

Galois representations and their deformations.

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Goal: study representations of Galois groups of p -adic or number fields with p -adic coefficients, by "deforming" representations with modulo p coefficients.

References:

- * Gouvêa, notes
- * Böckle, notes
- * Mazur's article "Deforming Galois representations"
- * Mézard, notes

01. Galois groups

K perfect field, L a normal extension of K . Define $\text{Gal}(L/K)$
 $= \{\sigma : L \rightarrow L \mid \sigma \text{ field automorphism}, \sigma|_K = \text{id}_K\}$

Topology:

- * if $L|K$ is finite then give $\text{Gal}(L|K)$ the discrete topology
- * if $L|K$ is infinite then give $\text{Gal}(L|K)$ the Krull topology:
 a basis of open neighborhoods of id_L is the collection of sets
 $\{\sigma \in \text{Gal}(L|K) \mid \sigma|_E = \text{id}_E\}$ where E varies over the finite
 subextensions $E|K$

As groups: $\text{Gal}(L|K) \cong \varprojlim_{\substack{E|K \text{ finite and normal} \\ E \subseteq L}} \text{Gal}(E|K)$

If $\text{Gal}(E|K)$ has the discrete topology, then this is an isomorphism of topological groups.

This makes $\text{Gal}(L|K)$ into a profinite group.

$\Rightarrow \text{Gal}(L|K)$ is compact and hausdorff.

\Rightarrow Open subgroups are the closed subgroups of finite index

Theorem. (**Galois correspondence**) The map

$$\begin{array}{ccc} \left\{ \text{subextensions } L \mid E \subset K \right\} & \longrightarrow & \left\{ \substack{\text{closed subgroups} \\ \text{of } \text{Gal}(L|K)} \right\} \\ E & \longmapsto & \text{Gal}(L|E) \end{array} \quad \text{is a bijection.}$$

The inverse is $H \mapsto E = L^H$.

This induces a bijection

$$\begin{array}{ccc} \left\{ \substack{L|E \subset K, \\ E \subset K \text{ finite}} \right\} & \longrightarrow & \left\{ \substack{\text{open subgroups} \\ \text{of } \text{Gal}(L|K)} \right\} \end{array}$$



When $L = K^{\text{alg}}$, we call $\text{Gal}(K^{\text{alg}}|K)$ the **absolute Galois group** of K , we write G_K .

Examples. * $K = \mathbb{F}_p$, we know that finite extensions are of the form \mathbb{F}_{p^n} and

$$\text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z} \quad \text{is an isomorphism.}$$

$$(x \mapsto x^p) \longmapsto 1$$

"Frobenius element"

$$\text{Gal}(\mathbb{F}_p^{\text{alg}}/\mathbb{F}_p) = \varprojlim_{n \rightarrow \infty} \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \xrightarrow{\sim} \varprojlim_{n \rightarrow \infty} \mathbb{Z}/n\mathbb{Z} =: \hat{\mathbb{Z}}$$

$$\begin{aligned} (\text{the maps are } \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) &\longrightarrow \text{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p) \quad m \mid n \text{ with } m \mid n \\ d &\longmapsto d|_{\mathbb{F}_{p^m}} \end{aligned}$$

$$\begin{aligned} \mathbb{Z}/n\mathbb{Z} &\longrightarrow \mathbb{Z}/m\mathbb{Z} \\ x &\longmapsto x \bmod m \end{aligned}$$

The Frobenius of $\text{Gal}(\mathbb{F}_p^{\text{alg}}/\mathbb{F}_p)$ is mapped to $1 \in \hat{\mathbb{Z}}$.

* $k = \mathbb{Q}_p$; we denote by \mathbb{Q}_p^{ur} the maximal unramified extension of \mathbb{Q}_p .

