We define $\hat{D}_{\bar\rho} : \hat{\mathcal{C}}_k \longrightarrow \underline{Set}$ as the functor

* $\hat{D}_{\bar\rho}(A) := \left\{ \begin{matrix} \text{deformations of} \\ \bar\rho \text{ to } A \end{matrix} \right\} \Big/ \text{strict equivalence}$

* $\hat{D}_{\bar\rho}(f)$ maps a deformation $\rho : G \to GL_n(A)$ to the class of $f \circ \rho$
  $f : A \to B$ morphism in $\hat{\mathcal{C}}_k$

(We obtain a functor $D_{\bar\rho} : \mathcal{C}_k \longrightarrow \underline{Set}$ by restricting $\hat{D}_{\bar\rho}$ to $\mathcal{C}_k$)

Goal: show that $D_{\bar\rho}$ is "pro-represented" by some $R \in \hat{\mathcal{C}}_k$, in the sense that $D_{\bar\rho} \cong \mathrm{Hom}_{\hat{\mathcal{C}}_k}(R, \cdot)$

02.05.18

Fix any $\Lambda \in \hat{\mathcal{C}}_k$. We define $\mathcal{C}_\Lambda$ as the category of local Artinian Λ-algebras with residue field k and local morphisms of Λ-algebras (that induce the identity on k).

* $\hat{\mathcal{C}}_\Lambda$ = category of complete local Noetherian Λ-algebras with residue field k. Morphisms as above.

The natural choice for Λ (when working in char $\Lambda = 0$) is the ring of Witt vectors $W(k)$ of k.
(= unique complete discrete valuation ring with residue field k and uniformizer p).
(Mézard's notes, Serre "Local fields")

We only need

$$W(\mathbb{F}_{p^n}) = \mathbb{Z}_{p^n} \quad \text{ring of integers in } \mathbb{Q}_{p^n}$$

$$\hat{\mathcal{C}}_{\mathbb{Z}_{p^n}} \ni \mathbb{Z}_{p^n}, \quad \mathbb{F}_{p^n}[[T]] \in \hat{\mathcal{C}}_{\mathbb{F}_{p^n}}, \text{ but } \notin \hat{\mathcal{C}}_{\mathbb{Z}_{p^n}}.$$

Deformation functors Fix $\bar\rho : G \to GL_n(k)$; We defined $D_{\bar\rho} : \mathcal{C}_k \longrightarrow \underline{Set}$, $\hat{D}_{\bar\rho} : \hat{\mathcal{C}}_k \to \underline{Set}$.

One defines

$D_{\bar\rho,\Lambda} : \mathcal{C}_\Lambda \longrightarrow \underline{Set}$ by $D_{\bar\rho,\Lambda}(A) = \left\{ \begin{matrix} \text{deformations of} \\ \bar\rho \text{ to } A \end{matrix} \right\} \Big/ \text{strict equivalence}$

$\hat{D}_{\bar\rho,\Lambda} : \hat{\mathcal{C}}_\Lambda \longrightarrow \underline{Set}$ in the same way.

Representable Functors.

Let $\mathcal{C}$ be any category, $\mathcal{F}: \mathcal{C} \to \underline{Set}$ any functor.

* $\mathcal{F}$ is called representable $:\Leftrightarrow \exists A \in \mathcal{C}: \mathcal{F} \cong Hom_{\mathcal{C}}(A, \cdot)$ as functors.

For our categories $\mathcal{C}_k, \hat{\mathcal{C}}_k, \ldots$ we define the following:

* $\mathcal{F}: \mathcal{C}_\Lambda \to \underline{Set}$ is pro-representable $:\Leftrightarrow \exists R \in \hat{\mathcal{C}}_\Lambda$ and an isomorphism of functors

$$Hom_{\hat{\mathcal{C}}}(R, \cdot) \longrightarrow \mathcal{F}$$

(as a functor $\mathcal{C}_\Lambda \to \underline{Set}$)

Why representability?

Assume $D_{\bar{\rho},\Lambda}$ is (pro-) representable by some R in $\hat{\mathcal{C}}_\Lambda$: means that there is a bijection $Hom_{\hat{\mathcal{C}}_\Lambda}(R, A) \longrightarrow D_{\bar{\rho},\Lambda}(A)$ for every $A \in \mathcal{C}_\Lambda$.

Take $A = R$. Then there is a strict equivalence class ρ_R in $D_{\bar{\rho},\Lambda}(R)$ corresponding to id_R.

Then the bijection

$$Hom_{\hat{\mathcal{C}}_\Lambda}(R, A) \longrightarrow D_{\bar{\rho},\Lambda}(A)$$

$$(f: R \to A) \longmapsto \text{class of } f \circ \rho_R : G \to GL_n(A)$$

We call (R, ρ_R) a universal couple, R is the universal deformation ring of $\bar{\rho}$, ρ_R is the universal deformation of $\bar{\rho}$.

