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Master's Thesis

An étale homotopy-theoretic reformulation of the Section Conjecture

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Abstract

We give a detailed exposition of Quick's reformulation of the Section Conjecture given in [Qui13a] and summarize the required homotopy-theoretical foundations (scattered over [Qui08], [Qui13b] and [Qui11]) beforehand. Along the way we correct a minor mistake regarding the set of connected components $\pi_0(\mathbf{B}\pi^{\mathrm{h}G})$ of the homotopy fixed point space of the classifying space $\mathbf{B}\pi$ of a profinite group π with respect to the natural action of a quotient $\pi \twoheadrightarrow G$ and explain why the necessary changes do not pose a problem to the arithmetic applications discussed later on.

Zusammenfassung

Wir geben eine detaillierte Darstellung von Quick's Umformulierung der Schnittvermutung, zuerst entwickelt in [Qui13a], und fassen hierfür zunächst die notwendigen homotopietheoretischen Grundlagen (zu finden in [Qui08], [Qui13b] und [Qui11]) zusammen. Außerdem korrigieren wir einen unerheblichen Fehler bezüglich der Zusammenhangskomponenten $\pi_0(\mathbf{B}\pi^{\mathrm{h}G})$ des Homotopie-Fixpunktraums des klassifizierenden Raums $\mathbf{B}\pi$ für eine proendliche Gruppe π bezüglich der natürlichen Wirkung eines Quotienten $\pi \twoheadrightarrow G$ und erklären, wieso die daraus resultierenden Änderungen die späteren arithmetischen Anwendungen nicht beeinträchtigen.

Contents

1 Introduction

Motivation: Anabelian Geometry

The basic question one asks in anabelian geometry is the following:

Question. How much information about a scheme is contained in its étale fundamental group?

More accurately, Grothendieck wrote in a letter to Faltings:

> "Eine allgemeine Grundidee ist, dass für gewisse, sog. *anabelsche*, Schemata X (von endlichem Typ) über k, die Geometrie von X vollständig durch die (profinite) Fundamentalgruppe $\pi_1^{\text{ét}}(X,\xi)$ bestimmt ist [...] zusammen mit der Extra-Struktur, die durch den Homomorphismus
>
> $$\pi_1^{\text{ét}}(X,\xi) \to \pi_1^{\text{ét}}(\operatorname{Spec}(k),\xi) = \operatorname{Gal}(k(\xi)/k)$$
>
> gegeben ist [...]."

He doesn't give a complete classification of those schemes which are to be considered "anabelian". In fact, he stated the following:

> "Völlige Klarheit habe ich nur im Fall $\dim X = 1$."

In dimension 1, *smooth, projective curves of genus $g \geq 2$ over a field k finitely generated over* $\mathbf{Q}$ should be anabelian. More generally, *hyperbolic curves* over such fields should be anabelian:

Definition ([Sti13, Definition 5]). A *hyperbolic curve* over a field k is a smooth, geometrically connected curve C over k, such that there exists a smooth, projective compactification $C \subseteq \bar{C}$ satisfying:

(a) $Y = \bar{C} \setminus C$ is étale over k

(b) $\chi_C = 2 - 2 \cdot g - \deg(Y) < 0$

In higher dimensions, schemes $X \to \operatorname{Spec}(k)$ that can be written as a chain

$$X = X_n \to X_{n-1} \to \ldots \to X_1 \to \operatorname{Spec}(k)$$

of smooth fibrations of anabelian curves should also be anabelian.

Regarding the behaviour of anabelian schemes, Grothendieck formulated the following three conjectures commonly referred to as "Isomorphy Conjecture", "Hom Conjecture" and "Section Conjecture" respectively:

Anabelian Conjectures (Grothendieck, '83)**.**

(Isom) The isomorphism type of X is determined by the isomorphism type of $\pi_1^{\text{ét}}(X)$.

(Hom) For two anabelian curves X, Y over k, the canonical map

$$\mathrm{Hom}_k(X,Y) \longrightarrow \mathrm{Hom}_{\mathrm{Gal}_k,\mathrm{out}}(\pi_1^{\text{ét}}(X), \pi_1^{\text{ét}}(Y))$$

induces a bijection

$$\mathrm{Hom}_k^{\mathrm{dom}}(X,Y) \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{op}}_{\mathrm{Gal}_k,\mathrm{out}}(\pi_1^{\text{ét}}(X), \pi_1^{\text{ét}}(Y)),$$

where the left-hand side means *dominant* morphisms of schemes and the right-hand side means *open* outer [1] group homomorphisms.

(Sect) For an anabelian [2] curve X over k, the canonical map

$$X(k) \longrightarrow \mathrm{Hom}_{\mathrm{Gal}_k,\mathrm{out}}(\mathrm{Gal}_k, \pi_1^{\text{ét}}(X,x))$$

is a bijection [3] .

[1]Recall that, given (profinite) groups G, G' and H together with augmentation maps $G \longrightarrow H \xleftarrow{\psi} G'$, the set of *outer* group homomorphisms of G and G' with respect to H is given by

$$\mathrm{Hom}_{H,\mathrm{out}}(G,G') := \mathrm{Hom}_H(G,G')/K',$$

where $\sigma \in K' := \ker\psi$ acts on $\mathrm{Hom}_H(G,G')$ via $\sigma \cdot \varphi := c_\sigma \circ \varphi$ with c_σ the conjugation automorphism $\tau \mapsto \sigma\tau\sigma^{-1}$ induced by σ.

[2]In fact, in the case of non-projective curves one needs to slightly modify the above conjecture: Rational points of the boundary $\partial C := \bar{C} \setminus C$ of C within its projective compactification give rise to sections, known as *cuspidal sections*, of $\pi_1^{\text{ét}}(C,c) \to \mathrm{Gal}_k$ that don't arise from rational points on C. One has to exclude sections of this form in the non-projective case — see [Sti13, "The Erratum"] for more details and precise references.

[3]The set $\mathrm{Hom}_{\mathrm{Gal}_k,\mathrm{out}}(\mathrm{Gal}_k, \pi_1^{\text{ét}}(X))$ classifies sections of the short exact sequence

$$0 \longrightarrow \pi_1^{\text{ét}}(\bar{X},x) \longrightarrow \pi_1^{\text{ét}}(X,x) \longrightarrow \mathrm{Gal}_k \longrightarrow 0$$

up to conjugation with elements of $\pi_1^{\text{ét}}(\bar{X},x)$.

Some Major Results in Anabelian Geometry

The first major achievement, due to Neukirch and Uchida, showed an enhanced version of the first of the above conjectures for number fields — even before Grothendieck formulated his conjectures in the first place:

Theorem (Neukirch-Uchida, '69). Let k_1 and k_2 be number fields with fixed separable closures $\bar{k}_1, \bar{k}_2$ and let

$$\sigma\colon \operatorname{Gal}(\bar{k}_1/k_1) \to \operatorname{Gal}(\bar{k}_2/k_2)$$

be an isomorphism of profinite groups. Then there exists a unique isomorphism $\alpha\colon \bar{k}_2 \to \bar{k}_1$ with $\alpha(k_2) = k_1$ inducing σ.

Proof. [NSW, Theorem 12.2.1]. □

The next major breakthrough, due to Tamagawa, proved a similarly strong form of the first of the above conjectures for *affine* hyperbolic curves:

Theorem (Tamagawa, '96). Let k be a field finitely generated over $\mathbf{Q}$. Let U_1 and U_2 be *affine* hyperbolic curves over k. The canonical map

$$\operatorname{Isom}_k(U_1, U_2) \longrightarrow \operatorname{Isom}_{\operatorname{Gal}_k,\mathrm{out}}(\pi_1^{\text{ét}}(U_1), \pi_1^{\text{ét}}(U_2))$$

is bijective.

Proof. [Tam97, Theorem 0.3]. □

In fact, in the same paper Tamagawa proved a similar result over *finite* fields for arbitrary affine curves as well:

Theorem (Tamagawa, '96). Let k_1, k_2 be finite fields. Let U_1 and U_2 be affine (not-necessarily hyperbolic) curves over k_1 and k_2 respectively. Then the canonical map

$$\operatorname{Isom}(U_1, U_2) \longrightarrow \operatorname{Isom}_{\mathrm{out}}(\pi_1^{\text{ét}}(U_1), \pi_1^{\text{ét}}(U_2))$$

is bijective [4]. In particular, if $\pi_1^{\text{ét}}(U_1)$ is isomorphic to $\pi_1^{\text{ét}}(U_2)$, then U_1 is isomorphic to U_2 as a scheme.

Proof. [Tam97, Theorem 0.6] □

[4]Here the "out" qualifier is to be understood as above for the case $H = 0$ the trivial group with the unique augmentations.

It was Mochizuki who realized that this last Theorem can be used to cover the case of not-necessarily affine hyperbolic curves (over number fields) as well:

Theorem (Mochizuki, '96)**.** Let k be a number field. Let U_1, U_2 be (not-necessarily affine) hyperbolic curves over k. Then the canonical map

$$\mathrm{Isom}_k(U_1, U_2) \longrightarrow \mathrm{Isom}_{\mathrm{Gal}_k,\mathrm{out}}(\pi_1^{\text{ét}}(U_1), \pi_1^{\text{ét}}(U_2))$$

is bijective.

Proof. [Moc96, Theorem 10.2] □

Shortly before publishing the above argument, Mochizuki actually showed a much stronger version of the above Theorem using *p-adic* techniques:

A generalization of the Hom-Conjecture for hyperbolic curves over *sub-p-adic* fields, i.e. fields that are finitely generated over some subfield of the p-adics $\mathbf{Q}_p$ for some prime p.

Theorem (Mochizuki, '95)**.** Let p be a prime number. Let k be a *sub-p-adic field*. Let Y be a smooth variety over k and let X be a hyperbolic curve over k. Then the canonical map

$$\mathrm{Hom}_k^{\mathrm{dom}}(Y, X) \longrightarrow \mathrm{Hom}^{\mathrm{op}}_{\mathrm{Gal}_k,\mathrm{out}}(\pi_1^{\text{ét}}(Y), \pi_1^{\text{ét}}(X))$$

is bijective.

Proof. This is a special case of [Moc95, Theorem 16.5]. □

The Section Conjecture

The third of the above conjectures is commonly referred to as *(Grothendieck's) Section Conjecture*.

While there has been a lot of effort and smaller progress, to this day the Section Conjecture has not seen a major breakthrough comparable to the results above.

In fact (to the author's knowledge), the only curves known to satisfy the Section Conjecture don't have any rational points. One should note, however, that a good understanding of the case of curves without rational points would suffice to prove the Section Conjecture in general — this is an observation essentially due to Tamagawa, see [Sti13, Corollary 101].

In [Qui13a], Quick suggested a new *étale homotopy-theoretic* perspective on the Section Conjecture. More concretely, Quick rephrased the Section Conjecture in terms of a map

$$\eta\colon \bar{\mathscr{X}}^G \to \bar{\mathscr{X}}^{\mathrm{h}G}.$$

Here $\bar{\mathscr{X}}$ is essentially given by (a rigidified model of) the *étale homotopy type* of the base change $\bar{X}$ of a variety X over k to the algebraic closure $\bar{k}$ and the spaces $\bar{\mathscr{X}}^G$ and $\bar{\mathscr{X}}^{\mathrm{h}G}$ denote the *fixed point space* and *homotopy fixed point space* of $\bar{\mathscr{X}}$ with respect to a canonical action of the absolute Galois group $G = \mathrm{Gal}_k$ of k respectively.

Thus, the map η should be thought of as a map comparing *fixed points* ($\approx X(k)$) to homotopy fixed points ($\approx \mathrm{Hom}_{G,\mathrm{out}}(G, \pi_1^{\text{ét}}(X,x))$).

Summary

The aim of this thesis is to work out the details of Quick's reformulation of the Section Conjecture given in [Qui13a].

- In Chapter 2 we give a short field guide for using pro-categories, introduce various notions of "spaces" that will be used throughout this thesis, give a very quick summary of the Kan-Quillen model structure on the category of simplicial sets and close with a reminder on the notions of internal categories, groupoids and nerves.
- Chapter 3 is devoted to studying the *homotopy theory of profinite spaces* as developed by Quick in [Qui08]. As our interest lies mainly in the arithmetic applications, we merely summarise the most important results and notions of [Qui08] in §1. In §2, we shortly discuss the homotopy theory of profinite groupoids and compare it to the one of profinite spaces, whereas in §3 we give a detailed exposition of the content of Chapter 2 in [Qui13a]. It is here where we discuss the general formalism enabling us to define the map η alluded to above (Section 3.7).
- Chapter 4 starts in §1 with a short introduction to Artin and Mazur's *étale homotopy type* with the aim to motivate the Friedlander's rigidified *Čech étale topological type* of a variety introduced in §2.
- Finally, Chapter 5 applies the results of Chapter 3 to the étale topological type of Chapter 4 to obtain arithmetic results: The map η "detects rational points" (Theorem 5.2.1) and can be used to reformulate Grothendieck's Section Conjecture (Theorem 5.3.1).

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2 Preliminaries

Conventions.

- The categories of *sets* (resp. *finite sets*) will be denoted by **sets** (resp. **fsets**).
- We use $\varinjlim$ and $\varprojlim$ to denote *filtered* colimits and *cofiltered* limits respectively.
- Colimits and Limits of arbitrary shape will be denoted by colim and lim respectively.
- Given a functor $F\colon \mathbf{C} \to \mathbf{D}$, we denote its *opposite functor* by $F^{\mathrm{op}}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{D}^{\mathrm{op}}$.
  Recall that taking the opposite of a functor gives an anti-equivalence

  $$\mathrm{Fun}(\mathbf{C},\mathbf{D})^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathbf{D}^{\mathrm{op}}), \quad F \mapsto F^{\mathrm{op}}$$

  along which we will sometimes identify the above two functor categories.
- Given a category $\mathbf{C}$ and an object $S \in \mathbf{C}$, we denote the corresponding slice categories by $\mathbf{C} \downarrow S$ and $S \downarrow \mathbf{C}$, respectively. Hence objects in $\mathbf{C} \downarrow S$ are morphisms $X \to S$ and morphisms are given by appropriate commutative triangles. Similarly, objects in $S \downarrow \mathbf{C}$ are morphisms $S \to X$ and morphisms are again given by the appropriate commutative triangles.
- If we have a category $\mathbf{M}$ equipped with some notion of "homotopy", like a model category for example, we usually denote the *homotopy* category of $\mathbf{M}$ by $\mathrm{ho}(\mathbf{M})$.

2.1 (Co-)Presheaves and (Co-)Yoneda

Definition 2.1.1. Let $\mathbf{C}$ be a small category.

(a) Write $\mathbf{C}^\wedge := \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{sets})$ for the *category of presheaves on* $\mathbf{C}$.

(b) Write $\mathbf{C}^\vee := \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{sets}^{\mathrm{op}})$ for the *category of copresheaves on* $\mathbf{C}$.

Definition 2.1.2. Let $\mathbf{C}$ be a small category.

(a) Write $h_{\mathbf{C}}\colon \mathbf{C} \hookrightarrow \mathbf{C}^\wedge$ for the yoneda embedding

$$X \mapsto h_X\colon \begin{cases} \mathbf{C}^{\mathrm{op}} \to \mathbf{sets} \\ Y \mapsto \mathrm{Hom}_{\mathbf{C}}(Y,X) \end{cases}$$

(b) Write $h^{\mathbf{C}}\colon \mathbf{C} \hookrightarrow \mathbf{C}^\vee$ for the coyoneda embedding

$$X \mapsto h^X\colon \begin{cases} \mathbf{C}^{\mathrm{op}} \to \mathbf{sets}^{\mathrm{op}} \\ Y \mapsto \mathrm{Hom}_{\mathbf{C}}(X,Y) \end{cases}$$

Lemma 2.1.3 ((Co-)Yoneda lemma). Let $\mathbf{C}$ be a locally small category and $X \in \mathbf{C}$.

(i) For any $F \in \mathbf{C}^{\wedge}$, the mapping $f \mapsto f_X(1_X)$ defines a bijection of sets

$$\mathrm{Hom}_{\mathbf{C}^{\wedge}}(h_X, F) \xrightarrow{\sim} FX$$

with inverse $s \mapsto \eta_s : f \mapsto F(f)(s)$.

(ii) For any $F \in \mathbf{C}^{\vee}$, the mapping $f \mapsto f_X(1_X)$ defines a bijection of sets

$$\mathrm{Hom}_{\mathbf{C}^{\vee}}(F, h^X) \xrightarrow{\sim} FX$$

with inverse $s \mapsto \eta^s : f \mapsto F(f)(s)$.

Moreover, both of the above bijections are natural in X and F respectively.

Remark 2.1.4.

(i) Note that, since $\mathrm{Fun}(\mathbf{C}, \mathbf{D})^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{D}^{\mathrm{op}})$, the functor $F \mapsto F^{\mathrm{op}}$ establishes an equivalence between the categories $\mathbf{C}^{\mathrm{op}\wedge}$ and $\mathbf{C}^{\vee\,\mathrm{op}}$.

(ii) Under the above equivalence, one has

$$(h_X)^{\mathrm{op}} = h^X \quad \text{and} \quad (h^X)^{\mathrm{op}} = h_X\,.$$

2.2 Pro-Categories

Summary. We quickly review some basics concerning the theory of *pro-categories* following (the dual of) [SGA IV, Éxpose I.8].

Setup. Throughout this section, let $\mathbf{C}$ denote a category.

Definition 2.2.1.

(a) A *pro-object* $\mathscr{X}$ *in* $\mathbf{C}$ is a cofinally small, cofiltered diagram $\mathscr{X} : \mathscr{I} \to \mathbf{C}$.

(b) Let $\mathscr{X}$ be a pro-object in $\mathbf{C}$.
The *formal limit* of $\mathscr{X}$ is given by

$$\text{«}\varprojlim\text{»}\,\mathscr{X} := \varprojlim\, h^{\mathbf{C}} \circ \mathscr{X} \in \mathbf{C}^{\vee}.$$

(c) A *morphism of pro-objects* $\mathscr{X} \to \mathscr{Y}$ is a morphism «$\varprojlim$»$\,\mathscr{X} \to$ «$\varprojlim$»$\,\mathscr{Y}$ (in $\mathbf{C}^{\vee}$) of their formal limits.

(d) The *pro-category of* $\mathbf{C}$ is the category $\mathrm{Pro}(\mathbf{C})$ with objects the pro-objects in $\mathbf{C}$ and morphisms of such as morphisms.

Notation 2.2.2. By abuse of notation, we often suppress the action of $\mathscr{X}$ on morphisms and simply denote pro-objects by $\mathscr{X} = (X_i)_i$ (or even (X_i)).

