Robust Real Rate Rules

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Abstract: Central banks wish to avoid self-fulfilling fluctuations. Monetary rules with a unit response to real rates achieve this under the weakest possible assumptions about the behaviour of households and firms. They are robust to household heterogeneity, hand-to-mouth consumers, non-rational household/firm expectations, active fiscal policy, missing transversality conditions and to any form of intertemporal or nominal-real link. These rules: allow the implementation of arbitrary inflation dynamics, including optimal policy; are easy to implement in practice, with bonds of any maturity; and can attain high welfare. The performance of these rules provides insights into monetary transmission—the Fisher equation is key.

Keywords: robust monetary rules, determinacy, Taylor principle, inflation dynamics, monetary transmission mechanism

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Today you start work as president of the Fictian Central Bank (FCB). As FCB president, you have a clear mandate to stabilize inflation, even if that results in unemployment or output losses. How should you act? You have studied New Keynesian macro, so you are inclined to follow some variant of the Taylor rule. You recall the prescription of the Taylor principle: the response of nominal rates to inflation should be greater than one to ensure determinacy and rule out self-fulfilling fluctuations in inflation. But you also remember reading other papers which talked of the Taylor principle being insufficient if there are hand-to-mouth households (Gali, Lopez-Salido & Valles 2004), firm-specific capital (Sveen & Weinke 2005), high government spending (Natvik 2009), or if the inflation target is positive (Ascari & Ropele 2009), particularly in the presence of trend growth and sticky wages (Khan, Phaneuf & Victor 2019). Indeed, you recollect that the Taylor principle inverts if there are sufficiently many hand-to-mouth households (Bilbiie 2008), certain financial frictions (Manea 2019), or non-rational expectations (Branch & McGough 2010; 2018). You also recall that if real government surpluses do not respond to government debt levels, then following the Taylor principle can lead to explosive inflation (Leeper & Leith 2016; Cochrane 2022). Is there a way you could act to ensure determinacy and stable inflation, even if one or more of these circumstances is true? This paper provides a family of "robust real rate rules" that manage to do this. We then reassess classic questions of monetary economics through the lens of these rules.

To illustrate the idea behind these rules, suppose that both nominal and real bonds are traded in an economy. If a unit of the former is purchased at t, it returns the principal plus a nominal yield of i_t in period $t+1$. If a unit of the latter is purchased at t, it returns the principal plus a nominal yield of $r_t + \pi_{t+1}$ in period $t+1$, where π_{t+1} is realized inflation between t and $t+1$. US Treasury Inflation Protected Securities (TIPS) are one example of such real bonds.

Abstracting for the moment from inflation risk premia, term premia and liquidity premia, arbitrage between the nominal and real bond markets implies that the Fisher equation must hold, i.e.:

$$i_t = r_t + \mathbb{E}_t \pi_{t+1}, \tag{1}$$

where $\mathbb{E}_t \pi_{t+1}$ is the full information rational expectation of period $t+1$'s inflation rate, given period t's information. Suppose further than the central bank observes both the nominal and real bond markets, and that it can intervene in the former. Then the central bank can choose to set nominal interest rates according to the simple rule:

$$i_t = r_t + \phi \pi_t, \tag{2}$$

where $\phi > 1$.[2] Combining these two equations gives that:

$$\mathbb{E}_t \pi_{t+1} = \phi \pi_t,$$

which has a unique non-explosive solution of $\pi_t = 0$.[3] Determinate inflation!

Why is this robust? Firstly, the rule does not require the aggregate Euler equation to hold, even approximately. For the Fisher equation (1) to hold (still ignoring risk/term/liquidity premia for now), there only need to be two deep pocketed, fully informed, rational agents. Arbitrage takes care of the rest. Even full information is not necessary. Since large markets aggregate information (Hellwig 1980; Lou et al. 2019), the Fisher equation can come to hold even when information about future inflation is dispersed amongst market participants.

Given that the rule does not require the aggregate Euler equation to hold, it is automatically robust to heterogeneity, hand-to-mouth agents and non-rational consumer expectations. The only expectations that matter are the expectations of

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[2] We ignore the zero lower bound for now. We provide rules that retain their good properties in the presence of the ZLB in Section 4.

[3] Here we sidestep the issues raised by Cochrane (2011) and follow the standard New Keynesian literature in assuming agents will always select non-explosive paths for inflation. The escape clause rules of Christiano & Takahashi (2018; 2020) are one way by which central banks could ensure coordination on the expectations consistent with non-explosive inflation. We give an alternative solution in Appendix D.

participants in the markets for nominal and real bonds. It is much more reasonable to assume that financial market outcomes lead to rational expectations than to assume rationality of households more generally.

Secondly, the rule does not require the aggregate Phillips equation to hold. The slope of the Phillips curve will have no impact on the dynamics of inflation. If the FCB president is unconcerned with output, they do not need to know if the Phillips curve holds, let alone its slope. Nor does it matter how firms form inflation expectations. Inflation is pinned down by the Fisher and monetary rules, so while non-rational firm expectations could affect output fluctuations, they will not alter the dynamics of inflation.

This may be surprising. How could price setters fail to determine inflation? The short answer is "Walras's law". To see how this plays out, suppose that today all firms decide to double their price. Financial market participants still expect zero inflation next period, because that is the only outcome consistent with non-explosive inflation in future. Thus, financial market participants always value nominal bonds the same as real bonds. But the central bank's monetary rule instructs it to attempt to produce nominal rates which are much higher than real rates, as today's inflation is high. So, the central bank wants to sell nominal bonds, i.e., to borrow money from financial market participants.

However, no amount of nominal bond selling will induce market participants to lower their valuation of nominal bonds below that of real bonds, though both valuations may fall together (i.e., both nominal and real rates rise). Thus, the central bank will end up reducing the money supply to zero. With households having zero cash, not all final goods will be sold. (For example, if there are cash goods and credit goods, only credit goods will be sold.) Thus, the final goods market will not clear. To obtain market clearing in final goods, at least some price setters must reduce their price until inflation is zero, so ensuring that the central bank sets nominal rates equal to real rates.

The possibility of decoupling inflation from the rest of the economy has wide ranging implications. For example, there is a tradition in monetary economics of examining model features producing amplification or dampening of monetary shocks. Under a real rate rule, assuming the Fisher equation holds, then no change to the model can ever produce amplification or dampening. Thus, such amplification/dampening results were always highly dependent on the particular monetary rule being used. With a greater than unit response to real rates, amplification can be flipped to dampening, and vice versa.

Likewise, a persistent question in monetary economics has been “which shocks drive inflation?”. Here too, the answer must be crucially sensitive to the monetary rule being used. Under a real rate rule, only monetary policy shocks or shocks to the Fisher equation can possibly move inflation. The central bank has the power to almost perfectly control inflation, so ultimate responsibility for inflation must rest with them.

The rest of this paper further examines “real rate rules”, along with the classic questions of monetary economics they help answer. The next section generalizes the simple rule of equation (2) along various dimensions, including examining rules that respond to other endogenous variables. We also look at the implication of real rate rules in simple New Keynesian models. Section 1 goes on to show that there are similar rules that determinately implement an arbitrary path for inflation, robustly across models. Hence, real rate rules can attain high welfare, and could explain observed inflation dynamics.

Next, Section 2 examines some potential challenges to the performance of real rate rules. We show they also work in fully non-linear models, that they are robust to wedges in the Fisher equation, and that they continue to work even in models in which inflation is determinate under a peg, except in knife edge cases.

Section 3 discusses how a real rate rule could be implemented in practice. We show that it is easy to adapt real rate rules to work with longer bonds. Finally, Section 4 looks at the consequences of the zero lower bound for the performance of these rules. Section 4 also discusses how explosive equilibria for inflation may be ruled out in the absence of any transversality constraint on inflation.

Literature. Rules like equation (2) have appeared in Adão, Correia & Teles (2011), Lubik, Matthes & Mertens (2019) and Holden (2021) amongst other places. However, in the prior literature they have chiefly been introduced for analytic convenience, rather than as serious proposals. One exception is the work of Cochrane (2017; 2022) who briefly discusses rules of this form within the context of a wider discussion of rules that hold $i_t - r_t$ constant (i.e. rules with $\phi = 0$). Cochrane (2018) further explores rules holding $i_t - r_t$ constant.

The "indexed payment on reserve" rules of Hall & Reis (2016) also rely on observable real rates, but use a different mechanism to achieve determinacy. They propose that the CB issues an asset ("reserves") with nominal return from \$1 of $\$(1 + r_t)\frac{p_{t+1}}{p_t^*}$ or $\$(1 + i_t)\frac{p_t}{p_t^*}$. Additionally, in older work, Hetzel (1990) proposes using the spread between nominal and real bonds to guide monetary policy, and Dowd (1994) proposes targeting the price of futures contracts on the price level, which has a similar flavour to our rules, since our rules effectively use expected inflation as the instrument of monetary policy.

There is also an established literature looking at rules tracking the efficient ("natural") real interest rate, see e.g. Cúrdia et al. (2015). This is a very different idea.

1 Generalizations and generality

This section considers assorted generalizations to real rate rules, and examines the sources of their robustness. We look at real rate rules 1) in the presence of monetary policy shocks, 2) in the three equation NK model, 3) with responses to other endogenous variables, and 4) with time varying inflation targets.

1.1 Monetary policy shocks

While the simple rule (2) always produces zero inflation, slight extensions of the rule allow inflation to move. For example, we may add a monetary policy shock, ζ_t to the rule, giving:

$$i_t = r_t + \phi\pi_t + \zeta_t. \tag{3}$$

Monetary policy shocks may perhaps reflect the central bank's limited information. If the central bank does not perfectly observe current inflation, and sets interest rates to $i_t = r_t + \phi\tilde{\pi}_t$, where $\tilde{\pi}_t$ is its signal about inflation, then it will end up setting a slightly different level for nominal rates than that dictated by the rule $i_t = r_t + \phi\pi_t$, effectively generating monetary policy shocks.[4]

The central bank might also deliberately decide to introduce monetary policy shocks correlated with the economy's structural shocks. For example, by lowering $i_t - r_t$ following a positive mark-up or cost-push shock, the central bank can lessen the movement in the output gap.[5] This has no effect on the determinacy region as structural shocks are exogenous. For now though, we assume that ζ_t is independent of other structural shocks.

From combining (3) with the Fisher equation (1) we have:

$$\mathbb{E}_t\pi_{t+1} = \phi\pi_t + \zeta_t,$$

which (with $\phi > 1$) has the unique solution $\pi_t = -\frac{1}{\phi-\rho_\zeta}\zeta_t$, if ζ_t follows an AR(1) process with persistence ρ_ζ.

A contractionary (positive) monetary policy shock results in a fall in inflation, as expected. If the central bank is more aggressive, so ϕ is larger, then inflation is less volatile. Only monetary policy shocks affect inflation. Of course, if there is a nominal

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[4] Lubik, Matthes & Mertens (2019) look at the determinacy consequences of a central bank that filters inflation signals in order to retrieve the optimal estimate. The determinacy problems they highlight all disappear if the central bank directly responds to its signal.

[5] Ireland (2007) presents evidence that the US Federal Reserve has reacted to mark-up shocks.

rigidity in the model, such as sticky prices or wages, monetary shocks may have an impact on real variables. But as long as the central bank follows rules like this, these real disruptions have no feedback to inflation. We can understand inflation without worrying about the rest of the economy.

In line with this, an extensive body of empirical evidence finds no role for the Phillips curve in forecasting inflation (see e.g. Atkeson & Ohanian 2001; Ang, Bekaert & Wei 2007; Stock & Watson 2009; Dotsey, Fujita & Stark 2018). In a recent contribution, Dotsey, Fujita & Stark (2018) find that in the post-1984 period, Phillips curve based forecasts perform worse than those of a simple IMA(1,1) model, both unconditionally and conditional on various measures of the state of the economy. This provides strong support for models in which the causation in the Phillips curve runs in only one direction: from inflation to the output gap.[6]

Additionally, Miranda-Agrippino & Ricco (2021) find that a contractionary monetary policy shock causes an immediate fall in the price level, while impacts on unemployment materialise much more slowly. Again, this suggests that causation in the Phillips curve runs from inflation to unemployment, not the other way round.

1.2 Robust real rate rules in the three equation NK world

To understand how our robust rule in equation (3) can explain causation running from inflation to the output gap in the Phillips curve, suppose the rest of the model comprises the Phillips curve:[7]

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + \kappa \omega_t, \tag{4}$$

[6] McLeay & Tenreyro (2019) provide an alternative explanation based on the fact that optimal policy prescribes a negative correlation between inflation and output, making difficult empirical identification of the Phillips curve.

[7] Throughout this paper, we multiply the mark-up shock by κ as the ratio of the response to x_t and the response to ω_t is not a function of either the (Calvo) price adjustment probability or the (Rotemberg) price adjustment cost. See Khan (2005) for derivations.

and the discounted/compounded Euler equation:

$$x_t = \delta \mathbb{E}_t x_{t+1} - \varsigma(r_t - n_t), \tag{5}$$

where x_t is the output gap, ω_t is a mark-up/cost-push shock, and n_t is the exogenous natural real rate of interest. This form of discounted/compounded Euler equation appears in Bilbiie (2019) and (under discounting) in McKay, Nakamura & Steinsson (2017). The latter paper shows it provides a good approximation to a heterogeneous agent model with incomplete markets. The standard Euler equation is recovered if $\delta = 1$ and ς is the elasticity of intertemporal substitution. This specification also nests the limited asset market participation or "TANK" model of Bilbiie (2008) when $\delta = 1$, but ς is allowed to be negative.

Since $\pi_t = -\frac{1}{\phi - \rho_\zeta}\zeta_t$, and ζ_t is AR(1) with persistence ρ_ζ, the Phillips curve (4) implies that $x_t = -\frac{1}{\kappa}\frac{1-\beta\rho_\zeta}{\phi-\rho_\zeta}\zeta_t - \omega_t$. The Phillips curve is determining the output gap, given the already determined level of inflation. Does x_t help forecast π_t here? Clearly no. $\mathbb{E}_t \pi_{t+1} = -\frac{1}{\phi-\rho_\zeta}\mathbb{E}_t \zeta_{t+1} = -\frac{\rho_\zeta}{\phi-\rho_\zeta}\zeta_t = \rho_\zeta \pi_t$. Once you know π_t, you already have all the information you need to form the optimal forecast of π_{t+1}. The correlation in π_t and x_t provides no extra information.[8]

This model also enables us to show the robustness of our rule's determinacy in practice. Note that with x_t expressed as a linear combination of exogenous variables, there is no need to solve the Euler equation (5) forward, so the degree of discounting (δ) can have no effect on determinacy. Not needing to solve the Euler equation forward also gives robustness to a missing transversality constraint on household assets. For example, if ω_t is independent across time, then the Euler equation implies $r_t = n_t + \frac{1}{\varsigma}\left[\frac{1}{\kappa}\frac{(1-\beta\rho_\zeta)(1-\delta\rho_\zeta)}{\phi-\rho_\zeta}\zeta_t + \omega_t\right]$. This contrasts with the results of Bilbiie (2019) who finds that when $\varsigma > 0$ and $\beta \leq 1$, the Taylor principle ($\phi > 1$) is only sufficient for determinacy in the discounting case ($\delta \leq 1$),[9] and with Bilbiie (2008) who finds that

[8] This result is robust to generalizing to an ARMA(1,1) process for ζ_t. See Appendix E.1.

[9] See equation (40) of Appendix C.1 of Bilbiie (2019).

when $\delta = 1$ and $\varsigma < 0$, the Taylor principle ($\phi > 1$) is neither necessary nor sufficient for determinacy.[10] Under our rule (3), the Taylor principle is necessary and sufficient for determinacy whether there is discounting or compounding, and whether ς is positive or negative (given $\phi \geq 0$).[11]

The rule is also robust to the presence of lags in the Euler or Phillips curve. For example, suppose the Phillips curve and Euler equation are instead given by:

$$\begin{aligned} \pi_t &= \tilde{\beta}(1-\varrho_\pi)\mathbb{E}_t\pi_{t+1} + \tilde{\beta}\varrho_\pi\pi_{t-1} + \kappa x_t + \kappa\omega_t, \\ x_t &= \tilde{\delta}(1-\varrho_x)\mathbb{E}_t x_{t+1} + \tilde{\delta}\varrho_x x_{t-1} - \varsigma(r_t - n_t), \end{aligned} \tag{6}$$

where $\tilde{\beta}$ and $\tilde{\delta}$ may not have the same structural interpretation as β and δ (depending on the precise micro-foundation). These equations have no impact on the solution for inflation, which remains $\pi_t = -\frac{1}{\phi-\rho_\zeta}\zeta_t$. Instead, the lag in the Euler equation changes the dynamics of real interest rate, with no impact on inflation or output gaps, while the lag in the Phillips curve affects both output gap and real rate dynamics, with no impact on inflation. For example, if ζ_t's law of motion is given by $\zeta_t = \rho_\zeta\zeta_{t-1} + \varepsilon_{\zeta,t}$, where $\mathbb{E}_{t-1}\varepsilon_{\zeta,t} = 0$, then:

$$x_t = \frac{1}{\kappa}\frac{1}{\phi-\rho_\zeta}\left[\left(\tilde{\beta}\varrho_\pi - \rho_\zeta\left(1-\tilde{\beta}(1-\varrho_\pi)\rho_\zeta\right)\right)\zeta_{t-1} - \left(1-\tilde{\beta}(1-\varrho_\pi)\rho_\zeta\right)\varepsilon_{\zeta,t}\right] - \omega_t.$$

As before, the output gap has a closed form solution in terms of the monetary policy and cost push shocks. Despite appearances, inflation is not a true endogenous state, as it must always equal $-\frac{1}{\phi-\rho_\zeta}\zeta_t$. Monetary policy shocks are still always contractionary, but they only have a short-lived impact on the output gap if ϱ_π is around $\frac{\rho_\zeta(1-\beta\rho_\zeta)}{\beta\left(1-\rho_\zeta^2\right)}$.

[10] See Proposition 7 of Appendix B.1 of Bilbiie (2008).

[11] In Appendix E.2 we prove that this is robust to monetary responses to the real rate which are not exactly equal to 1. This is also a corollary of the more general result proven in Appendix E.4.

1.3 Responding to other endogenous variables

The original Taylor rule contained a response to output. Even with a unit coefficient on the real interest rate, responding to output will change the determinacy conditions, though it still preserves some robustness. To see this, consider the monetary rule:

$$i_t = r_t + \phi_\pi \pi_t + \phi_x x_t + \zeta_t.$$

Assuming the lag-augmented NK Phillips curve (6) continues to hold, this monetary rule is equivalent to the rule:

$$i_t = r_t + \phi_\pi \pi_t + \kappa^{-1}\phi_x\big[\pi_t - \tilde{\beta}(1-\varrho_\pi)\mathbb{E}_t\pi_{t+1} - \tilde{\beta}\varrho_\pi\pi_{t-1}\big] - \phi_x\omega_t + \zeta_t.$$

(This is produced by using the Phillips curve to substitute out the output gap.) Combined with the Fisher equation, we have that:

$$\mathbb{E}_t\pi_{t+1} = \phi_\pi \pi_t + \kappa^{-1}\phi_x\big[\pi_t - \tilde{\beta}(1-\varrho_\pi)\mathbb{E}_t\pi_{t+1} - \tilde{\beta}\varrho_\pi\pi_{t-1}\big] - \phi_x\omega_t + \zeta_t.$$

This has a determinate solution if the quadratic:

$$\big[1 + \kappa^{-1}\phi_x\tilde{\beta}(1-\varrho_\pi)\big]A^2 - (\phi_\pi + \kappa^{-1}\phi_x)A + \kappa^{-1}\phi_x\tilde{\beta}\varrho_\pi = 0$$

has a unique solution for A inside the unit circle. It is sufficient that the quadratic is positive at $A = -1$ but negative at $A = 1$, which holds if and only if:

$$1 + \kappa^{-1}\phi_x\big(1+\tilde{\beta}\big) + \phi_\pi > 0,$$
$$1 - \kappa^{-1}\phi_x\big(1-\tilde{\beta}\big) - \phi_\pi < 0.$$

So, if $\kappa > 0$, $\phi_x \geq 0$ and $\tilde{\beta} \in [0,1]$ as expected, then it is sufficient that $\phi_\pi > 1$ as before.[12] This is still considerable robustness. Providing there is something like a Phillips curve linking inflation and the output gap, the standard $\phi_\pi > 1$ condition will be sufficient for determinacy. This would not hold with a more standard monetary rule without a response to real rates: in that case determinacy depends on $\tilde{\delta}$ and ς, as shown by the Bilbiie (2008; 2019) results discussed in the last subsection.

[12] This is stronger than necessary. The second condition states that $\phi_\pi + \kappa^{-1}\phi_x\big(1-\tilde{\beta}\big) > 1$ so a response to the output gap can substitute for a response to inflation. This condition is identical to that for the standard (purely forward looking) three equation NK model with Taylor type rule found in Woodford (2001).

Responding to real rates provides additional robustness even with a response to output as it disconnects the Euler equation from the rest of the model. The only remaining role of the Euler equation is to give a path for real rates, given the already determined paths of output and inflation.[13] The Fisher equation, not the Euler equation is central to monetary policy transmission under real rate rules.

For greater robustness, the central bank can replace the response to the output gap with a response to the cost push shock ω_t. With an appropriate response to ω_t, this is observationally equivalent to responding to the output gap, but ensures determinacy under the standard Taylor principle.

However, it may be hard for the central bank to observe the cost push shock. To get round this, suppose that the central bank knows that a Phillips curve in the form of equation (6) holds. (Our results would generalize to other links between real and nominal variables.) For now, suppose the central bank also knows the coefficients in equation (6). Then the central bank could use a rule of the form:

$$i_t = r_t + \phi_\pi \pi_t + \phi_x \big[x_t - \kappa^{-1} \big[\pi_t - \tilde{\beta}(1 - \varrho_\pi) \mathbb{E}_t \pi_{t+1} - \tilde{\beta} \varrho_\pi \pi_{t-1} \big] \big] + \zeta_t.$$

By equation (6), this implies that:

$$i_t = r_t + \phi_\pi \pi_t - \phi_x \omega_t + \zeta_t,$$

as desired. Of course, the central bank is also unlikely to know the exact coefficients in the Phillips curve. However, we show in Appendix E.3 that the central bank may learn these coefficients in real time, without changing the determinacy conditions, at least under reasonable parameter restrictions.[14]

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[13] This is analogous to how the Euler equation is slack when solving for optimal monetary policy. In that case, the combined Euler equation and Fisher equation give the level of nominal rates required to hit the optimal output gap and inflation. The author thanks Florin Bilbiie for this observation.

[14] It is sufficient (but not necessary) that $\phi_x \geq 0$, $\phi_\pi \geq 0$, $\kappa \geq 0$, $\tilde{\beta} \in [0,1]$, $\varrho_\pi \in [0,1)$, $\rho_\zeta \in [0,1)$ and $\phi_\pi > \max\left\{\frac{1}{\tilde{\beta}(1-\varrho_\pi)}, 2(1-\varrho_\pi), \frac{\phi_x(1+\tilde{\beta})}{\kappa}\right\}$.

If the central bank wishes to respond to other endogenous variables, a similar approach should be possible if they are aware of the broad form of the model's structural equations. However, the central bank may legitimately worry about having fundamental misconceptions about how the economy works. They can be reassured though that the Taylor principle will be enough for determinacy if the response to other endogenous variables is small enough, no matter the form of the model's other equations. We prove this in Appendix E.4. This also implies that a precise unit response to real rates is not needed for determinacy. Real rates are just another endogenous variable, so determinacy only requires a response that is sufficiently close to one.

Classic results on determinacy in monetary models can be reinterpreted through this lens. Even if the central bank is not responding to real interest rates, it is still likely to be responding to variables that are highly correlated with them. Determinate rules will be ones sufficiently close to a real rate rule.

For example, many models contain an Euler equation of the form:

$$1 = \beta(\exp r_t)\mathbb{E}_t\left(\frac{C_t}{C_{t+1}}\right)^{\frac{1}{\varsigma}},$$

where C_t is real consumption per capita and ς is the elasticity of intertemporal substitution. Additionally, in many models, in equilibrium, consumption growth roughly follows an ARMA(1,1) process:

$$g_t := \log\left(\frac{C_t}{C_{t-1}}\right) = (1-\rho_g)g + \rho_g g_{t-1} + \varepsilon_{g,t} + \theta_g \varepsilon_{g,t-1}, \qquad \varepsilon_{g,t} \sim N(0,\sigma_g^2).$$

(This is a good approximation to US post-war data.[15]) Combining these two equations gives that:

$$r_t = -\log\beta + \frac{1-\rho_g}{\varsigma}g - \frac{1}{2}\left(\frac{\sigma_g}{\varsigma}\right)^2 + \frac{\rho_g}{\varsigma}g_t + \frac{\theta_g}{\varsigma}\varepsilon_{g,t},$$

[15] Estimating on US data from 1947Q1 to 2021Q4 (BEA series: A794RX) with T-distributed shocks gives $\rho_g = 0.69$, $\theta_g = -0.50$ (p-values both below 10^{-5}). Using Gaussian shocks on less volatile sub-periods gives similar results.

implying that a (roughly) $\frac{\rho_g}{\varsigma}$ response to consumption growth can substitute for a (roughly) unit response to real rates.

Of course, output (growth, level or gap) is in turn highly correlated with consumption growth, so output (growth, level or gap) may also substitute for real rates. For example, in the Smets & Wouters (2007) model of the US economy, the monetary rule is of the form $i_t = \phi_\pi \pi_t + z_t + \zeta_t$, where z_t is a linear combination of other endogenous variables and ζ_t is the monetary shock. At the estimated posterior mode, the correlation between z_t and the real interest rate is 0.63, with both variables having standard deviation of 0.46%. Thus, the Smets & Wouters (2007) estimates imply that the Fed is already about two thirds of the way to using a simple robust real rate rule.

1.4 Implementing arbitrary inflation dynamics

Real rate rules can determinately implement any path for inflation, no matter the rest of the model. This implies they can also implement optimal policy, and so attain high welfare. It also implies that any observed inflation and interest rate dynamics are consistent with a real rate rule.

Let π_t^* be an exogenous stochastic process, perhaps a function of the economy's other shocks,[16] and consider the rule:

$$i_t = r_t + \mathbb{E}_t \pi_{t+1}^* + \phi(\pi_t - \pi_t^*). \tag{7}$$

From the Fisher equation (1), this implies:

$$\mathbb{E}_t(\pi_{t+1} - \pi_{t+1}^*) = \phi(\pi_t - \pi_t^*).$$

Again with $\phi > 1$, there is a unique solution, now with $\pi_t = \pi_t^*$. I.e., at all periods of time, and in all states of the world, realised inflation is equal to π_t^*. Effectively, the central bank is able to choose an arbitrary path for inflation as the unique, determinate equilibrium outcome.

[16] Ireland (2007) also allows the central bank's inflation target to respond to other structural shocks.

