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Galois representations and their deformations.

A. CONTI (3/225) andrea.conti@iwr.uni-heidelberg.de

Goal: study representations of Galois groups of p-adic or number fields with p-adic coefficients, by "deforming" representations with modulo p coefficients.

References:
* Gouvêa, notes
* Mézard, notes
* Böckle, notes
* Mazur's article "Deforming Galois representations"

01. Galois groups

K perfect field, L a normal extension of K. Define $\mathrm{Gal}(L/K)$
$= \{\sigma : L \to L \mid \sigma \text{ field automorphism}, \sigma|_K = \mathrm{id}_K\}$

· Topology:
* if $L|K$ is finite then give $\mathrm{Gal}(L|K)$ the discrete topology
* if $L|K$ is infinite then give $\mathrm{Gal}(L|K)$ the Krull topology:

a basis of open neighborhoods of id_L is the collection of sets
$\{\sigma \in \mathrm{Gal}(L/K) \mid \sigma|_E = \mathrm{id}_E\}$ where E varies over the finite subextensions $E|K$

As groups: $$\mathrm{Gal}(L/K) \cong \varprojlim_{\substack{E|K \text{ finite and normal} \\ E \subseteq L}} \mathrm{Gal}(E|K)$$

If $\mathrm{Gal}(E|K)$ has the discrete topology, then this is an isomorphism of topological groups.

This makes $\mathrm{Gal}(L/K)$ into a profinite group.

$\Rightarrow$ $\mathrm{Gal}(L|K)$ is compact and hausdorff.

$\Rightarrow$ Open subgroups are the closed subgroups of finite index

Theorem. (Galois correspondence) The map

$$\left\{ \begin{matrix} \text{subextensions} \\ L \mid E \mid K \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{closed subgroups} \\ \text{of } \mathrm{Gal}(L|K) \end{matrix} \right\}$$

$$E \longmapsto \mathrm{Gal}(L|E)$$ is a bijection.

The inverse is $H \mapsto E = L^H$.

This induces a bijection

$$\left\{ \begin{matrix} L|E|K, \\ E|K \text{ finite} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{open subgroups} \\ \text{of } \mathrm{Gal}(L|K) \end{matrix} \right\}$$

When $L = K^{alg}$, we call $\mathrm{Gal}(K^{alg}|K)$ the absolute Galois group of K, we write G_K.

Examples. * $K = \mathbb{F}_p$, we know that finite extensions are of the form $\mathbb{F}_{p^n}$ and

$$\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$$ is an isomorphism.

$$(x \mapsto x^p) \longmapsto 1$$

"Frobenius element"

$$\mathrm{Gal}(\mathbb{F}_p^{alg}/\mathbb{F}_p) = \varprojlim_{n\to\infty} \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \xrightarrow{\sim} \varprojlim_{n\to\infty} \mathbb{Z}/n\mathbb{Z} =: \hat{\mathbb{Z}}$$

(the maps are $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \to \mathrm{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p)$ m, n with $m \mid n$

$$\sigma \mapsto \sigma|_{\mathbb{F}_{p^m}}$$

$$\mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z}$$

$$x \longmapsto x \bmod m$$

The Frobenius of $\mathrm{Gal}(\mathbb{F}_p^{alg}/\mathbb{F}_p)$ is mapped to $1 \in \hat{\mathbb{Z}}$.

* $K = \mathbb{Q}_p$; we denote by $\mathbb{Q}_p^{ur}$ the maximal unramified extension of $\mathbb{Q}_p$.
($\Leftrightarrow$ maximal extension for which p is still a uniformizer (generator of the maximal ideal of the valuation ring))

Valuation ring $\mathbb{Z}_p^{ur}$ has residue field $\mathbb{Z}_p^{ur}/p\mathbb{Z}_p^{ur} \cong \mathbb{F}_p^{alg}$

There is a map $\mathrm{Gal}(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) \longrightarrow \mathrm{Gal}(\mathbb{F}_p^{alg}/\mathbb{F}_p)$

$$\sigma \longmapsto \sigma \text{ modulo } p$$

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This map is a group isomorphism.

$$\begin{array}{l} \mathbb{Q}_p^{alg} \\ \Big| \Big) \text{ "Inertia group" } (I_p) \\ \mathbb{Q}_p^{ur} \\ \Big| \Big) \mathrm{Gal} = \hat{\mathbb{Z}} \quad \text{(topologically generated by } \mathrm{Frob}_p \text{)} \\ \mathbb{Q}_p \end{array}$$

* $K = \mathbb{Q}$, $\mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$, let p be prime.
Choose an extension of the p-adic valuation on $\mathbb{Q}$ to $\mathbb{Q}^{alg}$ (not unique!)
($\leftrightarrow$ choose an embedding $\mathbb{Q}^{alg} \to \mathbb{Q}_p^{alg}$)

Write a map

$$\mathrm{Gal}(\mathbb{Q}_p^{alg}/\mathbb{Q}_p) \longrightarrow \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$$
$$\sigma \longmapsto \sigma|_{\mathbb{Q}^{alg}}$$

This map is an injective group homomorphism and it identifies $\mathrm{Gal}(\mathbb{Q}_p^{alg}/\mathbb{Q}_p)$ with a subgroup $D_p \leq \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$

$$D_p = \{ \sigma \in \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q}) \mid v(\sigma^{-1}(x)) = v(x) \quad \forall x \in \mathbb{Q}^{alg} \}$$

We have Injections

$$I_p \leq D_p \subseteq \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$$

02. Galois groups for extensions unramified outside a finite set.

K number field, S is a finite set of places of K.

Def. 1) K^S is the largest extension of K that is unramified at all places not in S.

2) $G_{K,S} := \mathrm{Gal}(K^S/K)$

Remark. $v \in S$, then we have

$$\begin{tikzcd} \mathrm{Gal}(K_v^{alg}/K_v) \arrow[r, hook] \arrow[rr, bend right, "\text{Injection ?}"'] & \mathrm{Gal}(K^{alg}/K) \arrow[r] & \mathrm{Gal}(K^S/K) \end{tikzcd}$$

Answer: Injection when $K = \mathbb{Q}$, $\# S \geq 2$

Exercise. An open subgroup $H \leq G_{K,S}$ has the form G_{K_1, S_1} where $K_1 | K$ is finite and S_1 is a set of places of K_1.
(places in S_1 have to lie over the places of S $\Rightarrow$ S_1 finite)

Theorem. (Hasse - Minkowski) Let K, S as before.
Let $d \in \mathbb{N}_{>0}$. Then there are only finitely many extensions of K of degree d and unramified outside S.

Corollary. $\mathrm{Hom}_{cont}(G_{K,S}, \mathbb{F}_p)$ is finite. (Morphisms of topological groups)

Corollary. For every open subgroup $H \leq G_{K,S}$, $\mathrm{Hom}_{cont}(H, \mathbb{F}_p)$ is finite.
$\underset{\text{Def.}}{\Longleftrightarrow}$ $G_{K,S}$ satisfies the "p-finiteness condition".

($\mathbb{F}_p$ has the discrete topology)

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03. Galois representations

Let G be a profinite group and let A be a topological ring.

Def. A continuous representation of G with A-coefficients is a continuous group homomorphism $\rho: G \longrightarrow GL_n(A)$ for some integer n.

Given representations $\rho_1, \rho_2 : G \longrightarrow GL_n(A)$, we say that they are equivalent iff. $\exists\, P \in GL_n(A)$: $P^{-1} \cdot \rho_1 \cdot P = \rho_2$

Another point of view: let M be a finite free A-module of rank n.
Then a continuous representation $\rho: G \to GL_n(A)$ gives a continuous action $G \curvearrowright M$

$$G \times M \longrightarrow M$$
$$(g, m) \longmapsto \rho(g)\,(m_i)_{i=1..n}$$

where $(m_i)_{i=1..n}$ are coordinates of m

Def.
We call ρ a Galois representation if G is:
* $Gal(K^{alg}/K)$ for a finite extension $K \mid \mathbb{Q}_p$
* $G_{K,S}$ for a number field K and a finite set of places K.

From now on G is one of those groups.

Choices of coefficient ring A:

1) $A = \mathbb{C}$
2) $A = \mathbb{F}_{p^n}$
3) $A = \mathcal{O}_E$ for $E \mid \mathbb{Q}_p$ finite
4) $A = E$, $E \mid \mathbb{Q}_p$ finite

① $A = \mathbb{C}$

Proposition. A representation $\rho : G \to GL_n(\mathbb{C})$ has finite image.

Proof. Consider an open neighborhood $\mathcal{U}$ of $\mathbb{1}_n \in GL_n(\mathbb{C})$. If $\mathcal{U}$ is sufficiently small, the only subgroup $\subseteq \mathcal{U}$ is $\{\mathbb{1}_n\}$ (Exercise). By continuity of ρ, $\exists\, V$ open neighborhood of $e \in G$ such that $\rho(V) \subseteq \mathcal{U}$. We can choose V' a neighborhood of e which is an open subgroup of G and s.t. $V' \subseteq V$. Then $\rho(V') \subseteq \rho(V) \subseteq \mathcal{U}$.
$\Rightarrow \rho(V') = \{\mathbb{1}_n\}$. Since V' is of finite index in G, $\rho(V')$ is of finite index in $\rho(G)$. ▨

Expl. Take $K \mid \mathbb{Q}$ Galois, finite. Then $Gal(K/\mathbb{Q}) \hookrightarrow GL_n(\mathbb{C})$.

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② Proposition. If $\rho : G \to GL_n(\mathbb{F}_p^{alg})$ is a continuous representation,
$GL_n(\mathbb{Q}_p)$

then it factors through $\rho' : G \to GL_n(\mathbb{F}_{p^m})$ for some finite extension
$\to GL_n(E)$

$\mathbb{F}_{p^m} / \mathbb{F}_p$

$E / \mathbb{Q}_p$

Proof. Similar for $\mathbb{F}_p$, more difficult for $\mathbb{Q}_p$. $\blacksquare$

Proof of the last Proposition.

Look at $\rho(G) \leq GL_n(\overline{\mathbb{Q}}_p)$. It is a compact Hausdorff topological group

$\Rightarrow$ Baire's Lemma holds for $\rho(G)$.

