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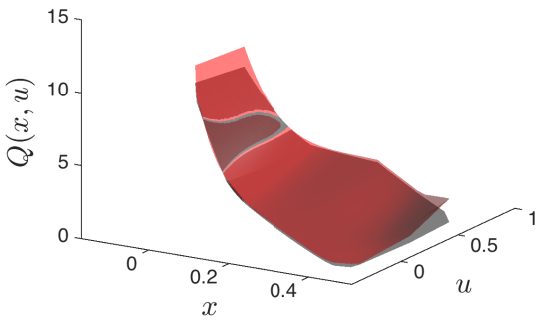

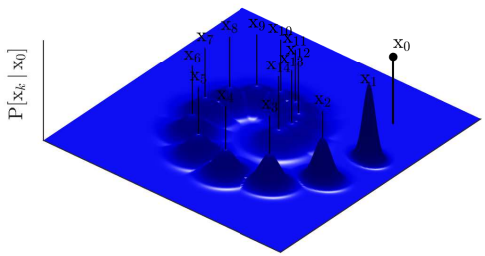

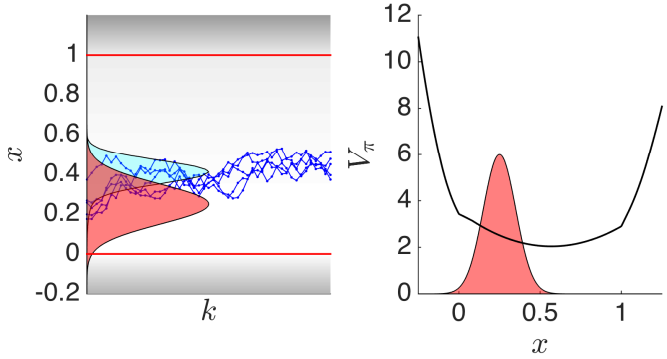

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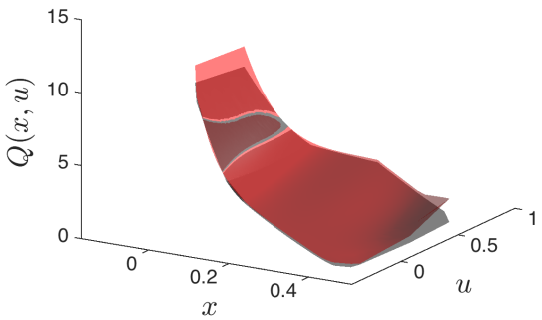

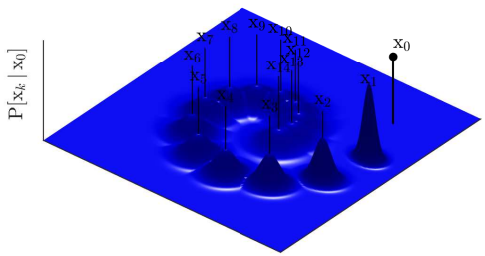

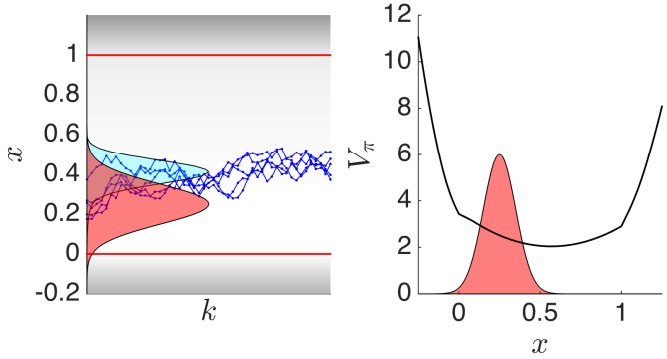

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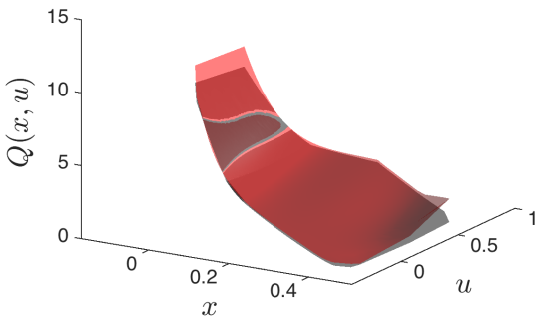

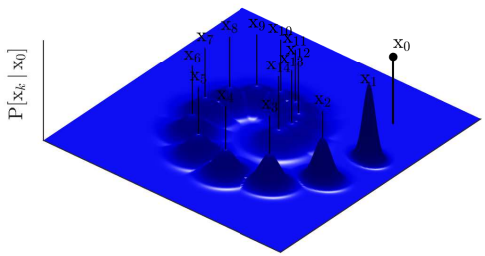

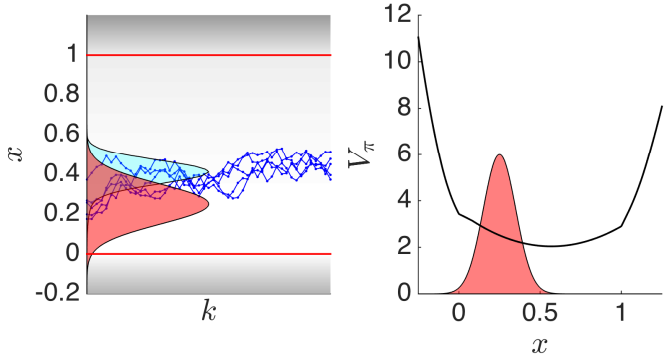

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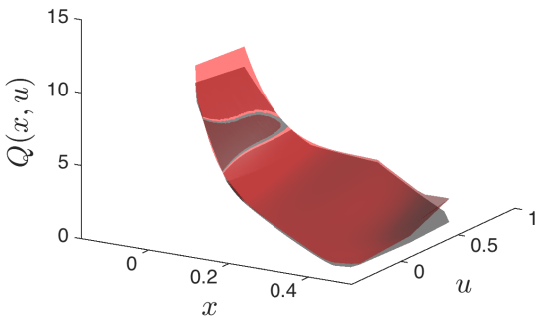