The only constraint is that the targeted path for inflation cannot be a function of endogenous variables. However, this is not much of a limitation, since in stationary equilibrium, endogenous variables must have a representation as a function of the infinite history of the economy's shocks. This means that by choosing π_t^* appropriately, rules in the form of (7) can mimic the outcomes of any other monetary policy regime.[17]

For example, suppose that the central bank were to set interest rates in a different (though time invariant) way, for example by using another rule, or by adopting optimal policy under either commitment or discretion, given some objective. For simplicity, suppose further that the economy's equilibrium conditions are linear, e.g., because we are working under a first order approximation. Let $(\varepsilon_{1,t}, \dots, \varepsilon_{N,t})_{t\in\mathbb{Z}}$ be the set of structural shocks in the economy,[18] all of which are assumed mean zero and independent both of each other, and over time. Finally, assume that the central bank's behaviour produces stationary inflation, $\tilde{\pi}_t$, with the $\sim$ denoting that this is inflation under the alternative monetary regime. Then, by linearity and stationarity, there must exist a constant $\tilde{\pi}^*$ and coefficients $(\theta_{1,k}, \dots, \theta_{N,k})_{k\in\mathbb{N}}$ such that:

$$\tilde{\pi}_t = \tilde{\pi}^* + \sum_{k=0}^{\infty} \sum_{n=1}^{N} \theta_{n,k} \varepsilon_{n,t-k},$$

with $\sum_{k=0}^{\infty} \theta_{n,k}^2 < \infty$ for $n = 1, \dots, N$. So, if the central bank sets:

$$\pi_t^* = \tilde{\pi}^* + \sum_{k=0}^{\infty} \sum_{n=1}^{N} \theta_{n,k} \varepsilon_{n,t-k},$$

[17] Other papers have examined the implementation of optimal policy in specific models using instrument rate rules (see e.g. Svensson & Woodford 2003; Dotsey & Hornstein 2006; Evans & Honkapohja 2006; Evans & McGough 2010). However, the various prior proposals do not enable the implementation of a certain inflation path robustly across models.

[18] This may include sunspot shocks if they are added following Farmer, Khramov & Nicolò (2015).

(exogenous!) and uses the rule (7), then for all t and in all states of the world, $\pi_t = \pi_t^* = \tilde{\pi}_t$. Moreover, this implies in turn that all the endogenous variables in the two economies must be identical in all periods and in all states of the world.[19]

This has two important implications. Firstly, it means that appropriately designed real rate rules can implement (timeless/unconditional/etc.) optimal policy, and thus attain the highest possible level of welfare. In Appendix C we look at welfare in New Keynesian models when the central bank is constrained to follow a real rate rule that produces simple inflation dynamics. We show that even with such a constraint, real rate rules can still come close to fully optimal policy.

Secondly, it means that it is impossibly to test empirically if a central bank is using a general real rate rule. Any dynamics of inflation and interest rates are consistent with a real rate rule like (7), for an appropriately chosen π_t^*. Thus, real rate rules are observationally equivalent to any other specification for central bank behaviour. While in the last subsection we found that the Fed was not exactly using a simple real rate rule, we now see that a slightly more sophisticated real rate rule could fully explain Fed behaviour.

The only slight difficulty with setting π_t^* as a function of structural shocks is that the central bank may struggle to observe these shocks. The central bank can certainly observe linear combinations of structural shocks, via estimating a VAR with sufficiently many lags. For variables that are plausibly contemporaneously exogenous, such as commodity prices for a small(ish) economy, this is already sufficient to recover the corresponding structural shock. To infer other shocks, the central bank needs to know more about the structure of the economy. However, we do not need to assume any more than is standard in rational expectations models. Forming rational expectations requires you to know the structure of the economy; if you know this

[19] Proven in Appendix E.5.

structure, then you know the mapping from the reduced form shocks estimated by a VAR to the model's structural shocks.[20] Additionally, it is common to assume that the central bank responds to an output gap constructed by comparing outcomes to an economy without price rigidity. This already requires the central bank to know the values of all parameters and structural shocks.

1.5 Adding interest rate smoothing

High degrees of interest rate smoothing are often thought to be a good description of actual central bank behaviour given the rarity of large interest rate changes. However, since the rule (7) can generate arbitrary inflation dynamics (and hence arbitrary nominal rate dynamics), we cannot conclude based on observed nominal rates that the central bank is actually smoothing rates. Nonetheless, interest rate smoothing is worth investigating in our context, as it can be a source of additional robustness.

For example, suppose that the central bank sets interest rates according to the fully smoothed real rate rule:

$$i_t - r_t = i_{t-1} - r_{t-1} + \mathbb{E}_t \pi^*_{t+1} - \mathbb{E}_{t-1} \pi^*_t + \theta(\pi_t - \pi^*_t),$$

where $\theta > 0$ and where π^*_t is the exogenous inflation target, as before. Under a real rate rule, the central bank should attempt to smooth $i_t - r_t$, not just i_t. This ensures real rates can still be substituted out from the Fisher equation. Hence, we have $i_{t-1} - r_{t-1}$ on the right-hand side.

Combining this monetary rule with the Fisher equation gives:

$$i_{t-1} - r_{t-1} + \mathbb{E}_t \pi^*_{t+1} - \mathbb{E}_{t-1} \pi^*_t + \theta(\pi_t - \pi^*_t) = i_t - r_t = \mathbb{E}_t \pi_{t+1}.$$

Now, from the lagged Fisher equation, $i_{t-1} - r_{t-1} = \mathbb{E}_{t-1} \pi_t$, so:

$$\theta(\pi_t - \pi^*_t) = \mathbb{E}_t(\pi_{t+1} - \pi^*_{t+1}) - \mathbb{E}_{t-1}(\pi_t - \pi^*_t).$$

[20] This mapping may not be unique valued if there are more shocks than observables. However, since we expect a relatively small number of shocks to explain the bulk of business cycle variance, this is unlikely to be problematic in practice.

To solve this equation, first let $p_t := \sum_{s=1}^{t}(\pi_t - \pi_t^*)$ be the price level relative to its target trend, normalized to $p_0 = 0$. Thus:

$$\theta(p_t - p_{t-1}) = \mathbb{E}_t(p_{t+1} - p_t) - \mathbb{E}_{t-1}(p_t - p_{t-1}).$$

Summing this equation over time (from period 1 to period t) then gives that:

$$\theta p_t = \mathbb{E}_t(p_{t+1} - p_t) - \mathbb{E}_0 p_1.$$

Hence, if we define $\hat{p}_t := p_t + \frac{1}{\theta}\mathbb{E}_0 p_1$, then:

$$(1 + \theta)\hat{p}_t = \mathbb{E}_t \hat{p}_{t+1}.$$

For $\theta > 0$, this has the unique equilibrium $\hat{p}_t = 0$, so $\pi_t = \pi_t^*$ for all t, as required.

In equilibrium then, our smoothed real rate rule produces the same inflation (and hence the same nominal rates) as our unsmoothed real rate rule, equation (7). However, it is more robust in one crucial respect. Whereas the rule in equation (7) required a response to current inflation of $\phi > 1$, the fully smoothed real rate rule just needs a response to current inflation of $\theta > 0$.

In practice, it may be hard for central banks to commit to responding more than one for one to inflation. Even if they manage this, it is likely to be hard for them to convince other economic agents that they really will be so aggressive all the time. Since inflation and nominal rates are identical for any $\phi > 1$, there is no way for these agents to observe ϕ. Even with $\phi < 1$, there are equilibria which look identical to the equilibria with $\phi > 1$. It is likely to be far easier for central banks to convince economic agents that they just respond positively to inflation. This is all that is needed for a fully smoothed real rate rule.

For the rest of this paper, we return to looking at unsmoothed rules. However, all our results would generalize to smoothed rules. There is a strong case for the preferability of such smoothing.

2 Challenges to real rate rules

We have established the excellent properties of real rate rules when the linear Fisher equation holds. However, the linear Fisher equation may fail to hold exactly due to risk premia or other wedges. We address risk premia in the first subsection here, via examining real rate rules in fully non-linear models, and then we look at other wedges in the following subsection. We show real rate rules still retain their robustness.

We then examine whether the possibility of inflation being determined independently of monetary policy represents a challenge to robust real rate rules. This is relevant under active fiscal policy, for example. We show that with long maturity debt, a solution with stable inflation and stable real variables always exists, independent of whether fiscal policy is active or passive. This implies that the fiscal theory of the price level fails to determine a unique outcome in general, a result which may be of independent interest.

2.1 Risk premia and non-linear models

Our examples so far have been linearized models. Linearization removes the risk premium that enters the Fisher equation due to inflation risk. It is thus important for us to verify that real rate rules still work in fully non-linear models.

Suppose that Ξ_t is the real stochastic discount factor (SDF) between period t and period $t+1$, and that I_t is the gross nominal interest rate (so $i_t = \log I_t$) and that R_t is the gross real interest rate (so $r_t = \log R_t$). Then the pricing equations for one-period nominal and real bonds imply:

$$I_t \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} = 1, \qquad R_t \mathbb{E}_t \Xi_{t+1} = 1.$$

The natural nonlinear version of equation (2) is the following rule:

$$I_t = R_t \Pi^* \left(\frac{\Pi_t}{\Pi^*} \right)^{\phi},$$

where we allow for a constant gross inflation target of Π^*.

Combining this rule with the bond pricing equations implies that:

$$\mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} = \frac{\mathbb{E}_t \Xi_{t+1}}{\Pi^*} \left(\frac{\Pi^*}{\Pi_t}\right)^{\phi},$$

so:

$$\mathbb{E}_t \frac{\Xi_{t+1}}{\mathbb{E}_t \Xi_{t+1}} \frac{\Pi^*}{\Pi_{t+1}} = \left(\frac{\Pi^*}{\Pi_t}\right)^{\phi}.$$

It is easy to see that $\Pi_t = \Pi^*$ is always one solution of this equation, as $\mathbb{E}_t \frac{\Xi_{t+1}}{\mathbb{E}_t \Xi_{t+1}} = 1$. Thus, robust real rate rules are always consistent with stable inflation, even in fully non-linear models.

Furthermore, under mild assumptions, there exists a constant $\overline{Z} \geq 1$ such that for all sufficiently high ϕ, $1 \leq \frac{\Pi^*}{\Pi_t} \leq \overline{Z}^{\frac{1}{\phi-1}}$. This upper bound tends to 1 as ϕ goes to ∞, thus for large ϕ, any solution must have $\Pi_t \approx \Pi^*$. This holds even if the SDF, Ξ_t, is a complicated function of inflation and its history. Under slightly stronger assumptions on the SDF, we can even guarantee that $\Pi_t = \Pi^*$ is the unique solution for all sufficiently large ϕ. These results are proven in Appendix A. For the sake of tractability, we return to the linearized world for the bulk of the rest of this paper.

2.2 Wedges in the Fisher equation

One natural concern is that real rate rules may lose their robust determinacy if the Fisher equation does not hold exactly. Risk premia are one source of a wedge in the Fisher equation, but we showed in the previous subsection that real rate rules continue to perform well in the presence of endogenous risk premia. However, there are other reasons why there may be a wedge in the Fisher equation. For example, nominal bonds may provide greater liquidity services than real bonds, and so nominal bonds may command a premium. Such a premium is documented by Fleckenstein, Longstaff & Lustig (2014), based on comparing synthetic treasury bonds constructed from TIPS and inflation swaps to actual treasury bonds. Furthermore, TIPS provide deflation protection, which may result in TIPS also commanding a premium, giving another source of a wedge in the Fisher equation. A Fisher equation wedge could even come from bounded rationality of market participants.

Suppose then that the linearized Fisher equation takes the form:

$$i_t = r_t + \mathbb{E}_t \pi_{t+1} + \nu_t,$$

where ν_t is a potentially endogenous wedge term. We assume though that ν_t is stationary, and that there exists some $\overline{\mu}_0, \overline{\mu}_1, \overline{\mu}_2, \overline{\gamma}_0, \overline{\gamma}_1, \overline{\gamma}_2 \geq 0$ such that for any stationary solution for π_t, $|\mathbb{E}\nu_t| \leq \overline{\mu}_0 + \overline{\mu}_1 |\mathbb{E}\pi_t| + \overline{\mu}_2 \operatorname{Var} \pi_t$ and $\operatorname{Var} \nu_t \leq \overline{\gamma}_0 + \overline{\gamma}_1 |\mathbb{E}\pi_t| + \overline{\gamma}_2 \operatorname{Var} \pi_t$, for all $t \in \mathbb{Z}$ and $j, k \in \mathbb{N}$. This assumption is extremely mild, as all of these coefficients may be arbitrarily large. For example, if ν_t were to come purely from an inflation risk premium, we would expect $\overline{\mu}_2 > 0$ and $\overline{\gamma}_0 > 0$ but all other coefficients to be zero. Alternatively, if ν_t were to come purely from the liquidity services provided by nominal bonds, we would expect $\overline{\mu}_0$, $\overline{\gamma}_0$ and $\overline{\mu}_1$ to be positive (the latter as the value of liquidity services might vary over the cycle), but all other coefficients to be zero.

Combining the modified Fisher equation with the simple rule in (2) gives:

$$\mathbb{E}_t \pi_{t+1} + \nu_t = \phi \pi_t,$$

so:

$$\begin{aligned} \pi_t &= \phi^{-1} \mathbb{E}_t \pi_{t+1} + \phi^{-1} \nu_t = \phi^{-2} \mathbb{E}_t \pi_{t+2} + \phi^{-2} \mathbb{E}_t \nu_{t+1} + \phi^{-1} \nu_t = \cdots \\ &= \mathbb{E}_t \sum_{k=0}^{\infty} \phi^{-k-1} \nu_{t+k} + \lim_{k \to \infty} \left[\phi^{-k} \mathbb{E}_t \pi_{t+k} \right] = \mathbb{E}_t \sum_{k=0}^{\infty} \phi^{-k-1} \nu_{t+k}, \end{aligned}$$

assuming as ever that we select the stationary equilibrium for inflation.[21] Thus, with $\phi > 1$:

$$|\mathbb{E}\pi_t| = \frac{|\mathbb{E}\nu_t|}{\phi - 1} \leq \frac{\overline{\mu}_0 + \overline{\mu}_1 |\mathbb{E}\pi_t| + \overline{\mu}_2 \operatorname{Var} \pi_t}{\phi - 1},$$

[21] Ireland (2015) finds a role for risk premia in explaining US inflation fluctuations, so it is empirically plausible that the Fisher equation wedge should appear in the solution for inflation.

and:[22]

$$\operatorname{Var}\pi_t = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\phi^{-j-1}\phi^{-k-1}\operatorname{Cov}\left(\mathbb{E}_t\nu_{t+j},\mathbb{E}_t\nu_{t+k}\right) \le \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\phi^{-j-1}\phi^{-k-1}\operatorname{Var}\nu_t$$
$$\le \frac{\overline{\gamma}_0+\overline{\gamma}_1|\mathbb{E}\pi_t|+\overline{\gamma}_2\operatorname{Var}\pi_t}{(\phi-1)^2}.$$

So, for sufficiently large ϕ:[23]

$$|\mathbb{E}\pi_t| \le \frac{[(\phi-1)^2-\overline{\gamma}_2]\overline{\mu}_0+\overline{\mu}_2\overline{\gamma}_0}{(\phi-1-\overline{\mu}_1)[(\phi-1)^2-\overline{\gamma}_2]-\overline{\mu}_2\overline{\gamma}_1} = O\left(\frac{1}{\phi}\right),$$
$$\operatorname{Var}\pi_t \le \frac{(\phi-1-\overline{\mu}_1)\overline{\gamma}_0+\overline{\mu}_0\overline{\gamma}_1}{(\phi-1-\overline{\mu}_1)[(\phi-1)^2-\overline{\gamma}_2]-\overline{\mu}_2\overline{\gamma}_1} = O\left(\frac{1}{\phi^2}\right).$$

Hence, as $\phi\to\infty$, $\mathbb{E}\pi_t\to 0$ and $\operatorname{Var}\pi_t\to 0$. While the central bank can no longer guarantee precisely zero inflation in the presence of an endogenous wedge, if they are aggressive enough, they can ensure the mean and variance of inflation are arbitrarily close to zero. Thus, wedges in the Fisher equation do not present a substantial challenge to the performance of real rate rules.

However, if the pricing of nominal bonds is indeed highly distorted by the liquidity services they provide (for example), then the central bank may attain lower inflation bias and variance for a given ϕ by intervening in inflation swap markets rather than nominal bond ones. In our notation, an inflation swap is a contract agreed in period t between two parties, A and B, in which party A promises to make a net payment of $\Pi_{t+1}-K_t$ to party B in period $t+1$, where K_t is the negotiated contract rate. Writing Ξ_{t+1} for the real SDF between periods t and $t+1$, this contract rate must solve:

$$\mathbb{E}_t\frac{\Xi_{t+1}}{\Pi_{t+1}}(\Pi_{t+1}-K_t)=0.$$

So, from log-linearizing:

$$k_t=\log K_t=\mathbb{E}_t\pi_{t+1},$$

to first order.

[22] Here we use the fact that by the Cauchy-Schwarz inequality, the law of total variance and stationarity: $\operatorname{Cov}\left(\mathbb{E}_t\nu_{t+j},\mathbb{E}_t\nu_{t+k}\right) \le \sqrt{(\operatorname{Var}\mathbb{E}_t\nu_{t+j})(\operatorname{Var}\mathbb{E}_t\nu_{t+k})} = \sqrt{(\operatorname{Var}\nu_{t+j}-\mathbb{E}\operatorname{Var}_t\nu_{t+j})(\operatorname{Var}\nu_{t+k}-\mathbb{E}\operatorname{Var}_t\nu_{t+k})} \le \operatorname{Var}\nu_t$.

[23] In particular, we need $\phi-1>\overline{\mu}_1$, $(\phi-1)^2>\overline{\gamma}_2$ and $(\phi-1-\overline{\mu}_1)[(\phi-1)^2-\overline{\gamma}_2]>\overline{\mu}_2\overline{\gamma}_1$.

The central bank can then use the inflation swap real rate rule:

$$k_t = \phi\pi_t.$$

Combined with the inflation swap pricing equation, this gives $\mathbb{E}_t\pi_{t+1} = \phi\pi_t$, just like when the central bank intervenes in nominal bond markets. The advantage of directly targeting inflation swap contract rates is that inflation swaps are unlikely to provide liquidity services, unlike nominal bonds, meaning the inflation swap pricing equation will be less distorted than the Fisher equation. One final benefit of directly targeting inflation swap contract rates is that inflation swaps do not include the deflation protection given by TIPS. This removes one additional source of distortion in the $i_t - r_t$ gap.

2.3 The fiscal theory of the price level and the risk of over determinacy

As long as the linear Fisher equation holds, robust real rate rules can never fail to rule out sunspots. However, in an economy in which the price level is determinate independent of monetary policy, they may still produce explosive inflation.[24] This is true of any monetary rule respecting the Taylor principle, not just the real rate rules we examine in this paper. Inflation becomes "over determined", and an explosive solution is all that remains.

For example, suppose that government debt is all one period and nominal, and that real government surpluses are not responsive to government debt levels, meaning fiscal policy is "active". Then the price level is pinned down by the government debt valuation equation (see e.g. Cochrane (2022)), in line with the fiscal theory of the price

[24] Note: it is certainly not the case though that in any model in which an interest rate peg is determinate, a real rate rule would produce explosive inflation. For example, in the New Keynesian model with a discounted Euler equation, from Subsection 1.2, if $\delta \in \left(-\frac{1+\beta+\kappa\varsigma}{1+\beta}, \frac{1-\beta-\kappa\varsigma}{1-\beta}\right)$ then an interest rate peg is determinate. We saw that the real rate rule is also determinate (and non-explosive) in this model.

level. In particular, to a first order approximation with flexible prices and constant real interest rates:[25]

$$\pi_t - \mathbb{E}_{t-1}\pi_t = -\varepsilon_{s,t}, \tag{8}$$

where $\varepsilon_{s,t}$ is an exogenous shock to the present value of real government surpluses, scaled by the value of outstanding real government debt, with $\mathbb{E}_{t-1}\varepsilon_{s,t} = 0$. Suppose in this world that the central bank did follow the basic real rate rule $i_t = r_t + \phi\pi_t + \varepsilon_{\zeta,t}$, where $\mathbb{E}_{t-1}\varepsilon_{\zeta,t} = 0$. Then, from the Fisher equation, $\mathbb{E}_{t-1}\pi_t = \phi\pi_{t-1} + \varepsilon_{\zeta,t-1}$, implying from (8) that:

$$\pi_t = \phi\pi_{t-1} + \varepsilon_{\zeta,t-1} - \varepsilon_{s,t}.$$

With $\phi > 1$, this is an explosive process.

How big a threat is this to the robustness of real rate rules? We need to understand under what conditions following the Taylor principle leads to explosive inflation. Suppose as before then that the central bank follows the simple real rate rule $i_t = r_t + \phi\pi_t + \varepsilon_{\zeta,t}$. We also assume the Fisher equation holds, but we make zero assumptions on the form of the rest of the model. First define the expectational error, $\eta_t := \pi_t - \mathbb{E}_{t-1}\pi_t$. By construction, $\mathbb{E}_{t-1}\eta_t = 0$. In a linearized model, in equilibrium η_t must be a linear combination of the model's structural shocks. So, we can always decompose η_t as $\eta_t = \alpha\varepsilon_{\zeta,t} + \nu_t$, where $\mathbb{E}_{t-1}\nu_t\varepsilon_{\zeta,t} = 0$ and $\mathbb{E}_{t-1}\nu_t = 0$. Thus, from the monetary rule:

$$\pi_t - \mathbb{E}_{t-1}\pi_t = \alpha(i_t - r_t - \phi\pi_t) + \nu_t.$$

Combining this with the Fisher equation then implies that:

$$(1 + \alpha\phi)\pi_t - \mathbb{E}_{t-1}\pi_t = \alpha\mathbb{E}_t\pi_{t+1} + \nu_t.$$

Then from taking expectations conditional on $t - 1$ information we have:

$$\alpha\phi e_{t-1} = \alpha\mathbb{E}_{t-1}e_t,$$

where $e_t := \mathbb{E}_t\pi_{t+1}$.

There are now two cases. If $\alpha \neq 0$, meaning that monetary policy shocks cause either unexpected inflation or disinflation, as in the data (see e.g., Miranda-Agrippino &

[25] See Cochrane (2022), Subsection 2.5 and following.

Ricco 2021), then $\phi e_t = \mathbb{E}_t e_{t+1}$. With $\phi > 1$, this has the unique non-explosive solution $e_t = 0$, implying that $\eta_t = \pi_t = \alpha\varepsilon_{\zeta,t} + \nu_t$. This is stable, determinate inflation.

However, if $\alpha = 0$, monetary policy shocks do not have any contemporaneous effect on inflation and:

$$\pi_t = \mathbb{E}_{t-1}\pi_t + \eta_t = i_{t-1} - r_{t-1} + \nu_t = \phi\pi_{t-1} + \varepsilon_{\zeta,t-1} + \nu_t,$$

from (in turn) the definition of η_t, the Fisher equation, the decomposition of η_t and the monetary rule. If $\phi > 1$, this is explosive "over determined" inflation.

This establishes that the only situation in which a real rate rule is inconsistent with stable inflation is if monetary policy shocks have no contemporaneous impact on inflation. This is important for two reasons.

Firstly, it suggests that only in an unlikely, knife edge, case will following the Taylor principle guarantee explosive inflation. A minor change in price/wage stickiness, debt maturity structure, or the introduction of a small cost channel of monetary policy will likely introduce at least some correlation between monetary policy shocks and current inflation, restoring the existence of an equilibrium with stable inflation. Of course, there may also still be other equilibria with explosive inflation, but if we continue to assume that agents always pick an equilibrium with stable inflation if one exists, then that will be the result.

For example, suppose that the government issues multi-period (geometric coupon) debt, and that both monetary and fiscal policy are active (i.e., real government surpluses do not respond to debt, and the monetary rule satisfies the Taylor principle). Based on results with one period debt, researchers have tended to assume that this "active-active" combination will inevitably produce explosive inflation. This is incorrect. In Appendix B.1 we examine the equilibria of a non-linear model with multi-period debt under flexible prices. We show that under active fiscal policy, there is a valid equilibrium in which real variables and inflation are stable and independent of surpluses, whether or not monetary policy is active. These equilibria feature a growing bubble in the price of government debt which is balanced by declining debt quantities. The initial debt price jumps to ensure the transversality condition is still satisfied, giving a "Fiscal Theory of the Debt Price". Under passive monetary policy, we find a continuum of equilibria, contrary to the usual claim that the active fiscal, passive monetary, combination ensures unique outcomes (which is again only true with one period debt). These equilibria feature arbitrarily high inflation. In Appendix B.2 we show that these results also hold in a linearised model with sticky prices.

Secondly, our previous result gives central banks a simple test of whether they live in a world in which following the Taylor principle always produces explosive inflation. The central bank can adopt a real rate rule, with ϕ not much larger than 1, and can deliberately introduce small monetary policy shocks. It then just needs to statistically test whether the correlation between its monetary shock and current inflation is zero. With ϕ sufficiently close to 1, the sample size for the test will be large enough to have high power before π_t is excessively high. If the correlation is non-zero, then following the Taylor principle will not produce explosive inflation, and the central bank can then adopt a larger ϕ should it desire. If the correlation is estimated at zero, then the central bank should adopt $\phi < 1$ as it must be in an economy like that under the fiscal theory of the price level with one period debt.

Miranda-Agrippino & Ricco (2021) find unambiguous evidence of a negative contemporaneous impact of US monetary shocks on inflation. Thus, if the Fed is currently using a real rate rule—something we cannot rule out, due to observational equivalence—it can be confident that setting $\phi > 1$ is consistent with stable inflation.

3 Practical implementation of real rate rules

Until recently, central banks concentrated their monetary interventions in overnight debt markets. However, with the rise of quantitative easing, many central banks have been purchasing substantial quantities of longer maturity sovereign debt. There is no reason then that central banks could not conduct open market operations to fix the interest rate on longer maturity bonds. This is convenient as in most countries, inflation protected securities are only issued a few times per year, and at long maturities, e.g., five years. As a result, markets in shorter maturity inflation protected securities may be illiquid or even unavailable, and it can be difficult to reconstruct the short end of the real yield curve.

Inflation indexation lags further complicate the use of short maturity inflation protected securities (see e.g. Gürkaynak, Sack & Wright (2010)). For example, with time measured in quarters, 3-month maturity US TIPS have a period $t+1$ realized yield of $r_t + \pi_t$, not $r_t + \pi_{t+1}$ as one would hope. By using longer maturity bonds, the impact of this indexation lag is greatly reduced. In this section, we examine the performance of real rate rules when the central bank implements them using multiperiod debt.

3.1 Set-up

We aim to describe a set-up with many of the frictions that would be problematic for a naïve implementation of real rate rules.

The central bank's trading desk would be tasked with maintaining a particular level of the gap between nominal and real rates according to the market for bonds of a certain maturity. We let T be the maturity length of these bonds, measured in periods The units of time do not need to coincide with the maturity of the bond. For example, T may be 60 if periods are months and five-year bonds are used.

We allow for the possibility that inflation is not observed contemporaneously. For example, US CPI is observed with a one-month lag. To capture this, while keeping to the convention that $\mathbb{E}_t v_t = v_t$ for all t-dated endogenous variables v_t, we assume that market participants and the central bank use the $t-S$ information set in period t (i.e. they know the values of all $t-S, t-S-1, \dots$ dated variables), for some $S \geq 0$. Thus, since the central bank does not know π_t at t, we instead assume that they respond to deviations of π_{t-S} from target, rather than π_t.

