# Galois representations and their deformations.

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Goal: study representations of Galois groups of p-adic or number fields with p-adic coefficients, by "deforming" representations with modulo p coefficients.

References:
* Gouvea, notes
* Mézard, notes
* Böckle, notes
* Mazur's article "Deforming Galois representations"

## 01. Galois groups

K perfect field, L a normal extension of K. Define $Gal(L/K)$
$= \{ \sigma : L \to L \mid \sigma \text{ field automorphism}, \sigma|_K = id_K \}$

· Topology:
* if $L|K$ is finite then give $Gal(L|K)$ the discrete topology
* if $L|K$ is infinite then give $Gal(L|K)$ the Krull topology:

a basis of open neighborhoods of $id_L$ is the collection of sets

$$\{ \sigma \in Gal(L/K) \mid \sigma|_E = id_E \}$$ where E varies over the finite subextensions $E|K$

As groups: $$Gal(L/K) \cong \varprojlim_{\substack{E|K \text{ finite and normal} \\ E \subseteq L}} Gal(E|K)$$

If $Gal(E|K)$ has the discrete topology, then this is an isomorphism of topological groups.

This makes $Gal(L/K)$ into a profinite group.

$\Rightarrow$ $Gal(L|K)$ is compact and hausdorff.

$\Rightarrow$ Open subgroups are the closed subgroups of finite index

**Theorem.** (Galois correspondence) The map

$$\left\{ \begin{matrix} \text{subextensions} \\ L | E | K \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{closed subgroups} \\ \text{of } Gal(L|K) \end{matrix} \right\}$$

$$E \longmapsto Gal(L|E) \qquad \text{is a bijection.}$$

The inverse is $H \mapsto E = L^H$.

This induces a bijection

$$\left\{ \begin{matrix} L|E|K, \\ E|K \text{ finite} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{open subgroups} \\ \text{of } Gal(L|K) \end{matrix} \right\}$$

When $L = K^{alg}$, we call $Gal(K^{alg}|K)$ the absolute Galois group of $K$, we write $G_K$.

Examples. * $K = \mathbb{F}_p$, we know that finite extensions are of the form $\mathbb{F}_{p^n}$ and

$$Gal(\mathbb{F}_{p^n} / \mathbb{F}_p) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z} \qquad \text{is an isomorphism.}$$

$$(x \mapsto x^p) \longmapsto 1$$

"Frobenius element"

$$Gal(\mathbb{F}_p^{alg} / \mathbb{F}_p) = \varprojlim_{n \to \infty} Gal(\mathbb{F}_{p^n} / \mathbb{F}_p) \xrightarrow{\sim} \varprojlim_{n \to \infty} \mathbb{Z}/n\mathbb{Z} =: \hat{\mathbb{Z}}$$

(the maps are $Gal(\mathbb{F}_{p^n} / \mathbb{F}_p) \longrightarrow Gal(\mathbb{F}_{p^m} / \mathbb{F}_p)$ $\quad m, n$ with $m | n$

$$\sigma \longmapsto \sigma|_{\mathbb{F}_{p^m}}$$

$$\mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z}$$

$$x \longmapsto x \bmod m$$

The Frobenius of $Gal(\mathbb{F}_p^{alg} / \mathbb{F}_p)$ is mapped to $1 \in \hat{\mathbb{Z}}$.

* $K = \mathbb{Q}_p$; we denote by $\mathbb{Q}_p^{ur}$ the maximal unramified extension of $\mathbb{Q}_p$.
($\Leftrightarrow$ maximal extension for which $p$ is still a uniformizer (generator of the maximal ideal of the valuation ring))

Valuation ring $\mathbb{Z}_p^{ur}$ has residue field $\mathbb{Z}_p^{ur} / p\mathbb{Z}_p^{ur} \cong \mathbb{F}_p^{alg}$

There is a map $Gal(\mathbb{Q}_p^{ur} / \mathbb{Q}_p) \longrightarrow Gal(\mathbb{F}_p^{alg} / \mathbb{F}_p)$

$$\sigma \longmapsto \sigma \text{ modulo } p$$

This map is a group isomorphism.

$$\begin{array}{l} \mathbb{Q}_p^{alg} \\ \Big| \Big) \text{ "Inertia group" } (I_p) \\ \mathbb{Q}_p^{ur} \\ \Big| \Big) \mathrm{Gal} = \hat{\mathbb{Z}} \quad \text{(topologically generated by } \mathrm{Frob}_p) \\ \mathbb{Q}_p \end{array}$$

* $K = \mathbb{Q}$, $\mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$, let $p$ be prime.
Choose an extension of the $p$-adic valuation on $\mathbb{Q}$ to $\mathbb{Q}^{alg}$ (not unique!)
($\leftrightarrow$ choose an embedding $\mathbb{Q}^{alg} \to \mathbb{Q}_p^{alg}$)

Write a map
$$\mathrm{Gal}(\mathbb{Q}_p^{alg}/\mathbb{Q}_p) \longrightarrow \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$$
$$\sigma \longmapsto \sigma|_{\mathbb{Q}^{alg}}$$

This map is an injective group homomorphism and it identifies $\mathrm{Gal}(\mathbb{Q}_p^{alg}/\mathbb{Q}_p)$ with a subgroup $D_p \leq \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$

$$D_p = \{ \sigma \in \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q}) \mid v(\sigma^{-1}(x)) = v(x) \quad \forall x \in \mathbb{Q}^{alg} \}$$

We have injections
$$I_p \leq D_p \subseteq \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$$

## 02. Galois groups for extensions unramified outside a finite set.

$K$ number field, $S$ is a finite set of places of $K$.

Def. 1) $K^S$ is the largest extension of $K$ that is unramified at all places not in $S$.