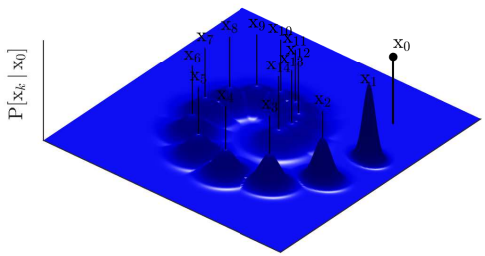

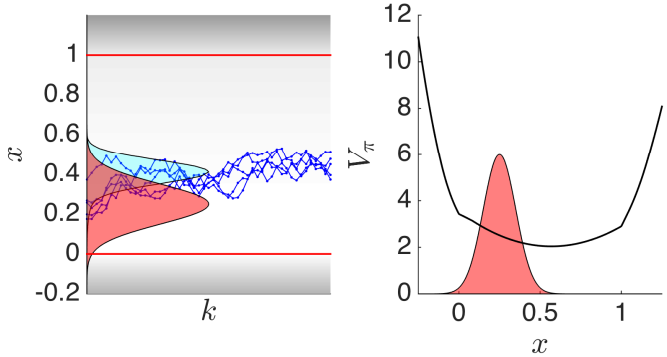

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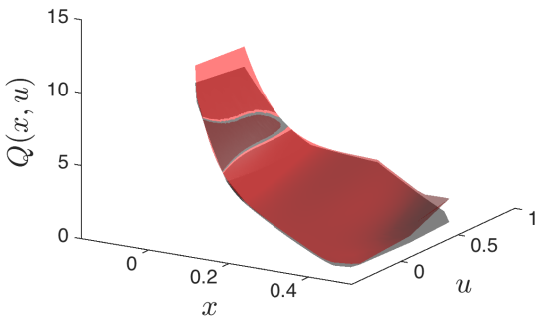

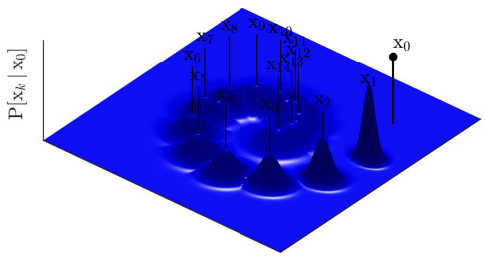

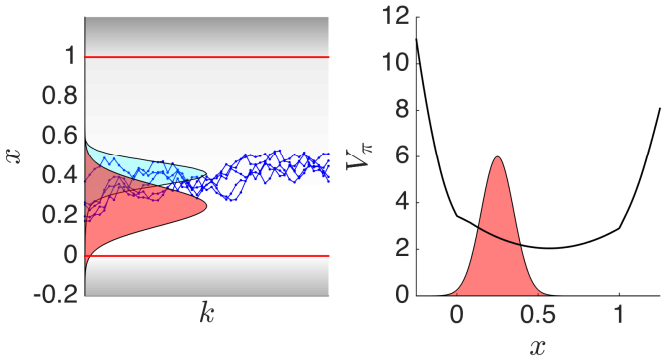

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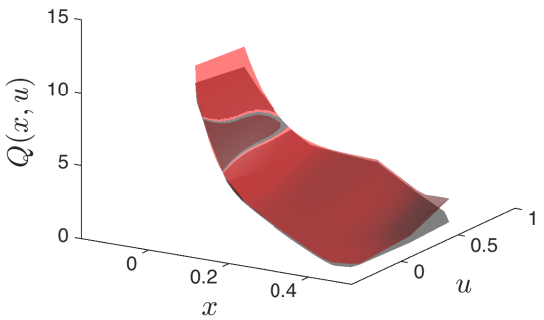

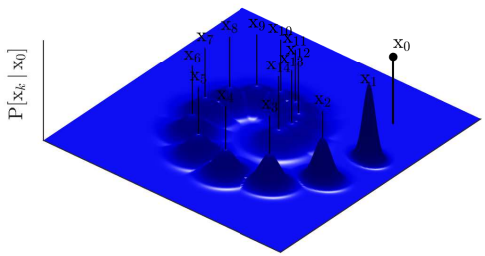

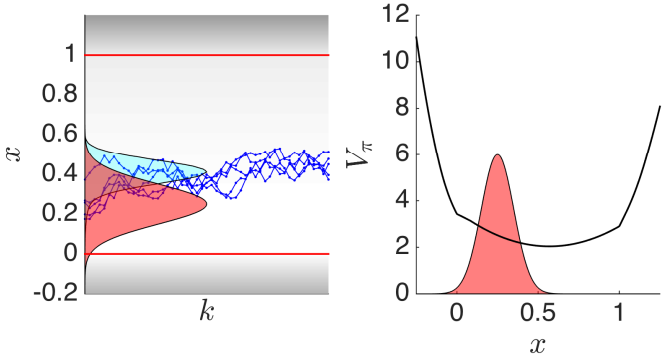

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"Classic" optimal control for given initial state $\mathbf{s}$:

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Can we define & treat optimality for dynamic systems in a more abstract way?

