi} = \mathbb{E}_t X_{t+1}$ satisfying (a), (b) and (c) above. Let $x_t := \log X_t$. Then from taking logs, we have:

$$\phi x_t = \log \mathbb{E}_t \exp x_{t+1} \geq \log \exp \mathbb{E}_t x_{t+1} = \mathbb{E}_t x_{t+1},$$

by Jensen's inequality. Therefore, by the law of iterated expectations, for any $k \in \mathbb{N}$:

$$\phi^k x_t \geq \mathbb{E}_t x_{t+k} = \mathbb{E}_t x_{t+k}.$$

As $k \to \infty$, the left-hand side tends to either plus infinity (if $x_t > 0$), zero (if $x_t = 0$), or minus infinity (if $x_t < 0$). On the other hand, as $k \to \infty$, the right-hand side tends to $\mathbb{E}x_t > -\infty$, by stationarity. Thus, we must have that $x_t \geq 0$ for all $t \in \mathbb{Z}$, else this equation would be violated. Hence, $X_t \geq 1$ for all $t \in \mathbb{Z}$.

Now note that by stationarity, the law of iterated expectations and Jensen's inequality:

$$\mathbb{E}X_t = \mathbb{E}X_{t+1} = \mathbb{E}\mathbb{E}_t X_{t+1} = \mathbb{E}X_t^{\phi} \geq (\mathbb{E}X_t)^{\phi},$$

so $1 \geq (\mathbb{E}X_t)^{\phi-1}$, meaning $\mathbb{E}X_t \leq 1$. However, since $X_t \geq 1$ for all $t \in \mathbb{Z}$, the only way we can have that $\mathbb{E}X_t \leq 1$ is if in fact $X_t = 1$ for all $t \in \mathbb{Z}$.

Therefore, $X_t \equiv 1$ is the unique solution to the original expectational difference equation satisfying (a), (b) and (c) above.

D.3 Uniqueness of the solution of a more general non-linear difference equation

Let $\underline{\phi} \geq 1$ and let $(Z_t)_{t\in\mathbb{Z}}$ be a stochastic process satisfying the following conditions:

i) $Z_t > 0$, for all $t \in \mathbb{Z}$,

ii) $\mathbb{E}_t Z_{t+1} = 1$, for all $t \in \mathbb{Z}$,

iii) $(Z_t)_{t\in\mathbb{Z}}$ is strictly stationary,

iv) there exists $\overline{Z} \geq 1$, independent of the stochastic process $(X_t)_{t\in\mathbb{Z}}$ (to be introduced), such that for all $\phi > \underline{\phi}$, and for all $t \in \mathbb{Z}$ and all $k \in \mathbb{N}$ with $k > 0$, $\mathbb{E}_t Z_{t+k}^{\frac{\phi}{\phi-1}} \leq \overline{Z}^{\frac{\phi}{\phi-1}}$.

The larger is $\underline{\phi}$, the weaker is the moment boundedness assumptions (iv). For example, if $\underline{\phi} = 2$, then this just requires bounded second moments.

Let $\underline{X} \in (0,1)$ and let $\phi > \underline{\phi}$. We seek to prove that the non-linear expectational difference equation:

$$X_t^{\phi} = \mathbb{E}_t Z_{t+1} X_{t+1},$$

has a unique solution that is:

a) bounded below by $\underline{X}$ (so $X_t > \underline{X} > 0$ for all $t \in \mathbb{Z}$),
b) strictly stationary (so for example $\mathbb{E}X_t = \mathbb{E}X_s$ for all $t, s, \in \mathbb{Z}$),
c) and has bounded unconditional mean, ϕ^{th} mean and log mean (i.e., $\mathbb{E}X_t < \infty$, $\mathbb{E}X_t^{\phi} < \infty$ and $|\mathbb{E}\log X_t| < \infty$ for all $t \in \mathbb{Z}$).

Clearly $X_t = 1$ is one such solution. Note that Z_t may be a function of X_t and its history, so Z_t and X_t are not guaranteed to be independent. The previous subappendix covers the case with $Z_t \equiv 1$ in which slightly weaker assumptions are needed.

First note that for all $t \in \mathbb{Z}$:

$$\begin{aligned} 1 = \mathbb{E}_t Z_{t+1} &= \mathbb{E}_t[Z_{t+1}1] = \mathbb{E}_t\big[Z_{t+1}\mathbb{E}_{t+1}[Z_{t+2}1]\big] = \mathbb{E}_t\big[\mathbb{E}_{t+1}[Z_{t+1}Z_{t+2}1]\big] \\ &= \mathbb{E}_t[Z_{t+1}Z_{t+2}1] = \mathbb{E}_t\big[Z_{t+1}Z_{t+2}\mathbb{E}_{t+2}[Z_{t+3}1]\big] = \cdots \\ &= \mathbb{E}_t\left[\prod_{j=1}^{k} Z_{t+j}\right], \qquad \forall k \in \mathbb{N}, \end{aligned}$$

by assumption (ii) and the law of iterated expectations.