Properties of (pro-) representable functors $\mathcal{C}_\Lambda \xrightarrow{\mathcal{F}} \underline{Set}$. In this section $\mathcal{F}$ is a pro-representable-

Functor $\mathcal{C}_\Lambda \to \underline{Set}$, $\mathcal{F} \cong Hom_{\hat{\mathcal{C}}_\Lambda}(R, \cdot)$

- left exact
- $\mathcal{F}(k) = \{*\}$, since the only element of $Hom_{\hat{\mathcal{C}}_\Lambda}(R, k)$ is $R \to R/m_R \; (\underset{\text{one } \underline{\underline{\text{fixed}}} \text{ identification}}{\cong} k)$
- ~~$\mathcal{F}$ is continuous~~ (later)
- $\mathcal{F}$ behaves well with respect to fiber products

$$k[\varepsilon] = k[T]/T^2, \quad \varepsilon = \overline{T} \quad (\text{ring with } \varepsilon^2 = 0)$$

$$k[\varepsilon] = \{ a + b\varepsilon \mid a, b \in k \}, \quad \varepsilon^2 = 0$$

- $F(k[\varepsilon])$ is a finite dimensional k-vector space.

Remark: fiber products don't always exist in $\hat{\mathcal{C}}_\Lambda$ $\quad \Lambda = W(k)$

Example $A = k[[X,Y]]$, $C = k[[X]]$, $B = k$

maps

$$\begin{matrix} A \to C \\ Y \mapsto 0 \end{matrix}, \quad \begin{matrix} B \to C \\ k \hookrightarrow k[[X]] \end{matrix}$$

Check: the fiber product $A \times_C B$ is not in the category $\hat{\mathcal{C}}_\Lambda$ (the fiber product as rings is not Noetherian ... this argument should not work)

Fiber products exist in $\mathcal{C}_\Lambda$.

F has the Mayer-Vietoris property if the map $F(A \times_C B) \to F(A) \times_{F(C)} F(B)$ is a bijection.

Remark: F pro-representable $\Rightarrow$ F has the Mayer-Vietoris-property.

- $F(k[\epsilon])$ is a k-vector space:

Why is it a k-vector space? It is when the map $F(k[\epsilon] \times_k k[\epsilon]) \to F(k[\epsilon]) \times_{F(k)} F(k[\epsilon])$ is a bijection. (induced by $\begin{matrix} k[\epsilon] \to k \leftarrow k[\epsilon] \\ \epsilon \mapsto 0 \leftarrow\!\shortmid \epsilon \end{matrix}$)

k-scalar multiplication is induced by

$$k \times k[\epsilon] \to k[\epsilon]$$
$$(\lambda, a + b\epsilon) \mapsto (a + \lambda b \epsilon)$$

addition is induced by

$$k[\epsilon] \times_k k[\epsilon] \to k[\epsilon]$$
$$(a + b_1\epsilon, a + b_2\epsilon) \mapsto a + (b_1 + b_2)\epsilon$$

In this case we call $F(k[\epsilon])$ the tangent space of F. Why?

When F is pro-representable, $F \cong \mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R, \cdot)$ then there is an isomorphism of k-vector spaces

$$t_R \longrightarrow F(k[\epsilon])$$

Recall: for R a local Noetherian Λ-algebra with maximal ideal m_R, then

$t_R^* = m_R / (m_R^2, m_\Lambda)$ is the cotangent space of R, $t_R = \mathrm{Hom}_k(t_R^*, k)$.

Idea of proof: $f \in F(k[\epsilon]) = \mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R, k[\epsilon])$

then $\underset{\text{for } r \in R}{f(r)} = \underset{\text{image mod } m_R}{\bar{r}} + f'(r)\epsilon$

Criteria for representability:

Criterion. (Grothendieck) Let $F: \mathcal{C}_\Lambda \longrightarrow \underline{\mathrm{Set}}$ be a functor such that $F(k) = \{*\}$.
Then F is pro-representable if and only if the following hold:
i) F has the Mayer-Vietoris property
ii) $F(k[\epsilon])$ is a finite dimensional vector space $/k$.

We will use a refined version:
Criterion. (Schlessinger)

Def. If $R, S \in \hat{\mathcal{C}}_\Lambda$, $f: R \to S$ a morphism. We say that f is small if it is surjective and if $\ker f$ is annihilated by m_R ($m_R \cdot \ker f = 0$) (Exp.: $k[\epsilon] \to k$, $\epsilon \mapsto 0$)

We introduce the Schlessinger conditions. $R_0, R_1, R_2 \in \mathcal{C}_\Lambda$, $\varphi_1: R_1 \to R_0$, $\varphi_2: R_2 \to R_0$,
$R_3 := R_1 \times_{R_0} R_2$; $(*): F(R_3) \longrightarrow F(R_1) \times_{F(R_0)} F(R_2)$

The conditions are:

H1: If $\varphi_2 : R_2 \to R_0$ is small, then $(*)$ is surjective

H2: If $R_0 = k$, $R_2 = k[\epsilon]$ with $\begin{array}{c} R_2 \to R_0 \\ \epsilon \mapsto 0 \end{array}$, then $(*)$ is bijective

H3: $F(k[\epsilon])$ is finite-dimensional over k.

H4: If $R_1 = R_2$, $\varphi_1 = \varphi_2$ is small, then $(*)$ is bijective.

Theorem of Schlessinger: If $F(k) = \{*\}$ and it satisfies $H_1 - H_4$, then F is pro-representable.