(\hookrightarrow maximal extension for which p is still a uniformizer (generator of the maximal ideal of the valuation ring))

Valuation ring \mathbb{Z}_p^{ur} has residue field $\mathbb{Z}_p^{\text{ur}}/\mathfrak{p}\mathbb{Z}_p^{\text{ur}} \cong \mathbb{F}_p^{\text{alg}}$

There is a map $\text{Gal}(\mathbb{Q}_p^{\text{ur}}/\mathbb{Q}_p) \rightarrow \text{Gal}(\mathbb{F}_p^{\text{alg}}/\mathbb{F}_p)$

$$d \longmapsto d \bmod p$$

This map is a group isomorphism.

$$\begin{array}{c} \mathbb{Q}_p^{\text{alg}} \\ | \quad) \quad \text{"Inertia group"} \quad (I_p) \\ \mathbb{Q}_p^{\text{ur}} \\ | \quad) \text{Gal} = \hat{\mathbb{Z}} \quad (\text{topologically generated by Frob}_p) \\ \mathbb{Q}_p \end{array}$$

* $K = \mathbb{Q}$, $\text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q})$, let p be prime.
 Choose an extension of the p -adic valuation on \mathbb{Q} to \mathbb{Q}^{alg} (not unique!)
 (\hookrightarrow choose an embedding $\mathbb{Q}^{\text{alg}} \rightarrow \mathbb{Q}_p^{\text{alg}}$)

Write a map $\text{Gal}(\mathbb{Q}_p^{\text{alg}}/\mathbb{Q}_p) \rightarrow \text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q})$

$$\sigma \mapsto \sigma|_{\mathbb{Q}^{\text{alg}}}$$

This map is an injective group homomorphism and it identifies $\text{Gal}(\mathbb{Q}_p^{\text{alg}}/\mathbb{Q}_p)$ with a subgroup $D_p \subseteq \text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q})$

$$D_p = \left\{ \sigma \in \text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q}) \mid v(\sigma^{-1}(x)) = v(x) \quad \forall x \in \mathbb{Q}^{\text{alg}} \right\}$$

We have injections

$$I_p \subseteq D_p \subseteq \text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q})$$

O2. Galois groups for extensions unramified outside a finite set.

K number field, S is a finite set of places of K .

Def. 1) K^S is the largest extension of K that is unramified at all places not in S .

$$2) G_{K,S} := \text{Gal}(K^S/K)$$

Remark. $v \notin S$, then we have $\text{Gal}(K_v^{\text{alg}}/K_v) \hookrightarrow \text{Gal}(K^{\text{alg}}/K) \rightarrow \text{Gal}(K^S/K)$

Answer: injection when $K = \mathbb{Q}$, $\#S \geq 2$

Injection?

Exercise. An open subgroup $H \subseteq G_{K,S}$ has the form G_{K_1, S_1} where $K_1 | K$ is finite and S_1 is a set of places of K_1 .

(places in S_1 have to lie over the places of S
 $\Rightarrow S_1$ finite)

Theorem. (Hasse-Minkowski) Let K, S as before.

Let $d \in \mathbb{N}_{>0}$. Then there are only finitely many extensions of K of degree d and unramified outside S .

Corollary. $\text{Hom}_{\text{cont}}(G_{K,S}, \mathbb{F}_p)$ is finite. (Morphisms of topological groups)

Corollary. For every open subgroup $H \subseteq G_{K,S}$, $\text{Hom}_{\text{cont}}(H, \mathbb{F}_p)$ is finite.

\Leftarrow Def. $G_{K,S}$ satisfies the " p -finiteness condition".

(\mathbb{F}_p has the discrete topology)

03. Galois representations

Let G be a profinite group and let A be a topological ring.

Def. A **continuous representation** of G with A -coefficients is a continuous group homomorphism $\rho: G \rightarrow GL_n(A)$ for some integer n .

Given representations $\rho_1, \rho_2: G \rightarrow GL_n(A)$, we say that they are equivalent iff. $\exists P \in GL_n(A): \rho_1^{-1} \cdot P \cdot \rho_2 = P$

Another point of view: let M be a finite free A -module of rank n .