We write $i_{t|t-S}$ for the nominal yield per-period on a T-period nominal bond at t, and $r_{t|t-S}$ for the real yield per-period on a T-period inflation protected bond at t. This notation captures the fact that period t nominal and real yields must be fixed in period $t-S$: market participants and the central bank only have access to the period $t-S$ information set at t, and these agents must know period t nominal and real rates. An economic agent who somehow knew the price level in real time would thus know nominal and real rates S periods in advance.

We allow for a wedge in the Fisher equation to capture inflation risk premia, liquidity premia, asymmetric term premia and even departures from full information rational expectations amongst market participants. Since only $t-S$ dated variables are known in period t, we denote the period t value of this shock by $\nu_{t|t-S}$. I.e., risk premia (etc.) will be determined S periods in advance, though market participants and the central bank will not act on this, as they use S period old data.

Furthermore, we allow for the possibility of an indexation lag in the return of the real bond. We assume that the lag is L periods. If periods are months, then L would be 3 for the US.

3.2 The generalized Fisher equation and monetary rule

Given all this, the Fisher equation coming from arbitrage between nominal and real bonds states that:

$$i_{t|t-S} - r_{t|t-S} = \nu_{t|t-S} + \mathbb{E}_{t-S}\frac{1}{T}\sum_{k=1}^{T}\pi_{t+k-L}.$$

The central bank's actions in period t cannot possibly impact $\pi_{t-1}, \pi_{t-2}, \ldots$ as these are already predetermined. Hence, for the central bank to be able to have some impact on $i_{t|t-S} - r_{t|t-S}$ we require that $T - L \geq 0$. So, for the US, the central bank would have to use bonds with maturity of at least three months.

Slightly generalizing our previous rule (7), we suppose that the central bank intervenes in T-period nominal bond markets to ensure that it is always the case that:

$$i_{t|t-S} - r_{t|t-S} = \bar{\nu}_{t|t-S} + \mathbb{E}_{t-S}\frac{1}{T}\sum_{k=1}^{T}\pi^*_{t+k-L} + \phi(\pi_{t-S} - \pi^*_{t-S}),$$

where $\bar{\nu}_{t|t-S}$ is the central bank's period t belief about the level of $\nu_{t|t-S}$, and where $\phi > 1$. $\bar{\nu}_{t|t-S}$ could also include a monetary policy shock component. We stress that the $t|t-S$ index here does not mean that the private sector knows monetary policy shocks S periods in advance, as the private sector (and the central bank) uses the $t-S$ information set at t.

Also note that while under conventional monetary policy, targeted nominal interest rates are (approximately) constant between monetary policy committee meetings, this may not be the case here. The rule effectively specifies a period t level for $i_{t|t-S} - r_{t|t-S}$, not for $i_{t|t-S}$. The level of $r_{t|t-S}$ may fluctuate (perhaps in part due to unexpected changes in $i_{t|t-S}$), so the central bank's trading desk could have to continuously tweak the level of $i_{t|t-S}$ to hold $i_{t|t-S} - r_{t|t-S}$ at its desired level. While this represents a departure from previous operating procedure, there is no reason why holding $i_{t|t-S} - r_{t|t-S}$ approximately constant should be any harder than holding $i_{t|t-S}$ approximately constant. This is thanks to real-time observability of $r_{t|t-S}$ via inflation protected bonds.

The central bank could also directly control $i_{t|t-S} - r_{t|t-S}$ by promising to freely exchange \$1 face value of real debt for $\$(1 + i_{t|t-S} - r_{t|t-S})$ face value of nominal debt, as suggested by Cochrane (2017; 2018). Alternatively, the central bank could buy or sell a long-short portfolio containing \$1 face value of nominal debt, and −\$1 face value of real debt to hold the portfolio's return fixed at $\$(i_{t|t-S} - r_{t|t-S})$.[26] Or, the central bank could directly pin down the contract rate on inflation swaps, as suggested in Subsection 2.2.

[26] The author thanks Peter Ireland for this suggestion.

3.3 Solution and robustness

Combining the multi-period Fisher equation and the monetary rule implies that the dynamics of inflation are governed by the single equation:

$$\mathbb{E}_{t-S}\frac{1}{T}\sum_{k=1}^{T}(\pi_{t+k-L}-\pi^*_{t+k-L})+(\nu_{t|t-S}-\bar{\nu}_{t|t-S})=\phi(\pi_{t-S}-\pi^*_{t-S}),$$

i.e.:

$$\mathbb{E}_{t}\frac{1}{T}\sum_{k=1}^{T}(\pi_{t+k+S-L}-\pi^*_{t+k+S-L})+(\nu_{t+S|t}-\bar{\nu}_{t+S|t})=\phi(\pi_{t}-\pi^*_{t}).$$

When $\nu_{t+S|t}-\bar{\nu}_{t+S|t}$ is exogenous, this expectational difference equation has a unique solution if and only if it has a unique solution when $\nu_{t+S|t}-\bar{\nu}_{t+S|t}=0$ for all t. In this case, via the substitution $\pi_t-\pi_t^*=c\lambda^t$ we have the characteristic equation:

$$\frac{1}{T}\sum_{k=1}^{T}\lambda^{k+S-L}=\phi.$$

The roots of this equation decide the determinacy of π_t. For determinacy, we need $\max\{0,-(1+S-L)\}$ roots strictly inside the unit circle, corresponding to the lags of inflation in our difference equation, and $\max\{0,T+S-L\}$ roots strictly outside the unit circle, corresponding to the leads of inflation in our difference equation. [27] This is indeed the case, as we prove in Appendix E.6 (given $\phi>1$). Thus, at least when $\nu_{t+S|t}-\bar{\nu}_{t+S|t}$ is exogenous, there is a unique solution for inflation.[28] In the special case in which the central bank observes ν_t so $\bar{\nu}_t=\nu_t$, then $|\pi_t-\pi_t^*|\to 0$ as $t\to\infty$. (There may not be equality for finite t due to the impact of initial conditions.)

In the general case in which $\nu_{t+S|t}-\bar{\nu}_{t+S|t}$ is potentially endogenous, as long as $\nu_{t+S|t}-\bar{\nu}_{t+S|t}$ is stationary, the solution must take the form:

$$\pi_t=\pi_t^*+\mathbb{E}_t\sum_{j=-\infty}^{\infty}A_j\big(\nu_{t+j+S|t+j}-\bar{\nu}_{t+j+S|t+j}\big).$$

[27] In fact, our assumptions that $T-L\geq 0$ and $S\geq 0$, imply $T+S-L\geq 0$.

[28] We do not have the indeterminacy issues for rules setting long-rates that were noted by McGough, Rudebusch & Williams (2005), due to the presence of the real rate in our rule.

Substituting this solution into our expectational difference equation, then taking $t + \min\{0, 1 + S - L\}$ dated expectations, and matching terms, gives that:

$$A_j = \frac{1}{\phi}\mathbb{1}[j = 0] + \frac{1}{\phi T}\sum_{k=1}^{T} A_{j-k-S+L}.$$

With $\phi > 1$, this difference equation has a unique solution for $\left(A_j\right)_{j\in\mathbb{Z}}$ in which $A_j \geq 0$ for all $j \in \mathbb{Z}$, as proven in Appendix E.7. Furthermore, if we define $B \coloneqq \sum_{j=-\infty}^{\infty} A_j$, then:

$$B = \frac{1}{\phi} + \sum_{j=-\infty}^{\infty}\frac{1}{\phi T}\sum_{k=1}^{T} A_{j-k-S+L} = \frac{1}{\phi} + \frac{1}{\phi T}\sum_{k=1}^{T}\sum_{j=-\infty}^{\infty} A_{j-k-S+L} = \frac{1}{\phi} + \frac{1}{\phi T}\sum_{k=1}^{T} B = \frac{1}{\phi}(1 + B).$$

Thus, $B = \frac{1}{\phi - 1}$. This is sufficient to establish that $\pi_t \approx \pi_t^*$ for large ϕ, even when $\nu_{t+S|t} - \bar{\nu}_{t+S|t}$ is endogenous, by an identical to argument to that of Subsection 2.2.

In particular, suppose we assume that $\nu_{t+S|t} - \bar{\nu}_{t+S|t}$ is stationary, and that there exists some $\bar{\mu}_0, \bar{\mu}_1, \bar{\mu}_2, \overline{\gamma}_0, \overline{\gamma}_1, \overline{\gamma}_2 \geq 0$ such that for any stationary solution for $\pi_t - \pi_t^*$, $\left|\mathbb{E}(\nu_{t+S|t} - \bar{\nu}_{t+S|t})\right| \leq \bar{\mu}_0 + \bar{\mu}_1|\mathbb{E}(\pi_t - \pi_t^*)| + \bar{\mu}_2 \operatorname{Var}(\pi_t - \pi_t^*)$ and $\operatorname{Var}(\nu_{t+S|t} - \bar{\nu}_{t+S|t}) \leq \overline{\gamma}_0 + \overline{\gamma}_1|\mathbb{E}(\pi_t - \pi_t^*)| + \overline{\gamma}_2 \operatorname{Var}(\pi_t - \pi_t^*)$, for all $t \in \mathbb{Z}$ and $j, k \in \mathbb{N}$. This assumption is very mild, as already discussed in Subsection 2.2. Then, following the argument of that subsection (and using the fact that $A_j \geq 0$ for all $j \in \mathbb{Z}$):

$$|\mathbb{E}(\pi_t - \pi_t^*)| \leq \frac{\bar{\mu}_0 + \bar{\mu}_1|\mathbb{E}(\pi_t - \pi_t^*)| + \bar{\mu}_2 \operatorname{Var}(\pi_t - \pi_t^*)}{\phi - 1},$$

and:

$$\operatorname{Var}(\pi_t - \pi_t^*) \leq \frac{\overline{\gamma}_0 + \overline{\gamma}_1|\mathbb{E}(\pi_t - \pi_t^*)| + \overline{\gamma}_2 \operatorname{Var}(\pi_t - \pi_t^*)}{(\phi - 1)^2}.$$

So, for sufficiently large ϕ:

$$|\mathbb{E}(\pi_t - \pi_t^*)| \leq \frac{[(\phi - 1)^2 - \overline{\gamma}_2]\bar{\mu}_0 + \bar{\mu}_2\overline{\gamma}_0}{(\phi - 1 - \bar{\mu}_1)[(\phi - 1)^2 - \overline{\gamma}_2] - \bar{\mu}_2\overline{\gamma}_1} = O\left(\frac{1}{\phi}\right),$$
$$\operatorname{Var}(\pi_t - \pi_t^*) \leq \frac{(\phi - 1 - \bar{\mu}_1)\overline{\gamma}_0 + \bar{\mu}_0\overline{\gamma}_1}{(\phi - 1 - \bar{\mu}_1)[(\phi - 1)^2 - \overline{\gamma}_2] - \bar{\mu}_2\overline{\gamma}_1} = O\left(\frac{1}{\phi^2}\right).$$

Hence, as $\phi \to \infty$, $\mathbb{E}(\pi_t - \pi_t^*) \to 0$ and $\operatorname{Var}(\pi_t - \pi_t^*) \to 0$. Thus, with ϕ large, even if the central bank imperfectly tracks the risk (etc.) premium ν_t, and even if their error is endogenous, it will still be the case that $\pi_t \approx \pi_t^*$ in all periods. I.e., even in the presence of unobservable endogenous wedges in the Fisher equation, the central bank can still determinately implement an arbitrary path for inflation. The presence of information or indexation lags makes no fundamental difference to this. While such lags may slow down the convergence of A_j to 0 as $j \to \pm\infty$, increasing the variance of $\pi_t - \pi_t^*$, still for a large enough ϕ, inflation will be very close to its target.

4 The zero lower bound

All our examples so far have ignored the zero lower bound (ZLB) on nominal interest rates. The zero lower bound is problematic for real rate rules as it prevents the central bank from fixing $i_t - r_t$ when $i_t = 0$. This means that at the ZLB, the Euler equation again becomes relevant for outcomes, reducing robustness. This section presents two solutions to restore robustness in the presence of the zero lower bound.

In Appendix D we also consider how the central bank can rule out the deflationary steady state that exists in the presence of the ZLB. That appendix also discusses how the central bank can rule out explosive paths for inflation. Such paths may not violate any transversality constraint, as pointed out by Cochrane (2011). Ruling out explosive paths for inflation is a very similar problem to that of ruling out ZLB traps, and similar solutions will work.

4.1 The problems caused by the ZLB for real rate rules

We can see the problems caused by the ZLB even in the simplest possible set-up used in this paper's introduction. In the presence of the zero lower bound, under the introduction's set-up, we have that:

$$\max\{0, r_t + \phi\pi_t\} = i_t = r_t + \mathbb{E}_t\pi_{t+1}.$$

While without the ZLB, we can cancel out the r_t in the monetary rule with the r_t from the Fisher equation, now this is no longer possible. Instead, we have that:

$$\max\{-r_t, \phi\pi_t\} = \mathbb{E}_t\pi_{t+1}.$$

Thus, real rates (and hence the Euler equation) potentially matter for inflation dynamics and determinacy. Holden (2021) points out that even if r_t is exogenous, with $r_t = 0$ for $t \neq 1$, and even if we assume that $\pi_t \to 0$ as $t \to \infty$, still this simple model has multiple solutions for a value of r_1 ($r_1 = 0$), and no solution for other values of r_1 ($r_1 < 0$).

Holden (2021) shows this multiplicity and non-existence of perfect foresight solutions is the rule for NK models with a ZLB, even with a terminal condition on inflation ensuring an eventual escape from the ZLB. Additionally, there are further solutions converging to a deflationary steady state with interest rates at zero. Furthermore, under rational expectations there are always at least as many solutions as under perfect foresight, as well as a continuum of further switching solutions (Holden 2021). Using a real rate rule does not appear to help any of these problems. Thus, while real rate rules may be more robust than other monetary rules when far from the ZLB, their performance is likely to be similar to traditional monetary rules close to the ZLB.

4.2 Price level real rate rules

One way to improve the performance of real rate rules near the ZLB is to replace the response to inflation with a response to the price level. Holden (2021) shows that responding to the price level is a robust way to ensure the existence of a unique solution with the ZLB, at least given that inflation does not converge to the deflationary steady state. We discuss how to rule out convergence to the deflationary steady state in Appendix D.

Price level rules rule out self-fulfilling temporary jumps to the ZLB as under a price level rule, the deflation during the bound period must be made up for by high inflation after exiting the bound. Thus, expected inflation is high in the last period at the bound, which via the Fisher equation, implies nominal interest rates should be high that period as well, unless real rates are still very low. This unwinds non-fundamental ZLB spells, as in a non-fundamental jump to the bound, real rates are unlikely to move enough to drive the economy to the ZLB on their own.

A variable target price level real rate rule takes the form:

$$i_t = \max\{0, r_t + \mathbb{E}_t p^*_{t+1} - p^*_t + \theta(p_t - p^*_t)\},$$

where p_t is the logarithm of the price level (so $\pi_t = p_t - p_{t-1}$), p^*_t is the exogenous price level target, and where $\theta > 0$. Away from the ZLB, combining this with the Fisher equation gives that:

$$r_t + \mathbb{E}_t p^*_{t+1} - p^*_t + \theta(p_t - p^*_t) = i_t = r_t + \mathbb{E}_t p_{t+1} - p_t,$$

so:

$$(1+\theta)(p_t - p^*_t) = \mathbb{E}_t(p_{t+1} - p^*_{t+1}).$$

With $\theta > 0$, this has the unique stationary solution $p_t = p^*_t$. Just like standard (inflation) real rate rules, price level real rate rules are robust, since away from the ZLB, price level determination is completely independent of the real interest rate or the rest of the model. Their advantage over standard real rate rules is in avoiding the multiplicity of transition paths highlighted by Holden (2021).

4.3 Perpetuity real rate rules

An even more robust approach is for the central bank to intervene in a market which does not have an equivalent to the ZLB. Perpetuities (also called "consols") are one such asset. For suppose that nominal interest rates were expected to be at i for all time. Then the price of a perpetuity would be $\frac{1}{i}$.[29] Thus, any finite, positive, perpetuity price is consistent with at least one path for future nominal interest rates. In other words, there is no upper or lower bound on the price of a perpetuity.

Note that the central bank does not strictly need the treasury to issue perpetuities in order to implement a perpetuity real rate rule. Since central banks in developed nations are generally believed to be extremely long-lived institutions, the central bank can issue perpetuities itself. As central banks can always print money to pay the coupon, central banks may be one of the only institutions that could be trusted to pay

[29] This is correct under continuous time with a continuous flow of coupons, and approximately correct under discrete time, as we will see below.

coupons for ever. Central banks may also decide to trust the perpetuities issued by some selected private banks, even if these will always carry some default risk. If the central bank views default as very unlikely in the short to medium term, then such default risk may not substantially distort pricing.

In the below, we will call standard perpetuities "nominal perpetuities". To implement a real rate rule on perpetuities, we will also need there to be a corresponding "real perpetuity" traded in the economy. In particular, we suppose that one unit of the period t nominal perpetuity bought at t returns \$1 at $t+1$, along with one unit of the period $t+1$ nominal perpetuity. On the other hand, one unit of the period t real perpetuity bought at t returns $\$\frac{P_{t+1}}{\Pi^{*t+1}}$ at $t+1$, along with one of the period $t+1$ real perpetuity, where P_{t+1} is the price level in period $t+1$ and $\Pi^* \geq 1$ is the target for the gross inflation rate. The nominal perpetuity trades at a price of $Q_{I,t}$ at t, whereas the real perpetuity trades at a price of $Q_{R,t}$ at t.

If we write Ξ_{t+1} for the real SDF between periods t and $t+1$, and $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$ for gross inflation between these periods, then the price of these two perpetuities must satisfy:

$$Q_{I,t} = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} [Q_{I,t+1} + 1], \qquad Q_{R,t} = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} \left[Q_{R,t+1} + \frac{P_{t+1}}{\Pi^{*t+1}} \right].$$

The real perpetuity price could be non-stationary due to the potential unit root in the logarithm of the price level, so it is helpful to define a detrended version. In particular, let:

$$\hat{Q}_{R,t} := Q_{R,t} \frac{\Pi^{*t}}{P_t} = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi^*} [\hat{Q}_{R,t+1} + 1].$$

Rewritten in this way, the analogy between the pricing of nominal and real perpetuities is clear. If $\Pi_t = \Pi^*$ for all t, then $Q_{I,t} = \hat{Q}_{R,t}$ for all t as well. If inflation and the SDF are stationary, then $\hat{Q}_{R,t}$ and $Q_{I,t}$ will admit a stationary solution.

We also assume that one period nominal bonds are traded in the economy, with gross return I_t. As in Subsection 2.1, the pricing for these bonds must satisfy:

$$I_t \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} = 1.$$

We can now redo the argument of this subsection's initial paragraph, slightly more formally. So, suppose that the gross nominal interest rate I_t is pegged at the constant level I (which may be inconsistent with the inflation target of Π^*). Then, the pricing equation for nominal perpetuities has a solution in which $Q_{I,t} = Q_I$ for all t, with $Q_I = I^{-1}[Q_I + 1]$, since $\mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} = I^{-1}$, for all t. Thus, $Q_I = \frac{1}{I-1}$. As $I \to 1$ (the ZLB), $Q_I \to \infty$, while as $I \to \infty$, $Q_I \to 0$. Thus, in line with our initial argument, any finite, positive, nominal perpetuity price is consistent with at least one possible path for nominal rates, no matter the dynamics of the real SDF. This ensures that the central bank can set the nominal perpetuity price to an arbitrary level, without any constraints. We do not need the real perpetuity price to be unbounded in this manner, as the central bank will not intervene in real perpetuity markets.

The reader might worry that a bound on nominal perpetuity prices could enter another way. Suppose that nominal perpetuity prices were known at least one period in advance (e.g., because there is no uncertainty), and that money is available to trade. Then it would be the case that $Q_{I,t+1} + 1 \geq Q_{I,t}$, else nominal perpetuities would have return strictly dominated by that of cash. This inequality is an immediate consequence of $I_t \geq 1$ though, when $Q_{I,t+1}$ is known at t. $I_t \geq 1$ implies $\frac{Q_{I,t}}{I_t} \leq Q_{I,t}$, so:

$$Q_{I,t} \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} = \frac{Q_{I,t}}{I_t} \leq Q_{I,t} = \mathbb{E}_t \frac{\Xi_{t+1}}{\Pi_{t+1}} [Q_{I,t+1} + 1],$$

which implies $Q_{I,t+1} + 1 \geq Q_{I,t}$ if $Q_{I,t+1}$ is known at t. Thus, the bound on one period nominal rates is all that really matters, and we have already showed that this bound does not imply a bound on $Q_{I,t}$. Intuitively, $Q_{I,t+1} + 1 \geq Q_{I,t}$ is not a constraint on $Q_{I,t}$ as $Q_{I,t+1}$ is endogenous.

We can now introduce our perpetuity real rate rule. We suppose that the central bank intervenes in nominal perpetuity markets to ensure:

$$Q_{I,t} = \hat{Q}_{R,t} \left(\frac{\Pi_t}{\Pi^*} \right)^{-\psi},$$

for some exponent $\psi \in \mathbb{R}$. While $\psi > 0$ may seem natural (so that high inflation results in low bond prices and thus high interest rates), we do not impose this.

We analyse the resulting dynamics via log-linearizing around the steady-state with inflation at Π^*.[30] In particular, suppose that:

$$Q_{I,t} = Q \exp q_{I,t}, \qquad \hat{Q}_{R,t} = Q \exp q_{R,t},$$
$$\Xi_t = \Xi \exp \xi_t, \qquad \Pi_t = \Pi^* \exp \pi_t,$$

where $Q := \frac{1}{I^*-1}$, with $I^* := \frac{\Pi^*}{\Xi}$. We assume $\Xi < 1$, so $I^* > 1$. Then to a first order approximation around $q_{I,t} = q_{R,t} = \xi_t = \pi_t = 0$:

$$q_{I,t} = \mathbb{E}_t \left[\xi_{t+1} - \pi_{t+1} + \frac{\Xi}{\Pi^*} q_{I,t+1} \right], \qquad q_{R,t} = \mathbb{E}_t \left[\xi_{t+1} + \frac{\Xi}{\Pi^*} q_{R,t+1} \right],$$
$$q_{I,t} = q_{R,t} - \psi \pi_t.$$

Thus:

$$\begin{aligned}\psi \pi_t = q_{R,t} - q_{I,t} &= \mathbb{E}_t \left[\xi_{t+1} + \frac{\Xi}{\Pi^*} q_{R,t+1} \right] - \mathbb{E}_t \left[\xi_{t+1} - \pi_{t+1} + \frac{\Xi}{\Pi^*} q_{I,t+1} \right] \\ &= \mathbb{E}_t \left[\pi_{t+1} + \frac{\Xi}{\Pi^*} (q_{R,t+1} - q_{I,t+1}) \right] = \mathbb{E}_t \left[\pi_{t+1} + \frac{\Xi}{\Pi^*} \psi \pi_{t+1} \right].\end{aligned}$$

Hence, if we define $\phi := \psi \left[1 + \frac{\Xi}{\Pi^*} \psi \right]^{-1}$, we then have that:

$$\phi \pi_t = \mathbb{E}_t \pi_{t+1},$$

just as when one period bonds are used. With $\phi > 1$, this has the unique stationary solution $\pi_t = 0$ (so $\Pi_t = \Pi^*$), as usual. The crucial difference is that with the perpetuity real rate rule, this is achieved without violating the ZLB.

As a final observation, note that our definition of ϕ implies that $\psi = -\phi \left[\frac{\Xi}{\Pi^*} \phi - 1 \right]^{-1}$, so, for sufficiently large ϕ ($\phi > I^* = \frac{\Pi^*}{\Xi}$) $\psi < -\frac{\Pi^*}{\Xi} < 0$. Thus, under a perpetuity real rate rule with sufficiently large ϕ, the central bank will raise nominal perpetuity prices in response to high inflation. This sign becomes more intuitive once money flows are considered. While if the central bank buys perpetuities, they are raising the money supply in the period of purchase, in every subsequent period they are reducing the money supply, as the private sector must pay coupons back to the central bank. Given the forward-looking nature of inflation determination, it is this long-run reduction which is crucial.

[30] While it would ideally be better to examine these determinacy questions in a fully non-linear model, this is not tractable. We take comfort from the fact that even Cochrane (2011) primarily relies on linearized models.

5 Conclusion

This paper's implications are stark. Under a real rate rule: the central bank can always achieve its target for inflation; any movement in inflation must be due to a monetary policy shock or a central bank choice to so move inflation; monetary policy works in spite of, not because of, real rate movements; causation in the Phillips curve runs exclusively from inflation to the output gap, not the other way round; household and firm decisions, constraints and inflation expectations are irrelevant for inflation dynamics; no model features can amplify or dampen inflation variance, except changes in the central bank's own behaviour. Real rate rules can implement optimal policy, or match observed dynamics. They continue to work in the presence of endogenous wedges in the Fisher equation, or active fiscal policy. They can be implemented using assets for which there is already a liquid market: either nominal and real long bonds, or inflation swaps.

To a policy maker, these conclusions may be shocking. However, for readers familiar with New Keynesian models, perhaps they are not completely surprising. In models in which an aggressive response to inflation produces determinacy, with an extremely aggressive response, the variance of inflation can be pushed down to near zero. And Rupert & Šustek (2019) argue that even in New Keynesian models with a standard monetary rule, monetary policy does not operate via real rates. Rather, real rate rules just crystallise the monetary policy transmission mechanism that is at work in all New Keynesian models. Monetary policy acts via the Fisher equation, and via the Taylor principle's promise to induce explosive inflation should inflation deviate from target. Accepting standard New Keynesian models means accepting this story.

Those for whom this is unpalatable will likely be drawn towards the fiscal theory of the price level. However, we showed that this theory fails to determine a unique outcome for inflation. It is even consistent with arbitrarily high inflation. Thus, the problems with Taylor principle equilibrium selection raised by Cochrane (2011) apply equally well to fiscal theory of the price level equilibrium selection. Solutions to these problems for New Keynesian models are given in Christiano & Takahashi (2018; 2020), and we give solutions specifically for real rate rules in Appendix D. Hence, Taylor principle equilibrium selection may be less problematic than that for the fiscal theory of the price level. In this case, real rate rules provide a robust way to implement monetary policy.

References

Adão, Bernardino, Isabel Correia & Pedro Teles. 2011. 'Unique Monetary Equilibria with Interest Rate Rules'. *Review of Economic Dynamics* 14 (3) (July): 432–442.

Ang, Andrew, Geert Bekaert & Min Wei. 2007. 'Do Macro Variables, Asset Markets, or Surveys Forecast Inflation Better?' *Journal of Monetary Economics* 54 (4) (May 1): 1163–1212.

Ascari, Guido & Tiziano Ropele. 2009. 'Trend Inflation, Taylor Principle, and Indeterminacy'. *Journal of Money, Credit and Banking* 41 (8): 1557–1584.

Atkeson, Andrew & Lee E Ohanian. 2001. 'Are Phillips Curves Useful for Forecasting Inflation?' Edited by Edward J Green & Richard M Todd. *Federal Reserve Bank of Minneapolis Quarterly Review* (Winter 2001): 12.

