Exercise: we saw that Dimension conjecture $\Rightarrow$ Leopoldt's conjecture

$*$ $n=2$.

$$\dim_{\text{Krull}}\left(R/\mathfrak{m}_\Lambda R \right) \geq d_1 - d_2$$

$K = \mathbb{Q}$, $p > 2$, $\operatorname{char} k = p$ $\qquad v \in S_\infty$

$\bar{\rho}: G_{\mathbb{Q},S} \longrightarrow GL_2(k)$ $\qquad H^0(G_v, \mathrm{Ad}_{\bar\rho})$

Def.: $\rho: G_{\mathbb{Q},S} \to GL_2(R)$, we say that

$*$ ρ is odd if $\rho(c) \sim \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ for some complex conjugate c (all)

$*$ ρ is even otherwise.

$$\dim\left(R/\mathfrak{m}_\Lambda R \right) \geq \begin{cases} 3, & \bar\rho \text{ odd} \\ 1, & \bar\rho \text{ even} \end{cases}$$

Explicit description of deformation spaces.

Global case and $K = \mathbb{Q}$, any set S of primes of $\mathbb{Q}$. Fix $\bar\rho: G_{\mathbb{Q},S} \to GL_n(k)$

k a finite field $\quad$ ($\Lambda \in \hat{\mathcal{C}}_k$, work in $\hat{\mathcal{C}}_\Lambda$)

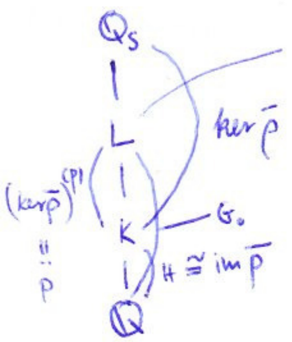
Question: describe $R^{\text{univ}}_{\bar\rho}$.

Consider $R \in \hat{\mathcal{C}}_\Lambda$, then $\Gamma_n(R)$ is a pro-p-group

$$\Gamma_n(R) = \ker\left(GL_n(R) \to GL_n(k) \right)$$

Take any deformation $\rho: G_{\mathbb{Q},S} \to GL_n(R)$ of $\bar\rho$; look at $\rho|_{\ker\bar\rho}: \ker\bar\rho \to \Gamma_n(R)$

$\Rightarrow$ $\rho|_{\ker\bar\rho}$ factors through $(\ker\bar\rho)^{(p)}$ (pro-p-completion)



largest pro-p extension of K unramified outside S

S_1 primes of K above S

$|$

S

$$\rho : G_{\mathbb{Q},S} \longrightarrow GL_n(R) \xrightarrow{\text{construct}} \varphi : P \longrightarrow \Gamma_n(R) \text{ group homomorphism}$$

Study three cases:

* $\bar{\rho}$ <u>tame</u> ($:\Leftrightarrow$ $p \nmid \# \operatorname{im}\bar{\rho}$)

* $n=2$: $\bar{\rho}$ full ($:\Leftrightarrow$ $\operatorname{im}\bar{\rho} \supseteq SL_2(\mathbb{F}_p)$ up to conjugation)

* $n=2$: $\operatorname{im}\bar{\rho}$ solvable and of order multiple of p.

<u>Some reminders on group theory</u>:

Thm. (Schur–Zassenhaus)

G profinite group, P normal pro-p subgroup of finite index prime to p.

Then $\exists A \leq G$ subgroup such that $A \xrightarrow{\sim} G/P$ is an isomorphism.

All such A's are conjugate by an element of P. One gets $G \cong P \rtimes G/P$

Def. $\bar{P} = P/\langle P^p, [P,P]\rangle$ where P is a pro-p group

maximal abelian p-elementary continuous quotient of P.

Thm. (Burnside basis theorem) If P is a pro-p group, $x_1, \dots, x_d$ are elements of P such that $\bar{x}_1, \dots, \bar{x}_d$ generate $\bar{P}$, then $x_1, \dots, x_d$ topologically generate P as an abstract group.

