galrep
16.05.18

Representability of $D_{\bar\rho}$ when $C_k(\bar\rho) = k$. Need to check H1 – H4 in Schlessinger's criterion.

Objects $R_0, R_1, R_2 \in \hat{\mathcal{C}}_\Lambda$ with morphisms $R_1 \longrightarrow R_0$, $R_2 \longrightarrow R_0$; set $R_3 := R_1 \times_{R_0} R_2$

$$b : D_{\bar\rho}(R_3) \longrightarrow D_{\bar\rho}(R_1) \times_{D_{\bar\rho}(R_0)} D_{\bar\rho}(R_2)$$

$E_i :=$ set of all deformations $\rho : G \to GL_n(R_i)$ of $\bar\rho$.

$$D_{\bar\rho}(R_i) = E_i /_{\text{strict equiv.}} = E_i / \Gamma_n(R_i) \qquad \left(\Gamma_n(R_i) := \ker(GL_n(R) \to GL_n(k)) \right)$$

Then write

$$b : E_3/\Gamma_n(R_3) \longrightarrow E_1/\Gamma_n(R_1) \times_{E_0/\Gamma_n(R_0)} E_2/\Gamma_n(R_2)$$

<u>H1</u>: if $R_2 \xrightarrow[\text{small}]{} R_0 \Rightarrow b$ surjective

Proof. Take ρ_1, ρ_2 representatives of classes in $E_1/\Gamma_n(R_1)$, $E_2/\Gamma_n(R_2)$. Assume that the images of ρ_1 and ρ_2 via $R_1 \to R_0$, $R_2 \to R_0$ are in the same class.

$\rho_{10} : G \to GL_n(R_0)$
$\rho_{20} : G \to GL_n(R_0)$

Then $\exists\, M_0 \in \Gamma_n(R_0)$ s.t. $M_0^{-1} \rho_{20} M_0 = \rho_{10}$.

Since $R_2 \to R_0$ is surjective, $\Gamma_n(R_2) \longrightarrow \Gamma_n(R_0)$ is surjective. So we can choose a lift M_2 of M_0 in $\Gamma_n(R_2)$. Then the representations $M_2^{-1}\rho_2 M_2$ and ρ_1 give the same representations in $GL_n(R_0)$. They glue to a representation $\rho_3 : G \to GL_n(R_3)$

$$(\rho_3(\cdot) = (\rho_1(\cdot), M_2^{-1}\rho_2(\cdot)M_2) \in GL_n(R_1 \times_{R_0} R_2)$$

$\blacksquare$

<u>H2</u>, <u>H4</u>: similar strategies (see Gouvea)

<u>H3</u>: $D_{\bar\rho}(\underbrace{k[\varepsilon]}_{=k[X]/X^2})$ is a finite-dimensional k-vector space.

Proof. $G_0 = \ker\bar\rho$ $(\rightsquigarrow (G:G_0) < \infty)$; $R \in \hat{\mathcal{C}}_\Lambda$ and $\rho : G \to GL_n(R)$ a lift of $\bar\rho$.

Then $\rho|_{G_0}$ takes values in $\Gamma_n(R)$. This is a pro-p group.

Take $R = k[\varepsilon]$. Then $\Gamma_n(k[\varepsilon]) = 1 + \underbrace{\varepsilon M_n(k)}_{\text{p-elementary finite group}}$

A lift $\rho : G \to GL_n(k[\varepsilon])$ of $\bar\rho$ defines a homomorphism

$$G_0 \to \Gamma_n(k[\varepsilon]), \qquad (\rho|_{G_0} = \rho'|_{G_0} \Leftrightarrow \rho \sim \rho' \text{ strictly equiv.})$$

but $\Gamma_n(k[\varepsilon])$ is a finite sum of $\mathbb{Z}/p\mathbb{Z}$'s ($:\Leftrightarrow$ p-elementary finite)

By p-finiteness $\mathrm{Hom}_{cont}(G_0, \Gamma_n(k[\varepsilon]))$ is finite.

By Schlessinger, $\exists$ universal deformation couple $(R^{univ}_{\bar\rho}, \rho^{univ})$ for $\bar\rho$.