Bilbiie, Florin. 2008. 'Limited Asset Markets Participation, Monetary Policy and (Inverted) Aggregate Demand Logic'. *Journal of Economic Theory* 140 (1) (May 1): 162–196.

———. 2019. *Monetary Policy and Heterogeneity: An Analytical Framework*. 2019 Meeting Papers. Society for Economic Dynamics.

Branch, William A. & Bruce McGough. 2010. 'Dynamic Predictor Selection in a New Keynesian Model with Heterogeneous Expectations'. *Journal of Economic Dynamics and Control* 34 (8) (August 1): 1492–1508.

———. 2018. 'Chapter 1 - Heterogeneous Expectations and Micro-Foundations in Macroeconomics'. In *Handbook of Computational Economics*, edited by Cars Hommes & Blake LeBaron, 4:3–62. Handbook of Computational Economics. Elsevier.

Christiano, Lawrence J. & Yuta Takahashi. 2018. 'Discouraging Deviant Behavior in Monetary Economics'. *NBER Working Paper Series* No. 24949.

———. 2020. 'Anchoring Inflation Expectations'.

Cochrane, John H. 2011. 'Determinacy and Identification with Taylor Rules'. *Journal of Political Economy* 119 (3): 565–615.

———. 2017. 'The Grumpy Economist: Target the Spread'. *The Grumpy Economist*.

———. 2018. 'The Zero Bound, Negative Rates, and Better Rules' (March 2): 27.

———. 2022. *The Fiscal Theory of the Price Level*. Princeton University Press.

Cúrdia, Vasco, Andrea Ferrero, Ging Cee Ng & Andrea Tambalotti. 2015. 'Has U.S. Monetary Policy Tracked the Efficient Interest Rate?' *Journal of Monetary Economics* 70: 72–83.

Damjanovic, Tatiana, Vladislav Damjanovic & Charles Nolan. 2008. 'Unconditionally Optimal Monetary Policy'. *Journal of Monetary Economics* 55 (3) (April 1): 491–500.

Dotsey, Michael, Shigeru Fujita & Tom Stark. 2018. 'Do Phillips Curves Conditionally Help to Forecast Inflation?' *International Journal of Central Banking*: 50.

Dotsey, Michael & Andreas Hornstein. 2006. 'Implementation of Optimal Monetary Policy'. *Economic Quarterly*: 113–133.

Dowd, Kevin. 1994. 'A Proposal to End Inflation'. *The Economic Journal* 104 (425): 828–840.

Eggertsson, Gauti B. & Michael Woodford. 2003. 'The Zero Bound on Interest Rates and Optimal Monetary Policy'. *Brookings Papers on Economic Activity* 34 (1): 139–235.

Evans, George W. & Seppo Honkapohja. 2001. *Learning and Expectations in Macroeconomics*. Frontiers of Economic Research. Princeton and Oxford: Princeton University Press.

———. 2006. 'Monetary Policy, Expectations and Commitment'. *The Scandinavian Journal of Economics* 108 (1): 15–38.

Evans, George W. & Bruce McGough. 2010. 'Implementing Optimal Monetary Policy in New-Keynesian Models with Inertia'. *The B.E. Journal of Macroeconomics* 10 (1).

Farmer, Roger E.A., Vadim Khramov & Giovanni Nicolò. 2015. 'Solving and Estimating Indeterminate DSGE Models'. *Journal of Economic Dynamics and Control* 54 (May 1): 17–36.

Fernández-Villaverde, Jesús, Grey Gordon, Pablo Guerrón-Quintana & Juan F. Rubio-Ramírez. 2015. 'Nonlinear Adventures at the Zero Lower Bound'. *Journal of Economic Dynamics and Control* 57 (C): 182–204.

Fleckenstein, Matthias, Francis A. Longstaff & Hanno Lustig. 2014. 'The TIPS-Treasury Bond Puzzle'. *The Journal of Finance* 69 (5): 2151–2197.

Gali, Jordi, J. David Lopez-Salido & Javier Valles. 2004. 'Rule-of-Thumb Consumers and the Design of Interest Rate Rules'. *Journal of Money, Credit, and Banking* 36 (4) (July 28): 739–763.

Gürkaynak, Refet S., Brian Sack & Jonathan H. Wright. 2010. 'The TIPS Yield Curve and Inflation Compensation'. *American Economic Journal: Macroeconomics* 2 (1): 70–92.

Hall, Robert E & Ricardo Reis. 2016. *Achieving Price Stability by Manipulating the Central Bank's Payment on Reserves*. Working Paper. National Bureau of Economic Research.

Hellwig, Martin F. 1980. 'On the Aggregation of Information in Competitive Markets'. *Journal of Economic Theory* 22 (3) (June 1): 477–498.

Hetzel, Robert L. 1990. 'Maintaining Price Stability: A Proposal'. *Economic Review of the Federal Reserve Bank of Richmond* (March): 3.

Holden, Tom D. 2021. 'Existence and Uniqueness of Solutions to Dynamic Models with Occasionally Binding Constraints'. *The Review of Economics and Statistics* (October 15): 1–45.

Ireland, Peter N. 2007. 'Changes in the Federal Reserve's Inflation Target: Causes and Consequences'. *Journal of Money, Credit and Banking* 39 (8): 1851–1882.

———. 2015. 'Monetary Policy, Bond Risk Premia, and the Economy'. *Journal of Monetary Economics* 76 (November 1): 124–140.

Justiniano, Alejandro, Giorgio E. Primiceri & Andrea Tambalotti. 2013. 'Is There a Trade-Off between Inflation and Output Stabilization?' *American Economic Journal: Macroeconomics* 5 (2) (April): 1–31.

Khan, Hashmat. 2005. 'Price-Setting Behaviour, Competition, and Markup Shocks in the New Keynesian Model'. *Economics Letters* 87 (3) (June 1): 329–335.

Khan, Hashmat, Louis Phaneuf & Jean Gardy Victor. 2019. 'Rules-Based Monetary Policy and the Threat of Indeterminacy When Trend Inflation Is Low'. *Journal of Monetary Economics* (March): S0304393219300479.

Leeper, E. M. & C. Leith. 2016. 'Chapter 30 - Understanding Inflation as a Joint Monetary–Fiscal Phenomenon'. In *Handbook of Macroeconomics*, edited by John B. Taylor & Harald Uhlig, 2:2305–2415. Elsevier.

Lou, Youcheng, Sahar Parsa, Debraj Ray, Duan Li & Shouyang Wang. 2019. 'Information Aggregation in a Financial Market with General Signal Structure'. *Journal of Economic Theory* 183 (September 1): 594–624.

Lubik, Thomas A., Christian Matthes & Elmar Mertens. 2019. *Indeterminacy and Imperfect Information*. Federal Reserve Bank of Richmond Working Papers.

Manea, Cristina. 2019. 'Collateral-Constrained Firms and Monetary Policy': 66.

McGough, Bruce, Glenn D. Rudebusch & John C. Williams. 2005. 'Using a Long-Term Interest Rate as the Monetary Policy Instrument'. *Journal of Monetary Economics* 52 (5) (July 1): 855–879.

McKay, Alisdair, Emi Nakamura & Jón Steinsson. 2017. 'The Discounted Euler Equation: A Note'. *Economica* 84 (336): 820–831.

McLeay, Michael & Silvana Tenreyro. 2019. 'Optimal Inflation and the Identification of the Phillips Curve'. In . NBER Chapters. National Bureau of Economic Research, Inc.

Miranda-Agrippino, Silvia & Giovanni Ricco. 2021. 'The Transmission of Monetary Policy Shocks'. *American Economic Journal: Macroeconomics* 13 (3) (July): 74–107.

Natvik, Gisle James. 2009. 'Government Spending and the Taylor Principle'. *Journal of Money, Credit and Banking* 41 (1): 57–77.

Rupert, Peter & Roman Šustek. 2019. 'On the Mechanics of New-Keynesian Models'. *Journal of Monetary Economics* 102. Carnegie-Rochester-NYU Conference Series on Public Policy. "A Conference Honoring the Contributions of Charles Plosser to Economics" Held at the University of Rochester Simon Business School, April 20-21, 2018 (April 1): 53–69.

Smets, Frank & Rafael Wouters. 2007. 'Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach'. *American Economic Review* 97 (3) (June): 586–606.

Stock, James & Mark W. Watson. 2009. 'Phillips Curve Inflation Forecasts'. In *Understanding Inflation and the Implications for Monetary Policy*, edited by Jeffrey Fuhrer, Yolanda Kodrzycki, Jane Little & Giovanni Olivei, 99–202. Cambridge: MIT Press.

Sveen, Tommy & Lutz Weinke. 2005. 'New Perspectives on Capital, Sticky Prices, and the Taylor Principle'. *Journal of Economic Theory* 123 (1). Monetary Policy and Capital Accumulation (July 1): 21–39.

Svensson, Lars E. O & Michael Woodford. 2003. 'Implementing Optimal Policy through Inflation-Forecast Targeting' (June).

Woodford, Michael. 1999. 'Commentary: How Should Monetary Policy Be Conducted in an Era of Price Stability?' In *New Challenges for Monetary Policy*. Jackson Hole, Wyoming: Federal Reserve Bank of Kansas City.

———. 2001. 'The Taylor Rule and Optimal Monetary Policy'. *The American Economic Review* 91 (2): 232–237.

———. 2003. *Interest and Prices. Foundations of a Theory of Monetary Policy*. Princeton University Press.

Online Appendix to: "Robust Real Rate Rules"

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The views expressed in this paper are those of the author and do not represent the views of the Deutsche Bundesbank, the Eurosystem or its staff.

Appendix A Non-linear expectational difference equations

We are interested in the non-linear expectational difference equation:

$$\left(\frac{\Pi^*}{\Pi_t}\right)^{\phi} = \mathbb{E}_t \frac{\Xi_{t+1}}{\mathbb{E}_t \Xi_{t+1}} \frac{\Pi^*}{\Pi_{t+1}}.$$

If we define $X_t := \frac{\Pi^*}{\Pi_t}$ and $Z_t := \frac{\Xi_{t+1}}{\mathbb{E}_t \Xi_{t+1}}$ then this difference equation is a particular example of the more general equation:

$$X_t^{\phi} = \mathbb{E}_t Z_{t+1} X_{t+1}.$$

We show in Appendix A.1 that if $Z_t = 1$ for all t, then this has a unique solution for $\phi > 1$, and we show in Appendix A.2 that it still has a unique solution for arbitrary Z_t under a few additional conditions, and that the solution is approximately unique under even milder conditions.

For the results of Appendix A.2 to apply, we need that Π_t is bounded above. This is true in any model with monopolistic competition in which at least some small fraction of firms do not adjust their price each period. This does not seem an unrealistic assumption, at least if the model's time periods are sufficiently short. Even under hyper-inflation, it is still unlikely that firms adjust prices many times per day.

Π_t is bounded above in such a model because the price level remains finite even if adjusting firms set an infinite price, as all demand switches to non-adjusting firms. For example, the model of Fernández-Villaverde et al. (2015) contains the equation:

$$1 = \theta \Pi_t^{\varepsilon - 1} + (1 - \theta) \widetilde{\Pi}_t^{1-\varepsilon},$$

where $\widetilde{\Pi}_t$ is the relative price of adjusting firms and $\varepsilon > 1$. This equation comes from the definition of the aggregate price. As $\widetilde{\Pi}_t \to \infty$, $\Pi_t \to \theta^{-\frac{1}{\varepsilon - 1}} < \infty$, thus inflation is always bounded above, as required.

A.1 Uniqueness of the solution of a simple non-linear expectational difference equation

Let $\phi > 1$. We seek to prove that the non-linear expectational difference equation:

$$X_t^{\phi} = \mathbb{E}_t X_{t+1},$$

has a unique solution that is:

a) positive (i.e., $X_t > 0$ for all $t \in \mathbb{Z}$),

b) strictly stationary (so for example $\mathbb{E}X_t = \mathbb{E}X_s$ for all $t, s, \in \mathbb{Z}$),

c) and has bounded unconditional mean and log mean (i.e., $\mathbb{E}X_t < \infty$ and $|\mathbb{E} \log X_t| < \infty$ for all $t \in \mathbb{Z}$).

Clearly $X_t = 1$ is one such solution.

Let X_t be a solution to $X_t^{\phi} = \mathbb{E}_t X_{t+1}$ satisfying (a), (b) and (c) above. Let $x_t := \log X_t$. Then from taking logs, we have:

$$\phi x_t = \log \mathbb{E}_t \exp x_{t+1} \geq \log \exp \mathbb{E}_t x_{t+1} = \mathbb{E}_t x_{t+1},$$

by Jensen's inequality. Therefore, by the law of iterated expectations, for any $k \in \mathbb{N}$:

$$\phi^k x_t \geq \mathbb{E}_t x_{t+k} = \mathbb{E}_t x_{t+k}.$$

As $k \to \infty$, the left-hand side tends to either plus infinity (if $x_t > 0$), zero (if $x_t = 0$), or minus infinity (if $x_t < 0$). On the other hand, as $k \to \infty$, the right-hand side tends to $\mathbb{E}x_t > -\infty$, by stationarity. Thus, we must have that $x_t \geq 0$ for all $t \in \mathbb{Z}$, else this equation would be violated. Hence, $X_t \geq 1$ for all $t \in \mathbb{Z}$.

Now note that by stationarity, the law of iterated expectations and Jensen's inequality:

$$\mathbb{E}X_t = \mathbb{E}X_{t+1} = \mathbb{E}\mathbb{E}_t X_{t+1} = \mathbb{E}X_t^{\phi} \geq (\mathbb{E}X_t)^{\phi},$$

so $1 \geq (\mathbb{E}X_t)^{\phi-1}$, meaning $\mathbb{E}X_t \leq 1$. However, since $X_t \geq 1$ for all $t \in \mathbb{Z}$, the only way we can have that $\mathbb{E}X_t \leq 1$ is if in fact $X_t = 1$ for all $t \in \mathbb{Z}$.

Therefore, $X_t \equiv 1$ is the unique solution to the original expectational difference equation satisfying (a), (b) and (c) above.

A.2 Uniqueness of the solution of a more general non-linear difference equation

Let $\underline{\phi} \geq 1$ and let $(Z_t)_{t\in\mathbb{Z}}$ be a stochastic process satisfying the following conditions:

i) $Z_t > 0$, for all $t \in \mathbb{Z}$,

ii) $\mathbb{E}_t Z_{t+1} = 1$, for all $t \in \mathbb{Z}$,

iii) $(Z_t)_{t\in\mathbb{Z}}$ is strictly stationary,

iv) there exists $\overline{Z} \geq 1$, independent of the stochastic process $(X_t)_{t\in\mathbb{Z}}$ (to be introduced), such that for all $\phi > \underline{\phi}$, and for all $t \in \mathbb{Z}$ and all $k \in \mathbb{N}$ with $k > 0$, $\mathbb{E}_t Z_{t+k}^{\frac{\phi}{\phi-1}} \leq \overline{Z^{\frac{\phi}{\phi-1}}}$.

The larger is $\underline{\phi}$, the weaker is the moment boundedness assumptions (iv). For example, if $\underline{\phi} = 2$, then this just requires bounded second moments.

Let $\underline{X} \in (0,1)$ and let $\phi > \underline{\phi}$. We seek to prove that the non-linear expectational difference equation:

$$X_t^{\phi} = \mathbb{E}_t Z_{t+1} X_{t+1},$$

has a unique solution that is:

a) bounded below by $\underline{X}$ (so $X_t > \underline{X} > 0$ for all $t \in \mathbb{Z}$),

b) strictly stationary (so for example $\mathbb{E}X_t = \mathbb{E}X_s$ for all $t, s, \in \mathbb{Z}$),

c) and has bounded unconditional mean, ϕ^{th} mean and log mean (i.e., $\mathbb{E}X_t < \infty$, $\mathbb{E}X_t^{\phi} < \infty$ and $|\mathbb{E}\log X_t| < \infty$ for all $t \in \mathbb{Z}$).

Clearly $X_t = 1$ is one such solution. Note that Z_t may be a function of X_t and its history, so Z_t and X_t are not guaranteed to be independent. The previous subappendix covers the case with $Z_t \equiv 1$ in which slightly weaker assumptions are needed.

First note that for all $t \in \mathbb{Z}$:

$$\begin{aligned} 1 = \mathbb{E}_t Z_{t+1} &= \mathbb{E}_t[Z_{t+1}1] = \mathbb{E}_t\big[Z_{t+1}\mathbb{E}_{t+1}[Z_{t+2}1]\big] = \mathbb{E}_t\big[\mathbb{E}_{t+1}[Z_{t+1}Z_{t+2}1]\big] \\ &= \mathbb{E}_t[Z_{t+1}Z_{t+2}1] = \mathbb{E}_t\big[Z_{t+1}Z_{t+2}\mathbb{E}_{t+2}[Z_{t+3}1]\big] = \cdots \\ &= \mathbb{E}_t\left[\prod_{j=1}^{k} Z_{t+j}\right], \qquad \forall k \in \mathbb{N}, \end{aligned}$$

by assumption (ii) and the law of iterated expectations.

Now let $x_t := \log X_t$ and $\underline{x} := \log \underline{X}$. Then from taking logs, we have:

$$\phi x_t = \log \mathbb{E}_t Z_{t+1} \exp x_{t+1} \geq \log \exp \mathbb{E}_t Z_{t+1} x_{t+1} = \mathbb{E}_t Z_{t+1} x_{t+1},$$

by Jensen's inequality, as $\mathbb{E}_t[Z_{t+1} \times (\cdot)]$ defines a measure since $\mathbb{E}_t Z_{t+1} = 1$. Therefore, by the law of iterated expectations, for any $k \in \mathbb{N}$:

$$\phi^k x_t \geq \mathbb{E}_t \left[\prod_{j=1}^{k} Z_{t+j}\right] x_{t+k} \geq \mathbb{E}_t \left[\prod_{j=1}^{k} Z_{t+j}\right] \underline{x} = \underline{x} > -\infty,$$

by the result of the previous paragraph. As $k \to \infty$, the left-hand side tends to either plus infinity (if $x_t > 0$), zero (if $x_t = 0$), or minus infinity (if $x_t < 0$). Thus, we must have that $x_t \geq 0$ for all $t \in \mathbb{Z}$, else this equation would be violated. Hence, $X_t \geq 1$ for all $t \in \mathbb{Z}$.

Now, define $\overline{z} := \log \overline{Z}$, and for all $t \in \mathbb{Z}$ and all $k \in \mathbb{N}$ with $k > 0$ define:

$$\tilde{z}_{t,t+k} := \log \left[\mathbb{E}_t Z_{t+k}^{\frac{\phi}{\phi-1}}\right]^{\frac{\phi-1}{\phi}} < \overline{z},$$

by our assumptions (iv). Then by repeatedly applying Hölder's inequality:

$$\begin{aligned}
X_t^{\phi} = \mathbb{E}_t Z_{t+1} X_{t+1} &\leq \left[\mathbb{E}_t Z_{t+1}^{\frac{\phi}{\phi-1}}\right]^{\frac{\phi-1}{\phi}} \left[\mathbb{E}_t X_{t+1}^{\phi}\right]^{\frac{1}{\phi}} \\
&\leq \left[\mathbb{E}_t Z_{t+1}^{\frac{\phi}{\phi-1}}\right]^{\frac{\phi-1}{\phi}} \left[\mathbb{E}_t \left[\left[\mathbb{E}_{t+1} Z_{t+2}^{\frac{\phi}{\phi-1}}\right]^{\frac{\phi-1}{\phi}} \left[\mathbb{E}_{t+1} X_{t+2}^{\phi}\right]^{\frac{1}{\phi}}\right]\right]^{\frac{1}{\phi}} \\
&\leq \left[\mathbb{E}_t Z_{t+1}^{\frac{\phi}{\phi-1}}\right]^{\frac{\phi-1}{\phi}} \left[\mathbb{E}_t Z_{t+2}^{\frac{\phi}{\phi-1}}\right]^{\frac{\phi-1}{\phi^2}} \left[\mathbb{E}_t X_{t+2}^{\phi}\right]^{\frac{1}{\phi^2}} \\
&\leq \cdots \\
&\leq \prod_{j=1}^{k} \left[\mathbb{E}_t Z_{t+j}^{\frac{\phi}{\phi-1}}\right]^{\frac{\phi-1}{\phi^j}} \left[\mathbb{E}_t X_{t+k}^{\phi}\right]^{\frac{1}{\phi^k}},
\end{aligned}$$

for all $k \in \mathbb{N}$ with $k > 0$. Thus, from taking logs and limits:

$$x_t \leq \sum_{j=1}^{\infty} \phi^{-j} \tilde{z}_{t,t+j} + \frac{1}{\phi} \lim_{k\to\infty} \left[\phi^{-k} \log \mathbb{E}_t X_{t+k}^{\phi}\right] = \sum_{j=1}^{\infty} \phi^{-j} \tilde{z}_{t,t+j} \leq \frac{\overline{z}}{\phi - 1},$$

where the equality follows from the fact that by stationarity, $\lim_{k\to\infty} \mathbb{E}_t X_{t+k}^{\phi} = \mathbb{E}X_t^{\phi} < \infty$. Thus, $X_t \leq \overline{Z}^{\frac{1}{\phi-1}}$ for all $t \in \mathbb{Z}$. By assumption $\overline{Z}$ is not a function of ϕ, so as $\phi \to \infty$, this upper bound on X_t tends to 1. Hence, for large ϕ, $X_t \approx 1$, giving approximate uniqueness.

We can derive even stronger results in the case in which $\underline{\phi} = 1$ (in our assumptions) and one additional assumption holds. First note that with $\underline{\phi} = 1$, from taking limits as $\phi \to 1$ in assumption (iv), we must have that $Z_t \leq \overline{Z}$ with probability one (for all $t \in \mathbb{Z}$).

Let Z_t^* be the value that would be taken by Z_t if it were the case that $X_t = 1$ for all $t \in \mathbb{Z}$. So, it is also the case that $Z_t^* \leq \overline{Z}$ with probability one (for all $t \in \mathbb{Z}$), by our assumption (iv). Suppose further that there exists $\kappa \geq 0$ such that:

$$\mathbb{E}|Z_t - Z_t^*| \leq \kappa \mathbb{E}(X_t - 1).$$

This is reasonable, since if $X_t \to 1$ (almost surely), we expect that $Z_t \to Z_t^*$ (almost surely) as well.

Now note that:

$$\mathbb{E}(X_t - 1) = \mathbb{E}\left[(\mathbb{E}_t Z_{t+1} X_{t+1})^{\frac{1}{\phi}} - 1\right] \leq \mathbb{E}\left[\frac{1}{\phi}(\mathbb{E}_t Z_{t+1} X_{t+1} - 1)\right] = \frac{1}{\phi}[\mathbb{E}Z_t X_t - 1],$$

(using stationarity and the law of iterated expectations in the final equality). Thus:

$$\begin{aligned}
\mathbb{E}(X_t - 1) &= \mathbb{E}\left[(\mathbb{E}_t Z_{t+1} X_{t+1})^{\frac{1}{\phi}} - 1\right] \leq \mathbb{E}\left[\frac{1}{\phi}(\mathbb{E}_t Z_{t+1} X_{t+1} - 1)\right] = \frac{1}{\phi}[\mathbb{E}Z_t X_t - 1] \\
&= \frac{1}{\phi}[\mathbb{E}Z_t X_t - \mathbb{E}Z_t^*] = \frac{1}{\phi}[\mathbb{E}(Z_t - Z_t^*)X_t + \mathbb{E}Z_t^*(X_t - 1)] \\
&\leq \frac{1}{\phi}[\mathbb{E}|Z_t - Z_t^*|X_t + \mathbb{E}Z_t^*(X_t - 1)] \leq \frac{1}{\phi}\left[\kappa\mathbb{E}(X_t - 1)\overline{Z}^{\frac{1}{\phi-1}} + \overline{Z}\mathbb{E}(X_t - 1)\right] \\
&= \frac{1}{\phi}\left[\kappa\overline{Z}^{\frac{1}{\phi-1}} + \overline{Z}\right]\mathbb{E}(X_t - 1),
\end{aligned}$$

(from, respectively, the convexity of $y \mapsto y^{\frac{1}{\phi}}$, stationarity and the law of iterated expectations, the fact that $\mathbb{E}Z_t^* = 1$, algebra, that $y \leq |y|$, our bounds on X_t, $\mathbb{E}|Z_t - Z_t^*|$ and Z_t^*, and more algebra). As $\phi \to \infty$, $\kappa\overline{Z}^{\frac{1}{\phi-1}} + \overline{Z} \to \kappa + \overline{Z} < \infty$, so for large ϕ it must be the case that $\frac{1}{\phi}\left[\kappa\overline{Z}^{\frac{1}{\phi-1}} + \overline{Z}\right] < 1$. Hence if ϕ is large enough for this to hold, then

$\mathbb{E}(X_t - 1) \leq 0$. However, since $X_t \geq 1$ for all $t \in \mathbb{Z}$, the only way we can have that $\mathbb{E}X_t \leq 1$ is if in fact $X_t = 1$ for all $t \in \mathbb{Z}$.

Therefore, for large enough ϕ, $X_t \equiv 1$ is the unique solution to the original expectational difference equation satisfying (a), (b) and (c) above.

Appendix B Fiscal Theory of the Price Level (FTPL) results

B.1 Exact equilibria under active fiscal policy with geometric coupon debt and flexible prices

Suppose the representative household supplies one unit of labour, inelastically. Production of the final good is given by:

$$y_t = l_t (= 1).$$

In period 0, the representative household maximises:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to the budget constraint:

$$P_t c_t + A_t + Q_t B_t + P_t \tau_t = P_t y_t + I_{t-1} A_{t-1} + B_{t-1}(1 + \omega Q_t),$$

where c_t is consumption, τ_t are real lump sum taxes, P_t is the price of the final good, A_t is the number of one period nominal bonds purchased by the household at t, which each return I_t in period $t+1$, Q_t is the price of a long bond and B_t are the number of units of this long bond purchased by the household at t. One unit of the period t long bond bought at t returns \$1 at $t+1$, along with ω units of the period $t+1$ bond.

The household first order conditions imply:

$$1 = \beta I_t \mathbb{E}_t \frac{P_t c_t}{P_{t+1} c_{t+1}},$$
$$Q_t = \beta \mathbb{E}_t \frac{P_t c_t}{P_{t+1} c_{t+1}} (1 + \omega Q_{t+1}).$$

The household transversality conditions are that:

$$\lim_{t\to\infty} \beta^t \frac{A_t}{P_t c_t} = 0,$$
$$\lim_{t\to\infty} \beta^t \frac{Q_t B_t}{P_t c_t} = 0.$$

The government fixes taxes at a constant positive level:

$$\tau_t = \tau, \qquad \tau > 0.$$

The government issues no one period bonds, so:

$$A_t = 0.$$

The central bank pegs nominal interest rates at:

$$I_t = \beta^{-1}.$$

(We will discuss active monetary policy later.)

The final goods market clears, so:

$$y_t = c_t = 1.$$

Thus, from the household budget constraint, we have the following government budget constraint:

$$Q_t B_t + P_t \tau = B_{t-1}(1 + \omega Q_t).$$

We look for an equilibrium in which $P_t = P$ for all $t \geq 0$. We do not impose a priori that $P = P_{-1}$.