Now let $x_t := \log X_t$ and $\underline{x} := \log \underline{X}$. Then from taking logs, we have:

$$\phi x_t = \log \mathbb{E}_t Z_{t+1} \exp x_{t+1} \geq \log \exp \mathbb{E}_t Z_{t+1} x_{t+1} = \mathbb{E}_t Z_{t+1} x_{t+1},$$

by Jensen's inequality, as $\mathbb{E}_t[Z_{t+1} \times (\cdot)]$ defines a measure since $\mathbb{E}_t Z_{t+1} = 1$. Therefore, by the law of iterated expectations, for any $k \in \mathbb{N}$:

$$\phi^k x_t \geq \mathbb{E}_t\left[\prod_{j=1}^{k} Z_{t+j}\right] x_{t+k} \geq \mathbb{E}_t\left[\prod_{j=1}^{k} Z_{t+j}\right] \underline{x} = \underline{x} > -\infty,$$

by the result of the previous paragraph. As $k \to \infty$, the left-hand side tends to either plus infinity (if $x_t > 0$), zero (if $x_t = 0$), or minus infinity (if $x_t < 0$). Thus, we must have that $x_t \geq 0$ for all $t \in \mathbb{Z}$, else this equation would be violated. Hence, $X_t \geq 1$ for all $t \in \mathbb{Z}$.

Now, define $\overline{z} := \log \overline{Z}$, and for all $t \in \mathbb{Z}$ and all $k \in \mathbb{N}$ with $k > 0$ define:

$$\tilde{z}_{t,t+k} := \log \left[\mathbb{E}_t Z_{t+k}^{\frac{\phi}{\phi-1}} \right]^{\frac{\phi-1}{\phi}} < \overline{z},$$

by our assumptions (iv). Then by repeatedly applying Hölder's inequality:

$$\begin{aligned}
X_t^{\phi} = \mathbb{E}_t Z_{t+1} X_{t+1} &\leq \left[\mathbb{E}_t Z_{t+1}^{\frac{\phi}{\phi-1}} \right]^{\frac{\phi-1}{\phi}} \left[\mathbb{E}_t X_{t+1}^{\phi} \right]^{\frac{1}{\phi}} \\
&\leq \left[\mathbb{E}_t Z_{t+1}^{\frac{\phi}{\phi-1}} \right]^{\frac{\phi-1}{\phi}} \left[\mathbb{E}_t \left[\left[\mathbb{E}_{t+1} Z_{t+2}^{\frac{\phi}{\phi-1}} \right]^{\frac{\phi-1}{\phi}} \left[\mathbb{E}_{t+1} X_{t+2}^{\phi} \right]^{\frac{1}{\phi}} \right] \right]^{\frac{1}{\phi}} \\
&\leq \left[\mathbb{E}_t Z_{t+1}^{\frac{\phi}{\phi-1}} \right]^{\frac{\phi-1}{\phi}} \left[\mathbb{E}_t Z_{t+2}^{\frac{\phi}{\phi-1}} \right]^{\frac{\phi-1}{\phi^2}} \left[\mathbb{E}_t X_{t+2}^{\phi} \right]^{\frac{1}{\phi^2}} \\
&\leq \cdots \\
&\leq \prod_{j=1}^{k} \left[\mathbb{E}_t Z_{t+j}^{\frac{\phi}{\phi-1}} \right]^{\frac{\phi-1}{\phi^j}} \left[\mathbb{E}_t X_{t+k}^{\phi} \right]^{\frac{1}{\phi^k}},
\end{aligned}$$

for all $k \in \mathbb{N}$ with $k > 0$. Thus, from taking logs and limits:

$$x_t \leq \sum_{j=1}^{\infty} \phi^{-j} \tilde{z}_{t,t+j} + \frac{1}{\phi} \lim_{k \to \infty} \left[\phi^{-k} \log \mathbb{E}_t X_{t+k}^{\phi} \right] = \sum_{j=1}^{\infty} \phi^{-j} \tilde{z}_{t,t+j} \leq \frac{\overline{z}}{\phi - 1},$$

where the equality follows from the fact that by stationarity, $\lim_{k\to\infty} \mathbb{E}_t X_{t+k}^{\phi} = \mathbb{E} X_t^{\phi} < \infty$. Thus, $X_t \leq \overline{Z}^{\frac{1}{\phi-1}}$ for all $t \in \mathbb{Z}$. By assumption $\overline{Z}$ is not a function of ϕ, so as $\phi \to \infty$, this upper bound on X_t tends to 1. Hence, for large ϕ, $X_t \approx 1$, giving approximate uniqueness.

We can derive even stronger results in the case in which $\underline{\phi} = 1$ (in our assumptions) and one additional assumption holds. First note that with $\underline{\phi} = 1$, from taking limits as $\phi \to 1$ in assumption (iv), we must have that $Z_t \leq \overline{Z}$ with probability one (for all $t \in \mathbb{Z}$).

Let Z_t^* be the value that would be taken by Z_t if it were the case that $X_t = 1$ for all $t \in \mathbb{Z}$. So, it is also the case that $Z_t^* \leq \overline{Z}$ with probability one (for all $t \in \mathbb{Z}$), by our assumption (iv). Suppose further that there exists $\kappa \geq 0$ such that:

$$\mathbb{E}|Z_t - Z_t^*| \leq \kappa \mathbb{E}(X_t - 1).$$

This is reasonable, since if $X_t \to 1$ (almost surely), we expect that $Z_t \to Z_t^*$

(almost surely) as well.