Then a continuous representation $\rho: G \rightarrow GL_n(A)$ gives a continuous action $G \curvearrowright M$

$$G \times M \rightarrow M$$

$$(g, m) \mapsto \rho(g)(m_i)_{i=1..n}$$

where $(m_i)_{i=1..n}$ are coordinates of m

Def.

We call ρ a **Galois representation** if G is:

- * $\text{Gal}(K^{\text{alg}}/K)$ for a finite extension $K \mid \mathbb{Q}_p$

- * $G_{K,S}$ for a number field K and a finite set of places S .

From now on G is one of those groups.

Choices of coefficient ring A :

1) $A = \mathbb{C}$

2) $A = \mathbb{F}_p$

3) $A = \mathcal{O}_E$ for $E \mid \mathbb{Q}_p$ finite

4) $A = E$, $E \mid \mathbb{Q}_p$ finite

① $A = \mathbb{C}$

Proposition. A representation $\rho: G \rightarrow GL_n(\mathbb{C})$ has finite image. \square

Proof. Consider an open neighborhood U of $\mathbb{1}_n \in GL_n(\mathbb{C})$. If U is sufficiently small, the

only subgroup $\subseteq U$ is $\{\mathbb{1}_n\}$ (Exercise). By continuity of ρ , $\exists V$ open neighborhood of $e \in G$ such that $\rho(V) \subseteq U$. We can choose V' a neighborhood of e which is an open subgroup of G and s.t. $V' \subseteq V$. Then $\rho(V') \subseteq \rho(V) \subseteq U$.

$\Rightarrow \rho(V') = \{\mathbb{1}_n\}$. Since V' is of finite index in G , $\rho(V')$ is of finite index in $\rho(G)$.

Expl. Take $K \mid \mathbb{Q}$ Galois, finite. Then $\text{Gal}(K \mid \mathbb{Q}) \hookrightarrow GL_n(\mathbb{C})$. \square

② Proposition. If $\rho: G \rightarrow \mathrm{GL}_n(\mathbb{F}_p^{\text{alg}})$ is a continuous representation,
 $(\mathrm{GL}_n(\mathbb{Q}_p))$
then it factors through $\rho': G \rightarrow \mathrm{GL}_n(\mathbb{F}_{p^m})$ for some finite extension
 $\mathbb{F}_{p^m} / \mathbb{F}_p$
 E / \mathbb{Q}_p

Proof. Similar for \mathbb{F}_p , more difficult for \mathbb{Q}_p . □

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Proof of the last Proposition.

Look at $\rho(G) \subseteq GL_n(\overline{\mathbb{Q}_p})$. It is a compact Hausdorff topological group

\Rightarrow Baire's Lemma holds for $\rho(G)$.

(Baire's Lemma: a countable union of nowhere dense closed subspaces of X is nowhere dense in X .)

Nowhere dense: it does not contain any open set of X)

$$GL_n(\overline{\mathbb{Q}_p}) = \bigcup_{\substack{E/\mathbb{Q}_p \\ \text{finite}}} GL_n(E) \quad \text{countable union of closed subsets.}$$

($\forall n \in \mathbb{N}$: there are only finitely many E/\mathbb{Q}_p st. $|E : \mathbb{Q}_p| = n$)

Write $\rho(G) = \bigcup_{\substack{E/\mathbb{Q}_p \\ \text{finite}}} (GL_n(E) \cap \rho(G))$. Either there exists E/\mathbb{Q}_p finite such

that $GL_n(E) \cap \rho(G)$ has finite index in $\rho(G)$

\Rightarrow We can choose $F \subseteq E$ finite such that $\rho(G) \subseteq GL_n(F)$ (finite index -)

Or for every E/\mathbb{Q}_p finite, $GL_n(E) \cap \rho(G)$ has infinite index in $\rho(G)$

$\Rightarrow GL_n(E) \cap \rho(G)$ is nowhere dense in $\rho(G)$ (Basis of open subgroups
 \Rightarrow open subgroups in compact spaces are of finite index)