With $P_t = P$ for $t \geq 0$, the household Euler equations simplify to (respectively):

$$1 = \beta I_t,$$

$$Q_t = \beta \mathbb{E}_t (1 + \omega Q_{t+1}).$$

The former equation is consistent with the CB's peg of $I_t = \beta^{-1}$.

We consider the following solution to the latter equation:

$$Q_t = \frac{\beta}{1 - \beta\omega} + \left(Q_0 - \frac{\beta}{1 - \beta\omega} \right) (\beta\omega)^{-t}.$$

We wish to find Q_0, which is free to jump. There are three cases to consider:

Case 1: $Q_0 < \frac{\beta}{1-\beta\omega}$. Then Q_t eventually goes to zero (and then negative), which certainly cannot be consistent with a world in which $I_t > 0$. Thus, this case is ruled out.

Case 2: $Q_0 = \frac{\beta}{1-\beta\omega}$. Then Q_t is constant, and the government budget constraint becomes:

$$B_t = \beta^{-1} B_{t-1} - \beta^{-1}(1 - \beta\omega) P\tau.$$

Thus:

$$B_t = P\tau \frac{1 - \beta\omega}{1 - \beta} + \left(B_{-1} - P\tau \frac{1 - \beta\omega}{1 - \beta} \right) \beta^{-t-1}$$

So:

$$\beta^t \frac{Q_t B_t}{P_t c_t} = \frac{\beta}{1-\beta\omega}\frac{1}{P}\left[P\tau\frac{1-\beta\omega}{1-\beta}\beta^t + \left(B_{-1} - P\tau\frac{1-\beta\omega}{1-\beta}\right)\beta^{-1}\right]$$
$$\to \frac{1}{1-\beta\omega}\frac{1}{P}\left(B_{-1} - P\tau\frac{1-\beta\omega}{1-\beta}\right)$$

as $t \to \infty$.

Thus, from the transversality constraint:

$$P = \frac{B_{-1}}{\tau}\frac{1-\beta}{1-\beta\omega}.$$

This is the standard FTPL equilibrium. **Equilibrium type 1!**

Case 3: $Q_0 > \frac{\beta}{1-\beta\omega}$.

Define:

$$q_t := Q_t(\beta\omega)^t,$$
$$b_t := B_t\omega^{-t}.$$

Then the government budget constraint states:

$$b_t = \left(1 + \frac{(\beta\omega)^t}{\omega q_t}\right)b_{t-1} - \frac{\beta^t P\tau}{q_t},$$

and the transversality constraint states:

$$\frac{1}{P}\lim_{t\to\infty} q_t b_t = 0.$$

By our solution for q_t, we know that $q_t \to Q_0 - \frac{\beta}{1-\beta\omega} > 0$. Thus, the transversality condition requires:

$$\lim_{t\to\infty} b_t = 0.$$

Now define:

$$\hat{b}_t := \frac{b_t}{\prod_{k=0}^{t}\left(1 + \frac{(\beta\omega)^k}{\omega q_k}\right)},$$

with $\hat{b}_{-1} = b_{-1} = \omega B_{-1}$.

The denominator in the definition of $\hat{b}_t$ is greater than 1, so if $b_t \to 0$ as $t \to \infty$, then certainly $\hat{b}_t \to 0$. Likewise, if $\hat{b}_t \to 0$ as $t \to \infty$, then also $b_t \to 0$, since for all t:

$$\begin{aligned}
&\prod_{k=0}^{t}\left(1+\frac{(\beta\omega)^k}{\omega q_k}\right) \le \prod_{k=0}^{\infty}\left(1+\frac{(\beta\omega)^k}{\omega q_k}\right)\\
&=\prod_{k=0}^{\infty}\left(1+\frac{1-\beta\omega}{\beta\omega+\omega\big((1-\beta\omega)Q_0-\beta\big)(\beta\omega)^{-k}}\right)\\
&=\exp\sum_{k=0}^{\infty}\log\left(1+\frac{1-\beta\omega}{\beta\omega+\omega\big((1-\beta\omega)Q_0-\beta\big)(\beta\omega)^{-k}}\right)\\
&\le \exp\int_{-1}^{\infty}\log\left(1+\frac{1-\beta\omega}{\beta\omega+\omega\big((1-\beta\omega)Q_0-\beta\big)(\beta\omega)^{-k}}\right)\\
&=\frac{(1+\omega Q_0)(1-\beta\omega)}{\omega\big((1-\beta\omega)Q_0-\beta\big)}\exp\left[\frac{1}{\log(\beta\omega)}\left[\operatorname{dilog}\left(\frac{1+\beta\omega(1+\omega Q_0)}{\beta\omega(1+\omega Q_0)}\right)+\operatorname{dilog}(\beta\omega)\right.\right.\\
&\qquad\qquad\left.\left.-\operatorname{dilog}\left(\frac{1+\beta\omega(1+\omega Q_0)}{1+\omega Q_0}\right)\right]\right]\\
&<\infty,
\end{aligned}$$

where $\operatorname{dilog}(x) := \int_1^x \frac{\log(z)}{1-z}dz$ for all x is the dilogarithm function.

Now, substituting the definition of $\hat{b}_t$ into the law of motion for b_t gives:

$$\hat{b}_t = \hat{b}_{t-1} - \frac{\beta^t P\tau}{q_t \prod_{k=0}^{t}\left(1+\frac{(\beta\omega)^k}{\omega q_k}\right)},$$

so:

$$\begin{aligned}
\hat{b}_t &= \hat{b}_{-1} - P\tau\sum_{j=0}^{t}\frac{\beta^j}{q_j\prod_{k=0}^{j}\left(1+\frac{(\beta\omega)^k}{\omega q_k}\right)}\\
&= \hat{b}_{-1} - P\tau\sum_{j=0}^{t}\frac{\prod_{k=0}^{j}\beta\left(1+\frac{1-\beta\omega}{\beta\omega+(\omega(1-\beta\omega)Q_0-\beta\omega)(\beta\omega)^{-k}}\right)^{-1}}{\beta\left[\frac{\beta}{1-\beta\omega}(\beta\omega)^j+\left(Q_0-\frac{\beta}{1-\beta\omega}\right)\right]}.
\end{aligned}$$

Note that for $k \ge 0$:

$$1 < 1+\frac{1-\beta\omega}{\beta\omega+(\omega(1-\beta\omega)Q_0-\beta\omega)(\beta\omega)^{-k}} \le 1+\frac{1}{\omega Q_0} < \frac{1}{\beta\omega},$$

so:

$$(\beta^2\omega)^{j+1} < \prod_{k=0}^{j}\beta\left(1+\frac{1-\beta\omega}{\beta\omega+(\omega(1-\beta\omega)Q_0-\beta\omega)(\beta\omega)^{-k}}\right)^{-1} < \beta^{j+1}.$$

Thus, since the denominator within the sum is converging to $\beta\big(Q_0-\frac{\beta}{1-\beta\omega}\big)$ the sum is finite and has a finite limit as $t \to \infty$.

Hence, one equilibrium is for $Q_0 > \frac{\beta}{1-\beta\omega}$ to be arbitrary and for P to be given by:

$$P = \frac{\hat{b}_{-1}}{\tau \sum_{j=0}^{\infty} \frac{\prod_{k=0}^{j} \beta \left(1 + \frac{1-\beta\omega}{\beta\omega + (\omega(1-\beta\omega)Q_0 - \beta\omega)(\beta\omega)^{-k}}\right)^{-1}}{\beta\left[\frac{\beta}{1-\beta\omega}(\beta\omega)^j + \left(Q_0 - \frac{\beta}{1-\beta\omega}\right)\right]}}.$$

Equilibrium type 2!

Alternatively, suppose P is given. When can we solve the previous equation to find Q_0? As $Q_0 \to \frac{\beta}{1-\beta\omega}$, the right-hand side of the previous equation tends to:

$$\frac{\hat{b}_{-1}}{\tau\omega}\frac{1-\beta}{1-\beta\omega} = \frac{B_{-1}}{\tau}\frac{1-\beta}{1-\beta\omega}.$$

As $Q_0 \to \infty$, this right-hand side tends to ∞. Thus, by the intermediate value theorem, for any $P \in \left[\frac{B_{-1}}{\tau}\frac{1-\beta}{1-\beta\omega}, \infty\right)$, there is a Q_0 that satisfies the transversality constraint. Hence, inflation is unbounded about in the initial period.

Equilibrium type 3!

Therefore, the FTPL implies a lower bound on the price level, not an upper bound, and so with passive monetary policy, there are multiple equilibria.

Now suppose that monetary policy is active, with:

$$I_t = \beta^{-1}\Pi_t^{\phi},$$

with $\phi > 1$ and $\Pi_t := \frac{P_t}{P_{t-1}}$. β^{-1} is the real interest rate in this model, so this is a non-linear real rate rule. Given that $c_t = 1$, the Euler equation for one period bonds implies the nonlinear Fisher equation:

$$1 = \beta I_t \mathbb{E}_t \frac{1}{\Pi_{t+1}},$$

so, for $t \geq 0$:

$$\mathbb{E}_t \frac{1}{\Pi_{t+1}} = \left(\frac{1}{\Pi_t}\right)^{\phi}.$$

$\Pi_t = 1$ is the unique stationary solution to this equation, by the results of Appendix A.1 (with $X_t := \frac{1}{\Pi_t}$). In this candidate equilibrium, $I_t = \beta^{-1}$, so Π_t and I_t have the same time series as under the passive policy in the special case in which $P = P_{-1}$.

Consequently, if $P_{-1} \geq \frac{B_{-1}}{\tau} \frac{1-\beta}{1-\beta\omega}$ then by the above results, there exists a Q_0 under which all equilibrium conditions and transversality conditions are satisfied. Thus, even with active monetary and active fiscal policy, there is still a stable equilibrium for inflation and real variables.

B.2 Linearised equilibria under active fiscal policy with geometric coupon debt and sticky prices

We just give the linearised equations of the model. These follow equations 5.17 to 5.21 of Cochrane (2022). All shocks (variables of the form $\varepsilon_{\cdot,t}$) are assumed to be mean zero and independent, both across time and across shocks.

Euler:

$$x_t = \mathbb{E}_t x_{t+1} - \sigma r_t.$$

Phillips:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t.$$

Fisher:

$$i_t = r_t + \mathbb{E}_t \pi_{t+1}.$$

Robust real rate rule:

$$i_t = r_t + \phi \pi_t + \varepsilon_{i,t}.$$

Exogenous real government surplus:

$$s_t = \varepsilon_{s,t}.$$

Debt evolution (v_t is the value of debt to GDP, e_t is the ex-post nominal return on government debt):

$$\rho v_t = v_{t-1} + e_t - \pi_t - s_t.$$

Equal returns:

$$\mathbb{E}_t e_{t+1} = i_t.$$

Bond pricing (ω controls the maturity structure. $\omega = 0$ is one period debt, $\omega = 1$ is a perpetuity):

$$e_t = \omega q_t - q_{t-1}.$$

We assume that $\omega > 0$. Then for any $\phi \neq 0$, the following solves these linear expectational difference equations:

$$\pi_t = -\frac{\varepsilon_{i,t}}{\phi}, \quad x_t = -\frac{\varepsilon_{i,t}}{\kappa\phi},$$
$$r_t = \frac{\varepsilon_{i,t}}{\sigma\kappa\phi}, \quad v_t = -\frac{\varepsilon_{i,t}}{\sigma\kappa\phi},$$
$$e_t = \varepsilon_{s,t} - \left(\frac{\rho}{\sigma\kappa\phi} + \frac{1}{\phi}\right)\varepsilon_{i,t} + \frac{\varepsilon_{i,t-1}}{\sigma\kappa\phi},$$
$$q_t = \frac{1}{\omega}\left[q_{t-1} + \varepsilon_{\mathrm{s},t} - \left(\frac{\rho}{\sigma\kappa\phi} + \frac{1}{\phi}\right)\varepsilon_{i,t} + \frac{\varepsilon_{i,t-1}}{\sigma\kappa\phi}\right].$$

As in the non-linear, flexible price case, the bond price is exploding. However, the real value of government debt remains stationary, which is sufficient for the transversality constraint to be satisfied. Inflation and all real variables are also stationary. Thus, if monetary policy is passive ($\phi \in (0,1)$), then the linearised model has multiple valid equilibria, this one, and the standard "FTPL" one in which q_t is stationary (see Cochrane (2022)). Conversely, if monetary policy is active ($\phi > 1$), then the model possesses a valid equilibrium with stationary inflation and real variables.

Appendix C Welfare in New Keynesian models

In Subsection 1.4, we established that a rule of our form could exactly mimic any other time invariant policy, if responses to structural shocks and their lags are allowed. Thus, rules of our form can mimic unconditionally optimal policy, optimal commitment policy from a timeless perspective, or optimal discretionary policy. Hence, rules of our form can achieve high welfare.

We begin this section by looking at unconditionally optimal time-invariant policy using our rules, in a simple NK model. We then go on to analyse the performance of our rules if further restrictions are placed upon them, such as only permitting the central bank to respond to current or sufficiently recent shocks. We show that optimal policy in estimated models of the US economy comes close to stabilizing inflation, with optimal inflation dynamics describable by an ARMA process with few MA terms.

C.1 Welfare in the basic three equation model

Any welfare analysis requires us to specify the rest of the model, as welfare is generally a function of output's variability, not just that of inflation. Thus, as a first example suppose that inflation and output are linked by the standard Phillips curve:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + \kappa \omega_t,$$

where x_t is the output gap, and ω_t is a mark-up shock, which is assumed IID with mean zero. Additionally, suppose that the policy maker wants to minimise the unconditional expectation of a quadratic loss function in inflation and the output gap. I.e., the period t policy maker minimises:

$$(1-\beta)\mathbb{E}\sum_{k=0}^{\infty}\beta^k\left(\pi_{t+k}^2+\lambda x_{t+k}^2\right),$$

for some $\lambda > 0$ and $\beta \in (0,1)$.

We suppose that the policy maker is constrained to choose a time-invariant (i.e., stationary) policy, thus the objective simplifies to:[32]

$$\mathbb{E}\left(\pi_t^2 + \lambda x_t^2\right).$$

As the policy maker only cares about inflation and output gaps, with the former being effectively under their control, and the latter only determined by inflation and mark-up shocks, the optimal policy must have the form:

$$\pi_t = \kappa \sum_{k=0}^{\infty} \theta_k \omega_{t-k},$$

for some $\theta_0, \theta_1, \dots$ to be determined. We have already shown that such a policy may be determinately implemented via a rule of the form of (6).

Substituting this policy into the Phillips curve then gives:

$$\sum_{k=0}^{\infty} \theta_k \omega_{t-k} = \beta \sum_{k=0}^{\infty} \theta_{k+1} \omega_{t-k} + x_t + \omega_t,$$

so:

$$x_t = \sum_{k=0}^{\infty} \left(\theta_k - \beta\theta_{k+1} - \mathbb{1}[k=0]\right)\omega_{t-k}.$$

Hence, the policy maker's objective is to choose $\theta_0, \theta_1, \dots$ to minimise:

$$\mathbb{E}\left(\pi_t^2 + \lambda x_t^2\right) = \mathbb{E}\left[\omega_t^2\right] \sum_{k=0}^{\infty} \left[\kappa^2 \theta_k^2 + \lambda\left(\theta_k - \beta\theta_{k+1} - \mathbb{1}[k=0]\right)^2\right].$$

The first order conditions then give:[33]

$$\theta_0 + \frac{\lambda}{\kappa^2}\left(\theta_0 - \beta\theta_1 - 1\right) = 0,$$

$$\theta_1 + \frac{\lambda}{\kappa^2}\left(\theta_1 - \beta\theta_2\right) - \beta\frac{\lambda}{\kappa^2}\left(\theta_0 - \beta\theta_1 - 1\right) = 0,$$

$$\forall k > 1, \qquad \theta_k + \frac{\lambda}{\kappa^2}\left(\theta_k - \beta\theta_{k+1}\right) - \beta\frac{\lambda}{\kappa^2}\left(\theta_{k-1} - \beta\theta_k\right) = 0.$$

Unsurprisingly, this agrees with the unconditionally optimal solution given in the prior literature (e.g. Damjanovic, Damjanovic & Nolan (2008)), which satisfies:

$$\pi_t + \frac{\lambda}{\kappa}\left(x_t - \beta x_{t-1}\right) = 0,$$

[32] See e.g. Damjanovic, Damjanovic & Nolan (2008).

[33] See Appendix E.8 for the solution of these conditions.

i.e.:

$$\kappa \sum_{k=0}^{\infty} \theta_k \omega_{t-k} + \frac{\lambda}{\kappa}\Bigg[\sum_{k=0}^{\infty}(\theta_k - \beta\theta_{k+1} - \mathbb{1}[k=0])\omega_{t-k} \\ - \beta \sum_{k=1}^{\infty}(\theta_{k-1} - \beta\theta_k - \mathbb{1}[k-1=0])\omega_{t-k}\Bigg] = 0.$$

To see the equivalence, note that from matching coefficients, this equation holds if and only if the above first order conditions hold. We will present a convenient representation of the solution to these equations below.

Additionally, note that as $\frac{\lambda}{\kappa^2} \to 0$, $\theta_k \to 0$ for all $k \in \mathbb{N}$. In other words, if the central bank does not care about the output gap, then they optimally choose to have constant inflation, i.e., to follow the rule from equation (2). The central bank also chooses constant inflation if the Phillips curve is vertical (i.e., $\kappa = \pm\infty$). In this case, neither inflation nor mark-up shocks have any impact on the output gap.

C.2 Optimal policy under limited central bank memory

The first order conditions derived above also enable us to easily solve for optimal unconditional policy under limited memory. For example, if the central bank does not "remember" $\omega_{t-1}, \omega_{t-2}, \ldots$, so uses a rule that is only a function of ω_t at t, then the optimal θ_0 will satisfy the above first order conditions with $\theta_1 = \theta_2 = \cdots = 0$. This means:

$$\theta_0 + \frac{\lambda}{\kappa^2}(\theta_0 - 1) = 0,$$

so $\theta_0 = \frac{\lambda}{\lambda+\kappa^2}$. It turns out that this exactly coincides with the solution under discretion.[34]

If the central bank can "remember" ω_{t-1}, so π_t is an MA(1), then the optimal solution will have:

$$\theta_0 + \frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$
$$\theta_1 + \frac{\lambda}{\kappa^2}\theta_1 - \beta\frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0.$$

[34] See Appendix E.9.

The solution has $\theta_0 \geq 0$ and $\theta_1 \leq 0$. Thus, the shock increases π_t while reducing $\mathbb{E}_t\pi_{t+1}$, thus dampening the required movement in x_t, from the Phillips curve. We will see that this is already enough to come close to the fully optimal policy.

Going one step further, if the central bank can also "remember" π_{t-1}, then they can choose interest rates to ensure π_t follows the ARMA(1,1) process:

$$\pi_t = \rho\pi_{t-1} + \kappa\theta_0\omega_t + \kappa\theta_1\omega_{t-1},$$

for some ρ, θ_0, θ_1 to be determined.[35] Since US inflation appears to be well approximated by an ARMA(1,1) (Stock & Watson 2009), this may be a reasonable model of Fed behaviour. This ARMA(1,1) process has the MA(∞) representation:

$$\pi_t = \kappa\theta_0\sum_{k=0}^{\infty}\rho^k\omega_{t-k} + \kappa\theta_1\sum_{k=0}^{\infty}\rho^k\omega_{t-1-k} = \kappa\theta_0\omega_t + \kappa(\rho\theta_0 + \theta_1)\sum_{k=1}^{\infty}\rho^{k-1}\omega_{t-k}. \qquad (9)$$

Substituting this policy into the Phillips curve gives:

$$\theta_0\omega_t + (\rho\theta_0 + \theta_1)\sum_{k=1}^{\infty}\rho^{k-1}\omega_{t-k} = \beta(\rho\theta_0 + \theta_1)\omega_t + \beta(\rho\theta_0 + \theta_1)\sum_{k=1}^{\infty}\rho^k\omega_{t-k} + x_t + \omega_t,$$

meaning:

$$x_t = [(1-\beta\rho)\theta_0 - \beta\theta_1 - 1]\omega_t + (1-\beta\rho)(\rho\theta_0 + \theta_1)\sum_{k=1}^{\infty}\rho^{k-1}\omega_{t-k}.$$

Hence, the policy maker's objective is to choose ρ, θ_0, θ_1 to minimise:

$$\begin{aligned}\mathbb{E}\left(\pi_t^2 + \lambda x_t^2\right) = \mathbb{E}[\omega_t^2]\Bigg[&\kappa^2\theta_0^2 + \lambda[(1-\beta\rho)\theta_0 - \beta\theta_1 - 1]^2 \\ &+ [\kappa^2(\rho\theta_0 + \theta_1)^2 + \lambda(1-\beta\rho)^2(\rho\theta_0 + \theta_1)^2]\frac{1}{1-\rho^2}\Bigg].\end{aligned}$$

Tedious algebra gives that the first order conditions have solution:[36]

$$\rho = \frac{\kappa^2 + (1+\beta^2)\lambda - \sqrt{(\kappa^2 + (1-\beta)^2\lambda)(\kappa^2 + (1+\beta)^2\lambda)}}{2\beta\lambda}, \qquad \theta_0 = \frac{\rho}{\beta}, \qquad \theta_1 = -\rho.$$

[35] The targeted inflation can respond to lagged targeted inflation without changing the determinacy properties of realised inflation (always equal to targeted inflation in equilibrium). Targeted inflation cannot respond to other endogenous variables without potentially changing these determinacy properties.

[36] There is an additional solution to the first order condition with $\rho = \frac{\kappa^2+(1+\beta^2)\lambda+\sqrt{(\kappa^2+(1-\beta)^2\lambda)(\kappa^2+(1+\beta)^2\lambda)}}{2\beta\lambda}$, but this is outside of the unit circle as: $\frac{\kappa^2+(1+\beta^2)\lambda+\sqrt{(\kappa^2+(1-\beta)^2\lambda)(\kappa^2+(1+\beta)^2\lambda)}}{2\beta\lambda} > \frac{\kappa^2+(1+\beta^2)\lambda+\sqrt{(\kappa^2+(1-\beta)^2\lambda)(\kappa^2+(1-\beta)^2\lambda)}}{2\beta\lambda} = \frac{\kappa^2+(1-\beta+\beta^2)\lambda}{\beta\lambda} > \frac{1-\beta+\beta^2}{\beta} = \frac{1}{\beta} + \beta - 1 > 1$. However, the given solution is inside the unit circle as

As $\lambda \to 0$, or $\kappa \to \infty$, $\rho \to 0$. As $\lambda \to \infty$, or $\kappa \to 0$, $\rho \to \beta$. Since there is no other solution for κ to the equation $\rho = \beta$ than $\kappa = 0$, we must have $\rho \le \beta$, so $\rho\theta_0 + \theta_1 \le 0$, meaning that the response of inflation to a positive mark-up shock is again negative after the first period. Since we have one extra degree of freedom, this must attain even higher welfare than the MA(1) solution. In fact, it attains the unconditionally optimal solution. Examination of the unconditionally optimal solution from Appendix E.8 reveals that it has the same form as equation (9), thus by a revealed preference argument, the two solutions must coincide. (For example, the solution for ρ agrees with the geometric decay rate of the MA coefficients at lags beyond the first of the fully optimal solution we found in Appendix E.8.)

Hence, in a world in which the only inefficient shocks are IID cost-push shocks, the central bank can attain the unconditionally optimal welfare by ensuring inflation follows an appropriate ARMA(1,1) process. This process will have an MA coefficient equal to $-\beta \approx -0.99$, and as long as the central bank cares about output stabilisation, it will have a high degree of persistence. This is very close to the IMA(1,1) processes estimated by Dotsey, Fujita & Stark (2018) for the post-1984 period.

To see the welfare attained by the other policies we have discussed, Figure 1 plots the policy frontiers attained by varying λ for each of the polices. In all cases, we follow Eggertsson & Woodford (2003) in setting $\beta = 0.99$ and $\kappa = 0.02$. The figure makes clear that the MA(1) policy (green) is a substantial improvement on the MA(0) (discretionary) policy (red). It also shows just how close Woodford's timeless perspective (1999)[37] (blue, hidden behind purple) comes to the unconditionally optimal policy.

$$\frac{\kappa^2+(1+\beta^2)\lambda-\sqrt{(\kappa^2+(1-\beta)^2\lambda)(\kappa^2+(1+\beta)^2\lambda)}}{2\beta\lambda} > \frac{\kappa^2+(1+\beta^2)\lambda-\sqrt{(\kappa^2+(1+\beta)^2\lambda)(\kappa^2+(1+\beta)^2\lambda)}}{2\beta\lambda} = -1,$$
$$\frac{\kappa^2+(1+\beta^2)\lambda-\sqrt{(\kappa^2+(1-\beta)^2\lambda)(\kappa^2+(1+\beta)^2\lambda)}}{2\beta\lambda} < \frac{\kappa^2+(1+\beta^2)\lambda-\sqrt{(\kappa^2+(1-\beta)^2\lambda)(\kappa^2+(1-\beta)^2\lambda)}}{2\beta\lambda} = 1.$$

[37] See Appendix E.10 for the derivation of this solution.

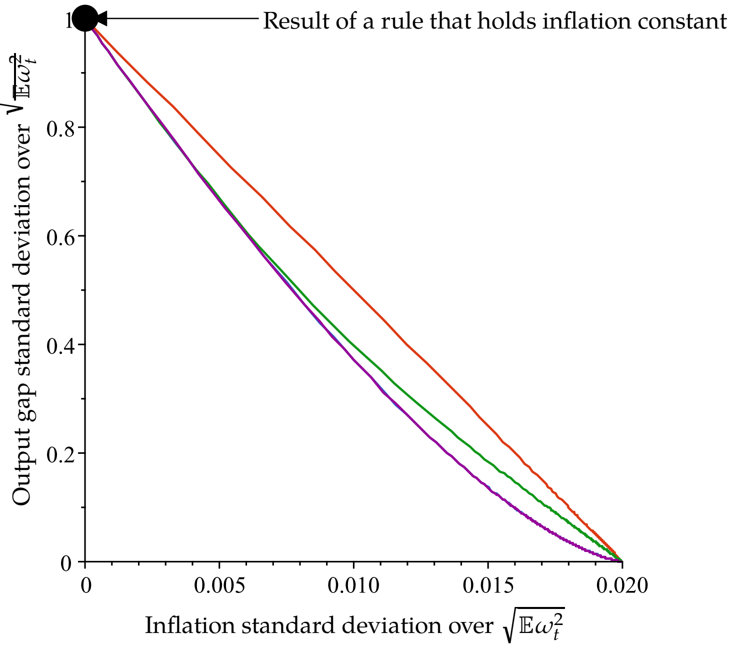

Figure 1: Policy frontiers (values attained by varying λ). $\beta = 0.99$, $\kappa = 0.02$.

Purple: Unconditionally optimal policy, equivalent to ARMA$(1,1)$ policy.

Blue (hidden behind purple): Timeless optimal solution.

Red: Policy just responding to current shocks, equivalent to discretion.

Green: Policy that responds to current and once lagged shocks.