We want to address:

- Any set/space of state & input (functional, discrete, games, etc.)
- Any kind of dynamics (stochastic, mixed-integer, etc.)
- Any kind of stage cost (discontinuous, taking infinite values, stochastic, etc.)
- Infinite-horizon problems, state-dependent termination (game, target)
- Obtain input policies as opposed to input profiles

Markov Decision Processes: the Bellman equations and Dynamic Programming

Sébastien Gros

Cybernetic, NTNU
Elec. Eng., Chalmers

TUM lectures on RL

Outline

Notation & assumptions

- We will work in discrete time for simplicity

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- What do we keep from "classic" optimal control?
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  - Performance is minimizing the sum of L

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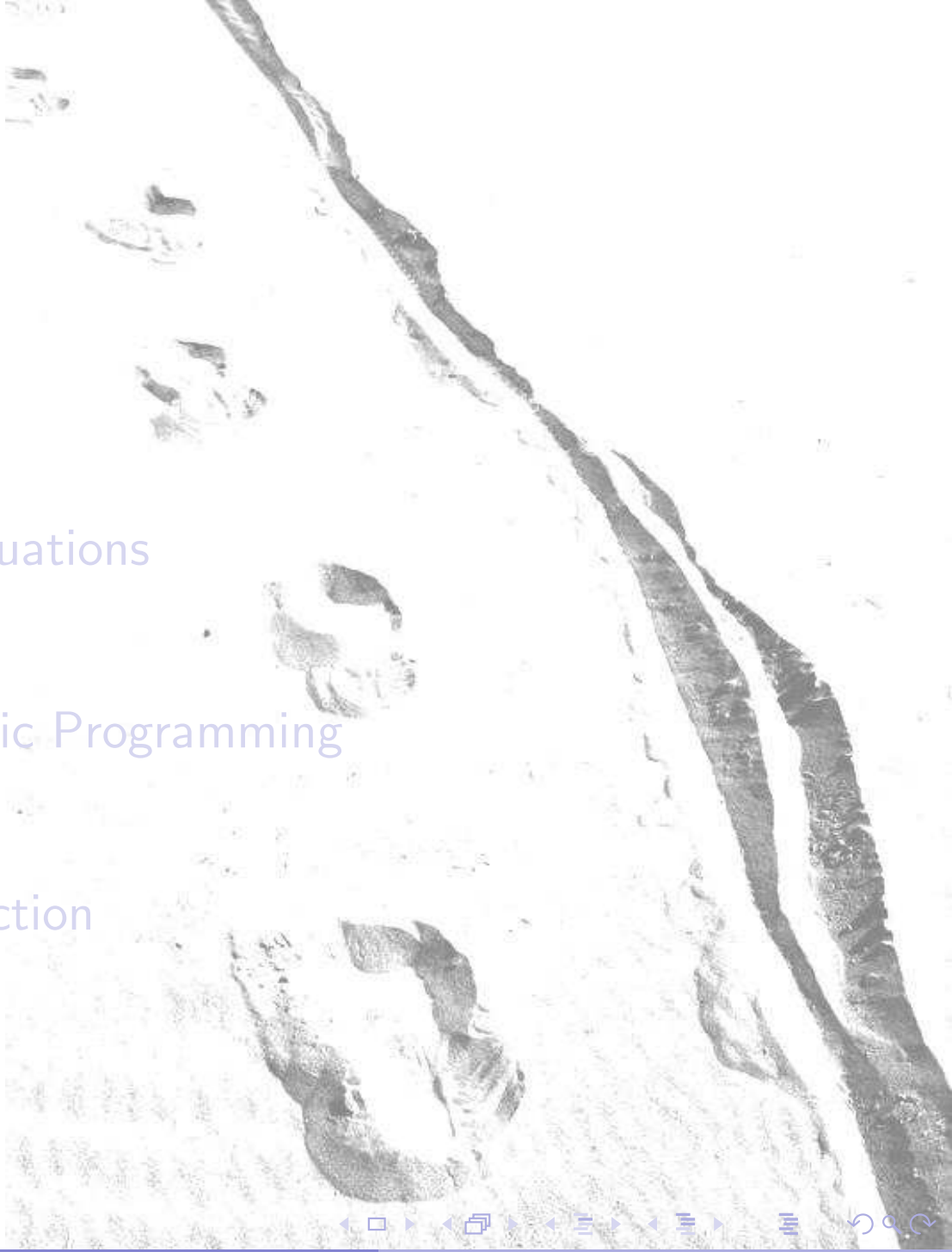

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- Process noise $\mathbf{x}_+ = \mathbf{f}(\mathbf{x}, \mathbf{u}) + \mathbf{e}$ where e.g. $\mathbf{e} \sim \mathcal{N}(0, \Sigma)$, then

$$\mathbb{P}[\mathbf{x}_+ \,|\, \mathbf{x}, \mathbf{u}] \propto e^{-\frac{1}{2}(\mathbf{x}_+ - \mathbf{f}(\mathbf{x},\mathbf{u}))^\top \Sigma (\mathbf{x}_+ - \mathbf{f}(\mathbf{x},\mathbf{u}))}$$

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Stage cost function: $L(\mathbf{x}, \mathbf{u}) \in \mathbb{R}$

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yields:

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MDP, return and discount - How to assess long-term performance?

The triplet $(\mathbb{P}[\mathbf{x}_+ \,|\, \mathbf{x}, \mathbf{u}], \gamma, L)$ define a Markov Decision Process

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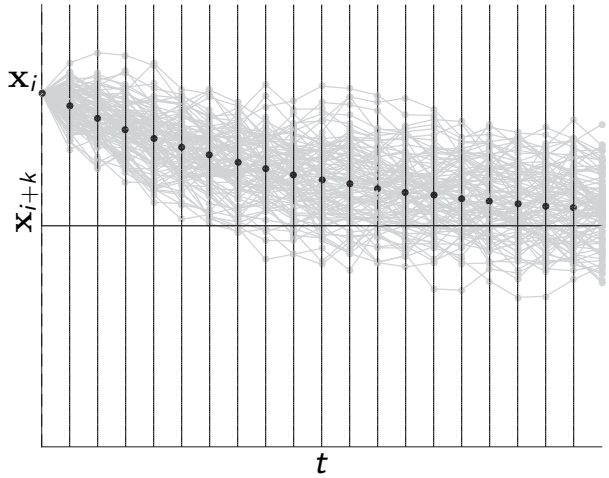