Now note that:

$$\begin{aligned}\mathbb{E}(X_t-1) &= \mathbb{E}\left[(\mathbb{E}_t Z_{t+1}X_{t+1})^{\frac{1}{\phi}}-1\right] \le \mathbb{E}\left[\frac{1}{\phi}(\mathbb{E}_t Z_{t+1}X_{t+1}-1)\right]\\ &= \frac{1}{\phi}[\mathbb{E}Z_tX_t-1],\end{aligned}$$

(using stationarity and the law of iterated expectations in the final equality). Thus:

$$\begin{aligned}\mathbb{E}(X_t-1) &= \mathbb{E}\left[(\mathbb{E}_t Z_{t+1}X_{t+1})^{\frac{1}{\phi}}-1\right] \le \mathbb{E}\left[\frac{1}{\phi}(\mathbb{E}_t Z_{t+1}X_{t+1}-1)\right] = \frac{1}{\phi}[\mathbb{E}Z_tX_t-1]\\ &= \frac{1}{\phi}[\mathbb{E}Z_tX_t-\mathbb{E}Z_t^*] = \frac{1}{\phi}[\mathbb{E}(Z_t-Z_t^*)X_t+\mathbb{E}Z_t^*(X_t-1)]\\ &\le \frac{1}{\phi}[\mathbb{E}|Z_t-Z_t^*|X_t+\mathbb{E}Z_t^*(X_t-1)]\\ &\le \frac{1}{\phi}\left[\kappa\mathbb{E}(X_t-1)\overline{Z}^{\frac{1}{\phi-1}}+\overline{Z}\mathbb{E}(X_t-1)\right]\\ &= \frac{1}{\phi}\left[\kappa\overline{Z}^{\frac{1}{\phi-1}}+\overline{Z}\right]\mathbb{E}(X_t-1),\end{aligned}$$

(from, respectively, the convexity of $y \mapsto y^{\frac{1}{\phi}}$, stationarity and the law of iterated expectations, the fact that $\mathbb{E}Z_t^*=1$, algebra, that $y \le |y|$, our bounds on X_t, $\mathbb{E}|Z_t-Z_t^*|$ and Z_t^*, and more algebra). As $\phi\to\infty$, $\kappa\overline{Z}^{\frac{1}{\phi-1}}+\overline{Z}\to\kappa+\overline{Z}<\infty$, so for large ϕ it must be the case that $\frac{1}{\phi}\left[\kappa\overline{Z}^{\frac{1}{\phi-1}}+\overline{Z}\right]<1$. Hence if ϕ is large enough for this to hold, then $\mathbb{E}(X_t-1)\le 0$. However, since $X_t\ge 1$ for all $t\in\mathbb{Z}$, the only way we can have that $\mathbb{E}X_t\le 1$ is if in fact $X_t=1$ for all $t\in\mathbb{Z}$.

Therefore, for large enough ϕ, $X_t\equiv 1$ is the unique solution to the original expectational difference equation satisfying (a), (b) and (c) above.

Appendix E Determinacy without the response to the change in relative inflation

We suppose that the central bank sets nominal interest rates using the rule:

$$\begin{aligned}i_{t|t-S} = \max\Bigg\{0, r_{t|t-S}+\bar{\nu}_{t|t-S}+(i_{t-1|t-1-S}-r_{t-1|t-1-S}-\bar{\nu}_{t-1|t-1-S})\\ +\mathbb{E}_{t-S}\frac{1}{T}\sum_{k=1}^{T}\breve{\pi}^*_{t+k-L}-\mathbb{E}_{t-1-S}\frac{1}{T}\sum_{k=1}^{T}\breve{\pi}^*_{t-1+k-L}+\theta(\pi_{t-S}-\breve{\pi}^*_{t-S})\Bigg\},\end{aligned}$$

with $\theta>0$. Define $\Delta_t := (\nu_{t+S|t}-\bar{\nu}_{t+S|t})-(\nu_{t-1+S|t-1}-\bar{\nu}_{t-1+S|t-1})$ (as in the main text), and:

$$\tilde{e}_t := \mathbb{E}_t \frac{1}{T} \sum_{k=1}^{T} (\pi_{t+k-L+S} - \breve{\pi}^*_{t+k-L+S}).$$

Then, combining the monetary rule with the multi-period Fisher equation from Subsection 5.2 gives:

$$\tilde{e}_t + \Delta_t = \tilde{e}_{t-1} + \theta(\pi_t - \breve{\pi}^*_t).$$

And substituting this back into the definition of e_t them implies:

$$\theta T \tilde{e}_t = \mathbb{E}_t \sum_{k=1}^{T} (\tilde{e}_{t+k-L+S} - \tilde{e}_{t+k-L+S-1} + \Delta_{t+k-L+S}).$$

When Δ_t is exogenous, this expectational difference equation has a unique solution if and only if it has a unique solution when $\Delta_t = 0$ for all t. In this case, via the substitution $e_t = c\lambda^t$ we have the characteristic polynomial, $\theta T \lambda^{L-S} = \lambda^T - 1$. (Note, our assumptions imply $T \geq L - S \geq 0$ and $T \geq 1$). The roots of this equation decide the determinacy of e_t (and hence π_t). For determinacy, we need $L - S \geq 0$ roots strictly inside the unit circle, corresponding to the lags of e_t in our difference equation, and $T - L + S \geq 0$ roots strictly outside the unit circle, corresponding to the leads of e_t in our difference equation.