(Baire's Lemma: a countable union of nowhere dense closed subspaces of X is nowhere dense in X.

Nowhere dense: it does not contain any open set of X)

$$GL_n(\overline{\mathbb{Q}}_p) = \bigcup_{\substack{E/\mathbb{Q}_p \\ \text{finite}}} GL_n(E)$$ countable union of closed subsets.

($\forall n \in \mathbb{N}$: there are only finitely many $E \mid \mathbb{Q}_p$ st. $[E:\mathbb{Q}_p] = n$)

Write $\rho(G) = \bigcup_{\substack{E \mid \mathbb{Q}_p \\ \text{finite}}} (GL_n(E) \cap \rho(G))$. Either there exists $E/\mathbb{Q}_p$ finite such that $GL_n(E) \cap \rho(G)$ has finite index in $\rho(G)$

$\Rightarrow$ We can choose $F \mid E$ finite such that $\rho(G) \subseteq GL_n(F)$ (finite index –)

Or for every $E/\mathbb{Q}_p$ finite, $GL_n(E) \cap \rho(G)$ has infinite index in $\rho(G)$

$\Rightarrow$ $GL_n(E) \cap \rho(G)$ is nowhere dense in $\rho(G)$ (Basis of open subgroups $\Rightarrow$ open subgroups in compact spaces are of finite index)

Now $\rho(G)$ is a countable union of nowhere dense sets $\Rightarrow$ Contradicts Baire's Lemma. ▨

Lemma. If $\rho: G \longrightarrow GL_n(K)$ is a continuous representation with coefficients in $K/\mathbb{Q}_p$ finite, then there exists a continuous representation $\rho': G \to GL_n(\mathcal{O}_K)$ such that if $\iota: GL_n(\mathcal{O}_K) \longrightarrow GL_n(K)$ is the inclusion $\rho \underset{\text{equivalent}}{\cong} \iota \circ \rho'$

Proof. Recall: an $\mathcal{O}_K$-lattice in K^n is a free $\mathcal{O}_K$-module L of rank n such that $L \otimes_{\mathcal{O}_K} K \cong K^n$.

Choosing a basis for K^n we obtain a continuous action of G on K^n via ρ.
Let L be any lattice in K^n. For $g \in GL_n(K)$ let $g(L) := \{g(x) \mid x \in L\}$

Exercise: $g(L)$ is a lattice, and $\mathrm{Stab}(L) = \{g \in GL_n(K) \mid g(L) \subseteq L\}$ is an open subgroup of $GL_n(K)$. EX

Look at $\rho^{-1}(\underbrace{\mathrm{Stab}(L)}_{\subseteq_{\text{open}} GL_n(K)}) \subseteq_{\text{open}} G \underset{G \text{ compact}}{\Longrightarrow} \rho^{-1}(\mathrm{Stab}(L))$ has finite index in G.

Choose a set $\{g_1, \dots, g_m\}$ of representatives for $G/\rho^{-1}(\mathrm{Stab}(L))$.

Then define $$L' := \sum_{i=1}^{m} \rho(g_i)(L).$$ We check that the lattice L' is G-stable.

(G-stable: $\rho(g) L' \subseteq L' \quad \forall g \in G$)

Choose an $\mathcal{O}_K$-basis for the lattice L', then the action of G on L' gives a (continuous) representation $\rho' : G \to GL_n(\mathcal{O}_K)$

By construction $\iota \circ \rho' \sim \rho$ ∎

Start with $\rho : G \to GL_n(K)$ continuous representation.

Then by the Lemma we can choose a conjugate of ρ with values in $GL_n(\mathcal{O}_K)$.

Then we can reduce modulo the maximal ideal $\mathfrak{m}_K \triangleleft \mathcal{O}_K$ and we obtain a "residual" representation $\bar{\rho} : G \to GL_n(\underbrace{\mathcal{O}_K/\mathfrak{m}_K}_{= \mathbb{F}_{p^m}})$ attached to ρ.

Def. If G acts on a finite free module M. Choose a filtration $M \supsetneq M_n \supsetneq \dots \supsetneq \{0\}$ in G-stable A-modules such that M_i/M_{i-1} is an irreducible $A[G]$-module.
(A: field)

(Does not admit any G-stable submodule)

Then the semi-simplification of M ist the $A[G]$-module $\bigoplus_{i=1}^{n} M_i/M_{i-1}$.

Example: If $\underset{g \in G}{\rho(g)} = \begin{pmatrix} \chi_1(g) & \delta(g) \\ 0 & \chi_2(g) \end{pmatrix} \to \bar{\rho}^{ss}(g) = \begin{pmatrix} \chi_1(g) & 0 \\ 0 & \chi_2(g) \end{pmatrix}$

χ_1, χ_2: Character of ρ

Remark: the representation $\bar{\rho}^{ss}$ attached to ρ is well-defined up to equivalence.

$$\begin{tikzcd} K \arrow[r] & \mathcal{O}_K \arrow[r] & \mathbb{F}_{p^m} \arrow[ll, bend left, dashed] \end{tikzcd}$$

Idea: fix $\bar{\rho} : G \to GL_n(\mathbb{F}_{p^m})$ and look at $\rho : G \to GL_n(\mathcal{O}_K)$ (with $\mathcal{O}_K/m_K = \mathbb{F}_{p^m}$) such that $\rho \bmod m_K = \bar{\rho}$.

Example of p-adic Galois representation. ("p-adic cyclotomic character")

(p prime, $n \in \mathbb{N}_{\geq 1}$)

$$\chi_n : G_{\mathbb{Q}} \longrightarrow Gal(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times \quad (= GL_1(\mathbb{Z}/p^n\mathbb{Z}))$$

This representations are compatible with the maps $(\mathbb{Z}/p^m\mathbb{Z})^\times \to (\mathbb{Z}/p^n\mathbb{Z})^\times$ for $m \geq n$.

$$\chi_m \text{ modulo } p^n = \chi_n.$$

We can take $\underbrace{\varprojlim_n \chi_n}_{=: \chi_{cyc}} : G_{\mathbb{Q}} \longrightarrow \varprojlim_n Gal(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) = \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})^\times = \mathbb{Z}_p^\times$

Write $\mathbb{Q}(\zeta_{p^\infty}) = \bigcup_{n \geq 1} \mathbb{Q}(\zeta_{p^n})$

We call χ_{cyc} the p-adic cyclotomic character. χ_{cyc} factors through

$G_{\mathbb{Q}, p^\infty} \longrightarrow \mathbb{Z}_p^\times$. It also factors through $G_{\mathbb{Q}}^{ab} \longrightarrow \mathbb{Z}_p^\times$.

Theorem: (Kronecker-Weber) The product of all cyclotomic characters gives an isomorphism

$$G_{\mathbb{Q}}^{ab} \xrightarrow{\sim} \prod_p \mathbb{Z}_p^\times.$$

Look at "deformation functors".

$*$ $\mathcal{C}$ category, $R \in \mathcal{C}$, then $\underbrace{\mathrm{Hom}_{\mathcal{C}}(R, \cdot)}^{=: h_R} : \mathcal{C} \to \underline{\mathrm{Set}}$ is the functor

$A \mapsto \mathrm{Hom}_{\mathcal{C}}(R, A)$, $f \in \mathrm{Mor}_{\mathcal{C}}(A, B) \mapsto \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(R, A) & \to & \mathrm{Hom}_{\mathcal{C}}(R, B) \\ g & \mapsto & f \circ g \end{array} \right.$

We will work with some categories of rings.

Fix a field k. We denote by $\mathcal{C}_k$ the category whose objects are Artinian, local rings with residue field k and morphisms are local ring morphisms, that induce the <u>identity</u> on k.

Examples. * $k = \mathbb{F}_p$, then $\mathbb{Z}/p^n\mathbb{Z} \in \mathcal{C}_{\mathbb{F}_p}$ $\forall n \in \mathbb{N}_{>0}$.

$$\begin{array}{c} \mathbb{Z}/p^n\mathbb{Z} \in \mathcal{C}_{\mathbb{F}_p} \\ \quad\ \ \ \rotatebox{90}{$\in$} \\ \mathbb{F}_p[T]/T^n \end{array}$$

* $\exists$ unique degree n unramified extension of $\mathbb{Q}_p$, we will denote it by $\mathbb{Q}_{p^n}$. We write $\mathbb{Z}_{p^n}$ for its valuation ring, then $\mathbb{Z}_{p^n}/p\mathbb{Z}_{p^n} = \mathbb{F}_{p^n}$

$\forall m \in \mathbb{N}_{>0}$: $\mathbb{Z}_{p^n}/p^m\mathbb{Z}_{p^n} \in \mathcal{C}_{\mathbb{F}_{p^m}}$

$$\mathbb{Z}_{p^n}/p^m\mathbb{Z}_{p^n} \longrightarrow \mathbb{Z}_{p^n}/p^m\mathbb{Z}_{p^n}$$

This is not a morphism in $\mathcal{C}_{\mathbb{F}_{p^m}}$ if $m \geq 2$.

$$x \text{ modulo } p^m \longmapsto \mathrm{Frob}_p(x) \text{ modulo } p^m$$

Let $\hat{\mathcal{C}}_k$ be the category whose objects are complete local Noetherian rings with residue field k, morphisms are local ring morphisms that induce the identity on k.

Example. $\mathbb{Z}_{p^n} \in \hat{\mathcal{C}}_{\mathbb{F}_{p^n}}$, $\mathbb{F}_{p^n}[[T]] \in \hat{\mathcal{C}}_{\mathbb{F}_{p^n}}$ | An object of $\mathcal{C}_k$ is also an object in $\hat{\mathcal{C}}_k$ (same for morphisms)

DEFORMATION FUNCTORS

Fix $n \geq 1$.

Let G be a profinite group, k a finite field. Fix a continuous representation $\bar{\rho}: G \to GL_n(k)$.

Def. For $A \in \hat{\mathcal{C}}_k$, a deformation (of $\bar{\rho}$ to A) is a continuous representation $\rho: G \to GL_n(A)$ such that ρ modulo $m_A = \bar{\rho}$.