Lemma 2.2.3. For a pro-object $\mathscr{X} = (X_i)_i$ and an object Y in $\mathbf{C}$, one has

$$(\text{«}\varprojlim\text{»}_i X_i)(Y) = \varinjlim_i \operatorname{Hom}_{\mathbf{C}}(X_i, Y).$$

Proof. Recalling that $\text{«}\varprojlim\text{»}_i X_i = \varprojlim_i h^{X_i}$ and that $h_{X_i}(Y) = \operatorname{Hom}_{\mathbf{C}}(X_i, Y)$, we conclude by the following computation:

$$\begin{aligned}
(\text{«}\varprojlim\text{»}_i X_i)(Y) &= \operatorname{Hom}_{\mathbf{C}^\vee}(\varprojlim_i h^{X_i}, h^Y) \\
&= \operatorname{Hom}_{\mathbf{C}^{\vee\mathrm{op}}}(h^Y, \varprojlim_i h^{X_i}) \\
&= \operatorname{Hom}_{\mathbf{C}^{\mathrm{op}\wedge}}(h_Y, \varinjlim_i h_{X_i}) \\
&= (\varinjlim_i h_{X_i})(Y) \\
&= \varinjlim_i (h_{X_i}(Y))
\end{aligned}$$

□

Lemma 2.2.4. Let $F\colon \mathbf{C} \to \mathbf{sets}$ be a functor and $\mathscr{X} = (X_i)_i$ a pro-object in $\mathbf{C}$. Then there is a natural bijection of sets

$$\operatorname{Hom}_{\mathrm{Fun}(\mathbf{C},\mathbf{sets})}(\text{«}\varprojlim\text{»}_i X_i, F) \xrightarrow{\sim} \varprojlim_i F(X_i).$$

Proof. Indeed, this is just the universal property of limits combined with the appropriate yoneda lemma:

$$\begin{aligned}
\operatorname{Hom}_{\mathrm{Fun}(\mathbf{C},\mathbf{sets})}(\text{«}\varprojlim\text{»}_i X_i, F) &= \operatorname{Hom}_{\mathbf{C}^\vee}(F, \text{«}\varprojlim\text{»}_i X_i) \\
&= \varprojlim_i \operatorname{Hom}_{\mathbf{C}^\vee}(F, h^{X_i}) \\
&= \varprojlim_i F(X_i).
\end{aligned}$$

□

Corollary 2.2.5. Let $\mathscr{X} = (X_i)_i$ and $\mathscr{Y} = (Y_j)_j$ be pro-objects in $\mathbf{C}$. Then there is a natural bijection

$$\operatorname{Hom}_{\mathrm{Pro}(\mathbf{C})}((X_i)_i, (Y_j)_j) \xrightarrow{\sim} \varprojlim_j \varinjlim_i \operatorname{Hom}_{\mathbf{C}}(X_i, Y_j).$$

Proof. Indeed,

$$\begin{aligned}
\operatorname{Hom}_{\mathrm{Pro}(\mathbf{C})}((X_i)_i, (Y_j)_j) &= \operatorname{Hom}_{\mathrm{Fun}(\mathbf{C},\mathbf{sets})}(\text{«}\varprojlim\text{»}_j Y_j, \text{«}\varprojlim\text{»}_i X_i) \\
&= \varprojlim_j ((\text{«}\varprojlim\text{»}_i X_i)(Y_j)) \\
&= \varprojlim_j \varinjlim_i \operatorname{Hom}_{\mathbf{C}}(X_i, Y_j).
\end{aligned}$$

□

Corollary 2.2.6. The functor $\mathbf{C} \hookrightarrow \mathrm{Pro}(\mathbf{C})$ carrying X to the constant diagram $\underline{X}\colon \mathbf{1} \mapsto X$ is a fully faithful embedding.

Proof. This boils down to the (co)yoneda embedding being fully faithful:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}(\underline{X}, \underline{Y}) &= \mathrm{Hom}_{\mathbf{C}^\vee}(h^X, h^Y)\\ &= \mathrm{Hom}_{\mathbf{C}}(X, Y).\end{aligned}$$

□

Convention. By abuse of notation, we will often confuse $\mathbf{C}$ with the essential image of the above fully faithful embedding $\mathbf{C} \hookrightarrow \mathrm{Pro}(\mathbf{C})$ without always explicitly mentioning it.

Corollary 2.2.5 gives us an easy way to construct morphisms in pro-categories:

Construction 2.2.7. Let $\mathscr{X} = (X_i)_{i\in\mathscr{I}}$ and $\mathscr{Y} = (Y_j)_{j\in\mathscr{J}}$ be pro-objects in $\mathbf{C}$. Assume given a functor $\alpha\colon \mathscr{J} \to \mathscr{I}$ together with a natural transformation $f : \mathscr{X} \circ \alpha \to \mathscr{Y}$. Then f defines an element

$$(f_j)_j \in \varprojlim_j \mathrm{Hom}_{\mathbf{C}}(X_{\alpha(j)}, Y_j).$$

The canonical map $\mathrm{Hom}_{\mathbf{C}}(X_{\alpha(j)}, Y_j) \to \operatorname*{colim}\limits_{\longrightarrow i} \mathrm{Hom}_{\mathbf{C}}(X_i, Y_j)$ now gives rise to a map

$$\varprojlim_j \mathrm{Hom}_{\mathbf{C}}(X_{\alpha(j)}, Y_j) \to \varprojlim_j \operatorname*{colim}_{\longrightarrow i} \mathrm{Hom}_{\mathbf{C}}(X_i, Y_j) = \mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}(\mathscr{X}, \mathscr{Y})$$

given explicitly by $(f_j)_j \mapsto ([f_j])_j$. Hence, the tuple (α, f) gives rise to a morphism of pro-objects $\mathscr{X} \to \mathscr{Y}$, that is usually plainly denoted by f or $(f_j)_j$.

Definition 2.2.8. Let $\mathscr{X}$ and $\mathscr{Y}$ be pro-objects in $\mathbf{C}$.

A *strict* morphism $\mathscr{X} \to \mathscr{Y}$ is a morphism of pro-objects obtained by a tuple (α, f) as in Construction 2.2.7.

Definition 2.2.9. Let $\mathbf{D}$ be a category admitting small cofiltered limits and $F\colon \mathbf{C} \to \mathbf{D}$ a functor.

The *extension by limits* of F is the functor $F^\vee$ obtained by right Kan extending F along the inclusion $\mathbf{C} \hookrightarrow \mathrm{Pro}(\mathbf{C})$, that is:

$$F^\vee\colon \mathscr{X} \mapsto \varprojlim F \circ \mathscr{X}.$$

With a little care, one even shows the following characterisation of pro-categories:

Property 2.2.10. Let $\mathbf{C}$ be a category and $\mathbf{A}$ any category admitting (small) cofiltered limits. Then the inclusion $\mathbf{C} \hookrightarrow \mathrm{Pro}(\mathbf{C})$ induces an equivalence of categories

$$\mathrm{Fun}'(\mathrm{Pro}(\mathbf{C}), \mathbf{A}) \xrightarrow{\simeq} \mathrm{Fun}(\mathbf{C}, \mathbf{A}),$$

where $\mathrm{Fun}'(\mathrm{Pro}(\mathbf{C}), \mathbf{A})$ denotes the full subcategory of $\mathrm{Fun}(\mathrm{Pro}(\mathbf{C}), \mathbf{A})$ spanned by the functors commuting with (small) cofiltered limits. In other words, every functor $\mathbf{C} \to \mathbf{A}$ admits an essentially unique extension $\mathrm{Pro}(\mathbf{C}) \to \mathbf{A}$ by means of forming cofiltered limits.

Since the term will be used one time later on, we also quickly recall the following notion:

Definition 2.2.11. Let $F\colon \mathbf{C} \to \mathbf{sets}$ be a functor.

(a) A *pro-representation* of F is a representation of $F^\vee\colon \mathrm{Pro}(\mathbf{C}) \to \mathbf{sets}$, the extension by limits of F, that is:
a pro-object $\mathscr{X}$ in $\mathbf{C}$ and an element $x \in F^\vee(\mathscr{X}) = \varprojlim F \circ \mathscr{X}$ such that the corresponding morphism $F^\vee \to h^{\mathscr{X}}$ is an isomorphism in $\mathrm{Pro}(\mathbf{C})^\vee$.

(b) F is *pro-representable* if there exists a pro-representation of F.

2.3 (Co-)Simplicial Objects

Definition 2.3.1. We define the category $\mathbf{\Delta}$ as follows:

(a) The objects of $\mathbf{\Delta}$ are linearly ordered sets $[n] := \{0 < 1 < \ldots < n\}$ for $n \geq 0$.

(b) A morphism $[m] \to [n]$ in $\mathbf{\Delta}$ is a *order-preserving* map of sets $[m] \to [n]$.

The category $\mathbf{\Delta}$ is referred to as the *simplex category*.

Definition 2.3.2. Let $\mathbf{C}$ be a category.

(a) The *category of simplicial objects in* $\mathbf{C}$, denoted by $\mathbf{C}_{\mathbf{\Delta}}$, is the category of $\mathbf{C}$-valued presheaves on $\mathbf{\Delta}$, that is:
$$\mathbf{C}_{\mathbf{\Delta}} := \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{C}).$$

(b) The *category of cosimplicial objects in* $\mathbf{C}$, denoted by $\mathbf{C}^{\mathbf{\Delta}}$, is the category of $\mathbf{C}$-valued presheaves on $\mathbf{\Delta}^{\mathrm{op}}$, that is:
$$\mathbf{C}^{\mathbf{\Delta}} := \mathrm{Fun}(\mathbf{\Delta}, \mathbf{C}).$$

Remark 2.3.3. Note that any functor $F\colon \mathbf{C} \to \mathbf{D}$ induces a functor $F_{\mathbf{\Delta}}\colon \mathbf{C}_{\mathbf{\Delta}} \to \mathbf{D}_{\mathbf{\Delta}}$ and a functor $F^{\mathbf{\Delta}}\colon \mathbf{C}^{\mathbf{\Delta}} \to \mathbf{D}^{\mathbf{\Delta}}$ by means of postcomposition with F.

2.4 Profinite Sets and Profinite Completion

Profinite Sets

Pro-objects in finite sets are called *profinite*:

Definition 2.4.1. The *category of profinite sets* is the category **ProFin** := Pro(**fsets**) of pro-objects in finite sets.

As is often the case for pro-objects in suitably "finite" objects, there is a corresponding topological notion:

Definition 2.4.2.

(a) A *stone space* is a compact, Hausdorff, totally disconnected topological space.

(b) The *category of stone spaces* is the full subcategory **St** of the category of topological spaces **Top** determined by the stone topological spaces.

Lemma 2.4.3 ([Joh86, p. 236])**.** The extension by limits of the inclusion **fsets** → **St** given by equipping a finite set with the discrete topology induces an equivalence of categories

$$\mathbf{ProFin} \to \mathbf{St}, (X_i)_i \mapsto \varprojlim_i X_i.$$

Profinite Completion of Sets

Let $\mathscr{R}(X)$ denote the set of all equivalence relations R on a set X such that X/R is a finite set endowed with the ordering given by inclusion.

Construction 2.4.4. Let X be a set. The *profinite completion* of X is the profinite set

$$\hat{X} := (X/R)_{R\in\mathscr{R}(X)}.$$

Let $f : X \to Y$ be a map of sets. Given any $R \in \mathscr{R}(Y)$, we obtain an induced equivalence relation $R_f \in \mathscr{R}(X)$ given by

$$R_f := \{(x_1, x_2) \in X \times X \mid f(x_1)Rf(x_2)\}.$$

Furthermore, we get an induced map $f_R : X/R_f \to Y/R$. The maps f_R assemble into a strict morphism

$$\hat{f} := (f_R)_{R\in\mathscr{R}(Y)} : \hat{X} \to \hat{Y}.$$

Thus, we have constructed a functor $(\hat{-})$: **sets** → **ProFin**.

Definition 2.4.5. The *profinite completion functor* is the functor $(\hat{-})$: **sets** → **ProFin** of Construction 2.4.4. The profinite set $\hat{X}$ is called *profinite completion of* X.

2.5 (Pointed) ((Pro(finite))) Spaces

Definition 2.5.1. We write ...

(a) ... $\mathbf{ss} := \mathbf{sets}_{\Delta}$ and call $\mathbf{ss}$ the category of *spaces*.

(b) ... $\hat{\mathbf{s}}\mathbf{s} := \mathbf{ProFin}_{\Delta}$ and call $\hat{\mathbf{s}}\mathbf{s}$ the category of *profinite spaces*.

(c) ... pro-$\mathbf{ss} := \mathrm{Pro}(\mathbf{ss})$ and call pro-$\mathbf{ss}$ the category of *pro-spaces*.

(d) ... $\mathbf{ss}_* := * \downarrow \mathbf{ss}$ and call $\mathbf{ss}_*$ the category of *pointed spaces*.

(e) ... $\hat{\mathbf{s}}\mathbf{s}_* := * \downarrow \hat{\mathbf{s}}\mathbf{s}$ and call $\hat{\mathbf{s}}\mathbf{s}_*$ the category of *pointed profinite spaces*.

(f) ... pro-$\mathbf{ss}_* := \mathrm{Pro}(\mathbf{ss}_*)$ and call pro-$\mathbf{ss}_*$ the category of *pointed pro-spaces*.

The Space underlying a Profinite Space

Since the category **sets** admits arbitrary small limits, the inclusion $\mathbf{fsets} \hookrightarrow \mathbf{sets}$ induces a forgetful functor

$$\mathbf{ProFin} \to \mathbf{sets}$$

by extension by limits (Definition 2.2.9). In virtue of Remark 2.3.3, this functor in turn induces a functor

$$|-|: \hat{\mathbf{s}}\mathbf{s} \to \mathbf{ss}$$

on simplicial objects.

Definition 2.5.2. Let X be a profinite space. The *underlying space* of X is the simplicial set $|X|$ constructed above.

Profinite Completion of Spaces

In virtue of Remark 2.3.3, the profinite completion functor $(\hat{-}): \mathbf{sets} \to \mathbf{ProFin}$ induces a profinite completion functor $\mathbf{ss} \to \hat{\mathbf{s}}\mathbf{s}$ that we will also denote by $(\hat{-})$.

Definition 2.5.3. Let X be a space. The *profinite completion* of X is the profinite space $\hat{X}$ constructed above.

Nerves and Realizations

Definition 2.5.4. Let $\mathbf{C}$ be a locally small category and $Q \in \mathbf{C}^{\Delta}$ a cosimplicial object in $\mathbf{C}$. The *Q-nerve* is the functor $\mathbf{N}^Q: \mathbf{C} \to \mathbf{ss}$ given by the composition

$$\mathbf{C} \hookrightarrow \mathbf{C}^{\wedge} \xrightarrow{-\circ Q^{\mathrm{op}}} \mathbf{\Delta}^{\wedge} = \mathbf{ss},$$

where the fist arrow is the yoneda embedding.

Whenever **C** admits appropriate colimits, the Q-nerve admits a left adjoint:

Lemma 2.5.5. Let **C** be a locally small category admitting small colimits and $Q \in \mathbf{C}^{\boldsymbol{\Delta}}$ a cosimplicial object in **C**. The Left Kan extension $|-|^Q = \mathrm{Lan}_{\mathbf{N}^Q}(\mathbf{1}_{\mathbf{C}})$ of the identity $\mathbf{1}_{\mathbf{C}}$ along the Q-nerve $\mathbf{N}^Q \colon \mathbf{C} \to \mathbf{ss}$ constitutes a left adjoint of $\mathbf{N}^Q$. More explicitly, one has

$$|X|^Q = \mathrm{colim}_{\Delta^n \to X} Q^n.$$

Proof. This is formal. A detailed proof (in slightly more generality) can for example be found in [SGA IV, Éxpose I.5]. In short, the argument reads as follows: Since one wants $|-|^Q$ to be left adjoint to $\mathbf{N}^Q$, one in particular wants

$$\begin{aligned}\mathrm{Hom}_{\mathbf{C}}(|\Delta^n|^Q, A) &= \mathrm{Hom}_{\mathbf{ss}}(\Delta^n, \mathbf{N}^Q(A)) \\ &= \mathbf{N}^Q(A)_n \\ &= \mathrm{Hom}_{\mathbf{C}}(Q^n, A).\end{aligned}$$

From here, one extends via colimits using that for any space X one has $X = \mathrm{colim}_{\Delta^n \to X} \Delta^n$ (as is true for general presheaf categories). □

Definition 2.5.6. Let **C** be a locally small category admitting small colimits and $Q \in \mathbf{C}^{\boldsymbol{\Delta}}$ a cosimplicial object in **C**. The functor $|-|^Q \colon \mathbf{ss} \to \mathbf{C}$ of Lemma 2.5.5 is referred to as the *Q-realization*.

Using the general machinery of nerve- and realization functors, one constructs plenty interesting examples:

Definition 2.5.7. Let $\boldsymbol{\Delta} \hookrightarrow \mathbf{Cat}$ be the canonical inclusion, considered as a cosimplicial object $Q \in \mathbf{Cat}^{\boldsymbol{\Delta}}$. The associated Q-nerve will subsequently plainly denoted by $\mathbf{N} \colon \mathbf{Cat} \to \mathbf{ss}$ and referred to as *the nerve functor*. Concretely, one has

$$\mathbf{N}(\mathbf{C}) \colon [n] \mapsto \mathrm{Hom}_{\mathbf{Cat}}([n], \mathbf{C}),$$

and the face and degeneracy maps are given by composition and adding in appropriate identities, respectively.

Definition 2.5.8. Let $Q \in \mathbf{Top}^{\boldsymbol{\Delta}}$ denote the cosimplicial object given by

$$[n] \mapsto |\Delta^n| := \{(x_0, \dots, x_n) \in \mathbf{R}^{n+1}_{\geq 0} \mid x_0 + \ldots + x_n = 1\}.$$

(a) The corresponding Q-nerve will subsequently denoted by $\mathbf{Sing} \colon \mathbf{Top} \to \mathbf{ss}$. Concretely, one has
$$\mathbf{Sing}(X) \colon [n] \mapsto \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X).$$

(b) As **Top** furthermore admits small colimits, **Sing** admits a left adjoint, called *geometric realization*, $|-| \colon \mathbf{ss} \to \mathbf{Top}$ given by

$$|X| = \mathrm{colim}_{\Delta^n \to X} |\Delta^n|.$$

Kan-Quillen Model Structure on ss

Quillen equipped the category **ss** of spaces with a cofibrantly generated model structure as follows:

Definition 2.5.9. Let $f : Y \to X$ be a morphism of spaces.

(a) f is a *cofibration* if it is a monomorphism $Y \hookrightarrow X$, i.e. a levelwise injection

$$f_n : Y_n \hookrightarrow X_n$$

for all $n \in \mathbf{N}$.

(b) f is a *weak equivalence* if it induces a weak homotopy equivalence $|Y| \to |X|$ on geometric realizations.

(c) f is a *fibration* if it is a Kan fibration, i.e. has the right lifting property against the horn inclusions:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & Y \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dotted, "\exists"] & X \end{tikzcd}$$

Quillen now proved the following:

Theorem 2.5.10 ([Qui67, §II.3])**.** The notions of cofibrations, weak equivalences and fibrations of Definition 2.5.9 equip **ss** with a cofibrantly generated model structure.

Convention. If not stated otherwise, we will equip **ss** with the Kan-Quillen model structure of Theorem 2.5.10. Furthermore, the pointed variant $\mathbf{ss}_*$ will be equipped with the induced model structure.

Definition 2.5.11 (π-finite space)**.** A space X is called π-finite if it satisfies the following conditions:

(a) X is connected.

(b) All homotopy groups of X are finite.

The full subcategory of **ss** determined by π-finite spaces will be denoted by $\mathbf{ss}^{\pi}$.

Remark 2.5.12. One possible definition of the n-th homotopy group of a pointed space $(X, x) \in \mathbf{ss}_*$ is given by

$$\pi_n(X, x) := \pi_n(|X|, |x|),$$

where the right-hand side denotes the usual homotopy group in algebraic topology. There is an intrinsic way to define the homotopy groups in purely "simplicial terms" though, that is very similar to the definition of profinite homotopy groups we give in Section 3.1.4.

2.6 Internal Categories, Groupoids and Nerves

We shortly recall the notions of *internal category*, *internal groupoid* and *internal nerve*, following [nLab:internal-categories].

Setup. Throughout this section, let $\mathbf{A}$ denote a category admitting fibre products.