2) $G_{K,S} := \mathrm{Gal}(K^S/K)$

Remark. $v \notin S$, then we have

$$\begin{tikzcd} \mathrm{Gal}(K_v^{alg}/K_v) \arrow[r, hook] \arrow[rr, bend right, "\text{Injection ?}"'] & \mathrm{Gal}(K^{alg}/K) \arrow[r] & \mathrm{Gal}(K^S/K) \end{tikzcd}$$

Answer: Injection when $K = \mathbb{Q}$, $\# S \geq 2$

Exercise. An open subgroup $H \leq G_{K,S}$ has the form $G_{K_1,S_1}$ where $K_1 | K$ is finite and $S_1$ is a set of places of $K_1$.

(places in $S_1$ have to lie over the places of $S$ $\Rightarrow$ $S_1$ finite)

Theorem. (Hasse - Minkowski) Let $K, S$ as before.
Let $d \in \mathbb{N}_{>0}$. Then there are only finitely many extensions of $K$ of degree $d$ and unramified outside $S$.

Corollary. $\mathrm{Hom}_{cont}(G_{K,S}, \mathbb{F}_p)$ is finite. (Morphisms of topological groups)

Corollary. For every open subgroup $H \leq G_{K,S}$, $\mathrm{Hom}_{cont}(H, \mathbb{F}_p)$ is finite.

$\underset{\text{Def.}}{\Longleftrightarrow}$ $G_{K,S}$ satisfies the "p-finiteness condition".

($\mathbb{F}_p$ has the discrete topology)

# 03. Galois representations

Let $G$ be a profinite group and let $A$ be a topological ring.

Def. A continuous representation of $G$ with $A$-coefficients is a continuous group homomorphism $\rho: G \longrightarrow GL_n(A)$ for some integer $n$.

Given representations $\rho_1, \rho_2 : G \longrightarrow GL_n(A)$, we say that they are equivalent iff. $\exists\, P \in GL_n(A)$ : $P^{-1} \cdot \rho_1 \cdot P = \rho_2$

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Another point of view: let $M$ be a finite free $A$-module of rank $n$.
Then a continuous representation $\rho: G \longrightarrow GL_n(A)$ gives a continuous action $G \curvearrowright M$

$$G \times M \longrightarrow M$$
$$(g, m) \longmapsto \rho(g)\,(m_i)_{i=1..n}$$

where $(m_i)_{i=1..n}$ are coordinates of $m$

Def.
We call $\rho$ a Galois representation if $G$ is:
- $\ast$ $Gal(K^{alg}/K)$ for a finite extension $K|\mathbb{Q}_p$
- $\ast$ $G_{K,S}$ for a number field $K$ and a finite set of places $K$.

From now on $G$ is one of those groups.

Choices of coefficient ring $A$:

1) $A = \mathbb{C}$
2) $A = \mathbb{F}_{p^n}$
3) $A = \mathcal{O}_E$ for $E|\mathbb{Q}_p$ finite
4) $A = E$, $E|\mathbb{Q}_p$ finite

① $A = \mathbb{C}$

Proposition. A representation $\rho : G \longrightarrow GL_n(\mathbb{C})$ has finite image.

Proof. Consider an open neighborhood $\overset{\mathcal{U}}{\vee}$ of $\mathbb{1}_n \in GL_n(\mathbb{C})$. If $\mathcal{U}$ is sufficiently small, the only subgroup $\subseteq \mathcal{U}$ is $\{\mathbb{1}_n\}$ (Exercise). By continuity of $\rho$, $\exists\, V$ open neighborhood of $e \in G$ such that $\rho(V) \subseteq \mathcal{U}$. We can choose $V'$ a neighborhood of $e$ which is an open subgroup of $G$ and s.t. $V' \subseteq V$. Then $\rho(V') \subseteq \rho(V) \subseteq \mathcal{U}$.
$\Rightarrow \rho(V') = \{\mathbb{1}_n\}$. Since $V'$ is of finite index in $G$, $\rho(V')$ is of finite index in $\rho(G)$. ▨

Expl. Take $K|\mathbb{Q}$ Galois, finite. Then $Gal(K|\mathbb{Q}) \hookrightarrow GL_n(\mathbb{C})$.

(2) Proposition. If $\rho : G \to GL_n(\mathbb{F}_p^{alg})$ is a continuous representation,
$GL_n(\mathbb{Q}_p)$

then it factors through $\rho' : G \to GL_n(\mathbb{F}_{p^m})$ for some finite extension
$\to GL_n(E)$

$\mathbb{F}_{p^m} / \mathbb{F}_p$

$E / \mathbb{Q}_p$

Proof. Similar for $\mathbb{F}_p$, more difficult for $\mathbb{Q}_p$. ∎

Proof of the last Proposition.

Look at $\rho(G) \leq GL_n(\overline{\mathbb{Q}}_p)$. It is a compact Hausdorff topological group

$\Rightarrow$ Baire's Lemma holds for $\rho(G)$.

( Baire's Lemma: a countable union of nowhere dense closed subspaces of $X$ is nowhere dense in $X$.

Nowhere dense: it does not contain any open set of $X$ )

$$GL_n(\overline{\mathbb{Q}}_p) = \bigcup_{\substack{E/\mathbb{Q}_p \\ \text{finite}}} GL_n(E)$$ countable union of closed subsets.

