Robust Real Rate Rules

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Abstract: Central banks wish to avoid self-fulfilling fluctuations. Monetary rules with a unit response to real rates achieve this under the weakest possible assumptions about the behaviour of households and firms. They are robust to household heterogeneity, hand-to-mouth consumers, non-rational household/firm expectations, active fiscal policy, missing transversality conditions and to any form of intertemporal or nominal-real links. They are easy to employ in practice, using inflation protected bonds to infer real rates. With a time-varying inflation target, they can implement arbitrary inflation dynamics, including optimal policy. They work thanks to the key role played by the Fisher equation in monetary transmission.

Keywords: robust monetary rules, determinacy, Taylor principle, inflation dynamics, monetary transmission mechanism

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Today you start work as president of the Fictian Central Bank (FCB). As FCB president, you have a clear mandate to stabilize inflation, even if that results in unemployment or output losses. How should you act? You have studied New Keynesian macro, so you are inclined to follow some variant of the Taylor rule. You recall the prescription of the Taylor principle: the response of nominal rates to inflation should be greater than one to ensure determinacy—the existence of a unique stable solution without self-fulfilling fluctuations. But you also remember reading other papers which talked of the Taylor principle being insufficient if there are hand-to-mouth households (Gali, Lopez-Salido & Valles 2004), firm-specific capital (Sveen & Weinke 2005), high government spending (Natvik 2009), or if the inflation target is positive (Ascari & Ropele 2009), particularly in the presence of trend growth and sticky wages (Khan, Phaneuf & Victor 2019). Indeed, you recollect that the Taylor principle inverts if there are sufficiently many hand-to-mouth households (Bilbiie 2008), certain financial frictions (Manea 2019), or non-rational expectations (Branch & McGough 2010; 2018). You also recall that if real government surpluses do not respond to government debt levels, then following the Taylor principle can lead to explosive inflation (Leeper & Leith 2016; Cochrane 2022). Is there a way you could act to ensure determinacy and stable inflation, even if one or more of these circumstances is true? This paper provides a family of "robust real rate rules" that manage to do this. We then reassess classic questions of monetary economics through the lens of these rules.

To illustrate the idea behind these rules, suppose that both nominal and real bonds are traded in an economy. If a unit of the former is purchased at t, it returns the principal plus a nominal yield of i_t in period $t+1$. If a unit of the latter is purchased at t, it returns the principal plus a nominal yield of $r_t + \pi_{t+1}$ in period $t+1$, where π_{t+1} is realized inflation between t and $t+1$. US Treasury Inflation Protected Securities (TIPS) are one example of such real bonds.

Abstracting for the moment from inflation risk premia, term premia and liquidity premia, arbitrage between the nominal and real bond markets implies that the Fisher equation must hold, i.e.:

$$i_t = r_t + \mathbb{E}_t \pi_{t+1}, \tag{1}$$

where $\mathbb{E}_t \pi_{t+1}$ is the full information rational expectation of period $t+1$'s inflation rate, given period t's information. Suppose further than the central bank observes both the nominal and real bond markets, and that it can intervene in the former. Then the central bank can choose to set nominal interest rates according to the simple rule:

$$i_t = r_t + \phi \pi_t, \tag{2}$$

where $\phi > 1$.[2] Combining these two equations gives that:

$$\mathbb{E}_t \pi_{t+1} = \phi \pi_t,$$

which has a unique non-explosive solution of $\pi_t = 0$.[3] Determinate inflation!

Why is this robust? Firstly, the rule does not require the aggregate Euler equation to hold, even approximately. For the Fisher equation (1) to hold (still ignoring risk/term/liquidity premia for now), there only need to be two deep pocketed, fully informed, rational agents. Arbitrage takes care of the rest. Even full information is not necessary. Since large markets aggregate information (Hellwig 1980; Lou et al. 2019), the Fisher equation can come to hold even when information about future inflation is dispersed amongst market participants.

Given that the rule does not require an aggregate Euler equation to hold, it is automatically robust to heterogeneity, hand-to-mouth agents and non-rational consumer expectations. The only expectations that matter are the expectations of

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[2] We ignore the zero lower bound (ZLB) for now. We provide rules that retain their good properties in the presence of the ZLB in Section 4.

[3] Here we sidestep the issues raised by Cochrane (2011) and follow the standard New Keynesian literature in assuming agents will always select non-explosive paths for inflation. The escape clause rules of Christiano & Takahashi (2018; 2020) are one way by which central banks could ensure coordination on the expectations consistent with non-explosive inflation, and the limited memory arguments of Angeletos & Lian (2021) give an alternative justification for conventional equilibrium selection. We give an alternative solution in Subsection 4.3.

participants in the markets for nominal and real bonds. It is much more reasonable to assume that financial market outcomes lead to rational expectations than to assume rationality of households more generally.

In fact, even financial market participants do not need to be fully rational. The combination of equations (1) and (2) is globally stable under learning, even when financial market participants start with a prior not centred at zero. In particular, suppose financial market participants approximate $\mathbb{E}_t\pi_{t+1}$ with $\frac{1}{t+w}\left[w\breve{\pi}+\sum_{s=1}^{t}\pi_s\right]$, where $\breve{\pi}$ is the mean of their prior beliefs, and $w \ge 0$ gives the weight placed on these priors. For convenience, define $\pi_0 := \frac{\breve{\pi}}{\phi}$. Then, for $t \ge 1$, π_t solves:

$$\phi\pi_t = \frac{1}{t+w}\left[w\breve{\pi}+\sum_{s=1}^{t}\pi_s\right] = \frac{t-1+w}{t+w}\phi\pi_{t-1}+\frac{1}{t+w}\pi_t,$$

which implies that if $\phi > 1$, then $\pi_t \to 0$ as $t \to \infty$.[4] This guarantee of global stability under learning is a large improvement over standard monetary rules, for which at best local stability can be proven (see e.g., Bullard & Mitra 2002).

Secondly, the rule of equation (2) does not require an aggregate Phillips curve to be present. The slope of the Phillips curve can have no impact on the dynamics of inflation. If a central bank is unconcerned with output, they do not even need to know if the Phillips curve holds, let alone its slope. Nor does it matter how firms form inflation expectations. Inflation is pinned down by the Fisher and monetary rules, so while non-rational firm expectations could affect output fluctuations, they will not alter the dynamics of inflation. The only requirement is that at least some prices are updated each period using current information.

The possibility of decoupling inflation from the rest of the economy has wide ranging implications. For example, there is a tradition in monetary economics of examining

[4] We have that:

$$\pi_t = \frac{t+w-1}{t+w-\phi^{-1}}\pi_{t-1} = \frac{\breve{\pi}}{\phi}\prod_{s=1}^{t}\frac{s+w-1}{s+w-\phi^{-1}} = \frac{\Gamma(t+w)}{\Gamma(w)}\frac{\Gamma\left(w+1-\phi^{-1}\right)}{\Gamma\left(t+w+1-\phi^{-1}\right)}\frac{\breve{\pi}}{\phi}.$$

Hence, by Gautschi's inequality, $\pi_t = O\left(\frac{(t+w+1)^{\left(\frac{1}{\phi}\right)}}{t+w}\right)$ as $t \to \infty$.

model features producing amplification or dampening of monetary shocks. Under a real rate rule, assuming the Fisher equation holds, then no change to the model can ever produce amplification or dampening, except perhaps a change to the monetary rule. Thus, such amplification/dampening results were always highly dependent on the particular monetary rule being used. With a greater than unit response to real rates, amplification can be flipped to dampening, and vice versa.

Likewise, a persistent question in monetary economics has been "which shocks drive inflation?". Here too, the answer must be crucially sensitive to the monetary rule being used. Under a real rate rule, only monetary policy shocks or shocks to the Fisher equation can possibly move inflation. The central bank has the power to almost perfectly control inflation, so ultimate responsibility for inflation must rest with them.

The rest of this paper further examines "real rate rules", along with the classic questions of monetary economics they help answer. The next section generalizes the simple rule of equation (2) along various dimensions, including examining rules that respond to other endogenous variables. We also look at the implication of real rate rules in simple New Keynesian models. Section 1 goes on to show that there are similar rules that determinately implement an arbitrary path for inflation, robustly across models. Hence, real rate rules can attain high welfare, and could explain observed inflation dynamics.

Next, Section 2 examines some potential challenges to the performance of real rate rules. We show they also work in fully non-linear models, that they are robust to wedges in the Fisher equation, and that they continue to work even in models in which inflation is determinate under a peg, except in knife edge cases.

Section 3 discusses how a real rate rule could be implemented in practice. We show that it is easy to adapt real rate rules to work with longer bonds. Finally, Section 4 looks at the consequences of the zero lower bound for the performance of these rules. It also gives one way to rule out equilibria with explosive inflation or permanent ZLB traps.

Prior literature. Rules like equation (2) have appeared in Adão, Correia & Teles (2011), Lubik, Matthes & Mertens (2019) and Holden (2021) amongst other places. However, in the prior literature they have chiefly been introduced for analytic convenience, rather than as serious proposals. One exception is the work of Cochrane (2017; 2022) who briefly discusses rules of this form within the context of a wider discussion of rules that hold $i_t - r_t$ constant (i.e. rules with $\phi = 0$). Cochrane (2018) further explores rules holding $i_t - r_t$ constant.

The "indexed payment on reserve" rules of Hall & Reis (2016) also rely on observable real rates, but use a different mechanism to achieve determinacy. They propose that the CB issues an asset ("reserves") with nominal return from \$1 of $\$(1 + r_t)\frac{p_{t+1}}{p_t^*}$ or $\$(1 + i_t)\frac{p_t}{p_t^*}$. Additionally, in older work, Hetzel (1990) proposes using the spread between nominal and real bonds to guide monetary policy, and Dowd (1994) proposes targeting the price of futures contracts on the price level, which has a similar flavour to our rules, since our rules effectively use expected inflation as the instrument of monetary policy.

There is also an established literature looking at rules tracking the efficient ("natural") real interest rate, see e.g. Cúrdia et al. (2015). This is a very different idea.

1 Generalizations and generality

This section considers assorted generalizations to real rate rules, and examines the sources of their robustness. We look at real rate rules 1) in the presence of monetary policy shocks, 2) in the three equation NK model, 3) with responses to other endogenous variables, 4) with time varying inflation targets and 5) with interest rate smoothing.

1.1 Monetary policy shocks

While the simple rule (2) always produces zero inflation, slight extensions of the rule allow inflation to move. For example, we may add a monetary policy shock, ζ_t to the rule, giving:

$$i_t = r_t + \phi\pi_t + \zeta_t. \tag{3}$$

One source of monetary policy shocks could be the central bank's limited information. If the central bank does not perfectly observe current inflation, and sets interest rates to $i_t = r_t + \phi\tilde{\pi}_t$, where $\tilde{\pi}_t$ is its signal about inflation, then it will end up setting a slightly different level for nominal rates than that dictated by the rule $i_t = r_t + \phi\pi_t$, effectively generating monetary policy shocks.[5] However, this kind of limited information is inconsistent with our simple model's assumptions. Real bonds purchased at $t-1$ give a return in period t which is a function of π_t. Hence, π_t must be available to all parties in period t. (It is not "true" inflation that matters, but whatever inflation measure is used in the real bond contract.) Of course, in reality inflation is released with a lag, and real bonds have additional indexation lag. We explicitly model these lags in Section 3, but our conclusions will remain the same.