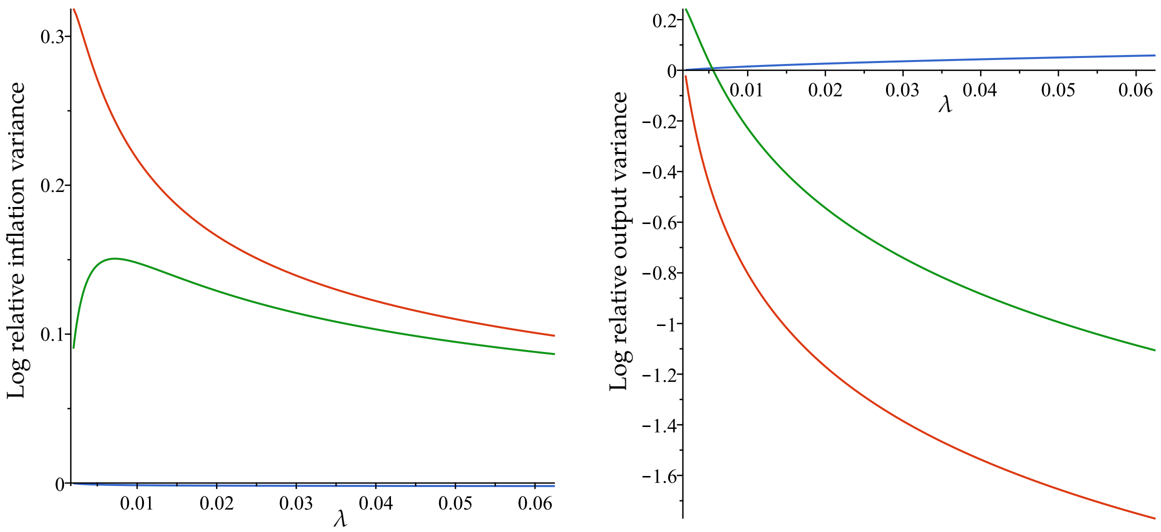

Figure 2: Logarithms of ratios of variance under a given policy to variance under unconditionally optimal policy. $\beta = 0.99$, $\kappa = 0.02$.

Blue: Timeless optimal solution.

Red: Policy just responding to current shocks, equivalent to discretion.

Green: Policy that responds to current and once lagged shocks.

Figure 2 shows how these differences across policies are driven by λ, by plotting the logarithm of the ratio of variance under a given policy to the variance under unconditionally optimal policy. We allow λ to vary from 0.002 (the value obtained by a second order approximation to the consumer's utility with $\kappa = 0.02$, if the elasticity of substitution across goods equals 10) to $\frac{1}{16}$ (corresponding to an equal weight on annual inflation and the output gap). Both the $\mathrm{MA}(0)$ and the $\mathrm{MA}(1)$ policy generate too much inflation variance and too little variance in output, relative to the unconditionally optimal solution. However, if the central bank can feasibly respond to ω_t and ω_{t-1} they can probably also respond to π_{t-1}, which is enough to deliver the unconditional optimum.

C.3 Welfare in larger NK models

Even in larger models, optimal inflation dynamics appear to be well approximated by an ARMA process with relatively few MA terms. Figure 3 shows the dynamics of observed and optimal inflation in the Justiniano, Primiceri & Tambalotti (2013) model. (This is a medium-scale New Keynesian DSGE model broadly similar to the model of Smets & Wouters (2007).) While actual inflation is highly persistent, with the same shocks hitting the economy, optimal inflation is far less persistent, with the sample autocorrelation essentially insignificant at 95% after four lags.

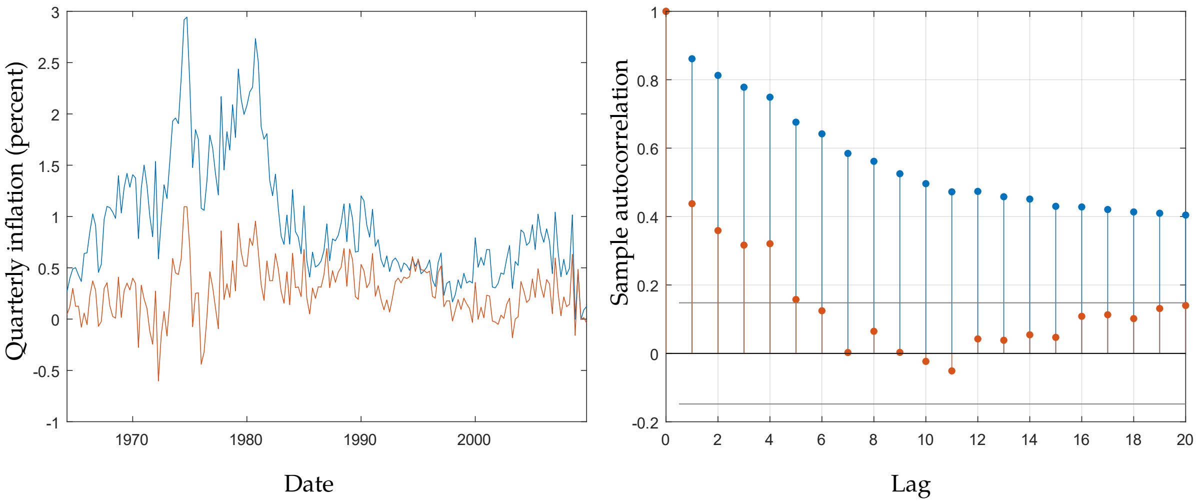

Figure 3: Behaviour of realised inflation (blue) and optimal inflation (red) in the Justiniano, Primiceri & Tambalotti (2013) model.

Left panel shows the timeseries. Right panel shows their sample autocorrelation.

Note that for any $\rho \in (-1,1)$, the solution for optimal inflation has a multiple shock, ARMA$(1,\infty)$ representation of the form:

$$\pi_t - \pi = \rho(\pi_{t-1} - \pi) + \sum_{k=0}^{\infty} \sum_{n=1}^{N} \theta_{n,k}^{(\rho)} \varepsilon_{n,t-k},$$

where $\varepsilon_{1,t}, \dots, \varepsilon_{N,t}$ are the model's structural shocks. We can approximate this process by truncating the MA terms at some point, e.g. by considering the multiple shock ARMA$(1,K)$ process:

$$\pi_t^{(K)} - \pi = \rho\left(\pi_{t-1}^{(K)} - \pi\right) + \sum_{k=0}^{K} \sum_{n=1}^{N} \theta_{n,k}^{(\rho)} \varepsilon_{n,t-k}.$$

In Figure 4 we plot the proportion of the variance of optimal inflation that is explained by this truncated process for $K = 0, \dots, 16$, and $\rho \in \{0, 0.61\}$.[38] A multiple shock ARMA(1,1) process already explains over 90% of the variance of optimal inflation, while a multiple shock ARMA(1,2) explains over 95%. Thus, optimal inflation in plausible models can be well approximated by relatively simple inflation dynamics.

[38] $\rho = 0.61$ is the value of ρ that minimises the variance of $\sum_{k=0}^{\infty} \sum_{n=1}^{N} \theta_{n,k}^{(\rho)} \varepsilon_{n,t-k}$. I.e. it is the value of ρ that would be estimated by OLS using an infinite sample of observations from optimal inflation.

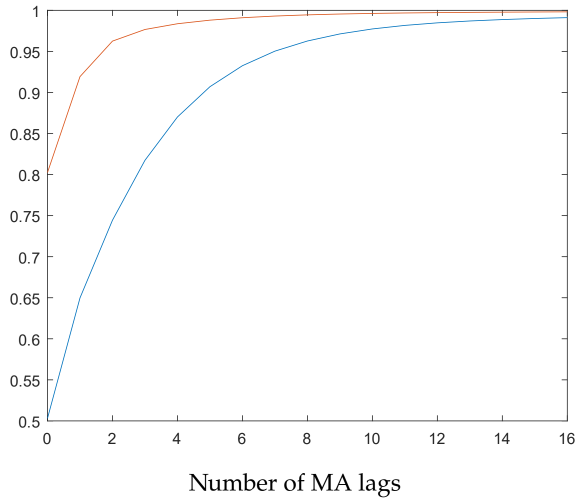

Figure 4: Proportion of the variance of optimal inflation in the Justiniano, Primiceri & Tambalotti (2013) model explained by truncating the number of MA lags. Blue: $\rho = 0$. Red: $\rho = 0.61$.

Appendix D Standing offers to buy/sell perpetuities for equilibrium selection

The possibility of trading perpetuities can be a powerful tool for central banks, even if they do not wish to use them for day-to-day policy. Nominal perpetuity prices are functions of the entire expected future path of nominal rates, and hence they embed information on the economy's selected equilibrium. If the economy is in an explosive inflation equilibrium, then nominal rates will also be exploding, so perpetuity prices will be extremely low. Conversely, if the economy is stuck at the ZLB, then perpetuity prices will be extremely high, or even infinite. This suggests that standing offers to buy perpetuities at a low price could rule out extreme hyperinflations, while standing offers to sell perpetuities at a high price could rule out becoming permanently stuck at the ZLB. This gives a route to answer Cochrane's (2011) argument that there is nothing to rule out non-stationary equilibria under monetary rules satisfying the Taylor-principle.

D.1 Central bank behaviour

We suppose that the central bank uses a real rate rule like equation (7) to set one period nominal rates, but subject to the ZLB, meaning they set:

$$i_t = \max\{0, r_t + \mathbb{E}_t \pi^*_{t+1} + \phi(\pi_t - \pi^*_t)\}.$$

They also maintain a standing offer to buy (nominal) perpetuities at a price $Q \exp b_t$ and a standing offer to sell such perpetuities at a price $Q \exp s_t$, where $b_t \leq s_t$ and where Q is as defined in Subsection 4.3. The central bank hopes to achieve an equilibrium in which $\pi_t = \pi^*_t$ whenever $r_t + \mathbb{E}_t \pi^*_{t+1} \geq 0$.[39] (When $r_t + \mathbb{E}_t \pi^*_{t+1} < 0$, then if $\pi_t = \pi^*_t$, the ZLB binds, so by the Fisher equation, $\mathbb{E}_t \pi_{t+1} = -r_t$ which will not in general equal $\mathbb{E}_t \pi^*_{t+1}$. Thus, generically, inflation must be away from target in either period t or period $t+1$.)

[39] There may be multiple equilibria in which $\pi_t = \pi^*_t$ whenever $r_t + \mathbb{E}_t \pi^*_{t+1} \geq 0$, as r_t is endogenous. We assume the central bank has picked one of these.

We write i_t^* for the nominal interest rate in the desired equilibrium at t, r_t^* for the real interest rate in the desired equilibrium at t, and we define $q_{I,t}^*$ to be such that $Q \exp q_{I,t}^*$ is the perpetuity price in the desired equilibrium at t. Now, from combining the log-linearized pricing equation for nominal perpetuities from Subsection 4.3 with a log-linearization of the nominal bond pricing equation, we have that:

$$q_{I,t} = -(i_t - i^*) + \exp(-i^*)\, \mathbb{E}_t q_{I,t+1},$$

where i^* is the steady state of i_t^*. Hence, the desired nominal perpetuity price satisfies:

$$q_{I,t}^* = -(i_t^* - i^*) + \exp(-i^*)\, \mathbb{E}_t q_{I,t+1}^*.$$

Given their knowledge of the economy, the central bank should be able to calculate an approximation to $q_{I,t}^*$.[40] We write $\tilde{q}_{I,t}^*$ for their estimate of this quantity. We assume that there is some value $\chi \geq 0$ such that $\tilde{q}_{I,t}^*$ is always within χ of the true value, $q_{I,t}^*$, for all t. Since χ may be large, this is a mild assumption. We then suppose that the central bank sets $b_t = \tilde{q}_{I,t}^* - \chi$ and $s_t = \tilde{q}_{I,t}^* + \chi$. Thus, $b_t \leq q_{I,t}^* \leq s_t$, and hence the central bank's offer to buy at $Q \exp b_t$ and sell at $Q \exp s_t$ are not inconsistent with the desired equilibrium perpetuity price of $Q \exp q_{I,t}^*$. We assume that π_t^* and r_t^* are stationary with finite unconditional means, so that $q_{I,t}^*$, b_t and s_t are also stationary with finite unconditional means.

D.2 Equilibria

The central bank's buy and sell offers will ensure that it is always the case that $b_t \leq q_{I,t} \leq s_t$. However, if $\chi > 0$, so $b_t < s_t$, then it is still possible for there to be multiple equilibria consistent with this. For example, an equilibrium in which the economy is at the bound for the next ten periods may result in quite similar perpetuity prices to one in which it is at the bound for eleven periods. Hence, for large χ, both of these equilibria may be consistent with $q_{I,t}$ lying between b_t and s_t. This could be ruled out

[40] Getting the precise value may be difficult since $q_{I,t}^*$ depends on the path of real interest rates under the desired equilibrium, which is only calculable from a full model of the economy.

either via switching to a price level real rate rule, giving a unique transition path back to steady state, or by the central bank improving their estimate of $q^*_{I,t}$, reducing χ.

Crucially though, even with large χ, these buy and sell offers are sufficient to rule out permanent ZLB traps and explosive inflation. We show that if these buy and sell offers are maintained, then equilibria with permanent ZLB traps or explosive inflation are impossible. Suppose without loss of generality that we are currently in period 0. In an explosive equilibrium, $\mathbb{E}_0(\pi_t - \pi^*_t) = \phi^t(\pi_0 - \pi^*_0) > 0$. Thus, with r_t at least asymptotically bounded in mean,[41] there exists $\kappa > 0$ such that $\mathbb{E}_0(i_t - i^*) \ge \kappa\phi^t$ for large t. Thus, for large t:

$$\mathbb{E}_0 q_{I,t} = -\mathbb{E}_0(i_t - i^*) + \exp(-i^*)\,\mathbb{E}_0 q_{I,t+1} \le -\kappa\phi^t + \exp(-i^*)\,\mathbb{E}_0 s_{t+1}.$$

Hence, as $t \to \infty$, $\mathbb{E}_0 q_{I,t} \to -\infty$, since $\mathbb{E}_0 s_{t+1} \to \mathbb{E} s_t < \infty$ as $t \to \infty$. This gives a contradiction from the fact that $\mathbb{E}_0 q_{I,t} \ge \mathbb{E}_0 b_t \to \mathbb{E} b_t > -\infty$ as $t \to \infty$. Likewise, if $\sup_{s \ge t} i_s \to 0$, almost surely conditional on period 0 information, as $t \to \infty$, then $\mathbb{E}_0 Q_{I,t} \to \infty$ as $t \to \infty$, so $\mathbb{E}_0 q_{I,t} \to \infty$ as well, giving the required contradiction from $\mathbb{E}_0 q_{I,t} \le \mathbb{E}_0 s_t \to \mathbb{E} s_t < \infty$.[42] Therefore, equilibria with explosive inflation or permanent ZLB spells are impossible if the central bank's buy and sell offers are maintained. At all times, the desired equilibrium is consistent with these standing offers though, since inflation can always jump back to its target, no matter the history.

The desired equilibrium acquires additional robustness from the fact that if price setters choose a non-equilibrium level for π_t, then goods markets will not clear, as argued in the paper's introduction. So, as long as financial market participants believe in the central bank's commitment to the perpetuity price bounds, if firms choose market clearing prices, then we will have $\pi_t = \pi^*_t$ whenever $r^*_t + \mathbb{E}_t \pi^*_{t+1} \ge 0$.

[41] This is plausible if we think prices and wages become flexible for sufficiently high inflation, for example.

[42] Under the log-linearized approximation, we only have that $\mathbb{E}_0 q_{I,t} \ge (1 - \exp(-i^*))^{-1}$, which is large but finite. In practice, χ should be far less than $(1 - \exp(-i^*))^{-1}$, so our results would also apply to the approximation.

D.3 Further discussion

Cochrane (2011) reviews assorted prior proposals to rule out undesirable equilibria via central bank interventions in other markets. He argues that a common problem is that they only rule out equilibria by ensuring that the two assets targeted by the central bank have mutually inconsistent prices in some states. It seems reasonable to be sceptical of such approaches if the central bank ever has to set inconsistent prices in equilibrium, which is not the case for our approach.

However, out of equilibrium, there is no fundamental problem with specifying that the central bank should *try* to set mutually inconsistent prices, even if they are doomed to fail. This would happen in our context if a moment of collective insanity led price setters to choose an extremely high value of π_t, despite the fact that goods markets will not clear in this case. Then the central bank would try to acquire an infinite long position in perpetuities and/or an infinite short position in nominal bonds. Of course, ex ante this would be crazy behaviour from the central bank and would surely lead to a crisis. But there is a big difference between how you would respond to someone threatening to blow themselves up if you do not do what they want, and how you would respond to someone who has already wired the explosive to go off if you do the wrong thing. With the standing offers to buy and sell perpetuities institutionalised before the out of equilibrium event materialised, these offers would acquire sufficient credibility to rule out the undesirable equilibria.

Appendix E Proofs and supplemental results

E.1 Phillips curve based forecasting with ARMA(1,1) policy shocks

As before, we have the monetary rule:

$$i_t = r_t + \phi\pi_t + \zeta_t,$$

which combined with the Fisher equation gives:

$$\mathbb{E}_t\pi_{t+1} = \phi\pi_t + \zeta_t.$$

Suppose ζ_t follows the ARMA(1,1) process:

$$\zeta_t = \rho_\zeta\zeta_{t-1} + \varepsilon_{\zeta,t} + \theta_\zeta\varepsilon_{\zeta,t-1}, \qquad \varepsilon_{\zeta,t} \sim N(0,\sigma_\zeta^2)$$

with $\rho_\zeta, \theta_\zeta \in (-1,1)$. Then from matching coefficients, with $\phi > 1$ we have the unique solution:

$$\pi_t = -\frac{1}{\phi - \rho_\zeta}\left[\zeta_t + \frac{\theta_\zeta}{\phi}\varepsilon_{\zeta,t}\right].$$

Thus:

$$\pi_t - \rho_\zeta\pi_{t-1} = -\frac{1}{\phi - \rho_\zeta}\left(1 + \frac{\theta_\zeta}{\phi}\right)\left[\varepsilon_{\zeta,t} + \frac{\phi - \rho_\zeta}{\phi + \theta_\zeta}\theta_\zeta\varepsilon_{\zeta,t-1}\right],$$

so π_t also follows an ARMA(1,1) process. Suppose for now that $-\rho_\zeta \le \theta_\zeta$, which is likely to be satisfied in reality as we expect ρ_ζ to be large and positive, while θ_ζ should be close to zero. (For example, Dotsey, Fujita & Stark (2018) find that an IMA(1,1) model fits inflation well, in which case $-\rho_\zeta = -1 < \theta_\zeta$ as required.) Then $0 < \frac{\phi-\rho_\zeta}{\phi+\theta_\zeta} < 1$, so $\left|\frac{\phi-\rho_\zeta}{\phi+\theta_\zeta}\theta_\zeta\right| < 1$ meaning the process for inflation is invertible. With inflation following an invertible linear process, the full-information optimal forecast of π_{t+1} is a linear combination of $\pi_t, \pi_{t-1}, \dots$. In particular, as before x_t is not useful.

In the unlikely case in which $-\rho_\zeta > \theta_\zeta$, of if the forecaster's information set $\mathcal{I}_t$ is smaller than $\{\pi_t, x_t, \pi_{t-1}, x_{t-1}, \dots\}$,[43] then x_t may contain some useful information. Combining the solution for inflation with the Phillips curve:

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \kappa x_t + \kappa\omega_t,$$

[43] We nonetheless assume that π_t and x_t are in $\mathcal{I}_t$.

gives:

$$x_t = -\frac{1}{\kappa}\left[\frac{1-\beta\rho_\zeta}{\phi-\rho_\zeta}\left(\zeta_t + \frac{\theta_\zeta}{\phi}\varepsilon_{\zeta,t}\right) - \beta\frac{\theta_\zeta}{\phi}\varepsilon_{\zeta,t}\right] - \omega_t$$
$$= \frac{1}{\kappa}\left[(1-\beta\rho_\zeta)\pi_t + \beta\frac{\theta_\zeta}{\phi}\varepsilon_{\zeta,t}\right] - \omega_t.$$

In this case, it is possible that $\mathbb{E}[\pi_{t+1}|\mathcal{I}_t] \neq \mathbb{E}[\pi_{t+1}|\mathcal{I}_{t-1},\pi_t]$ as x_t provides an independent signal about $\varepsilon_{\zeta,t}$.

There are two important special cases. If $\omega_t = 0$, and the forecaster knows this, then:

$$\varepsilon_{\zeta,t} = \frac{\phi}{\beta\theta_\zeta}[\kappa x_t - (1-\beta\rho_\zeta)\pi_t],$$

so:

$$\zeta_t = -\left(\phi - \frac{1}{\beta}\right)\pi_t - \frac{\kappa}{\beta}x_t,$$

which enables the forecaster to form the full-information optimal forecast:

$$\mathbb{E}_t\pi_{t+1} = -\frac{1}{\phi-\rho_\zeta}(\rho_\zeta\zeta_t + \theta_\zeta\varepsilon_{\zeta,t}) = \frac{1}{\beta}(\pi_t - \kappa x_t).$$

(This formula also follows immediately from the Phillips curve.) Note that the output gap has what Dotsey, Fujita & Stark (2018) call the "wrong" sign, meaning Phillips curve based forecasting regressions may have surprising results. However, in the general case in which ω_t has positive variance, then output's signal about $\varepsilon_{\zeta,t}$ will be polluted by the noise from ω_t, making it much less informative. Indeed, with ϕ large, as we expect, then $\frac{\theta_\zeta}{\phi}\varepsilon_{\zeta,t}$ will have low variance, making it more likely that it is drowned out by the noise from ω_t.

The second important special case is when $\varepsilon_{\zeta,t} = 0$, and again the forecaster knows this. In this case, much as in the main text:

$$\mathbb{E}_t\pi_{t+1} = \rho_\zeta\pi_t - \frac{1}{\phi-\rho_\zeta}\left(1+\frac{\theta_\zeta}{\phi}\right)\left[\mathbb{E}_t\varepsilon_{\zeta,t+1} + \frac{\phi-\rho_\zeta}{\phi+\theta_\zeta}\theta_\zeta\varepsilon_{\zeta,t}\right] = \rho_\zeta\pi_t,$$

so x_t is unhelpful.

The general case will inherit aspects of these two special cases, as well as the case in which π_t's stochastic process was invertible. Inflation and its lags will certainly help forecast inflation, but the output gap may also provide a little extra information, possibly with the "wrong" sign.

E.2 Robustness to non-unit responses to real interest rates

Suppose that the central bank is unable to respond with a precise unit coefficient to real interest rates, so instead follows the monetary rule:

$$i_t = (1 + \gamma)r_t + \phi\pi_t + \zeta_t,$$

where $\gamma \in \mathbb{R}$ is some small value giving the departure from unit responses.

For simplicity, suppose the rest of the model takes the same form as in Subsection 1.2, with:

$$x_t = \delta\mathbb{E}_t x_{t+1} - \varsigma(r_t - n_t),$$

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \kappa x_t + \kappa\omega_t,$$

$$i_t = r_t + \mathbb{E}_t\pi_{t+1}.$$

We suppose $\phi > 1$, but do not make any assumptions on the signs of $\delta, \beta, \kappa, \varsigma, \gamma$, beyond assuming that $\varsigma \neq 0$ (so monetary policy has some effect on the output gap) and $\kappa \neq 0$ (so monetary policy has some effect on inflation, via the output gap).

Combining the monetary rule with the Fisher equation gives:

$$\mathbb{E}_t\pi_{t+1} = \gamma r_t + \phi\pi_t + \zeta_t,$$

so:

$$r_t = \frac{1}{\gamma}(\mathbb{E}_t\pi_{t+1} - \phi\pi_t - \zeta_t),$$

meaning:

$$x_t = \delta\mathbb{E}_t x_{t+1} - \frac{\varsigma}{\gamma}(\mathbb{E}_t\pi_{t+1} - \phi\pi_t) + \varsigma n_t + \frac{\varsigma}{\gamma}\zeta_t.$$

Then, since:

$$\mathbb{E}_t\pi_{t+1} = \frac{1}{\beta}\pi_t - \frac{\kappa}{\beta}x_t - \frac{\kappa}{\beta}\omega_t,$$

we have that:

$$\mathbb{E}_t x_{t+1} = \left(\frac{1}{\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}\right)x_t - \frac{\varsigma}{\delta\gamma}\left(\phi - \frac{1}{\beta}\right)\pi_t - \frac{\varsigma}{\delta\gamma}\left(\gamma n_t + \zeta_t + \frac{\kappa}{\beta}\omega_t\right).$$

Woodford (2003) (Addendum to Chapter 4, Proposition C.1) proves that this model is determinate if and only if both eigenvalues of the matrix:

$$M := \begin{bmatrix} \frac{1}{\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta} & -\frac{\varsigma}{\delta\gamma}\left(\phi - \frac{1}{\beta}\right) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}$$

are outside of the unit circle, which in turn is proven to hold if and only if EITHER: Case I: $1 < \det M$, $0 < 1 + \det M - \operatorname{tr} M$, and $0 < 1 + \det M + \operatorname{tr} M$, OR Case II: $0 > 1 + \det M - \operatorname{tr} M$, and $0 > 1 + \det M + \operatorname{tr} M$. Note:

$$\det M = \frac{1}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}\phi,$$
$$\operatorname{tr} M = \frac{1}{\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta} + \frac{1}{\beta}.$$

Thus, Case I requires:

$$1 < \det M = \frac{1}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}\phi,$$
$$0 < 1 + \det M - \operatorname{tr} M = \frac{(1-\beta)(1-\delta)}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}(\phi - 1),$$
$$\text{and } 0 < 1 + \det M + \operatorname{tr} M = \frac{(1+\beta)(1+\delta)}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}(1 + \phi).$$

And Case II requires:

$$0 > 1 + \det M - \operatorname{tr} M = \frac{(1-\beta)(1-\delta)}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}(\phi - 1),$$
$$\text{and } 0 > 1 + \det M + \operatorname{tr} M = \frac{(1+\beta)(1+\delta)}{\beta\delta} - \frac{\varsigma\kappa}{\gamma\beta\delta}(1 + \phi).$$

To see when these conditions are satisfied, first suppose that $\frac{\varsigma\kappa}{\gamma\beta\delta} < 0$, so $\frac{\varsigma\kappa}{\gamma\beta\delta} = -\frac{|\varsigma\kappa|}{|\gamma||\beta\delta|}$. Then if γ is sufficiently small in magnitude, it is immediately clear that all three conditions of Case I are satisfied, since $\phi > 0$, $\phi - 1 > 0$ and $1 + \phi > 0$. In particular, in this case we need:

$$|\gamma| < |\varsigma\kappa| \min\left\{ \begin{matrix} \frac{\phi}{\max\{0, -(\operatorname{sign}(\beta\delta) - |\beta\delta|)\}}, \\ \frac{\phi - 1}{\max\{0, -(\operatorname{sign}(\beta\delta))(1-\beta)(1-\delta)\}}, \\ \frac{1 + \phi}{\max\{0, -(\operatorname{sign}(\beta\delta))(1+\beta)(1+\delta)\}} \end{matrix} \right\}.$$

Alternatively, suppose that $\frac{\varsigma\kappa}{\gamma\beta\delta} > 0$, so $\frac{\varsigma\kappa}{\gamma\beta\delta} = \frac{|\varsigma\kappa|}{|\gamma||\beta\delta|}$. Then, similarly, if γ is sufficiently small in magnitude, both conditions of Case II are satisfied, since $\phi - 1 > 0$ and $1 + \phi > 0$. In particular, in this case we need:

$$|\gamma| < |\varsigma\kappa| \min \left\{ \begin{matrix} \dfrac{\phi - 1}{\max\{0, (\operatorname{sign}(\beta\delta))(1-\beta)(1-\delta)\}}, \\ \dfrac{1+\phi}{\max\{0, (\operatorname{sign}(\beta\delta))(1+\beta)(1+\delta)\}} \end{matrix} \right\}.$$

Thus, it is always sufficient for determinacy that:

$$|\gamma| < |\varsigma\kappa| \min \left\{ \begin{matrix} \dfrac{\phi}{\max\{0, -(\operatorname{sign}(\beta\delta) - |\beta\delta|)\}}, \\ \dfrac{\phi - 1}{|(1-\beta)(1-\delta)|}, \\ \dfrac{1+\phi}{|(1+\beta)(1+\delta)|} \end{matrix} \right\}.$$

Since the right-hand side is strictly positive, there is a positive measure of γ for which we have determinacy.