Proof of Schlessinger's criterion gives a surjection $\Lambda[[T_1, \dots, T_d]] \to R^{univ}_{\bar\rho}$

By looking at tangent spaces, one sets, for the minimal d: $d = \dim_k t_{R^{univ}_{\bar\rho}}$

$$= \dim_k D_{\bar\rho}(k[\varepsilon])$$

* Can we give d in terms of $\bar\rho$?
* $R^{univ}_{\bar\rho} = \Lambda[[T_1, \dots, T_d]]/I$, we will describe $\dim R^{univ}_{\bar\rho}$ in terms of $\bar\rho$.

Explicit case: $n = 1$.

Fix $\bar\rho : G \to k^\times = GL_1(k)$. We describe the universal deformation couple for $\bar\rho$. There is a lift $\rho_0 : G \to W(k)^\times$ given by $(W(\mathbb{F}_{p^n}) = \mathbb{Z}_{p^n})$ the Teichmüller lift $k^\times \to W(k)^\times$.

Take another lift $\rho : G \to A^\times$ for $A \in \hat{\mathcal{C}}_\Lambda$. (one has a map $W(k)^\times \to A^\times$)

Then $\rho_0^{-1} \cdot \rho : G \to A^\times$ takes values in $1 + m_A$.

$\rho_0^{-1}(g) = \rho_0(g)^{-1}$. $\cdot$ not $\circ$

$1 + m_A$ is a pro-p-group, so $\rho_0^{-1} \circ \rho$ factors through the pro-p-completion Γ of G.

$$\left(\Gamma := \varprojlim_{H \text{ quotient of } G \text{ that is finite p-Group}} H\right)$$

$$\begin{tikzcd} G \arrow[r, "\rho_0^{-1}\rho"] \arrow[d, "\gamma"'] & 1+m_A \\ \Gamma \arrow[ur, "\exists f_\rho"'] & \end{tikzcd}$$ continuous group hom.

By continuity f_ρ can be extended to

$$f_\rho : \Lambda[\![\Gamma]\!] \longrightarrow A \quad , \Lambda\text{-alg. hom.}$$

$$\Lambda[\![\Gamma]\!] := \varprojlim_{\Gamma' \text{ open normal subgrp. of } \Gamma} \Lambda[\Gamma/\Gamma']$$ ($\Lambda[\Gamma/\Gamma']$: group-algebra)

Remark. * $\Lambda[\![\Gamma]\!] \in \hat{\mathcal{C}}_\Lambda$.

One checks that

G satisfies p-finiteness $\Rightarrow$ Γ is top. f.g. by $x_1, \ldots, x_k$ $\Rightarrow$ $\Lambda[\![\Gamma]\!]$ is a quotient of $\Lambda[\![x_1, \ldots, x_n]\!]$

Proposition. The universal deformation couple for $\bar{\rho}$ is $\left(\Lambda[\![\Gamma]\!], \rho^{univ}\right)$, where $\rho^{univ} = \rho_0 \cdot [\gamma]$

where $[\gamma]$ is the composition $G \to \Gamma \to \Lambda[\![\Gamma]\!]^\times$

Proof. We already defined a Λ-algebra morphism $f_\rho : \Lambda[\![\Gamma]\!] \to A$ (for any deformation $\rho : G \to A^\times$)

We check: for $g \in G$

$$f_\rho \circ \rho^{univ}(g) = f_\rho(\rho_0(g)[\gamma(g)]) = \rho_0(g)\, f_\rho([\gamma(g)]) = \underbrace{\rho_0(g)\rho_0^{-1}(g)}_{=1}\rho(g) = \rho(g).$$

Check that f_ρ is unique.

$\blacksquare$

Remark. * $R_{\bar\rho}^{univ}$ is independent of $\bar\rho$.

Take any $\bar\rho : G \to GL_n(k)$ (with $C_k(\bar\rho) = k$). Then there is a universal def. couple $(R_{\bar\rho}^{univ}, \rho^{univ})$. Look at $\det \circ \rho^{univ} : G \to R_{\bar\rho}^{univ,\times}$. It is a deformation of $\det \circ \bar\rho$. $\Rightarrow$ There exists a Λ-alg. morphism $\Lambda[\![\Gamma]\!] \longrightarrow R_{\bar\rho}^{univ}$ inducing $\det \circ \rho^{univ}$

It is a special case of "functoriality".

* Take any algebraic morphism $GL_m \xrightarrow{\delta} GL_n$. (of algebraic groups)

Take $\bar\rho_m : G \to GL_m(k)$, then we get $\bar\rho_n : G \to GL_n(k) := \delta_k \circ \bar\rho_m$.