The central bank might also deliberately decide to introduce monetary policy shocks correlated with the economy's structural shocks. For example, by lowering $i_t - r_t$ following a positive mark-up or cost-push shock, the central bank can lessen the movement in the output gap.[6] This has no effect on the determinacy region as structural shocks are exogenous. For now though, we assume that ζ_t is independent of other structural shocks.

From combining (3) with the Fisher equation (1) we have:

$$\mathbb{E}_t\pi_{t+1} = \phi\pi_t + \zeta_t,$$

which (with $\phi > 1$) has the unique solution $\pi_t = -\frac{1}{\phi-\rho_\zeta}\zeta_t$, if ζ_t follows an AR(1) process with persistence ρ_ζ.

[5] Lubik, Matthes & Mertens (2019) look at the determinacy consequences of a central bank that filters inflation signals in order to retrieve the optimal estimate. The determinacy problems they highlight all disappear if the central bank directly responds to its signal.

[6] Ireland (2007) presents evidence that the US Federal Reserve has reacted to mark-up shocks.

A contractionary (positive) monetary policy shock results in a fall in inflation, as expected. If the central bank is more aggressive, so ϕ is larger, then inflation is less volatile. Only monetary policy shocks affect inflation. Of course, if there is a nominal rigidity in the model, such as sticky prices or wages, monetary shocks may have an impact on real variables. But as long as the central bank follows rules like this, these real disruptions have no feedback to inflation. We can understand inflation without worrying about the rest of the economy.

In line with this, an extensive body of empirical evidence finds no role for the Phillips curve in forecasting inflation (see e.g. Atkeson & Ohanian 2001; Ang, Bekaert & Wei 2007; Stock & Watson 2009; Dotsey, Fujita & Stark 2018). In a recent contribution, Dotsey, Fujita & Stark (2018) find that in the post-1984 period, Phillips curve based forecasts perform worse than those of a simple IMA(1,1) model, both unconditionally and conditional on various measures of the state of the economy. This provides strong support for models in which the causation in the Phillips curve runs in only one direction: from inflation to the output gap.[7]

Additionally, Miranda-Agrippino & Ricco (2021) find that a contractionary monetary policy shock causes an immediate fall in the price level, while impacts on unemployment materialise much more slowly. Again, this suggests that causation in the Phillips curve runs from inflation to unemployment, not the other way round.

1.2 Robust real rate rules in the three equation NK world

To understand how our robust rule in equation (3) can explain causation running from inflation to the output gap in the Phillips curve, suppose the rest of the model comprises the Phillips curve:[8]

[7] McLeay & Tenreyro (2019) provide an alternative explanation based on the fact that optimal policy prescribes a negative correlation between inflation and output, making difficult empirical identification of the Phillips curve.

[8] Throughout this paper, we multiply the mark-up shock by κ as the ratio of the response to x_t and the response to

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + \kappa \omega_t, \tag{4}$$

and the discounted/compounded Euler equation:

$$x_t = \delta \mathbb{E}_t x_{t+1} - \varsigma (r_t - n_t), \tag{5}$$

where x_t is the output gap, ω_t is a mark-up/cost-push shock, and n_t is the exogenous natural real rate of interest. This form of discounted/compounded Euler equation appears in Bilbiie (2019) and (under discounting) in McKay, Nakamura & Steinsson (2017). The latter paper shows it provides a good approximation to a heterogeneous agent model with incomplete markets. The standard Euler equation is recovered if $\delta = 1$ and ς is the elasticity of intertemporal substitution. This specification also nests the limited asset market participation or "TANK" model of Bilbiie (2008) when $\delta = 1$, but ς is allowed to be negative.

Since $\pi_t = -\frac{1}{\phi - \rho_\zeta} \zeta_t$, and ζ_t is AR(1) with persistence ρ_ζ, the Phillips curve (4) implies that $x_t = -\frac{1}{\kappa} \frac{1-\beta\rho_\zeta}{\phi-\rho_\zeta} \zeta_t - \omega_t$. The Phillips curve is determining the output gap, given the already determined level of inflation. Does x_t help forecast π_t here? Clearly no. $\mathbb{E}_t \pi_{t+1} = -\frac{1}{\phi-\rho_\zeta} \mathbb{E}_t \zeta_{t+1} = -\frac{\rho_\zeta}{\phi-\rho_\zeta} \zeta_t = \rho_\zeta \pi_t$. Once you know π_t, you already have all the information you need to form the optimal forecast of π_{t+1}. The correlation in π_t and x_t provides no extra information.[9]

This model also enables us to show the robustness of our rule's determinacy in practice. Note that with x_t expressed as a linear combination of exogenous variables, there is no need to solve the Euler equation (5) forward, so the degree of discounting (δ) can have no effect on determinacy. Not needing to solve the Euler equation forward also gives robustness to a missing transversality constraint on household assets. For example, if ω_t is independent across time, then the Euler equation implies $r_t = n_t + \frac{1}{\varsigma}\left[\frac{1}{\kappa}\frac{(1-\beta\rho_\zeta)(1-\delta\rho_\zeta)}{\phi-\rho_\zeta}\zeta_t + \omega_t\right]$. This contrasts with the results of Bilbiie (2019) who finds

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ω_t is not a function of either the (Calvo) price adjustment probability or the (Rotemberg) price adjustment cost. See Khan (2005) for derivations.

[9] This result is robust to generalizing to an ARMA(1,1) process for ζ_t. See Appendix E.1.

that when $\varsigma > 0$ and $\beta \leq 1$, the Taylor principle ($\phi > 1$) is only sufficient for determinacy in the discounting case ($\delta \leq 1$),[10] and with Bilbiie (2008) who finds that when $\delta = 1$ and $\varsigma < 0$, the Taylor principle ($\phi > 1$) is neither necessary nor sufficient for determinacy.[11] Under our rule (3), the Taylor principle is necessary and sufficient for determinacy whether there is discounting or compounding, and whether ς is positive or negative (given $\phi \geq 0$).[12]

The rule is also robust to the presence of lags in the Euler or Phillips curve. For example, suppose the Phillips curve and Euler equation are instead given by:

$$\begin{aligned} \pi_t &= \tilde{\beta}(1-\varrho_\pi)\mathbb{E}_t\pi_{t+1} + \tilde{\beta}\varrho_\pi\pi_{t-1} + \kappa x_t + \kappa\omega_t, \\ x_t &= \tilde{\delta}(1-\varrho_x)\mathbb{E}_t x_{t+1} + \tilde{\delta}\varrho_x x_{t-1} - \varsigma(r_t - n_t), \end{aligned} \tag{6}$$

where $\tilde{\beta}$ and $\tilde{\delta}$ may not have the same structural interpretation as β and δ (depending on the precise micro-foundation). These equations have no impact on the solution for inflation, which remains $\pi_t = -\frac{1}{\phi-\rho_\zeta}\zeta_t$. Instead, the lag in the Euler equation changes the dynamics of real interest rate, with no impact on inflation or output gaps, while the lag in the Phillips curve affects both output gap and real rate dynamics, with no impact on inflation. For example, if ζ_t's law of motion is given by $\zeta_t = \rho_\zeta\zeta_{t-1} + \varepsilon_{\zeta,t}$, where $\mathbb{E}_{t-1}\varepsilon_{\zeta,t} = 0$, then:

$$x_t = \frac{1}{\kappa}\frac{1}{\phi-\rho_\zeta}\left[\left(\tilde{\beta}\varrho_\pi - \rho_\zeta\left(1-\tilde{\beta}(1-\varrho_\pi)\rho_\zeta\right)\right)\zeta_{t-1} - \left(1-\tilde{\beta}(1-\varrho_\pi)\rho_\zeta\right)\varepsilon_{\zeta,t}\right] - \omega_t.$$

As before, the output gap has a closed form solution in terms of the monetary policy and cost push shocks. Despite appearances, inflation is not a true endogenous state, as it must always equal $-\frac{1}{\phi-\rho_\zeta}\zeta_t$. Monetary policy shocks are still always contractionary, but they only have a short-lived impact on the output gap if ϱ_π is around $\frac{\rho_\zeta(1-\beta\rho_\zeta)}{\beta\left(1-\rho_\zeta^2\right)}$.

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[10] See equation (40) of Appendix C.1 of Bilbiie (2019).

[11] See Proposition 7 of Appendix B.1 of Bilbiie (2008).

[12] In Appendix E.2 we prove that this is robust to monetary responses to the real rate which are not exactly equal to 1. This is also a corollary of the more general result proven in Appendix E.4.

1.3 Responding to other endogenous variables

The original Taylor rule contained a response to output. Even with a unit coefficient on the real interest rate, responding to output will change the determinacy conditions, though it still preserves some robustness. To see this, consider the monetary rule:

$$i_t = r_t + \phi_\pi \pi_t + \phi_x x_t + \zeta_t.$$

Assuming the lag-augmented NK Phillips curve (6) continues to hold, this monetary rule is equivalent to the rule:

$$i_t = r_t + \phi_\pi \pi_t + \kappa^{-1}\phi_x\big[\pi_t - \tilde{\beta}(1-\varrho_\pi)\mathbb{E}_t\pi_{t+1} - \tilde{\beta}\varrho_\pi\pi_{t-1}\big] - \phi_x\omega_t + \zeta_t.$$

(This is produced by using the Phillips curve to substitute out the output gap.) Combined with the Fisher equation, we have that:

$$\mathbb{E}_t\pi_{t+1} = \phi_\pi \pi_t + \kappa^{-1}\phi_x\big[\pi_t - \tilde{\beta}(1-\varrho_\pi)\mathbb{E}_t\pi_{t+1} - \tilde{\beta}\varrho_\pi\pi_{t-1}\big] - \phi_x\omega_t + \zeta_t.$$

This has a determinate solution if the quadratic:

$$\big[1 + \kappa^{-1}\phi_x\tilde{\beta}(1-\varrho_\pi)\big]A^2 - (\phi_\pi + \kappa^{-1}\phi_x)A + \kappa^{-1}\phi_x\tilde{\beta}\varrho_\pi = 0$$

has a unique solution for A inside the unit circle. It is sufficient that the quadratic is positive at $A = -1$ but negative at $A = 1$, which holds if and only if:

$$1 + \kappa^{-1}\phi_x(1+\tilde{\beta}) + \phi_\pi > 0, \qquad 1 - \kappa^{-1}\phi_x(1-\tilde{\beta}) - \phi_\pi < 0.$$

So, if $\kappa > 0$, $\phi_x \geq 0$ and $\tilde{\beta} \in [0,1]$ as expected, then it is sufficient that $\phi_\pi > 1$ as before.[13] This is still considerable robustness. Providing there is something like a Phillips curve linking inflation and the output gap, the standard $\phi_\pi > 1$ condition will be sufficient for determinacy. This would not hold with a more standard monetary rule without a response to real rates: in that case determinacy depends on $\tilde{\delta}$ and ς, as shown by the Bilbiie (2008; 2019) results discussed in the last subsection.