Now $\rho(G)$ is a countable union of nowhere dense sets \Rightarrow Contradicts Baire's Lemma. □

Lemma. If $\rho: G \rightarrow GL_n(K)$ is a continuous representation with coefficients in K/\mathbb{Q}_p

finite, then there exists a continuous representation $\rho': G \rightarrow GL_n(\mathcal{O}_K)$ such that

if $i: GL_n(\mathcal{O}_K) \rightarrow GL_n(K)$ is the inclusion $\rho' \cong i \circ \rho$
equivalent

Proof. Recall: an \mathcal{O}_K -lattice in K^n is a free \mathcal{O}_K -module L of rank n such that

$$L \otimes_{\mathcal{O}_K} K \cong K^n.$$

Choosing a basis for K^n we obtain a continuous action of G on K^n via ρ .

Let L be any lattice in K^n . For $g \in GL_n(K)$ let $g(L) := \{g(x) \mid x \in L\}$

Exercise: $g(L)$ is a lattice, and $\text{Stab}(L) = \{g \in GL_n(K) \mid g(L) \subseteq L\}$ is an open subgroup of $GL_n(K)$. Ex

Look at $\underbrace{\rho^{-1}(\text{Stab}(L))}_{\substack{\text{open} \\ \subseteq GL_n(K)}} \subseteq G \Rightarrow \underbrace{\rho^{-1}(\text{Stab}(L))}_{G \text{ compact}} \text{ has finite index in } G$.

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Choose a set $\{g_1, \dots, g_m\}$ of representatives for $\frac{G}{\tilde{\rho}^{-1}(\text{stab}(L))}$.

Then define

$$L' := \sum_{i=1}^m g(g_i)(L). \quad \text{We check that the lattice } L' \text{ is } G\text{-stable.}$$

$$(G\text{-stable: } g(g) L' \subseteq L' \quad \forall g \in G)$$

Choose an \mathcal{O}_K -basis for the lattice L' , then the action of G on L' gives a (continuous) representation $\rho': G \rightarrow \text{GL}_n(\mathcal{O}_K)$

$$\text{By construction } \iota \circ \rho' \sim \rho \quad \blacksquare$$

Start with $\rho: G \rightarrow \text{GL}_n(K)$ continuous representation.

Then by the Lemma we can choose a conjugate of ρ with values in $\text{GL}_n(\mathcal{O}_K)$.

Then we can reduce modulo the maximal ideal $m_K \subset \mathcal{O}_K$ and we obtain a "residual" representation $\bar{\rho}: G \rightarrow \text{GL}_n(\underbrace{\mathcal{O}_K/m_K}_{=\mathbb{F}_p})$ attached to ρ .

Def. If G acts on a finite free module M . Choose a filtration $M \supseteq M_n \supseteq \dots \supseteq \{0\}$

in G -stable A -modules such that $\frac{M_i}{M_{i-1}}$ is an irreducible $A[G]$ -module.
(A : field)

(Does not admit any G -stable submodule)

Then the semi-simplification of M is the $A[G]$ -module $\bigoplus_{i=1}^n \frac{M_i}{M_{i-1}}$.

$$\text{Example: If } \rho(g) = \begin{pmatrix} \chi_1(g) & \delta(g) \\ 0 & \chi_2(g) \end{pmatrix} \rightarrow \bar{\rho}^{\text{ss}}(g) = \begin{pmatrix} \chi_1(g) & 0 \\ 0 & \chi_2(g) \end{pmatrix}$$

χ_1, χ_2 : Character of ρ

Remark: the representation $\bar{\rho}^{\text{ss}}$ attached to ρ is well-defined up to equivalence.

$$K \longrightarrow \mathcal{O}_K \longrightarrow \mathbb{F}_{p^m}$$

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Idea: fix $\bar{\rho}: G \rightarrow \mathrm{GL}_n(\mathbb{F}_{p^m})$ and look at $\rho: G \rightarrow \mathrm{GL}_n(\mathcal{O}_K)$
 (with $\mathcal{O}_K/m_K = \mathbb{F}_{p^m}$) such that $\rho \bmod m_K = \bar{\rho}$.