E.3 Real-time learning of Phillips curve coefficients

We start by assuming that the central bank knows the Phillips curve coefficients. A close examination of this case will lead to a natural learning scheme for when the central bank does not know these coefficients.

As in the main text, suppose the central bank is using the rule:

$$i_t = r_t + \phi_\pi \pi_t + \phi_x \big[x_t - \kappa^{-1} \big[\pi_t - \tilde{\beta}(1 - \varrho_\pi) \mathbb{E}_t \pi_{t+1} - \tilde{\beta} \varrho_\pi \pi_{t-1} \big] \big] + \zeta_t,$$

and that the model also contains the Phillips curve:

$$\pi_t = \tilde{\beta}(1 - \varrho_\pi) \mathbb{E}_t \pi_{t+1} + \tilde{\beta} \varrho_\pi \pi_{t-1} + \kappa x_t + \kappa \omega_t,$$

and the Fisher equation:

$$i_t = r_t + \mathbb{E}_t \pi_{t+1}.$$

We suppose that ζ_t follows the ARMA(1,1) process:

$$\zeta_t = \rho_\zeta \zeta_{t-1} + \varepsilon_{\zeta,t} + \theta_\zeta \varepsilon_{\zeta,t-1}, \qquad \varepsilon_{\zeta,t} \sim N(0, \sigma_\zeta^2),$$

with $\rho_\zeta, \theta_\zeta \in (-1,1)$, and for simplicity, we suppose that $\omega_t = \varepsilon_{\omega,t}$, where $\varepsilon_{\omega,t} \sim N(0, \sigma_\omega^2)$.

From combining all the above equations, we have that if $\phi_\pi > 1$, there is a unique solution with:

$$\pi_t = -\frac{1}{\phi_\pi - \rho_\zeta}\left[\zeta_t + \frac{\theta_\zeta}{\phi_\pi}\varepsilon_{\zeta,t}\right] + \frac{\phi_x}{\phi_\pi}\varepsilon_{\omega,t}.$$

Thus, if we define:

$$\begin{aligned}
m_0 &:= \frac{\sigma_\zeta^2}{\kappa(\phi_\pi - \rho_\zeta)}\left[\tilde{\beta}(1-\varrho_\pi)(\rho_\zeta + \theta_\zeta) - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)\right],\\
m_1 &:= \frac{\sigma_\zeta^2}{\kappa(\phi_\pi - \rho_\zeta)}\left[\left[\tilde{\beta}(1-\varrho_\pi)\rho_\zeta - 1\right](\rho_\zeta + \theta_\zeta) + \tilde{\beta}\varrho_\pi\left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)\right],\\
m_2 &:= \frac{\sigma_\zeta^2}{\kappa(\phi_\pi - \rho_\zeta)}\left[\left[\tilde{\beta}(1-\varrho_\pi)\rho_\zeta - 1\right]\rho_\zeta + \tilde{\beta}\varrho_\pi\right](\rho_\zeta + \theta_\zeta),
\end{aligned}$$

then by the Phillips curve $m_0 = \mathbb{E}x_t\varepsilon_{\zeta,t}$, $m_1 = \mathbb{E}x_t\varepsilon_{\zeta,t-1}$ and $m_2 = \mathbb{E}x_t\varepsilon_{\zeta,t-2}$. Also note that:

$$\begin{aligned}
\kappa &= \frac{\sigma_\zeta^2}{\phi_\pi - \rho_\zeta}\frac{\left(\rho_\zeta + \theta_\zeta - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)^2}{\rho_\zeta\left(\rho_\zeta + \theta_\zeta - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)m_0 - \left((\rho_\zeta + \theta_\zeta)m_1 - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)m_2\right)},\\
\tilde{\beta} &= \frac{\left(\rho_\zeta + \theta_\zeta - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)\left(m_0 - (\rho_\zeta m_1 - m_2)\right) - \frac{\phi_\pi + \theta_\zeta}{(\rho_\zeta + \theta_\zeta)\phi_\pi}\left((\rho_\zeta + \theta_\zeta)m_1 - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)m_2\right)}{\rho_\zeta\left(\rho_\zeta + \theta_\zeta - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)m_0 - \left((\rho_\zeta + \theta_\zeta)m_1 - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)m_2\right)},\\
\varrho_\pi &= -\frac{\left(\rho_\zeta + \theta_\zeta - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)(\rho_\zeta m_1 - m_2)}{\left(\rho_\zeta + \theta_\zeta - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)\left(m_0 - (\rho_\zeta m_1 - m_2)\right) - \frac{\phi_\pi + \theta_\zeta}{(\rho_\zeta + \theta_\zeta)\phi_\pi}\left((\rho_\zeta + \theta_\zeta)m_1 - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)m_2\right)}.
\end{aligned}$$

In other words, once the central bank knows m_0, m_1 and m_2 they can infer the parameters of the Phillips curve from the known properties of their monetary rule and monetary shock. This is essentially an instrumental variables regression. We are using $\varepsilon_{\zeta,t}$, $\varepsilon_{\zeta,t-1}$ and $\varepsilon_{\zeta,t-2}$ as instruments for $\mathbb{E}_t\pi_{t+1}$, π_t and π_{t-1} in a regression of the output gap on those variables. This works as long as $\theta_\zeta \neq 0$, else $\mathbb{E}_t\pi_{t+1}$ and π_t are colinear.

If the central bank does not know the true values of κ, $\tilde{\beta}$ and ϱ_π, we suppose they dynamically update estimates of m_0, m_1 and m_2 using the following decreasing gain learning rules (for $t > 0$):

$$\begin{aligned}
m_{0,t} &= m_{0,t-1} + t^{-1}(x_t\varepsilon_{\zeta,t} - m_{0,t-1}),\\
m_{1,t} &= m_{1,t-1} + t^{-1}(x_t\varepsilon_{\zeta,t-1} - m_{1,t-1}),\\
m_{2,t} &= m_{2,t-1} + t^{-1}(x_t\varepsilon_{\zeta,t-2} - m_{2,t-1}),
\end{aligned}$$

where $\iota \in (0,1]$ is a gain parameter. Then they can use the monetary rule:

$$i_t = r_t + \phi_\pi \pi_t + \phi_x[x_t + q_{1,t-1}\mathbb{E}_t\pi_{t+1} + q_{0,t-1}\pi_t + q_{-1,t-1}\pi_{t-1}] + \zeta_t,$$

where:

$$q_{1,t} := \frac{\phi_\pi - \rho_\zeta}{\sigma_\zeta^2} \frac{\left(\rho_\zeta + \theta_\zeta - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right) m_{0,t} - \frac{\phi_\pi + \theta_\zeta}{(\rho_\zeta + \theta_\zeta)\phi_\pi}\left((\rho_\zeta + \theta_\zeta) m_{1,t} - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right) m_{2,t}\right)}{\left(\rho_\zeta + \theta_\zeta - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)^2},$$

$$q_{0,t} := -\frac{\phi_\pi - \rho_\zeta}{\sigma_\zeta^2} \frac{\rho_\zeta\left(\rho_\zeta + \theta_\zeta - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right) m_{0,t} - \left((\rho_\zeta + \theta_\zeta) m_{1,t} - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right) m_{2,t}\right)}{\left(\rho_\zeta + \theta_\zeta - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)^2},$$

$$q_{-1,t} := -\frac{\phi_\pi - \rho_\zeta}{\sigma_\zeta^2} \frac{\left(\rho_\zeta + \theta_\zeta - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)(\rho_\zeta m_{1,t} - m_{2,t})}{\left(\rho_\zeta + \theta_\zeta - \left(1 + \frac{\theta_\zeta}{\phi_\pi}\right)\rho_\zeta\right)^2}.$$

This is reasonable, as if $m_{0,t-1} \approx m_0$, $m_{1,t-1} \approx m_1$ and $m_{2,t-1} \approx m_2$ then $q_{1,t-1} \approx \kappa^{-1}\tilde{\beta}(1 - \varrho_\pi)$, $q_{0,t-1} \approx -\kappa^{-1}$ and $q_{-1,t-1} \approx \kappa^{-1}\tilde{\beta}\varrho_\pi$, so this monetary rule is approximately the same as the full information one previously considered. Using lagged estimates ($q_{1,t-1}$ not $q_{1,t}$ etc.) in the monetary rule reflects central bank information (processing) delays and simplifies the model's solution. It is also a common assumption in the reduced form learning literature (Evans & Honkapohja 2001).

With the new monetary rule, the model is no-longer linear. As a result, the exact solution is analytically intractable. However, we are only really interested in asymptotic dynamics. If $m_{0,t} \to m_0$, $m_{1,t} \to m_1$ and $m_{2,t} \to m_2$ as $t \to \infty$ then we know the asymptotic solution will be the stable full information one we found previously. We will analyse the system's behaviour with help from the stochastic approximation tools frequently used in the reduced form learning literature (Evans & Honkapohja 2001). These tools only require a zeroth order approximation in t^{-1} to the dynamics of x_t and π_t.[44] Intuitively, this is because x_t (hence π_t) enters the law of motion for $m_{0,t}$, $m_{1,t}$ and $m_{2,t}$ multiplied by t^{-1}, so a zeroth order approximation to the dynamics of x_t

[44] Given certain regularity conditions on the higher order terms. These conditions will be satisfied here, at least providing we restrict $m_{0,t}$, $m_{1,t}$ and $m_{2,t}$ to a small enough open set around m_0, m_1 and m_2, using a so called projection facility.

and π_t in t^{-1} delivers a first order approximation to the dynamics of $m_{0,t}$, $m_{1,t}$ and $m_{2,t}$ in t^{-1}.

We conjecture a time-varying coefficients solution with:

$$\pi_t = A_{t-1}\zeta_t + B_{t-1}\varepsilon_{\zeta,t} + C_{t-1}\varepsilon_{\omega,t} + D_{t-1}\pi_{t-1} + O(t^{-1}),$$

where we conjecture $A_t = A_{t-1} + O(t^{-1})$, $B_t = B_{t-1} + O(t^{-1})$, $C_t = C_{t-1} + O(t^{-1})$ and $D_t = D_{t-1} + O(t^{-1})$. Substituting this into the monetary rule, Fisher equation and Phillips curve implies:

$$\begin{aligned}&\left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi) - \phi_x q_{1,t-1}\right]A_t\left(\rho_\zeta\zeta_t + \theta_\zeta\varepsilon_{\zeta,t}\right)\\ &\qquad = \left[\phi_\pi + \phi_x\kappa^{-1} + \phi_x q_{0,t-1} - \left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi) - \phi_x q_{1,t-1}\right]D_t\right]\left[A_{t-1}\zeta_t\right.\\ &\qquad \left. + B_{t-1}\varepsilon_{\zeta,t} + C_{t-1}\varepsilon_{\omega,t} + D_{t-1}\pi_{t-1}\right] + \phi_x\left[q_{-1,t-1} - \kappa^{-1}\tilde{\beta}\varrho_\pi\right]\pi_{t-1} - \phi_x\varepsilon_{\omega,t}\\ &\qquad + \zeta_t + O(t^{-1}).\end{aligned}$$

Matching terms and using $A_t = A_{t-1} + O(t^{-1})$ and $D_t = D_{t-1} + O(t^{-1})$ then gives that:

$$\begin{aligned}&\left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi) - \phi_x q_{1,t}\right]A_t\rho_\zeta\\ &\qquad = \left[\phi_\pi + \phi_x\kappa^{-1} + \phi_x q_{0,t} - \left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi) - \phi_x q_{1,t}\right]D_t\right]A_t + 1\\ &\qquad + O(t^{-1}),\\ &\left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi) - \phi_x q_{1,t}\right]A_t\theta_\zeta\\ &\qquad = \left[\phi_\pi + \phi_x\kappa^{-1} + \phi_x q_{0,t} - \left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi) - \phi_x q_{1,t}\right]D_t\right]B_t + O(t^{-1}),\\ &0 = \left[\phi_\pi + \phi_x\kappa^{-1} + \phi_x q_{0,t} - \left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi) - \phi_x q_{1,t-1}\right]D_t\right]C_t - \phi_x + O(t^{-1}),\\ &0 = \left[\phi_\pi + \phi_x\kappa^{-1} + \phi_x q_{0,t} - \left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi) - \phi_x q_{1,t}\right]D_t\right]D_t + \phi_x\left[q_{-1,t} - \kappa^{-1}\tilde{\beta}\varrho_\pi\right]\\ &\qquad + O(t^{-1}).\end{aligned}$$

The final equation has two roots, but we know we need to pick the one that gives $D_t \to 0$ as $\phi_x \to 0$. Now if $q_{0,t}$ is sufficiently close to q_0, then $\phi_\pi + \phi_x\kappa^{-1} + \phi_x q_{0,t} > 0$, so:

$$D_t = \frac{(\phi_\pi + \phi_x\kappa^{-1} + \phi_x q_{0,t}) - \sqrt{\begin{array}{c}(\phi_\pi + \phi_x\kappa^{-1} + \phi_x q_{0,t})^2 \cdots\\ +4\phi_x\left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi) - \phi_x q_{1,t}\right]\left[q_{-1,t} - \kappa^{-1}\tilde{\beta}\varrho_\pi\right]\end{array}}}{2\left[1 + \phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi) - \phi_x q_{1,t}\right]}$$
$$+ O(t^{-1}),$$

and:

$$A_t = \Big[\big[1 + \phi_x \kappa^{-1} \tilde{\beta}(1 - \varrho_\pi) - \phi_x q_{1,t}\big]\big(D_t + \rho_\zeta\big) - \big(\phi_\pi + \phi_x \kappa^{-1} + \phi_x q_{0,t}\big)\Big]^{-1} + O(t^{-1}),$$

$$B_t = \frac{\theta_\zeta\big[1 + \phi_x \kappa^{-1} \tilde{\beta}(1 - \varrho_\pi) - \phi_x q_{1,t}\big]A_t}{\phi_\pi + \phi_x \kappa^{-1} + \phi_x q_{0,t} - \big[1 + \phi_x \kappa^{-1} \tilde{\beta}(1 - \varrho_\pi) - \phi_x q_{1,t}\big]D_t} + O(t^{-1}),$$

$$C_t = \frac{\phi_x}{\phi_\pi + \phi_x \kappa^{-1} + \phi_x q_{0,t} - \big[1 + \phi_x \kappa^{-1} \tilde{\beta}(1 - \varrho_\pi) - \phi_x q_{1,t}\big]D_t} + O(t^{-1}).$$

Since $q_{1,t} = q_{1,t-1} + O(t^{-1})$, $q_{0,t} = q_{0,t-1} + O(t^{-1})$ and $q_{-1,t} = q_{-1,t-1} + O(t^{-1})$, as required we have that $A_t = A_{t-1} + O(t^{-1})$, $B_t = B_{t-1} + O(t^{-1})$, $C_t = C_{t-1} + O(t^{-1})$ and $D_t = D_{t-1} + O(t^{-1})$.

Using this result again, we then have that:

$$\begin{aligned} x_t = \kappa^{-1}\Big[&\big[1 - \tilde{\beta}(1 - \varrho_\pi)\big(D_{t-1} + \rho_\zeta\big)\big]A_{t-1}\zeta_t \\ &+ \big[B_{t-1} - \tilde{\beta}(1 - \varrho_\pi)\big(A_{t-1}\theta_\zeta + B_{t-1}D_{t-1}\big)\big]\varepsilon_{\zeta,t} \\ &+ \Big[\big[1 - \tilde{\beta}(1 - \varrho_\pi)D_{t-1}\big]C_{t-1} - \kappa\Big]\varepsilon_{\omega,t} \\ &+ \Big[\big[1 - \tilde{\beta}(1 - \varrho_\pi)D_{t-1}\big]D_{t-1} - \tilde{\beta}\varrho_\pi\Big]\pi_{t-1}\Big] + O(t^{-1}). \end{aligned}$$

Plugging this into the law of motion for $m_{0,t}$, $m_{1,t}$ and $m_{2,t}$ gives a purely backward looking non-linear system in the endogenous states $m_{0,t}$, $m_{1,t}$, $m_{2,t}$ and π_t. This system is of the correct form to be analysed by the stochastic approximation results given in Evans & Honkapohja (2001).

To apply these results, first suppose that for all t, $m_{0,t} = \widehat{m}_0$, $m_{1,t} = \widehat{m}_1$ and $m_{2,t} = \widehat{m}_2$, for some values $\widehat{m}_0$, $\widehat{m}_1$ and $\widehat{m}_2$. Then $q_{1,t} = \hat{q}_1$, $q_{0,t} = \hat{q}_0$ and $q_{-1,t} = \hat{q}_{-1}$ for all t, where:

$$\hat{q}_1 := \frac{\phi_\pi - \rho_\zeta}{\sigma_\zeta^2} \frac{\Big(\rho_\zeta + \theta_\zeta - \Big(1 + \frac{\theta_\zeta}{\phi_\pi}\Big)\rho_\zeta\Big)\widehat{m}_0 - \frac{\phi_\pi + \theta_\zeta}{(\rho_\zeta + \theta_\zeta)\phi_\pi}\Big((\rho_\zeta + \theta_\zeta)\widehat{m}_1 - \Big(1 + \frac{\theta_\zeta}{\phi_\pi}\Big)\widehat{m}_2\Big)}{\Big(\rho_\zeta + \theta_\zeta - \Big(1 + \frac{\theta_\zeta}{\phi_\pi}\Big)\rho_\zeta\Big)^2},$$

$$\hat{q}_0 := -\frac{\phi_\pi - \rho_\zeta}{\sigma_\zeta^2} \frac{\rho_\zeta\Big(\rho_\zeta + \theta_\zeta - \Big(1 + \frac{\theta_\zeta}{\phi_\pi}\Big)\rho_\zeta\Big)\widehat{m}_0 - \Big((\rho_\zeta + \theta_\zeta)\widehat{m}_1 - \Big(1 + \frac{\theta_\zeta}{\phi_\pi}\Big)\widehat{m}_2\Big)}{\Big(\rho_\zeta + \theta_\zeta - \Big(1 + \frac{\theta_\zeta}{\phi_\pi}\Big)\rho_\zeta\Big)^2},$$

$$\hat{q}_{-1} := -\frac{\phi_\pi - \rho_\zeta}{\sigma_\zeta^2} \frac{\Big(\rho_\zeta + \theta_\zeta - \Big(1 + \frac{\theta_\zeta}{\phi_\pi}\Big)\rho_\zeta\Big)(\rho_\zeta\widehat{m}_1 - \widehat{m}_2)}{\Big(\rho_\zeta + \theta_\zeta - \Big(1 + \frac{\theta_\zeta}{\phi_\pi}\Big)\rho_\zeta\Big)^2}.$$

Thus, for all t, $A_t = \hat{A}$, $B_t = \hat{B}$, $C_t = \hat{C}$ and $D_t = \widehat{D}$, where:

$$\widehat{D} = \frac{(\phi_\pi + \phi_x\kappa^{-1} + \phi_x\hat{q}_0) - \sqrt{\begin{matrix}(\phi_\pi + \phi_x\kappa^{-1} + \phi_x\hat{q}_0)^2 \cdots \\ +4\phi_x\big[1 + \phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi) - \phi_x\hat{q}_1\big]\big[\hat{q}_{-1} - \kappa^{-1}\tilde{\beta}\varrho_\pi\big]\end{matrix}}}{2\big[1 + \phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi) - \phi_x\hat{q}_1\big]},$$

and:

$$\begin{aligned}
\hat{A} &= \Big[\big[1 + \phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi) - \phi_x\hat{q}_1\big]\big(\widehat{D} + \rho_\zeta\big) - (\phi_\pi + \phi_x\kappa^{-1} + \phi_x\hat{q}_0)\Big]^{-1}, \\
\hat{B} &= \frac{\theta_\zeta\big[1 + \phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi) - \phi_x\hat{q}_1\big]\hat{A}}{\phi_\pi + \phi_x\kappa^{-1} + \phi_x\hat{q}_0 - \big[1 + \phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi) - \phi_x\hat{q}_1\big]\widehat{D}}, \\
\hat{C} &= \frac{\phi_x}{\phi_\pi + \phi_x\kappa^{-1} + \phi_x\hat{q}_0 - \big[1 + \phi_x\kappa^{-1}\tilde{\beta}(1-\varrho_\pi) - \phi_x\hat{q}_1\big]\widehat{D}}.
\end{aligned}$$

So:

$$\pi_t = \hat{A}\zeta_t + \hat{B}\varepsilon_{\zeta,t} + \hat{C}\varepsilon_{\omega,t} + \widehat{D}\pi_{t-1},$$

and:

$$\begin{aligned}
x_t &= \kappa^{-1}\Big[\big[1 - \tilde{\beta}(1-\varrho_\pi)\big(\widehat{D} + \rho_\zeta\big)\big]\hat{A}\zeta_t + \big[\hat{B} - \tilde{\beta}(1-\varrho_\pi)\big(\hat{A}\theta_\zeta + \hat{B}\widehat{D}\big)\big]\varepsilon_{\zeta,t} \\
&\qquad + \Big[\big[1 - \tilde{\beta}(1-\varrho_\pi)\widehat{D}\big]\hat{C} - \kappa\Big]\varepsilon_{\omega,t} + \Big[\big[1 - \tilde{\beta}(1-\varrho_\pi)\widehat{D}\big]\widehat{D} - \tilde{\beta}\varrho_\pi\Big]\pi_{t-1}\Big] \\
&= \kappa^{-1}\Big[\big[1 - \tilde{\beta}(1-\varrho_\pi)\big(\widehat{D} + \rho_\zeta\big)\big]\hat{A}\big[\rho_\zeta\big[\rho_\zeta\zeta_{t-2} + \varepsilon_{\zeta,t-1} + \theta_\zeta\varepsilon_{\zeta,t-2}\big] + \varepsilon_{\zeta,t} + \theta_\zeta\varepsilon_{\zeta,t-1}\big] \\
&\qquad + \big[\hat{B} - \tilde{\beta}(1-\varrho_\pi)\big(\hat{A}\theta_\zeta + \hat{B}\widehat{D}\big)\big]\varepsilon_{\zeta,t} + \Big[\big[1 - \tilde{\beta}(1-\varrho_\pi)\widehat{D}\big]\hat{C} - \kappa\Big]\varepsilon_{\omega,t} \\
&\qquad + \Big[\big[1 - \tilde{\beta}(1-\varrho_\pi)\widehat{D}\big]\widehat{D} - \tilde{\beta}\varrho_\pi\Big]\Big[\hat{A}\big[\rho_\zeta\zeta_{t-2} + \varepsilon_{\zeta,t-1} + \theta_\zeta\varepsilon_{\zeta,t-2}\big] + \hat{B}\varepsilon_{\zeta,t-1} \\
&\qquad + \hat{C}\varepsilon_{\omega,t-1} + \widehat{D}\big[\hat{A}\zeta_{t-2} + \hat{B}\varepsilon_{\zeta,t-2} + \hat{C}\varepsilon_{\omega,t-2} + \widehat{D}\pi_{t-3}\big]\Big]\Big].
\end{aligned}$$

Hence:

$$\begin{aligned}
\mathbb{E}x_t\varepsilon_{\zeta,t} &= \sigma_\zeta^2\kappa^{-1}\Big[\big[1 - \tilde{\beta}(1-\varrho_\pi)\big(\widehat{D} + \rho_\zeta + \theta_\zeta\big)\big]\hat{A} + \big[1 - \tilde{\beta}(1-\varrho_\pi)\widehat{D}\big]\hat{B}\Big], \\
\mathbb{E}x_t\varepsilon_{\zeta,t-1} &= \sigma_\zeta^2\kappa^{-1}\Big[\big[1 - \tilde{\beta}(1-\varrho_\pi)\big(\widehat{D} + \rho_\zeta\big)\big]\hat{A}\big(\rho_\zeta + \theta_\zeta\big) \\
&\qquad + \Big[\big[1 - \tilde{\beta}(1-\varrho_\pi)\widehat{D}\big]\widehat{D} - \tilde{\beta}\varrho_\pi\Big]\big(\hat{A} + \hat{B}\big)\Big], \\
\mathbb{E}x_t\varepsilon_{\zeta,t-2} &= \sigma_\zeta^2\kappa^{-1}\Big[\big[1 - \tilde{\beta}(1-\varrho_\pi)\big(\widehat{D} + \rho_\zeta\big)\big]\hat{A}\rho_\zeta\big(\rho_\zeta + \theta_\zeta\big) \\
&\qquad + \Big[\big[1 - \tilde{\beta}(1-\varrho_\pi)\widehat{D}\big]\widehat{D} - \tilde{\beta}\varrho_\pi\Big]\big[\hat{A}\big(\rho_\zeta + \theta_\zeta\big) + \widehat{D}\big(\hat{A} + \hat{B}\big)\big]\Big].
\end{aligned}$$

Now denote by $\mathcal{T}$ the map taking the vector:

$$\widehat{m} := \begin{bmatrix} \widehat{m}_0 \\ \widehat{m}_1 \\ \widehat{m}_2 \end{bmatrix}$$

to the vector:

$$\mathcal{T}(\widehat{m}) := \begin{bmatrix} \mathbb{E}x_t\varepsilon_{\zeta,t} \\ \mathbb{E}x_t\varepsilon_{\zeta,t-1} \\ \mathbb{E}x_t\varepsilon_{\zeta,t-2} \end{bmatrix}.$$

Stochastic approximation theory relates the stability of our nonlinear difference equation to the stability of the ODE:

$$\frac{d\widehat{m}(\tau)}{d\tau} = \mathcal{T}\big(\widehat{m}(\tau)\big) - \widehat{m}(\tau).$$

The $\mathcal{T}$ map here plays the role usually played by the mapping from the perceived law of motion to the actual law of motion in the reduced form learning literature (Evans & Honkapohja 2001).

