Proof of the last Proposition.

Look at $\rho(G) \leq GL_n(\overline{\mathbb{Q}_p})$. It is a compact Hausdorff topological group

$\Rightarrow$ Baire's Lemma holds for $\rho(G)$.

(Baire's Lemma: a countable union of nowhere dense closed subspaces of X is nowhere dense in X.

Nowhere dense: it does not contain any open set of X)

$$GL_n(\overline{\mathbb{Q}_p}) = \bigcup_{\substack{E/\mathbb{Q}_p \\ \text{finite}}} GL_n(E)$$ countable union of closed subsets.

($\forall n \in \mathbb{N}$: there are only finitely many $E|\mathbb{Q}_p$ s.t. $[E:\mathbb{Q}_p] = n$)

Write $\rho(G) = \bigcup_{\substack{E|\mathbb{Q}_p \\ \text{finite}}} (GL_n(E) \cap \rho(G))$. Either there exists $E/\mathbb{Q}_p$ finite such

that $GL_n(E) \cap \rho(G)$ has finite index in $\rho(G)$

$\Rightarrow$ We can choose $F|E$ finite such that $\rho(G) \subseteq GL_n(F)$ (finite index $\_$)

Or for every $E/\mathbb{Q}_p$ finite, $GL_n(E) \cap \rho(G)$ has infinite index in $\rho(G)$

$\Rightarrow$ $GL_n(E) \cap \rho(G)$ is nowhere dense in $\rho(G)$ (Basis of open subgroups $\Rightarrow$ open subgroups in compact spaces are of finite index)

Now $\rho(G)$ is a countable union of nowhere dense sets $\Rightarrow$ Contradicts Baire's Lemma.

$\boxtimes$

Lemma. If $\rho: G \to GL_n(K)$ is a continuous representation with coefficients in $K/\mathbb{Q}_p$ finite, then there exists a continuous representation $\rho': G \to GL_n(\mathcal{O}_K)$ such that if $\iota: GL_n(\mathcal{O}_K) \to GL_n(K)$ is the inclusion $\rho \underset{\uparrow}{\cong} \iota \circ \rho'$
equivalent

Proof. Recall: an $\mathcal{O}_K$-lattice in K^n is a free $\mathcal{O}_K$-module L of rank n such that $L \otimes_{\mathcal{O}_K} K \cong K^n$.

Choosing a basis for K^n we obtain a continuous action of G on K^n via ρ.
Let L be any lattice in K^n. For $g \in GL_n(K)$ let $g(L) := \{g(x) \mid x \in L\}$

Exercise: $g(L)$ is a lattice, and $\mathrm{Stab}(L) = \{g \in GL_n(K) \mid g(L) \subseteq L\}$ is an open subgroup of $GL_n(K)$. EX

Look at $\rho^{-1}(\underbrace{\mathrm{Stab}(L)}_{\subseteq_{\text{open}} GL_n(K)}) \underset{\text{open}}{\subseteq} G \underset{G \text{ compact}}{\Longrightarrow} \rho^{-1}(\mathrm{Stab}(L))$ has finite index in G.

Choose a set $\{g_1, \ldots, g_m\}$ of representatives for $G / \rho^{-1}(\mathrm{Stab}(L))$.

Then define $$L' := \sum_{i=1}^{m} \rho(g_i)(L).$$ We check that the lattice L' is G-stable.

(G-stable: $\rho(g) L' \subseteq L' \quad \forall g \in G$)

Choose an $\mathcal{O}_K$-basis for the lattice L', then the action of G on L' gives a (continuous) representation $\rho' : G \to GL_n(\mathcal{O}_K)$

By construction $\iota \circ \rho' \sim \rho$

$\blacksquare$

Start with $\rho : G \to GL_n(K)$ continuous representation.

Then by the Lemma we can choose a conjugate of ρ with values in $GL_n(\mathcal{O}_K)$.

