The conditions are:

H1: If $\varphi_2 : R_2 \to R_0$ is small, then $(*)$ is surjective

H2: If $R_0 = k$, $R_2 = k[\epsilon]$ with $\begin{array}{c} R_2 \to R_0 \\ \epsilon \mapsto 0 \end{array}$, then $(*)$ is bijective

H3: $F(k[\epsilon])$ is finite-dimensional over k.

H4: If $R_1 = R_2$, $\varphi_1 = \varphi_2$ is small, then $(*)$ is bijective.

Theorem of Schlessinger: If $F(k) = \{*\}$ and it satisfies $H_1 - H_4$, then F is pro-representable.

09.05.18

Continuity of Functors

$F : \hat{\mathcal{C}}_\Lambda \to \underline{\text{Set}}$ functor. We say that F is continuous if the natural map

$$F(A) \longrightarrow \varprojlim_n F(A/m_A^n) \quad \text{is a bijection } \forall A \in \hat{\mathcal{C}}_\Lambda$$

Exercise:
* F representable $\Rightarrow$ F continuous
* $\hat{D}_{\bar{\rho},\Lambda}$ is continuous
* If we extend $D_{\bar{\rho},\Lambda}$ by continuity ($D_{\bar{\rho},\Lambda}(A) := \varprojlim_n D_{\bar{\rho},A}(A/m_A^n)$ for $A \in \hat{\mathcal{C}}_\Lambda$), we get $\hat{D}_{\bar{\rho},\Lambda}$
* If $D_{\bar{\rho},\Lambda}$ is pro-represented by some $R \in \hat{\mathcal{C}}_\Lambda$, then $\hat{D}_{\bar{\rho},\Lambda} \cong \operatorname{Hom}_{\hat{\mathcal{C}}_\Lambda}(R, \cdot)$

Existence of Universal Deformation Pairs

G, k, $\Lambda \in \hat{\mathcal{C}}_k$, fix $\bar{\rho} : G \to GL_n(k)$

We want to show $D_{\bar{\rho},\Lambda}$ is pro-representable by proving it satisfies $H_1 - H_4$ of Schlessingers Criterion.

Theorem. Assume G satisfies the p-finiteness condition

(Φ_p) $\mathrm{Hom}_{cont}(H, \mathbb{Z}/p\mathbb{Z})$ is a finite set $\forall$ open subgroups $H \leq G$

Then $D_{\bar{\rho},\Lambda} : \mathcal{C}_\Lambda \to \underline{\mathrm{Set}}$ satisfies H1, H2, H3. of Schlessinger's criterion.

Let $C_k(\bar{\rho}) = \mathrm{Hom}_{\bar{\rho}}(k^n, k^n) \left(= \left\{ \begin{array}{l} M \in M^{n\times n}(k) \text{ s.t.} \\ \bar{\rho}\cdot M = M \cdot \bar{\rho} \end{array} \right. \right)$

If $C_k(\bar{\rho}) = k$ $\binom{\text{scalar}}{\text{matrices}}$ then $D_{\bar{\rho},\Lambda}$ also satisfies H4.

(Mazur proved Theorem for $\bar{\rho}$ abs. irreducible, Ramakrishna proved for $C_k(\bar{\rho}) = k$)

Corollary. If G satisfies p-finiteness and $C_k(\bar{\rho}) = k$ then $D_{\bar{\rho},\Lambda}$ is pro-representable:

there exists a couple (R^{univ}, ρ^{univ}) with the following universal property:

$R^{univ} \in \hat{\mathcal{C}}_\Lambda$, $\rho^{univ} : G \to GL_n(R^{univ})$

$\forall A \in \hat{\mathcal{C}}_\Lambda$ and every deformation $\rho : G \to GL_n(A)$ of $\bar{\rho}$, there exists an unique Λ-algebra morphism $f_\rho : R^{univ} \longrightarrow A$ s.t. $\rho \underset{\text{str. equiv.}}{\cong} f_\rho \circ \rho^{univ}$

Remarks: 1) When is $C_k(\bar{\rho}) = k$? Schur's. Lemma $\Rightarrow$ Condition holds for $\bar{\rho}$ abs. irreducible

Take $\bar{\rho}(g) = \begin{pmatrix} \chi_1(g) & * \\ 0 & \chi_2(g) \end{pmatrix}$, χ_1, χ_2 distinct characters $G \to k^\times$, $* \neq 0$ $\}$ $\bar{\rho} : G \to GL_n(k)$, $GL_n(k)$ irred.

Then $C_k(\bar{\rho}) = k$.