Internal Categories and Functors

Definition 2.6.1. A *category* $\mathbf{C}$ *internal to* $\mathbf{A}$ consists of

(a) an *object of objects* $\mathrm{ob}(\mathbf{C}) \in \mathbf{A}$

(b) an *object of morphisms* $\mathrm{mor}(\mathbf{C}) \in \mathbf{A}$

together with

(i) *source* and *target* morphisms $s, t\colon \mathrm{mor}(\mathbf{C}) \to \mathrm{ob}(\mathbf{C})$

(ii) an *identity-assigning morphism* $e\colon \mathrm{ob}(\mathbf{C}) \to \mathrm{mor}(\mathbf{C})$

(iii) a *composition morphism* $c\colon \mathrm{mor}(\mathbf{C}) \times_{t,\mathrm{ob}(\mathbf{C}),s} \mathrm{mor}(\mathbf{C}) \to \mathrm{mor}(\mathbf{C})$

such that the following diagrams, expressing the usual laws of a category, commute:

- laws specifying the source and target of the identity morphisms:

$$\begin{tikzcd}
\mathrm{mor}(\mathbf{C}) \arrow[dr, "s"'] & \mathrm{ob}(\mathbf{C}) \arrow[l, "e"'] \arrow[r, "e"] \arrow[d, "\mathbf{1}"] & \mathrm{mor}(\mathbf{C}) \arrow[dl, "t"] \\
& \mathrm{ob}(\mathbf{C}) &
\end{tikzcd}$$

- laws specifying the source and target of composite morphisms:

$$\begin{tikzcd}
\mathrm{mor}(\mathbf{C}) \times_{t,\mathrm{ob}(\mathbf{C}),s} \mathrm{mor}(\mathbf{C}) \arrow[r, "c"] \arrow[d, "\mathbf{pr}_1"'] & \mathrm{mor}(\mathbf{C}) \arrow[d, "s"] \\
\mathrm{mor}(\mathbf{C}) \arrow[r, "s"'] & \mathrm{ob}(\mathbf{C})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{mor}(\mathbf{C}) \times_{t,\mathrm{ob}(\mathbf{C}),s} \mathrm{mor}(\mathbf{C}) \arrow[r, "c"] \arrow[d, "\mathbf{pr}_2"'] & \mathrm{mor}(\mathbf{C}) \arrow[d, "t"] \\
\mathrm{mor}(\mathbf{C}) \arrow[r, "t"'] & \mathrm{ob}(\mathbf{C})
\end{tikzcd}$$

- the law expressing the associativity of composition:

$$\begin{tikzcd}
(\mathrm{mor}(\mathbf{C}) \times_{\mathrm{ob}(\mathbf{C})} \mathrm{mor}(\mathbf{C})) \times_{\mathrm{ob}(\mathbf{C})} \mathrm{mor}(\mathbf{C}) \arrow[r, "c \times_{\mathrm{ob}(\mathbf{C})} \mathbf{1}_{\mathrm{mor}(\mathbf{C})}"] \arrow[d] & \mathrm{mor}(\mathbf{C}) \times_{\mathrm{ob}(\mathbf{C})} \mathrm{mor}(\mathbf{C}) \arrow[dd, "c"] \\
\mathrm{mor}(\mathbf{C}) \times_{\mathrm{ob}(\mathbf{C})} (\mathrm{mor}(\mathbf{C}) \times_{\mathrm{ob}(\mathbf{C})} \mathrm{mor}(\mathbf{C})) \arrow[d, "\mathbf{1}_{\mathrm{mor}(\mathbf{C})} \times_{\mathrm{ob}(\mathbf{C})} c"'] & \\
\mathrm{mor}(\mathbf{C}) \times_{\mathrm{ob}(\mathbf{C})} \mathrm{mor}(\mathbf{C}) \arrow[r, "c"'] & \mathrm{mor}(\mathbf{C}),
\end{tikzcd}$$

where the left-upper downwards-pointing morphism switches the factors.

- left and right unit laws for composition:

$$\begin{tikzcd}
\mathrm{ob}(\mathbf{C}) \times_{\mathrm{ob}(\mathbf{C})} \mathrm{mor}(\mathbf{C}) \arrow[r, "e \times_{\mathrm{ob}(\mathbf{C})} \mathbf{1}"] \arrow[dr, "\mathbf{pr}_2"'] & \mathrm{mor}(\mathbf{C}) \times_{\mathrm{ob}(\mathbf{C})} \mathrm{mor}(\mathbf{C}) \arrow[d, "c"] & \mathrm{mor}(\mathbf{C}) \times_{\mathrm{ob}(\mathbf{C})} \mathrm{ob}(\mathbf{C}) \arrow[l, "\mathbf{1} \times_{\mathrm{ob}(\mathbf{C})} e"'] \arrow[dl, "\mathbf{pr}_1"] \\
& \mathrm{mor}(\mathbf{C}) &
\end{tikzcd}$$

As one might expect, there is a corresponding notion of functors:

Definition 2.6.2. Let $\mathbf{C}$ and $\mathbf{C}'$ be categories internal to $\mathbf{A}$.
An *internal functor* $F : \mathbf{C} \to \mathbf{C}'$ consists of

(a) a morphism $\mathrm{ob}(F)\colon \mathrm{ob}(\mathbf{C}) \to \mathrm{ob}(\mathbf{C}')$ of objects in $\mathbf{A}$

(b) a morphism $\mathrm{mor}(F)\colon \mathrm{mor}(\mathbf{C}) \to \mathrm{mor}(\mathbf{C}')$ of objects in $\mathbf{A}$

such that the following diagrams, expressing the usual laws of a functor, commute:

- Compatibility with composition:

$$\begin{tikzcd}
\mathrm{mor}(\mathbf{C}) \times_{\mathrm{ob}(\mathbf{C})} \mathrm{mor}(\mathbf{C}) \arrow[r, "c"] \arrow[d, "\mathrm{mor}(F) \times_{\mathrm{ob}(F)} \mathrm{mor}(F)"'] & \mathrm{mor}(\mathbf{C}) \arrow[d, "\mathrm{mor}(F)"] \\
\mathrm{mor}(\mathbf{C}') \times_{\mathrm{ob}(\mathbf{C}')} \mathrm{mor}(\mathbf{C}') \arrow[r, "c'"'] & \mathrm{mor}(\mathbf{C}')
\end{tikzcd}$$

- Compatibility with identities:

$$\begin{tikzcd}
\mathrm{ob}(\mathbf{C}) \arrow[r, "e"] \arrow[d, "\mathrm{ob}(F)"'] & \mathrm{mor}(\mathbf{C}) \arrow[d, "\mathrm{mor}(F)"] \\
\mathrm{ob}(\mathbf{C}') \arrow[r, "e'"'] & \mathrm{mor}(\mathbf{C}')
\end{tikzcd}$$

Example 2.6.3. A category internal to **sets** is nothing but a *small* category. A functor internal to **sets** is just an ordinary functor.

Internal Groupoids

An internal groupoid is an internal category such that every morphism is invertible:

Definition 2.6.4. A *groupoid internal to* **A** consists of

(a) a category Γ internal to **A**

(b) a morphism $i\colon \operatorname{mor}(\Gamma) \to \operatorname{mor}(\Gamma)$

such that the following diagrams commute:

- Inverting a morphism swaps source and target:

$$\begin{tikzcd}
 & \operatorname{mor}(\Gamma) \arrow[dl, "s"'] \arrow[d, "i"] \arrow[dr, "t"] & \\
\operatorname{ob}(\Gamma) & \operatorname{mor}(\Gamma) \arrow[l, "t"'] \arrow[r, "s"] & \operatorname{ob}(\Gamma)
\end{tikzcd}$$

- Right invertibility:

$$\begin{tikzcd}
\operatorname{mor}(\Gamma) \arrow[r, "\Delta"] \arrow[d, "s"'] & \operatorname{mor}(\Gamma) \times_{t,\operatorname{ob}(\Gamma),t} \operatorname{mor}(\Gamma) \arrow[r, "\mathbf{1}\times i"] & \operatorname{mor}(\Gamma) \times_{t,\operatorname{ob}(\Gamma),s} \operatorname{mor}(\Gamma) \arrow[d, "c"] \\
\operatorname{ob}(\Gamma) \arrow[rr, "e"'] & & \operatorname{mor}(\Gamma)
\end{tikzcd}$$

- Left invertibility:

$$\begin{tikzcd}
\operatorname{mor}(\Gamma) \arrow[r, "\Delta"] \arrow[d, "t"'] & \operatorname{mor}(\Gamma) \times_{s,\operatorname{ob}(\Gamma),s} \operatorname{mor}(\Gamma) \arrow[r, "i\times \mathbf{1}"] & \operatorname{mor}(\Gamma) \times_{t,\operatorname{ob}(\Gamma),s} \operatorname{mor}(\Gamma) \arrow[d, "c"] \\
\operatorname{ob}(\Gamma) \arrow[rr, "e"'] & & \operatorname{mor}(\Gamma)
\end{tikzcd}$$

Internal Nerves

Definition 2.6.5. Let **C** be a category internal to **A**.

The *internal nerve* of **A** is the simplicial object $\mathbf{N}^{\mathbf{A}}(\mathbf{C}) \in \mathbf{A}_{\boldsymbol{\Delta}}$ in **A** with n-simplices given by

$$\mathbf{N}^{\mathbf{A}}(\mathbf{C})\colon [n] \mapsto \underbrace{\operatorname{mor}(\mathbf{C}) \times_{t,\operatorname{ob}(\mathbf{C}),s} \operatorname{mor}(\mathbf{C}) \times_{t,\operatorname{ob}(\mathbf{C}),s} \ldots \times_{t,\operatorname{ob}(\mathbf{C}),s} \operatorname{mor}(\mathbf{C})}_{n \text{ times}} \in \mathbf{A},$$

and with face and degeneracy maps given by composition and adding appropriate identities.

Profinite Categories, Groupoids and Nerves

Definition 2.6.6.

(a) A *profinite category* is a category internal to **ProFin**.

(b) Given profinite categories $\mathbf{C}$ and $\mathbf{C}'$, a *continuous* functor $F\colon \mathbf{C} \to \mathbf{C}'$ is an internal functor.

(c) The resulting category will be denoted by $\hat{\mathbf{Cat}}$ and referred to as *category of (small) profinite categories*.

Definition 2.6.7. The *profinite nerve* $\hat{\mathbf{N}}\colon \hat{\mathbf{Cat}} \to \hat{\mathbf{ss}}$ is the internal nerve functor associated to $\hat{\mathbf{Cat}}$.

The profinite nerve doesn't lose any information about a profinite category:

Lemma 2.6.8. The profinite nerve functor $\hat{\mathbf{N}}\colon \hat{\mathbf{Cat}} \to \hat{\mathbf{ss}}$ is fully faithful.

Proof. The profinite topology is compatible with the proof for the ordinary nerve functor $\mathbf{N}\colon \mathbf{Cat} \to \mathbf{ss}$ and can be used essentially verbatim – see for example [Kerodon, Tag 002Y]. ☐

Convention. In virtue of Lemma 2.6.8, we will subsequently identify a profinite category $\mathbf{C}$ with it's nerve $\hat{\mathbf{N}}(\mathbf{C})$ without always mentioning it explicitly.

Example 2.6.9. Let G and G' be profinite groups.

(i) G defines a profinite groupoid with one object $*$, morphism space G, identity map $e\colon * \mapsto 1_G$ and composition given by multiplication. This groupoid will be denoted by $\mathbf{B}G$.

(ii) The morphisms of profinite groupoids $\mathbf{B}G \to \mathbf{B}G'$ are exactly the morphisms of profinite groups $G \to G'$.

(iii) As $\mathrm{ob}(\mathbf{B}G) = *$ and $\mathrm{mor}(\mathbf{B}G) = G$, the resulting profinite classifying space $\mathbf{B}G$ is given in degree n by the profinite set

$$(\mathbf{B}G)_n = \underbrace{G \times G \times \ldots \times G}_{n \text{ times}}.$$

3 Homotopy Theory

3.1 The Homotopy Theory of Profinite Spaces

Summary. In [Qui08], Quick endowed $\hat{\mathbf{s}}\mathbf{S}$ with a simplicial, left proper and fibrantly generated model structure. For the convenience of the reader, we shortly summarize the results of [Qui08], [Qui13b] and [Qui11] relevant to the applications on étale homotopy that will be discussed later on.

3.1.1 Weak Equivalences of Profinite Spaces

In order to understand the notion of a *weak equivalence* between two profinite spaces, we first need to introduce suitable notions of (profinite) fundamental groups and of local coefficient systems:

Profinite Fundamental Groups

Setup. Let X denote a profinite space.

Profinite spaces have a theory of covering spaces very similar to the one of ordinary spaces (see [GZ67, Appendix I, 2.1]):

Definition 3.1.1 ([Blo19, Definition 4.69]).

(a) A *finite cover* of X is a morphism $p\colon E \to X$ of profinite spaces such that for each commutative diagram

$$\begin{tikzcd} \Delta^0 \arrow[r, "e"] \arrow[d] & E \arrow[d, "p"] \\ \Delta^n \arrow[r, "\sigma"'] & X \end{tikzcd}$$

there exists a *unique* filler $s\colon \Delta^n \to E$ such that the diagram

$$\begin{tikzcd} \Delta^0 \arrow[r, "e"] \arrow[d] & E \arrow[d, "p"] \\ \Delta^n \arrow[r, "\sigma"'] \arrow[ur, dotted, "\exists! s"] & X \end{tikzcd}$$

commutes and such that the map $p_0\colon E_0 \to X_0$ is a finite cover of ordinary topological spaces.

(b) The *category of finite covers of* X, denoted by $\mathbf{FCov}/X$, is the full subcategory of $\hat{\mathbf{s}}\mathbf{S} \downarrow X$ determined by finite covers of X.

Remark 3.1.2. Note that the notion of finite cover as in Definition 3.1.1 slightly differs from Quick's original definition in [Qui08, §2.1]: There, Quick didn't require the map $E_0 \to X_0$ to be a finite cover of ordinary topological spaces. Blom showed in his master's thesis [Blo19] that Quick's definition doesn't guarantee that the category of finite covers will form a Galois category and verifies that his adjusted definition fixes this issue ([Blo19, §4.6]).

Definition 3.1.3. Let $g\colon Y \to X$ be a morphism of profinite spaces. Then pulling back along g defines a functor

$$\mathbf{FCov}/g\colon \mathbf{FCov}/X \to \mathbf{FCov}/Y, (p\colon E \to X) \mapsto (g^*p\colon g^*E = E \times_X Y \to Y)$$

called *pullback functor*.

The lifting property of covers readily implies that one has $E = \mathrm{sk}_0(E)$ whenever $E \to \Delta^0$ is a cover. We can hence identify $\mathbf{FCov}/\Delta^0$ with the category **ProFin** of profinite sets.

Definition 3.1.4. Let $x\colon \Delta^0 \to X$ be a vertex of X. The *fibre functor* of X at x is the functor

$$(-)_x = \mathbf{FCov}/x\colon \mathbf{FCov}/X \to \mathbf{ProFin}, E \mapsto E_x,$$

given by pulling back along x.

One now shows that $(\mathbf{FCov}/X, (-)_x)$ is a *Galois Category* in the sense of [SGA I, Éxpose V]. The fibre functor $(-)_x\colon \mathbf{FCov}/X \to \mathbf{fsets}$ is hence pro-representable by the *universal (pro-)cover* $\hat{X}(x) = (X_i(x))_{i\in I}$.

Definition 3.1.5.

(a) Let $x\colon \Delta^0 \to X$ be a vertex of X. The *profinite fundamental group* $\hat{\pi}_1(X,x)$ of X at x is the profinite group

$$\hat{\pi}_1(X,x) := \mathrm{Aut}(\hat{X}(x)),$$

where $\hat{X}(x) = (X_i(x))_i$ is the universal (pro-)cover of X with respect to the fibre functor $(-)_x$.

(b) The *profinite fundamental groupoid* $\hat{\Pi}X$ of X is the fundamental groupoid of the Galois Category $\mathbf{FCov}/X$ in the sense of [SGA I, Éxpose V].

Remark 3.1.6. Recall that for the universal (pro-)cover $\hat{X}(x) = (X_i(x))_i$, one has $\mathrm{Hom}(\hat{X}(x), X_i(x)) = \mathrm{Hom}(X_i(x), X_i(x)) = \mathrm{Aut}(X_i(x))$, since all the $X_i(x)$ are *Galois* covers. Hence in the limit one has that

$$\begin{aligned}\mathrm{Hom}(\hat{X}(x), \hat{X}(x)) &= \varprojlim_i \mathrm{Hom}(\hat{X}(x), X_i(x)) \\ &= \varprojlim_i \mathrm{Aut}(X_i(x)).\end{aligned}$$

In particular, $\hat{\pi}_1(X,x) = \mathrm{Aut}(\hat{X}(x)) = \varprojlim_i \mathrm{Aut}(X_i(x))$ is seen to be profinite.

Quick then shows that the profinite fundamental group behaves well with respect to profinite completion of spaces:

Proposition 3.1.7 ([Qui08, Propostion 2.1])**.** For a pointed space (X,x), the canonical map $\pi_1(X,x) \to \hat{\pi}_1(\hat{X},\hat{x})$ induces an isomorphism

$$\pi_1^\wedge(X,x) \to \hat{\pi}_1(\hat{X},\hat{x})$$

of profinite groups, where $\pi_1^\wedge(X,x)$ denotes the profinite group completion of $\pi_1(X,x)$.

Local Coefficient Systems

Definition 3.1.8. Let $\mathbf{C}$ be a category.

(a) A *local coefficient system* $\mathscr{M}$ *with values in* $\mathbf{C}$ on a profinite groupoid Γ is a functor

$$\mathscr{M} : \Gamma^{\mathrm{op}} \to \mathbf{C}.$$

(b) A *morphism of local coefficient systems* $(\mathscr{M},\Gamma) \to (\mathscr{M}',\Gamma')$ is a morphism of *underlying* groupoids $f : \Gamma \to \Gamma'$ (i.e. an ordinary functor $\Gamma \to \Gamma'$) and a natural transformation $\mathscr{M}' \to \mathscr{M} \circ f^{\mathrm{op}}$.

(c) A *local coefficient system with values in* $\mathbf{C}$ on a profinite space X is a local coefficient system with values in $\mathbf{C}$ on $\hat{\Pi}X$.

(d) A *topological local coefficient system* is a local coefficient system with values in the category of topological abelian groups.

Construction 3.1.9.

(i) Let X be a profinite space, Γ a profinite groupoid and $\varphi : X \to \Gamma$ a morphism of profinite spaces. Associating to each $\gamma \in \Gamma$ the space $\hat{X}_\gamma$ given by the pullback square

$$\begin{tikzcd} \hat{X}_\gamma \arrow[r] \arrow[d] & \Gamma \downarrow \gamma \arrow[d] \\ X \arrow[r, "\varphi"] & \Gamma \end{tikzcd}$$

defines a functor $\Gamma \to \hat{\mathbf{s}}\mathbf{s}$, called *covering systems for* φ.
If $\Gamma = \hat{\Pi}X$ and φ is the canonical map, then $\hat{X}_\gamma$ is the universal covering $(\hat{X},x)$ for $x=\gamma$.

(ii) If $Y : \Gamma \to \hat{\mathbf{s}}\mathbf{s}$ is any functor and if $\mathscr{M}$ is a topological local coefficient system on Γ, there is an associated cochain complex $\mathrm{hom}_\Gamma(Y,\mathscr{M})$ given in degree n by the group $\mathrm{hom}_\Gamma(Y(-)_n,\mathscr{M})$ of *continuous* natural transformations $Y(-)_n \to \mathscr{M}(-)$.
The differentials of $\mathrm{hom}_\Gamma(Y,\mathscr{M})$ are given by the alternating sum of the face maps.

(iii) We write $\mathrm{C}^*_\Gamma(X;\mathscr{M}) := \mathrm{hom}_\Gamma(\hat{X},\mathscr{M})$.

Definition 3.1.10. For a topological local coefficient system $\mathscr{M}$ on Γ and a map $X \to \Gamma$ in $\hat{\mathbf{s}}\mathbf{s}$, we define the *continuous cohomology of* X *with coefficients in* $\mathscr{M}$, denoted by $\mathrm{H}^*_\Gamma(X;\mathscr{M})$, to be the cohomology of the cochain complex $\mathrm{C}^*_\Gamma(X;\mathscr{M})$.

If $\Gamma = \hat{\Pi}X$, we write $\mathrm{H}^*(X;\mathscr{M}) := \mathrm{H}^*_{\hat{\Pi}X}(X;\mathscr{M})$.

Weak Equivalences

Definition 3.1.11 ([Qui08, Definition 2.6]). A morphism $f : Y \to X$ in $\hat{\mathbf{s}}\mathbf{s}$ is a *weak equivalence* if:

(a) $\hat{\pi}_0(f) \colon \hat{\pi}_0(Y) \to \hat{\pi}_0(X)$ is an isomorphism of profinite sets.