Assume $\bar\rho_m, \bar\rho_n$ are absolutely irreducible. Then $\delta_{R^{univ}_{\bar\rho_n}} \circ \rho_n^{univ}$ is a deformation of $\bar\rho_m$. By the universal property we get $R^{univ}_{\bar\rho_m} \xrightarrow{r(\delta)} R^{univ}_{\bar\rho_n}$ s.t.

$$r(\delta) \circ \rho_m^{univ} = \rho_n^{univ}$$

* $m = n$, $GL_{n/\Lambda} \xrightarrow{\delta} GL_{n/\Lambda}$ given by conjugation with fixed $g \in GL_n(\Lambda)$.

Take $\bar\rho_m : G \to GL_m(k)$, then $\exists$ a morphism $R_{\bar\rho_m \otimes \bar\rho_n} \to R_{\bar\rho_m} \hat{\otimes}_\Lambda R_{\bar\rho_n}$

$\bar\rho_n : G \to GL_n(k)$

completed tensor product

__Dimension of the deformation rings__

$$\Lambda[[T_1, \dots, T_d]]/I \cong R^{univ}_{\bar\rho}$$

__Reminder__: Group cohomology

* G pro-finite group, M topological abelian group with a continuous action of G.
  $(G, \cdot)$ $(M, +)$

Can define a functor $\underset{\in \underline{Ab}}{\underline{M}} \mapsto M^G = \{m \in M \mid gm = m \ \forall g \in G\}$

$H^0(G,M) = M^G$, $H^i(G,M) \rightsquigarrow$ Right-derived functor of $M \mapsto M^G$.

We will compute cohomology with the following complex.

$$M \to K^1 \xrightarrow{d^1} K^2 \xrightarrow{d^2} \dots \to K^i \xrightarrow{d^i} \dots$$

with $K^i := \{ f : G^i \to M \text{ continuous functions}\}$ with $d^i f(g_1, \dots, g_{i+1})$

$$= g_1 f(g_2, \dots, g_{i+1}) + \sum_{j=1}^{i} (-1)^j f(g_1, \dots, g_j g_{j+1}, \dots, g_{i+1}) + (-1)^{i+1} f(g_1, \dots, g_i)$$

Then check: $H^i(G,M) = \dfrac{\text{Cochains in } K^i}{\text{Coboundaries in } K^i}$

Explicitly: $H^0(G,M) = M^G$

$Z^1(G,M) = \{ f : G \to M \mid f(g_1 g_2) = g_2 f(g_1) + f(g_2) \}$

$B^1(G,M) = \{ f: G \to M \mid \exists\, m \in M \text{ s.t. } f(g) = (g-1)m \ \ \forall g \in G \}$

<u>Cohomological interpretation of the tangent space.</u>

Fix $\bar{\rho}: G \to GL_n(k)$ $\quad (C_k(\bar{\rho}) = k)$. We said that $d = \dim_k D_{\bar{\rho}}(k[\varepsilon])$

$$= \dim_k t_{R^{univ}_{\bar{\rho}}}$$

$$\left(t_{R^{univ}_{\bar{\rho}}} = \mathrm{Hom}_\Lambda (R^{univ}_{\bar{\rho}}, k[\varepsilon]) \right)$$

Start with $\rho: G \to GL_n(k[\varepsilon])$ a deformation of $\bar{\rho}$. Then $\forall g \in G$:

$$\rho(g) = \bar{\rho}(g)\,(1 + \underset{\substack{\cap \\ M_n(k)}}{c_g}\, \varepsilon) \qquad (k \hookrightarrow k[\varepsilon] = k + \varepsilon k)$$

check : ρ homomorphism

$\Rightarrow c : G \to M_n(k),\ g \mapsto c_g$ is a 1-cocycle for the adjoint action $G \curvearrowright M_n(k)$ of $\bar{\rho}$.

[Adjoint representation: $G \curvearrowright M_n(k)$

$g \cdot m := \bar{\rho}(g)^{-1}\, m\, \bar{\rho}(g)$. Write $\mathrm{Ad}_{\bar{\rho}}$

Proposition: The previous construction gives an isomorphism (of vector spaces)

$$D_{\bar{\rho}}(k[\varepsilon]) \longrightarrow H^1(G, \mathrm{Ad}_{\bar{\rho}})$$

Corollary. $d = \dim_k (H^1(G, \mathrm{Ad}_{\bar{\rho}}))$.