Responding to real rates provides additional robustness even with a response to output as it disconnects the Euler equation from the rest of the model. The only

[13] This is stronger than necessary. The second condition states that $\phi_\pi + \kappa^{-1}\phi_x(1-\tilde{\beta}) > 1$ so a response to the output gap can substitute for a response to inflation. This condition is identical to that for the standard (purely forward looking) three equation NK model with Taylor type rule found in Woodford (2001).

remaining role of the Euler equation is to give a path for real rates, given the already determined paths of output and inflation.[14] The Fisher equation, not the Euler equation is central to monetary policy transmission under real rate rules.

For greater robustness, the central bank can replace the response to the output gap with a response to the cost push shock ω_t. With an appropriate response to ω_t, this is observationally equivalent to responding to the output gap, but ensures determinacy under the standard Taylor principle.

However, it may be hard for the central bank to observe the cost push shock. To get round this, suppose that the central bank knows that a Phillips curve in the form of equation (6) holds. (Our results would generalize to other links between real and nominal variables.) For now, suppose the central bank also knows the coefficients in equation (6). Then the central bank could use a rule of the form:

$$i_t = r_t + \phi_\pi \pi_t + \phi_x \big[x_t - \kappa^{-1} \big[\pi_t - \tilde{\beta}(1-\varrho_\pi) \mathbb{E}_t \pi_{t+1} - \tilde{\beta} \varrho_\pi \pi_{t-1} \big] \big] + \zeta_t.$$

By equation (6), this implies that:

$$i_t = r_t + \phi_\pi \pi_t - \phi_x \omega_t + \zeta_t,$$

as desired. Of course, the central bank is also unlikely to know the exact coefficients in the Phillips curve. However, we show in Appendix E.3 that the central bank may learn these coefficients in real time, without changing the determinacy conditions, at least under reasonable parameter restrictions.[15]

If the central bank wishes to respond to other endogenous variables, a similar approach should be possible if they are aware of the broad form of the model's structural equations. However, the central bank may legitimately worry about having fundamental misconceptions about how the economy works. They can be reassured

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[14] This is analogous to how the Euler equation is slack when solving for optimal monetary policy. In that case, the combined Euler equation and Fisher equation give the level of nominal rates required to hit the optimal output gap and inflation. The author thanks Florin Bilbiie for this observation.

[15] It is sufficient (but not necessary) that $\phi_x \geq 0$, $\phi_\pi \geq 0$, $\kappa \geq 0$, $\tilde{\beta} \in [0,1]$, $\varrho_\pi \in [0,1)$, $\rho_\zeta \in [0,1)$ and $\phi_\pi > \max\left\{ \frac{1}{\tilde{\beta}(1-\varrho_\pi)}, 2(1-\varrho_\pi), \frac{\phi_x(1+\tilde{\beta})}{\kappa} \right\}$.

though that the Taylor principle is sufficient for determinacy if the response to other endogenous variables is small enough, no matter the form of the model's other equations. We prove this in Appendix E.4. This also implies that a precise unit response to real rates is not needed for determinacy. Real rates are just another endogenous variable, so determinacy only requires a response sufficiently close to one.

Classic results on determinacy in monetary models can be reinterpreted through this lens. Even if the central bank is not responding to real interest rates, it is still likely to be responding to variables that are highly correlated with them. Determinate rules will be ones sufficiently close to a real rate rule.

For example, many models contain an Euler equation of the form:

$$1 = \beta(\exp r_t)\mathbb{E}_t\left(\frac{C_t}{C_{t+1}}\right)^{\frac{1}{\varsigma}},$$

where C_t is real consumption per capita and ς is the elasticity of intertemporal substitution. Additionally, in many models, in equilibrium, consumption growth roughly follows an ARMA(1,1) process:

$$g_t := \log\left(\frac{C_t}{C_{t-1}}\right) = (1-\rho_g)g + \rho_g g_{t-1} + \varepsilon_{g,t} + \theta_g \varepsilon_{g,t-1}, \qquad \varepsilon_{g,t} \sim N(0, \sigma_g^2).$$

(This is a good approximation to US post-war data.[16]) Combining these two equations gives that:

$$r_t = -\log\beta + \frac{1-\rho_g}{\varsigma}g - \frac{1}{2}\left(\frac{\sigma_g}{\varsigma}\right)^2 + \frac{\rho_g}{\varsigma}g_t + \frac{\theta_g}{\varsigma}\varepsilon_{g,t},$$

implying that a (roughly) $\frac{\rho_g}{\varsigma}$ response to consumption growth can substitute for a (roughly) unit response to real rates.

Of course, output (growth, level or gap) is in turn highly correlated with consumption growth, so output (growth, level or gap) may also substitute for real rates. For example, in the Smets & Wouters (2007) model of the US economy, the monetary rule is of the form $i_t = \phi_\pi \pi_t + z_t + \zeta_t$, where z_t is a linear combination of other endogenous

[16] Estimating on US data from 1947Q1 to 2021Q4 (BEA series: A794RX) with T-distributed shocks gives $\rho_g = 0.69$, $\theta_g = -0.50$ (p-values both below 10^{-5}). Using Gaussian shocks on less volatile sub-periods gives similar results.

variables and ζ_t is the monetary shock. At the estimated posterior mode, the correlation between z_t and the real interest rate is 0.63, with both variables having standard deviation of 0.46%. Thus, the Smets & Wouters (2007) estimates imply that the Fed is already about two thirds of the way to using a simple robust real rate rule.

There is one final way of allowing an interest rate response to other endogenous variables that is both simple and robust. Rather than placing the endogenous variables directly within the rule, the central bank can follow a time-varying inflation target which is a function of these endogenous variables. We examine this approach in the next subsection.

1.4 Implementing arbitrary inflation dynamics

Instead of responding directly to other endogenous variable or exogenous shocks, the central bank could instead adopt a time-varying inflation target. With this target responding to other endogenous variables or shocks, very similar results can be obtained. In this subsection, we show that real rate rules can determinately implement any target path for inflation, no matter the rest of the model. This implies they can also implement optimal policy, and so attain high welfare. It also implies that any observed inflation and interest rate dynamics are consistent with a real rate rule.

Let π_t^* be the central bank's inflation target. This may be a function of other endogenous variables, and of the economy's shocks.[17] For example, in order to dampen the output response to mark-up shocks, the central bank could set π_t^* either as a decreasing function of x_t, or as an increasing function of ω_t. The central bank should publish this target each period, else the limited information of market participants could lead to additional volatility.

With a time-varying inflation target, the real rate rule becomes:

$$i_t = r_t + \mathbb{E}_t \pi_{t+1}^* + \phi(\pi_t - \pi_t^*). \tag{7}$$

[17] Ireland (2007) also allows the central bank's inflation target to respond to other structural shocks.

From the Fisher equation (1), this implies:

$$\mathbb{E}_t(\pi_{t+1} - \pi^*_{t+1}) = \phi(\pi_t - \pi^*_t).$$

Again with $\phi > 1$, there is a unique solution for $\pi_t - \pi^*_t$, now with $\pi_t = \pi^*_t$ for all t. I.e., at all periods of time, and in all states of the world, realised inflation is equal to π^*_t. Effectively, the central bank is able to choose an arbitrary path for inflation as the unique, determinate equilibrium outcome.

There are only two constraints on the targeted path for inflation. The first is that the central bank must be capable of calculating a reasonable approximation to $\mathbb{E}_t\pi^*_{t+1}$. One way to ensure this is to make π^*_t only a function of $t - 1$ dated variables. Alternatively, the central bank could respond to variables for which there are liquid futures or option markets, or the central bank could form these expectations using a forecasting model. Errors in these forecasts will show up as monetary policy shocks, increasing the variance of $\pi_t - \pi^*_t$, but this can be dampened with large ϕ.

The second constraint on the target inflation path is that it should not induce explosive dynamics in inflation. For example, if $\pi^*_t := 2\pi_{t-1} + \varepsilon_{*,t}$, for some target shock $\varepsilon_{*,t}$, then with $\pi_t = \pi^*_t$, $\pi_t = 2\pi_{t-1} + \varepsilon_{*,t}$, which is an explosive process. One way to ensure this is for π^*_t to only be a function of exogenous variables, but this is far from necessary. However, even were π^*_t constrained to not be a function of endogenous variables, this would still not be much of a limitation, since in stationary equilibrium, endogenous variables must have a representation as a function of the infinite history of the economy's shocks. This means that even with an exogenous π^*_t, rules in the form of (7) can mimic the outcomes of any other monetary policy regime. We show this formally in Appendix E.5.

This has two important implications. Firstly, it means that appropriately designed real rate rules can implement (timeless/unconditional/etc.) optimal policy, and thus attain the highest possible level of welfare.[18] In Appendix C we look at welfare in New Keynesian models when the central bank is constrained to follow a real rate rule that produces simple inflation dynamics. We show that even with such a constraint, real rate rules can still come close to fully optimal policy.

Secondly, it means that it is impossible to test empirically if a central bank is using a general real rate rule. Any dynamics of inflation and interest rates are consistent with a real rate rule like (7), for an appropriately chosen π_t^*. Thus, real rate rules are observationally equivalent to any other specification for central bank behaviour. While in the last subsection we found that the Fed was not exactly using a simple real rate rule, we now see that a slightly more sophisticated real rate rule could fully explain Fed behaviour.

1.5 Adding interest rate smoothing

High degrees of interest rate smoothing are often thought to be a good description of actual central bank behaviour given the rarity of large interest rate changes. However, since the rule (7) can generate arbitrary inflation dynamics (and hence arbitrary nominal rate dynamics), we cannot conclude based on observed nominal rates that the central bank is actually smoothing rates. Nonetheless, interest rate smoothing is worth investigating in our context, as it can be a source of additional robustness.

For example, suppose that the central bank sets interest rates according to the fully smoothed real rate rule:

$$i_t - r_t = i_{t-1} - r_{t-1} + \mathbb{E}_t \pi_{t+1}^* - \mathbb{E}_{t-1} \pi_t^* + \theta(\pi_t - \pi_t^*),$$

where $\theta > 0$ and where π_t^* is the inflation target, as before. Under a real rate rule, the central bank should attempt to smooth $i_t - r_t$, not just i_t. This ensures real rates can still be substituted out from the Fisher equation. Hence, we have $i_{t-1} - r_{t-1}$ on the

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[18] Other papers have examined the implementation of optimal policy in specific models using instrument rate rules (see e.g. Svensson & Woodford 2003; Dotsey & Hornstein 2006; Evans & Honkapohja 2006; Evans & McGough 2010). Ours is unique in enabling the implementation of a certain inflation path robustly across models.

right-hand side.