We will prove that the polynomial:

$$\lambda^T - 1 = \theta T \lambda^{L-S}$$

has $L - S$ roots strictly inside the unit circle and $T - L + S$ roots strictly outside of the unit circle, if either $L - S = 0$ or $\theta > \frac{2}{T}$.

First, note that in the special case of $L - S = 0$, the result is trivial, as the polynomial becomes $\lambda^T = 1 + \theta T$, so $|\lambda| = (1 + \theta T)^{\frac{1}{T}} > 1$ as required. (This case overlaps with the result of the main text.)

Next note that as $\theta \to \infty$, $L - S$ roots go to 0, so at least for large θ, $L - S$ roots are strictly inside the unit circle, as needed. What happens to the other $T - L + S$ roots as $\theta \to \infty$? If $T = L - S$, then there are no such roots, so assume $T > L - S$. To examine what happens to these roots, first define $\kappa := \theta^{-\frac{1}{T-L+S}}$, so $\kappa \to 0$ as $\theta \to \infty$, and:

$$(\lambda^T - 1) - T\kappa^{-(T-L+S)}\lambda^{L-S} = 0.$$

Next, suppose $\lambda = z\kappa^{-1}$ where $z = z_0 + O(\kappa)$ as $\kappa \to 0$, so:

$$\left(\left(z_0^T + O(\kappa)\right)\kappa^{-T} - 1\right) - T\left(z_0^{L-S} + O(\kappa)\right)\kappa^{-T} = 0,$$

as $\kappa \to 0$. Multiplying by κ^T then gives:

$$z_0^T - \kappa^T - T z_0^{L-S} + O(\kappa) = 0,$$
as $\kappa \to 0$. Hence as $T \geq 1$, we must have that $z_0^T - T z_0^{L-S} = 0$, so:
$$z_0 \in \left\{ \exp\left(\frac{\log(T) + 2i\bar{\pi}k}{T - L + S} \right) \middle| k \in \{0, \dots, T - L + S - 1\} \right\},$$
where $\bar{\pi}$ is the mathematical constant usually denoted by π, and $i := \sqrt{-1}$. Thus, as $\lambda = z_0 \kappa^{-1} + O(1)$, as $\kappa \to 0$ (meaning $\theta \to \infty$), $|\lambda| \to \infty$. So, as required, for large enough θ, the other $T - L + S$ roots are strictly outside the unit circle. I.e., we are guaranteed determinacy for sufficiently large θ.

Now, suppose θ is large enough to give determinacy, and consider what happens as θ is continuously reduced towards zero. There are two possibilities, either for some critical θ a root crosses the unit circle, or there is determinacy for all positive θ. In the former case, there must be some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that the critical value of θ is:
$$\theta^* := \frac{\lambda^T - 1}{T \lambda^{L-S}}.$$
But then by the triangle inequality:
$$\theta^* = |\theta^*| \leq \frac{|\lambda|^T + 1}{T |\lambda|^{L-S}} = \frac{2}{T}.$$
Hence, in either case, for any $\theta > \frac{2}{T}$, we must have $L - S$ roots strictly inside the unit circle and $T - L + S$ roots strictly outside of the unit circle, as required.

Appendix F Fiscal Theory of the Price Level (FTPL) results

F.1 Exact equilibria under active fiscal policy with geometric coupon debt and flexible prices

Suppose the representative household supplies one unit of labour, inelastically. Production of the final good is given by $y_t = l_t (= 1)$. In period 0, the representative household maximises $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t$, subject to the budget constraint:
$$P_t c_t + A_t + Q_t B_t + P_t \tau_t = P_t y_t + I_{t-1} A_{t-1} + (1 + \omega Q_t) B_{t-1},$$
where c_t is consumption, τ_t are real lump sum taxes, P_t is the price of the final good, A_t is the number of one period nominal bonds purchased by the household at t, which each return I_t in period $t+1$, Q_t is the price of a long

(geometric coupon) bond and B_t are the number of units of this long bond purchased by the household at t. One unit of the period t long bond bought at t returns \$1 at $t+1$, along with ω units of the period $t+1$ bond.

The household first order conditions imply:

$$1 = \beta I_t \mathbb{E}_t \frac{P_t c_t}{P_{t+1} c_{t+1}}, \qquad Q_t = \beta \mathbb{E}_t \frac{P_t c_t}{P_{t+1} c_{t+1}} (1 + \omega Q_{t+1}).$$

The household transversality conditions are that:

$$\lim_{t\to\infty} \beta^t \frac{A_t}{P_t c_t} = 0, \qquad \lim_{t\to\infty} \beta^t \frac{Q_t B_t}{P_t c_t} = 0.$$

The government fixes taxes at a constant positive level $\tau_t = \tau$, where $\tau > 0$. The government issues no one period bonds, so $A_t = 0$. The central bank pegs nominal interest rates at $I_t = \beta^{-1}$. (We will discuss active monetary policy later.)

The final goods market clears, so $y_t = c_t = 1$. Thus, from the household budget constraint, we have the following government budget constraint:

$$Q_t B_t + P_t \tau = B_{t-1}(1 + \omega Q_t).$$

We look for an equilibrium in which $P_t = P$ for all $t \geq 0$. We do not impose a priori that $P = P_{-1}$.