Example of p -adic Galois representation. (" p -adic cyclotomic character")
 (p prime, $n \in \mathbb{N}_{\geq 1}$)

$$\chi_n: G_{\mathbb{Q}} \longrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times (= \mathrm{GL}_1(\mathbb{Z}/p^n\mathbb{Z}))$$

This representations are compatible with the maps $(\mathbb{Z}/p^m\mathbb{Z})^\times \rightarrow (\mathbb{Z}/p^n\mathbb{Z})^\times$ for $m \geq n$.

$$\chi_m \bmod p^n = \chi_n.$$

We can take $\varprojlim_n \chi_n : G_{\mathbb{Q}} \rightarrow \varprojlim_n \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) = \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})^\times = \mathbb{Z}_p^\times$

$$\text{Write } \mathbb{Q}(\zeta_{p^\infty}) = \bigcup_{n \geq 1} \mathbb{Q}(\zeta_{p^n})$$

We call χ_{cyc} the p -adic cyclotomic character. χ_{cyc} factors through

$$G_{\mathbb{Q}, p^\infty} \longrightarrow \mathbb{Z}_p^\times. \text{ It also factors through } G_{\mathbb{Q}}^{\mathrm{ab}} \longrightarrow \mathbb{Z}_p^\times.$$

Theorem: (Kronecker-Weber) The product of all cyclotomic characters gives an isomorphism

$$G_{\mathbb{Q}}^{\mathrm{ab}} \xrightarrow{\sim} \prod_p \mathbb{Z}_p^\times.$$

Look at "deformation functors". $=: h_R$

* \mathcal{C} -category, $R \in \mathcal{C}$, then $\mathrm{Hom}_{\mathcal{C}}(R, -): \mathcal{C} \rightarrow \underline{\mathrm{Set}}$ is the functor

$$A \mapsto \mathrm{Hom}_{\mathcal{C}}(R, A), \quad f \in \mathrm{Mor}_{\mathcal{C}}(A, B) \mapsto \begin{cases} \mathrm{Hom}_{\mathcal{C}}(R, A) \rightarrow \mathrm{Hom}_{\mathcal{C}}(R, B) \\ g \mapsto f \circ g \end{cases}$$

We will work with some categories of rings.

Fix a field k . We denote by \mathcal{C}_k the category whose objects are Artinian, local rings with residue field k and morphisms are local ring morphisms, that induce the identity on k .

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Examples. * If $k = \mathbb{F}_p$, then $\mathbb{Z}_{p^n} \in \mathcal{C}_{\mathbb{F}_p}$ $\forall n \in \mathbb{N}_{>0}$.

$$\mathbb{F}_p[[T]] / T^n$$

* \exists unique degree n unramified extension of \mathbb{Q}_p , we will denote it by \mathbb{Q}_{p^n} . We write \mathbb{Z}_{p^n} for its valuation ring, then $\mathbb{Z}_{p^n}/p^n \mathbb{Z}_{p^n} = \mathbb{F}_{p^n}$

$\forall m \in \mathbb{N}_{>0}$: $\mathbb{Z}_{p^n}/p^m \mathbb{Z}_{p^n} \in \mathcal{C}_{\mathbb{F}_{p^m}}$

$$\mathbb{Z}_{p^n}/p^m \mathbb{Z}_{p^n} \longrightarrow \mathbb{Z}_{p^n}/p^m \mathbb{Z}_{p^n} \quad \text{This is not a morphism in } \mathcal{C}_{\mathbb{F}_{p^m}} \text{ if } m \geq 2.$$

$$x \pmod{p^m} \mapsto \text{Frob}_p(x) \pmod{p^m}$$

Let $\hat{\mathcal{C}}_k$ be the category whose objects are complete local Noetherian rings with residue field k , morphisms are local ring morphisms that induce the identity on k .