(b) For every $y \in Y_0$, the induced map
$$f_* \colon \hat{\pi}_1(Y, y) \to \hat{\pi}_1(X, f(y))$$
is an isomorphism of profinite groups.

(c) For every local coefficient system $\mathscr{M}$ of finite abelian groups on X and for every $q \geq 0$, the induced map
$$f^* \colon \mathrm{H}^q(X; \mathscr{M}) \to \mathrm{H}^q(Y; f^*\mathscr{M})$$
is an isomorphism.

3.1.2 $\hat{\mathbf{s}}\mathbf{s}$ as a Model Category

Let $\mathscr{T}$ denote the set of isomorphism classes of finite sets and $\mathscr{G}$ the set of isomorphism classes of finite groups.

Definition 3.1.12. Let P and Q be the following two sets of morphisms of profinite spaces:

(a) P consisting of $\mathbf{E}G \to \mathbf{B}G$, $\mathbf{B}G \to *$, $\mathbf{L}(A, n) \to \mathbf{K}(A, n+1)$, $\mathbf{K}(A, n) \to *$, $\mathbf{K}(S, 0) \to *$ for every finite set $S \in \mathscr{T}$, every finite group $G \in \mathscr{G}$, every finite abelian group $A \in \mathscr{G}$ and every $n \geq 0$.

(b) Q consisting of $\mathbf{E}G \to *$, $\mathbf{L}(A, n) \to *$ for every finite group $G \in \mathscr{G}$, every finite abelian group $A \in \mathscr{G}$ and every $n \geq 0$.

Remark 3.1.13. Let G be a group, A an abelian group and $n \geq 0$ an integer. Recall the definition of the following spaces:

(i) The space $\mathbf{E}G \in \mathbf{ss}$ is given in degree n by
$$(\mathbf{E}G)_n = \underbrace{G \times G \times \ldots \times G}_{n+1 \text{ times}},$$
and has face and degeneracy maps given by composition and adding identities. Moreover, $\mathbf{E}G$ comes equipped with a free G-action given by
$$g \cdot (g_0, \ldots, g_n) := (gg_0, \ldots, gg_n)$$
the quotient space of which canonically identifies with $\mathbf{B}G$, explaining the map $\mathbf{E}G \to \mathbf{B}G$ above.

(ii) The *Eilenberg MacLane-Space* $\mathbf{K}(A,n) \in \mathbf{ss}$ is obtained by applying the Dold-Kan correspondence to the complex $A[n]_\bullet$ given by

$$A[n]_k = \begin{cases} A & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

so that in particular

$$\mathrm{Hom}_{\mathrm{ho}(\mathbf{ss})}(X, \mathbf{K}(A,n)) = \mathrm{H}^n(X;A).$$

(iii) The space $\mathbf{L}(A,n) \in \mathbf{ss}$ is given by applying the Dold-Kan correspondence to the complex $l(A,n)_\bullet$ given by

$$l(A,n)_k = \begin{cases} A & \text{if } k = n, n+1 \\ 0 & \text{otherwise} \end{cases}$$

where the differential $l(A,n)_{n+1} \to l(A,n)_n$ is given by the identity $\mathbf{1}_A \colon A \to A$.

(iv) The map

$$\mathbf{L}(A,n) \to \mathbf{K}(A,n+1)$$

is the one obtained by applying the Dold-Kan correspondence to the canonical map of complexes

$$l(A,n)_\bullet \to A[n+1]_\bullet.$$

Theorem 3.1.14 ([Qui08, Theorem 2.12]). There is a left proper, fibrantly generated model structure on $\hat{\mathbf{s}}\mathbf{s}$ with weak equivalences as in Definition 3.1.11 for which P is the set of generating fibrations and Q is the set of generating trivial fibrations. The cofibrations are the maps isomorphic to levelwise monomorphisms.

Convention. We will always equip $\hat{\mathbf{s}}\mathbf{s}$ with Quick's model structure of Theorem 3.1.14. Moreover, the pointed variant $\hat{\mathbf{s}}\mathbf{s}_*$ as well as slice categories of the form $\hat{\mathbf{s}}\mathbf{s} \downarrow X$ will be equipped with the induced model structures, respectively.

Remark 3.1.15. Note that, in general, $\mathrm{ho}(\hat{\mathbf{s}}\mathbf{s} \downarrow X)$ is not equivalent to $\mathrm{ho}(\hat{\mathbf{s}}\mathbf{s}) \downarrow X$. One merely has a canonical functor

$$\mathrm{ho}(\hat{\mathbf{s}}\mathbf{s} \downarrow X) \to \mathrm{ho}(\hat{\mathbf{s}}\mathbf{s}) \downarrow X$$

induced by the localization functor $\hat{\mathbf{s}}\mathbf{s} \to \mathrm{ho}(\hat{\mathbf{s}}\mathbf{s})$.

3.1.3 Simplicial Enrichment

Summary. The categories $\hat{\mathbf{s}}\mathbf{s}$ and $\hat{\mathbf{s}}\mathbf{s}_*$ can be endowed with natural simplicial enrichments turning them into simplicial model categories. Although we will mainly need this in the case of $\hat{\mathbf{s}}\mathbf{s}_*$, we will shortly summarize the definition of the mapping spaces and the tensor and cotensor objects in both cases.

For the convenience of the reader, we furthermore recall the definition of tensoring and cotensoring in enriched category theory (see [Bor94, §6.5]):

Definition 3.1.16. Let $\mathscr{V}$ be a monoidal category and $\mathbf{C}$ a $\mathscr{V}$-enriched category.

(a) A *tensoring* of $\mathbf{C}$ over $\mathscr{V}$ is a functor

$$-\otimes -\colon \mathscr{V}\times \mathbf{C}\to \mathbf{C}$$

endowed with $\mathscr{V}$-isomorphisms

$$\mathrm{Map}_{\mathbf{C}}(v\otimes c, c') \xrightarrow{\sim} \mathrm{Map}_{\mathscr{V}}(v, \mathrm{Map}_{\mathbf{C}}(c,c'))$$

natural in $v\in\mathscr{V}$ and $c,c'\in\mathbf{C}$.

(b) A *cotensoring* of $\mathbf{C}$ over $\mathscr{V}$ is a functor

$$[-,-]\colon \mathscr{V}^{\mathrm{op}}\times \mathbf{C}\to \mathbf{C}$$

endowed with $\mathscr{V}$-isomorphisms

$$\mathrm{Map}_{\mathscr{V}}(v, \mathrm{Map}_{\mathbf{C}}(c,c')) \xrightarrow{\sim} \mathrm{Map}_{\mathbf{C}}(c, [v,c'])$$

natural in $v\in\mathscr{V}$ and $c,c'\in\mathbf{C}$.

$\hat{\mathbf{s}}\mathbf{s}$ as a Simplicial Model Category

Reference. [Qui13b, §2.1]

Definition 3.1.17. The *mapping space* of two profinite spaces X and Y is the simplicial set

$$\mathrm{Map}_{\hat{\mathbf{s}}\mathbf{s}}(X,Y)\colon [n]\mapsto \mathrm{Hom}_{\hat{\mathbf{s}}\mathbf{s}}(X\times\Delta^n, Y)\in \mathbf{sets}.$$

The above mapping space construction is obviously functorial in X and Y and gives hence rise to a functor

$$\mathrm{Map}_{\hat{\mathbf{s}}\mathbf{s}}(-,-)\colon \hat{\mathbf{s}}\mathbf{s}^{\mathrm{op}}\times\hat{\mathbf{s}}\mathbf{s}\to \mathbf{ss}.$$

Finally, here is how to construct (co-)tensorings of $\hat{\mathbf{s}}\mathbf{s}$ over $\mathbf{ss}$:

Construction 3.1.18. Let X be a profinite space and let K be a simplicial set.

$K \otimes X$:

(i) If K is *finite*, i.e. has finitely many non-degenerate simplices only, then

$$K \otimes X : [n] \mapsto X_n \times K_n.$$

(ii) If K is an arbitrary simplicial set, then K is the filtered colimit over its finite simplicial subsets K_α and we define

$$K \otimes X := \varinjlim_{\alpha} (K_\alpha \otimes X).$$

$[K,X]$: Recall that any profinite space is canonically isomorphic to a limit $\lim_\beta X_\beta$ of simplicial finite sets ([Qui13b, §2.1]). For a finite simplicial set K,

$$\mathrm{Hom}_{\hat{\mathbf{s}}\mathbf{s}}(K,X) = \lim_\beta \mathrm{Hom}_{\hat{\mathbf{s}}\mathbf{s}}(K,X_\beta)$$

carries a natural structure of profinite set.

(i) If K is finite, then the cotensor $[K,X]$ is the profinite space given by

$$[n] \mapsto \mathrm{Hom}_{\hat{\mathbf{s}}\mathbf{s}}(K \times \Delta^n, X) = \lim_\beta \mathrm{Hom}_{\hat{\mathbf{s}}\mathbf{s}}(K \times \Delta^n, X_\beta).$$

(ii) If K is an arbitrary simplicial set, then K is again the filtered colimit over its finite simplicial subsets K_α and we define

$$[K,X] := \varprojlim_{\alpha} [K_\alpha, X].$$

$\hat{\mathbf{s}}\mathbf{s}_*$ as a Simplicial Model Category

Reference. [Qui13b, §2.2]

The simplicial structure on $\hat{\mathbf{s}}\mathbf{s}_*$ makes use of the *smash product*:

Definition 3.1.19. Let (X,x) and (Y,y) be two pointed profinite spaces.

(a) The *wedge* of X and Y is the pointed profinite space $X \vee Y$ in the following pushout square:

$$\begin{tikzcd} \Delta^0 \sqcup \Delta^0 \arrow[r, "x+y"] \arrow[d, "\exists!"'] & X \sqcup Y \arrow[d] \\ \Delta^0 \arrow[r] & X \vee Y \end{tikzcd}$$

It comes with a canonical map $(X \vee Y, [x]=[y]) \to (X \times Y, x \times y)$ induced by the inclusions $X \xrightarrow{\sim} X \times \{y\} \subseteq X \times Y$ and $Y \xrightarrow{\sim} \{x\} \times Y \subseteq X \times Y$ respectively.

(b) The *smash product* of two pointed profinite spaces X and Y is the pointed profinite space

$$X \wedge Y := (X \times Y)/(X \vee Y).$$

We can make any profinite space X a pointed profinite space by freely adjoining a base point:

$$X_+ := X \sqcup \Delta^0.$$

The smash product now allows us to define mapping spaces for pointed profinite spaces:

Definition 3.1.20. The *mapping space* of two pointed profinite spaces (X,x) and (Y,y) is the simplicial set

$$\mathrm{Map}_{\hat{\mathbf{ss}}_*}(X,Y)\colon [n] \mapsto \mathrm{Hom}_{\hat{\mathbf{ss}}_*}(X \wedge \Delta^n_+, Y) \in \mathbf{sets}.$$

We yet again obtain a bifunctor

$$\mathrm{Map}_{\hat{\mathbf{ss}}_*}(-,-)\colon \hat{\mathbf{ss}}_*^{\mathrm{op}} \times \hat{\mathbf{ss}}_* \to \mathbf{ss}.$$

In the pointed case the construction of (co)tensorings is an easy adaption of the unpointed case:

Construction 3.1.21. Let (X,x) be a pointed profinite space and let K be a simplicial set.

$K \otimes X$:

(i) If K is *finite*, then

$$K \otimes X := X \wedge K_+.$$

(ii) If K is an arbitrary simplicial set, then K is the filtered colimit over its finite simplicial subsets K_α and we define

$$K \otimes X := \varinjlim_{\alpha} (K_\alpha \otimes X).$$

$[K,X]$:

(i) If K is *finite*, then $[K,X]$ is the profinite space given by

$$[n] \mapsto [K,X]_n := \mathrm{Hom}_{\hat{\mathbf{ss}}_*}(K_+ \wedge \Delta^n_+, X),$$

which is pointed via the map $K_+ \to \Delta^0 \xrightarrow{x} X$.

(ii) For arbitrary simplicial sets K we yet again extend by taking appropriate limits:

$$[K,X] := \varprojlim_{\alpha} [K_\alpha, X].$$

There also is an appropriate variant of the cotensor construction in the case where K already is pointed:

Notation 3.1.22. Let (X,x) be a pointed profinite space and let (K,k) be a pointed simplicial set.

(i) If K is *finite*, then $[K,X]$ denotes the profinite space given by

$$[n] \mapsto \mathrm{Hom}_{\hat{\mathbf{ss}}_*}(K \wedge \Delta^n_+, X).$$

(ii) For arbitrary pointed simplicial sets K we yet again extend by taking appropriate limits:

$$[K,X] := \varprojlim_{\alpha} [K_\alpha, X],$$

where α this time runs through the *pointed* simplicial finite subsets of K.

3.1.4 Homotopy Groups of Profinite Spaces

Definition 3.1.23 ([Qui08, Definition 2.15]).

(a) The *simplicial circle* is the simplicial pointed finite set

$$\mathbf{S}^1 := \Delta^1/\partial\Delta^1.$$

(b) The *simplicial (profinite) loop space* of a pointed profinite space X is the space

$$\mathbf{\Omega}X := [\mathbf{S}^1, X] \in \hat{\mathbf{s}}\mathbf{s}_*,$$

where we view $\mathbf{S}^1$ as a *pointed* finite set.

(c) Let (X, x) be a pointed profinite space and $n \geq 2$.
The *nth profinite homotopy group of X at x* is the profinite group

$$\hat{\pi}_n(X, x) := \hat{\pi}_0(\mathbf{\Omega}^n(\mathbf{R}_* X)),$$

where $\mathbf{R}_* X$ denotes a fibrant replacement of X in $\hat{\mathbf{s}}\mathbf{s}_*$.

Remark 3.1.24. Note that since $\mathbf{S}^1$ is a pointed, finite simplicial set, one actually has

$$\mathbf{\Omega}X = [\mathbf{S}^1, X] = \mathrm{Map}_{\hat{\mathbf{s}}\mathbf{s}_*}(\mathbf{S}^1, X).$$

Hence a 0-simplex in $\mathbf{\Omega}X$ can be identified with a 1-simplex σ of X such that $d_0\sigma = x = d_1\sigma$, i.e. a *loop at x*. Under this identification, the group structure on $\hat{\pi}_n(X, x)$ comes from the concatenation of loops in the profinite space $\mathbf{\Omega}^{n-1}X$.

Lemma 3.1.25 ([Qui13b, Lemma 2.9]). Let X be a fibrant pointed profinite space. Then for $n \geq 0$ we have a natural isomorphism

$$|\hat{\pi}_n(X)| \xrightarrow{\sim} \pi_n(|X|)$$

of abstract groups.

Lemma 3.1.26 ([Qui11, Corollary 3.15]). Let X be a π-finite space. Then the unit map $X \to |\hat{X}|$ is a weak equivalence, i.e. induces isomorphisms

$$\pi_n(X) \xrightarrow{\sim} \pi_n(|\hat{X}|)$$

for all $n \geq 0$.

3.1.5 Homotopy Limits of Profinite Spaces

In this subsection we shortly summarize properties of homotopy limits that are specific to $\hat{\mathbf{s}}$ and that will be used later on in the applications to étale homotopy theory.

Construction 3.1.27. Let $X(-)\colon \mathscr{I} \to \hat{\mathbf{s}}_*$ be a small diagram. The equalizer in $\hat{\mathbf{s}}_*$ of the diagram

$$\prod_{i\in\mathscr{I}}[\mathbf{N}(\mathscr{I}\downarrow i), X(i)] \rightrightarrows \prod_{\alpha\colon i\to i'\in\mathscr{I}}[\mathbf{N}(\mathscr{I}\downarrow i), X(i')]$$

where the two maps are induced by the two maps

$$[\mathbf{N}(\mathscr{I}\downarrow i), X(i)] \xrightarrow{X(\alpha)} [\mathbf{N}(\mathscr{I}\downarrow i), X(i')]$$

and

$$[\mathbf{N}(\mathscr{I}\downarrow i'), X(i')] \xrightarrow{\mathbf{N}(\mathscr{I}\downarrow\alpha)} [\mathbf{N}(\mathscr{I}\downarrow i), X(i')]$$

of pointed profinite spaces, respectively, will be denoted by

$$[\mathbf{N}(\mathscr{I}\downarrow -), X(-)] \in \hat{\mathbf{s}}_*.$$

Definition 3.1.28. Let $\mathscr{I}$ be a small category and $X(-)\colon \mathscr{I} \to \hat{\mathbf{s}}_*$ be a functor. The *homotopy limit* of $X(-)$ is the pointed profinite space

$$\operatorname{holim}_{i\in\mathscr{I}} X(i) := [\mathbf{N}(\mathscr{I}\downarrow -), X(-)] \in \hat{\mathbf{s}}_*$$

of Construction 3.1.27.

Remark 3.1.29. Note that any natural transformation $f : Y(-) \to X(-)$ induces a map of pointed profinite spaces

$$\operatorname{holim}_{i\in\mathscr{I}} f(i)\colon \operatorname{holim}_{i\in\mathscr{I}} Y(i) \to \operatorname{holim}_{i\in\mathscr{I}} X(i).$$

Lemma 3.1.30 ([Qui13b, Lemma 2.13])**.** Let $\mathscr{I}$ be a small category.

(i) If, for a diagram $X(-)\colon \mathscr{I} \to \hat{\mathbf{s}}_*$, $X(i)$ is fibrant for every $i \in \mathscr{I}$, then $\operatorname{holim}_{i\in\mathscr{I}} X(i)$ is fibrant in $\hat{\mathbf{s}}_*$.

(ii) Let $f(-)\colon Y(-) \to X(-)$ be a natural transformation of functors from $\mathscr{I}$ to the full subcategory of fibrant pointed profinite spaces. Let

$$f := \operatorname{holim}_{i\in\mathscr{I}} f(i)\colon \operatorname{holim}_{i\in\mathscr{I}} Y(i) \to \operatorname{holim}_{i\in\mathscr{I}} X(i)$$

be the induced map on homotopy limits. If $f(i)\colon Y(i) \to X(i)$ is a weak equivalence in $\hat{\mathbf{s}}_*$ for every $i \in \mathscr{I}$, then f is a weak equivalence in $\hat{\mathbf{s}}_*$.

Proposition 3.1.31.

(i) The forgetful functor $|{-}|\colon \hat{\mathbf{ss}} \to \mathbf{ss}$ preserves fibrations and weak equivalences between fibrant objects.

(ii) The forgetful functor $|{-}|\colon \hat{\mathbf{ss}}_* \to \mathbf{ss}_*$ preserves homotopy limits.

Proof.

(i) That $|{-}|$ preserves fibrations and weak equivalences between fibrant objects is proven in [Qui08, Proposition 2.28].

(ii) $|{-}|$ commutes with homotopy limits since homotopy limits of spaces are also computed via the Bousfield-Kan formula of Construction 3.1.27 and as $|{-}|$ commutes with limits and cotensor objects.

□

3.2 Some Homotopy Theory of Profinite Groupoids

Summary. We develop some basic homotopy theory *intrinsic* to the theory of profinite groupoids and compare it to the homotopy theory of their associated profinite spaces.

Absolute Case

Definition 3.2.1. Let $f, g\colon \Gamma \to \Gamma'$ be two morphisms of profinite groupoids.

(a) A *homotopy* $h\colon f \Rightarrow g$ is a continuous natural transformation of functors.

(b) f is *homotopic* to g, denoted by $f \simeq g$, if there exists a homotopy $f \Rightarrow g$.

Remark 3.2.2. Since continuous natural transformations between profinite groupoids necessarily are natural isomorphisms, the homotopy relation of Definition 3.2.1 defines an equivalence relation (that also is compatible with the composition of morphisms).

Definition 3.2.3. The *homotopy* category of the category of profinite groupoids, denoted by $\mathrm{ho}(\hat{\mathbf{G}}\mathbf{pd})$, has profinite groupoids as objects and morphisms of profinite groupoids *up to homotopy* as morphisms.