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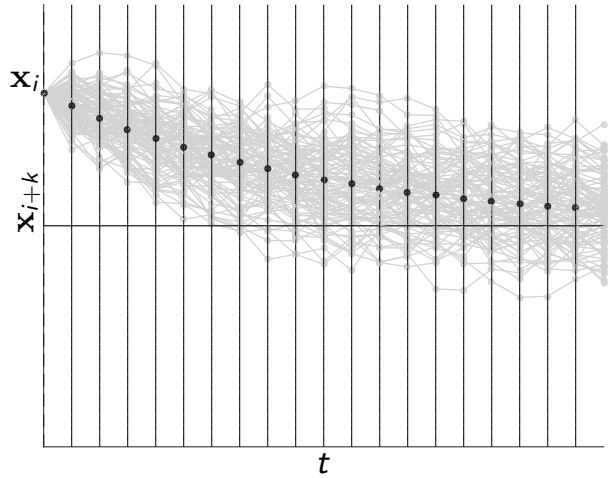


Cost-to-go (a.k.a. return)

$$J = \mathbb{E}\left[\sum_{k=0}^{\infty} \gamma^k L(\mathbf{x}_{i+k}, \mathbf{u}_{i+k})\right]$$

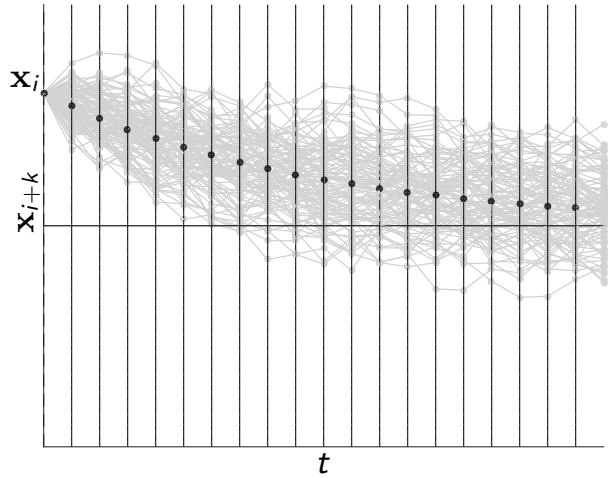
where

- $\mathbf{x}_{i+k}, \mathbf{u}_{i+k}$ is the state trajectory and input sequence
- $\gamma \leq 1$ is the discount factor

MDP, return and discount - How to assess long-term performance?

The triplet $(\mathbb{P}[\mathbf{x}_+ \,|\, \mathbf{x}, \mathbf{u}], \gamma, L)$ define a Markov Decision Process

- State and input $\mathbf{x}, \mathbf{u}$
- State transition (model) $\mathbb{P}[\mathbf{x}_+ \,|\, \mathbf{x}, \mathbf{u}]$
- Stage cost L
- Discount γ



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Policy (decisions) taken such that J is minimized

What is $\mathbb{E}$ expecting over (distribution)?

Policy - How to define decisions?

Deterministic

$$\mathbf{u} = \boldsymbol{\pi}(\mathbf{x})$$

Meaning: what input $\mathbf{u}$ do we select *provided* that we are in state $\mathbf{x}$

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Example LQR:

$$L(\mathbf{x}, \mathbf{u}) = \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} + \frac{1}{2}\mathbf{u}^\top R\mathbf{u}$$

$$\mathbf{x}_+ = A\mathbf{x} + B\mathbf{u}$$

yields: $\mathbf{u} = -K\mathbf{x}$
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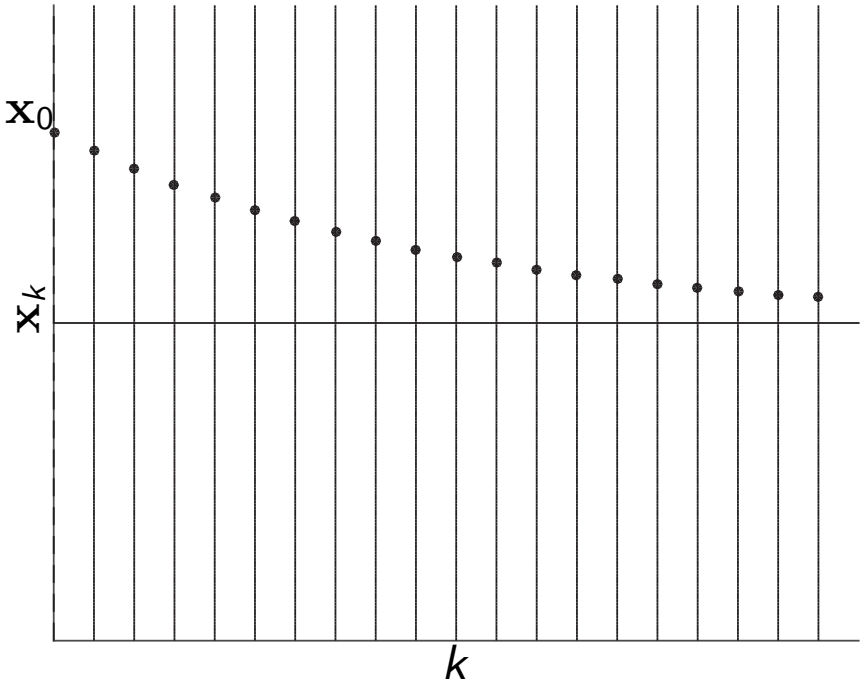
Remarks

- MDPs always have a deterministic optimal policy
- Optimal policy may not be unique
- If several actions are equally good, then a random choice between them is still optimum, hence some MDPs admit stochastic optimal policies

Unpacking the Markov Chain

Unpacking the Markov Chain



Deterministic state transition

$$\mathbf{x}_+ = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

$$\mathbb{P}[\mathbf{x}_+ \mid \mathbf{x}, \mathbf{u}] = \delta(\mathbf{x}_+ - \mathbf{f}(\mathbf{x}, \mathbf{u}))$$