Combining this monetary rule with the Fisher equation gives:

$$i_{t-1} - r_{t-1} + \mathbb{E}_t \pi^*_{t+1} - \mathbb{E}_{t-1} \pi^*_t + \theta(\pi_t - \pi^*_t) = i_t - r_t = \mathbb{E}_t \pi_{t+1}.$$

Now, from the lagged Fisher equation, $i_{t-1} - r_{t-1} = \mathbb{E}_{t-1}\pi_t$, so:

$$\theta(\pi_t - \pi^*_t) = \mathbb{E}_t(\pi_{t+1} - \pi^*_{t+1}) - \mathbb{E}_{t-1}(\pi_t - \pi^*_t).$$

To solve this equation, first let $p_t := \sum_{s=1}^{t}(\pi_t - \pi^*_t)$ be the price level relative to its target trend, normalized to $p_0 = 0$. Thus:

$$\theta(p_t - p_{t-1}) = \mathbb{E}_t(p_{t+1} - p_t) - \mathbb{E}_{t-1}(p_t - p_{t-1}).$$

Summing this equation over time (from period 1 to period t) then gives that:

$$\theta p_t = \mathbb{E}_t(p_{t+1} - p_t) - \mathbb{E}_0 p_1.$$

Hence, if we define $\hat{p}_t := p_t + \frac{1}{\theta}\mathbb{E}_0 p_1$, then:

$$(1 + \theta)\hat{p}_t = \mathbb{E}_t \hat{p}_{t+1}.$$

For $\theta > 0$, this has the unique equilibrium $\hat{p}_t = 0$, so $\pi_t = \pi^*_t$ for all t, as required.

In equilibrium then, our smoothed real rate rule produces the same inflation (and hence the same nominal rates) as our unsmoothed real rate rule, equation (7). However, it is more robust in one crucial respect. Whereas the rule in equation (7) required a response to current inflation of $\phi > 1$, the fully smoothed real rate rule just needs a response to current inflation of $\theta > 0$.

In practice, it may be hard for central banks to commit to responding more than one for one to inflation. Even if they manage this, it is likely to be hard for them to convince other economic agents that they really will be so aggressive all the time. Since inflation and nominal rates are identical for any $\phi > 1$, there is no way for these agents to observe ϕ. Even with $\phi < 1$, there are equilibria which look identical to the equilibria with $\phi > 1$. It is likely to be far easier for central banks to convince economic agents that they just respond positively to inflation. This is all that is needed for a fully smoothed real rate rule.

For the rest of this paper, we return to looking at unsmoothed rules. However, all our results would generalize to smoothed rules. There is a strong case for the preferability of such smoothing.

2 Challenges to real rate rules

We have established the excellent properties of real rate rules when the linear Fisher equation holds. However, the linear Fisher equation may fail to hold exactly due to risk premia or other wedges. We address risk premia in the first subsection here, via examining real rate rules in fully non-linear models, and then we look at other wedges in the following subsection. We show real rate rules still retain their robustness.

We then examine whether the possibility of inflation being determined independently of monetary policy represents a challenge to robust real rate rules. This is relevant under active fiscal policy, for example. We show that with long maturity debt, a solution with stable inflation and stable real variables always exists, independent of whether fiscal policy is active or passive. This implies that the fiscal theory of the price level fails to determine a unique outcome in general, a result which may be of independent interest.

Finally, we verify that it is actually possible for a central bank to apply a real rate rule out of equilibrium. This concern disappears once out of equilibrium behaviour is fully specified.

2.1 Risk premia and non-linear models

Our examples so far have been linearized models. Linearization removes the risk premium that enters the Fisher equation due to inflation risk. It is thus important for us to verify that real rate rules still work in fully non-linear models.

Suppose that Ξ_t is the real stochastic discount factor (SDF) between period t and period $t+1$, and that I_t is the gross nominal interest rate (so $i_t = \log I_t$) and that R_t is the gross real interest rate (so $r_t = \log R_t$). Then the pricing equations for one-period nominal and real bonds imply:

$$I_t \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} = 1, \qquad R_t \mathbb{E}_t \Xi_{t+1} = 1.$$

The nonlinear version of equation (2) is the following rule:

$$I_t = R_t \Pi^* \left(\frac{\Pi_t}{\Pi^*}\right)^{\phi},$$

where we allow for a constant gross inflation target of Π^*. Combining this rule with the bond pricing equations implies that:

$$\mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} = \frac{\mathbb{E}_t \Xi_{t+1}}{\Pi^*} \left(\frac{\Pi^*}{\Pi_t}\right)^{\phi},$$

so:

$$\mathbb{E}_t \frac{\Xi_{t+1}}{\mathbb{E}_t \Xi_{t+1}} \frac{\Pi^*}{\Pi_{t+1}} = \left(\frac{\Pi^*}{\Pi_t}\right)^{\phi}.$$

It is easy to see that $\Pi_t = \Pi^*$ is always one solution of this equation, as $\mathbb{E}_t \frac{\Xi_{t+1}}{\mathbb{E}_t \Xi_{t+1}} = 1$. Thus, robust real rate rules are always consistent with stable inflation, even in fully non-linear models.

Furthermore, under mild assumptions, there exists a constant $\overline{Z} \geq 1$ such that for all sufficiently high ϕ, $1 \leq \frac{\Pi^*}{\Pi_t} \leq \overline{Z}^{\frac{1}{\phi-1}}$. This upper bound tends to 1 as ϕ goes to ∞, thus for large ϕ, any solution must have $\Pi_t \approx \Pi^*$. This holds even if the SDF, Ξ_t, is a complicated function of inflation and its history. Under slightly stronger assumptions on the SDF, we can even guarantee that $\Pi_t = \Pi^*$ is the unique solution for all sufficiently large ϕ. These results are proven in Appendix A. For the sake of tractability, we return to the linearized world for the bulk of the rest of this paper.

2.2 Wedges in the Fisher equation

One natural concern is that real rate rules may lose their robust determinacy if the Fisher equation does not hold exactly. Risk premia are one source of a wedge in the Fisher equation, but we showed in the previous subsection that real rate rules continue to perform well in the presence of endogenous risk premia. However, there are other reasons why there may be a wedge in the Fisher equation. For example, nominal bonds may provide greater liquidity services than real bonds, and so nominal bonds may command a premium. Such a premium is documented by Fleckenstein, Longstaff & Lustig (2014), based on comparing synthetic treasury bonds constructed from TIPS and inflation swaps to actual treasury bonds. Furthermore, TIPS provide deflation protection, which may result in TIPS also commanding a premium, giving another source of a wedge in the Fisher equation. A Fisher equation wedge could even come from bounded rationality of market participants.

Suppose then that the linearized Fisher equation takes the form:

$$i_t = r_t + \mathbb{E}_t \pi_{t+1} + \nu_t,$$

where ν_t is a potentially endogenous wedge term. We assume though that ν_t is stationary, and that there exists some $\overline{\mu}_0, \overline{\mu}_1, \overline{\mu}_2, \overline{\gamma}_0, \overline{\gamma}_1, \overline{\gamma}_2 \geq 0$ such that for any stationary solution for π_t, $|\mathbb{E}\nu_t| \leq \overline{\mu}_0 + \overline{\mu}_1 |\mathbb{E}\pi_t| + \overline{\mu}_2 \operatorname{Var} \pi_t$ and $\operatorname{Var} \nu_t \leq \overline{\gamma}_0 + \overline{\gamma}_1 |\mathbb{E}\pi_t| + \overline{\gamma}_2 \operatorname{Var} \pi_t$, for all $t \in \mathbb{Z}$. This assumption is extremely mild, as all of these coefficients may be arbitrarily large. For example, if ν_t were to come purely from an inflation risk premium, we would expect $\overline{\mu}_2 > 0$ and $\overline{\gamma}_0 > 0$ but all other coefficients to be zero. Alternatively, if ν_t were to come purely from the liquidity services provided by nominal bonds, we would expect $\overline{\mu}_0$, $\overline{\gamma}_0$ and $\overline{\mu}_1$ to be positive (the latter as the value of liquidity services might vary over the cycle), but all other coefficients to be zero.

Combining the modified Fisher equation with the simple rule in (2) gives:

$$\mathbb{E}_t \pi_{t+1} + \nu_t = \phi \pi_t,$$

so:

$$\begin{aligned} \pi_t &= \phi^{-1} \mathbb{E}_t \pi_{t+1} + \phi^{-1} \nu_t = \phi^{-2} \mathbb{E}_t \pi_{t+2} + \phi^{-2} \mathbb{E}_t \nu_{t+1} + \phi^{-1} \nu_t = \cdots \\ &= \mathbb{E}_t \sum_{k=0}^{\infty} \phi^{-k-1} \nu_{t+k} + \lim_{k \to \infty} \left[\phi^{-k} \mathbb{E}_t \pi_{t+k} \right] = \mathbb{E}_t \sum_{k=0}^{\infty} \phi^{-k-1} \nu_{t+k}, \end{aligned}$$

assuming as ever that we select the stationary equilibrium for inflation.[19] Thus, with $\phi > 1$:

[19] Ireland (2015) finds a role for risk premia in explaining US inflation fluctuations, so it is empirically plausible that the Fisher equation wedge should appear in the solution for inflation.

$$|\mathbb{E}\pi_t| = \frac{|\mathbb{E}\nu_t|}{\phi - 1} \le \frac{\overline{\mu}_0 + \overline{\mu}_1|\mathbb{E}\pi_t| + \overline{\mu}_2 \operatorname{Var} \pi_t}{\phi - 1},$$

and:[20]

$$\begin{aligned}\operatorname{Var} \pi_t &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \phi^{-j-1}\phi^{-k-1} \operatorname{Cov}\left(\mathbb{E}_t \nu_{t+j}, \mathbb{E}_t \nu_{t+k}\right) \le \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \phi^{-j-1}\phi^{-k-1} \operatorname{Var} \nu_t \\ &\le \frac{\overline{\gamma}_0 + \overline{\gamma}_1|\mathbb{E}\pi_t| + \overline{\gamma}_2 \operatorname{Var} \pi_t}{(\phi - 1)^2}.\end{aligned}$$

So, for sufficiently large ϕ:[21]

$$\begin{aligned}|\mathbb{E}\pi_t| &\le \frac{[(\phi-1)^2 - \overline{\gamma}_2]\overline{\mu}_0 + \overline{\mu}_2\overline{\gamma}_0}{(\phi - 1 - \overline{\mu}_1)[(\phi-1)^2 - \overline{\gamma}_2] - \overline{\mu}_2\overline{\gamma}_1} = O\left(\frac{1}{\phi}\right), \\ \operatorname{Var} \pi_t &\le \frac{(\phi - 1 - \overline{\mu}_1)\overline{\gamma}_0 + \overline{\mu}_0\overline{\gamma}_1}{(\phi - 1 - \overline{\mu}_1)[(\phi-1)^2 - \overline{\gamma}_2] - \overline{\mu}_2\overline{\gamma}_1} = O\left(\frac{1}{\phi^2}\right).\end{aligned}$$

Hence, as $\phi \to \infty$, $\mathbb{E}\pi_t \to 0$ and $\operatorname{Var} \pi_t \to 0$. While the central bank can no longer guarantee precisely zero inflation in the presence of an endogenous wedge, if they are aggressive enough, they can ensure the mean and variance of inflation are arbitrarily close to zero. Thus, wedges in the Fisher equation do not present a substantial challenge to the performance of real rate rules.