Lemma 3.2.4. Let G and G' be profinite groups. Then the canonical map

$$\mathrm{Hom}(G, G') \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbf{G}}\mathbf{pd}}(\mathbf{B}G, \mathbf{B}G')$$

induces a bijection

$$\mathrm{Hom}_{\mathrm{out}}(G, G') \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{ho}(\hat{\mathbf{G}}\mathbf{pd})}(\mathbf{B}G, \mathbf{B}G').$$

Proof. Given two group homomorphisms $\varphi, \psi\colon G \to G'$, a homotopy $h\colon \varphi \Rightarrow \psi$ consists of choosing an element $g' \in G'$ such that for all $g \in G$ the diagram

$$\begin{tikzcd} * \arrow[r, "g'"] \arrow[d, "\varphi(g)"'] & * \arrow[d, "\psi(g)"] \\ * \arrow[r, "g'"'] & * \end{tikzcd}$$

commutes, i.e. such that

$$\varphi(g) = g'\psi(g)g'^{-1}$$

holds for all $g \in G$. Hence φ and ψ are homotopic if, and only if there exists a $g' \in G'$ such that

$$\varphi = c_{g'} \circ \psi.$$

□

Relative Case

We now turn to the question what happens in the *relative* situation, where the groups G and G' are equipped with an augmentation map $G \to H \leftarrow G'$.

Definition 3.2.5. Let $s\colon \Gamma \to \Gamma_0$ and $s'\colon \Gamma' \to \Gamma_0$ be two profinite groupoids over Γ_0. Let $f, g\colon \Gamma \to \Gamma'$ be two morphisms of profinite groupoids over Γ_0.

(a) A *homotopy* $h\colon f \Rightarrow g$ *over* Γ_0 is a homotopy $f \Rightarrow g$ in the sense of Definition 3.2.1 with the property that the whiskered transformation $s'h\colon s = s' \circ f \Rightarrow s' \circ g = s$ equals the identity transformation $\mathbf{1}_s\colon s \Rightarrow s$.

(b) f is *homotopic* to g *over* Γ_0, denoted by $f \simeq_{\Gamma_0} g$ if there exists a homotopy $f \Rightarrow g$ over Γ_0.

Remark 3.2.6. As the homotopies over Γ_0 are again given by isomorphisms, we yet again obtain an equivalence relation (that is compatible with composition).

Definition 3.2.7. Let Γ_0 be a profinite groupoid. The *homotopy category* of $\hat{\mathbf{G}}\mathbf{pd} \downarrow \Gamma_0$, denoted by $\mathrm{ho}(\hat{\mathbf{G}}\mathbf{pd} \downarrow \Gamma_0)$, is the category with profinite groupoids over Γ_0 as objects and morphisms of such *up to homotopy over* Γ_0 as morphisms.

In the case of profinite groups, the homotopy category again admits an explicit, group-theoretic interpretation:

Proposition 3.2.8. Let $\varphi\colon G \to H \leftarrow G'\colon \psi$ be a span of profinite groups. Then there is a canonical bijection

$$\mathrm{Hom}_H(G, G')_\Delta \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{ho}(\hat{\mathbf{G}}\mathbf{pd}\downarrow \mathbf{B}H)}(\mathbf{B}G, \mathbf{B}G'),$$

where $\Delta := \psi^{-1}(\mathbf{Z}_H(\mathrm{im}(\varphi)))$, and where $\mathbf{Z}_H$ denotes the centraliser in H.

Proof. Let $f_0, f_1\colon G \to G'$ be two morphisms of profinite groups over H. Then these are homotopic as morphisms $\mathbf{B}G \to \mathbf{B}G'$ over $\mathbf{B}H$ if and only if there exists a continuous natural transformation $h\colon f_0 \Rightarrow f_1$ with the property $\psi h = \mathbf{1}_\varphi$. Such a transformation corresponds to an element $g' \in G'$ with the property that for all $g \in G$ the diagram

$$\begin{tikzcd} * \arrow[r, "g'"] \arrow[d, "f_0(g)"'] & * \arrow[d, "f_1(g)"] \\ * \arrow[r, "g'"'] & * \end{tikzcd}$$

commutes, i.e. such that $f_0 = c_{g'} \circ f_1$, and such that

$$c_{\psi(g')} \circ \varphi = \varphi,$$

i.e. such that $g' \in \psi^{-1}(\mathbf{Z}_H(\mathrm{im}(\varphi))) = \Delta$. □

Comparison with the Homotopy Theory of Profinite Spaces

Proposition 3.2.9. Let $s' \colon \Gamma' \to \Gamma_0$ be a profinite groupoid over Γ_0 such that the induced morphism of profinite spaces $\Gamma' \to \Gamma_0$ is a fibration. Then the map

$$\mathrm{Hom}_{\hat{\mathbf{G}}\mathbf{pd}\downarrow\Gamma_0}(\Gamma,\Gamma') \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbf{s}}\mathbf{s}\downarrow\Gamma_0}(\Gamma,\Gamma')$$

induces a bijection

$$\mathrm{Hom}_{\mathrm{ho}(\hat{\mathbf{G}}\mathbf{pd}\downarrow\Gamma_0)}(\Gamma,\Gamma') \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{ho}(\hat{\mathbf{s}}\mathbf{s}\downarrow\Gamma_0)}(\Gamma,\Gamma').$$

Proof. Since $\Gamma' \to \Gamma_0$ is assumed to be a fibration of profinite spaces, we have

$$\mathrm{Hom}_{\mathrm{ho}(\hat{\mathbf{s}}\mathbf{s}\downarrow\Gamma_0)}(\Gamma,\Gamma') = \pi_0(\mathrm{Map}_{\hat{\mathbf{s}}\mathbf{s}\downarrow\Gamma_0}(\Gamma,\Gamma')),$$

i.e., two morphisms $f_0, f_1 \colon \Gamma \to \Gamma'$ over Γ_0 induce the same morphism in the homotopy category $\mathrm{ho}(\hat{\mathbf{s}}\mathbf{s}\downarrow\Gamma_0)$ if and only if there exists a morphism of profinite spaces $h \colon \Gamma\times\Delta^1 \to \Gamma'$ *over* Γ_0 such that the diagram

$$\begin{tikzcd} \Gamma\times\{0\} \arrow[d] \arrow[dr, "f_0"] & \\ \Gamma\times\Delta^1 \arrow[r, "h"] & \Gamma' \\ \Gamma\times\{1\} \arrow[u] \arrow[ur, "f_1"'] & \end{tikzcd}$$

commutes. As $\hat{\mathrm{N}}$ is fully faithful (Lemma 2.6.8) and commutes with products, such diagrams correspond to continuous natural transformations $h \colon f_0 \Rightarrow f_1$ of profinite functors such that $s'h = \mathbf{1}_{\Gamma\to\Gamma_0}$, which is to say: homotopies $f_0 \Rightarrow f_1$ over Γ_0. □

Remark 3.2.10. Note that choosing $\Gamma_0 = *$ in Proposition 3.2.9 gives the expected “absolute” version.

Lemma 3.2.11. Let $G \twoheadrightarrow H$ be a surjective morphism of profinite groups. Then the induced map $\mathbf{B}G \to \mathbf{B}H$ is a fibration of profinite spaces.

Proof. Let $\Delta := \ker(G \twoheadrightarrow H)$. Then Δ acts on $\mathbf{B}G$ and we have

$$(\mathbf{B}G)/\Delta = \mathbf{B}(G/\Delta) = \mathbf{B}H,$$

i.e. $\mathbf{B}G \to \mathbf{B}H$ is a principal Δ-fibration with base $\mathbf{B}H$ and hence a fibration in $\hat{\mathbf{s}}\mathbf{s}$ ([Qui08, Corollary 2.25]). □

Corollary 3.2.12. Let $\varphi\colon G \to H \twoheadleftarrow G'\colon \psi$ be a span of profinite groups such that ψ is surjective. Then the canonical map

$$\mathrm{Hom}_H(G,G') \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbf{s}}\mathbf{s}\downarrow \mathbf{B}H}(\mathbf{B}G,\mathbf{B}G')$$

induces a bijection

$$\mathrm{Hom}_H(G,G')_\Delta \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{ho}(\hat{\mathbf{s}}\mathbf{s}\downarrow \mathbf{B}H)}(\mathbf{B}G,\mathbf{B}G'),$$

where $\Delta := \psi^{-1}(\mathbf{Z}_H(\mathrm{im}(\varphi)))$.

Proof. Combine Proposition 3.2.9, Proposition 3.2.8 and Lemma 3.2.11. □

3.3 Profinite Models for Spaces

Summary. In this section, we construct for any π-finite space $X \in \mathbf{ss}$ (see Definition 2.5.11) a profinite space $FX \in \hat{\mathbf{s}}\mathbf{s}$ together with a natural comparison map of spaces $X \to |FX|$. The profinite space FX has the same homotopy type as X, as is witnessed by the map $X \to |FX|$ (see Theorem 3.3.1), and thus serves as a *profinite model* of X.

Theorem 3.3.1. Let G be a finite group.

There exists a functor $F\colon \mathbf{ss}^\pi \downarrow \mathbf{B}G \to \hat{\mathbf{s}}\mathbf{s} \downarrow \mathbf{B}G$ taking values in *fibrant* profinite spaces and a *natural weak equivalence*

$$\varphi\colon \iota \overset{\simeq}{\Longrightarrow} |F-|,$$

where $\iota\colon \mathbf{ss}^\pi \downarrow \mathbf{B}G \to \mathbf{ss} \downarrow \mathbf{B}G$ denotes the inclusion.

Hence for any π-finite space X over $\mathbf{B}G$, φ in particular induces isomorphisms

$$\pi_*(\varphi_X)\colon \pi_*(X) \to \pi_*(|FX|)$$

on homotopy groups.

Proof. Since $\mathbf{B}G$ is a simplicial finite set, the map $X \to \mathbf{B}G$ induces a map $\hat{X} \to \mathbf{B}G$ in $\hat{\mathbf{s}}\mathbf{s}$, by the universal property of profinite completion. Let $Z \mapsto \mathbf{R}_{\mathbf{B}G}Z$ be a fixed functorial fibrant replacement in $\hat{\mathbf{s}}\mathbf{s} \downarrow \mathbf{B}G$. Define FX to be the fibrant replacement of the profinite completion of X:

$$FX := \mathbf{R}_{\mathbf{B}G}\hat{X}$$

The map $\varphi_X\colon X \to |FX|$ is the composition of the counit $X \to |\hat{X}|$ and $|\hat{X}| \to |\mathbf{R}_{\mathbf{B}G}\hat{X}|$ over $\mathbf{B}G$. We thus only need to check that φ_X is a weak equivalence of underlying simplicial sets. Denote a fixed functorial fibrant replacement in $\hat{\mathbf{s}}\mathbf{s}$ by $Z \mapsto \mathbf{R}Z$. Recall that according to Lemma 3.1.25, $|\hat{\pi}_n(Z)| = \pi_n(|\mathbf{R}Z|)$. Since X is assumed to be π-finite, the map $X \to |\mathbf{R}\hat{X}|$ is a weak equivalence of simplicial sets in virtue of Lemma 3.1.26. Finally, the commutative triangle

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "w"'] & & {|\hat{X}|} \arrow[dl, "w"] \\ & {|\mathbf{R}X|} & \end{tikzcd}$$

implies that the map $X \to |\hat{X}|$ is a weak equivalence by the 2-out-of-3 property. Hence φ_X is a weak equivalence, as it is the composition of the weak equivalence $X \to |\hat{X}|$ followed by the weak equivalence $|\hat{X}| \to |\mathbf{R}_{\mathbf{B}G}\hat{X}|$. The stated functoriality is just expressing the fact that $(\hat{-})$ as well as $\mathbf{R}_{\mathbf{B}G}(-)$ are functorial. □

Remark 3.3.2. The space FX of Theorem 3.3.1 functions as a *profinite model* of the π-finite space X.

3.4 Continuous Homotopy Fixed Points

Setup. Let G be a profinite group.

Fix a functorial fibrant replacement $X \mapsto \mathbf{R}_{\mathbf{B}G}X$ in $\hat{\mathbf{s}}\mathbf{s} \downarrow \mathbf{B}G$.

Definition 3.4.1. Let $X \in \hat{\mathbf{s}}\mathbf{s} \downarrow \mathbf{B}G$. The *(continuous) homotopy-fixed point space of X* is the derived space of sections of $X \to \mathbf{B}G$:

$$X^{\mathrm{h}G} := \mathbf{R}\mathrm{Map}_{\mathbf{B}G}(\mathbf{B}G, X) \in \mathbf{ss}$$

Remark 3.4.2. Note that since $\mathbf{B}G$ is cofibrant in $\hat{\mathbf{s}}\mathbf{s} \downarrow \mathbf{B}G$, we can compute $X^{\mathrm{h}G}$ explicitly by

$$X^{\mathrm{h}G} = \mathbf{R}\mathrm{Map}_{\mathbf{B}G}(\mathbf{B}G, X) = \mathrm{Map}_{\mathbf{B}G}(\mathbf{B}G, \mathbf{R}_{\mathbf{B}G}X)$$

3.5 Homotopy Fixed Points and Sections

Summary. We relate the set $\mathrm{S}(\pi)$ of sections of a surjective group homomorphism $\pi \twoheadrightarrow G$ of profinite groups with the homotopy-fixed point space $\mathbf{B}\pi^{\mathrm{h}G}$.

Setup. Let $\bar{\pi}$ be a profinite group and let

$$1 \longrightarrow \bar{\pi} \longrightarrow \pi \longrightarrow G \longrightarrow 1$$

be a fixed extension of G by $\bar{\pi}$.

We denote the set of $\bar{\pi}$-conjugacy classes of continuous sections of the above sequence by $\mathrm{S}(\pi)$.

Proposition 3.5.1. The mapping $[f] \mapsto \hat{\pi}_1(f)$ defines a bijection of sets

$$\pi_0(\mathbf{B}\pi^{\mathrm{h}G}) \xrightarrow{\sim} \mathrm{Hom}_G(G, \pi)_\Delta,$$

where Δ is the preimage of the center $\mathbf{Z}(G)$ of G along $\pi \twoheadrightarrow G$.

Proof. Since $\pi \twoheadrightarrow G$ is surjective, $\mathbf{B}\pi \to \mathbf{B}G$ is a fibration (Lemma 3.2.11), hence

$$\begin{aligned} \pi_0(\mathbf{B}\pi^{\mathrm{h}G}) &= \pi_0(\mathbf{R}\mathrm{Map}_{\hat{\mathbf{s}}\mathbf{s}\downarrow\mathbf{B}G}(\mathbf{B}G, \mathbf{B}\pi)) \\ &= \pi_0(\mathrm{Map}_{\hat{\mathbf{s}}\mathbf{s}\downarrow\mathbf{B}G}(\mathbf{B}G, \mathbf{B}\pi)) \\ &= \mathrm{Hom}_{\mathrm{ho}(\hat{\mathbf{s}}\mathbf{s}\downarrow\mathbf{B}G)}(\mathbf{B}G, \mathbf{B}\pi). \end{aligned}$$

In virtue of Proposition 3.2.8, this latter set is in bijection with $\mathrm{Hom}_G(G, \pi)_\Delta$, where $\Delta = (\pi \twoheadrightarrow G)^{-1}(\mathbf{Z}_G(G))$. □

Corollary 3.5.2. If G is centerless, then the mapping $[f] \mapsto \hat{\pi}_1(f)$ defines a bijection of sets

$$\pi_0(\mathbf{B}\pi^{\mathrm{h}G}) \xrightarrow{\sim} \mathrm{S}(\pi).$$

Proof. In this case $\Delta = \ker(\pi \twoheadrightarrow G) = \bar{\pi}$, hence $\mathrm{Hom}_G(G,\pi)_\Delta = \mathrm{Hom}_G(G,\pi)_{\bar{\pi}} = \mathrm{S}(\pi)$. □

Remark 3.5.3. In [Qui13a, Proposition 2.9], the content of Corollary 3.5.2 is missing the assumption of G being centerless.

3.6 Profinite Models for Pro-Spaces

Summary. For our later applications to étale homotopy theory, we need to enhance the construction of profinite models of π-finite spaces of Section 3.3 to pro-objects of π-finite spaces. This generalization is obtained in this section.

Construction 3.6.1. Let $G = \lim_k G(k)$ be a profinite group given as the limit of finite groups $G(k)$ indexed over a cofiltered category K. Let $\mathscr{X} = (\mathscr{X}(i))_{i\in I}$ be a pro-space such that every $\mathscr{X}(i)$ is a π-finite space. Let furthermore $\mathscr{X}$ be equipped with a *strict* morphism $\mathscr{X} = (\mathscr{X}(i))_{i\in I} \to (\mathbf{B}G(k))_{k\in K} = \mathbf{B}G$ of pro-objects in $\hat{\mathbf{s}}\mathbf{s}$. We thus have a functor $\alpha\colon K \to I$ and natural maps $\mathscr{X}(\alpha(k)) \to \mathbf{B}G(k)$ in $\hat{\mathbf{s}}\mathbf{s}$ for every $k \in K$. Since $\mathscr{X}(i)$ satisfies the assumptions of Theorem 3.3.1 for each i, we can apply the functor $\mathscr{X}(i) \mapsto F\mathscr{X}(i)$ to obtain a pro-object $(F\mathscr{X}(i))_{i\in I} \in \mathrm{Pro}(\hat{\mathbf{s}}\mathbf{s}_*)$ in the category of pointed profinite spaces, together with a strict morphism

$$(F\mathscr{X}(i))_{i\in I} \to (\mathbf{B}G(k))_{k\in K} = \mathbf{B}G.$$

Since taking homotopy limits is functorial with respect to strict morphisms — Remark 3.1.29, we get an induced map in $\hat{\mathbf{s}}\mathbf{s}_*$

$$\varphi\colon\ \mathrm{holim}_{i\in I} F\mathscr{X}(i) \to \mathrm{holim}_{k\in K} \mathbf{B}G(k) = \mathbf{B}G.$$

Since homotopy limits in simplicial model categories preserve fibrations [Hir03, Theorem 18.5.1], φ is a fibration in $\hat{\mathbf{s}}\mathbf{s}_*$.

Remark 3.6.2. Note that the equality $\mathrm{holim}_{k\in K} \mathbf{B}G(k) = \mathbf{B}G$ in Construction 3.6.1 holds as the spaces $\mathbf{B}G(k)$ are fibrant, since the $G(k)$ are finite, and hence

$$\begin{aligned}\mathrm{holim}_{k\in K} \mathbf{B}G(k) &= \lim_{k\in K} \mathbf{B}G(k)\\ &= \mathbf{B}G.\end{aligned}$$

Proposition 3.6.3. Let $X\colon \mathscr{I} \to \hat{\mathbf{s}}\mathbf{s}_*$ be a small cofiltered diagram such that each $X(i)$ is fibrant and such that each $|X(i)|$ is π-finite. Then for all $n \geq 1$ there is a natural isomorphism

$$\hat{\pi}_n(\mathrm{holim}_i X(i)) \xleftarrow{\sim} \varprojlim_i \pi_n(|X(i)|)$$

of profinite groups.

Proof.

(i) For any fibrant pointed profinite space Y, Lemma 3.1.25 provides us an isomorphism

$$|\hat{\pi}_n(Y)| \cong \pi_n(|Y|)$$

of abstract groups along which we equip $\pi_n(|Y|)$ with the structure of a profinite group. The spaces $X(i)$ are assumed to be fibrant and hence, according to Lemma 3.1.30 (i), $\mathrm{holim}_i X(i)$ is too.

(ii) In virtue of Proposition 3.1.31, $|-|$ commutes with homotopy limits and preserves fibrant objects. Hence

$$|\mathrm{holim}_i X(i)| = \mathrm{holim}_i |X(i)|.$$

and the spaces $|X(i)|, i \in \mathscr{I}$ are fibrant.

(iii) As all $|X(i)|$ are fibrant, [BK72, Ch. XI §7] supplies a Bousfield-Kan-type spectral sequence involving derived limits of the form

$$\mathrm{E}_2^{s,t} = \begin{cases} \varprojlim_i^s \pi_t(|X(i)|) & \text{if } 0 \leq s \leq t \\ 0 & \text{otherwise} \end{cases}$$

converging to $\pi_{s+t}(\mathrm{holim}_i |X(i)|)$.

