$B^1(G,M) = \{ f: G \to M \mid \exists\, m \in M \text{ s.t. } f(g) = (g-1)m \;\; \forall g \in G \}$

Cohomological interpretation of the tangent space.

Fix $\bar\rho : G \to GL_n(k)$ $\quad (C_k(\bar\rho) = k)$. We said that $d = \dim_k D_{\bar\rho}(k[\varepsilon])$

$$= \dim_k t_{R^{univ}_{\bar\rho}}$$

$$\left(t_{R^{univ}_{\bar\rho}} = \mathrm{Hom}_\Lambda (R^{univ}_{\bar\rho}, k[\varepsilon]) \right)$$

Start with $\rho : G \to GL_n(k[\varepsilon])$ a deformation of $\bar\rho$. Then $\forall g \in G$:

$$\rho(g) = \bar\rho(g)\,(1 + \underset{\underset{M_n(k)}{\cap}}{c_g}\,\varepsilon) \qquad (k \hookrightarrow k[\varepsilon] = k + \varepsilon k)$$

check : ρ homomorphism

$\Rightarrow$ $c : G \to M_n(k),\ g \mapsto c_g$ is a 1-cocycle for the adjoint action $G \curvearrowright M_n(k)$ of $\bar\rho$.

Adjoint representation: $G \curvearrowright M_n(k)$

$g \cdot m := \bar\rho(g)^{-1}\, m\, \bar\rho(g)$. Write $\mathrm{Ad}_{\bar\rho}$

Proposition: The previous construction gives an isomorphism (of vector spaces)

$$D_{\bar\rho}(k[\varepsilon]) \longrightarrow H^1(G, \mathrm{Ad}_{\bar\rho})$$

Corollary. $d = \dim_k (H^1(G, \mathrm{Ad}_{\bar\rho}))$.

23.05.18

Obstructions. Take $R_0, R_1 \in \hat{\mathcal{C}}_\Lambda$ with a morphism of Λ-algebras $R_1 \xrightarrow{\varphi} R_0$ surjective
$I := \ker\varphi$. and $\ker\varphi \cdot m_{R_1} = 0$.

Fix $\bar\rho : G \to GL_n(k)$. Consider a deformation $\rho_0 : G \to GL_n(R_0)$ of $\bar\rho$.

Can we lift ρ_0 to $\rho_1 : G \to GL_n(R_1)$?

$$\begin{tikzcd} GL_n(R_1) \arrow[r, "\varphi"] & GL_n(R_0) \\ G \arrow[u, dashed, "\rho_1"] \arrow[ur, "\rho_0"'] & \end{tikzcd}$$

Take σ any map (of sets) $G \xrightarrow{\sigma} GL_n(R_1)$ inducing ρ_0. For $g_1, g_2 \in G$ set

$$c_\sigma(g_1, g_2) = \sigma(g_1 g_2)\,\sigma(g_2)^{-1}\,\sigma(g_1)^{-1}.$$

We know σ lifts ρ_1 so $c_\sigma(g_1,g_2) \equiv 1 \pmod{I}$ $\rightsquigarrow$ We can write

$$c_\sigma(g_1,g_2) = 1 + \underset{\underset{M_n(I)}{\cap}}{d_\sigma(g_1,g_2)}$$

<u>Check</u>: $\begin{aligned} G \times G &\to I \\ (g_1,g_2) &\mapsto d_\sigma(g_1,g_2) \end{aligned}$ defines a cocycle in $Z^2(G, \mathrm{Ad}_{\rho_I})$

$I \cdot m_{R_1} = 0 \Rightarrow I$ has a k-Vector space structure $(\exists k \to I)$

ρ_I = composition of $\bar\rho$ with $k \to I$

Ad_{ρ_I} = G acting on $M_n(I)$ via conjugation with ρ_I.

If σ' is another set-theoretic lift $G \to GL_n(R_1)$ of ρ_0, then $d_\sigma(g_1,g_2)\, d_{\sigma'}(g_1,g_2)^{-1}$ is a coboundary

$\to$ The class $\mathcal{O}(\rho_0)$ of d_σ in $H^2(G, \mathrm{Ad}_{\rho_I})$ only depends on ρ_0.

If $\exists$ a group homomorphism $\sigma : G \to GL_n(R_1)$ lifting ρ_0, then $\mathcal{O}(\rho_0)$ is trivial.

If $\mathcal{O}(\rho_0)$ is trivial, then ρ_0 admits a lift $G \to GL_n(R_1)$ (group homomorph.!)

<u>Conclusion</u>: ρ_0 admits a lift to $R_1 \iff \mathcal{O}(\rho_0) = 0$.