However, if the pricing of nominal bonds is indeed highly distorted by the liquidity services they provide (for example), then the central bank may attain lower inflation bias and variance for a given ϕ by intervening in inflation swap markets rather than nominal bond ones. In our notation, an inflation swap is a contract agreed in period t between two parties, A and B, in which party A promises to make a net payment of $\Pi_{t+1} - K_t$ to party B in period $t + 1$, where K_t is the negotiated contract rate. Writing Ξ_{t+1} for the real SDF between periods t and $t + 1$, this contract rate must solve:

$$\mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} (\Pi_{t+1} - K_t) = 0.$$

So, from log-linearizing:

$$k_t = \log K_t = \mathbb{E}_t \pi_{t+1},$$

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[20] Here we use the fact that by the Cauchy-Schwarz inequality, the law of total variance and stationarity: $\operatorname{Cov}\left(\mathbb{E}_t \nu_{t+j}, \mathbb{E}_t \nu_{t+k}\right) \le \sqrt{\left(\operatorname{Var} \mathbb{E}_t \nu_{t+j}\right)\left(\operatorname{Var} \mathbb{E}_t \nu_{t+k}\right)} = \sqrt{\left(\operatorname{Var} \nu_{t+j} - \mathbb{E} \operatorname{Var}_t \nu_{t+j}\right)\left(\operatorname{Var} \nu_{t+k} - \mathbb{E} \operatorname{Var}_t \nu_{t+k}\right)} \le \operatorname{Var} \nu_t$.

[21] In particular, we need $\phi - 1 > \overline{\mu}_1$, $(\phi - 1)^2 > \overline{\gamma}_2$ and $(\phi - 1 - \overline{\mu}_1)[(\phi - 1)^2 - \overline{\gamma}_2] > \overline{\mu}_2\overline{\gamma}_1$.

to first order.

The central bank can then use the inflation swap real rate rule:

$$k_t = \phi \pi_t.$$

Combined with the inflation swap pricing equation, this gives $\mathbb{E}_t \pi_{t+1} = \phi \pi_t$, just like when the central bank intervenes in nominal bond markets. The advantage of directly targeting inflation swap contract rates is that inflation swaps are unlikely to provide liquidity services, unlike nominal bonds, meaning the inflation swap pricing equation will be less distorted than the Fisher equation. One final benefit of directly targeting inflation swap contract rates is that inflation swaps do not include the deflation protection given by TIPS. This removes one additional source of distortion in the $i_t - r_t$ gap.

2.3 The fiscal theory of the price level and the risk of over determinacy

As long as the linear Fisher equation holds, robust real rate rules can never fail to rule out sunspots. However, in an economy in which the price level is determinate independent of monetary policy, they may still produce explosive inflation.[22] This is true of any monetary rule respecting the Taylor principle, not just the real rate rules we examine in this paper. Inflation becomes "over determined", and an explosive solution is all that remains.

For example, suppose that government debt is all one period and nominal, and that real government surpluses are not responsive to government debt levels, meaning fiscal policy is "active". Then the price level is pinned down by the government debt valuation equation (see e.g. Cochrane (2022)), in line with the fiscal theory of the price level (FTPL). In particular, to a first order approximation with flexible prices and

[22] Note: it is certainly not the case though that in any model in which an interest rate peg is determinate, a real rate rule would produce explosive inflation. For example, in the New Keynesian model with a discounted Euler equation, from Subsection 1.2, if $\delta \in \left(-\frac{1+\beta+\kappa\varsigma}{1+\beta}, \frac{1-\beta-\kappa\varsigma}{1-\beta}\right)$ then an interest rate peg is determinate. We saw that the real rate rule is also determinate (and non-explosive) in this model.

constant real interest rates:[23]

$$\pi_t - \mathbb{E}_{t-1}\pi_t = -\varepsilon_{s,t}, \tag{8}$$

where $\varepsilon_{s,t}$ is an exogenous shock to the present value of real primary government surpluses, scaled by the value of outstanding real government debt, with $\mathbb{E}_{t-1}\varepsilon_{s,t} = 0$. Suppose in this world that the central bank did follow the basic real rate rule $i_t = r_t + \phi\pi_t + \varepsilon_{\zeta,t}$, where $\phi > 1$ and $\mathbb{E}_{t-1}\varepsilon_{\zeta,t} = 0$. Then, from the Fisher equation, $\mathbb{E}_{t-1}\pi_t = \phi\pi_{t-1} + \varepsilon_{\zeta,t-1}$, implying from (8) that:

$$\pi_t = \phi\pi_{t-1} + \varepsilon_{\zeta,t-1} - \varepsilon_{s,t}.$$

With $\phi > 1$, this is an explosive process. We know from Subsection 1.1 that if there were to be a stationary solution for π_t, it must have $\pi_t = -\frac{1}{\phi}\varepsilon_{\zeta,t}$. But this is inconsistent with equation (8) as long as $\varepsilon_{\zeta,t} \neq \phi\varepsilon_{s,t}$, so only the non-stationary solution remains.

However, this is a knife edge result. For example, suppose that the government issues multi-period (geometric coupon) debt, and that both monetary and fiscal policy are active (i.e., real primary government surpluses do not respond to debt, and the monetary rule satisfies the Taylor principle). Based on results with one period debt, researchers have tended to assume that this "active-active" combination will inevitably produce explosive inflation. This is incorrect.

In Appendix B.1 we examine the equilibria of a non-linear model with multi-period debt under flexible prices. We show that under active fiscal policy, there is a valid equilibrium in which real variables and inflation are stable and independent of surpluses, whether or not monetary policy is active. These equilibria feature a growing bubble in the price of government debt which is balanced by declining debt quantities. The initial debt price jumps to ensure the transversality condition is still satisfied, giving a "Fiscal Theory of the Debt Price". These equilibria exist as long as the geometric decay factor for the bond coupons is not precisely equal to zero (the one-period debt case). Furthermore, under passive monetary policy, we find a continuum

[23] See Cochrane (2022), Subsection 2.5 and following.

of equilibria, contrary to the usual claim that the active fiscal, passive monetary, combination ensures unique outcomes (which is again only true with one period debt). These equilibria feature arbitrarily high inflation.

These results are not specific to the particular model set-up we use in Appendix B.1. Firstly, in Appendix B.2 we show that these results also hold in a linearised model with sticky prices. Then, in Appendix B.3 we show that generically, any model achieving determinacy via an FTPL-type mechanism must admit a stable solution under a real rate rule. There are only two main restrictions for this result. Firstly, the potentially explosive variables such as bond prices must not feed back to the real economy. Secondly, the equations determining the potentially explosive variables must not be too forward looking. Both of these assumptions are satisfied by standard FTPL models under geometric coupon debt.[24] Therefore, only in knife edge cases will following the Taylor principle guarantee explosive inflation.

2.4 Setting nominal rates out of equilibrium

Real rate rules work so well thanks to the cancellation of the real rates in both the rule and the Fisher equation. The reader may worry that this cancellation masks a type of singularity that would prevent the central bank from setting rates according to a real rate rule.

To see the apparent problem, we suppose that the economy is currently in period 0, and that all in future periods, the central bank's behaviour will be given by the simple real rate rule of equation (2). We assume the Fisher equation (1) holds in all periods. Then for $t > 0$:

$$r_t + \mathbb{E}_t \pi_{t+1} = i_t = r_t + \phi \pi_t.$$

[24] Note that the geometric coupon bond first order condition $Q_t = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}}(1 + \omega Q_{t+1})$ can be rewritten as the two equations $E_t = \frac{1+\omega Q_t}{Q_{t-1}}$, and $1 = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} E_{t+1}$. Here Q_t is potentially explosive, but is determined by a backward-looking equation, while E_t is asymptotically stable.

Our discussion up to now would naturally lead the reader to conclude that $\pi_t = 0$ for all $t > 0$, unconditional on whatever happens in period 0. Suppose this were true. Then, the period 0 Fisher equation would imply that $i_0 = r_0$. Thus, apparently, nothing the central bank could do in period 0 could ever produce $i_0 \neq r_0$. In particular, it seems that the central bank cannot apply a real rate rule in period 0 if $\pi_0 \neq 0$. This is incorrect, as if a real rate rule applies from period 0 onwards, it is only the case that $\pi_t = 0$ for all $t > 0$ if it happens that $\pi_0 = 0$.

This confusion stems from us having given an incomplete description of equilibrium up to now. A full equilibrium description specifies the outcome for every possible history, not just those on the equilibrium path. We glossed over off equilibrium behaviour till now to simplify our presentation, but these details do matter.

A full description of the standard equilibrium of the Fisher equation (1) and real rate rule (2) is as follows. Suppose the rule was introduced in period 0. Then, for all $t \geq 0$, if $\pi_s = 0$ for all $s \in \{0,1,\dots,t-1\}$, then $\pi_t = 0$. Otherwise, $\pi_t = \phi\pi_{t-1}$. In period 0, the first condition holds vacuously, so on the equilibrium path, $\pi_0 = 0$, and hence $\pi_t = 0$ for all $t \in \mathbb{N}$. However, suppose that off the equilibrium path, $\pi_0 \neq 0$. Then $\pi_1 = \phi\pi_0$, and hence the period 0 Fisher equation states that $i_0 - r_0 = \phi\pi_0$. Thus, $i_0 - r_0$ is not fixed; it is a function of period 0 inflation, something that the central bank can affect in period 0 via open market operations. There is no singularity.[25]

The only way the singularity could reappear would be if financial market participants did not have rational expectations. For example, suppose they had learned that $\pi_t = 0$

[25] It is worth noting that there are other equilibria of equations (1) and (2) that imply an identical equilibrium path but generate more plausible behaviour off this path. Suppose that in period 0 when the rule is introduced, the economy starts in state A. Suppose also that each period a biased coin is tossed which comes up heads with probability $q \in (0,1]$. If the economy is in state A in period t, then $\pi_t = 0$, whereas if the economy is in state B in period t, then $\pi_t = \frac{\phi}{q}\pi_{t-1}$. For $t > 0$, the economy is in state A at t if and only if either (i) the economy was in state A at $t-1$ and $\pi_{t-1} = 0$, or (ii) the coin comes up tails. Otherwise, the economy is in state B at t. Thus, in state B, $\mathbb{E}_t\pi_{t+1} = q\frac{\phi}{q}\pi_t + (1-q)0 = \phi\pi_t$, as required. Hence, explosions need not last for ever following a deviation.

for all t, without understanding why this is. Then it would again be the case that the Fisher equation would imply $i_t = r_t$ for all t, even out of equilibrium. A slight tweak to real rate rules can address this without otherwise compromising their performance.