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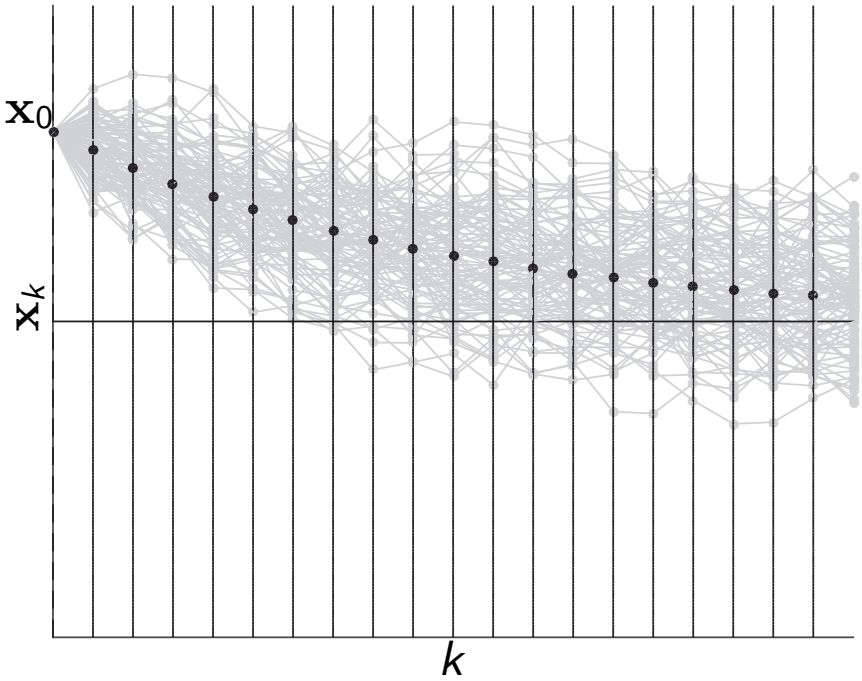

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$$\mathbf{x}, \mathbf{u} \to \mathbf{x}_+ \quad \text{defined by} \quad \mathbb{P}\left[\mathbf{x}_+ \,|\, \mathbf{x}, \mathbf{u}\right]$$

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$$\mathbf{u} = \boldsymbol{\pi}(\mathbf{x}), \quad \text{or} \quad \boldsymbol{\pi}[\mathbf{u} \,|\, \mathbf{x}]$$

Transtion $\mathbf{x}_0 \to \mathbf{x}_k$ is stochastic:

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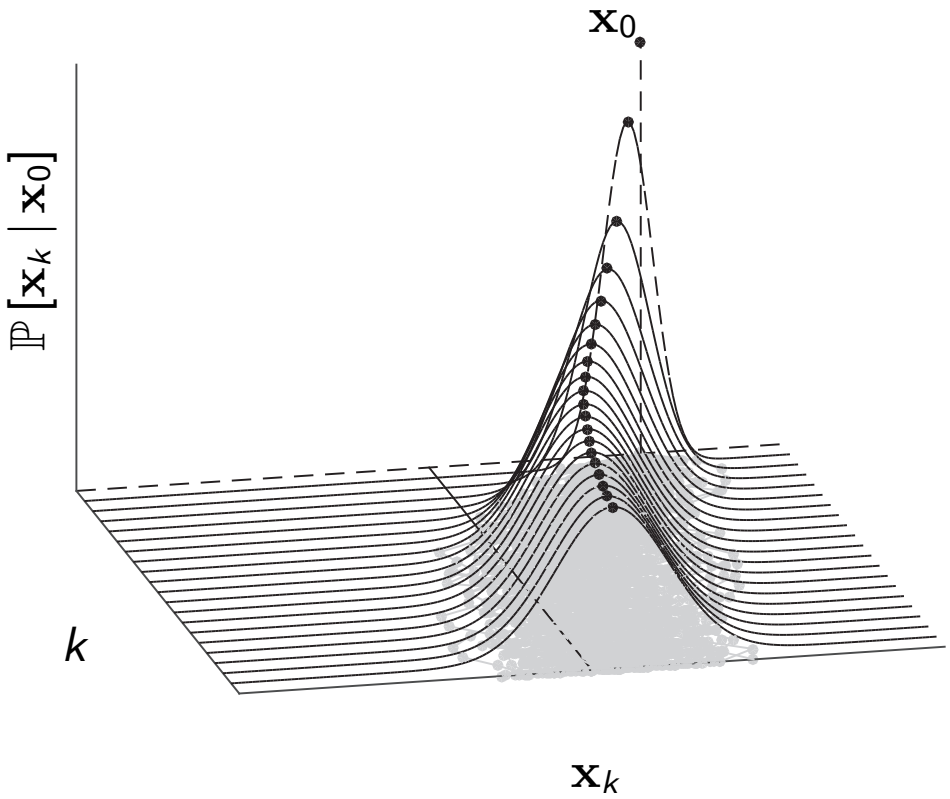

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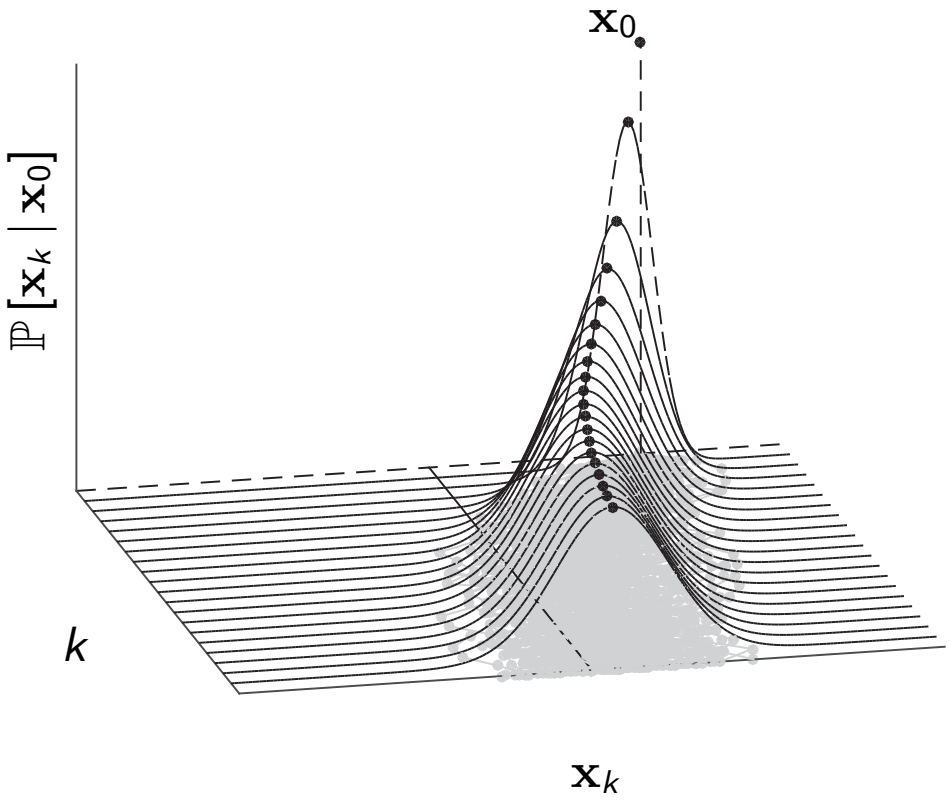