With $P_t = P$ for $t \geq 0$, the household Euler equations simplify to (respectively):

$$1 = \beta I_t, \qquad Q_t = \beta \mathbb{E}_t (1 + \omega Q_{t+1}).$$

The former equation is consistent with the CB's peg of $I_t = \beta^{-1}$.

We consider the following solution to the latter equation:

$$Q_t = \frac{\beta}{1 - \beta\omega} + \left(Q_0 - \frac{\beta}{1 - \beta\omega}\right)(\beta\omega)^{-t}.$$

We wish to find Q_0, which is free to jump. There are three cases to consider:

Case 1: $Q_0 < \frac{\beta}{1-\beta\omega}$. Then Q_t eventually goes to zero (and then negative), which certainly cannot be consistent with a world in which $I_t > 0$. Thus, this case is ruled out.

Case 2: $Q_0 = \frac{\beta}{1-\beta\omega}$. Then Q_t is constant, and the government budget constraint becomes:

$$B_t = \beta^{-1} B_{t-1} - \beta^{-1}(1 - \beta\omega) P \tau.$$

Thus:

$$B_t = P\tau \frac{1-\beta\omega}{1-\beta} + \left(B_{-1} - P\tau \frac{1-\beta\omega}{1-\beta}\right)\beta^{-t-1}$$

So:

$$\begin{aligned}\beta^t \frac{Q_t B_t}{P_t c_t} &= \frac{\beta}{1-\beta\omega}\frac{1}{P}\left[P\tau \frac{1-\beta\omega}{1-\beta}\beta^t + \left(B_{-1} - P\tau \frac{1-\beta\omega}{1-\beta}\right)\beta^{-1}\right] \\ &\to \frac{1}{1-\beta\omega}\frac{1}{P}\left(B_{-1} - P\tau \frac{1-\beta\omega}{1-\beta}\right)\end{aligned}$$

as $t \to \infty$. Thus, from the transversality constraint, $P = \frac{B_{-1}}{\tau}\frac{1-\beta}{1-\beta\omega}$. This is the standard FTPL equilibrium. **Equilibrium type 1!**

Case 3: $Q_0 > \frac{\beta}{1-\beta\omega}$.

Define $q_t := Q_t(\beta\omega)^t$, and $b_t := B_t\omega^{-t}$. Then the government budget constraint states:

$$b_t = \left(1 + \frac{(\beta\omega)^t}{\omega q_t}\right) b_{t-1} - \frac{\beta^t P\tau}{q_t},$$

and the transversality constraint states $\frac{1}{P}\lim_{t\to\infty} q_t b_t = 0$. By our solution for q_t, we know that $q_t \to Q_0 - \frac{\beta}{1-\beta\omega} > 0$. Thus, the transversality condition requires $\lim_{t\to\infty} b_t = 0$. Now define:

$$\hat{b}_t := \frac{b_t}{\prod_{k=0}^t \left(1 + \frac{(\beta\omega)^k}{\omega q_k}\right)},$$

with $\hat{b}_{-1} = b_{-1} = \omega B_{-1}$. The denominator in the definition of $\hat{b}_t$ is greater than 1, so if $b_t \to 0$ as $t \to \infty$, then certainly $\hat{b}_t \to 0$. Likewise, if $\hat{b}_t \to 0$ as $t \to \infty$, then also $b_t \to 0$, since for all t:

$$\begin{aligned}&\prod_{k=0}^{t}\left(1 + \frac{(\beta\omega)^k}{\omega q_k}\right) \le \prod_{k=0}^{\infty}\left(1 + \frac{(\beta\omega)^k}{\omega q_k}\right) \\ &= \prod_{k=0}^{\infty}\left(1 + \frac{1-\beta\omega}{\beta\omega + \omega\big((1-\beta\omega)Q_0 - \beta\big)(\beta\omega)^{-k}}\right) \\ &= \exp \sum_{k=0}^{\infty} \log\left(1 + \frac{1-\beta\omega}{\beta\omega + \omega\big((1-\beta\omega)Q_0 - \beta\big)(\beta\omega)^{-k}}\right) \\ &\le \exp \int_{-1}^{\infty} \log\left(1 + \frac{1-\beta\omega}{\beta\omega + \omega\big((1-\beta\omega)Q_0 - \beta\big)(\beta\omega)^{-k}}\right) \\ &= \frac{(1+\omega Q_0)(1-\beta\omega)}{\omega\big((1-\beta\omega)Q_0 - \beta\big)} \exp\left[\frac{1}{\log(\beta\omega)}\left[\operatorname{dilog}\left(\frac{1+\beta\omega(1+\omega Q_0)}{\beta\omega(1+\omega Q_0)}\right) + \operatorname{dilog}(\beta\omega)\right.\right. \\ &\qquad\qquad \left.\left. - \operatorname{dilog}\left(\frac{1+\beta\omega(1+\omega Q_0)}{1+\omega Q_0}\right)\right]\right] < \infty,\end{aligned}$$

where $\operatorname{dilog}(x) := \int_1^x \frac{\log(z)}{1-z} dz$ for all x is the dilogarithm function.