We say that $\rho_1, \rho_2 : G \to GL_n(A)$ are strictly equivalent iff. $\exists M \in \ker(GL_n(A) \to GL_n(k))$ such that $M^{-1}\rho_1 M = \rho_2$.

Remark: if $A \xrightarrow{f} B$ is a morphism in $\hat{\mathcal{C}}_k$ and $\rho_1, \rho_2 : G \to GL_n(A)$ are strictly equivalent representations, then $f\rho_1, f\rho_2 : G \to GL_n(B)$ are strictly equivalent.

We define $\hat{D}_{\bar\rho} : \hat{\mathcal{C}}_k \longrightarrow \underline{Set}$ as the functor

* $\hat{D}_{\bar\rho}(A) := \left\{ \begin{matrix} \text{deformations of} \\ \bar\rho \text{ to } A \end{matrix} \right\} \Big/ \text{strict equivalence}$

* $\hat{D}_{\bar\rho}(f)$ maps a deformation $\rho : G \to GL_n(A)$ to the class of $f \circ \rho$
$f : A \to B$ morphism in $\hat{\mathcal{C}}_k$

(We obtain a functor $D_{\bar\rho} : \mathcal{C}_k \longrightarrow \underline{Set}$ by restricting $\hat{D}_{\bar\rho}$ to $\mathcal{C}_k$)

Goal: show that $D_{\bar\rho}$ is "pro-represented" by some $R \in \hat{\mathcal{C}}_k$, in the sense that $D_{\bar\rho} \cong \mathrm{Hom}_{\hat{\mathcal{C}}_k}(R, \cdot)$

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Fix any $\Lambda \in \hat{\mathcal{C}}_k$. We define $\mathcal{C}_\Lambda$ as the category of local Artinian Λ-algebras with residue field k and local morphisms of Λ-algebras (that induce the identity on k).

* $\hat{\mathcal{C}}_\Lambda$ = category of complete local Noetherian Λ-algebras with residue field k. Morphisms as above.

The natural choice for Λ (when working in char $\Lambda = 0$) is the ring of Witt vectors $W(k)$ of k.
(= unique complete discrete valuation ring with residue field k and uniformizer p).
(Mézard's notes, Serre "Local fields")

We only need
$$W(\mathbb{F}_{p^n}) = \mathbb{Z}_{p^n} \quad \text{ring of integers in } \mathbb{Q}_{p^n}$$

$$\hat{\mathcal{C}}_{\mathbb{Z}_{p^n}} \ni \mathbb{Z}_{p^n}, \quad \mathbb{F}_{p^n}[[T]] \in \hat{\mathcal{C}}_{\mathbb{F}_{p^n}}, \text{ but } \notin \hat{\mathcal{C}}_{\mathbb{Z}_{p^n}}.$$

<u>Deformation functors</u> Fix $\bar\rho : G \to GL_n(k)$; We defined $D_{\bar\rho} : \mathcal{C}_k \longrightarrow \underline{Set}$, $\hat{D}_{\bar\rho} : \hat{\mathcal{C}}_k \to \underline{Set}$.

One defines
$$D_{\bar\rho,\Lambda} : \mathcal{C}_\Lambda \longrightarrow \underline{Set} \quad \text{by} \quad D_{\bar\rho,\Lambda}(A) = \left\{ \begin{matrix} \text{deformations of} \\ \bar\rho \text{ to } A \end{matrix} \right\} \Big/ \text{strict equivalence}$$

$\hat{D}_{\bar\rho,\Lambda} : \hat{\mathcal{C}}_\Lambda \longrightarrow \underline{Set}$ in the same way.

Representable Functors.

Let $\mathcal{C}$ be any category, $F: \mathcal{C} \to \underline{Set}$ any functor.

* F is called representable $:\Leftrightarrow \exists A \in \mathcal{C}: \; F \cong Hom_{\mathcal{C}}(A, \cdot)$ as functors.

For our categories $\mathcal{C}_k, \hat{\mathcal{C}}_k, \ldots$ we define the following:

* $F: \mathcal{C}_\Lambda \to \underline{Set}$ is pro-representable $:\Leftrightarrow \exists R \in \hat{\mathcal{C}}_\Lambda$ and an isomorphism of functors

$$Hom_{\hat{\mathcal{C}}}(R, \cdot) \longrightarrow F$$

(as a functor $\mathcal{C}_\Lambda \to \underline{Set}$)

Why representability?

Assume $D_{\bar{\rho},\Lambda}$ is (pro-)representable by some R in $\hat{\mathcal{C}}_\Lambda$: means that there is a bijection $Hom_{\hat{\mathcal{C}}_\Lambda}(R, A) \longrightarrow D_{\bar{\rho},\Lambda}(A)$ for every $A \in \mathcal{C}_\Lambda$.

Take $A = R$. Then there is a strict equivalence class ρ_R in $D_{\bar{\rho},\Lambda}(R)$ corresponding to id_R.

Then the bijection

$$\begin{aligned} Hom_{\hat{\mathcal{C}}_\Lambda}(R, A) &\longrightarrow D_{\bar{\rho},\Lambda}(A) \\ (f: R \to A) &\longmapsto \text{class of } f \circ \rho_R : G \to GL_n(A) \end{aligned}$$

We call (R, ρ_R) a universal couple, R is the universal deformation ring of $\bar{\rho}$, ρ_R is the universal deformation of $\bar{\rho}$.

Properties of (pro-) representable functors $\mathcal{C}_\Lambda \xrightarrow{F} \underline{Set}$. In this section F is a pro-representable-Functor $\mathcal{C}_\Lambda \to \underline{Set}$, $F \cong Hom_{\hat{\mathcal{C}}_\Lambda}(R, \cdot)$

- left exact
- $F(k) = \{*\}$, since the only element of $Hom_{\hat{\mathcal{C}}_\Lambda}(R, k)$ is $R \to R/m_R$ $(\cong k)$ one $\underline{\underline{\text{fixed}}}$ identification
- ~~F is continuous~~ (later)
- F behaves well with respect to fiber products

$k[\varepsilon] = k[T]/T^2$, $\varepsilon = \bar{T}$ (ring with $\varepsilon^2 = 0$)

$$k[\varepsilon] = \{ a + b\varepsilon \mid a, b \in k \}, \ \varepsilon^2 = 0$$

- $F(k[\varepsilon])$ is a finite dimensional k-vector space.

Remark: fiber products don't always exist in $\hat{\mathcal{C}}_\Lambda$ $\Lambda = W(k)$

Example $A = k[\![X,Y]\!]$, $C = k[\![X]\!]$, $B = k$

maps

$$\begin{array}{c} A \longrightarrow C \\ Y \longmapsto 0 \end{array}, \quad \begin{array}{c} B \longrightarrow C \\ k \hookrightarrow k[\![X]\!] \end{array}$$

Check: the fiber product $A \times_C B$ is not in the category $\hat{\mathcal{C}}_\Lambda$ (the fiber product as rings is not Noetherian ... this argument should not work)

Fiber products exist in $\mathcal{C}_\Lambda$.

F has the Mayer-Vietoris property if the map $F(A \times_C B) \longrightarrow F(A) \times_{F(C)} F(B)$ is a bijection.

Remark: F pro-representable $\Rightarrow$ F has the Mayer-Vietoris-property.

- $F(k[\epsilon])$ is a k-vector space:

Why is it a k-vector space? It is when the map $F(k[\epsilon] \times_k k[\epsilon]) \hookrightarrow F(k[\epsilon]) \times_{F(k)} F(k[\epsilon])$ is a bijection. (induced by $\begin{array}{c} k[\epsilon] \to k \leftarrow k[\epsilon] \\ \epsilon \longmapsto 0 \longleftarrow \epsilon \end{array}$)

k-scalar multiplication is induced by

$$k \times k[\epsilon] \longrightarrow k[\epsilon]$$
$$(\lambda, a + b\epsilon) \longmapsto (a + \lambda b \epsilon)$$

addition is induced by

$$k[\epsilon] \times_k k[\epsilon] \longrightarrow k[\epsilon]$$
$$(a + b_1\epsilon, a + b_2\epsilon) \longmapsto a + (b_1 + b_2)\epsilon$$

In this case we call $F(k[\epsilon])$ the tangent space of F. Why?

When F is pro-representable, $F \cong \mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R, \cdot)$ then there is an isomorphism of k-vector spaces

$$t_R \longrightarrow F(k[\epsilon])$$

Recall: for R a local Noetherian Λ-algebra with maximal ideal m_R, then $t_R^* = m_R/(m_R^2, m_\Lambda)$ is the cotangent space of R, $t_R = \mathrm{Hom}_k(t_R^*, k)$.

Idea of proof: $f \in F(k[\epsilon]) = \mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R, k[\epsilon])$

then $\underset{\text{for } r\in R}{f(r)} = \underset{\substack{\text{image} \\ \text{mod } m_R}}{\bar{r}} + f'(r)\epsilon$

Criteria for representability:

Criterion. (Grothendieck) Let $F: \mathcal{C}_\Lambda \longrightarrow \underline{\mathrm{Set}}$ be a functor such that $F(k) = \{*\}$. Then F is pro-representable if and only if the following hold:

i) F has the Mayer-Vietoris property

ii) $F(k[\epsilon])$ is a finite dimensional vector space $/k$.

We will use a refined version:

Criterion. (Schlessinger)

Def. If $R, S \in \hat{\mathcal{C}}_\Lambda$, $f: R \to S$ a morphism. We say that f is small if it is surjective and if $\ker f$ is annihilated by m_R ($m_R \cdot \ker f = 0$) $\left(\text{Exp.: } \begin{array}{c} k[\epsilon] \to k \\ \epsilon \mapsto 0 \end{array}\right)$

We introduce the Schlessinger conditions. $R_0, R_1, R_2 \in \mathcal{C}_\Lambda$, $\varphi_1: R_1 \to R_0$, $\varphi_2: R_2 \to R_0$,

$R_3 := R_1 \times_{R_0} R_2$. ; $(*)$: $F(R_3) \longrightarrow F(R_1) \times_{F(R_0)} F(R_2)$

The conditions are:

H1: If $\varphi_2 : R_2 \to R_0$ is small, then $(*)$ is surjective

H2: If $R_0 = k$, $R_2 = k[\epsilon]$ with $\begin{matrix} R_2 \to R_0 \\ \epsilon \mapsto 0 \end{matrix}$, then $(*)$ is bijective

H3: $F(k[\epsilon])$ is finite-dimensional over k.