( $\forall n \in \mathbb{N}$: there are only finitely many $E \mid \mathbb{Q}_p$ s.t. $[E:\mathbb{Q}_p] = n$ )

Write $\rho(G) = \bigcup_{\substack{E \mid \mathbb{Q}_p \\ \text{finite}}} (GL_n(E) \cap \rho(G))$. Either there exists $E/\mathbb{Q}_p$ finite such

that $GL_n(E) \cap \rho(G)$ has finite index in $\rho(G)$

$\Rightarrow$ We can choose $F \mid E$ finite such that $\rho(G) \subseteq GL_n(F)$ ( finite index ... )

Or for every $E/\mathbb{Q}_p$ finite, $GL_n(E) \cap \rho(G)$ has infinite index in $\rho(G)$

$\Rightarrow$ $GL_n(E) \cap \rho(G)$ is nowhere dense in $\rho(G)$ ( Basis of open subgroups $\Rightarrow$ open subgroups in compact spaces are of finite index)

Now $\rho(G)$ is a countable union of nowhere dense sets $\Rightarrow$ Contradicts Baire's Lemma.

$\square$

**Lemma.** If $\rho: G \longrightarrow GL_n(K)$ is a continuous representation with coefficients in $K/\mathbb{Q}_p$ finite, then there exists a continuous representation $\rho': G \longrightarrow GL_n(\mathcal{O}_K)$ such that if $\iota: GL_n(\mathcal{O}_K) \longrightarrow GL_n(K)$ is the inclusion $\rho \underset{\substack{\uparrow \\ \text{equivalent}}}{\cong} \iota \circ \rho'$

Proof. Recall: an $\mathcal{O}_K$-lattice in $K^n$ is a free $\mathcal{O}_K$-module $L$ of rank $n$ such that $L \otimes_{\mathcal{O}_K} K \cong K^n$.

Choosing a basis for $K^n$ we obtain a continuous action of $G$ on $K^n$ via $\rho$.

Let $L$ be any lattice in $K^n$. For $g \in GL_n(K)$ let $g(L) := \{g(x) \mid x \in L\}$

Exercise: $g(L)$ is a lattice, and $\mathrm{Stab}(L) = \{g \in GL_n(K) \mid g(L) \subseteq L\}$ is an open subgroup of $GL_n(K)$. EX

Look at $\rho^{-1}(\underbrace{\mathrm{Stab}(L)}_{\underset{\text{open}}{\subseteq} GL_n(K)}) \underset{\text{open}}{\subseteq} G \underset{G \text{ compact}}{\Longrightarrow} \rho^{-1}(\mathrm{Stab}(L))$ has finite index in $G$.

Choose a set $\{g_1, \ldots, g_m\}$ of representatives for $G/\rho^{-1}(\mathrm{Stab}(L))$.

Then define $L' := \sum_{i=1}^{m} \rho(g_i)(L)$. We check that the lattice $L'$ is $G$-stable.

( $G$-stable: $\rho(g) L' \subseteq L' \quad \forall g \in G$ )

Choose an $\mathcal{O}_K$-basis for the lattice $L'$, then the action of $G$ on $L'$ gives a (continuous) representation $\rho' : G \to GL_n(\mathcal{O}_K)$

By construction $\iota \circ \rho' \sim \rho$

$\blacksquare$

Start with $\rho : G \to GL_n(K)$ continuous representation.

Then by the Lemma we can choose a conjugate of $\rho$ with values in $GL_n(\mathcal{O}_K)$.

Then we can reduce modulo the maximal ideal $\mathfrak{m}_K \triangleleft \mathcal{O}_K$ and we obtain a "residual" representation $\bar{\rho} : G \to GL_n(\underbrace{\mathcal{O}_K/\mathfrak{m}_K}_{=\mathbb{F}_{p^m}})$ attached to $\rho$.

**Def.** If $G$ acts on a finite free module $M$. Choose a filtration $M \supsetneq M_n \supsetneq \ldots \supsetneq \{0\}$ in $G$-stable $A$-modules ($A$: field) such that $M_i/M_{i-1}$ is an irreducible $A[G]$-module.

( Does not admit any $G$-stable submodule )

Then the semi-simplification of $M$ ist the $A[G]$-module $\bigoplus_{i=1}^{n} M_i/M_{i-1}$.

Example: If $\underset{g \in G}{\rho(g)} = \begin{pmatrix} \chi_1(g) & \delta(g) \\ 0 & \chi_2(g) \end{pmatrix} \to \bar{\rho}^{ss}(g) = \begin{pmatrix} \chi_1(g) & 0 \\ 0 & \chi_2(g) \end{pmatrix}$

$\chi_1, \chi_2$: Character of $\rho$

Remark: the representation $\bar{\rho}^{ss}$ attached to $\rho$ is well-defined up to equivalence.

$$\begin{tikzcd} K \arrow[r] & \mathcal{O}_K \arrow[r] & \mathbb{F}_{p^m} \arrow[ll, bend left, dashed] \end{tikzcd}$$

Idea: fix $\bar{\rho} : G \to GL_n(\mathbb{F}_{p^m})$ and look at $\rho : G \to GL_n(\mathcal{O}_K)$ (with $\mathcal{O}_K/m_K = \mathbb{F}_{p^m}$) such that $\rho \bmod m_K = \bar{\rho}$.

Example of $p$-adic Galois representation. ("$p$-adic cyclotomic character")

($p$ prime, $n \in \mathbb{N}_{\geq 1}$)

$$\chi_n : G_{\mathbb{Q}} \longrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times \quad (= GL_1(\mathbb{Z}/p^n\mathbb{Z}))$$

This representations are compatible with the maps $(\mathbb{Z}/p^m\mathbb{Z})^\times \to (\mathbb{Z}/p^n\mathbb{Z})^\times$ for $m \geq n$.

$$\chi_m \text{ modulo } p^n = \chi_n .$$

We can take $\underbrace{\varprojlim_n \chi_n}_{=:\,\chi_{cyc}} : G_{\mathbb{Q}} \longrightarrow \varprojlim_n \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) = \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})^\times = \mathbb{Z}_p^\times$

Write $\mathbb{Q}(\zeta_{p^\infty}) = \bigcup_{n \geq 1} \mathbb{Q}(\zeta_{p^n})$

We call $\chi_{cyc}$ the $p$-adic cyclotomic character. $\chi_{cyc}$ factors through

$G_{\mathbb{Q},p^\infty} \longrightarrow \mathbb{Z}_p^\times$. It also factors through $G_{\mathbb{Q}}^{ab} \longrightarrow \mathbb{Z}_p^\times$.