Now, substituting the definition of $\hat{b}_t$ into the law of motion for b_t gives:

$$\hat{b}_t = \hat{b}_{t-1} - \frac{\beta^t P\tau}{q_t \prod_{k=0}^{t}\left(1+\frac{(\beta\omega)^k}{\omega q_k}\right)},$$

so:

$$\begin{aligned}\hat{b}_t &= \hat{b}_{-1} - P\tau\sum_{j=0}^{t}\frac{\beta^j}{q_j \prod_{k=0}^{j}\left(1+\frac{(\beta\omega)^k}{\omega q_k}\right)}\\ &= \hat{b}_{-1} - P\tau\sum_{j=0}^{t}\frac{\prod_{k=0}^{j}\beta\left(1+\frac{1-\beta\omega}{\beta\omega+(\omega(1-\beta\omega)Q_0-\beta\omega)(\beta\omega)^{-k}}\right)^{-1}}{\beta\left[\frac{\beta}{1-\beta\omega}(\beta\omega)^j+\left(Q_0-\frac{\beta}{1-\beta\omega}\right)\right]}.\end{aligned}$$

Note that for $k \geq 0$:

$$1 < 1+\frac{1-\beta\omega}{\beta\omega+(\omega(1-\beta\omega)Q_0-\beta\omega)(\beta\omega)^{-k}} \leq 1+\frac{1}{\omega Q_0} < \frac{1}{\beta\omega},$$

so:

$$(\beta^2\omega)^{j+1} < \prod_{k=0}^{j}\beta\left(1+\frac{1-\beta\omega}{\beta\omega+(\omega(1-\beta\omega)Q_0-\beta\omega)(\beta\omega)^{-k}}\right)^{-1} < \beta^{j+1}.$$

Thus, since the denominator within the sum is converging to $\beta\left(Q_0-\frac{\beta}{1-\beta\omega}\right)$ the sum is finite and has a finite limit as $t\to\infty$.

Hence, one equilibrium is for $Q_0 > \frac{\beta}{1-\beta\omega}$ to be arbitrary and for P to be given by:

$$P = \frac{\hat{b}_{-1}}{\tau\sum_{j=0}^{\infty}\frac{\prod_{k=0}^{j}\beta\left(1+\frac{1-\beta\omega}{\beta\omega+(\omega(1-\beta\omega)Q_0-\beta\omega)(\beta\omega)^{-k}}\right)^{-1}}{\beta\left[\frac{\beta}{1-\beta\omega}(\beta\omega)^j+\left(Q_0-\frac{\beta}{1-\beta\omega}\right)\right]}}.$$

Equilibrium type 2!

Alternatively, suppose P is given. When can we solve the previous equation to find Q_0? As $Q_0 \to \frac{\beta}{1-\beta\omega}$, the right-hand side of the previous equation tends to:

$$\frac{\hat{b}_{-1}}{\tau\omega}\frac{1-\beta}{1-\beta\omega} = \frac{B_{-1}}{\tau}\frac{1-\beta}{1-\beta\omega}.$$

As $Q_0 \to \infty$, this right-hand side tends to ∞. Thus, by the intermediate value theorem, for any $P \in \left[\frac{B_{-1}}{\tau}\frac{1-\beta}{1-\beta\omega},\infty\right)$, there is a Q_0 that satisfies the transversality constraint. Hence, inflation is unbounded above in the initial period.

Equilibrium type 3!

Therefore, the FTPL implies a lower bound on the price level, not an upper bound, and so with passive monetary policy, there are multiple equilibria.

Now suppose that monetary policy is active, with $I_t = \beta^{-1}\Pi_t^{\phi}$, with $\phi > 1$ and $\Pi_t := \frac{P_t}{P_{t-1}}$. β^{-1} is the real interest rate in this model, so this is a non-linear real rate rule. Given that $c_t = 1$, the Euler equation for one period bonds implies the nonlinear Fisher equation:

$$1 = \beta I_t \mathbb{E}_t \frac{1}{\Pi_{t+1}},$$

so, for $t \geq 0$:

$$\mathbb{E}_t \frac{1}{\Pi_{t+1}} = \left(\frac{1}{\Pi_t}\right)^{\phi}.$$

$\Pi_t = 1$ is the unique stationary solution to this equation, by the results of Appendix D.2 (with $X_t := \frac{1}{\Pi_t}$). In this candidate equilibrium, $I_t = \beta^{-1}$, so Π_t and I_t have the same time series as under the passive policy in the special case in which $P = P_{-1}$. Consequently, if $P_{-1} \geq \frac{B_{-1}}{\tau}\frac{1-\beta}{1-\beta\omega}$ then by the above results, there exists a Q_0 under which all equilibrium conditions and transversality conditions are satisfied. Thus, even with active monetary and active fiscal policy, there is still a stable equilibrium for inflation and real variables.

F.2 Linearised equilibria under active fiscal policy with geometric coupon debt and sticky prices

We now examine the fiscal theory of the price level in a richer model with sticky prices. We just give the linearised equations of the model. These follow equations 5.17 to 5.21 of Cochrane (2023), and the reader is referred there for the derivations. All shocks (variables of the form $\varepsilon_{\cdot,t}$) are assumed to be mean zero and independent, both across time and across shocks. The equations follow:

- Euler: $x_t = \mathbb{E}_t x_{t+1} - \sigma r_t$.
- Phillips: $\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t$.
- Fisher: $i_t = r_t + \mathbb{E}_t \pi_{t+1}$.
- Robust real rate rule: $i_t = r_t + \phi\pi_t + \varepsilon_{i,t}$.
- Exogenous real government surplus: $s_t = \varepsilon_{s,t}$.