H4: If $R_1 = R_2$, $\varphi_1 = \varphi_2$ is small, then $(*)$ is bijective.

Theorem of Schlessinger: If $F(k) = \{*\}$ and it satisfies $H_1 - H_4$, then F is pro-representable.

09.05.18

Continuity of Functors

$F: \hat{\mathcal{C}}_\Lambda \to \underline{\text{Set}}$ functor. We say that F is continuous if the natural map

$$F(A) \longrightarrow \varprojlim_n F(A/m_A^n) \quad \text{is a bijection } \forall A \in \hat{\mathcal{C}}_\Lambda$$

Exercise:
* F representable $\Rightarrow$ F continuous
* $\hat{D}_{\bar{\rho},\Lambda}$ is continuous
* If we extend $D_{\bar{\rho},\Lambda}$ by continuity ($D_{\bar{\rho},\Lambda}(A) := \varprojlim_n D_{\bar{\rho},A}(A/m_A^n)$ for $A \in \hat{\mathcal{C}}_\Lambda$), we get $\hat{D}_{\bar{\rho},\Lambda}$
* If $D_{\bar{\rho},\Lambda}$ is pro-represented by some $R \in \hat{\mathcal{C}}_\Lambda$, then $\hat{D}_{\bar{\rho},\Lambda} \cong \mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R, \cdot)$

Existence of Universal Deformation Pairs

G, k, $\Lambda \in \hat{\mathcal{C}}_k$, fix $\bar{\rho}: G \to GL_n(k)$

We want to show $D_{\bar{\rho},\Lambda}$ is pro-representable by proving it satisfies $H_1 - H_4$ of Schlessingers Criterion.

Theorem. Assume G satisfies the p-finiteness condition

(Φ_p) $\mathrm{Hom}_{cont}(H, \mathbb{Z}/p\mathbb{Z})$ is a finite set $\forall$ open subgroups $H \leq G$

Then $D_{\bar\rho,\Lambda} : \mathcal{C}_\Lambda \to \underline{Set}$ satisfies H1, H2, H3. of Schlessinger's criterion.

Let $C_k(\bar\rho) = \mathrm{Hom}_{\bar\rho}(k^n, k^n) \left(= \left\{ \begin{array}{l} M \in M^{n\times n}(k) \text{ s.t.} \\ \bar\rho \cdot M = M \cdot \bar\rho \end{array} \right\} \right)$

If $C_k(\bar\rho) = k$ $\left(\begin{array}{c}\text{scalar} \\ \text{matrices}\end{array}\right)$ then $D_{\bar\rho,\Lambda}$ also satisfies H4.

(Mazur proved Theorem for $\bar\rho$ abs. irreducible, Ramakrishna proved for $C_k(\bar\rho) = k$)

Corollary. If G satisfies p-finiteness and $C_k(\bar\rho) = k$ then $D_{\bar\rho,\Lambda}$ is pro-representable:

there exists a couple (R^{univ}, ρ^{univ}) with the following universal property:

$R^{univ} \in \hat{\mathcal{C}}_\Lambda$, $\rho^{univ} : G \to GL_n(R^{univ})$

$\forall A \in \hat{\mathcal{C}}_\Lambda$ and every deformation $\rho : G \to GL_n(A)$ of $\bar\rho$, there exists an unique Λ-algebra morphism $f_\rho : R^{univ} \to A$ s.t. $\rho \underset{\substack{\text{str.}\\ \text{equiv.}}}{\cong} f_\rho \circ \rho^{univ}$

Remarks: 1) When is $C_k(\bar\rho) = k$? Schur's. Lemma $\Rightarrow$ condition holds for $\bar\rho$ abs. irreducible

Take $\bar\rho(g) = \begin{pmatrix} \chi_1(g) & * \\ 0 & \chi_2(g) \end{pmatrix}$, χ_1, χ_2 distinct characters $G \to k^\times$, $* \neq 0$ $\underbrace{\quad}_{\begin{array}{c}\bar\rho : G \to GL_n(k) \\ GL_n(k) \text{ irred.}\end{array}}$

Then $C_k(\bar\rho) = k$.

2) What to do when $C_k(\bar\rho) \neq k$?

* Look at "versal" deformations
* Look at "framed" deformations

Versal deformations:

Def. Let $F, G : \mathcal{C}_\Lambda \to \underline{Set}$ be functors s.t. $F(k) = \{*\}$, $G(k) = \{*\}$.

A natural transformation $\eta : F \to G$ is formally smooth $:\iff \forall A, B \in \mathcal{C}_\Lambda$

$\forall$ surjections $A \to B$ the natural map $F(A) \to F(B) \times_{G(B)} G(A)$ is surjective.

$$\left(\begin{tikzcd} & F(A) \arrow[dl] \arrow[dr, "\eta_A"] & \\ F(B) \arrow[dr, "\eta_B"] & & G(A) \arrow[dl] \\ & G(B) & \end{tikzcd}\right)$$

Proposition.
If F, G are pro-representable,
$F = \mathrm{Hom}(R_F, \cdot)$, $G = \mathrm{Hom}(R_G, \cdot)$ for some $R_F, R_G \in \hat{\mathcal{C}}_\Lambda$

then:
a transformation $F \to G$ is formally smooth

$\iff$ it is induced by a morphism $R_G \to R_F$ that makes R_F a ring of formal power series over R_G.

$$\left(R_F = R[\![T_1, \ldots, T_n]\!] \right)$$

Proved in Mézard's notes, in Schlessinger's paper (Lemma 2.5) ■

Def. We say that a functor $F : \mathcal{C}_\Lambda \to \underline{Set}$ admits a versal deformation if there exists a couple $(\underset{\hat{\mathcal{C}}_\Lambda}{\overset{\cap}{R}}, \underset{F(R)}{\overset{\cap}{\rho}})$ such that the morphism of functors

$$\mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R, A) \to F(A) \quad \text{defined for } A \in \hat{\mathcal{C}}_\Lambda \quad (f : R \to A) \mapsto F(f)(\rho)$$

is formally smooth.

Theorem. (Schlessinger) Assume G satisfies ϕ_p. Then $D_{\bar{\rho}, \Lambda}$ admits a versal deformation (R^{ver}, ρ^{ver}).
This means that for $\forall A \in \mathcal{C}_\Lambda$ and $\forall$ deformations ρ of $\bar{\rho}$ to A there exists $f_\rho \in \mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R^{ver}, A)$ s.t. $f_\rho \circ \rho^{ver} \underset{str.}{\cong} \rho$, but f_ρ is not uniquely determined.

Framed deformations:

Define a framed deformation functor $D^{\square}_{\bar{\rho},\Lambda} : \mathcal{C}_\Lambda \to \underline{\mathrm{Set}}$ as

$$D^{\square}_{\bar{\rho},\Lambda}(A) = \left\{ \begin{array}{l} \text{set of deformations} \\ \text{of } \bar{\rho} \text{ to } \rho : G \to GL_n(A) \end{array} \right\}$$

Deformations as actions on A-modules

See $\bar{\rho} : G \longrightarrow GL_n(k)$ as datum of:

* a continuous action of G on a n-dimensional k-vector space $V_{\bar{\rho}}$
* a choice of basis β for $V_{\bar{\rho}}$

We call a deformation of $\bar{\rho}$ the datum of
(framed deformation)

(1) a continuous action of G on a free A-module V_A of rank n

such that $V_A \otimes_A k \underset{G\text{-modules}}{\cong} V_{\bar{\rho}}$

(2) a choice of a lift of the basis β of $V_{\bar{\rho}}$ to an A-basis of V_A

(same as choosing a lift of $\bar{\rho}$ to $\rho : G \to GL_n(A)$)

$D_{\bar{\rho},\Lambda}(A) =$ set of choices of (1),

$D^{\square}_{\bar{\rho},\Lambda}(A) =$ set of choices of (1) and (2).

Theorem. Assume G has property Φ_p. Then the framed deformation functor $D^{\square}_{\bar{\rho},\Lambda}$ is pro-representable. (No need of Schlessinger)

Proposition. The transformation of functors $D^{\square}_{\bar{\rho},\Lambda} \longrightarrow D_{\bar{\rho},\Lambda}$ defined by

$$(\rho : G \to GL_n(A)) \longmapsto [\rho]_{\text{str. equivalence}}$$

is formally smooth.

Proof. exercise. $D^{\square}_{\bar\rho,\Lambda}(A) \longrightarrow D^{\square}_{\bar\rho,\Lambda}(B) \times_{D_{\bar\rho,\Lambda}(B)} D_{\bar\rho,\Lambda}(A)$ (surjection $A \to B$) $\blacksquare$

Corollary. if $\mathcal{Z}_k(\bar\rho) = k$ ($\rightarrow D^{\square}_{\bar\rho,\Lambda}$ and $D_{\bar\rho,\Lambda}$ are represented as $\mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R^{\square}_{univ}, \cdot)$, $\mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R_{univ}, \cdot)$), then $R^{\square}_{univ} = R_{univ}[\![T_1, \dots, T_n]\!]$.

Proof of pro-representability of $D^{\square}_{\bar\rho,\Lambda}$.

We look for an object $R \in \hat{\mathcal{C}}_\Lambda$ such that $D_{\bar\rho,\Lambda}(A) = \mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R, A)$ $\forall A \in \hat{\mathcal{C}}_\Lambda$.

1) We prove the result for G finite. (fix $\rho: G \to GL_n(k)$)

Define Λ-algebra $\Lambda[G,n]$ by giving

* generators: X^g_{ij}, $g \in G$, $i,j \in \{1,\dots,n\}$
* relations: $X^{gh}_{ij} = \sum_{l=1}^{n} X^g_{il} X^h_{lj}$ $\forall g,h \in G$ $\forall i,j \in \{1,\dots,n\}$ ($e \in G$ unity)

$$X^e_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Then there is a bijection, for every Λ-algebra A

$$b_A: \mathrm{Hom}_\Lambda(\Lambda[G,n], A) \longrightarrow \mathrm{Hom}_{\underline{Grp}}(G, GL_n(A))$$
$$f \longmapsto (g \mapsto (f(X^g_{ij}))_{i,j})$$

<u>Check</u>: this is a bijection. $(\Lambda[-,n] \dashv GL_n)$

Consider $\bar\rho \in \mathrm{Hom}_{\underline{Grp}}(G, GL_n(k))$, then $b_k^{-1}(\bar\rho) =: f_{\bar\rho}$ is a Λ-algebra-morphism $\Lambda[G,n] \to k$. Set $m_{\bar\rho} = \ker f_{\bar\rho}$.