Then we can reduce modulo the maximal ideal $m_K \triangleleft \mathcal{O}_K$ and we obtain a "residual" representation $\bar{\rho} : G \to GL_n(\underbrace{\mathcal{O}_K / m_K}_{= \mathbb{F}_{p^m}})$ attached to ρ.

Def. If G acts on a finite free module M. Choose a filtration $M \supsetneq M_n \supsetneq \ldots \supsetneq \{0\}$ in G-stable A-modules such that M_i / M_{i-1} is an irreducible $A[G]$-module.
(A: field)

(Does not admit any G-stable submodule)

Then the semi-simplification of M ist the $A[G]$-module $\bigoplus_{i=1}^{n} M_i / M_{i-1}$.

Example: If $\underset{g \in G}{\rho(g)} = \begin{pmatrix} \chi_1(g) & \delta(g) \\ 0 & \chi_2(g) \end{pmatrix} \to \bar{\rho}^{ss}(g) = \begin{pmatrix} \chi_1(g) & 0 \\ 0 & \chi_2(g) \end{pmatrix}$

χ_1, χ_2: Character of ρ

Remark: the representation $\bar{\rho}^{ss}$ attached to ρ is well-defined up to equivalence.

$$\begin{tikzcd} K \arrow[r] & \mathcal{O}_K \arrow[r] & \mathbb{F}_{p^m} \arrow[ll, dashed, bend left] \end{tikzcd}$$

Idea: fix $\bar{\rho} : G \to GL_n(\mathbb{F}_{p^m})$ and look at $\rho : G \to GL_n(\mathcal{O}_K)$ (with $\mathcal{O}_K / m_K = \mathbb{F}_{p^m}$) such that $\rho \bmod m_K = \bar{\rho}$.

Example of p-adic Galois representation. ("p-adic cyclotomic character")

(p prime, $n \in \mathbb{N}_{\geq 1}$)

$$\chi_n : G_{\mathbb{Q}} \longrightarrow Gal(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times \quad (= GL_1(\mathbb{Z}/p^n\mathbb{Z}))$$

This representations are compatible with the maps $(\mathbb{Z}/p^m\mathbb{Z})^\times \to (\mathbb{Z}/p^n\mathbb{Z})^\times$ for $m \geq n$.

$$\chi_m \text{ modulo } p^n = \chi_n .$$

We can take $\underbrace{\varprojlim_n \chi_n}_{=: \chi_{cyc}} : G_{\mathbb{Q}} \longrightarrow \varprojlim_n Gal(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) = \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})^\times = \mathbb{Z}_p^\times$

Write $\mathbb{Q}(\zeta_{p^\infty}) = \bigcup_{n \geq 1} \mathbb{Q}(\zeta_{p^n})$

We call χ_{cyc} the p-adic cyclotomic character. χ_{cyc} factors through

$G_{\mathbb{Q}, p^\infty} \longrightarrow \mathbb{Z}_p^\times$. It also factors through $G_{\mathbb{Q}}^{ab} \longrightarrow \mathbb{Z}_p^\times$.

Theorem: (Kronecker-Weber) The product of all cyclotomic characters gives an isomorphism

$$G_{\mathbb{Q}}^{ab} \xrightarrow{\sim} \prod_p \mathbb{Z}_p^\times .$$

Look at "deformation functors".

* $\mathcal{C}$ category, $R \in \mathcal{C}$, then $\underbrace{Hom_{\mathcal{C}}(R, \cdot)}^{=: h_R} : \mathcal{C} \longrightarrow \underline{Set}$ is the functor

$A \longmapsto Hom_{\mathcal{C}}(R, A)$, $f \in Mor_{\mathcal{C}}(A, B) \longmapsto \left\{ \begin{array}{ccc} Hom_{\mathcal{C}}(R,A) & \longrightarrow & Hom_{\mathcal{C}}(R,B) \\ g & \longmapsto & f \circ g \end{array} \right.$

We will work with some categories of rings.

Fix a field k. We denote by $\mathcal{C}_k$ the category whose objects are Artinian, local rings with residue field k and morphisms are local ring morphisms, that induce the <u>identity</u> on k.