(iv) As all the occuring spaces are assumed to be π-finite, all the higher derived limits

$$\varprojlim_i^s \pi_t(|X(i)|) \quad s \geq 1$$

vanish ([SS16, Lemma A.7]). Hence the spectral sequence degenerates to a single row, giving the desired isomorphism

$$\varprojlim_i \pi_n(|X(i)|) \xrightarrow{\sim} \pi_n(\mathrm{holim}_i |X(i)|) = \hat{\pi}_n(\mathrm{holim}_i X(i)).$$

□

Remark 3.6.4.

(i) The *π-finiteness* assumption on $|X(i)|$ in Proposition 3.6.3 is actually superfluous: A close inspection of the proof of [SS16, Lemma A.7] reveals that the higher derived limits of arbitrary cofiltered diagrams of profinite groups vanish, so that the proof of Proposition 3.6.3 also applies in this more general situation.

(ii) For a detailed exposition of "derived limits" of (not-necessarily abelian) groups, we refer the reader to [BK72, Ch. XI §6.5].

Lemma 3.6.5. For each $n \geq 1$, the homotopy group $\hat{\pi}_n(\operatorname{holim}_i F\mathscr{X}(i))$ is naturally isomorphic, as a profinite group, to $\varprojlim_i \pi_n(\mathscr{X}(i))$.

Proof. As each $\mathscr{X}(i)$ is connected, the $F\mathscr{X}(i)$ are connected, pointed and fibrant profinite spaces. Since $|\mathscr{X}(i)|$ is furthermore assumed to be π-finite, we can apply Proposition 3.6.3 to obtain an isomorphism of profinite groups

$$\hat{\pi}_n(\operatorname{holim}_i F\mathscr{X}(i)) \xleftarrow{\sim} \varprojlim_i \hat{\pi}_n(F\mathscr{X}(i)).$$

Lemma 3.1.25 and Theorem 3.3.1 now, for each i, supply an isomorphism

$$|\hat{\pi}_n(F\mathscr{X}(i))| \xrightarrow{\sim} \pi_n(|F\mathscr{X}(i)|) \xleftarrow{\sim} \pi_n(\mathscr{X}(i))$$

of abstract groups. Hence, since the $X(i)$ are π-finite, $\hat{\pi}_n(F\mathscr{X}(i))$ is a *finite* profinite group, so that the above isomorphism is seen to be topological. Putting everything together, we obtain a chain of isomorphisms

$$\begin{aligned}
\hat{\pi}_n(\operatorname{holim}_i F\mathscr{X}(i)) &\xleftarrow{\sim} \varprojlim_i \hat{\pi}_n(F\mathscr{X}(i)) \\
&\xrightarrow{\sim} \varprojlim_i \pi_n(|F\mathscr{X}(i)|) \\
&\xleftarrow{\sim} \varprojlim_i \pi_n(\mathscr{X}(i)).
\end{aligned}$$

of profinite groups as desired. □

Definition 3.6.6. We call $\mathscr{X}_{\mathrm{pf}} := \operatorname{holim}_{i \in I} F\mathscr{X}(i) \in \hat{\mathbf{s}}\mathbf{s}$ together with the map

$$\varphi\colon \mathscr{X}_{\mathrm{pf}} \to \operatorname{holim}_{k \in K} \mathbf{B}G(k) = \mathbf{B}G$$

a *profinite model* over $\mathbf{B}G$ of the pro-space $\mathscr{X} = (\mathscr{X}(i))_{i \in I}$.

Using the profinite model, we can define continuous homotopy-fixed points for pro-spaces as well:

Definition 3.6.7. Let $\mathscr{X}$ be a pro-space over $\mathbf{B}G$ as in Construction 3.6.1.

The *(continuous) homotopy-fixed point space* of $\mathscr{X}$ is the homotopy-fixed point space of its profinite model:

$$\mathscr{X}^{\mathrm{h}G} := (\mathscr{X}_{\mathrm{pf}})^{\mathrm{h}G} = \operatorname{Map}_{\mathbf{B}G}(\mathbf{B}G, \mathscr{X}_{\mathrm{pf}}).$$

3.7 Group Actions on Pro-Spaces

Summary.

- We introduce a notion of action of a profinite group G on a pro-space $\bar{\mathscr{X}}$ motivated by the kind of action the absolute Galois group Gal_k admits on the étale topological type of a variety over k.
- Given a pro-space $\bar{\mathscr{X}}$ and a profinite group G acting on $\bar{\mathscr{X}}$, we construct a space
$$\bar{\mathscr{X}}^G \in \mathbf{ss},$$
which should be thought of as the *fixed point space* of $\bar{\mathscr{X}}$ under the action of G.
- For any morphism of pro-spaces $\bar{\mathscr{X}} \to \mathscr{X}$, we construct a map of spaces
$$\eta \colon \bar{\mathscr{X}}^G \to \mathscr{X}^{\mathrm{h}G}$$
comparing fixed points to homotopy-fixed points.

Setup. Let $G = (G/U)_U$ be a profinite group indexed by the cofiltered system of its open normal subgroups $U \trianglelefteq G$ and let $\bar{\mathscr{X}} = (\bar{\mathscr{X}}(j))_{j \in \mathscr{J}}$ be a pro-space. Assume that G acts on $\bar{\mathscr{X}}$ via *strict* automorphisms:

Any $g \in G$ comes with a functor $g \colon \mathscr{J} \to \mathscr{J}$ and a natural isomorphism
$$T(g) \colon \bar{\mathscr{X}} \circ g \xrightarrow{\sim} \bar{\mathscr{X}}.$$

Definition 3.7.1. The *fixed point space* of $\bar{\mathscr{X}}$ under the action of G is
$$\bar{\mathscr{X}}^G := \mathrm{Map}_{\text{pro-}\mathbf{ss}}(*, \bar{\mathscr{X}})^G \in \mathbf{ss},$$
where the action of G on $\mathrm{Map}_{\text{pro-}\mathbf{ss}}(*, \bar{\mathscr{X}}) = \lim_{j \in \mathscr{J}} \bar{\mathscr{X}}(j)$ is given by postcomposition with the automorphisms $T(g)$.

We now construct the comparison map
$$\eta \colon \bar{\mathscr{X}}^G \to \mathscr{X}^{\mathrm{h}G}$$
already mentioned in this sections summary.

Construction 3.7.2. The construction of η is divided into several steps:

(i) Since $\mathrm{Hom}_{\text{pro-}\mathbf{ss}}(\mathbf{E}G, \bar{\mathscr{X}}) = \varprojlim_j \varinjlim_U \mathrm{Hom}_{\mathbf{ss}}(\mathbf{E}G/U, \bar{\mathscr{X}}(j))$, any map $f \colon \mathbf{E}G \to \bar{\mathscr{X}}$ is given by a compatible family $(f_j)_{j \in \mathscr{J}}$ with $f_j \in \mathrm{Hom}_{\mathbf{ss}}(\mathbf{E}(G/U), \bar{\mathscr{X}}(j))$.
Considering $\mathbf{E}G$ to be given by $\mathbf{E}G = (\mathbf{E}(G/U))_U$ with $U \trianglelefteq G$ the open and normal subgroups, G acts on $\mathbf{E}G$ by strict automorphisms: $g \in G$ acts trivially on the indexing category $\{U \trianglelefteq G\}$ and $T(g)_U \colon \mathbf{E}G/U \to \mathbf{E}G/U$ is given via multiplication by g. Since G acts trivially on the indexing category of $\mathbf{E}G$, we can define an action of G on a map $f = (f_j)_j \colon \mathbf{E}G \to \bar{\mathscr{X}}$ via
$$g \cdot f = (g \cdot f_j \colon \mathbf{E}(G/U) \to \bar{\mathscr{X}}(j))_j,$$

where $g \cdot f_j$ denotes the composition

$$\mathbf{E}G/U \xrightarrow{g^{-1}} \mathbf{E}G/U \xrightarrow{f_{g(j)}} \bar{\mathscr{X}}(g(j)) \xrightarrow{T(g)_j} \bar{\mathscr{X}}(j).$$

One readily checks that this action extends to an action of G on the whole mapping space $\mathrm{Map}_{\text{pro-}\mathbf{ss}}(\mathbf{E}G, \bar{\mathscr{X}})$.
Precomposing with the G-equivariant map $\mathbf{E}G \to *$ in pro-$\mathbf{ss}$ hence induces a map of spaces

$$\bar{\mathscr{X}}^G = \mathrm{Map}_{\text{pro-}\mathbf{ss}}(*, \bar{\mathscr{X}})^G \to \mathrm{Map}_{\text{pro-}\mathbf{ss}}(\mathbf{E}G, \bar{\mathscr{X}})^G.$$

(ii) Now given any G-equivariant map $f : \mathbf{E}G \to \bar{\mathscr{X}}$, i.e. a 0-simplex of $\mathrm{Map}_{\text{pro-}\mathbf{ss}}(\mathbf{E}G, \bar{\mathscr{X}})^G$, we obtain the G-equivariant map $1 \times f : \mathbf{E}G \to \mathbf{E}G \times \bar{\mathscr{X}}$. Quotiening out the G-action gives rise to a section

$$\mathbf{B}G = \mathbf{E}G/G \to (\mathbf{E}G \times \bar{\mathscr{X}})/G = \mathbf{E}G \times_G \bar{\mathscr{X}}$$

of the map $\mathbf{E}G \times_G \bar{\mathscr{X}} \to \mathbf{B}G$ obtained by the projection $\mathbf{E}G \times \bar{\mathscr{X}} \to \mathbf{E}G$.
Thus, we have constructed a map of spaces

$$\mathrm{Map}_{\text{pro-}\mathbf{ss}}(\mathbf{E}G, \bar{\mathscr{X}})^G \to \mathrm{Map}_{\text{pro-}\mathbf{ss}\downarrow\mathbf{B}G}(\mathbf{B}G, \mathbf{E}G \times_G \bar{\mathscr{X}}).$$

(iii) Now let $p : \bar{\mathscr{X}} \to \mathscr{X}$ be a strict morphism of pro-spaces over $\mathbf{B}G$ with $\mathscr{X}$ satisfying the hypotheses of Construction 3.6.1, so that we can form a profinite model $\mathscr{X}_{\mathrm{pf}}$. Equipping $\mathscr{X}$ with the trivial G-action, p induces a G-equivariant map

$$1 \times p : \mathbf{E}G \times \bar{\mathscr{X}} \to \mathbf{E}G \times \mathscr{X}$$

which in turn induces a map

$$\mathbf{E}G \times_G \bar{\mathscr{X}} = (\mathbf{E}G \times \bar{\mathscr{X}})/G \to (\mathbf{E}G \times \mathscr{X})/G = \mathbf{B}G \times \mathscr{X} \to \mathscr{X}.$$

Postcomposing with this last map hence induces a map of spaces

$$\mathrm{Map}_{\text{pro-}\mathbf{ss}\downarrow\mathbf{B}G}(\mathbf{B}G, \mathbf{E}G \times_G \bar{\mathscr{X}}) \to \mathrm{Map}_{\text{pro-}\mathbf{ss}\downarrow\mathbf{B}G}(\mathbf{B}G, \mathscr{X}).$$

(iv) Since taking profinite models $(-)_{\mathrm{pf}}$ is functorial, we furthermore obtain a map

$$\mathrm{Map}_{\text{pro-}\mathbf{ss}\downarrow\mathbf{B}G}(\mathbf{B}G, \mathscr{X}) \to \mathrm{Map}_{\hat{\mathbf{ss}}\downarrow\mathbf{B}G}(\mathbf{B}G, \mathscr{X}_{\mathrm{pf}}) = \mathscr{X}^{\mathrm{h}G}.$$

(v) The map $\eta : \bar{\mathscr{X}}^G \to \mathscr{X}^{\mathrm{h}G}$ is the composition

$$\begin{tikzcd}
\bar{\mathscr{X}}^G \arrow[rr, "\eta"] \arrow[d] & & \mathscr{X}^{\mathrm{h}G} \\
\mathrm{Map}_{\text{pro-}\mathbf{ss}}(\mathbf{E}G, \bar{\mathscr{X}})^G \arrow[dr] & & \mathrm{Map}_{\text{pro-}\mathbf{ss}\downarrow\mathbf{B}G}(\mathbf{B}G, \mathscr{X}) \arrow[u] \\
& \mathrm{Map}_{\text{pro-}\mathbf{ss}\downarrow\mathbf{B}G}(\mathbf{B}G, \mathbf{E}G \times_G \bar{\mathscr{X}}) \arrow[ur] &
\end{tikzcd}$$

4 Étale Homotopy

4.1 Artin and Mazur's Étale Homotopy Type

Summary. In this section, we shortly recall Artin and Mazur's construction and some basic properties of the Étale Homotopy Type of [AM86]. In more detail:

- Section 4.1.1 introduces *pro-homotopy types* and their homotopy, homology and cohomology (pro-)groups.
- Section 4.1.2 defines the notion of a *hypercovering* of a (pointed) site and recapitulates the "homotopy theory" of such hypercoverings.
- Section 4.1.3 discusses the Verdier functor $\mathbf{C} \mapsto \Pi\mathbf{C}$ of [AM86, §9].
- Section 4.1.4 defines the étale homotopy type $\Pi^{\text{ét}}X$ of a (suitable) scheme X as the pro-space obtained by applying the Verdier functor to the small étale site $\mathbf{Ét}(X)$ of X, and discusses various results of [AM86] related to the étale homotopy type.

Setup. Throughout this section, let $\mathbf{C}$ denote a site admitting fibre products and a terminal object.

4.1.1 Pro-Homotopy Types

Summary. The étale homotopy type, in its original form, is not an object of some homotopy category of a model category, but rather a pro-object in the homotopy category $\mathrm{ho}(\mathbf{ss})$ of the category of simplicial sets. In this section, we give a short overview of how one can work with such "pro-homotopy types".

Definition 4.1.1.

(a) The *category of pro-homotopy types* is the pro-category of the homotopy category of $\mathbf{ss}$:

$$\text{pro-}\mathrm{ho}(\mathbf{ss}) := \mathrm{Pro}(\mathrm{ho}(\mathbf{ss}))$$

(b) The *category of pointed pro-homotopy types* is the pro-category of the homotopy category of $\mathbf{ss}_*$:

$$\text{pro-}\mathrm{ho}(\mathbf{ss}_*) := \mathrm{Pro}(\mathrm{ho}(\mathbf{ss}_*))$$

There are a natural notions of homotopy and (co-)homology (pro-)groups of (pointed) pro-homotopy types:

Definition 4.1.2. Let $\mathscr{H} = (\mathscr{H}(i))_i \in \text{pro-ho}(\mathbf{ss}_*)$ be a pointed pro-homotopy type.

(a) The *n-th (pro-) homotopy group* of $\mathscr{H}$ is given by

$$\pi_n(\mathscr{H}) := (\pi_n(\mathscr{H}(i)))_i \in \mathrm{Pro}(\mathbf{Grp}).$$

For $n \geq 2$ it is a pro-abelian group.

(b) For an abelian group $A \in \mathbf{Ab}$, the *n-th homology (pro-)group of $\mathscr{H}$ with values in A* is given by

$$\mathrm{H}_n(\mathscr{H}; A) := (\mathrm{H}_n(\mathscr{H}(i); A))_i \in \mathrm{Pro}(\mathbf{Ab}).$$

(c) For an abelian group $A \in \mathbf{Ab}$, the *n-th cohomology group of $\mathscr{H}$ with values in A* is given by

$$\mathrm{H}^n(\mathscr{H}; A) := \varinjlim_i \mathrm{H}^n(\mathscr{H}(i); A) \in \mathbf{Ab}.$$

Remark 4.1.3. We define cohomology groups of pro-homotopy types through filtered colimits as these are *exact* in **Ab** and should commute with cohomology anyway.

$\sharp$-Isomorphisms of Pro-Homotopy Types

Summary. The notion of isomorphism internal to the category pro-ho(**ss**) of pro-homotopy groups is too strict. For example, the appropriate analogue of "Whitehead's Theorem" *ceases to hold* for pro-homotopy types:

A map $\mathscr{H} \to \mathscr{H}'$ in $\text{pro-ho}(\mathbf{ss}_*)$ inducing isomorphisms

$$\pi_*(\mathscr{H}) \xrightarrow{\sim} \pi_*(\mathscr{H}')$$

on all homotopy pro-groups is not necessarily an isomorphism. Artin and Mazur resolved this issue by introducing the notion of a $\sharp$-*isomorphism* ([AM86, §4]).

Construction 4.1.4. Let $\mathscr{H} = (\mathscr{H}(i))_i$ be a (pointed) pro-homotopy type. The various coskeletons $\mathrm{cosk}_n(\mathscr{H}(i))$ gather in a functorial way into the pro-homotopy type

$$\mathscr{H}^\sharp := (\mathrm{cosk}_n(\mathscr{H}(i)))_{(n,i)} \in \text{pro-ho}(\mathbf{ss}_*).$$

Definition 4.1.5. A morphism $f: \mathscr{H} \to \mathscr{H}'$ of pointed pro-homotopy types is a $\sharp$-*isomorphism* (or also *weak equivalence of pro-homotopy types*) if it induces an isomorphism $f^\sharp: \mathscr{H}^\sharp \to \mathscr{H}'^\sharp$ of pro-homotopy types upon sharpification.

Example 4.1.6. Let $\mathscr{H}$ be a pointed pro-homotopy type. Then the natural map

$$\mathscr{H} \to \mathscr{H}^\sharp$$

of pro-homotopy types evidently induces isomorphisms

$$\pi_*(\mathscr{H}) \to \pi_*(\mathscr{H}^\sharp)$$

but can only be an actual isomorphism in $\text{pro-ho}(\mathbf{ss}_*)$ when $\mathscr{H} = \mathrm{cosk}_n(\mathscr{H})$ for some finite n.

Artin and Mazur now prove the following characterisation:

Theorem 4.1.7 ([AM86, Corollary 4.4]). Let $f : \mathscr{H} \to \mathscr{H}'$ be a morphism in pro-ho($\mathbf{ss}_*$). The following are equivalent:

(i) f is a $\sharp$-isomorphism.

(ii) $\operatorname{cosk}_n(f)\colon \operatorname{cosk}_n(\mathscr{H}) \to \operatorname{cosk}_n(\mathscr{H}')$ is an isomorphism for each $n \geq 0$.

(iii) $\pi_n(f)\colon \pi_n(\mathscr{H}) \to \pi_n(\mathscr{H}')$ is an isomorphism for each $n \geq 0$.

4.1.2 Hypercoverings

Definition 4.1.8. Let $p\colon \mathbf{sets} \to \mathbf{C}$ be a point of $\mathbf{C}$.

A *pointed* simplicial object with values in $(\mathbf{C}, p)$ is a tuple (X, x) consisting of a simplicial object $X \in \mathbf{C}_{\boldsymbol{\Delta}}$ together with a choice of point $x \in p^{\#}(X_0)$.

Definition 4.1.9. A *hypercovering* of $\mathbf{C}$ is a (pointed) simplicial object $X \in \mathbf{C}_{\boldsymbol{\Delta}}$ such that for all $n \geq 0$:

(a) The unique morphism $X_0 \to *$ to the final object of $\mathbf{C}$ is a covering.

(b) The morphism $X_{n+1} \to (\operatorname{cosk}_n(X))_{n+1}$ is a covering.

Definition 4.1.10. Let $X \in \mathbf{C}_{\boldsymbol{\Delta}}$ be a simplicial object of $\mathbf{C}$ and Δ^1 the simplicial unit interval. Then $X \otimes T \in \mathbf{C}_{\boldsymbol{\Delta}}$ is the simplicial object given by

$$(X \otimes \Delta^1)_n := \sqcup_{\sigma \in \Delta^1_n} X_n^{\sigma},$$

and with face and degeneracy maps acting on the superscript σ:

$$d_k\colon K_n^{\sigma} \to K_{n-1}^{d_k\sigma}, \quad s_k\colon K_n^{\sigma} \to K_{n+1}^{s_k\sigma}$$

Definition 4.1.11. Let $f_0, f_1\colon Y \to X$ be morphisms of simplicial objects $X, Y \in \mathbf{C}_{\boldsymbol{\Delta}}$ of $\mathbf{C}$.