Thm. (Boston) G profinite, $P \leq G$ normal pro-p subgroup of finite index, coprime to p.

A a subgrp of G s.t. $A \cong G/P$. Then $\bar{P}$ has a structure of $\mathbb{F}_p[A]$-module.

Let $\bar{V}$ an $\mathbb{F}_p[A]$-submodule of $\bar{P}$. Then there exists an A-invariant subgroup V of G such that the image of V in $\bar{P}$ is $\bar{V}$.

Def. Let $A \leq GL_n(k)$ be a subgroup of order prime to p and let $Ad|_A$ be its adjoint representation. Let V be a $k[A]$-module.

(By Maschke's theorem $Ad|_A$ and V decompose as direct sums of irreducible $k[A]$-modules)

We say that V is prime-to-adjoint if V and $k[A]$ have no common irreducible $k[A]$-submodules.

"prime to adjoint"-principle (named by Böckle!)

Exercise: X fin. generated subgroup of $\Gamma_n(R)$ $\overset{\hat{\mathcal{C}}_\Lambda}{\cup}$

$A \leq GL_n(R)$ finite subgroup of order prime to p normalizing X.

Then: X is prime-to-adjoint for A $\iff$ X is trivial.

Hint: $K_r := \ker(\Gamma_n(R) \to \Gamma_n(R/m_R^r))$ $\quad X \cap K_r / X \cap K_{r+1}$

Explicit deformation of tame representations.

$\bar{\rho}: G_{\mathbb{Q},S} \to GL_n(k)$ tame, so $p \nmid \# \mathrm{im}\bar{\rho} = \# H$

One constructs a lift of $\bar{\rho}$ to $W(k)$

Schur-Zassenhaus $\underset{\text{apply to}}{\rightarrow}$ $*$ $G_0 \geq P$ p-Sylow normal, $H \cong G/P$

$\underset{\text{one gets}}{\Rightarrow}$ $G_0 \cong P \rtimes H$

One has $\pi: GL_n(W(k)) \to GL_n(k) \geq \mathrm{im}\bar{\rho}$

Look at $\pi^{-1}(\mathrm{im}\bar{\rho}) \geq \Gamma_n(W(k))$ pro-p subgroup, index is $\#\mathrm{im}\bar{\rho}$

$\Rightarrow \pi^{-1}(\mathrm{im}\bar{\rho}) = \Gamma_n(W(k)) \rtimes \mathrm{im}\bar{\rho}$.

We write $G : H \xrightarrow[\text{given by } \bar{\rho}]{\cong} \operatorname{im}\bar{\rho} \longrightarrow \pi^{-1}(\operatorname{im}\bar{\rho}) \subseteq GL_n(W(k))$

For any $R \in \hat{\mathcal{C}}_\Lambda$, we get $G_R : H \longrightarrow GL_n(R)$ by composing with $W(k) \to R$.

Define a functor

$$E_{\bar{\rho}} : \hat{\mathcal{C}}_\Lambda \longrightarrow \underline{\mathrm{Set}} \quad ; \quad E_{\bar{\rho}}(R) = \mathrm{Hom}_H(P, \Gamma_n(R)).$$

We compare $E_{\bar{\rho}}$ with $D_{\bar{\rho}}$.

* For $\rho : G_0 \longrightarrow GL_n(R)$ ($G \to GL_n(k)$ factors through G_0!), then

$$\rho|_P : P \to \Gamma_n(R) \quad \text{is } H\text{-equivariant (check)}$$

$$\rightsquigarrow \quad \rho|_P \in E_{\bar{\rho}}(R)$$

On the other hand:

* Take $\varphi \in \mathrm{Hom}_H(P, \Gamma_n(R))$, then

$$\rho : G_0 \cong P \rtimes H \xrightarrow[(\varphi, \bar{\rho})]{} \Gamma_n(R) \rtimes \operatorname{im}\bar{\rho} \hookrightarrow GL_n(R)$$

<u>check:</u> ρ is a representation, because φ is H-equivariant.