2) What to do when $C_k(\bar{\rho}) \neq k$?

* Look at "versal" deformations

* Look at "framed" deformations

Versal deformations:

Def. Let $F, G : \mathcal{C}_\Lambda \to \underline{Set}$ be functors s.t. $F(k) = \{*\}$, $G(k) = \{*\}$.

A natural transformation $\eta : F \to G$ is formally smooth $:\Longleftrightarrow$ $\forall A, B \in \mathcal{C}_\Lambda$

$\forall$ surjections $A \to B$ the natural map $F(A) \to F(B) \times_{G(B)} G(A)$ is surjective.

$$\left(\begin{tikzcd} & F(A) \arrow[dl] \arrow[dr, "\eta_A"] & \\ F(B) \arrow[dr, "\eta_B"'] & & G(A) \arrow[dl] \\ & G(B) & \end{tikzcd}\right)$$

Proposition.
If F, G are pro-representable,
$F = \mathrm{Hom}(R_F, \cdot)$, $G = \mathrm{Hom}(R_G, \cdot)$ for some $R_F, R_G \in \hat{\mathcal{C}}_\Lambda$

then:
a transformation $F \to G$ is formally smooth

$\Longleftrightarrow$ it is induced by a morphism $R_G \to R_F$ that makes R_F a ring of formal power series over R_G.

$$\left(R_F = R[\![T_1, \dots, T_n]\!] \right)$$

Proved in Mézard's notes, in Schlessinger's paper (Lemma 2.5) $\blacksquare$

Def. We say that a functor $F : \mathcal{C}_\Lambda \to \underline{Set}$ admits a versal deformation if there exists a couple $(\underset{\in \hat{\mathcal{C}}_\Lambda}{R}, \underset{\in F(R)}{\rho})$ such that the morphism of functors

$$\mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R, A) \to F(A) \quad \text{defined for } A \in \hat{\mathcal{C}}_\Lambda \quad (f : R \to A) \mapsto F(f)(\rho)$$

is formally smooth.

Theorem. (Schlessinger) Assume G satisfies ϕ_p. Then $D_{\bar\rho, \Lambda}$ admits a versal deformation (R^{ver}, ρ^{ver}).
This means that for $\forall A \in \mathcal{C}_\Lambda$ and $\forall$ deformations ρ of $\bar\rho$ to A there exists $f_\rho \in \mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R^{ver}, A)$ s.t. $f_\rho \circ \rho^{ver} \underset{\text{str.}}{\equiv} \rho$, but f_ρ is not uniquely determined.

Framed deformations:

Define a framed deformation functor $D^{\square}_{\bar{\rho},\Lambda} : \mathcal{C}_{\Lambda} \longrightarrow \underline{Set}$ as

$$D^{\square}_{\bar{\rho},\Lambda}(A) = \left\{ \begin{array}{l} \text{set of deformations} \\ \text{of } \bar{\rho} \text{ to } \rho : G \to GL_n(A) \end{array} \right\}$$

Deformations as actions on A-modules

See $\bar{\rho} : G \longrightarrow GL_n(k)$ as datum of:

* a continuous action of G on a n-dimensional k-vector space $V_{\bar{\rho}}$
* a choice of basis β for $V_{\bar{\rho}}$

We call a deformation of $\bar{\rho}$ the datum of
(framed deformation)

(1) a continuous action of G on a free A-module V_A of rank n

such that $V_A \otimes_A k \underset{G\text{-modules}}{\cong} V_{\bar{\rho}}$

(2) a choice of a lift of the basis β of $V_{\bar{\rho}}$ to an A-basis of V_A

(same as choosing a lift of $\bar{\rho}$ to $\rho : G \to GL_n(A)$)

$D_{\bar{\rho},\Lambda}(A)$ = set of choices of (1),

$D^{\square}_{\bar{\rho},\Lambda}(A)$ = set of choices of (1) and (2).

Theorem. Assume G has property Φ_p. Then the framed deformation functor $D^{\square}_{\bar{\rho},\Lambda}$ is pro-representable. (No need of Schlessinger)

Proposition. The transformation of functors $D^{\square}_{\bar{\rho},\Lambda} \longrightarrow D_{\bar{\rho},\Lambda}$ defined by

$$\big(\rho : G \to GL_n(A)\big) \longmapsto [\rho]_{\text{str. equivalence}}$$

is formally smooth.