Set $R :=$ $m_{\bar\rho}$-adic completion of $\Lambda[G,n]$ $= \varprojlim_n \Lambda[G,n]/m^n_{\bar\rho}$

Show that R has the universal property: take a deformation $\rho: G \to GL_n(A)$ of $\bar\rho$ to some $A \in \hat{\mathcal{C}}_\Lambda$. Then $f_\rho := b_A^{-1}(\rho): \Lambda[G,n] \to A$, then $f_\rho(m_{\bar\rho}) \subseteq m_A$.

In particular f_ρ can be extended by continuity to $f_\rho : R \to A$. Then one checks

$$\begin{tikzcd} G \arrow[r, "\rho^{univ}"] \arrow[dr, "\rho"'] & GL_n(R) \arrow[d, phantom, "\circlearrowleft"] \arrow[dl, "f_\rho"] \\ & GL_n(A) \end{tikzcd}$$

Define ρ^{univ} as the homomorphism $G \to GL_n(R)$ attached to $\Lambda[G,n] \to R$ by b_R

Also, f_ρ with this property is unique.

$\Rightarrow$ (R, ρ^{univ}) is an universal deformation couple for $\bar\rho$

2) G profinite, then $G = \varprojlim\limits_{\substack{H \trianglelefteq G \\ G/H \text{ finite} \\ H \subseteq \ker \bar\rho}} G/H$

(are exactly the H's such that $\bar\rho$ factors through $\bar\rho_H : G/H \to GL_n(k)$)

For each such H there is a representing pair $(R_{\bar\rho_H}, \bar\rho_H)$ for the functor $D^{\square}_{\bar\rho_H,\Lambda}$. The set $\{R_{\bar\rho_H}\}_H$ is a projective system. and

$$R_{\bar\rho} := \varprojlim_H R_{\bar\rho_H} \text{ is in the category } \hat{\mathcal{C}}_\Lambda. \quad \text{(uses } \Phi_p\text{-assumption on } G!)$$

Then $R_{\bar\rho}$ pro-represents $D^{\square}_{\bar\rho,\Lambda}$:

$$\begin{aligned} D^{\square}_{\bar\rho,\Lambda}(A) &= \varprojlim_H D^{\square}_{\bar\rho_H,\Lambda}(A) = \varprojlim_H \operatorname{Hom}(R_{\bar\rho_H}, A) \\ &\cong \operatorname{Hom}(\varprojlim_H R_{\bar\rho_H}, A) \\ &= \operatorname{Hom}(R_{\bar\rho}, A). \end{aligned}$$

galrep
16.05.18

Representability of $D_{\bar\rho}$ when $C_k(\bar\rho) = k$. Need to check H1 – H4 in Schlessinger's criterion.

Objects $R_0, R_1, R_2 \in \hat{\mathcal{C}}_\Lambda$ with morphisms $\begin{matrix} R_1 \to R_0 \\ R_2 \to R_0 \end{matrix}$; set $R_3 := R_1 \times_{R_0} R_2$

$$b : D_{\bar\rho}(R_3) \longrightarrow D_{\bar\rho}(R_1) \times_{D_{\bar\rho}(R_0)} D_{\bar\rho}(R_2)$$

$E_i :=$ set of all deformations $\rho : G \to GL_n(R_i)$ of $\bar\rho$.

$$D_{\bar\rho}(R_i) = E_i /_{\text{strict equiv.}} = E_i / \Gamma_n(R_i) \qquad \left(\Gamma_n(R_i) := \ker(GL_n(R) \to GL_n(k)) \right)$$

Then write

$$b : E_3/\Gamma_n(R_3) \longrightarrow E_1/\Gamma_n(R_1) \times_{E_0/\Gamma_n(R_0)} E_2/\Gamma_n(R_2)$$

<u>H1</u>: if $R_2 \underset{\text{small}}{\longrightarrow} R_0 \Rightarrow b$ surjective

Proof. Take ρ_1, ρ_2 representatives of classes in $E_1/\Gamma_n(R_1)$, $E_2/\Gamma_n(R_2)$. Assume that the images of ρ_1 and ρ_2 via $\begin{matrix} R_1 \searrow \\ R_2 \nearrow \end{matrix} R_0$ are in the same class.

$\downarrow$

$\rho_{10} : G \to GL_n(R_0)$ Then $\exists\, M_0 \in \Gamma_n(R_0)$ s.t. $M_0^{-1} \rho_{20} M_0 = \rho_{10}$.

$\rho_{20} : G \to GL_n(R_0)$

Since $R_2 \to R_0$ is surjective, $\Gamma_n(R_2) \to \Gamma_n(R_0)$ is surjective. So we can choose a lift M_2 of M_0 in $\Gamma_n(R_2)$. Then the representations $M_2^{-1}\rho_2 M_2$ and ρ_1 give the same representations in $GL_n(R_0)$. They glue to a representation $\rho_3 : G \to GL_n(R_3)$

$$(\rho_3(\cdot) = (\rho_1(\cdot), M_2^{-1}\rho_2(\cdot)M_2) \in GL_n(R_1 \times_{R_0} R_2)$$

∎

<u>H2</u>, <u>H4</u>: similar strategies (see Gouvea)

<u>H3</u>: $D_{\bar\rho}(\underbrace{k[\varepsilon]}_{= k[x]/x^2})$ is a finite-dimensional k-vector space.

Proof. $G_0 = \ker\bar\rho$ ($\leadsto (G:G_0) < \infty$) ; $R \in \hat{\mathcal{C}}_\Lambda$ and $\rho : G \to GL_n(R)$ a lift of $\bar\rho$.

Then $\rho|_{G_0}$ takes values in $\Gamma_n(R)$. This is a pro-p group.

Take $R = k[\varepsilon]$. Then $\Gamma_n(k[\varepsilon]) = 1 + \underbrace{\varepsilon M_n(k)}_{\text{p-elementary finite group}}$

A lift $\rho : G \to GL_n(k[\varepsilon])$ of $\bar{\rho}$ defines a homomorphism

$$G_0 \to \Gamma_n(k[\varepsilon]), \qquad (\rho|_{G_0} = \rho'|_{G_0} \iff \rho \sim \rho' \text{ strictly equiv.})$$

but $\Gamma_n(k[\varepsilon])$ is a finite sum of $\mathbb{Z}/p\mathbb{Z}$'s $\quad (:\iff$ p-elementary finite$)$

By p-finiteness $\quad \mathrm{Hom}_{cont}(G_0, \Gamma_n(k[\varepsilon]))$ is finite.

By Schlessinger, $\exists$ universal deformation couple $(R_{\bar{\rho}}^{univ}, \rho^{univ})$ for $\bar{\rho}$.

Proof of Schlessinger's criterion gives a surjection $\Lambda[\![T_1, \dots, T_d]\!] \to R_{\bar{\rho}}^{univ}$

By looking at tangent spaces, one gets, for the minimal d: $d = \dim_k t_{R_{\bar{\rho}}^{univ}}$

$$= \dim_k D_{\bar{\rho}}(k[\varepsilon])$$

* Can we give d in terms of $\bar{\rho}$?
* $R_{\bar{\rho}}^{univ} = \Lambda[\![T_1, \dots, T_d]\!]/I$, we will describe $\dim R_{\bar{\rho}}^{univ}$ in terms of $\bar{\rho}$.

Explicit case: $n = 1$.

Fix $\bar{\rho} : G \to k^\times = GL_1(k)$. We describe the universal deformation couple for $\bar{\rho}$. There is a lift $\rho_0 : G \to W(k)^\times$ given by $\quad (W(\mathbb{F}_{p^n}) = \mathbb{Z}_{p^n})$
the Teichmüller lift $k^\times \to W(k)^\times$.

Take another lift $\rho : G \to A^\times$ for $A \in \hat{\mathcal{C}}_\Lambda$. (one has a map $W(k)^\times \to A^\times$)

Then $\rho_0^{-1} \cdot \rho : G \to A^\times$ takes values in $1 + m_A$.
$\rho_0^{-1}(g) = \rho_0(g)^{-1}$. $\quad \cdot$ not $\circ$

$1 + m_A$ is a pro-p-group, so $\rho_0^{-1} \circ \rho$ factors through the pro-p-completion Γ of G.

$$\Big(\Gamma := \varprojlim_{H \text{ quotient of } G \text{ that is finite p-Group}} H \Big)$$

$$\begin{tikzcd} G \arrow[r, "\rho_0^{-1}\rho"] \arrow[d, "\gamma"'] & 1+m_A \\ \Gamma \arrow[ur, "\exists f_\rho"'] & \end{tikzcd}$$

$\exists f_\rho$ continuous group hom.

By continuity f_ρ can be extended to

$$f_\rho : \Lambda[\![\Gamma]\!] \longrightarrow A \quad , \Lambda\text{-alg. hom.}$$

$$\Lambda[\![\Gamma]\!] := \varprojlim_{\Gamma' \text{ open normal subgrp. of } \Gamma} \underbrace{\Lambda[\Gamma/\Gamma']}_{\text{group algebra}}$$

Remark. * $\Lambda[\![\Gamma]\!] \in \hat{\mathcal{C}}_\Lambda$.