Example. $\mathbb{Z}_{p^n} \in \hat{\mathcal{C}}_{\mathbb{F}_{p^n}}$ | An object of \mathcal{C}_k is also an object in $\hat{\mathcal{C}}_k$
 $\mathbb{F}_{p^n}[[T]]$ | (same for morphisms)

DEFORMATION FUNCTORS

Fix $n \geq 1$.

Let G be a profinite group, k a finite field. Fix a continuous representation

$$\bar{\rho}: G \rightarrow \text{GL}_n(k).$$

Def. For $A \in \hat{\mathcal{C}}_k$, a deformation (of $\bar{\rho}$ to A) is a continuous representation $\rho: G \rightarrow \text{GL}_n(A)$ such that $\rho \pmod{m_A} = \bar{\rho}$.

We say that $\rho_1, \rho_2: G \rightarrow \text{GL}_n(A)$ are strictly equivalent iff. $\exists M \in \text{ker}(\text{GL}_n(A) \rightarrow \text{GL}_n(k))$ such that $M^{-1} \rho_1 M = \rho_2$.

Remark: if $A \xrightarrow{f} B$ is a morphism in $\hat{\mathcal{C}}_k$ and $\rho_1, \rho_2: G \rightarrow \text{GL}_n(A)$ are strictly equivalent representations, then $f \rho_1, f \rho_2: G \rightarrow \text{GL}_n(B)$ are strictly equivalent.

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We define $\hat{D}_{\bar{g}} : \hat{\mathcal{E}}_k \rightarrow \underline{\text{Set}}$ as the functor

* $\hat{D}_{\bar{g}}(A) := \left\{ \begin{array}{l} \text{deformations of} \\ \bar{g} \text{ to } A \end{array} \right\}$
strict equivalence

* $\hat{D}_{\bar{g}}(f)$ maps a deformation $\bar{g} : G \rightarrow GL_n(A)$ to the class of $f \circ g$
 $f : A \rightarrow B$ morphism in $\hat{\mathcal{E}}_k$

(We obtain a functor $D_{\bar{g}} : \mathcal{E}_k \rightarrow \underline{\text{Set}}$ by restricting $\hat{D}_{\bar{g}}$ to \mathcal{E}_k)

Goal: show that $D_{\bar{g}}$ is "pro-represented" by some $R \in \hat{\mathcal{E}}_k$, in the sense

that $D_{\bar{g}} \cong \text{Hom}_{\hat{\mathcal{E}}_k}(R, \cdot)$

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Fix any $\Lambda \in \hat{\mathcal{E}}_k$. We define \mathcal{E}_Λ as the category of local Artinian Λ -algebras with residue field k and local morphisms of Λ -algebras (that induce the identity on k)

* $\hat{\mathcal{E}}_\Lambda =$ category of complete local Noetherian Λ -algebras with residue field k . Morphisms as above.

The natural choice for Λ (when working in $\text{char } k = 0$) is the ring of Witt vectors $W(k)$ of k .
(= unique complete discrete valuation ring with residue field k and uniformizer p).
(Mézard's notes, Serre "Local fields")

We only need

$W(\mathbb{F}_{p^n}) = \mathbb{Z}_{p^n}$ ring of integers in \mathbb{Q}_{p^n}

$\hat{\mathcal{E}}_{\mathbb{Z}_{p^n}} \ni \mathbb{Z}_{p^n}, \quad \mathbb{F}_{p^n}[[T]] \in \hat{\mathcal{E}}_{\mathbb{F}_{p^n}}, \text{ but } \notin \hat{\mathcal{E}}_{\mathbb{Z}'_{p^n}}$

Deformation functors Fix $\bar{g} : G \rightarrow GL_n(k)$; We defined $D_{\bar{g}} : \mathcal{E}_k \rightarrow \underline{\text{Set}}$, $\hat{D}_{\bar{g}} : \hat{\mathcal{E}}_k \rightarrow \underline{\text{Set}}$.
one defines