Theorem: (Kronecker-Weber) The product of all cyclotomic characters gives an isomorphism

$$G_{\mathbb{Q}}^{ab} \xrightarrow{\sim} \prod_p \mathbb{Z}_p^\times .$$

Look at "deformation functors".

$*$ $\mathcal{C}$ category, $R \in \mathcal{C}$, then $\overbrace{\mathrm{Hom}_{\mathcal{C}}(R,\cdot)}^{=:\,h_R} : \mathcal{C} \to \underline{\mathrm{Set}}$ is the functor

$A \mapsto \mathrm{Hom}_{\mathcal{C}}(R,A)$, $f \in \mathrm{Mor}_{\mathcal{C}}(A,B) \mapsto \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(R,A) & \to & \mathrm{Hom}_{\mathcal{C}}(R,B) \\ g & \mapsto & f \circ g \end{array} \right.$

We will work with some categories of rings.

Fix a field $k$. We denote by $\mathcal{C}_k$ the category whose objects are Artinian, local rings with residue field $k$ and morphisms are local ring morphisms, that induce the <u>identity</u> on $k$.

Examples. * $k = \mathbb{F}_p$, then $\mathbb{Z}/p^n\mathbb{Z} \in \mathcal{C}_{\mathbb{F}_p}$ $\forall n \in \mathbb{N}_{>0}$.

$$\mathbb{F}_p[T]/T^n \in \mathcal{C}_{\mathbb{F}_p}$$

* $\exists$ unique degree $n$ unramified extension of $\mathbb{Q}_p$, we will denote it by $\mathbb{Q}_{p^n}$. We write $\mathbb{Z}_{p^n}$ for its valuation ring, then $\mathbb{Z}_{p^n}/p\mathbb{Z}_{p^n} = \mathbb{F}_{p^n}$

$\forall m \in \mathbb{N}_{>0}$: $\mathbb{Z}_{p^n}/p^m\mathbb{Z}_{p^n} \in \mathcal{C}_{\mathbb{F}_{p^m}}$

$$\mathbb{Z}_{p^n}/p^m\mathbb{Z}_{p^n} \longrightarrow \mathbb{Z}_{p^n}/p^m\mathbb{Z}_{p^n}$$

This is not a morphism in $\mathcal{C}_{\mathbb{F}_{p^m}}$ if $m \geq 2$.

$$x \text{ modulo } p^m \longmapsto \mathrm{Frob}_p(x) \text{ modulo } p^m$$

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Let $\hat{\mathcal{C}}_k$ be the category whose objects are complete local Noetherian rings with residue field $k$, morphisms are local ring morphisms that induce the identity on $k$.

Example. $\mathbb{Z}_{p^n} \in \hat{\mathcal{C}}_{\mathbb{F}_{p^n}} \ni \mathbb{F}_{p^n}[[T]]$

An object of $\mathcal{C}_k$ is also an object in $\hat{\mathcal{C}}_k$ (same for morphisms)

## DEFORMATION FUNCTORS

Fix $n \geq 1$.

Let $G$ be a profinite group, $k$ a finite field. Fix a continuous representation $\bar{\rho}: G \to GL_n(k)$.

Def. For $A \in \hat{\mathcal{C}}_k$, a deformation (of $\bar{\rho}$ to $A$) is a continuous representation $\rho: G \to GL_n(A)$ such that $\rho$ modulo $m_A = \bar{\rho}$.

We say that $\rho_1, \rho_2 : G \to GL_n(A)$ are strictly equivalent iff. $\exists M \in \ker(GL_n(A) \to GL_n(k))$ such that $M^{-1}\rho_1 M = \rho_2$.

Remark: if $A \xrightarrow{f} B$ is a morphism in $\hat{\mathcal{C}}_k$ and $\rho_1, \rho_2 : G \to GL_n(A)$ are strictly equivalent representations, then $f\rho_1, f\rho_2 : G \to GL_n(B)$ are strictly equivalent.

We define $\hat{D}_{\bar\rho} : \hat{\mathcal{C}}_k \longrightarrow \underline{Set}$ as the functor

* $\hat{D}_{\bar\rho}(A) := \left\{ \begin{matrix} \text{deformations of} \\ \bar\rho \text{ to } A \end{matrix} \right\} \Big/ \text{strict equivalence}$

* $\hat{D}_{\bar\rho}(f)$ maps a deformation $\rho : G \to GL_n(A)$ to the class of $f \circ \rho$
$f : A \to B$ morphism in $\hat{\mathcal{C}}_k$

(We obtain a functor $D_{\bar\rho} : \mathcal{C}_k \longrightarrow \underline{Set}$ by restricting $\hat{D}_{\bar\rho}$ to $\mathcal{C}_k$)

Goal: show that $D_{\bar\rho}$ is "pro-represented" by some $R \in \hat{\mathcal{C}}_k$, in the sense that $D_{\bar\rho} \cong \mathrm{Hom}_{\hat{\mathcal{C}}_k}(R, \cdot)$

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Fix any $\Lambda \in \hat{\mathcal{C}}_k$. We define $\mathcal{C}_\Lambda$ as the category of local Artinian $\Lambda$-algebras with residue field $k$ and local morphisms of $\Lambda$-algebras (that induce the identity on $k$).