In particular, suppose that the central bank uses the modified real rate rule:

$$i_t = r_t + \phi\pi_t + \psi\pi_{t-1}.$$

Then, from the Fisher equation:

$$\mathbb{E}_t\pi_{t+1} = \phi\pi_t + \psi\pi_{t-1}.$$

If $\phi > |1-\psi|$, this has a unique stationary solution in which $\pi_t = \varrho\pi_{t-1}$, where $\varrho = \frac{\phi-\sqrt{\phi^2+4\psi}}{2} \in (-1,1)$.[26] Hence, on the equilibrium path, the period t Fisher equation implies that $i_t - r_t = \varrho\pi_t$. Importantly, if $\psi \neq 0$, then $\varrho \neq 0$, so even with expectations fixed at their values on the equilibrium path, the central bank can still affect $i_t - r_t$ at t via affecting π_t through open market operations. Since ψ may be arbitrarily small, by continuity, the other desirable properties of real rate rules will be unaffected. Under learning, financial market participants would have learned that $\pi_t \approx \varrho\pi_{t-1}$, and so this mechanism would still work. We show in Appendix D.1 that real rate rules responding to the price level are also robust to this additional concern.

3 Practical implementation of real rate rules

Until recently, central banks concentrated their monetary interventions in overnight debt markets. However, with the rise of quantitative easing, many central banks have been purchasing substantial quantities of longer maturity sovereign debt. There is no reason then that central banks could not conduct open market operations to fix the interest rate on longer maturity bonds. This is convenient as in most countries, inflation protected securities are only issued a few times per year, and at long maturities, e.g., five years. As a result, markets in shorter maturity inflation protected securities may be illiquid or even unavailable, and it can be difficult to reconstruct the short end of the real yield curve.

[26] Proven in Appendix E.6.

Inflation indexation lags further complicate the use of short maturity inflation protected securities (see e.g. Gürkaynak, Sack & Wright (2010)). For example, with time measured in quarters, 3-month maturity US TIPS have a period $t+1$ realized yield of $r_t + \pi_t$, not $r_t + \pi_{t+1}$ as one would hope. By using longer maturity bonds, the impact of this indexation lag is greatly reduced. In this section, we examine the performance of real rate rules when the central bank implements them using multiperiod debt.

3.1 Set-up

We aim to describe a set-up with many of the frictions that would be problematic for a naïve implementation of real rate rules. The central bank's trading desk would be tasked with maintaining a particular level of the gap between nominal and real rates according to the market for bonds of a certain maturity. We let T be the maturity length of these bonds, measured in periods The units of time do not need to coincide with the maturity of the bond. For example, T may be 60 if periods are months and five-year bonds are used.

We allow for the possibility that inflation is not observed contemporaneously. For example, US CPI is observed with a one-month lag. To capture this, while keeping to the convention that $\mathbb{E}_t v_t = v_t$ for all t-dated endogenous variables v_t, we assume that market participants and the central bank use the $t-S$ information set in period t (i.e. they know the values of all $t-S, t-S-1, \ldots$ dated variables), for some $S \geq 0$. Thus, since the central bank does not know π_t at t, we instead assume that they respond to deviations of π_{t-S} from target, rather than π_t.

We write $i_{t|t-S}$ for the nominal yield per-period on a T-period nominal bond at t, and $r_{t|t-S}$ for the real yield per-period on a T-period inflation protected bond at t. This notation captures the fact that period t nominal and real yields must be fixed in period $t-S$: market participants and the central bank only have access to the period $t-S$ information set at t, and these agents must know period t nominal and real rates. An economic agent who somehow knew the price level in real time would thus know nominal and real rates S periods in advance.

We allow for a wedge in the Fisher equation to capture inflation risk premia, liquidity premia, asymmetric term premia and even departures from full information rational expectations amongst market participants. Since only $t - S$ dated variables are known in period t, we denote the period t value of this shock by $\nu_{t|t-S}$. I.e., risk premia (etc.) will be determined S periods in advance, though market participants and the central bank will not act on this, as they use S period old data.

Furthermore, we allow for the possibility of an indexation lag in the return of the real bond. We assume that the lag is L periods. If periods are months, then L would be 3 for the US.

3.2 The generalized Fisher equation and monetary rule

Given all this, the Fisher equation coming from arbitrage between nominal and real bonds states that:

$$i_{t|t-S} - r_{t|t-S} = \nu_{t|t-S} + \mathbb{E}_{t-S} \frac{1}{T} \sum_{k=1}^{T} \pi_{t+k-L}.$$

The central bank's actions in period t cannot possibly impact $\pi_{t-1}, \pi_{t-2}, \ldots$ as these are already predetermined. Hence, for the central bank to be able to have some impact on $i_{t|t-S} - r_{t|t-S}$ we require that $T - L \geq 0$. So, for the US, the central bank would have to use bonds with maturity of at least three months.

Slightly generalizing our previous rule (7), we suppose that the central bank intervenes in T-period nominal bond markets to ensure that it is always the case that:

$$i_{t|t-S} - r_{t|t-S} = \bar{\nu}_{t|t-S} + \mathbb{E}_{t-S} \frac{1}{T} \sum_{k=1}^{T} \pi^*_{t+k-L} + \phi(\pi_{t-S} - \pi^*_{t-S}),$$

where $\bar{\nu}_{t|t-S}$ is the central bank's period t belief about the level of $\nu_{t|t-S}$, and where $\phi > 1$. $\bar{\nu}_{t|t-S}$ could also include a monetary policy shock component. We stress that the $t|t-S$ index here does not mean that the private sector knows monetary policy shocks S periods in advance, as the private sector (and the central bank) uses the $t - S$

information set at t.

Also note that while under conventional monetary policy, targeted nominal interest rates are (approximately) constant between monetary policy committee meetings, this may not be the case here. The rule effectively specifies a period t level for $i_{t|t-S} - r_{t|t-S}$, not for $i_{t|t-S}$. The level of $r_{t|t-S}$ may fluctuate (perhaps in part due to unexpected changes in $i_{t|t-S}$), so the central bank's trading desk could have to continuously tweak the level of $i_{t|t-S}$ to hold $i_{t|t-S} - r_{t|t-S}$ at its desired level. While this represents a departure from previous operating procedure, there is no reason why holding $i_{t|t-S} - r_{t|t-S}$ approximately constant should be any harder than holding $i_{t|t-S}$ approximately constant. This is thanks to real-time observability of $r_{t|t-S}$ via inflation protected bonds.

The central bank could also directly control $i_{t|t-S} - r_{t|t-S}$ by promising to freely exchange \$1 face value of real debt for $\$(1 + i_{t|t-S} - r_{t|t-S})$ face value of nominal debt, as suggested by Cochrane (2017; 2018). Alternatively, the central bank could buy or sell a long-short portfolio containing \$1 face value of nominal debt, and −\$1 face value of real debt to hold the portfolio's return fixed at $\$(i_{t|t-S} - r_{t|t-S})$.[27] Or, the central bank could directly pin down the contract rate on inflation swaps, as suggested in Subsection 2.2.

3.3 Solution and robustness

Combining the multi-period Fisher equation and the monetary rule implies that the dynamics of inflation are governed by the single equation:

$$\mathbb{E}_t \frac{1}{T} \sum_{k=1}^{T} (\pi_{t+k+S-L} - \pi^*_{t+k+S-L}) + (\nu_{t+S|t} - \bar{\nu}_{t+S|t}) = \phi(\pi_t - \pi^*_t).$$

When $\nu_{t+S|t} - \bar{\nu}_{t+S|t}$ is exogenous, this expectational difference equation has a unique solution if and only if it has a unique solution when $\nu_{t+S|t} - \bar{\nu}_{t+S|t} = 0$ for all t. In this

[27] The author thanks Peter Ireland for this suggestion.

case, via the substitution $\pi_t - \pi_t^* = c\lambda^t$ we have the characteristic equation $\frac{1}{T}\sum_{k=1}^{T}\lambda^{k+S-L} = \phi$. The roots of this equation decide the determinacy of π_t. For determinacy, we need $\max\{0, -(1+S-L)\}$ roots strictly inside the unit circle, corresponding to the lags of inflation in our difference equation, and $\max\{0, T+S-L\}$ roots strictly outside the unit circle, corresponding to the leads of inflation in our difference equation. [28] This is indeed the case, as we prove in Appendix E.7 (given $\phi > 1$). Thus, at least when $\nu_{t+S|t} - \bar{\nu}_{t+S|t}$ is exogenous, there is a unique solution for inflation.[29] In the special case in which the central bank observes ν_t so $\bar{\nu}_t = \nu_t$, then $|\pi_t - \pi_t^*| \to 0$ as $t \to \infty$. (There may not be equality for finite t due to the impact of initial conditions.)

In the general case in which $\nu_{t+S|t} - \bar{\nu}_{t+S|t}$ is potentially endogenous, as long as $\nu_{t+S|t} - \bar{\nu}_{t+S|t}$ is stationary, the solution must take the form:

$$\pi_t = \pi_t^* + \mathbb{E}_t \sum_{j=-\infty}^{\infty} A_j \left(\nu_{t+j+S|t+j} - \bar{\nu}_{t+j+S|t+j}\right).$$

Substituting this solution into our expectational difference equation, then taking $t + \min\{0, 1+S-L\}$ dated expectations, and matching terms, gives that:

$$A_j = \frac{1}{\phi}\mathbb{1}[j=0] + \frac{1}{\phi T}\sum_{k=1}^{T} A_{j-k-S+L}.$$

With $\phi > 1$, this difference equation has a unique solution for $(A_j)_{j\in\mathbb{Z}}$ in which $A_j \geq 0$ for all $j \in \mathbb{Z}$, as proven in Appendix E.8.

Furthermore, under essentially identical conditions to those used in Subsection 2.2, we have that $\pi_t \approx \pi_t^*$ for large ϕ, even when $\nu_{t+S|t} - \bar{\nu}_{t+S|t}$ is endogenous. These conditions are very mild, as already discussed in that subsection. Thus, with ϕ large, even if the central bank imperfectly tracks the risk (etc.) premium ν_t, and even if their error is endogenous, it will still be the case that $\pi_t \approx \pi_t^*$ in all periods. I.e., even in the presence of unobservable endogenous wedges in the Fisher equation, the central bank can still

[28] In fact, our assumptions that $T - L \geq 0$ and $S \geq 0$, imply $T + S - L \geq 0$.

[29] We do not have the indeterminacy issues for rules setting long-rates that were noted by McGough, Rudebusch & Williams (2005), due to the presence of the real rate in our rule.

determinately implement an arbitrary path for inflation. The presence of information or indexation lags makes no fundamental difference to this. While such lags may slow down the convergence of A_j to 0 as $j \to \pm\infty$, increasing the variance of $\pi_t - \pi_t^*$, still for a large enough ϕ, inflation will be very close to its target.