Examples. * $k = \mathbb{F}_p$, then $\mathbb{Z}/p^n\mathbb{Z} \in \mathcal{C}_{\mathbb{F}_p}$ $\forall n \in \mathbb{N}_{>0}$.

$$\mathbb{F}_p[T]/T^n \in \mathcal{C}_{\mathbb{F}_p}$$

* $\exists$ unique degree n unramified extension of $\mathbb{Q}_p$, we will denote it by $\mathbb{Q}_{p^n}$. We write $\mathbb{Z}_{p^n}$ for its valuation ring, then $\mathbb{Z}_{p^n}/p\mathbb{Z}_{p^n} = \mathbb{F}_{p^n}$

$\forall m \in \mathbb{N}_{>0}$: $\mathbb{Z}_{p^n}/p^m\mathbb{Z}_{p^n} \in \mathcal{C}_{\mathbb{F}_{p^m}}$

$$\mathbb{Z}_{p^n}/p^m\mathbb{Z}_{p^n} \longrightarrow \mathbb{Z}_{p^n}/p^m\mathbb{Z}_{p^n}$$

This is not a morphism in $\mathcal{C}_{\mathbb{F}_{p^m}}$ if $m \geq 2$.

$$x \text{ modulo } p^m \longmapsto \mathrm{Frob}_p(x) \text{ modulo } p^m$$

Let $\hat{\mathcal{C}}_k$ be the category whose objects are complete local Noetherian rings with residue field k, morphisms are local ring morphisms that induce the identity on k.

Example. $\mathbb{Z}_{p^n} \in \hat{\mathcal{C}}_{\mathbb{F}_{p^n}}$, $\mathbb{F}_{p^n}[[T]] \in \hat{\mathcal{C}}_{\mathbb{F}_{p^n}}$

An object of $\mathcal{C}_k$ is also an object in $\hat{\mathcal{C}}_k$ (same for morphisms)

DEFORMATION FUNCTORS

Fix $n \geq 1$.

Let G be a profinite group, k a finite field. Fix a continuous representation $\bar{\rho}: G \to GL_n(k)$.

Def. For $A \in \hat{\mathcal{C}}_k$, a deformation (of $\bar{\rho}$ to A) is a continuous representation $\rho: G \to GL_n(A)$ such that ρ modulo $m_A = \bar{\rho}$.

We say that $\rho_1, \rho_2 : G \to GL_n(A)$ are strictly equivalent iff. $\exists M \in \ker(GL_n(A) \to GL_n(k))$ such that $M^{-1}\rho_1 M = \rho_2$.

Remark: if $A \xrightarrow{f} B$ is a morphism in $\hat{\mathcal{C}}_k$ and $\rho_1, \rho_2 : G \to GL_n(A)$ are strictly equivalent representations, then $f\rho_1, f\rho_2 : G \to GL_n(B)$ are strictly equivalent.

We define $D_{\bar{\rho}} : \hat{\mathcal{C}}_k \longrightarrow \underline{\mathrm{Set}}$ as the functor

* $D_{\bar{\rho}}(A) := \left\{ \begin{matrix} \text{deformations of} \\ \bar{\rho} \text{ to } A \end{matrix} \right\} \Big/ {\text{strict equivalence}}$

* $D_{\bar{\rho}}(f)$ maps a deformation $\rho : G \longrightarrow GL_n(A)$ to the class of $f \circ \rho$
  $f : A \to B$ morphism in $\hat{\mathcal{C}}_k$

(We obtain a functor $\mathcal{C}_k \longrightarrow \underline{\mathrm{Set}}$ by restricting $D_{\bar{\rho}}$ to $\mathcal{C}_k$)

Goal: show that $D_{\bar{\rho}}$ is "pro-represented" by some $R \in \hat{\mathcal{C}}_k$, in the sense that $D_{\bar{\rho}} \cong \mathrm{Hom}_{\hat{\mathcal{C}}_k}(R, \cdot)$