* $\hat{\mathcal{C}}_\Lambda$ = category of complete local Noetherian $\Lambda$-algebras with residue field $k$. Morphisms as above.

The natural choice for $\Lambda$ (when working in char $\Lambda = 0$) is the ring of Witt vectors $W(k)$ of $k$.
(= unique complete discrete valuation ring with residue field $k$ and uniformizer $p$).
(Mézard's notes, Serre "Local fields")

We only need
$$W(\mathbb{F}_{p^n}) = \mathbb{Z}_{p^n} \quad \text{ring of integers in } \mathbb{Q}_{p^n}$$

$$\hat{\mathcal{C}}_{\mathbb{Z}_{p^n}} \ni \mathbb{Z}_{p^n}, \quad \mathbb{F}_{p^n}[[T]] \in \hat{\mathcal{C}}_{\mathbb{F}_{p^n}}, \text{ but } \notin \hat{\mathcal{C}}_{\mathbb{Z}_{p^n}}.$$

<u>Deformation functors</u> Fix $\bar\rho : G \to GL_n(k)$; We defined $D_{\bar\rho} : \mathcal{C}_k \longrightarrow \underline{Set}$, $\hat{D}_{\bar\rho} : \hat{\mathcal{C}}_k \to \underline{Set}$.

One defines
$$D_{\bar\rho,\Lambda} : \mathcal{C}_\Lambda \longrightarrow \underline{Set} \quad \text{by} \quad D_{\bar\rho,\Lambda}(A) = \left\{ \begin{matrix} \text{deformations of} \\ \bar\rho \text{ to } A \end{matrix} \right\} \Big/ \text{strict equivalence}$$

$\hat{D}_{\bar\rho,\Lambda} : \hat{\mathcal{C}}_\Lambda \longrightarrow \underline{Set}$ in the same way.

# Representable Functors.

Let $\mathcal{C}$ be any category, $\mathcal{F}: \mathcal{C} \to \underline{Set}$ any functor.

* $\mathcal{F}$ is called representable $:\Leftrightarrow \exists A \in \mathcal{C}: \mathcal{F} \cong Hom_{\mathcal{C}}(A, \cdot)$ as functors.

For our categories $\mathcal{C}_k, \hat{\mathcal{C}}_k, \ldots$ we define the following:

* $\mathcal{F}: \mathcal{C}_\Lambda \to \underline{Set}$ is pro-representable $:\Leftrightarrow \exists R \in \hat{\mathcal{C}}_\Lambda$ and an isomorphism of functors

$$Hom_{\hat{\mathcal{C}}}(R, \cdot) \longrightarrow \mathcal{F}$$

(as a functor $\mathcal{C}_\Lambda \to \underline{Set}$ )

Why representability?

Assume $D_{\bar{\rho},\Lambda}$ is (pro-) representable by some $R$ in $\hat{\mathcal{C}}_\Lambda$: means that there is a bijection $Hom_{\hat{\mathcal{C}}_\Lambda}(R, A) \longrightarrow D_{\bar{\rho},\Lambda}(A)$ for every $A \in \mathcal{C}_\Lambda$.

Take $A = R$. Then there is a strict equivalence class $\rho_R$ in $D_{\bar{\rho},\Lambda}(R)$ corresponding to $id_R$.

Then the bijection

$$Hom_{\hat{\mathcal{C}}_\Lambda}(R, A) \longrightarrow D_{\bar{\rho},\Lambda}(A)$$

$$(f: R \to A) \longmapsto \text{class of } f \circ \rho_R : G \to GL_n(A)$$

We call $(R, \rho_R)$ a universal couple, $R$ is the universal deformation ring of $\bar{\rho}$, $\rho_R$ is the universal deformation of $\bar{\rho}$.

Properties of (pro-) representable functors $\mathcal{C}_\Lambda \xrightarrow{\mathcal{F}} \underline{Set}$. In this section $\mathcal{F}$ is a pro-representable-Functor $\mathcal{C}_\Lambda \to \underline{Set}$, $\mathcal{F} \cong Hom_{\hat{\mathcal{C}}_\Lambda}(R, \cdot)$

- left exact
- $\mathcal{F}(k) = \{*\}$, since the only element of $Hom_{\hat{\mathcal{C}}_\Lambda}(R, k)$ is $R \to R/m_R$ ($\cong k$) one fixed identification
- ~~$\mathcal{F}$ is continuous~~ (later)
- $\mathcal{F}$ behaves well with respect to fiber products

$$k[\varepsilon] = k[T]/T^2 \,, \quad \varepsilon = \overline{T} \qquad (\text{ring with } \varepsilon^2 = 0)$$

$$k[\varepsilon] = \{ a + b\varepsilon \mid a, b \in k \}, \quad \varepsilon^2 = 0$$

- $F(k[\varepsilon])$ is a finite dimensional $k$-vector space.

Remark: fiber products don't always exist in $\hat{\mathcal{C}}_\Lambda$ $\qquad \Lambda = W(k)$

Example $A = k[[X, Y]]$, $C = k[[X]]$, $B = k$

maps

$$\begin{array}{ll} A \longrightarrow C, & B \longrightarrow C \\ Y \longmapsto 0 & k \hookrightarrow k[[X]] \end{array}$$

Check: the fiber product $A \times_C B$ is not in the category $\hat{\mathcal{C}}_\Lambda$ (the fiber product as rings is not Noetherian ... this argument should not work)

Fiber products exist in $\mathcal{C}_\Lambda$.

$F$ has the Mayer-Vietoris property if the map $F(A \times_C B) \longrightarrow F(A) \times_{F(C)} F(B)$ is a bijection.