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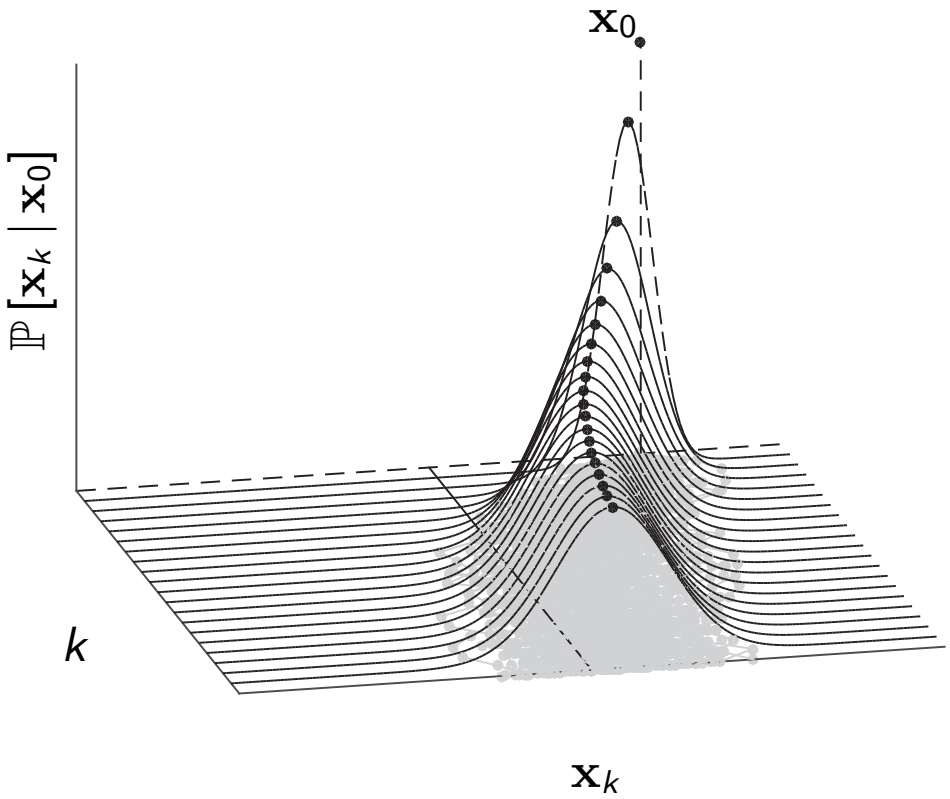
Transtion $\mathbf{x}_0 \to \mathbf{x}_k$ is stochastic:

$$\mathbb{P}[\mathbf{x}_k \,|\, \mathbf{x}_0]$$

1-step distribution:

$$\mathbb{P}[\mathbf{x}_1 \,|\, \mathbf{x}_0,\ \mathbf{u}_0 = \boldsymbol{\pi}(\mathbf{x}_0)] \equiv \mathbb{P}[\mathbf{x}_1 \,|\, \mathbf{x}_0,\ \boldsymbol{\pi}(\mathbf{x}_0)]$$

Unpacking the Markov Chain



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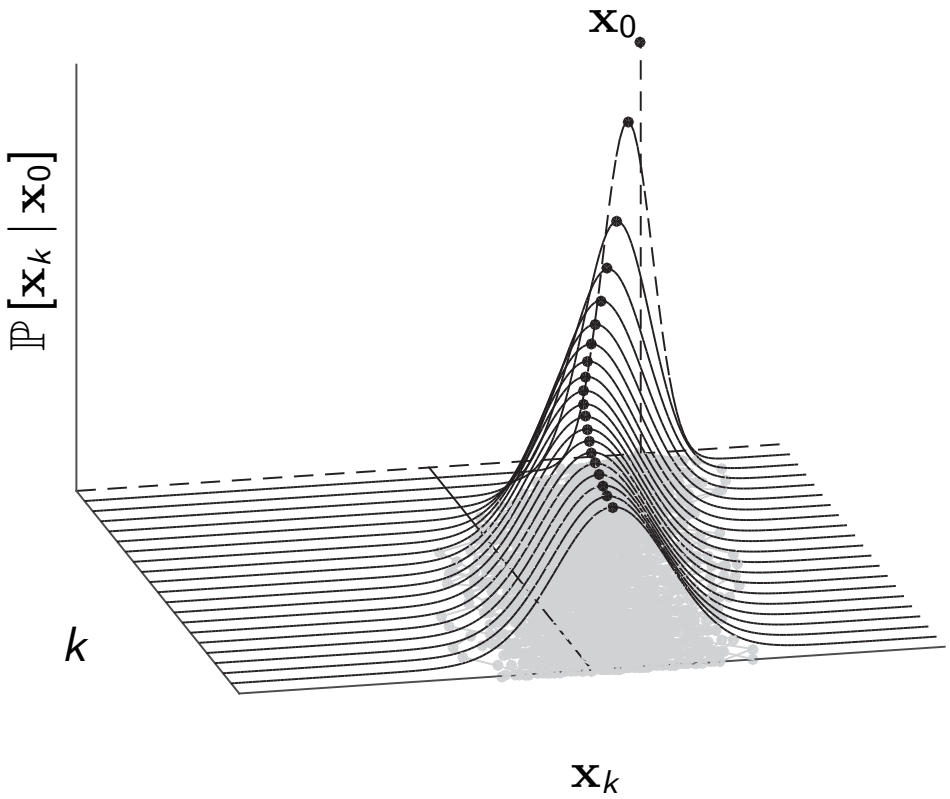
$$\mathbb{P}[\mathbf{x}_k \,|\, \mathbf{x}_0]$$