Remark: if $H^2(G, \mathrm{Ad}_{\bar\rho}) = 0$, then $H^2(G, \mathrm{Ad}_{\bar\rho_I}) = 0$

$$H^2(G, \mathrm{Ad}_{\bar\rho_I}) \cong H^2(G, \mathrm{Ad}_{\bar\rho}) \otimes_k I.$$

$\Rightarrow$ for any $R_1 \xrightarrow{\varphi} R_0$ and any ρ_0 there exists a lift of ρ_0 to R_1.

When $H^2(G, \mathrm{Ad}_{\bar\rho}) = 0$ we say that the deformation problem is unobstructed.

Interpretation of obstructions via group extensions.

Same situation: $R_1 \xrightarrow{\varphi} R_0$, $I = \ker\varphi$, $I \cdot m_{R_1} = 0$.

$Ad_{\tilde\rho_I}$ is a G-module. In general: Take a G-module A. Then there is a group isomorphism

$$O_{pext}(G,A) \longrightarrow H^2(G,A)$$

* $O_{pext}(G,A)$ is the set of A-extensions of G:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[dr] & E \arrow[r] \arrow[d, "h"] & G \arrow[r] & 1 \\ & & E' \arrow[ur] & & \end{tikzcd}$$

exact sequence of groups up to isomorph.

E, E' are isomorphic if $\exists h: E \to E'$ isomorph. making the diagrams commute.

* Baer sum: gives $O_{pext}(G,A)$ an abelian group structure.

* $E \in O_{pext}(G,A)$ maps to $0 \in H^2(G,A)$ $\iff$ E is a split extension.

Back to $R_1 \xrightarrow{\varphi} R_0$, $\underbrace{\ker\varphi}_{=:I} \cdot m_{R_1} = 0$. Then Ad_{ρ_I} is a G-module.

Choose $\rho_0 : G \to GL_n(R_0)$.

identify $Ad_{\rho_I} \cong 1 + Ad_{\rho_I}$ $\quad (1+a)(1+b) = 1 + a + b + \underbrace{ab}_{=0 \text{ since } I \in m_{R_1},\ I m_{R_1} = 0}$

$$\begin{tikzcd} 1 \arrow[r] & Ad_{\rho_I} \arrow[r] & GL_n(R_1) \arrow[r] & GL_n(R_0) \arrow[r] & 1 \\ 1 \arrow[r] & Ad_{\rho_I} \arrow[r] \arrow[u] & G \times_{GL_n(R_0)} GL_n(R_1) \arrow[r] \arrow[u] & G \arrow[r] \arrow[u, "\rho_0"] & 1 \end{tikzcd} \quad (*)$$

This Ad_{ρ_I}-extension of G gives a class $O(\rho_0) \in H^2(G, Ad_{\rho_I})$

Remark: The extension $(*)$ splits $\iff$ $\exists$ repres. $\rho_1 : G \to GL_n(R_1)$ lifting ρ_0.

Proof. if $(*)$ splits and $s : G \to G \times_{GL_n(R_0)} GL_n(R_1)$ is a section the define

$$\rho_1 := \pi_2 \circ s$$

(π_2: projection to $GL_n(R_1)$)

if ρ_1 exists then $s(g) := (g, \rho_1(g))$ is a section for $(*)$.

We get: ρ_1 exists $\iff \mathcal{O}(\rho_0)=0$.

Exercise: find something similar for H^1.

Theorem. (Mazur) $d_1 := \dim_k H^1(G, \mathrm{Ad}_{\bar\rho})$, $d_2 := \dim_k H^2(G, \mathrm{Ad}_{\bar\rho})$

Then:

$$\dim_{\text{Krull}}\left(R_{\bar\rho}^{\text{univ}} / m_\Lambda R_{\bar\rho}^{\text{univ}} \right) \geq d_1 - d_2$$

In particular, in the unobstructed case $(d_2=0)$, $R_{\bar\rho}^{\text{univ}} \cong \Lambda[[T_1, \dots, T_{d_1}]]$.

Proof. $R := R_{\bar\rho}^{\text{univ}}$

...

$$\Lambda[[T_1, \dots, T_{d_1}]] \longrightarrow R \longrightarrow 0 \qquad \text{modulo } m_\Lambda:$$

$$0 \longrightarrow \underset{\substack{\text{defined} \\ \text{as kernel}}}{J} \longrightarrow \underbrace{k[[T_1, \dots, T_{d_1}]]}_{=:F} \longrightarrow R/m_\Lambda R \longrightarrow 0$$

Goal: prove that J has at most d_2 generators. $m_F :=$ max. ideal of F,

reduce the sequence mod $m_F J$.

$$0 \longrightarrow J/m_F J \longrightarrow F/m_F J \longrightarrow R/m_\Lambda R \longrightarrow 0$$

still an isomorphism on tangent spaces.

Consider ρ' the image of the universal deformation ρ^{univ} modulo $m_\Lambda R$.