$D_{\bar{g}, \Lambda} : \mathcal{E}_\Lambda \rightarrow \underline{\text{Set}}$ by $D_{\bar{g}, \Lambda}(A) = \left\{ \begin{array}{l} \text{deformations of} \\ \bar{g} \text{ to } A \end{array} \right\}$
strict equivalence

$\hat{D}_{\bar{g}, \Lambda} : \hat{\mathcal{E}}_\Lambda \rightarrow \underline{\text{Set}}$ in the same way.

Representable Functors.

Let \mathcal{C} be any category, $F: \mathcal{C} \rightarrow \underline{\text{Set}}$ any functor.

* F is called representable : $\Leftrightarrow \exists A \in \mathcal{C}: F \cong \text{Hom}_{\mathcal{C}}(A, \cdot)$, as functors.

For our categories $\mathcal{E}_k, \hat{\mathcal{E}}_k, \dots$ we define the following:

* $F: \mathcal{E}_1 \rightarrow \underline{\text{Set}}$ is pro-representable : $\Leftrightarrow \exists R \in \hat{\mathcal{E}}_1$ and an isomorphism of functors

$$\begin{aligned} \text{Hom}_{\hat{\mathcal{E}}_1}(R, \cdot) &\longrightarrow F \\ (\text{as a functor} \\ \mathcal{E}_1 \rightarrow \underline{\text{Set}}) \end{aligned}$$

Why representability?

Assume $D_{\bar{S}, 1}$ is (pro-)representable by some R in $\hat{\mathcal{E}}_1$: means that there is a

bijection $\text{Hom}_{\hat{\mathcal{E}}_1}(R, A) \rightarrow D_{\bar{S}, 1}(A)$ for every $A \in \mathcal{E}_1$.

Take $A = R$. Then there is a strict equivalence class ρ_R in $D_{\bar{S}, 1}(R)$ corresponding to id_R .

Then the bijection

$$\begin{aligned} \text{Hom}_{\hat{\mathcal{E}}_1}(R, A) &\longrightarrow D_{\bar{S}, 1}(A) \\ (f: R \rightarrow A) &\longmapsto \text{class of} \\ f \circ \rho_R: G &\rightarrow GL_n(A) \end{aligned}$$

We call (R, ρ_R) a universal couple, R is the universal deformation ring of \bar{S} , ρ_R is the universal deformation of $\bar{\rho}$.

(pro-)

Properties of representable functors $\mathcal{E}_1 \xrightarrow{F} \underline{\text{Set}}$. In this section F is a pro-representable

- left exact
- $F(k) = \{*\}$, since the only

element of $\text{Hom}_{\hat{\mathcal{E}}_1}(R, k)$ is $R \xrightarrow{\sim} R/m_R \cong k$

one fixed identification

- ~~F is continuous (later)~~

- F behaves well with respect to fiber products

$$k[\varepsilon] = \frac{k[T]}{T^2}, \quad \varepsilon = T \quad (\text{ring with } \varepsilon^2=0)$$

$$k[\varepsilon] = \{a+b\varepsilon \mid a, b \in k\}, \quad \varepsilon^2=0$$

- $F(k[\varepsilon])$ is a finite dimensional k -vector space.

Remark: fiber products don't always exist in $\hat{\mathcal{C}}_A$ $A = W(k)$

Example $A = k[[X, Y]], \quad C = k[[X]], \quad B = k$

maps

$$\begin{array}{ccc} A \rightarrow C & , & B \rightarrow C \\ Y \mapsto 0 & & \\ k \hookrightarrow k[[X]] & & \end{array}$$

Check: the fiber product $A \times_B C$ is not in the category $\hat{\mathcal{C}}_A$ (the fiber product as rings is not Noetherian ... this argument should not work)

Fiber products exist in $\hat{\mathcal{C}}_A$.