2-step distribution:

$$\mathbb{P}[\mathbf{x}_2 \,|\, \mathbf{x}_0] = \int \mathbb{P}[\mathbf{x}_2 \,|\, \mathbf{x}_1,\, \boldsymbol{\pi}(\mathbf{x}_1)]\, \mathbb{P}[\mathbf{x}_1 \,|\, \mathbf{x}_0,\, \mathbf{u}_0]\, \mathrm{d}\mathbf{x}_1$$

Unpacking the Markov Chain



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Transtion $\mathbf{x}_0 \to \mathbf{x}_k$ is stochastic:

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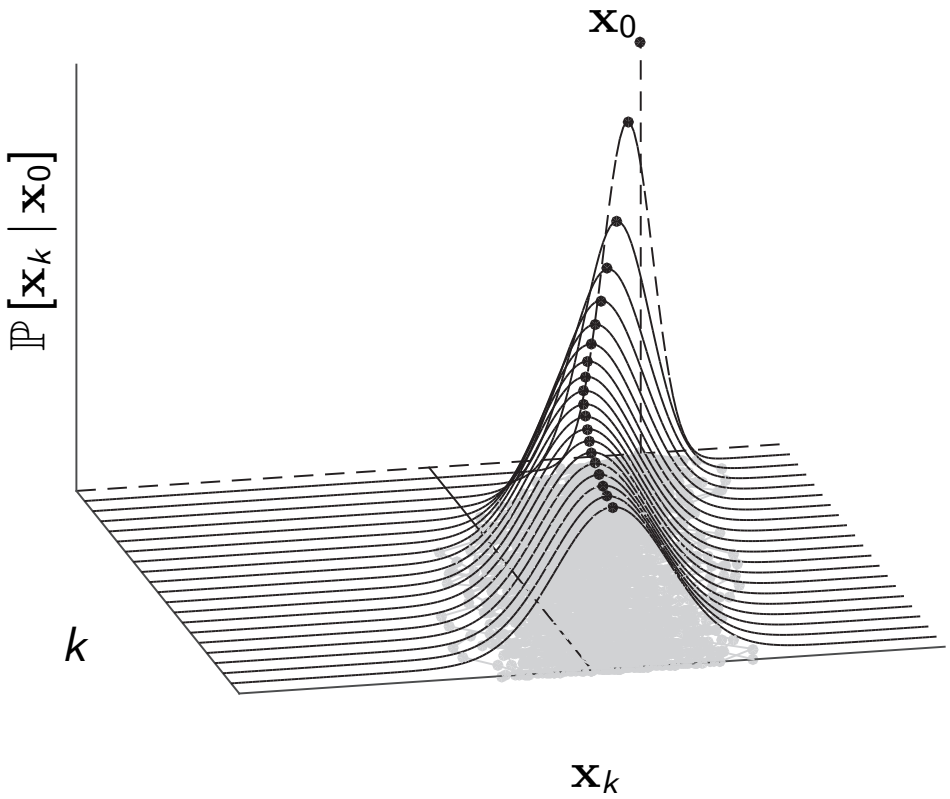
3-step distribution:

$$\mathbb{P}[\mathbf{x}_3 \,|\, \mathbf{x}_0] = \int \mathbb{P}[\mathbf{x}_3 \,|\, \mathbf{x}_2,\, \boldsymbol{\pi}(\mathbf{x}_2)]\, \mathbb{P}[\mathbf{x}_2 \,|\, \mathbf{x}_1,\, \boldsymbol{\pi}(\mathbf{x}_1)]\, \mathbb{P}[\mathbf{x}_1 \,|\, \mathbf{x}_0,\, \boldsymbol{\pi}(\mathbf{x}_0)]\, \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2$$

Unpacking the Markov Chain



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Transtion $\mathbf{x}_0 \to \mathbf{x}_k$ is stochastic:

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n-step distribution:

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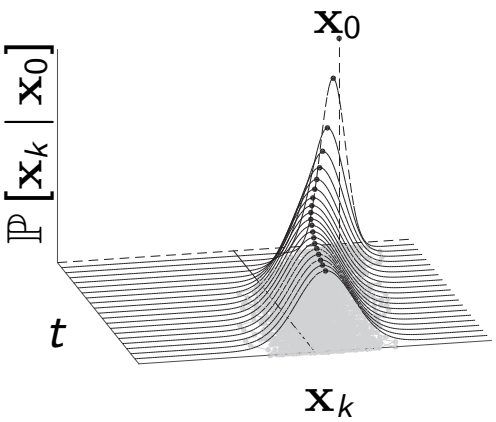

Stationary distribution: how often are the different states visited under a given policy, after the Markov chain reaches its stationary dynamics?

$$\mathbb{P}[\mathbf{x}_\infty \mid \mathbf{x}_0]$$

Markov chain distribution: $\tau_{\boldsymbol{\pi}}$ describes the distribution of trajectories, i.e. the probability distribution that specific trajectories $\mathbf{x}_{0,\ldots,\infty}$ are observed under policy $\boldsymbol{\pi}$