- Debt evolution (v_t is the value of debt to GDP, e_t is the ex-post nominal return on government debt): $\beta v_t = v_{t-1} + e_t - \pi_t - s_t$.
- Equal returns: $\mathbb{E}_t e_{t+1} = i_t$.
- Bond pricing (ω controls the maturity structure. $\omega = 0$ is one period debt, $\omega = 1$ is a perpetuity): $e_t = \omega q_t - q_{t-1}$.

We assume that $\omega > 0$. Then for any $\phi \neq 0$, the following solves these linear expectational difference equations:

$$\pi_t = -\frac{\varepsilon_{i,t}}{\phi}, \qquad x_t = -\frac{\varepsilon_{i,t}}{\kappa\phi}, \qquad r_t = \frac{\varepsilon_{i,t}}{\sigma\kappa\phi}, \qquad v_t = -\frac{\varepsilon_{i,t}}{\sigma\kappa\phi},$$

$$e_t = \varepsilon_{s,t} - \left(\frac{\beta}{\sigma\kappa\phi} + \frac{1}{\phi}\right)\varepsilon_{i,t} + \frac{\varepsilon_{i,t-1}}{\sigma\kappa\phi},$$

$$q_t = \frac{1}{\omega}\left[q_{t-1} + \varepsilon_{\mathrm{s},t} - \left(\frac{\beta}{\sigma\kappa\phi} + \frac{1}{\phi}\right)\varepsilon_{i,t} + \frac{\varepsilon_{i,t-1}}{\sigma\kappa\phi}\right].$$

As in the non-linear, flexible price case, the bond price is exploding. However, the real value of government debt remains stationary, which is sufficient for the transversality constraint to be satisfied. Inflation and all real variables are also stationary. Thus, if monetary policy is passive ($\phi \in (0,1)$), then the linearised model has multiple valid equilibria, this one, and the standard "FTPL" one in which q_t is stationary (see Cochrane (2023)). Conversely, if monetary policy is active ($\phi > 1$), then the model possesses a valid equilibrium with stationary inflation and real variables.

F.3 Stability under real rate rules for generic models

When is the real rate rule $i_t = r_t + \phi\pi_t + \varepsilon_{\zeta,t}$ with $\phi > 1$ consistent with stable real variables?

We need to impose at least some additional structure on the rest of the model in order to make progress on this question for general models. In particular, we assume that the other endogenous variables of the model can be partitioned into two groups, z_t and q_t, where z_t may affect q_t but not vice versa. The variables in z_t must be stationary in equilibrium, but always have a unique stationary solution if π_t is stationary. The variables in q_t need not be stationary in equilibrium. These restrictions are satisfied by models of the fiscal theory of the price level, for example, in which case hours, output, consumption, investment, debt-to-GDP, inflation, nominal & real rates and so on will be in z_t, while bond prices and quantities will be in q_t. That bond prices and quantities need not be stationary under the fiscal theory of the price level was carefully established from transversality conditions in Appendix F.1, under the assumption of geometric coupon debt. The calculations of Appendices F.1 and F.2 also show that only the value of government debt matters for "z_t" variables, not its decomposition into bond prices and quantities.

Then, without loss of generality, the linearized model (without the monetary rule) must have a representation in the following form:[10]

$$0 = A_{zz}\mathbb{E}_t z_{t+1} + B_{zz}z_t + C_{zz}z_{t-1} + d_z\pi_t + E_z\nu_t, \tag{16}$$

$$0 = A_{qq}\mathbb{E}_t q_{t+1} + B_{qq}q_t + C_{qq}q_{t-1} + A_{qz}\mathbb{E}_t z_{t+1} + B_{qz}z_t + C_{qz}z_{t-1} + d_q\pi_t + E_q\nu_t, \tag{17}$$

where ν_t is a vector of exogenous shocks with $\mathbb{E}_{t-1}\nu_t = 0$, and where the coefficient matrices are such that there is a unique matrix F_z with eigenvalues in the unit circle such that $F_z = -(A_{zz}F_z + B_{zz})^{-1}C_{zz}$. This condition on F_z imposes that z_t has a stationary solution if π_t is stationary: in other words, it ensures there is no real indeterminacy in the model. Note that q_t (and its lags and leads) do not enter the equation for z_t, by our assumption that q_t does not affect z_t.