(a) A *homotopy* $H\colon f_0 \Rightarrow f_1$ is a morphism

$$H\colon Y \otimes \Delta^1 \to X$$

of simplicial objects such that the diagram

$$\begin{tikzcd}
Y \otimes \{0\} \arrow[d, hook] \arrow[dr, "f_0"] & \\
Y \otimes \Delta^1 \arrow[r, "H"] & X \\
Y \otimes \{1\} \arrow[u, hook] \arrow[ur, "f_1"'] &
\end{tikzcd}$$

commutes.

(b) f_0 and f_1 are *homotopic*, if there exists a finite chain of homotopies connecting f_0 and f_1.

Definition 4.1.12. The *category of hypercoverings of* $\mathbf{C}$ is the category $\mathbf{HR}(\mathbf{C})$ with objects the hypercoverings in $\mathbf{C}$ and homotopy classes of maps of simplicial objects as morphisms.

Lemma 4.1.13 ([AM86, Corollary 8.13])**.** The category $\mathbf{HR}(\mathbf{C})$ is filtered.

4.1.3 The Verdier Functor

Definition 4.1.14. A category $\mathbf{C}$ is *distributive* if it satisfies the following conditions:

(a) $\mathbf{C}$ admits finite fibre products.

(b) Fibre products commute with all coproducts (that are representable in $\mathbf{C}$).

Definition 4.1.15. Let $\mathbf{C}$ be a distributive category.

(a) An object $X \in \mathbf{C}$ is *connected* if it has no non-trivial coproduct decomposition.

(b) $\mathbf{C}$ is *locally connected* if every object has a coproduct decomposition into connected objects.

(c) $\mathbf{C}$ is *connected* if it is locally connected and if it admits a final object $*$ that is furthermore connected.

Remark 4.1.16. By the distributivity, decompositions into connected objects are essentially unique. Hence, in a locally connected category $\mathbf{C}$, mapping an object X to its set of connected components $\pi_0(X)$ is functorial.

Definition 4.1.17. Let $\mathbf{C}$ be a locally connected category. The *connected component functor (of* $\mathbf{C}$*)* is the functor $\pi_0\colon \mathbf{C} \to \mathbf{sets}$ carrying an object X to its set of connected components.

Construction 4.1.18. Let $\mathbf{C}$ be a locally connected site. Applying the connected component functor π_0 to the category $\mathbf{HR}(\mathbf{C})$ defines a functor

$$\mathbf{HR}(\mathbf{C}) \to \mathrm{ho}(\mathbf{ss}), \quad X \mapsto \pi_0(X) := \{\pi_0(X_n)\}_n.$$

Since, according to Lemma 4.1.13, $\mathbf{HR}(\mathbf{C})$ is filtered, this functor is a pro-object in $\mathrm{ho}(\mathbf{ss})$, i.e. a pro-homotopy type. The rule associating to a locally connected site $\mathbf{C}$ the above pro-object is readily seen to be functorial with respect to morphisms of sites.

Definition 4.1.19. The *Verdier functor* is the functor $\mathbf{C} \mapsto \Pi\mathbf{C} := (\pi_0(X))_{X\in\mathbf{HR}(\mathbf{C})} \in \text{pro-}\mathrm{ho}(\mathbf{ss})$ of Construction 4.1.18.

Remark 4.1.20. Note that one really only gets a pro-homotopy type $\Pi\mathbf{C} \in \text{pro-}\mathrm{ho}(\mathbf{ss})$ and not an actual pro-space living in pro-$\mathbf{ss}$. This is because $\mathbf{HR}(\mathbf{C})$ is only filtered when one looks at morphisms of hypercoverings up to homotopy, so that, for a hypercovering X, $\pi_0(X)$ also is well-defined only as an object of $\mathrm{ho}(\mathbf{ss})$, not $\mathbf{ss}$. A "rigidified" version of $\Pi\mathbf{C}$, at least in the case of the étale homotopy type, resulting in an actual pro-space will be discussed in Section 4.2.

4.1.4 The Étale Homotopy Type

Summary. In this section, we finally provide a definition of the étale homotopy type $\Pi^{\text{ét}}X \in \text{pro-ho}(\mathbf{ss})$ of a (locally Noetherian) scheme X. We furthermore discuss various results about the étale homotopy type, including:

- The definition of *higher* étale homotopy groups $\pi_n^{\text{ét}}(X,x), n \geq 2$.
- A comparison between $\pi_1(\Pi^{\text{ét}}X, x)$ and $\pi_1^{\text{SGA3}}(X,x)$.
- A comparison between étale cohomology of X and the cohomology of $\Pi^{\text{ét}}X$.
- A result stating that $\Pi^{\text{ét}}X$ is profinite (in a suitable sense), whenever X is geometrically unibranch.
- The Generalized Riemann Existence Theorem comparing, for an algebraic variety X over $\mathbf{C}$, the homotopy type of $X(\mathbf{C})$ with the étale homotopy type $\Pi^{\text{ét}}X$.

Definition 4.1.21. A scheme X is said to be *ét-locally connected*, if the small étale site of X is locally connected in the sense of Definition 4.1.15.

Remark 4.1.22. Locally Noetherian schemes are ét-locally connected ([AM86, Proposition 9.5]).

Definition 4.1.23. Let X be a ét-locally connected scheme. The *étale homotopy type of X* is the pro-homotopy type

$$\Pi^{\text{ét}}X := \Pi\, \mathbf{\acute{E}t}(X) \in \text{pro-ho}(\mathbf{ss})$$

obtained by applying the Verdier functor to the small étale site $\mathbf{\acute{E}t}(X)$ of X.

Finally, there is a pointed variant of the étale homotopy type:

Definition 4.1.24.

(a) A *pointed* scheme is a tuple (X,x) consisting of a scheme X and a *geometric* point x of X, that is: a morphism $x\colon \operatorname{Spec}(\Omega) \to X$ for some algebraically closed field Ω.

(b) Any geometric point x of X gives rise to a point of the small étale site $\mathbf{\acute{E}t}(X)$ of X. Thus in such a situation, the étale homotopy type is naturally a pointed pro-homotopy type

$$\Pi^{\text{ét}}X \in \text{pro-ho}(\mathbf{ss}_*).$$

Higher Étale Homotopy Groups

Applying the notions of Section 4.1.1 to the étale homotopy type gives a natural definition of *higher* étale homotopy groups:

Definition 4.1.25. Let X be a ét-locally connected scheme and x a geometric point of X. The *n-th étale homotopy group of X with basepoint x* is the n-th (pro-)homotopy group of $\Pi^{\text{ét}}X$ with respect to the basepoint induced by x:

$$\pi_n^{\text{ét}}(X,x) := \pi_n(\Pi^{\text{ét}}X, x) \in \text{Pro}(\mathbf{Grp})$$

For $n \geq 2$, $\pi_n^{\text{ét}}(X,x)$ is of course abelian.

Relationship with $\pi_1^{\mathrm{SGA3}}(X)$

When X is Noetherian, Artin and Mazur prove that the first homotopy group of $\Pi^{\text{ét}}X$ is what one might hope:

Theorem 4.1.26 ([AM86, Corollary 10.7]). If X is a Noetherian scheme, then $\pi_1(\Pi^{\text{ét}}X)$ is naturally isomorphic to the "pro-groupe fondamentale enlargi" $\pi_n^{\mathrm{SGA3}}(X)$ of [SGA III, Éxpose X.6].

Relationship with Étale Cohomology

Furthermore, they show that $\Pi^{\text{ét}}X$ refines étale cohomology:

Theorem 4.1.27 ([AM86, Corollary 10.8]). Let X be a ét-locally connected scheme. The category of locally constant abelian sheaves on $\textbf{Ét}(X)$ is equivalent to the category of local coefficient systems on $\Pi^{\text{ét}}X$ and there is a canonical isomorphism

$$\mathrm{H}^*_{\text{ét}}(X;\mathscr{F}) \cong \mathrm{H}^*(\Pi^{\text{ét}}X;\mathscr{F}),$$

where $\mathscr{F}$ denotes both, the locally constant sheaf and the corresponding local coefficient system.

Profiniteness of $\Pi^{\text{ét}}X$

In §11 of [AM86], it is shown that $\Pi^{\text{ét}}X$ is *profinite* for a large class of schemes X.

Definition 4.1.28. A scheme X is *geometrically unibranch* if every X' admitting an étale morphism $X' \to X$ is already irreducible, if it is connected.

Example 4.1.29 ([Stacks, Tag 0BQ3]). Every normal scheme is geometrically unibranch.

Theorem 4.1.30 ([AM86, Theorem 11.1]). Let X be a pointed, connected, geometrically unibranch, Noetherian scheme. Then the étale homotopy type $\Pi^{\text{ét}}X$ of X is *profinite*, i.e. all the pro-groups $\pi_*^{\text{ét}}(X)$ are *profinite groups*.

Remark 4.1.31. In the situation of Theorem 4.1.30, one in particular sees that $\pi_1^{\mathrm{SGA3}}(X)$ is profinite. Hence, since $\pi_1^{\text{ét}}(X) = (\pi_1^{\mathrm{SGA3}}(X))^\wedge$,

$$\pi_1(\Pi^{\text{ét}}X) = \pi_1^{\mathrm{SGA3}}(X) = \pi_1^{\text{ét}}(X),$$

is nothing but the ordinary étale fundamental group.

Riemann Existence Theorem

Theorem 4.1.32 (Generalized Riemann Existence Theorem, [AM86, Theorem 12.9]). Let X be a connected and pointed scheme of finite type over the field of complex numbers $\mathbf{C}$. Equip the set of $\mathbf{C}$-valued points $X(\mathbf{C})$ of X with the analytic topology.

Then the canonical map $\Pi(X(\mathbf{C})) \to \Pi^{\text{ét}}(X)$ induces an isomorphism upon profinite completion.

4.1.5 Étale $\mathbf{K}(\pi, 1)$-spaces

Definition 4.1.33. Let G be a profinite group. The category G - **fsets** has as objects finite sets X equipped with a left G-action $G \times X \to X$ that is continuous with respect to the product topology when X is endowed with the discrete topology. A morphism $X \to X'$ is a *G-equivariant* map of sets. Declaring a family of maps $\{\varphi_i : X_i \to X\}_i$ to be a covering if it is *jointly surjective* (i.e. if $X = \bigcup_i \varphi_i(X_i)$) endows G - **fsets** with the structure of a site.

Construction 4.1.34. Let X be an ét-locally connected scheme.

- Since $\mathbf{\acute{E}t}(X)$ is locally connected, so is $\mathbf{F\acute{E}t}(X)$ — the site of *finite* étale covers of X. Choosing a geometric point $x \in X(\Omega)$ of X induces an equivalence of categories

  $$\mathbf{F\acute{E}t}(X) \xrightarrow{\simeq} \pi\text{-}\mathbf{fsets}, Y \mapsto Y_x$$

  between the category of finite étale covers of X and the category of finite sets with a continuous action by $\pi := \pi_1^{\text{ét}}(X, x)$.
- Endowing $\mathbf{F\acute{E}t}(X)$ with the topology induced by $\mathbf{\acute{E}t}(X)$, the above equivalence of categories is compatible with the installed topologies.
- Hence $\Pi(\mathbf{F\acute{E}t}(X)) \to \Pi(\pi\text{-}\mathbf{fsets})$ is a weak equivalence.
- By [AM86, Example 9.11], the latter $\Pi(\pi\text{-}\mathbf{fsets})$ is a $\mathbf{K}(\pi, 1)$:

  $$\pi_*(\Pi(\pi\text{-}\mathbf{fsets})) = \begin{cases} \pi, & * = 1 \\ 0, & * > 1 \end{cases}$$

- The inclusion $\mathbf{F\acute{E}t}(X) \hookrightarrow \mathbf{\acute{E}t}(X)$ of categories defines a morphism of sites $\mathbf{\acute{E}t}(X) \to \mathbf{F\acute{E}t}(X)$. Applying the Verdier functor, we obtain a canonical morphism

  $$\Pi^{\text{ét}} X \to \mathbf{K}(\pi_1^{\text{ét}}(X, x), 1)$$

  of pro-homotopy types.

Definition 4.1.35. A ét-locally connected scheme X is an *étale* $\mathbf{K}(\pi, 1)$ if for one (and thus every) geometric point x of X the above constructed morphism

$$\Pi^{\text{ét}} X \to \mathbf{K}(\pi_1^{\text{ét}}(X, x), 1)$$

is a weak equivalence of pro-homotopy types.

Corollary 4.1.36. Let X be a connected and ét-locally connected scheme. The following are equivalent:

(i) X is an étale $\mathbf{K}(\pi, 1)$.
(ii) $\pi_n^{\text{ét}}(X, x) = 0$ for all $n \geq 2$ and all $x \in X(\bar{k})$.

Proof. Since we already have a map

$$\Pi^{\text{ét}} X \to \mathbf{K}(\pi_1^{\text{ét}}(X, x), 1)$$

we can detect weak equivalences on homotopy groups — Theorem 4.1.7. □

4.2 Rigid Čech Types over a Field

In this section, let k be a field, $\bar{k}$ a fixed algebraic closure of k, let X, Y be schemes of finite type over k and $f : Y \to X$ a morphism over k.

Definition 4.2.1.

(a) A *rigid covering* $\alpha \colon U \to X$ of X over k is a disjoint union of pointed, étale and separated maps

$$\coprod_{x \in X(\bar{k})} (\alpha_x \colon (U_x, u_x) \to (X, x)),$$

where each U_x is connected and where $u_x \in U_x(\bar{k})$ is such that $\alpha_x(u_x) = x$.

(b) A *morphism of rigid coverings* $\phi : (\beta \colon V \to Y) \to (\alpha \colon U \to X)$ *over* f is a morphism of schemes $\phi \colon V \to U$ over f such that $\phi(v_x) = u_x$ for all $x \in X(\bar{k})$.

(c) The *category of rigid coverings of* X *over* k, denoted by $\mathbf{RC}(X/k)$, is the category with objects given by rigid coverings of X over k and morphisms given by morphisms of rigid coverings over $\mathbf{1}_X$.

Definition 4.2.2.

(a) If $\alpha \colon U \to X$ and $\beta \colon V \to Y$ are rigid coverings of X and Y over k respectively, then their *rigid product*

$$\alpha \overset{\mathrm{R}}{\underset{k}{\times}} \beta \colon U \overset{\mathrm{R}}{\underset{k}{\times}} V \to X \times_k Y$$

is given as the disjoint union

$$\coprod_{(x,y) \in (X \times_k Y)(\bar{k})} ((\alpha_x \times_k \beta_y)\big| \colon (U_x \times_k V_y)_0 \to X \times_k Y),$$

where $(U_x \times_k V_y)_0$ denotes the connected component of $U_x \times_k V_y$ containing the geometric point (u_x, v_y).

In other words, $\alpha \overset{\mathrm{R}}{\underset{k}{\times}} \beta$ is given as the restriction of $\alpha \times_k \beta$ to the open and closed subscheme $\coprod_{(x,y) \in (X \times_k Y)(\bar{k})} (U_x \times_k V_y)_0$.

(b) If $\alpha \colon U \to X$ is a rigid covering of X, then the *rigid pullback* $f^*(\alpha)$ of α along f is given as the disjoint union of pointed maps

$$\coprod_{y \in Y(\bar{k})} ((U_{f(y)} \times_X Y)_y \to Y),$$

where $(U_{f(y)} \times_X Y)_y$ is the connected component of $U_{f(y)} \times_X Y$ containing the geometric point $(f(y), y)$.

Definition 4.2.3. Let $U \to X$ be a rigid covering of X over k. The *Čech nerve* of $U \to X$ is the simplicial scheme

$$\check{\mathrm{N}}_X(U) := \operatorname{cosk}_0^X(U),$$

given concretely in degree n by

$$\check{\mathrm{N}}_X(U)_n = \underbrace{U \times_X U \times_X \ldots \times_X U}_{n+1 \text{ times}}.$$

Since a map of connected, separated and étale schemes over X is uniquely determined by the image of a geometric point, there is at most one morphism of rigid coverings between any two coverings in $\mathbf{RC}(X/k)$. It follows that $\mathbf{RC}(X/k)$ is cofiltered.

Definition 4.2.4.

(a) The *rigid Čech étale topological type* of X over k is the pro-space

$$(X/k)_{\text{rét}} \colon \mathbf{RC}(X/k) \to \mathbf{ss}$$

given by $(U \to X) \mapsto \pi_0(\check{\mathrm{N}}_X(U))$.

(b) A map $f : Y \to X$ over k induces a strict morphism of pro-spaces

$$f_{\text{rét}} \colon (Y/k)_{\text{rét}} \to (X/k)_{\text{rét}}$$

by means of the rigid pullback functor $f^* \colon \mathbf{RC}(X/k) \to \mathbf{RC}(Y/k)$.
Concretely, the projection $U \times_X Y \to U$ induces a map

$$\pi_0(\check{\mathrm{N}}_Y(f^*(U \to X))) \to \pi_0(\check{\mathrm{N}}_X(U))$$

which assembles into a natural transformation $(Y/k)_{\text{rét}} \circ f^* \to (X/k)_{\text{rét}}$.

(c) The assignment

$$X \mapsto (X/k)_{\text{rét}}$$

gives a functor $(-)_{\text{rét}} \colon \mathbf{Sch}_k^{\text{ft}} \to \text{pro-}\mathbf{ss}$ from the category of schemes of finite type over k to the category of pro-spaces.

The next proposition shows that for quasi-projective varieties X over k, $(X/k)_{\text{rét}}$ has the correct homotopy type:

Proposition 4.2.5. If X is a quasi-projective scheme over a Noetherian ring, then there is a canonical weak equivalence

$$\Pi^{\text{ét}} X \xrightarrow{\simeq} (X/k)_{\text{rét}}$$

of pro-homotopy types.

Proof. In view of the remark on page 102 in [Fri82] explaining how to simplify the definition of the rigid étale topological type when working with schemes of finite type over a field, this is [Fri82, Proposition 8.2]. $\square$

Lemma 4.2.6. Let L/k be a finite Galois extension of k in $\bar{k}$.
Then $\check{\mathrm{N}}_k(L)_n = \bigsqcup_{i=1}^{[L:k]^n} \mathrm{Spec}(L)$.

Proof. We have

$$\begin{aligned}
\check{\mathrm{N}}_k(L)_n &= \bigtimes_k^{n+1} \mathrm{Spec}(L) \\
&= \underbrace{\mathrm{Spec}(L) \times_k \mathrm{Spec}(L) \times_k \ldots \times_k \mathrm{Spec}(L)}_{n+1 \text{ times}} \qquad (4.1)\\
&= \mathrm{Spec}(L \otimes_k L \otimes_k \ldots \otimes_k L).
\end{aligned}$$

Since L/k is finite separable, we find a primitive element $x \in L$ with minimal polynomial $f \in k[X]$ such that $L = k[X]/(f)$. Hence

$$\begin{aligned}
\underbrace{L \otimes_k L \otimes_k \ldots \otimes_k L}_{n+1 \text{ times}} &= (k[X]/(f) \otimes_k L) \otimes_k \ldots \otimes_k L \\
&= (L[X]/(f)) \otimes_k L \otimes_k \ldots \otimes_k L \\
&= (\underbrace{L \times L \times \ldots \times L}_{[L:k] \text{ times}}) \otimes_k \underbrace{L \otimes_k \ldots \otimes_k L}_{n-1 \text{ times}} \\
&= (\underbrace{L \otimes_k L \times L \otimes_k L \times \ldots \times L \otimes_k L}_{[L:k] \text{ times}}) \otimes_k \underbrace{L \otimes_k \ldots \otimes_k L}_{n-2 \text{ times}} \\
&= (\underbrace{L \times L \times \ldots \times L}_{[L:k]^2 \text{ times}}) \otimes_k \underbrace{L \otimes_k \ldots \otimes_k L}_{n-2 \text{ times}} \\
&\ \vdots \\
&= \underbrace{L \times L \times \ldots \times L}_{[L:k]^n \text{ times}}.
\end{aligned}$$

□

Corollary 4.2.7. Let L/k be a finite Galois extension of k in $\bar{k}$.
Then $\pi_0(\check{\mathrm{N}}_k(L)) \cong \mathbf{B}\,\mathrm{Gal}(L/k)$.