We have a natural transformation $E_{\bar{\rho}} \to D_{\bar{\rho}}$

Theorem. (Boston) 1) The morphism $E_{\bar{\rho}} \longrightarrow D_{\bar{\rho}}$ is an isomorphism when $C_k(\bar{\rho}) = k$.

2) In general $E_{\bar{\rho}} \to D_{\bar{\rho}}$ is formally smooth.

3) $E_{\bar{\rho}}$ is always representable.

Proof. Show $E_{\bar{\rho}}(R) \longrightarrow D_{\bar{\rho}}(R)$ is a bijection.

* surjective: take ρ, then $\varphi = \rho|_P$ is an element of $E_{\bar{\rho}}(R)$ and $\rho_\varphi = \rho$.
* injective: if φ_1, φ_2 give ρ_{φ_1} strict equiv. ρ_{φ_2}, then $\varphi_1 = \varphi_2$.

$$\rho_{\varphi_1} = \underset{\substack{\cap \\ \Gamma_n(R)}}{\gamma} \rho_{\varphi_2} \gamma^{-1} \iff (\varphi_1, \bar{\rho}) = \gamma(\varphi_2, \bar{\rho})\gamma^{-1}$$

$\Rightarrow$ γ commutes with $\operatorname{im}\bar{\rho}$

$\Rightarrow$ γ scalar because $C_k(\bar{\rho}) = k$.

2) Exercise.

3) Similar to representability of $D_{\bar{\rho}}^{\square}$.

<u>Sketch</u>: $x_1, \dots, x_d$ top. generators of P (top. fin. generated because of p-finiteness)

$$W(k)[[T_{ij}^{(r)}]]_{\substack{1 \le i,j \le n \\ 1 \le r \le d}}$$

free pro-p-group on d generators

$$1 \longrightarrow N \longrightarrow \mathcal{F} \longrightarrow P \longrightarrow 1$$

$$(*) \quad \mathcal{F} \xrightarrow{\alpha} \Gamma_n\left(W(k)[[T_{ij}^{(r)}]]_{i,j,r}\right)$$

$$x_r \longmapsto \begin{pmatrix} 1 + T_{11}^{(r)} & & T_{ij}^{(r)} \\ & \ddots & \\ T_{ij}^{(r)} & & 1 + T_{nn}^{(r)} \end{pmatrix}$$

(N acts trivially, morph. H-equivariant) $\longrightarrow$ all relations generate an ideal I

Universal deformation ring is $W(k)[[T_{ij}^{(r)}]]/I$

Universal deformation is $\varphi : P \longrightarrow \Gamma_n(R)$ induced by α in $(*)$.

<u>Remark:</u> $\dim_k t_{D_{\bar{\rho}}} = \dim_k\left(\operatorname{Hom}_H(P, \Gamma_n(k[\varepsilon]))\right) = \dim_k\left(\operatorname{Hom}_H(\bar{P}, \operatorname{Ad}_{\bar{\rho}})\right)$

$C_k(\bar{\rho}) = k$.

An explicit deformation ring:

Theorem. (Boston–Mazur) $Z_S = \{ x \in k \mid$

fractional ideal $(x) = I^p$ for some fractional ideal I of K

$\underset{\substack{\cap \\ k}}{x_v} = g_v^p$ for some $g_v \in K_v$ $\forall v \in S_1 \}^{S_H}$, $B_S = Z_S/(k^\times)^p$ $\mathbb{F}_p[H]$-module.

$$V = \mathrm{coker}\left(\mu_p(K) \longrightarrow \prod_{v \in S_1} \mu_p(K_v) \right)$$

As an $\mathbb{F}_p[H]$-module

$$\bar{\rho} = \mathrm{Ind}_{H_\infty}^{H} \mathbb{F}_p \oplus \mathbb{F}_p \oplus V \oplus B_S \qquad \mathrm{ad}_{\bar{\rho}} \underset{\mathbb{F}_p[H]}{=} \mathrm{ad}^0_{\bar{\rho}} \oplus \mathbb{F}_p$$

H_∞ = subgp of H gen. by a comp. conjugation