One checks that

G satisfies p-finiteness $\Rightarrow$ Γ is top. f.g. by $x_1, \dots, x_k$ $\Rightarrow$ $\Lambda[\![\Gamma]\!]$ is a quotient of $\Lambda[\![x_1, \dots, x_n]\!]$

Proposition. The universal deformation couple for $\bar{\rho}$ is

$$(\Lambda[\![\Gamma]\!], \rho^{univ}), \text{ where } \rho^{univ} = \rho_0 \cdot [\gamma]$$

where $[\gamma]$ is the composition $G \to \Gamma \to \Lambda[\![\Gamma]\!]^\times$

Proof. We already defined a Λ-algebra morphism $f_\rho : \Lambda[\![\Gamma]\!] \to A$ (for any deformation $\rho : G \to A^\times$)

We check: for $g \in G$

$$f_\rho \circ \rho^{univ}(g) = f_\rho(\rho_0(g)[\gamma(g)]) = \rho_0(g)\, f_\rho([\gamma(g)]) = \underbrace{\rho_0(g)\rho_0^{-1}(g)}_{=1}\rho(g)$$
$$= \rho(g).$$

Check that f_ρ is unique.

$\blacksquare$

Remark. * $R^{univ}_{\bar{\rho}}$ is independent of $\bar{\rho}$.

Take any $\bar{\rho} : G \to GL_n(k)$ (with $C_k(\bar{\rho}) = k$). Then there is a universal def. couple $(R^{univ}_{\bar{\rho}}, \rho^{univ})$. Look at $\det \circ \rho^{univ} : G \to R^{univ,\times}_{\bar{\rho}}$. It is a deformation of $\det \circ \bar{\rho}$. $\to$ There exists a Λ-alg. morphism $\Lambda[\![\Gamma]\!] \longrightarrow R^{univ}_{\bar{\rho}}$

inducing $\det \circ \rho^{univ}$

It is a special case of "functoriality".

* Take any algebraic morphism $GL_m \xrightarrow{\delta} GL_n$. (of algebraic groups)

Take $\bar{\rho}_m : G \to GL_m(k)$, then we get $\bar{\rho}_n : G \to GL_n(k) := \delta_k \circ \bar{\rho}_m$.

Assume $\bar{\rho}_m, \bar{\rho}_n$ are absolutely irreducible. Then $\delta_{R^{univ}_{\bar{\rho}_n}} \circ \rho_n^{univ}$ is a deformation of $\bar{\rho}_m$. By the universal property we get $R^{univ}_{\bar{\rho}_m} \xrightarrow{r(\delta)} R^{univ}_{\bar{\rho}_n}$ s.t.

$$r(\delta) \circ \rho_m^{univ} = \rho_n^{univ}$$

* $m = n$, $GL_{n/\Lambda} \xrightarrow{\delta} GL_{n/\Lambda}$ given by conjugation with fixed $g \in GL_n(\Lambda)$.

Take $\bar{\rho}_m : G \to GL_m(k)$, then $\exists$ a morphism $R_{\bar{\rho}_m \otimes \bar{\rho}_n} \to R_{\bar{\rho}_m} \hat{\otimes}_\Lambda R_{\bar{\rho}_n}$
$\bar{\rho}_n : G \to GL_n(k)$

completed tensor product

<u>Dimension of the deformation rings</u>

$$\Lambda[[T_1, \dots, T_d]]/I \cong R^{univ}_{\bar{\rho}}$$

<u>Reminder</u>: Group cohomology

* G pro-finite group, M topological abelian group with a continuous action of G.
$(G, \cdot)$ $(M, +)$

Can define a functor $\underset{\in \underline{Ab}}{\underline{M}} \mapsto M^G = \{m \in M \mid gm = m \ \forall g \in G\}$

$H^0(G,M) = M^G$, $H^i(G,M) \rightsquigarrow$ Right-derived functor of $M \mapsto M^G$.

We will compute cohomology with the following complex.

$$M \to K^1 \xrightarrow{d^1} K^2 \xrightarrow{d^2} \dots \to K^i \xrightarrow{d^i} \dots$$

with $K^i := \{f : G^i \to M$ continuous functions$\}$ with $d^i f(g_1, \dots, g_{i+1})$

$$= g_1 f(g_2, \dots, g_{i+1}) + \sum_{j=1}^{i} (-1)^j f(g_1, \dots, g_j g_{j+1}, \dots, g_{i+1}) + (-1)^{i+1} f(g_1, \dots, g_i)$$

Then check: $H^i(G,M) = \dfrac{\text{Cochains in } K^i}{\text{Coboundaries in } K^i}$

Explicitly: $H^0(G,M) = M^G$
$Z^1(G,M) = \{f : G \to M \mid f(g_1 g_2) = g_2 f(g_1) + f(g_2)\}$

$B^1(G,M) = \{ f: G \to M \mid \exists\, m \in M \text{ s.t. } f(g) = (g-1)m \ \ \forall g \in G \}$

<u>Cohomological interpretation of the tangent space.</u>

Fix $\bar{\rho}: G \to GL_n(k)$ $\quad (C_k(\bar{\rho}) = k)$. We said that $d = \dim_k D_{\bar{\rho}}(k[\varepsilon])$

$$= \dim_k t_{R^{univ}_{\bar{\rho}}}$$

$$\left(t_{R^{univ}_{\bar{\rho}}} = \mathrm{Hom}_\Lambda (R^{univ}_{\bar{\rho}}, k[\varepsilon]) \right)$$

Start with $\rho: G \to GL_n(k[\varepsilon])$ a deformation of $\bar{\rho}$. Then $\forall g \in G$:

$$\rho(g) = \bar{\rho}(g)(1 + \underset{\underset{M_n(k)}{\cap}}{c_g}\, \varepsilon) \qquad (k \hookrightarrow k[\varepsilon] = k + \varepsilon k)$$

Check: ρ homomorphism

$\Rightarrow$ $c: G \to M_n(k)$, $g \mapsto c_g$ is a 1-cocycle for the adjoint action $G \curvearrowright M_n(k)$ of $\bar{\rho}$.

[Adjoint representation: $G \curvearrowright M_n(k)$
$g \cdot m := \bar{\rho}(g)^{-1} m \bar{\rho}(g)$. Write $\mathrm{Ad}_{\bar{\rho}}$

Proposition: The previous construction gives an isomorphism (of vector spaces)

$$D_{\bar{\rho}}(k[\varepsilon]) \longrightarrow H^1(G, \mathrm{Ad}_{\bar{\rho}})$$

Corollary. $\quad d = \dim_k (H^1(G, \mathrm{Ad}_{\bar{\rho}}))$.

23.05.18

<u>Obstructions</u> $\quad$ Take $R_0, R_1 \in \hat{\mathcal{C}}_\Lambda$ with a morphism of Λ-algebras $R_1 \xrightarrow{\varphi} R_0$ surjective and $\ker\varphi \cdot m_{R_1} = 0$.
$I := \ker\varphi$.

Fix $\bar{\rho}: G \to GL_n(k)$. Consider a deformation $\rho_0: G \to GL_n(R_0)$ of $\bar{\rho}$.

Can we lift ρ_0 to $\rho_1: G \to GL_n(R_1)$?

$$\begin{tikzcd} GL_n(R_1) \arrow[r, "\varphi"] & GL_n(R_0) \\ G \arrow[u, dashed, "\rho_1"] \arrow[ur, "\rho_0"'] & \end{tikzcd}$$

Take σ any map (of sets) $G \xrightarrow{\sigma} GL_n(R_1)$ inducing ρ_0. For $g_1, g_2 \in G$ set

$$c_\sigma(g_1, g_2) = \sigma(g_1 g_2)\, \sigma(g_2)^{-1} \sigma(g_1)^{-1}.$$

We know σ lifts ρ_1 so $c_\sigma(g_1,g_2) \equiv 1 \pmod{I}$ $\rightsquigarrow$ We can write

$$c_\sigma(g_1,g_2) = 1 + \underset{\substack{\cap \\ M_n(I)}}{d_\sigma(g_1,g_2)}$$

<u>Check</u>: $\begin{aligned} G\times G &\to I \\ (g_1,g_2) &\mapsto d_\sigma(g_1,g_2)\end{aligned}$ defines a cocycle in $Z^2(G, \mathrm{Ad}_{\rho_I})$

$I\cdot m_{R_1} = 0 \Rightarrow I$ has a k-vector space structure $(\exists\, k \to I)$

ρ_I = composition of $\bar\rho$ with $k \to I$

Ad_{ρ_I} = G acting on $M_n(I)$ via conjugation with ρ_I.

If σ' is another set-theoretic lift $G \to GL_n(R_1)$ of ρ_0, then $d_\sigma(g_1,g_2)\, d_{\sigma'}(g_1,g_2)^{-1}$ is a coboundary

$\to$ The class $\mathcal{O}(\rho_0)$ of d_σ in $H^2(G, \mathrm{Ad}_{\rho_I})$ only depends on ρ_0.

If $\exists$ a group homomorphism $\sigma: G \to GL_n(R_1)$ lifting ρ_0, then $\mathcal{O}(\rho_0)$ is trivial.

If $\mathcal{O}(\rho_0)$ is trivial, then ρ_0 admits a lift $G \to GL_n(R_1)$ (group homomorph.!)

<u>Conclusion</u>: ρ_0 admits a lift to R_1 $\Leftrightarrow$ $\mathcal{O}(\rho_0) = 0$.

Remark: if $H^2(G, \mathrm{Ad}_{\bar\rho}) = 0$, then $H^2(G, \mathrm{Ad}_{\bar\rho_I}) = 0$

$$H^2(G, \mathrm{Ad}_{\bar\rho_I}) \cong H^2(G, \mathrm{Ad}_{\bar\rho}) \otimes_k I.$$

$\Rightarrow$ for any $R_1 \xrightarrow{\varphi} R_0$ and any ρ_0 there exists a lift of ρ_0 to R_1.

When $H^2(G, \mathrm{Ad}_{\bar\rho}) = 0$ we say that the deformation problem is unobstructed.

Interpretation of obstructions via group extensions.

Same situation: $R_1 \xrightarrow{\varphi} R_0$, $I = \ker\varphi$, $I \cdot m_{R_1} = 0$.

$Ad_{\bar\rho_I}$ is a G-module. In general: Take a G-module A. Then there is a group isomorphism

$$\mathcal{O}_{pext}(G,A) \longrightarrow H^2(G,A)$$

* $\mathcal{O}_{pext}(G,A)$ is the set of A-extensions of G:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[dr] & E \arrow[r] \arrow[d, "h"] & G \arrow[r] & 1 \\ & & E' \arrow[ur] & & \end{tikzcd}$$

exact sequence of groups up to isomorph.