Proof. Indeed, using that $\check{\mathrm{N}}_k(L)_n = \bigsqcup_{i=1}^{[L:k]^n} \mathrm{Spec}(L)$ according to Lemma 4.2.6, we have

$$\begin{aligned}
\pi_0(\check{\mathrm{N}}_k(L)_n) &= \bigsqcup_{i=1,\ldots,[L:k]^n} \{*\} \\
&= \bigtimes_{i=1}^{n} \bigsqcup_{\sigma \in \mathrm{Gal}(L/k)} \{*\} \\
&\cong \bigtimes_{i=1}^{n} \mathrm{Gal}(L/k) \\
&= \underbrace{\mathrm{Gal}(L/k) \times \mathrm{Gal}(L/k) \times \ldots \times \mathrm{Gal}(L/k)}_{n \text{ times}}.
\end{aligned}$$

□

Corollary 4.2.8. Let k be a field with absolute Galois group Gal_k. The rigid Čech étale type of k is canonically isomorphic to the classifying pro-space $\mathbf{B}\,\mathrm{Gal}_k$ of Gal_k as a pro-space:

$$(\mathrm{Spec}(k)/k)_{\text{rét}} \cong \mathbf{B}\,\mathrm{Gal}_k \in \text{pro-}\,\mathbf{ss}$$

Proof. Since the rigid covers of $\mathrm{Spec}(k)$ over k given by a finite Galois extension L/k in a fixed algebraic closure $\bar{k}$ are cofinal among all rigid covers of $\mathrm{Spec}(k)$, we conclude by Corollary 4.2.7. □

Example 4.2.9. Let X be a geometrically connected variety over k. The structure morphism $p\colon X \to \mathrm{Spec}(k)$ induces a morphism of pro-spaces

$$p_{\text{rét}}\colon (X/k)_{\text{rét}} \to (\mathrm{Spec}(k)/k)_{\text{rét}} = \mathbf{B}\,\mathrm{Gal}_k\,.$$

The pullback $p^*(\mathrm{Spec}(L) \to \mathrm{Spec}(k))$ of a rigid cover $\mathrm{Spec}(L) \to \mathrm{Spec}(k)$ induced by a finite Galois extension L/k in $\bar{k}$ is given by

$$U_L := \coprod_{x \in X(\bar{k})} ((X \otimes_k L, x_L) \to (X, x),$$

where $x_L \in (X \otimes_k L)(\bar{k})$ denotes the canonical lift of x along $\mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(L)$. The canonical isomorphism $(X \otimes_k L) \times_X (X \otimes_k L) \cong X \otimes_k (L \otimes_k L)$ induces a functorial map of simplicial sets

$$\pi_0(\check{\mathrm{N}}_X(U_L)) \to \pi_0(\check{\mathrm{N}}_k(L)) \cong \mathbf{B}\,\mathrm{Gal}(L/k)$$

which determines $p_{\text{rét}} \in \lim_{L/k} \mathrm{Hom}_{\mathbf{ss}}(\pi_0(\check{\mathrm{N}}_X(U_L)), \mathbf{B}\,\mathrm{Gal}(L/k))$.

Example 4.2.10. Let X again be a geometrically connected variety over k. Every element $\sigma \in \mathrm{Gal}_k$ defines a morphism

$$\sigma\colon \bar{X} = X \otimes_k \bar{k} \to X \otimes_k \bar{k} = \bar{X}.$$

The map

$$\sigma_{\text{rét}}\colon (\bar{X}/k)_{\text{rét}} \to (\bar{X}/k)_{\text{rét}}$$

induced on rigid Čech étale types sends a rigid cover

$$\coprod_{\bar{x} \in \bar{X}(\bar{k})} (U_{\bar{x}}, u_{\bar{x}}) \to (\bar{X}, \bar{x})$$

to the rigid cover

$$\coprod_{\bar{x} \in \bar{X}(\bar{k})} (U_{\sigma(\bar{x})} \times_{\bar{X}} \bar{X})_{\bar{x}} \to \bar{X},$$

where $U_{\sigma(\bar{x})} \times_{\bar{X}} \bar{X}$ denotes the pullback of $U_{\sigma(\bar{x})} \to \bar{X}$ along $\bar{X} \xrightarrow{\sigma} \bar{X}$ and where $(U_{\sigma(\bar{x})} \times_{\bar{X}} \bar{X})_{\bar{x}}$ denotes the connected component of $U_{\sigma(\bar{x})} \times_{\bar{X}} \bar{X}$ containing $\bar{x}$. Hence $\sigma_{\text{rét}}$ is given explicitly on 0-simplices by sending the connected component $(U_{\sigma(x)} \times_{\bar{X}} \bar{X})_x$ to the component $U_{\sigma(x)}$. If we identify the pro-set of 0-simplices of $(\bar{X}/k)_{\text{rét}}$ with $X(\bar{k})$, then $\sigma_{\text{rét}}$ is thus given by the natural action of σ on $X(\bar{k})$. In particular: A 0-simplex of $(\bar{X}/k)_{\text{rét}}$ is fixed under the action of all $\sigma \in \mathrm{Gal}_k$ if and only if its a rational point of X.

4.3 Profinite Models for Étale Types

Setup.

- Let k be a field, $\bar{k}$ a fixed algebraic closure of k and Gal_k the absolute Galois group of k with respect to $\bar{k}$.
- Let X be a geometrically connected, geometrically unibranch and quasi-projective variety over k.
- We denote the rigid Čech étale topological type $(X/k)_{\text{rét}}$ by $\mathscr{X}$ and write $\mathscr{I}$ for the indexing category $\mathbf{RC}(X/k)$.

Observation. Using the isomorphism $(\mathrm{Spec}(k)/k)_{\text{rét}} \cong \mathbf{B}\,\mathrm{Gal}_k$ of Corollary 4.2.8, we can view $\mathscr{X}$ as a prospace over $\mathbf{B}\,\mathrm{Gal}_k$. Since X is assumed to be quasi-projective, Proposition 4.2.5 applies. Hence, since according to Theorem 4.1.30 the étale homotopy type of a geometrically unibranch and Noetherian scheme is profinite, each $\mathscr{X}(i)$ is a π-finite space. Furthermore, by Example 4.2.10, each $\mathscr{X}(i)$ is equipped with a morphism to $\mathbf{B}\Gamma$ for some finite quotient Γ of Gal_k. We are thus in a situation where the theory of *profinite models* of Section 3.6 applies, so that we obtain a profinite space $\mathscr{X}_{\text{pf}}$ over $\mathbf{B}\,\mathrm{Gal}_k$.

By Proposition 4.2.5 and Theorem 3.3.1, we obtain the following:

Corollary 4.3.1. The fibrant profinite space $\mathscr{X}_{\text{pf}}$ over $\mathbf{B}\,\mathrm{Gal}_k$ has the same homotopy type as the étale homotopy type:

The levelwise map of underlying pro-spaces $(\mathscr{X}(i))_{i\in\mathscr{I}} \to (F\mathscr{X}(i))_{i\in\mathscr{I}}$ induces isomorphisms

$$\pi_n^{\text{ét}}(X) \xrightarrow{\sim} \hat{\pi}_n(\mathscr{X}_{\text{pf}})$$

of profinite groups for all $n \geq 1$.

5 Diophantine Applications

Summary. In this chapter we apply the machinery of homotopy-fixed point spaces developed in Chapter 3 to the étale topological type of Chapter 4. We obtain ...

(i) ... a notion of *étale homotopy-fixed point space* for normal, quasi-projective varieties.

(ii) ... a criterion for the existence of rational points on those varieties in terms of a comparison of fixed points and homotopy-fixed points (Section 5.2).

(iii) ... a reformulation of Grothendieck's Section Conjecture for (again normal, quasi-projective) étale $\mathbf{K}(\pi, 1)$-varieties.

In fact, in all of the above it suffices to require the varieties under consideration to be *geometrically unibranch* instead of normal.

Setup. Except for the reformulation of the Section Conjecture, where we'll need stronger hypotheses, we fix the following situation:

- Let k be a field, $\bar{k}$ a fixed algebraic closure of k and $G := \mathrm{Gal}_k$ the absolute Galois group of k.
- Let X be a geometrically connected, geometrically unibranch and quasi-projective scheme of finite type over k.
- Let $\bar{X} := X \otimes_k \bar{k}$ denote the base change of X to $\bar{k}$ and denote the canonical projection $\bar{X} \to X$ by p.
- Let $\mathscr{X} := (X/k)_{\text{rét}}$ and $\bar{\mathscr{X}} := (\bar{X}/k)_{\text{rét}}$ denote the rigid Čech étale types of X and $\bar{X}$ over k respectively, and let p also denote the morphism $\bar{\mathscr{X}} \to \mathscr{X}$ induced by $p : \bar{X} \to X$.

5.1 Étale Homotopy-Fixed Point Spaces

As a first application, we obtain a notion of *étale homotopy-fixed point spaces* of varieties over k:

Observation. The pro-space $\bar{\mathscr{X}}$ is equipped with a natural action of G of the form discussed in Section 3.7. Let $x : \operatorname{Spec}(\bar{k}) \to X$ be any geometric point of X. It makes $\mathscr{X}$ into a pointed pro-space. We can thus form the profinite model $\mathscr{X}_{\text{pf}}$ over $\mathbf{B}G$ of $\mathscr{X}$ as discussed in Section 4.3.

Definition 5.1.1. The *étale homotopy-fixed point space of $\bar{X}$ over k* is the homotopy-fixed point space of the profinite model $\mathscr{X}_{\text{pf}}$ of $\mathscr{X}$:

$$\bar{\mathscr{X}}^{\mathrm{h}G} := \mathscr{X}_{\text{pf}}^{\mathrm{h}G} = \mathbf{R}\mathrm{Map}_{\mathbf{B}G}(\mathbf{B}G, \mathscr{X}_{\text{pf}}) \in \mathbf{ss}$$

5.2 Existence of Rational Points as a Homotopy-Fixed Point Problem

Section 3.7 provides a map

$$\eta\colon \bar{\mathscr{X}}^{G} \to \bar{\mathscr{X}}^{\mathrm{h}G}$$

of spaces comparing the fixed point space of $\bar{X}$ with the étale homotopy-fixed point space of $\bar{X}$.

Observation. We now relate $\pi_0\eta$ with the set of rational points $X(k)$:

(i) By means of the functoriality of $(-/k)_{\text{rét}}$, any *rational* point $x\colon \operatorname{Spec}(k) \to X$ gives rise to a section

$$\mathbf{B}G = (\operatorname{Spec}(k)/k)_{\text{rét}} \to \mathscr{X}$$

of the map $\mathscr{X} \to \mathbf{B}G$ induced by the structure morphism $X \to \operatorname{Spec}(k)$.

(ii) Taking profinite models, we hence obtain a well-defined map of sets

$$X(k) \to \operatorname{Hom}_{\mathrm{ho}(\hat{\mathbf{s}}\mathbf{s}\downarrow\mathbf{B}G)}(\mathbf{B}G, \mathscr{X}_{\mathrm{pf}}) \cong \pi_0(\bar{\mathscr{X}}^{\mathrm{h}G})$$

given explicitly by

$$X(k) \ni a \mapsto [(a_{\text{rét}})_{\mathrm{pf}}] \in \pi_0\bar{\mathscr{X}}^{\mathrm{h}G}.$$

(iii) Recall that the 0-simplices of $\bar{\mathscr{X}}$ can be identified with the constant pro-set $X(\bar{k})$ and that the action of G agrees with the usual action of G on $X(\bar{k})$. Hence $(\bar{\mathscr{X}}^{G})_0 = X(k)$ so that $\pi_0\bar{\mathscr{X}}^{G}$ can be identified with a quotient of $X(k)$. We thus have a natural *surjective* map of sets

$$X(k) \twoheadrightarrow \pi_0\bar{\mathscr{X}}^{G}, \quad a \mapsto [a].$$

(iv) The resulting diagram

$$\begin{tikzcd} X(k) \arrow[r] \arrow[d, two heads] & \pi_0\bar{\mathscr{X}}^{\mathrm{h}G} \\ \pi_0\bar{\mathscr{X}}^{G} \arrow[ur, "\pi_0\eta"'] & \end{tikzcd}$$

commutes. Indeed, recalling the construction of η in Construction 3.7.2, we compute the “zig-zag” of the above diagram:

(a) First, $a \in X(k)$ is sent to $[* \to \mathbf{B}G \xrightarrow{(a\otimes_k\bar{k})_{\text{rét}}} \bar{\mathscr{X}}] \in \pi_0\bar{\mathscr{X}}^{G} = \pi_0\mathrm{Map}_{\text{pro-}\mathbf{ss}\downarrow\mathbf{B}G}(*, \bar{\mathscr{X}})$.

(b) Composing with $\mathbf{E}G \to *$ we obtain the G-equivariant map

$$[\mathbf{E}G \to * \to \mathbf{B}G \xrightarrow{(a\otimes_k\bar{k})_{\text{rét}}} \bar{\mathscr{X}}] \in \pi_0\mathrm{Map}_{\text{pro-}\mathbf{ss}\downarrow\mathbf{B}G}(\mathbf{E}G, \bar{\mathscr{X}})^{G}.$$

To simplify notation, we will denote the above G-equivariant map $\mathbf{E}G \to \bar{\mathscr{X}}$ by $(a \otimes_k \bar{k})^{\mathbf{E}G}_{\text{rét}}$.

(c) Taking the product with $\mathbf{E}G$ and quotiening out the G-action gives

$$[\mathbf{B}G \xrightarrow{\mathbf{1}_{\mathbf{E}G}\times_G(a\otimes_k\bar{k})^{\mathbf{E}G}_{\text{rét}}} \mathbf{E}G \times_G \bar{\mathscr{X}}] \in \pi_0\mathrm{Map}_{\text{pro-}\mathbf{ss}\downarrow\mathbf{B}G}(\mathbf{B}G, \mathbf{E}G \times_G \bar{\mathscr{X}}).$$

(d) Postcomposing with $\mathbf{E}G \times_G p_{\text{rét}}$ and the projection $\mathbf{E}G \times_G \mathscr{X} = \mathbf{B}G \times \mathscr{X} \to \mathscr{X}$ results in

$$[\mathbf{B}G \xrightarrow{\mathbf{1}_{\mathbf{E}G} \times_G (a \otimes_k \bar{k})^{\mathbf{E}G}_{\text{rét}}} \mathbf{E}G \times_G \bar{\mathscr{X}} \xrightarrow{\mathbf{E}G \times_G p_{\text{rét}}} \mathbf{E}G \times_G \mathscr{X} \to \mathscr{X}] \in \pi_0 \mathrm{Map}_{\text{pro-}\mathbf{ss}\downarrow \mathbf{B}G}(\mathbf{B}G, \mathscr{X}).$$

But since we project to the second factor, this composition just equals

$$[\mathbf{B}G \xrightarrow{(p \circ (a \otimes_k \bar{k}))_{\text{rét}}} \mathscr{X}],$$

which, using that $p(a \otimes_k \bar{k}) = a$, is nothing but

$$[\mathbf{B}G \xrightarrow{a_{\text{rét}}} \mathscr{X}].$$

(e) Finally, we take profinite models and obtain

$$[(a_{\text{rét}})_{\text{pf}}] \in \pi_0 \mathrm{Map}_{\hat{\mathbf{s}}\mathbf{s}\downarrow \mathbf{B}G}(\mathbf{B}G, \mathscr{X}_{\text{pf}}) = \pi_0 \bar{\mathscr{X}}^{\text{h}G}.$$

Hence $\pi_0 \eta([a]) = [(a_{\text{rét}})_{\text{pf}}]$ as claimed.

All in all, since $X(k) \twoheadrightarrow \pi_0 \bar{\mathscr{X}}^G$ is surjective, we have proven the following theorem:

Theorem 5.2.1. The map $\pi_0 \eta \colon \pi_0 \bar{\mathscr{X}}^G \to \pi_0 \bar{\mathscr{X}}^{\text{h}G}$ induced by η *detects rational points*, that is:

Every homotopy-fixed point $a^{\text{h}G} \in \bar{\mathscr{X}}_0^{\text{h}G}$ lying in the same connected component of $\bar{\mathscr{X}}^{\text{h}G}$ as a homotopy-fixed point of the form $\eta_0(a^G)$ for some actual fixed point $a^G \in \bar{\mathscr{X}}_0^G$, witnesses the existence of a rational point $a \in X(k)$ on X lying in the connected component of a^G.

5.3 Reformulating Grothendieck's Section Conjecture

Finally, we obtain the desired reformultation of Grothendieck's Section Conjecture:

Setup. In addition on the requirements on X declared right at the beginning of Chapter 5, here we furthermore assume X to be an *étale* $\mathbf{K}(\pi,1)$*-variety* (Definition 4.1.35). Additionally, we require the ground field k to have a *centerless* absolute Galois group $G = \mathrm{Gal}_k$.

Theorem 5.3.1. An étale $\mathbf{K}(\pi,1)$-variety as above satisfies the surjectivity portion of Grothendieck's Section Conjecture if and only if the map

$$\pi_0\eta\colon \pi_0\bar{\mathscr{X}}^G \longrightarrow \pi_0\bar{\mathscr{X}}^{\mathrm{h}G}$$

is surjective.

Proof. Choose a geometric point $x \in X(\bar{k})$. Since X is assumed to be an étale $\mathbf{K}(\pi,1)$, $(X/k)_{\mathrm{rét}}$ is weakly equivalent to $\mathbf{B}\pi$ for $\pi = \pi_1^{\text{ét}}(X,x)$. Hence also

$$\bar{\mathscr{X}}^{\mathrm{h}G} = \mathbf{R}\mathrm{Map}_{\hat{\mathbf{ss}}\downarrow \mathbf{B}G}(\mathbf{B}G, \bar{\mathscr{X}}_{\mathrm{pf}}) \xrightarrow{\simeq} \mathbf{R}\mathrm{Map}_{\hat{\mathbf{ss}}\downarrow \mathbf{B}G}(\mathbf{B}G, \mathbf{B}\pi) = \mathbf{B}\pi^{\mathrm{h}G} \in \mathbf{ss}.$$

Postcomposing with the bijection from Corollary 3.5.2, we obtain a bijection of sets

$$\pi_0(\bar{\mathscr{X}}^{\mathrm{h}G}) \xrightarrow{\sim} \pi_0(\mathbf{B}\pi^{\mathrm{h}G}) \xrightarrow{\sim} \mathrm{S}(\pi)$$

along which we identify those two sets.

By our results of Section 5.2, we have a commutative triangle

$$\begin{tikzcd} X(k) \arrow[r, "\kappa"] \arrow[d, two heads] & \mathrm{S}(\pi) \\ \pi_0\bar{\mathscr{X}}^G, \arrow[ur, "\pi_0\eta"'] & \end{tikzcd}$$

where $\kappa\colon X(k) \to \mathrm{S}(\pi)$ denotes the profinite Kummer map of anabelian geometry.

Hence κ is surjective if and only if $\pi_0\eta$ is. $\square$

In particular, since absolute Galois groups of number fields and p-adic local fields are *slim* ([Moc12, Theorem 1.7]), hence centerless, and since smooth projective curves of genus $g \geq 2$ over such fields are étale $\mathbf{K}(\pi,1)$-varieties ([Sch96, Proposition 15]) and satisfy the injectivity portion of Grothendieck's Section Conjecture ([Sti13, Corollary 74]), we obtain the following Corollary:

Corollary 5.3.2. A smooth projective curve X of genus $g \geq 2$ over a number field or p-adic local field k satisfies Grothendiecks's Section Conjecture if and only if the map

$$\pi_0\eta\colon \pi_0\bar{\mathscr{X}}^G \longrightarrow \pi_0\bar{\mathscr{X}}^{\mathrm{h}G}$$

is surjective.

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