$: G \longrightarrow GL_n(R/m_\Lambda R)$

$\rightsquigarrow$ By the previous construction, one associates to ρ' a class

$$\mathcal{O}(\rho') \in H^2(G, \mathrm{Ad}_{\bar\rho}) \otimes_k J/m_F J.$$

Show $\exists$ injection $\mathrm{Hom}_k(J/m_F J, k) \hookrightarrow H^2(G, \mathrm{Ad}_{\bar\rho})$ (then we are done).

$$f \longmapsto (1 \otimes f)(\mathcal{O}(\rho'))$$

Prove that it is an injective: Assume $f \neq 0$, consider $\ker f \leq J/m_F J$.

Reduce modulo $\ker f$:

$$0 \to (J/m_F J)/\ker f \to \underbrace{(F/m_F J)/\ker f}_{=:A} \to R/m_\Lambda R \to 0$$

* $A \to R/m_\Lambda R$ is still an isomorphism of tangent spaces.
* the image of $\mathcal{O}(\rho')$ in $H^2(G, Ad_{\bar\rho}) \otimes (J/m_F J)/\ker f$ is trivial.

$\Rightarrow \rho'$ admits a lift to A

$\Rightarrow$ the exact sequence splits by the univ. prop. of ρ'.

$\Rightarrow \ker f = J/m_F J$ $\quad$ $\lightning$ to $f \neq 0$.

In many cases we have "=" & $\dim_{Krull}(R/m_\Lambda R) = d_1 - d_2$.

Dimension conjecture: "=" always holds.

Mazur calls this "generalized Leopoldt's conjecture".

Statement we use today: take K a number field, S set of places,

$S \supseteq$ p-adic and infinite places, $G_{K,S}$.

$r_1 = \#$ real embeddings of K, $r_2 = \frac{1}{2}\#$ complex embeddings

Leopoldt's conjecture $\Leftrightarrow$ the number of $\mathbb{Z}_p$ extensions of K unramified is

$\leftarrow Gal = \mathbb{Z}_p$

$$r_2 + 1 = \# Hom_{cont}(G_{K,S}, \mathbb{Z}_p)$$

$\Leftrightarrow$ outside S is $r_2 + 1$.

Example: K total real, K has a $\mathbb{Z}_p$-extension (p-adic cyclotomic ext.),

Leopoldt's $\Rightarrow$ it's the only $\mathbb{Z}_p$-extension.

<u>Show</u>: Dimension conjecture $\Rightarrow$ Leopoldt's conjecture

$\Updownarrow$

$$\dim(R/\mathfrak{m}_\Lambda R) = d_1 - d_2$$

For every place $v \in S$ of K, write $G_v \subseteq G_{K,S}$ for a decomp. group at v

Proposition. For $\bar\rho : G_{K,S} \to GL_n(k)$, we have $d_1 - d_2 = 1 + dn^2 - \sum\limits_{v \in S_\infty} \dim_k H^0(G_v, Ad_{\bar\rho})$
with $d = [K:\mathbb{Q}]$ (S_∞: infinite places)

"Proof": Tate's Global Euler Char. formular:
take M a finite $G_{K,S}$-Module, then: (assume $\#M$ is an S-unit)

$$\frac{\# H^0(G_{K,S}, M) \cdot \# H^2(G_{K,S}, M)}{\# H^1(G_{K,S}, M)} = \frac{\prod_{v \in S_\infty} \# H^0(G_v, M)}{\# M^d}$$

Apply it to $M = Ad_{\bar\rho}$. (We assumed $C_k(\bar\rho) = k$)

$d_0 = \dim_k H^0(G_{K,S}, Ad_{\bar\rho})$

$\rightsquigarrow$ $\underset{\overset{\|}{1}}{d_0} + d_2 - d_1 = \sum\limits_{v \in S_\infty} \dim_k H^0(G_v, Ad_{\bar\rho}) - n^2 d$ $\boxtimes$

Apply to special cases:

1) $n = 1$, any $\bar\rho : G_{K,S} \to k^\times$

For every $v \in S^\infty$ $H^0(G_v, Ad_{\bar\rho}) = 1$.

$d_1 - d_2 = 1 + d - \underbrace{\# S_\infty}_{= r_1 + r_2}$ $\quad d = r_1 + 2r_2$

$= 1 + r_2$

What is $d_1 - d_2$? By the dimension conjecture $d_1 - d_2 = \dim R_{\bar\rho}^{univ} / \mathfrak{m}_\Lambda R_{\bar\rho}^{univ}$

use $R_{\bar\rho}^{univ} = \Lambda [\![\widehat{G_{K,S}}^{(p)}]\!] = \# \widehat{\mathrm{Hom}}_{cont}(G_{K,S}, \mathbb{Z}_p)$