E, E' are isomorphic if $\exists h: E \to E'$ isomorph. making the diagrams commute.

* Baer sum: gives $\mathcal{O}_{pext}(G,A)$ an abelian group structure.

* $E \in \mathcal{O}_{pext}(G,A)$ maps to $0 \in H^2(G,A)$ $\iff$ E is a split extension.

Back to $R_1 \xrightarrow{\varphi} R_0$, $\underbrace{\ker\varphi}_{=:I} \cdot m_{R_1} = 0$. Then Ad_{ρ_I} is a G-module.

Choose $\rho_0 : G \to GL_n(R_0)$.

identify $Ad_{\rho_I} \cong 1 + Ad_{\rho_I}$ $\quad (1+a)(1+b) = 1+a+b+\underbrace{ab}_{\substack{=0 \text{ since} \\ I \in m_{R_1}, \\ I m_{R_1} = 0.}}$

$$\begin{tikzcd} 1 \arrow[r] & Ad_{\rho_I} \arrow[r] & GL_n(R_1) \arrow[r] & GL_n(R_0) \arrow[r] & 1 \\ 1 \arrow[r] & Ad_{\rho_I} \arrow[r] \arrow[u] & G \times_{GL_n(R_0)} GL_n(R_1) \arrow[r] \arrow[u] & G \arrow[r] \arrow[u, "\rho_0"] & 1 \end{tikzcd} \quad (*)$$

This Ad_{ρ_I}-extension of G gives a class $\mathcal{O}(\rho_0) \in H^2(G, Ad_{\rho_I})$

Remark: The extension $(*)$ splits $\iff$ $\exists$ repres. $\rho_1 : G \to GL_n(R_1)$ lifting ρ_0.

Proof. if $(*)$ splits and $s: G \to G \times_{GL_n(R_0)} GL_n(R_1)$ is a section the define

$$\rho_1 := \underbrace{\pi_2}_{\text{projection to } GL_n(R_1)} \circ\, s$$

if ρ_1 exists then $s(g) := (g, \rho_1(g))$ is a section for $(*)$.

We get: ρ_1 exists $\iff$ $\mathcal{O}(\rho_0) = 0$.

Exercise: find something similar for H^1.

Theorem. (Mazur) $d_1 := \dim_k H^1(G, \mathrm{Ad}_{\bar\rho})$, $d_2 := \dim_k H^2(G, \mathrm{Ad}_{\bar\rho})$

Then:

$$\dim_{\mathrm{Krull}} \left(R_{\bar\rho}^{\mathrm{univ}} / m_\Lambda R_{\bar\rho}^{\mathrm{univ}} \right) \geq d_1 - d_2$$

In particular, in the unobstructed case $(d_2 = 0)$, $R_{\bar\rho}^{\mathrm{univ}} \cong \Lambda[[T_1, \dots, T_{d_1}]]$.

Proof. $R := R_{\bar\rho}^{\mathrm{univ}}$

...

$$\Lambda[[T_1, \dots, T_{d_1}]] \longrightarrow R \longrightarrow 0 \qquad \text{modulo } m_\Lambda:$$

$$0 \longrightarrow \underset{\substack{\text{defined} \\ \text{as kernel}}}{J} \longrightarrow \underbrace{k[[T_1, \dots, T_{d_1}]]}_{=: F} \longrightarrow R/m_\Lambda R \longrightarrow 0$$

Goal: prove that J has at most d_2 generators. $m_F :=$ max. ideal of F,

reduce the sequence mod $m_F J$.

$$0 \longrightarrow J/m_F J \longrightarrow F/m_F J \longrightarrow R/m_\Lambda R \longrightarrow 0$$

still an isomorphism on tangent spaces.

Consider ρ' the image of the universal deformation ρ^{univ} modulo $m_\Lambda R$.

$G \longrightarrow GL_n(R/m_\Lambda R)$

$\rightsquigarrow$ By the previous construction, one associates to ρ' a class

$$\mathcal{O}(\rho') \in H^2(G, \mathrm{Ad}_{\bar\rho}) \otimes_k J/m_F J.$$

Show $\exists$ injection $\mathrm{Hom}_k(J/m_F J, k) \hookrightarrow H^2(G, \mathrm{Ad}_{\bar\rho})$ (then we are done).

$$f \longmapsto (1 \otimes f)(\mathcal{O}(\rho'))$$

Prove that it is an injective: Assume $f \neq 0$, consider $\ker f \subseteq J/m_F J$.

Reduce modulo $\ker f$:

$$0 \to (J/m_F J)/_{\ker f} \longrightarrow \underbrace{(F/m_F J)/_{\ker f}}_{=: A} \longrightarrow R/m_\Lambda R \to 0$$

* $A \to R/m_\Lambda R$ is still an isomorphism of tangent spaces.
* the image of $\mathcal{O}(\rho')$ in $H^2(G, \mathrm{Ad}_{\bar\rho}) \otimes (J/m_F J)/_{\ker f}$ is trivial.

$\Rightarrow \rho'$ admits a lift to A

$\Rightarrow$ the exact sequence splits by the univ. prop. of ρ'.

$\Rightarrow \ker f = J/m_F J$ ↯ to $f \neq 0$. ▨

In many cases we have "=": $\dim_{\mathrm{Krull}}(R/m_\Lambda R) = d_1 - d_2$.

<u>Dimension conjecture</u>: "=" always holds.

Mazur calls this "generalized Leopoldt's conjecture".

Statement we use today: take K a number field, S set of places,

$S \supseteq$ p-adic and infinite places, $G_{K,S}$.

$r_1 = \#$ real embeddings of K, $r_2 = \frac{1}{2}\#$ complex embeddings

Leopoldt's conjecture $\Leftrightarrow$ the number of $\mathbb{Z}_p$ extensions of K unramified is

$\hookleftarrow \mathrm{Gal} = \mathbb{Z}_p$

$r_2 + 1 = \#\mathrm{Hom}_{\mathrm{cont}}(G_{K,S}, \mathbb{Z}_p)$

$\Leftrightarrow$ outside S is $r_2 + 1$.

<u>Example</u>: K total real, K has a $\mathbb{Z}_p$-extension (p-adic cyclotomic ext.),

Leopoldt's $\Rightarrow$ it's the only $\mathbb{Z}_p$-extension.

Show: Dimension conjecture $\Rightarrow$ Leopoldt's conjecture

$\Updownarrow$

$\dim(R/m_\Lambda R) = d_1 - d_2$

For every place $v \in S$ of K, write $G_v \subseteq G_{K,S}$ for a decomp. group at v

Proposition. For $\bar\rho : G_{K,S} \to GL_n(k)$, we have $d_1 - d_2 = 1 + dn^2 - \sum_{v \in S_\infty} \dim_k H^0(G_v, Ad_{\bar\rho})$ with $d = [K:\mathbb{Q}]$ (S_∞: infinite places)

"Proof": Tate's Global Euler Char. formular:

take M a finite $G_{K,S}$-Module, then: (assume $\# M$ is an S-unit)

$$\frac{\# H^0(G_{K,S}, M) \cdot \# H^2(G_{K,S}, M)}{\# H^1(G_{K,S}, M)} = \frac{\prod_{v \in S_\infty} \# H^0(G_v, M)}{\# M^d}$$

Apply it to $M = Ad_{\bar\rho}$. (We assumed $C_k(\bar\rho) = k$)

$d_0 = \dim_k H^0(G_{K,S}, Ad_{\bar\rho})$

$\rightsquigarrow$ $\underset{=1}{d_0} + d_2 - d_1 = \sum_{v \in S_\infty} \dim_k H^0(G_v, Ad_{\bar\rho}) - n^2 d$ $\boxtimes$

Apply to special cases:

1) $n = 1$, any $\bar\rho : G_{K,S} \to k^\times$

For every $v \in S^\infty$ $H^0(G_v, Ad_{\bar\rho}) = 1$.

$d_1 - d_2 = 1 + d - \underbrace{\# S_\infty}_{= r_1 + r_2}$ $\quad d = r_1 + 2r_2$

$= 1 + r_2$

What is $d_1 - d_2$? By the dimension conjecture $d_1 - d_2 = \dim R^{univ}_{\bar\rho} / m_\Lambda R^{univ}_{\bar\rho}$

use $R^{univ}_{\bar\rho} = \Lambda[\![\widehat{G}^{(p)}_{K,S}]\!]$ $= \# \mathrm{Hom}_{\widehat{cont}}(G_{K,S}, \mathbb{Z}_p)$

Exercise: we saw that Dimension conjecture ⟹ Leopoldt's conjecture

* $n=2$. $\dim_{\text{Krull}}\left(R/_{m_\Lambda} R \right) \geq d_1 - d_2$

$K = \mathbb{Q}$, $p>2$, char $k = p$ $\qquad v \in S_\infty$

$\bar\rho : G_{\mathbb{Q},S} \longrightarrow GL_2(k)$ $\qquad H^0(G_v, \mathrm{Ad}_{\bar\rho})$

Def.: $\rho : G_{\mathbb{Q},S} \to GL_2(R)$, we say that

* ρ is odd if $\rho(c) \sim \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ for some complex conjugate c (all)

* ρ is even otherwise.

$$\dim \left(R/_{m_\Lambda} R \right) \geq \begin{cases} 3, & \bar\rho \text{ odd} \\ 1, & \bar\rho \text{ even} \end{cases}$$

<u>Explicit description of deformation spaces</u>

Global case and $K = \mathbb{Q}$, any set S of primes of $\mathbb{Q}$. Fix $\bar\rho : G_{\mathbb{Q},S} \to GL_n(k)$

k a finite field ($\Lambda \in \hat{\mathcal{E}}_k$, work in $\hat{\mathcal{E}}_\Lambda$)

Question: describe $R^{\text{univ}}_{\bar\rho}$.