4 The zero lower bound

All our examples so far have ignored the zero lower bound (ZLB) on nominal interest rates. The zero lower bound is problematic for real rate rules as it prevents the central bank from fixing $i_t - r_t$ when $i_t = 0$. This means that at the ZLB, the Euler equation again becomes relevant for outcomes, reducing robustness. This section presents a simple solution to restore robustness in the presence of the zero lower bound. In Appendix D we give two other potential solutions: price level real rate rules, and perpetuity real rate rules.

Furthermore, in Subsection 4.3 we show that when households hold perpetuities, appropriately constructed real rate rules can rule out both permanent ZLB traps as well as explosive paths for inflation, answering Cochrane (2011).

4.1 The problems caused by the ZLB for real rate rules

We can see the problems caused by the ZLB even in the simplest possible set-up used in this paper's introduction. In the presence of the zero lower bound, under the introduction's set-up, we have that:

$$\max\{0, r_t + \phi\pi_t\} = i_t = r_t + \mathbb{E}_t\pi_{t+1}.$$

While without the ZLB, we can cancel out the r_t in the monetary rule with the r_t from the Fisher equation, now this is no longer possible. Instead, we have that:

$$\max\{-r_t, \phi\pi_t\} = \mathbb{E}_t\pi_{t+1}.$$

Thus, real rates (and hence the Euler equation) potentially matter for inflation dynamics and determinacy. Holden (2021) points out that even if r_t is exogenous, with $r_t = 0$ for $t \neq 1$, and even if we assume that $\pi_t \to 0$ as $t \to \infty$, still there are multiple solutions for a value of r_1 ($r_1 = 0$), and no solution for other values of r_1 ($r_1 < 0$).

Holden (2021) shows this multiplicity and non-existence of perfect foresight solutions is the rule for NK models with a ZLB, even with a terminal condition on inflation ensuring an eventual escape from the ZLB. Additionally, there are further solutions converging to a deflationary steady state with interest rates at zero (Benhabib, Schmitt-Grohé & Uribe 2001). Furthermore, under rational expectations there are always at least as many solutions as under perfect foresight, as well as a continuum of further switching solutions (Holden 2021).

4.2 Modified inflation targets

One of the sources of non-existence is that the monetary rule is implicitly targeting an infeasible level for inflation when real rates are very low. A modified inflation target can fix this. In particular, along the lines of the rule from equation (7), consider the rule:

$$i_t = \max\{0, r_t + \mathbb{E}_t \breve{\pi}^*_{t+1} + \phi(\pi_t - \breve{\pi}^*_t)\},$$

where:

$$\breve{\pi}^*_t := \max\{-r_{t-1} + \epsilon, \pi^*_t\},$$

with π^*_t the original inflation target, and $\epsilon > 0$ is some small constant.

Under this modified rule, $\pi_t = \breve{\pi}^*_t$ for all t is an equilibrium. To see that it is indeed an equilibrium, note that with $\pi_t = \breve{\pi}^*_t$ for all t, the monetary rule gives that:

$$i_t - r_t = \max\{-r_t, \mathbb{E}_t \breve{\pi}^*_{t+1}\} = \mathbb{E}_t \breve{\pi}^*_{t+1} = \mathbb{E}_t \pi_{t+1}$$

as $\mathbb{E}_t \breve{\pi}^*_{t+1} = \mathbb{E}_t \max\{-r_t + \epsilon, \pi^*_{t+1}\} > -r_t$, so the Fisher equation holds as required. Hence, under this monetary rule, no matter what the rest of the model is like, there is a closed form solution for inflation in terms of observables (r_{t-1} and π^*_t). In particular, unlike under the simple rule from the previous subsection, there is always a solution. The existence of a closed form solution is particularly desirable as it is likely to be easier for agents to coordinate on simple solutions.

Additionally, under this solution, π_t, is bounded below by $-r_{t-1}$. This prevents the severe deflations that can accompany shocks taking the economy to the ZLB under standard monetary rules. It also removes all of the deflationary bias that usually accompanies the ZLB (Hills, Nakata & Schmidt 2019). Instead, the definition of $\breve{\pi}_t^*$ implies that $\mathbb{E}\pi_t \geq \mathbb{E}\pi_t^*$, so there is a mild inflationary bias.

The $\pi_t \equiv \breve{\pi}_t^*$ solution is also unique, conditional on $\breve{\pi}_t^*$ and on an approximate terminal condition. For the intuition for this, first suppose that the rule is introduced in period 0, and that there is no uncertainty after this point. Then, from the Fisher equation and monetary rule, for all $t \in \mathbb{N}$:

$$\pi_{t+1} - \breve{\pi}_{t+1}^* = \max\{-r_t - \breve{\pi}_{t+1}^*, \phi(\pi_t - \breve{\pi}_t^*)\} = \max\{-\max\{\epsilon, r_t + \pi_{t+1}^*\}, \phi(\pi_t - \breve{\pi}_t^*)\}.$$

Thus, since $\max\{\epsilon, r_t + \pi_{t+1}^*\} > 0$, if $\pi_t < \breve{\pi}_t^*$, then $\pi_{t+1} < \breve{\pi}_{t+1}^*$, and if $\pi_t > \breve{\pi}_t^*$, then $\pi_{t+1} > \breve{\pi}_{t+1}^*$. So, there are three classes of perfect foresight equilibria in period 0. The desired one in which $\pi_t = \breve{\pi}_t^*$, one in which inflation explodes to infinity, and one which is at the ZLB infinitely often (at least assuming π_t^* is bounded above). If a terminal condition rules out the explosive and liquidity trap equilibrium classes, then only the $\pi_t \equiv \breve{\pi}_t^*$ solution remains.

In the presence of uncertainty, essentially the same argument goes through. In particular, suppose that in period t, agents believe that for some $k > 0$, $|\pi_{t+k} - \breve{\pi}_{t+k}^*| < \epsilon$ with probability one. Thus $\pi_{t+k} > \breve{\pi}_{t+k}^* - \epsilon \geq -r_{t+k-1}$, so by the Fisher equation $i_{t+k-1} = r_{t+k-1} + \mathbb{E}_{t+k-1}\pi_{t+k} > 0$. Hence, by the monetary rule, in fact $\pi_{t+k-1} = \breve{\pi}_{t+k-1}^*$ (so $|\pi_{t+k-1} - \breve{\pi}_{t+k-1}^*| < \epsilon$), with probability one. By backwards induction, this means that for all $j \in \{0, \dots, k-1\}$, $\pi_{t+j} = \breve{\pi}_{t+j}^*$. Thus, if agents believe the $\pi_t \equiv \breve{\pi}_t^*$ solution will hold approximately at some point in future, then it will hold exactly today and in all intervening times. Furthermore, if agents believe there is some $K \geq 0$ such that there are infinitely many $k > K$ with $|\pi_{t+k} - \breve{\pi}_{t+k}^*| < \epsilon$ with probability one, then for all $j \in \mathbb{N}$, $\pi_{t+j} = \breve{\pi}_{t+j}^*$. In other words, if agents believe the solution approximately converges to the $\pi_t \equiv \breve{\pi}_t^*$ one, then it actually equals this one in all periods.

Conditional on a terminal condition of this form, the only remaining source of potential multiplicity is the bound in the definition of $\breve{\pi}_t^*$. Even if we assume that π_t^*

is exogenous, r_t is not, so if r_t is sufficiently responsive to $\mathbb{E}_t \pi_{t+1}$, in theory there could be one solution in which $\pi_{t+1} = \breve{\pi}^*_{t+1} = \pi^*_{t+1} > -r_t + \epsilon$ and one solution in which $\pi_{t+1} = \breve{\pi}^*_{t+1} = -r_t + \epsilon > \pi^*_{t+1}$. However, this does not occur for standard models. In Appendix E.10, we show that with the rest of the model given by equations (4) and (5), with π^*_t exogenous, $\beta\delta \geq 0$ and $\kappa\varsigma > 0$, there is a unique perfect foresight solution satisfying the terminal condition $\breve{\pi}^*_t \to \pi^*_t$ as $t \to \infty$. Furthermore, this uniqueness is robust, in the sense that no continuous change to the model or its parameters could overturn the uniqueness. Thus, our simple change to the inflation target delivers robust uniqueness for real rate rules, even in the presence of the ZLB.

4.3 Equilibrium selection with perpetuities

The modified inflation target real rate rules of the previous subsection delivered uniqueness conditional on a terminal condition ruling out inflation explosions or permanent ZLB episodes. In this final subsection, we examine how these two classes of undesirable equilibria may be avoided. This will enable us to answer Cochrane's (2011) argument that there is nothing to rule out non-stationary equilibria under monetary rules satisfying the Taylor-principle, and Benhabib, Schmitt-Grohé & Uribe's (2001) argument that there is nothing to rule out permanent ZLB spells under such rules.

We suppose that perpetuities (also called "consols") are traded in the economy. While actual perpetuities are rare, households may be able to approximate the flow of coupons from a perpetuity via holding a portfolio of government debt of different maturities. Additionally, there are many regular transfers from government to households or firms, such as unemployment benefits. While it is hard for households to capitalize and trade their flow of unemployment benefits, long-term government contracts (in defence, aerospace, etc.) certainly can be capitalized and traded. As long as such contracts enable a flow of firm profits, their value will have a perpetuity-like component.

Perpetuity prices are functions of the entire expected future path of nominal rates, and hence they embed information on the economy's selected equilibrium. Crucially, if the economy is stuck at the ZLB, then perpetuity prices will be extremely high, or even infinite. For the sake of exposition, we will derive results for the more general class of geometric coupon bonds, and later specialise to the perpetuity case.

We assume that one unit of the period t geometric coupon bond bought at t returns \$1 at $t+1$, along with $\omega \in (0,1]$ units of the period $t+1$ geometric coupon bond. The $\omega = 1$ case corresponds to a perpetuity. The geometric coupon bond trades at a price of Q_t at t. Thus, if Ξ_{t+1} is the real SDF between periods t and $t+1$, and $\Pi_{t+1} := \exp \pi_{t+1}$ is gross inflation between these periods, then the price of the bond must satisfy:

$$Q_t = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} [\omega Q_{t+1} + 1].$$

We assume that the government and central bank are the only institutions trusted enough to issue geometric coupon bonds, since private companies generally have shorter lives than governments. Thus, the total stock of such bonds, B_t is in the government and/or central bank's control. We assume there is some $\underline{B} > 0$ such that in all states of the world $B_t \geq \underline{B}\omega^t$. For this it is enough that the government issued geometric coupon bonds at some point in the past, with the commitment to never buy all of them back. Since it is optimal for governments to fund themselves with perpetuities (Debortoli, Nunes & Yared 2017; 2022), this does not seem an unreasonable commitment. Then, the household's period t transversality condition on geometric coupon bond holdings states that:

$$0 = \lim_{s\to\infty} \mathbb{E}_t \left[\prod_{k=1}^{s} \frac{\Xi_{t+k}}{\Pi_{t+k}} \right] Q_{t+s} B_{t+s} \geq \underline{B} \lim_{s\to\infty} \mathbb{E}_t \left[\prod_{k=1}^{s} \frac{\Xi_{t+k}}{\Pi_{t+k}} \right] Q_{t+s} \omega^{t+s} \geq 0,$$

and hence $\lim_{s\to\infty} \mathbb{E}_t \left[\prod_{k=1}^{s} \frac{\Xi_{t+k}}{\Pi_{t+k}} \right] \omega^s Q_{t+s} = 0$. Thus, for all t:

$$Q_t = \mathbb{E}_t \sum_{s=1}^{\infty} \left[\prod_{k=1}^{s} \frac{\Xi_{t+k}}{\Pi_{t+k}} \right] \omega^{s-1} = \mathbb{E}_t \sum_{s=0}^{\infty} \left[\prod_{k=0}^{s} \frac{1}{I_{t+k}} \right] \omega^s,$$

where, as usual, I_t is the gross interest rate on a one period nominal bond (so $I_t \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} = 1$).