Proof. exercise. $D^{\square}_{\bar\rho,\Lambda}(A) \longrightarrow D^{\square}_{\bar\rho,\Lambda}(B) \times_{D_{\bar\rho,\Lambda}(B)} D_{\bar\rho,\Lambda}(A)$ (surjection $A \to B$) ▨

Corollary. if $\mathcal{C}_k(\bar\rho) = k$ ($\rightsquigarrow D^{\square}_{\bar\rho,\Lambda}$ and $D_{\bar\rho,\Lambda}$ are represented as $\mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R^{\square}_{univ}, \cdot)$, $\mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R_{univ}, \cdot)$), then $R^{\square}_{univ} = R_{univ}[\![T_1, \dots, T_n]\!]$.

Proof of pro-representability of $D^{\square}_{\bar\rho,\Lambda}$.

We look for an object $R \in \hat{\mathcal{C}}_\Lambda$ such that $D_{\bar\rho,\Lambda}(A) = \mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R, A)$ $\forall A \in \hat{\mathcal{C}}_\Lambda$.

1) We prove the result for G finite. (fix $\rho: G \to GL_n(k)$)

Define Λ-algebra $\Lambda[G,n]$ by giving

* generators: X^g_{ij}, $g \in G$, $i,j \in \{1,\dots,n\}$
* relations: $X^{gh}_{ij} = \sum_{l=1}^{n} X^g_{il} X^h_{lj}$ $\forall g,h \in G$ $\forall i,j \in \{1,\dots,n\}$ ($e \in G$ unity)

$$X^e_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Then there is a bijection, for every Λ-algebra A

$$b_A : \mathrm{Hom}_\Lambda(\Lambda[G,n], A) \longrightarrow \mathrm{Hom}_{\underline{Grp}}(G, GL_n(A))$$
$$f \longmapsto (g \mapsto (f(X^g_{ij}))_{i,j})$$

<u>Check</u>: this is a bijection. ($\Lambda[-,n] \dashv GL_n$)

Consider $\bar\rho \in \mathrm{Hom}_{\underline{Grp}}(G, GL_n(k))$, then $b_k^{-1}(\bar\rho) =: f_{\bar\rho}$ is a Λ-algebra-morphism $\Lambda[G,n] \to k$. Set $m_{\bar\rho} = \ker f_{\bar\rho}$.

Set $R :=$ $m_{\bar\rho}$-adic completion of $\Lambda[G,n] = \varprojlim_n \Lambda[G,n]/m_{\bar\rho}^n$

Show that R has the universal property: take a deformation $\rho: G \to GL_n(A)$ of $\bar\rho$ to some $A \in \hat{\mathcal{C}}_\Lambda$. Then $f_\rho := b_A^{-1}(\rho): \Lambda[G,n] \to A$, then $f_\rho(m_{\bar\rho}) \subseteq m_A$.

In particular f_ρ can be extended by continuity to $f_\rho : R \to A$. Then one checks

$$\begin{tikzcd} G \arrow[rr, "\rho^{univ}"] \arrow[dr, "\rho"'] & & GL_n(R) \arrow[dl, "f_\rho"] \\ & GL_n(A) \arrow[u, phantom, "\circlearrowright"] & \end{tikzcd}$$

Define ρ^{univ} as the homomorphism $G \to GL_n(R)$ attached to $\Lambda[G, n] \to R$ by b_R

Also, f_ρ with this property is unique.

$\Rightarrow$ (R, ρ^{univ}) is an universal deformation couple for $\bar\rho$

2) G profinite, then $G = \varprojlim\limits_{\substack{H \trianglelefteq G \\ G/H \text{ finite} \\ H \subseteq \ker \bar\rho}} G/H$

(are exactly the H's such that $\bar\rho$ factors through $\bar\rho_H : G/H \to GL_n(k)$)

For each such H there is a representing pair $(R_{\bar\rho_H}, \bar\rho_H)$ for the functor $D^{\square}_{\bar\rho_H, \Lambda}$. The set $\{R_{\bar\rho_H}\}_H$ is a projective system. and

$$R_{\bar\rho} := \varprojlim_H R_{\bar\rho_H} \text{ is in the category } \hat{\mathcal{C}}_\Lambda.$$ (uses Φ_p-assumption on G!)

Then $R_{\bar\rho}$ pro-represents $D^{\square}_{\bar\rho, \Lambda}$:

$$D^{\square}_{\bar\rho, \Lambda}(A) = \varprojlim_H D^{\square}_{\bar\rho_H, \Lambda}(A) = \varprojlim_H \mathrm{Hom}(R_{\bar\rho_H}, A)$$

$$\cong \mathrm{Hom}(\varprojlim_H R_{\bar\rho_H}, A)$$

$$= \mathrm{Hom}(R_{\bar\rho}, A).$$