Consider $R \in \hat{\mathcal{E}}_\Lambda$, then $\Gamma_n(R)$ is a pro-p-group

$$\Gamma_n(R) = \ker\left(GL_n(R) \to GL_n(k) \right)$$

Take any deformation $\rho : G_{\mathbb{Q},S} \to GL_n(R)$ of $\bar\rho$; look at $\rho|_{\ker\bar\rho} : \ker\bar\rho \to \Gamma_n(R)$

⟹ $\rho|_{\ker\bar\rho}$ factors through $(\ker\bar\rho)^{(p)}$ (pro-p-completion)

$$\begin{tikzcd}
\mathbb{Q}_S \arrow[d, dash] \\
L \arrow[d, dash] \\
K \arrow[d, dash, "H \cong \mathrm{im}\,\bar\rho"] \\
\mathbb{Q}
\end{tikzcd}$$

$\mathbb{Q}_S$: largest pro-p extension of K unramified outside S

$\ker\bar\rho$ ($\mathbb{Q}_S$ over K), G_0, $(\ker\bar\rho)^{(p)} = P$ (L over K)

S_1 primes of K above S

$$\begin{array}{c} S_1 \\ | \\ S \end{array}$$

$\rho : G_{\mathbb{Q},S} \longrightarrow GL_n(R) \xrightarrow{\text{construct}} \varphi : P \longrightarrow \Gamma_n(R)$ group homomorphism

Study three cases:

* $\bar{\rho}$ <u>tame</u> ($:\Leftrightarrow$ $p \nmid \# \operatorname{im} \bar{\rho}$)

* $n=2$: $\bar{\rho}$ full ($:\Leftrightarrow$ $\operatorname{im}\bar{\rho} \supseteq SL_2(\mathbb{F}_p)$ up to conjugation)

* $n=2$: $\operatorname{im}\bar{\rho}$ solvable and of order multiple of p.

<u>Some reminders on group theory</u>:

Thm. (Schur–Zassenhaus)

G profinite group, P normal pro-p subgroup of finite index prime to p.

Then $\exists A \leq G$ subgroup such that $A \xrightarrow{\sim} G/P$ is an isomorphism.

All such A's are conjugate by an element of P. One gets $G \cong P \rtimes G/P$

Def. $\bar{P} = P/\overline{\langle P^p, [P,P] \rangle}$ where P is a pro-p group

maximal abelian p-elementary continuous quotient of P.

Thm. (Burnside basis theorem) If P is a pro-p group, $x_1, \dots, x_d$ are elements of P such that $\bar{x}_1, \dots, \bar{x}_d$ generate $\bar{P}$, then $x_1, \dots, x_d$ topologically generate P as an abstract group.

Thm. (Boston) G profinite, $P \leq G$ normal pro-p subgroup of finite index, coprime to p.

A a subgrp of G s.t. $A \cong G/P$. Then $\bar{P}$ has a structure of $\mathbb{F}_p[A]$-module.

Let $\bar{V}$ an $\mathbb{F}_p[A]$-submodule of $\bar{P}$. Then there exists an A-invariant subgroup V of G such that the image of V in $\bar{P}$ is $\bar{V}$.

Def. Let $A \leq GL_n(k)$ be a subgroup of order prime to p and let $Ad|_A$ be its adjoint representation. Let V be a $k[A]$-module.

(By Maschke's theorem $Ad|_A$ and V decompose as direct sums of irreducible $k[A]$-modules)

We say that V is prime-to-adjoint if V and $k[A]$ have no common irreducible $k[A]$-submodules.

"prime to adjoint"-principle (named by Böckle!)

Exercise: X fin. generated subgroup of $\Gamma_n(R)$, $R \in \hat{\mathcal{C}}_\Lambda$

$\underset{\text{finite subgroup}}{A \leq GL_n(R)}$ of order prime to p normalizing X.

Then: X is prime-to-adjoint for A $\iff$ X is trivial.

Hint: $K_r := \ker(\Gamma_n(R) \to \Gamma_n(R/m_R^r))$ $\quad X \cap K_r / X \cap K_{r+1}$

Explicit deformation of tame representations.

$\bar{\rho}: G_{\mathbb{Q},S} \to GL_n(k)$ tame, so $p \nmid \# \operatorname{im}\bar{\rho} = \# H$

One constructs a lift of $\bar{\rho}$ to $W(k)$

Schur-Zassenhaus $\xrightarrow{\text{apply to}}$ $*$ $G_0 \supseteq P$ p-Sylow normal, $H \cong G/P$

$\underset{\text{one gets}}{\Rightarrow}$ $G_0 \cong P \rtimes H$

One has $\pi: GL_n(W(k)) \longrightarrow GL_n(k) \supseteq \operatorname{im}\bar{\rho}$

Look at $\pi^{-1}(\operatorname{im}\bar{\rho}) \supseteq \Gamma_n(W(k))$ pro-p subgroup, index is $\# \operatorname{im}\bar{\rho}$

$\Rightarrow \pi^{-1}(\operatorname{im}\bar{\rho}) = \Gamma_n(W(k)) \rtimes \operatorname{im}\bar{\rho}$.

We write $G: H \underset{\text{given by } \bar{\rho}}{\xrightarrow{\cong}} \operatorname{im}\bar{\rho} \longrightarrow \pi^{-1}(\operatorname{im}\bar{\rho}) \subseteq GL_n(W(k))$

For any $R \in \hat{\mathcal{C}}_\Lambda$, we get $G_R: H \longrightarrow GL_n(R)$ by composing with $W(k) \to R$.

Define a functor

$$E_{\bar{\rho}}: \hat{\mathcal{C}}_\Lambda \longrightarrow \underline{\text{Set}} \quad ; \quad E_{\bar{\rho}}(R) = \operatorname{Hom}_H(P, \Gamma_n(R)).$$

We compare $E_{\bar{\rho}}$ with $D_{\bar{\rho}}$.

* For $\rho: G_0 \longrightarrow GL_n(R)$ ($G \to GL_n(k)$ factors through G_0!), then

$$\rho|_P: P \to \Gamma_n(R) \quad \text{is } H\text{-equivariant (check)}$$

$$\rightsquigarrow \quad \rho|_P \in E_{\bar{\rho}}(R)$$

On the other hand:

* Take $\varphi \in \operatorname{Hom}_H(P, \Gamma_n(R))$, then

$$\rho: G_0 \cong P \rtimes H \xrightarrow[(\varphi, \bar{\rho})]{} \Gamma_n(R) \rtimes \operatorname{im}\bar{\rho} \hookrightarrow GL_n(R)$$

<u>check:</u> ρ is a representation, because φ is H-equivariant.

We have a natural transformation $E_{\bar{\rho}} \to D_{\bar{\rho}}$

Theorem. (Boston) 1) The morphism $E_{\bar{\rho}} \longrightarrow D_{\bar{\rho}}$ is an isomorphism when $C_k(\bar{\rho}) = k$.

2) In general $E_{\bar{\rho}} \to D_{\bar{\rho}}$ is formally smooth.

3) $E_{\bar{\rho}}$ is always representable.

Proof. Show $E_{\bar{\rho}}(R) \longrightarrow D_{\bar{\rho}}(R)$ is a bijection.

* surjective: take ρ, then $\varphi = \rho|_P$ is an element of $E_{\bar{\rho}}(R)$ and $\rho_\varphi = \rho$.
* injective: if φ_1, φ_2 give ρ_{φ_1} strict equiv. ρ_{φ_2}, then $\varphi_1 = \varphi_2$.

$$\rho_{\varphi_1} = \underset{\underset{\Gamma_n(R)}{\cap}}{\gamma}\, \rho_{\varphi_2} \gamma^{-1} \iff (\varphi_1, \bar\rho) = \gamma(\varphi_2, \bar\rho)\gamma^{-1}$$

$\Rightarrow$ γ commutes with $\operatorname{im}\bar\rho$

$\Rightarrow$ γ scalar because $C_k(\bar\rho) = k$.

2) Exercise.

3) Similar to representability of $D_{\bar\rho}^{\square}$.

<u>Sketch</u>: $x_1, \dots, x_d$ top. generators of P (top. fin. generated because of p-finiteness)

$$W(k)[\![T_{ij}^{(r)}]\!]_{\substack{1\le i,j\le n\\ 1\le r\le d}}$$

free pro-p-group on d generators

$$1 \longrightarrow N \longrightarrow F \longrightarrow P \longrightarrow 1$$

$$(*) \quad F \xrightarrow{\alpha} \Gamma_n\left(W(k)[\![T_{ij}^{(r)}]\!]_{i,j,r}\right)$$

$$x_r \longmapsto \begin{pmatrix} 1+T_{11}^{(r)} & & T_{ij}^{(r)} \\ & \ddots & \\ T_{ij}^{(r)} & & 1+T_{nn}^{(r)} \end{pmatrix}$$

(N acts trivially, morph. H-equivariant) $\longrightarrow$ all relations generate an ideal I

Universal deformation ring is $W(k)[\![T_{ij}^{(r)}]\!]/I$

Universal deformation is $\varphi: P \longrightarrow \Gamma_n(R)$ induced by α in $(*)$.

<u>Remark:</u> $\dim_k t_{D_{\bar\rho}} = \dim_k(\operatorname{Hom}_H(P, \Gamma_n(k[\varepsilon]))) = \dim_k(\operatorname{Hom}_H(\bar P, \mathrm{Ad}_{\bar\rho}))$

$C_k(\bar\rho) = k$.

An explicit deformation ring:

Theorem. (Boston–Mazur) $Z_S = \{x \in \underset{\cup}{k} \mid$

fractional ideal $(x) = I^p$ for some fractional ideal I of K

$x_v = g_v^p$ for some $g_v \in K_v$ $\forall v \in S_1 \}^{5_H}$, $B_S = Z_S/(K^\times)^p$ $\mathbb{F}_p[H]$-module.

$$V = \operatorname{coker}\left(\mu_p(K) \longrightarrow \prod_{v \in S_1} \mu_p(K_v) \right)$$

As an $\mathbb{F}_p[H]$-module

$$\bar{\rho} = \operatorname{Ind}_{H_\infty}^{H} \mathbb{F}_p \oplus \mathbb{F}_p \oplus V \oplus B_S \qquad \mathrm{ad}_{\bar{\rho}} \underset{\mathbb{F}_p[H]}{=} \mathrm{ad}^0_{\bar{\rho}} \oplus \mathbb{F}_p$$

H_∞ = subgp of H gen. by a comp. conjugation