Now suppose that $I_{t+k} = 1$ (with probability one, conditional on period t information) for all sufficiently high k. Then $Q_{t+s} = \frac{1}{1-\omega}$ (with conditional probability one) for all sufficiently high s. So, the transversality condition holds if and only if:

$$0 = \lim_{s\to\infty} \mathbb{E}_t \left[\prod_{k=1}^{s} \frac{\Xi_{t+k}}{\Pi_{t+k}} \right] \frac{\omega^s}{1-\omega} = \lim_{s\to\infty} \frac{\omega^s}{1-\omega},$$

i.e., if and only if $|\omega| < 1$. In particular, it is violated if the bond is a perpetuity, meaning $\omega = 1$.[30]

In other words, permanent stays at the ZLB do in fact violate a transversality constraint when the stock of perpetuities is positive. Intuitively, with households having infinite nominal wealth, they wish to spend some of that wealth today on real goods, which ends up violating (real) goods market clearing. The only way goods market clearing could be restored is if inflation is infinite when nominal wealth is. We show this carefully in Appendix E.11. However, under standard assumptions on money demand, infinite inflation is only possible with infinite money supply growth, which is likely to be physically impossible for a central bank. Thus, as long as infinite inflation is ruled out by this consideration or some other, there is no equilibrium with a permanent ZLB stay.

We now use this fact to construct a monetary rule with both global uniqueness and local determinacy, the latter helping ensure learnability. We assume that the central bank sets nominal interest rates via a tweaked non-linear version of the modified inflation target real rate rule of the previous subsection. Our first tweak is that for simplicity, we assume that the inflation target is set one period in advance. Our second tweak is to introduce "punishment" in the form of a switch to the ZLB following large deviations. To define a large deviation, we will introduce an upper bound $\bar{I} > 1$ on nominal interest rates, and we will construct the modified inflation target to ensure

[30] The necessity of the transversality constraint is non-obvious in the $\omega = 1$ case. However, in Appendix E.11 we show that the problem with perpetuities can be transformed into a "cake eating" type problem with one period bonds, for which the transversality constraint is trivially necessary, even when $\omega = 1$.

gross nominal interest rates are strictly inside $(1, \bar{I})$ in equilibrium.

We suppose that the central bank sets:

$$I_t = \begin{cases} \max\left\{1, R_t \breve{\Pi}_t^* \left(\frac{\Pi_t}{\breve{\Pi}_{t-1}^*}\right)^\phi\right\}, & \text{if } I_{t-1} \in (1, \bar{I}) \\ 1, & \text{otherwise} \end{cases},$$

where:

$$\breve{\Pi}_t^* := \max\left\{\frac{\mathcal{E}}{R_t}, \min\left\{\frac{\bar{I}}{\mathcal{E} R_t}, \Pi_t^*\right\}\right\},$$

with $\phi > 1$ and $\mathcal{E} := \exp \epsilon \in \left(1, \sqrt{\bar{I}}\right)$. It is easy to see that $\Pi_t = \breve{\Pi}_{t-1}^*$ for all t is consistent with this rule and the standard nominal and real bond pricing equations:

$$I_t \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} = 1, \qquad R_t \mathbb{E}_t \Xi_{t+1} = 1.$$

In the local vicinity of this equilibrium path, we have $I_t = R_t \breve{\Pi}_t^* \left(\frac{\Pi_t}{\breve{\Pi}_{t-1}^*}\right)^\phi$, which implies:

$$\mathbb{E}_t \frac{\Xi_{t+1}}{\mathbb{E}_t \Xi_{t+1}} \frac{\breve{\Pi}_t^*}{\Pi_{t+1}} = \left(\frac{\breve{\Pi}_{t-1}^*}{\Pi_t}\right)^\phi.$$

This has a unique stationary solution under mild conditions by the results of Appendix A.

To analyse potential deviations from this equilibrium, we switch to an economy without uncertainty for simplicity. This is in line with Cochrane (2011) which is primarily conducted under this simplifying assumption.

First, suppose that for some reason, for some $t = t_0$, $\Pi_t > \breve{\Pi}_{t-1}^*$, but $I_{t-1} \in (1, \bar{I})$. Then:

$$\frac{\Pi_{t+1}}{\breve{\Pi}_t^*} = \max\left\{\frac{1}{R_t \breve{\Pi}_t^*}, \left(\frac{\Pi_t}{\breve{\Pi}_{t-1}^*}\right)^\phi\right\} \geq \left(\frac{\Pi_t}{\breve{\Pi}_{t-1}^*}\right)^\phi,$$

and so $\frac{\Pi_t}{\breve{\Pi}_{t-1}^*}$ explodes upwards as $t \to \infty$. Now for all t, $\breve{\Pi}_t^* \geq \frac{\mathcal{E}}{R_t}$, hence $\frac{\Pi_t}{\breve{\Pi}_{t-1}^*} \leq \frac{R_{t-1}\Pi_t}{\mathcal{E}} < I_{t-1}$. Thus, I_t must also (start to) explode upwards as $t \to \infty$. So, eventually, for some $t_1 \geq t_0$, $I_{t_1} > \bar{I}$. Thus $I_{t_1+1} = I_{t_1+2} = \cdots = 1$ according to the monetary rule. But this is only consistent with household optimality if Π_t is infinite at least once in $[t_0, \dots, t_1]$, which in turn is physically impossible. Hence, there is no equilibrium with such a deviation.

Now, suppose that for some reason, for some $t = t_0$, $\Pi_t < \breve{\Pi}_{t-1}^*$, but $I_{t-1} \in (1, \bar{I})$. Then:

$$\frac{\Pi_{t+1}}{\breve{\Pi}_t^*} = \max\left\{\frac{1}{R_t\breve{\Pi}_t^*}, \left(\frac{\Pi_t}{\breve{\Pi}_{t-1}^*}\right)^{\phi}\right\},$$

and so $\frac{\Pi_t}{\breve{\Pi}_{t-1}^*}$ either explodes downwards towards zero forever as $t \to \infty$ or hits $I_{t_1} = 1$ at some $t_1 \geq t_0$. Now for all t, $\breve{\Pi}_t^* \leq \frac{\bar{I}}{\mathcal{E}R_t}$, hence $\frac{\Pi_t}{\breve{\Pi}_{t-1}^*} \geq \frac{\mathcal{E}R_{t-1}\Pi_t}{\bar{I}} = \frac{\mathcal{E}}{\bar{I}}I_{t-1}$. Thus, in fact I_t must hit $I_{t_1} = 1$ at some $t_1 \geq t_0$. Thus, just as before, $I_{t_1+1} = I_{t_1+2} = \cdots = 1$, which is inconsistent with equilibrium, ruling out the initial deviation.

Therefore, if households hold perpetuities this tweaked real rate rules succeeds in producing global uniqueness. Admittedly, the punishment reduces its robustness, but for moderately high $\mathcal{E}$ and $\bar{I}$, and high ϕ, accidentally falling into the punishment regime would be very unlikely, even with additional uncertainty coming from wedges in the Fisher equation.

Of course, if there is something else in the economy ruling out explosive paths for inflation, then the punishment regime is unnecessary, and the central bank could just use the rule:

$$I_t = \max\left\{1, R_t\breve{\Pi}_t^*\left(\frac{\Pi_t}{\breve{\Pi}_{t-1}^*}\right)^{\phi}\right\}, \qquad \breve{\Pi}_t^* := \max\left\{\frac{\mathcal{E}}{R_t}, \Pi_t^*\right\}.$$

With households holding perpetuities, this still has no equilibria that are permanently stuck at the ZLB. Sticky prices are sufficient to rule out explosive equilibria, both as inflation is bounded above under standard price stickiness specifications (see Appendix A), and because under sticky prices, exploding inflation implies exploding real costs of this inflation. While prices may become more flexible at high inflation rates, there are practical limits on how often prices can change even under extreme hyperinflation. The price must at least remain constant for the time between picking an item off the shelf and arriving with it at the check-out. If this is correct, then even without a punishment regime, trade in perpetuities is sufficient to ensure a unique long-run equilibrium with inflation at target.

5 Conclusion

This paper's implications are stark. Under a real rate rule: the central bank can always achieve its inflation target, no matter the rest of the economy; any movement in inflation must be due to a monetary policy shock or a central bank choice to so move inflation; monetary policy works in spite of, not because of, real rate movements; causation in the Phillips curve (if it exists) runs exclusively from inflation to the output gap, not the other way round; household and firm decisions, constraints and inflation expectations are irrelevant for inflation dynamics; and nothing can amplify or dampen the impact of shocks on inflation, except changes in the central bank's own behaviour. With a time-varying inflation target, real rate rules can determinately implement optimal monetary policy, or match observed dynamics. They continue to work in the presence of the ZLB, endogenous wedges in the Fisher equation, or active fiscal policy. They can be implemented using assets for which there is already a liquid market: either nominal and real long bonds, or inflation swaps.

To a policy maker, these conclusions may be shocking. However, for readers familiar with New Keynesian models, perhaps they are not completely surprising. In models in which an aggressive response to inflation produces determinacy, with an extremely aggressive response, the variance of inflation can be pushed down to near zero. And Rupert & Šustek (2019) argue that even in New Keynesian models with a standard monetary rule, monetary policy does not operate via real rates. Rather, real rate rules just crystallise the monetary policy transmission mechanism that is at work in all New Keynesian models. Monetary policy acts via the Fisher equation, and via the Taylor principle's promise to induce explosive inflation should inflation deviate from target. Accepting standard New Keynesian models means accepting this story.

Those for whom this is unpalatable will likely be drawn towards the fiscal theory of the price level. However, we showed that this theory fails to determine a unique outcome for inflation. It is even consistent with arbitrarily high inflation. Thus, the problems with Taylor principle equilibrium selection raised by Cochrane (2011) apply equally well to fiscal theory of the price level equilibrium selection. We gave one solution to these problems, and Christiano & Takahashi (2018; 2020) and Angeletos & Lian (2021) give others. Hence, Taylor principle equilibrium selection may be less problematic than that for the fiscal theory of the price level. In this case, real rate rules provide a robust way to implement monetary policy.

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