Remark: $F$ pro-representable $\Rightarrow$ $F$ has the Mayer-Vietoris-property.

- $F(k[\epsilon])$ is a $k$-vector space:

Why is it a $k$-vector space? It is when the map $F(k[\epsilon] \times_k k[\epsilon]) \hookrightarrow F(k[\epsilon]) \times_{F(k)} F(k[\epsilon])$ is a bijection. (induced by $\begin{array}{c} k[\epsilon] \to k \leftarrow k[\epsilon] \\ \epsilon \longmapsto 0 \longleftarrow \epsilon \end{array}$)

$k$-scalar multiplication is induced by

$$k \times k[\epsilon] \longrightarrow k[\epsilon]$$
$$(\lambda, a + b\epsilon) \longmapsto (a + \lambda b \epsilon)$$

addition is induced by

$$k[\epsilon] \times_k k[\epsilon] \longrightarrow k[\epsilon]$$
$$(a + b_1\epsilon, a + b_2\epsilon) \longmapsto a + (b_1 + b_2)\epsilon$$

In this case we call $F(k[\epsilon])$ the tangent space of $F$. Why?

When $F$ is pro-representable, $F \cong \mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R, \cdot)$ then there is an isomorphism of $k$-vector spaces

$$t_R \longrightarrow F(k[\epsilon])$$

Recall: for $R$ a local Noetherian $\Lambda$-algebra with maximal ideal $m_R$, then $t_R^* = m_R/(m_R^2, m_\Lambda)$ is the cotangent space of $R$, $t_R = \mathrm{Hom}_k(t_R^*, k)$.

Idea of proof: $f \in F(k[\epsilon]) = \mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R, k[\epsilon])$

then $\underset{\text{for } r \in R}{f(r)} = \underset{\substack{\text{image} \\ \text{mod } m_R}}{\bar{r}} + f'(r)\epsilon$

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Criteria for representability:

Criterion. (Grothendieck) Let $F: \mathcal{C}_\Lambda \longrightarrow \underline{\mathrm{Set}}$ be a functor such that $F(k) = \{*\}$. Then $F$ is pro-representable if and only if the following hold:

i) $F$ has the Mayer-Vietoris property

ii) $F(k[\epsilon])$ is a finite dimensional vector space $/k$.

We will use a refined version:

Criterion. (Schlessinger)

Def. If $R, S \in \hat{\mathcal{C}}_\Lambda$, $f: R \to S$ a morphism. We say that $f$ is small if it is surjective and if $\ker f$ is annihilated by $m_R$ $(m_R \cdot \ker f = 0)$ $\left(\text{Exp.: } \begin{array}{c} k[\epsilon] \to k \\ \epsilon \mapsto 0 \end{array}\right)$

We introduce the Schlessinger conditions. $R_0, R_1, R_2 \in \mathcal{C}_\Lambda$, $\varphi_1: R_1 \to R_0$, $\varphi_2: R_2 \to R_0$,

$R_3 := R_1 \times_{R_0} R_2$ ; $(*)$: $F(R_3) \longrightarrow F(R_1) \times_{F(R_0)} F(R_2)$

 

The conditions are:

H1: If $\varphi_2 : R_2 \to R_0$ is small, then (*) is surjective

H2: If $R_0 = k$, $R_2 = k[\epsilon]$ with $\begin{array}{c} R_2 \to R_0 \\ \epsilon \mapsto 0 \end{array}$, then (*) is bijective

H3: $F(k[\epsilon])$ is finite-dimensional over $k$.

H4: If $R_1 = R_2$, $\varphi_1 = \varphi_2$ is small, then (*) is bijective.

Theorem of Schlessinger: If $F(k) = \{*\}$ and it satisfies $H_1 - H_4$, then $F$ is pro-representable.

09.05.18

## Continuity of Functors

$F : \hat{\mathcal{C}}_\Lambda \to \underline{\text{Set}}$ functor. We say that $F$ is continuous if the natural map

$$F(A) \longrightarrow \varprojlim_n F(A/m_A^n) \quad \text{is a bijection } \forall A \in \hat{\mathcal{C}}_\Lambda$$

Exercise:
* $F$ representable $\Rightarrow$ $F$ continuous
* $\hat{D}_{\bar{\rho},\Lambda}$ is continuous
* If we extend $D_{\bar{\rho},\Lambda}$ by continuity ( $D_{\bar{\rho},\Lambda}(A) := \varprojlim_n D_{\bar{\rho},A}(A/m_A^n)$ for $A \in \hat{\mathcal{C}}_\Lambda$), we get $\hat{D}_{\bar{\rho},\Lambda}$
* If $D_{\bar{\rho},\Lambda}$ is pro-represented by some $R \in \hat{\mathcal{C}}_\Lambda$, then $\hat{D}_{\bar{\rho},\Lambda} \cong \mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R, \cdot)$

## Existence of Universal Deformation Pairs

$G$, $k$, $\Lambda \in \hat{\mathcal{C}}_k$, fix $\bar{\rho} : G \to GL_n(k)$

We want to show $D_{\bar{\rho},\Lambda}$ is pro-representable by proving it satisfies $H_1 - H_4$ of Schlessingers Criterion.

**Theorem.** Assume $G$ satisfies the $p$-finiteness condition

$(\Phi_p)$ $\mathrm{Hom}_{\mathrm{cont}}(H, \mathbb{Z}/p\mathbb{Z})$ is a finite set $\forall$ open subgroups $H \leq G$

Then $D_{\bar\rho,\Lambda} : \mathcal{C}_\Lambda \to \underline{\mathrm{Set}}$ satisfies H1, H2, H3. of Schlessinger's criterion.