We conjecture that:

$$m := \begin{bmatrix} m_0 \\ m_1 \\ m_2 \end{bmatrix}$$

is a locally asymptotically stable point of this ODE. To check this, note that tedious algebra gives that:

$$\left.\frac{\partial\mathcal{T}(\widehat{m})}{\partial\widehat{m}}\right|_{\widehat{m}=m} = \frac{\phi_x}{\kappa\phi_\pi}\begin{bmatrix} 1 & \phi_\pi^{-1} - \tilde{\beta}(1-\varrho_\pi) & \dfrac{\phi_\pi^{-1} - \tilde{\beta}(1-\varrho_\pi)}{\phi_\pi - \rho_\zeta} \\ -\tilde{\beta}\varrho_\pi & 1 - \phi_\pi^{-1}\tilde{\beta}\varrho_\pi & \dfrac{\phi_\pi\big[\phi_\pi^{-1} - \tilde{\beta}(1-\varrho_\pi)\big] - \phi_\pi^{-1}\tilde{\beta}\varrho_\pi}{\phi_\pi - \rho_\zeta} \\ 0 & -\tilde{\beta}\varrho_\pi & \dfrac{\phi_\pi\big[1 - \tilde{\beta}(1-\varrho_\pi)\rho_\zeta\big] - \tilde{\beta}\varrho_\pi}{\phi_\pi - \rho_\zeta} \end{bmatrix}.$$

For simplicity, we assume $\phi_x \geq 0$, $\phi_\pi \geq 0$, $\kappa \geq 0$, $\tilde{\beta} \geq 0$, $\varrho_\pi \in [0,1)$, $\rho_\zeta \in [0,1)$ and $\phi_\pi \geq \big[\tilde{\beta}(1-\varrho_\pi)\big]^{-1}$. Under these assumptions, the off-diagonal elements of this matrix are all non-positive. Other cases may also go through, but for the sake of brevity we concentrate on this most relevant case. Given these assumptions, applying the Gershgorin circle theorem to the columns of this matrix gives the following upper bound on the real part of the eigenvalues of $\left.\frac{\partial\mathcal{T}(\widehat{m})}{\partial\widehat{m}}\right|_{\widehat{m}=m}$:

$$\frac{\phi_x}{\kappa\phi_\pi}\max\left\{\begin{matrix} 1 + \tilde{\beta}\varrho_\pi, \phi_\pi^{-1}\big[\tilde{\beta}(\phi_\pi - \varrho_\pi) + \phi_\pi - 1\big], \\ \dfrac{(1-\phi_\pi^{-1})(\phi_\pi - \tilde{\beta}\varrho_\pi) + \tilde{\beta}(1-\varrho_\pi)\big[1 + \phi_\pi(1-\rho_\zeta)\big] - \phi_\pi^{-1}}{\phi_\pi - \rho_\zeta} \end{matrix}\right\}.$$

The first and second arguments in curly brackets here are both less than $1+\tilde{\beta}$. Taking the derivative of the third argument in curly brackets with respect to ρ_ζ produces an expression whose sign is not a function of ρ_ζ. Thus, the third argument in curly brackets is maximized at either $\rho_\zeta = 0$ or $\rho_\zeta = 1$. In the former case, the argument is less or equal to $1+\tilde{\beta}$ providing $\tilde{\beta} \le 1$. In the latter case, the argument is less or equal to $1+\tilde{\beta}$ providing that $2(1-\varrho_\pi) \le \phi_\pi$. Therefore, if $\phi_x \ge 0$, $\phi_\pi \ge 0$, $\kappa \ge 0$, $\tilde{\beta} \in [0,1]$, $\varrho_\pi \in [0,1)$, $\rho_\zeta \in [0,1)$ and:

$$\phi_\pi > \max\left\{\frac{1}{\tilde{\beta}(1-\varrho_\pi)}, 2(1-\varrho_\pi), \frac{\phi_x(1+\tilde{\beta})}{\kappa}\right\},$$

then all of the eigenvalues of $\left.\frac{\partial \mathcal{T}(\widehat{m})}{\partial \widehat{m}}\right|_{\widehat{m}=m}$ are less than one. Consequently, in this case the ODE is locally asymptotically stable, so the stochastic approximation results of Evans & Honkapohja (2001) apply. In particular, if we suppose that $\widehat{m}_0$, $\widehat{m}_1$ and $\widehat{m}_2$ are constrained to remain within a sufficiently small ball around m_0, m_1 and m_2, then the central bank's estimates of the Phillips curve parameters will converge to their true values, and the model's dynamics will converge to the determinate ones under rational expectations.

E.4 Responding to other endogenous variables in a general model

Now, suppose the central bank uses the rule:

$$i_t = r_t + \phi_\pi \pi_t + \iota \phi_z^\top z_t + \phi_\nu^\top \nu_t.$$

Here, z_t is a vector of other endogenous variables, with $z_{t,1} = r_t$, $\iota > 0$ is a scalar governing the strength of response to all of them, and ν_t is an arbitrary exogenous stochastic process (potentially vector valued). As usual, we assume $\phi_\pi > 1$.

Without loss of generality, we suppose that the other endogenous variables satisfy the general linear expectational difference equation:

$$0 = A\mathbb{E}_t z_{t+1} + Bz_t + Cz_{t-1} + d\pi_t + E\nu_t,$$

where the coefficient matrices are such that there is a unique matrix F with eigenvalues in the unit circle such that $F = -(AF + B)^{-1}C$.[45] This condition on F just states that there is no real indeterminacy in the model. Once inflation is determined, so too is z_t. Having the same shock process entering both the monetary rule and the model's other equations is without loss of generality as it is multiplied by $\phi_\nu^\top$ and E respectively.

Now define:

$$G := -A(AF + B)^{-1}.$$

Let L be the lag operator, then note that:

$$(I - GL^{-1})(AF + B)(I - FL) = AL^{-1} + B + CL.$$

Thus, by the model's real determinacy, all of G's eigenvalues must also be inside the unit circle.

In terms of the lag operator, the model to be solved is then:

$$\mathbb{E}_t(1 - \phi_\pi^{-1}L^{-1})\pi_t = -\iota\phi_\pi^{-1}\phi_z^\top z_t - \phi_\pi^{-1}\phi_\nu^\top \nu_t,$$
$$\mathbb{E}_t(I - GL^{-1})(AF + B)(I - FL)z_t = -d\pi_t - E\nu_t.$$

Note for future reference that since ϕ_π^{-1}, G and F all have all their eigenvalues in the unit circle, $(1 - \phi_\pi^{-1}L^{-1})$, $(I - GL^{-1})$ and $(I - FL)$ are all invertible.

We conjecture a series solution of the form:

$$\pi_t = \sum_{k=0}^{\infty} \iota^k \pi_t^{(k)}, \qquad z_t = \sum_{k=0}^{\infty} \iota^k z_t^{(k)}.$$

Matching terms gives that $\pi_t^{(0)}$ solves:

$$\mathbb{E}_t(1 - \phi_\pi^{-1}L^{-1})\pi_t^{(0)} = -\phi_\pi^{-1}\phi_\nu^\top \nu_t,$$

implying that $\pi_t^{(0)}$ is determinate with:

$$\pi_t^{(0)} = -\mathbb{E}_t(1 - \phi_\pi^{-1}L^{-1})^{-1}\phi_\pi^{-1}\phi_\nu^\top \nu_t.$$

Similarly, from matching terms in the law of motion for z_t, we have that:

$$\mathbb{E}_t(I - GL^{-1})(AF + B)(I - FL)z_t^{(0)} = -d\pi_t^{(0)} - E\nu_t$$

[45] The lack of terms in $\mathbb{E}_t\pi_{t+1}$ and π_{t-1} is without loss of generality, as such responses can be included by adding an auxiliary variable $z_{t,j}$ with an equation of the form $z_{t,j} = \pi_t$.

so $z_t^{(0)}$ is also determinate (by our assumption on A, B and C) with:

$$z_t^{(0)} = -(I - FL)^{-1}(AF + B)^{-1}\mathbb{E}_t(I - GL^{-1})^{-1}\left(d\pi_t^{(0)} - E\nu_t\right).$$

Note that $\pi_t^{(0)}$ can be treated as exogenous for solving for $z_t^{(0)}$, as the causation only runs one way, from $\pi_t^{(0)}$ to $z_t^{(0)}$.

Now suppose that we have established that $\pi_t^{(k)}$ and $z_t^{(k)}$ are determinate for some $k \in \mathbb{N}$, with a determined solution not a function of higher order terms. (We have already proven the base case of $k = 0$.) We seek to prove that $\pi_t^{(k+1)}$ and $z_t^{(k+1)}$ are also determinate. Matching terms again gives that:

$$\mathbb{E}_t(1 - \phi_\pi^{-1}L^{-1})\pi_t^{(k+1)} = -\phi_\pi^{-1}\phi_z^\top z_t^{(k)},$$

so $\pi_t^{(k+1)}$ is also determinate, with:

$$\pi_t^{(k+1)} = -\mathbb{E}_t(1 - \phi_\pi^{-1}L^{-1})^{-1}\phi_\pi^{-1}\phi_z^\top z_t^{(k)},$$

where we used the inductive hypothesis that $z_t^{(k)}$ is already determined, and so it is effectively exogenous for the purpose of determining $\pi_t^{(k+1)}$. Then from matching terms in the law of motion for z_t:

$$\mathbb{E}_t(I - GL^{-1})(AF + B)(I - FL)z_t^{(k+1)} = -d\pi_t^{(k+1)},$$

so $z_t^{(k+1)}$ is also determinate, with:

$$z_t^{(k+1)} = -(I - FL)^{-1}(AF + B)^{-1}\mathbb{E}_t(I - GL^{-1})^{-1}d\pi_t^{(k+1)},$$

much as before. This completes our proof by induction, establishing that there is a series solution of the given form.

The only remaining thing to check is that the series does indeed converge for sufficiently small ι. This follows immediately from the product structure of the solution above, which means that the variances of $z_t^{(k)}$ and $\pi_t^{(k)}$ must be $O\left(h^k\right)$ for some $h \geq 1$. Hence for sufficiently small ι, the model is determinate. I.e., given the Taylor principle is satisfied, a sufficiently small response to other endogenous variables will not break determinacy.

E.5 If inflation is identical, other endogenous variables are identical

Let x_t and $\tilde{x}_t$ be vectors stacking the endogenous variables other than inflation in the economy with our rule and the economy with the alternative rule, respectively. We assume without loss of generality that they are all zero in steady state. By linearity, the equations other than the monetary rule or monetary policy first order condition must have the form:

$$0 = \mathcal{A}x_{t-1} + \mathscr{a}\pi_{t-1} + \mathcal{B}x_t + \mathscr{b}\pi_t + \mathcal{C}\mathbb{E}x_{t+1} + \mathscr{c}\mathbb{E}\pi_{t+1} + \sum_{n=1}^{N} \mathscr{d}_n \varepsilon_{n,t}\,, \tag{10}$$

in the economy with our rule, and they must have the form:

$$0 = \mathcal{A}\tilde{x}_{t-1} + \mathscr{a}\tilde{\pi}_{t-1} + \mathcal{B}\tilde{x}_t + \mathscr{b}\tilde{\pi}_t + \mathcal{C}\mathbb{E}\tilde{x}_{t+1} + \mathscr{c}\mathbb{E}\tilde{\pi}_{t+1} + \sum_{n=1}^{N} \mathscr{d}_n \varepsilon_{n,t},$$

in the economy with the alternative rule. (Here, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are square matrices, while $\mathscr{a}$, $\mathscr{b}$ and $\mathscr{c}$ are scalars, and $\mathscr{d}_1, \dots, \mathscr{d}_N$ are vectors.) Since $\pi_t \equiv \tilde{\pi}_t$, $x_t \equiv \tilde{x}_t$ must solve equation (10). It will be the unique solution providing the model has no source of indeterminacy other than perhaps monetary policy. For example, in a three equation NK model, given that $\pi_t \equiv \tilde{\pi}_t$, the Phillips curve implies that the output gap must agree in the two economies, thus the Euler equation then implies that the interest rate must also agree.

E.6 Roots of the characteristic equation arising from multiperiod bonds

We are interested in the roots for $\lambda \in \mathbb{C}$ of the characteristic equation:

$$\frac{1}{T}\sum_{k=1}^{T} \lambda^{k+S-L} = \phi,$$

for $T, S, L \in \mathbb{N}$ and $\phi > 1$. We wish to prove that this equation has $\max\{0, -(1+S-L)\}$ roots strictly inside the unit circle and $\max\{0, T+S-L\}$ roots strictly outside of the unit circle. We proceed by cases. (These cases have some overlap, which is inconsequential.)

Case 1: $1 + S - L \geq 0$

Note that in this case, $k + S - L \geq 0$ for all $k \in \{1, \dots, T\}$. Thus if $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, then by the triangle inequality:

$$\left|\frac{1}{T}\sum_{k=1}^{T} \lambda^{k+S-L}\right| \leq \frac{1}{T}\sum_{k=1}^{T}\left|\lambda^{k+S-L}\right| \leq \frac{1}{T}\sum_{k=1}^{T} 1 = 1 < \phi.$$

Hence, in this case, there cannot be any roots weakly inside the unit circle. Thus, by the fundamental theorem of algebra, the equation has $\max\{0, T + S - L\}$ roots all strictly outside the unit circle.

Case 2: $T + S - L \leq 0$

In this case, $-(k + S - L) \geq 0$ for all $k \in \{1, \dots, T\}$. Suppose $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$ and define $\kappa := \lambda^{-1}$, so $|\kappa| \leq 1$. Then again by the triangle inequality:

$$\left|\frac{1}{T}\sum_{k=1}^{T} \lambda^{k+S-L}\right| = \left|\frac{1}{T}\sum_{k=1}^{T} \kappa^{-(k+S-L)}\right| \leq \frac{1}{T}\sum_{k=1}^{T}\left|\kappa^{-(k+S-L)}\right| \leq \frac{1}{T}\sum_{k=1}^{T} 1 = 1 < \phi.$$

Hence, in this case, there cannot be any roots weakly outside the unit circle. Thus, by the fundamental theorem of algebra, the equation has $\max\{0, -(1 + S - L)\}$ roots all strictly inside the unit circle.

Case 3: $1 + S - L < 0$ and $T + S - L > 0$

Multiplying the original equation by $\lambda^{-(1+S-L)}$ gives:

$$\frac{1}{T}\sum_{k=1}^{T} \lambda^{k-1} = \phi\lambda^{-(1+S-L)}.$$

This equation has precisely the same roots of the original, since the original contained a term in λ^{1+S-L} and $1 + S - L < 0$. Now if $|\lambda| = 1$, then by the triangle inequality:

$$\left|\frac{1}{T}\sum_{k=1}^{T} \lambda^{k-1}\right| \leq \frac{1}{T}\sum_{k=1}^{T}\left|\lambda^{k-1}\right| = \frac{1}{T}\sum_{k=1}^{T} 1 = 1.$$

Also, if $|\lambda| = 1$, then:

$$\left|-\phi\lambda^{-(1+S-L)}\right| = \phi > 1.$$

Thus, for all $\lambda \in \mathbb{C}$ with $|\lambda| = 1$:

$$\left|\frac{1}{T}\sum_{k=1}^{T} \lambda^{k-1}\right| < \left|-\phi\lambda^{-(1+S-L)}\right|.$$

This means that the original equation cannot have any roots with $|\lambda| = 1$. Additionally, by Rouché's theorem, this implies that the polynomial $\lambda \mapsto \frac{1}{T}\sum_{k=1}^{T}\lambda^{k-1} - \phi\lambda^{-(1+S-L)}$ has the same number of zeros strictly inside the unit circle as the polynomial $\lambda \mapsto -\phi\lambda^{-(1+S-L)}$ (counting multiplicities). The latter polynomial has $-(1+S-L)$ roots inside the unit circle (all equal to zero). Therefore, both $\frac{1}{T}\sum_{k=1}^{T}\lambda^{k-1} = \phi\lambda^{-(1+S-L)}$ and our original equation have $-(1+S-L) = \max\{0, -(1+S-L)\}$ roots strictly inside the unit circle.

Finally, note that $\frac{1}{T}\sum_{k=1}^{T}\lambda^{k-1} = \phi\lambda^{-(1+S-L)}$ is a polynomial of degree $\max\{T-1, -(1+S-L)\}$, hence it has $\max\{T-1, -(1+S-L)\}$ roots in total, by the fundamental theorem of algebra. Hence our original equation has:

$$\begin{aligned}\max\{T-1, -(1+S-L)\} - [-(1+S-L)] &= T - 1 + (1+S-L) = T + S - L \\ &= \max\{0, T+S-L\}\end{aligned}$$

roots strictly outside the unit circle.

This completes the proof.

E.7 Uniqueness and positivity of the multiperiod bond solution

We are interested in the solution of the difference equation:

$$A_j = \frac{1}{\phi}\mathbb{1}[j=0] + \frac{1}{\phi T}\sum_{k=1}^{T} A_{j-k-S+L}.$$

To understand this difference equation, first let $\ell^\infty(\mathbb{Z})$ be the space of bounded sequences with indices in $\mathbb{Z}$. This is a complete normed space under the sup-norm. Then define an operator $\mathcal{T}: \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ by:

$$\left(\mathcal{T}(\tilde{A})\right)_j = \frac{1}{\phi}\mathbb{1}[j=0] + \frac{1}{\phi T}\sum_{k=1}^{T} \tilde{A}_{j-k-S+L},$$

for all $\tilde{A} \in \ell^\infty(\mathbb{Z})$ and $j \in \mathbb{Z}$.

Note that for $A^{(1)}, A^{(2)} \in \ell^\infty(\mathbb{Z})$ and $j \in \mathbb{Z}$:

$$\left(\mathcal{T}(A^{(1)}) - \mathcal{T}(A^{(2)})\right)_j = \frac{1}{\phi T}\sum_{k=1}^{T}\left(A^{(1)}_{j-k-S+L} - A^{(2)}_{j-k-S+L}\right),$$

so:

$$\left|\left(\mathcal{T}(A^{(1)}) - \mathcal{T}(A^{(2)})\right)_j\right| \leq \frac{1}{\phi T}\sum_{k=1}^{T}\left|A^{(1)}_{j-k-S+L} - A^{(2)}_{j-k-S+L}\right| \leq \frac{1}{\phi T}\sum_{k=1}^{T}\left\|A^{(1)} - A^{(2)}\right\|_\infty$$
$$= \frac{1}{\phi}\left\|A^{(1)} - A^{(2)}\right\|_\infty.$$

This means that for all $A^{(1)}, A^{(2)} \in \ell^\infty(\mathbb{Z})$:

$$\left\|\mathcal{T}(A^{(1)}) - \mathcal{T}(A^{(2)})\right\|_\infty \leq \frac{1}{\phi}\left\|A^{(1)} - A^{(2)}\right\|_\infty,$$

and hence that $\mathcal{T}$ is a contraction mapping, as $\phi > 1$. Thus, by the Banach fixed-point theorem, $\mathcal{T}$ has a unique fixed point, which must be our desired $A = \left(A_j\right)_{j\in\mathbb{Z}}$. Furthermore, the Banach fixed point theorem implies that if we define $A_j^{(0)} := 0$ for all $j \in \mathbb{Z}$, and $A^{(n+1)} := \mathcal{T}(A^{(n)})$ for all $n \in \mathbb{N}$, then $A^{(n)} \to A$ (under the sup norm) as $n \to \infty$.

Now, suppose $\tilde{A} \in \ell^\infty(\mathbb{Z})$ with $\tilde{A}_j \geq 0$ for all $j \in \mathbb{Z}$. Then, by the definition of $\mathcal{T}$, $\left(\mathcal{T}(\tilde{A})\right)_j \geq 0$ for all $j \in \mathbb{Z}$. Hence, as $A_j^{(0)} \geq 0$ for all $j \in \mathbb{Z}$, by induction, $A_j^{(n)} \geq 0$ for all $j \in \mathbb{Z}$. Therefore, as $A^{(n)} \to A$ as $n \to \infty$, $A_j \geq 0$ for all $j \in \mathbb{Z}$.

E.8 Solution properties of first welfare example

Recall, that for $k > 1$ the solution must satisfy the recurrence relation:

$$\theta_k + \frac{\lambda}{\kappa^2}(\theta_k - \beta\theta_{k+1}) - \beta\frac{\lambda}{\kappa^2}(\theta_{k-1} - \beta\theta_k) = 0.$$

The characteristic equation of this recurrence relationship has roots:

$$\frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta^2\frac{\lambda}{\kappa^2}\right) \pm \sqrt{\left(1 + \frac{\lambda}{\kappa^2} + \beta^2\frac{\lambda}{\kappa^2}\right)^2 - \left(2\beta\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}$$
$$= \frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta^2\frac{\lambda}{\kappa^2}\right) \pm \sqrt{\left(1 + (1+\beta)^2\frac{\lambda}{\kappa^2}\right)\left(1 + (1-\beta)^2\frac{\lambda}{\kappa^2}\right)}}{2\beta\frac{\lambda}{\kappa^2}}.$$

The positive root satisfies:

$$\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)+\sqrt{\left(1+(1+\beta)^2\frac{\lambda}{\kappa^2}\right)\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)}}{2\beta\frac{\lambda}{\kappa^2}}$$

$$>\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)+\sqrt{\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)}}{2\beta\frac{\lambda}{\kappa^2}}$$

$$=\frac{1+\frac{\lambda}{\kappa^2}-\beta(1-\beta)\frac{\lambda}{\kappa^2}}{\beta\frac{\lambda}{\kappa^2}}>\frac{1+\frac{\lambda}{\kappa^2}-(1-\beta)\frac{\lambda}{\kappa^2}}{\beta\frac{\lambda}{\kappa^2}}=1+\frac{1}{\beta\frac{\lambda}{\kappa^2}}>1.$$

The negative root satisfies:

$$\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)^2-\left(2\beta\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}$$

$$>\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}=0,$$

and:

$$\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+(1+\beta)^2\frac{\lambda}{\kappa^2}\right)\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)}}{2\beta\frac{\lambda}{\kappa^2}}$$

$$<\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)\left(1+(1-\beta)^2\frac{\lambda}{\kappa^2}\right)}}{2\beta\frac{\lambda}{\kappa^2}}=1.$$

Hence, the positive root is greater than 1, while the negative root is in $(0,1)$. Thus for $k \geq 1$:

$$\theta_k=\theta_1\left[\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta^2\frac{\lambda}{\kappa^2}\right)^2-\left(2\beta\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}\right]^{k-1}.$$

Hence, θ_0, θ_1 and θ_2 are the unique solution of the three linear (in θ_0, θ_1 and θ_2) equations:

$$\theta_0 + \frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$
$$\theta_1 + \frac{\lambda}{\kappa^2}(\theta_1 - \beta\theta_2) - \beta\frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$
$$\theta_2 = \theta_1\left[\frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta^2\frac{\lambda}{\kappa^2}\right) - \sqrt{\left(1 + \frac{\lambda}{\kappa^2} + \beta^2\frac{\lambda}{\kappa^2}\right)^2 - \left(2\beta\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}\right].$$

E.9 Solution under discretion of first welfare example

Under discretion, we have the standard first order condition:

$$\pi_t + \frac{\lambda}{\kappa}x_t = 0,$$

i.e.:

$$\kappa\sum_{k=0}^{\infty}\theta_k\omega_{t-k} + \frac{\lambda}{\kappa}\sum_{k=0}^{\infty}(\theta_k - \beta\theta_{k+1} - \mathbb{1}[k=0])\omega_{t-k} = 0,$$

so:

$$\theta_0 + \frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$
$$\forall k \geq 1, \qquad \theta_k + \frac{\lambda}{\kappa^2}(\theta_k - \beta\theta_{k+1}) = 0.$$

The latter recurrence relation has the general solution $\theta_k = \theta_1\left(\frac{\kappa^2}{\beta\lambda} + \frac{1}{\beta}\right)^{k-1}$, which is explosive as $\beta < 1$. Thus, we must have $\theta_1 = \theta_2 = \cdots = 0$. Hence, $\theta_0 = \frac{\lambda}{\lambda+\kappa^2}$.

E.10 Solution under the timeless perspective of first welfare example

The timeless perspective (Woodford 1999) leads to the first order condition:

$$\pi_t + \frac{\lambda}{\kappa}(x_t - x_{t-1}) = 0,$$

i.e.:

$$\kappa\sum_{k=0}^{\infty}\theta_k\omega_{t-k} + \frac{\lambda}{\kappa}\left[\sum_{k=0}^{\infty}(\theta_k - \beta\theta_{k+1} - \mathbb{1}[k=0])\omega_{t-k} - \sum_{k=1}^{\infty}(\theta_{k-1} - \beta\theta_k - \mathbb{1}[k-1=0])\omega_{t-k}\right] = 0,$$

so:

$$\theta_0 + \frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$
$$\theta_1 + \frac{\lambda}{\kappa^2}(\theta_1 - \beta\theta_2) - \frac{\lambda}{\kappa^2}(\theta_0 - \beta\theta_1 - 1) = 0,$$
$$\forall k > 1, \qquad \theta_k + \frac{\lambda}{\kappa^2}(\theta_k - \beta\theta_{k+1}) - \frac{\lambda}{\kappa^2}(\theta_{k-1} - \beta\theta_k) = 0.$$

The roots of the characteristic equation corresponding to the latter recurrence relation are:

$$\frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right) \pm \sqrt{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right)^2 - 4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}.$$

The positive root satisfies:

$$\frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right) + \sqrt{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right)^2 - 4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}} > \frac{\frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}}{2\beta\frac{\lambda}{\kappa^2}} = \frac{1+\beta}{2\beta} > 1.$$

The negative root satisfies:

$$\begin{aligned}&\frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right) - \sqrt{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right)^2 - 4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}} \\ &\qquad > \frac{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right) - \sqrt{\left(1 + \frac{\lambda}{\kappa^2} + \beta\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}} = 0,\end{aligned}$$

and:

$$\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)^2-4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}$$

$$=\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{1+(1-\beta)^2\left(\frac{\lambda}{\kappa^2}\right)^2+2(1+\beta)\frac{\lambda}{\kappa^2}}}{2\beta\frac{\lambda}{\kappa^2}}$$

$$<\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{1+(1-\beta)^2\left(\frac{\lambda}{\kappa^2}\right)^2+2(1-\beta)\frac{\lambda}{\kappa^2}}}{2\beta\frac{\lambda}{\kappa^2}}$$

$$=\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+(1-\beta)\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}=\frac{2\beta\frac{\lambda}{\kappa^2}}{2\beta\frac{\lambda}{\kappa^2}}=1.$$

Hence, the positive root is greater than 1, while the negative root is in $(0,1)$. Thus for $k\geq 1$:

$$\theta_k=\theta_1\left[\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)^2-4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}\right]^{k-1}.$$

Hence, θ_0, θ_1 and θ_2 are the unique solution of the three linear (in θ_0, θ_1 and θ_2) equations:

$$\theta_0+\frac{\lambda}{\kappa^2}(\theta_0-\beta\theta_1-1)=0,$$

$$\theta_1+\frac{\lambda}{\kappa^2}(\theta_1-\beta\theta_2)-\frac{\lambda}{\kappa^2}(\theta_0-\beta\theta_1-1)=0,$$

$$\theta_2=\theta_1\left[\frac{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)-\sqrt{\left(1+\frac{\lambda}{\kappa^2}+\beta\frac{\lambda}{\kappa^2}\right)^2-4\beta\left(\frac{\lambda}{\kappa^2}\right)^2}}{2\beta\frac{\lambda}{\kappa^2}}\right].$$