F has the Mayer-Vietoris property if the map $F(A \times_B C) \rightarrow F(A) \times_{F(B)} F(C)$ is a bijection.

Remark: F pro-representable $\Rightarrow F$ has the Mayer-Vietoris - property.

- $F(k[\varepsilon])$ is a k -vector space:

Why is it a k -vector space? It is when the map $F(k[\varepsilon] \times_k k[\varepsilon])) \xrightarrow{\sim} F(k[\varepsilon]) \times_{F(k)} F(k[\varepsilon])$ is a bijection. (induced by $k[\varepsilon] \xrightarrow{\sim} k \xleftarrow{\sim} k[\varepsilon]$)
 $\varepsilon \mapsto 0 \leftrightarrow \varepsilon$

k -scalar multiplication is induced by

$$\begin{aligned} k \times k[\varepsilon] &\longrightarrow k[\varepsilon] \\ (\lambda, a+b\varepsilon) &\mapsto (a+\lambda b)\varepsilon \end{aligned}$$

; addition is induced by

$$\begin{cases} k[\varepsilon] \times_k k[\varepsilon] \longrightarrow k[\varepsilon] \\ (a+b\varepsilon, c+d\varepsilon) \mapsto a+(b+c)\varepsilon \end{cases}$$

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In this case we call $F(k[\epsilon])$ the tangent space of F . Why?

When F is pro-representable, $F \cong \text{Hom}_{\mathcal{C}_k}(R, \cdot)$ then there is an isomorphism of k -vector spaces

$$t_R \longrightarrow F(k[\epsilon])$$

Recall: for R a local Noetherian A -algebra with maximal ideal m_R , then

$t_R^* = \frac{m_R}{(m_R^2, m_R)}$ is the cotangent space of R , $t_R = \text{Hom}_R(t_R^*, k)$.

Idea of proof: $f \in F(k[\epsilon]) = \text{Hom}_{\mathcal{C}_k}(R, k[\epsilon])$

$$\text{then } f(r) = \bar{r} + f'(r) \in$$

for $r \in R$ $\begin{matrix} \text{image} \\ \text{mod } m_R \end{matrix}$

Criteria for representability:

Criterion. (Grothendieck) Let $F: \mathcal{C}_k \rightarrow \underline{\text{Set}}$ be a functor such that $F(k) = \{*\}$.

Then F is pro-representable if and only if the following hold:

- i) F has the Mayer-Vietoris property
- ii) $F(k[\epsilon])$ is a finite dimensional vector space $/k$.

We will use a refined version:

Criterion. (Schlessinger)

Def. If $R, S \in \mathcal{C}_k$, $f: R \rightarrow S$ a morphism. We say that f is small if it is surjective and if $\ker f$ is annihilated by m_R ($m_R \cdot \ker f = 0$) (Exp.: $k[\epsilon] \xrightarrow{\epsilon \mapsto 0} k$)

We introduce the Schlessinger conditions. $R_0, R_1, R_2 \in \mathcal{C}_k$, $\varphi_1: R_1 \rightarrow R_0$, $\varphi_2: R_2 \rightarrow R_0$, $R_3 := R_1 \times_{R_0} R_2 \dots$; $(*) : F(R_3) \rightarrow F(R_1) \times_{F(R_0)} F(R_2)$

The conditions are:

H1: If $\varphi_2 : R_2 \rightarrow R_0$ is small, then $(*)$ is surjective

H2: If $R_0 = k$, $R_2 = k[\epsilon]$ with $R_2 \xrightarrow{\epsilon \mapsto 0} R_0$, then $(*)$ is bijective

H3: $F(k[\epsilon])$ is finite-dimensional over k .

H4: If $R_1 = R_2$, $\varphi_1 = \varphi_2$ is small, then $(*)$ is bijective.

Theorem of Schlessinger: If $F(k) = \{*\}$ and it satisfies H1-H4, then F is pro-representable.