Let $C_k(\bar\rho) = \mathrm{Hom}_{\bar\rho}(k^n, k^n) \left( = \left\{ \begin{array}{c} M \in M^{n\times n}(k) \text{ s.t.} \\ \bar\rho \cdot M = M \cdot \bar\rho \end{array} \right\} \right)$

If $C_k(\bar\rho) = k$ $\left(\begin{array}{c}\text{scalar}\\ \text{matrices}\end{array}\right)$ then $D_{\bar\rho,\Lambda}$ also satisfies H4.

(Mazur proved Theorem for $\bar\rho$ abs. irreducible, Ramakrishna proved for $C_k(\bar\rho) = k$)

**Corollary.** If $G$ satisfies $p$-finiteness and $C_k(\bar\rho) = k$ then $D_{\bar\rho,\Lambda}$ is pro-representable:

there exists a couple $(\underset{\hat{\mathcal{C}}_\Lambda}{\underset{\cap}{R^{univ}}}, \underset{: G \to GL_n(R^{univ})}{\underset{|}{\rho^{univ}}})$ with the following universal property:

$\forall A \in \hat{\mathcal{C}}_\Lambda$ and every deformation $\rho : G \to GL_n(A)$ of $\bar\rho$, there exists an unique $\Lambda$-algebra morphism $f_\rho : R^{univ} \to A$ s.t. $\rho \underset{\text{str. equiv.}}{\cong} f_\rho \circ \rho^{univ}$

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Remarks: 1) When is $C_k(\bar\rho) = k$? Schur's. Lemma $\Rightarrow$ Condition holds for $\bar\rho$ abs. irreducible

Take $\bar\rho(g) = \begin{pmatrix} \chi_1(g) & * \\ 0 & \chi_2(g) \end{pmatrix}$, $\chi_1, \chi_2$ distinct characters $G \to k^\times$, $* \neq 0$ $\Leftarrow$ $\bar\rho : G \to GL_n(k)$ $\Downarrow$ $GL_n(k)$ irred.

Then $C_k(\bar\rho) = k$.

2) What to do when $C_k(\bar\rho) \neq k$?
* Look at "versal" deformations
* Look at "framed" deformations

## Versal deformations:

**Def.** Let $F, G : \mathcal{C}_\Lambda \to \underline{\mathrm{Set}}$ be functors s.t. $F(k) = \{*\}$, $G(k) = \{*\}$.

A natural transformation $\eta : F \to G$ is formally smooth $:\Longleftrightarrow$ $\forall A, B \in \mathcal{C}_\Lambda$

$\forall$ surjections $A \twoheadrightarrow B$ the natural map $F(A) \longrightarrow F(B) \times_{G(B)} G(A)$ is surjective.

$$\begin{tikzcd} & F(A) \arrow[dl] \arrow[dr, "\eta_A"] & \\ F(B) \arrow[dr, "\eta_B"'] & & G(A) \arrow[dl] \\ & G(B) & \end{tikzcd}$$

**Proposition.**
If $F, G$ are pro-representable,

$F = \mathrm{Hom}(R_F, \cdot)$, $G = \mathrm{Hom}(R_G, \cdot)$ for some $R_F, R_G \in \hat{\mathcal{C}}_\Lambda$

then:
a transformation $F \to G$ is formally smooth

$\Longleftrightarrow$ it is induced by a morphism $R_G \to R_F$ that makes $R_F$ a ring of formal power series over $R_G$.

$$\left( R_F = R[\![T_1, \ldots, T_n]\!] \right)$$

Proved in Mézard's notes, in Schlessinger's paper (Lemma 2.5) ■

**Def.** We say that a functor $F : \mathcal{C}_\Lambda \to \underline{\mathrm{Set}}$ admits a versal deformation if there exists a couple $(R, \rho)$, $R \in \hat{\mathcal{C}}_\Lambda$, $\rho \in F(R)$, such that the morphism of functors

$$\mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R, A) \longrightarrow F(A) \quad \text{defined for } A \in \hat{\mathcal{C}}_\Lambda \quad (f : R \to A) \longmapsto F(f)(\rho)$$

is formally smooth.

**Theorem.** (Schlessinger) Assume $G$ satisfies $\Phi_p$. Then $D_{\bar\rho, \Lambda}$ admits a versal deformation $(R^{ver}, \rho^{ver})$.
This means that for $\forall A \in \hat{\mathcal{C}}_\Lambda$ and $\forall$ deformations $\rho$ of $\bar\rho$ to $A$ there exists $f_\rho \in \mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R^{ver}, A)$ s.t. $f_\rho \circ \rho^{ver} \underset{\text{str.}}{\cong} \rho$, but $f_\rho$ is not uniquely determined.

**Framed deformations:**

Define a framed deformation functor $D^{\square}_{\bar{\rho},\Lambda} : \mathcal{C}_{\Lambda} \to \underline{\mathrm{Set}}$ as

$$D^{\square}_{\bar{\rho},\Lambda}(A) = \left\{ \begin{array}{l} \text{set of deformations} \\ \text{of } \bar{\rho} \text{ to } \rho : G \to GL_n(A) \end{array} \right\}$$

**Deformations as actions on A-modules**

See $\bar{\rho} : G \longrightarrow GL_n(k)$ as datum of:

* a continuous action of $G$ on a $n$-dimensional $k$-vector space $V_{\bar{\rho}}$
* a choice of basis $\beta$ for $V_{\bar{\rho}}$

We call a deformation of $\bar{\rho}$ the datum of
(framed deformation)

(1) a continuous action of $G$ on a free $A$-module $V_A$ of rank $n$

such that $V_A \otimes_A k \underset{G\text{-modules}}{\cong} V_{\bar{\rho}}$

(2) a choice of a lift of the basis $\beta$ of $V_{\bar{\rho}}$ to an $A$-basis of $V_A$

(same as choosing a lift of $\bar{\rho}$ to $\rho : G \to GL_n(A)$)

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$D_{\bar{\rho},\Lambda}(A) = $ set of choices of (1) ,

$D^{\square}_{\bar{\rho},\Lambda}(A) = $ set of choices of (1) and (2).