We want to see if $\pi_t = -\frac{1}{\phi}\varepsilon_{\zeta,t}$ is consistent with (16) and , (17). This is the only possible stationary solution for inflation under the real rate rule $i_t = r_t + \phi\pi_t + \varepsilon_{\zeta,t}$ with $\phi > 1$. From this solution for π_t, (9) and the definition of F_z:

$$z_t = F_z z_{t-1} + (A_{zz}F_z + B_{zz})^{-1}\left(\frac{1}{\phi}d_z\varepsilon_{\zeta,t} - E_z\nu_t\right).$$

Hence, from (10):

$$\begin{aligned}0 = {} & A_{qq}\mathbb{E}_t q_{t+1} + B_{qq}q_t + C_{qq}q_{t-1} + \Big(\big(A_{qz}F_z + B_{qz}\big)F_z + C_{qz}\Big)z_{t-1} \\ & + \big(A_{qz}F_z + B_{qz}\big)(A_{zz}F_z + B_{zz})^{-1}\left(\frac{1}{\phi}d_z\varepsilon_{\zeta,t} - E_z\nu_t\right) - \frac{1}{\phi}d_q\varepsilon_{\zeta,t} + E_q\nu_t.\end{aligned}$$

If there is a real matrix F_q solving $F_q = -\big(A_{qq}F_q + B_{qq}\big)^{-1}C_{qq}$ then q_t admits a solution of the form:

$$q_t = F_q q_{t-1} + Gz_{t-1} + h\varepsilon_{\zeta,t} + J\nu_t,$$

for some matrices G and J and some vector h. This may be explosive, but that is allowed by our assumptions. (In the fiscal theory of the price level contexts, this

[10] The lack of terms in $\mathbb{E}_t\pi_{t+1}$ and π_{t-1} is without loss of generality, as such responses can be included by adding an auxiliary variable $z_{t,j}$ with an equation of the form $z_{t,j} = \pi_t$.

corresponds to explosions in bond prices and quantities of opposite signs, producing stable debt values.) In this case, there is no inconsistency with the solution for inflation implied by our real rate rule. So, the answer to the question "is a real rate rule consistent with stable z_t variables?" is the same as the answer to the question "does $A_{qq}F_q^2 + B_{qq}F_q + C_{qq}$ have a real solution for F_q?".

When $A_{qq} = 0$, this is simple. A real solution exists if and only if B_{qq} is full rank. Generically, matrices are full rank, so except in knife edge cases, a real solution exists when $A_{qq} = 0$. Furthermore, by continuity, for almost all A_{qq} with sufficiently small norm, a real solution must exist. Under standard models of the fiscal theory of the price level, $A_{qq} = 0$, since the geometric coupon bond first order condition $Q_t = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}}(1 + \omega Q_{t+1})$ can be rewritten as the two equations $E_t = \frac{1+\omega Q_t}{Q_{t-1}}$, and $1 = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} E_{t+1}$ (E_t is in z_t, while Q_t is in q_t, also see Appendices F.1 and F.2). Thus, generically, all models sufficiently close to a standard fiscal theory of the price level model must have a real solution for F_q. Therefore, for all such models, a real rate rule is consistent with a stationary path for z_t variables.

References

Bianchi, Francesco, Cosmin Ilut & Hikaru Saijo. 2023. 'Diagnostic Business Cycles'. *The Review of Economic Studies* (February 21): rdad024.

Bilbiie, Florin O. 2008. 'Limited Asset Markets Participation, Monetary Policy and (Inverted) Aggregate Demand Logic'. *Journal of Economic Theory* 140 (1) (May 1): 162–196.

———. 2019. *Monetary Policy and Heterogeneity: An Analytical Framework*. 2019 Meeting Papers. Society for Economic Dynamics.

Bordalo, Pedro, Nicola Gennaioli & Andrei Shleifer. 2018. 'Diagnostic Expectations and Credit Cycles'. *The Journal of Finance* 73 (1): 199–227.

Branch, William A. & Bruce McGough. 2009. 'A New Keynesian Model with Heterogeneous Expectations'. *Journal of Economic Dynamics and Control* 33 (5). Complexity in Economics and Finance (May 1): 1036–1051.

Bullard, James & Kaushik Mitra. 2002. 'Learning about Monetary Policy Rules'. *Journal of Monetary Economics* 49 (6) (September): 1105–1129.

Cochrane, John H. 2023. *The Fiscal Theory of the Price Level*. Princeton University Press.

Conlisk, John. 1974. 'Stability in a Random Coefficient Model'. *International Economic Review* 15 (2): 529–533.

Evans, George W. & Seppo Honkapohja. 2001. *Learning and Expectations in Macroeconomics*. Frontiers of Economic Research. Princeton and Oxford: Princeton University Press.

Fernández-Villaverde, Jesús, Grey Gordon, Pablo Guerrón-Quintana & Juan F. Rubio-Ramírez. 2015. 'Nonlinear Adventures at the Zero Lower Bound'. *Journal of Economic Dynamics and Control* 57 (C): 182–204.

L'Huillier, Jean-Paul, Sanjay R. Singh & Donghoon Yoo. 2021. 'Incorporating Diagnostic Expectations into the New Keynesian Framework'. *Working Papers*. Working Papers (February 9).

Marcet, Albert & Thomas J Sargent. 1989. 'Convergence of Least Squares Learning Mechanisms in Self-Referential Linear Stochastic Models'. *Journal of Economic Theory* 48 (2) (August 1): 337–368.

Smets, Frank & Rafael Wouters. 2007. 'Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach'. *American Economic Review* 97 (3) (June): 586–606.

Woodford, Michael. 2001. 'The Taylor Rule and Optimal Monetary Policy'. *The American Economic Review* 91 (2): 232–237.

———. 2019. 'Monetary Policy Analysis When Planning Horizons Are Finite'. *NBER Macroeconomics Annual* 33 (January): 1–50.