---

Theorem. Assume $G$ has property $\Phi_p$. Then the framed deformation functor $D^{\square}_{\bar{\rho},\Lambda}$ is pro-representable. (No need of Schlessinger)

Proposition. The transformation of functors $D^{\square}_{\bar{\rho},\Lambda} \longrightarrow D_{\bar{\rho},\Lambda}$ defined by

$$\big(\rho : G \to GL_n(A)\big) \longmapsto [\rho]_{\text{str. equivalence}}$$

is formally smooth.

Proof. exercise $D^{\square}_{\bar\rho,\Lambda}(A) \longrightarrow D^{\square}_{\bar\rho,\Lambda}(B) \times_{D_{\bar\rho,\Lambda}(B)} D_{\bar\rho,\Lambda}(A)$ (surjection $A \to B$) ▨

Corollary. if $\mathcal{Z}_k(\bar\rho) = k$ ($\rightsquigarrow D^{\square}_{\bar\rho,\Lambda}$ and $D_{\bar\rho,\Lambda}$ are represented as $\mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R^{\square}_{univ}, \cdot)$, $\mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R_{univ}, \cdot)$), then $R^{\square}_{univ} = R_{univ}[\![T_1, \dots, T_n]\!]$.

Proof of pro-representability of $D^{\square}_{\bar\rho,\Lambda}$.

We look for an object $R \in \hat{\mathcal{C}}_\Lambda$ such that $D_{\bar\rho,\Lambda}(A) = \mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R, A)$ $\forall A \in \hat{\mathcal{C}}_\Lambda$.

1) We prove the result for $G$ finite. (fix $\rho: G \to GL_n(k)$)

Define $\Lambda$-algebra $\Lambda[G,n]$ by giving

* generators: $X^g_{ij}$, $g \in G$, $i,j \in \{1,\dots,n\}$
* relations: $X^{gh}_{ij} = \sum_{l=1}^{n} X^g_{il} X^h_{lj}$ $\forall g,h \in G$ $\forall i,j \in \{1,\dots,n\}$ ($e \in G$ unity)

$$X^e_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Then there is a bijection, for every $\Lambda$-algebra $A$

$$b_A : \mathrm{Hom}_\Lambda(\Lambda[G,n], A) \longrightarrow \mathrm{Hom}_{\underline{Grp}}(G, GL_n(A))$$
$$f \longmapsto (g \mapsto (f(X^g_{ij}))_{i,j})$$

<u>Check</u>: this is a bijection. ($\Lambda[-,n] \dashv GL_n$)

Consider $\bar\rho \in \mathrm{Hom}_{\underline{Grp}}(G, GL_n(k))$, then $b_k^{-1}(\bar\rho) =: f_{\bar\rho}$ is a $\Lambda$-algebra-morphism $\Lambda[G,n] \to k$. Set $m_{\bar\rho} = \ker f_{\bar\rho}$.

Set $R := m_{\bar\rho}$-adic completion of $\Lambda[G,n] = \varprojlim_n \Lambda[G,n]/m_{\bar\rho}^n$

Show that $R$ has the universal property: take a deformation $\rho: G \to GL_n(A)$ of $\bar\rho$ to some $A \in \hat{\mathcal{C}}_\Lambda$. Then $f_\rho := b_A^{-1}(\rho): \Lambda[G,n] \to A$, then $f_\rho(m_{\bar\rho}) \subseteq m_A$.

In particular $f_\rho$ can be extended by continuity to $f_\rho : R \to A$. Then one checks

$$\begin{tikzcd} G \arrow[r, "\rho^{univ}"] \arrow[dr, "\rho"'] & GL_n(R) \arrow[d, phantom, "\circlearrowleft"] \arrow[dl, "f_\rho"] \\ & GL_n(A) \end{tikzcd}$$

Define $\rho^{univ}$ as the homomorphism $G \to GL_n(R)$ attached to $\Lambda[G,n] \to R$ by $b_R$

Also, $f_\rho$ with this property is unique.

$\Rightarrow$ $(R, \rho^{univ})$ is an universal deformation couple for $\bar\rho$

2) $G$ profinite, then $G = \varprojlim_{\substack{H \trianglelefteq G \\ G/H \text{ finite} \\ H \subseteq \ker \bar\rho}} G/H$

(are exactly the $H$'s such that $\bar\rho$ factors through $\bar\rho_H : G/H \to GL_n(k)$ )

For each such $H$ there is a representing pair $(R_{\bar\rho_H}, \bar\rho_H)$ for the functor $D^\square_{\bar\rho_H, \Lambda}$. The set $\{R_{\bar\rho_H}\}_H$ is a projective system. and

$R_{\bar\rho} := \varprojlim_H R_{\bar\rho_H}$ is in the category $\hat{\mathcal{C}}_\Lambda$. (uses $\Phi_p$-assumption on $G$ ! )

Then $R_{\bar\rho}$ pro-represents $D^\square_{\bar\rho, \Lambda}$ :

$$\begin{aligned} D^\square_{\bar\rho,\Lambda}(A) = \varprojlim_H D^\square_{\bar\rho_H,\Lambda}(A) &= \varinjlim_H \mathrm{Hom}(R_{\bar\rho_H}, A) \\ &\cong \mathrm{Hom}(\varprojlim_H R_{\bar\rho_H}, A) \\ &= \mathrm{Hom}(R_{\bar\rho}, A). \end{aligned}$$