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- Expectation over distribution $\tau_{\boldsymbol{\pi}}$, e.g. performance of policy $\boldsymbol{\pi}$:

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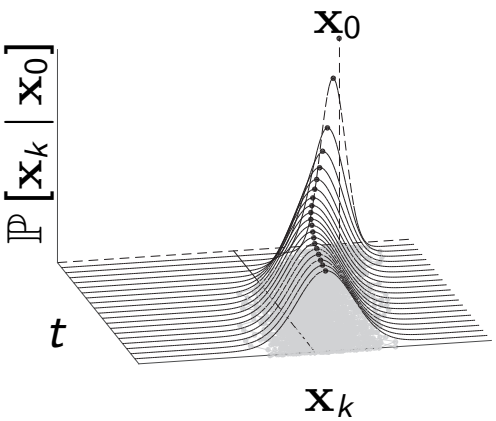

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$$J(\boldsymbol{\pi}) = \mathbb{E}_{\tau_{\boldsymbol{\pi}}}[L(\mathbf{x}, \boldsymbol{\pi}(\mathbf{x}))] = \int \sum_{k=0}^{\infty} \gamma^k L(\mathbf{x}_k, \boldsymbol{\pi}(\mathbf{x}_k)) \mathbb{P}[\mathbf{x}_k \mid \mathbf{x}_0] \mathbb{P}[\mathbf{x}_0] \, \mathrm{d}\mathbf{x}_0 \mathrm{d}\mathbf{x}_k$$

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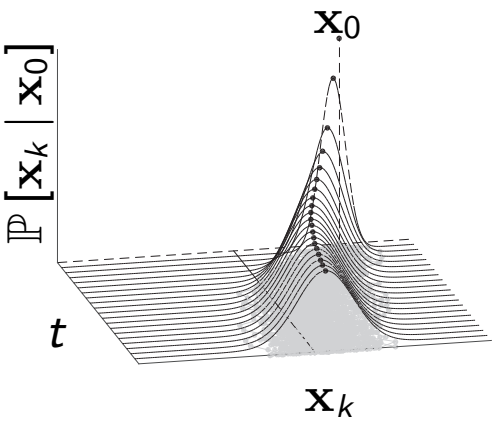

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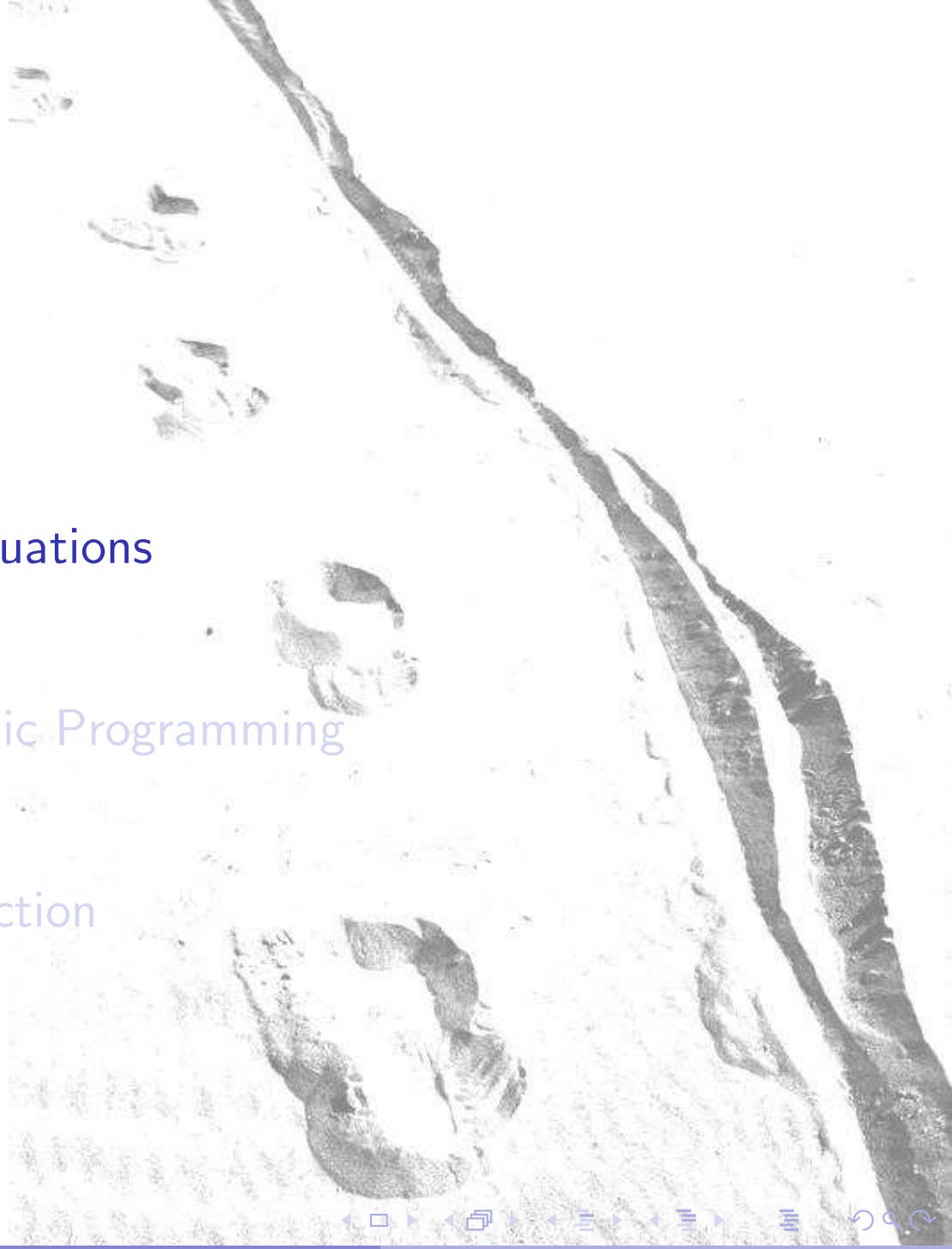
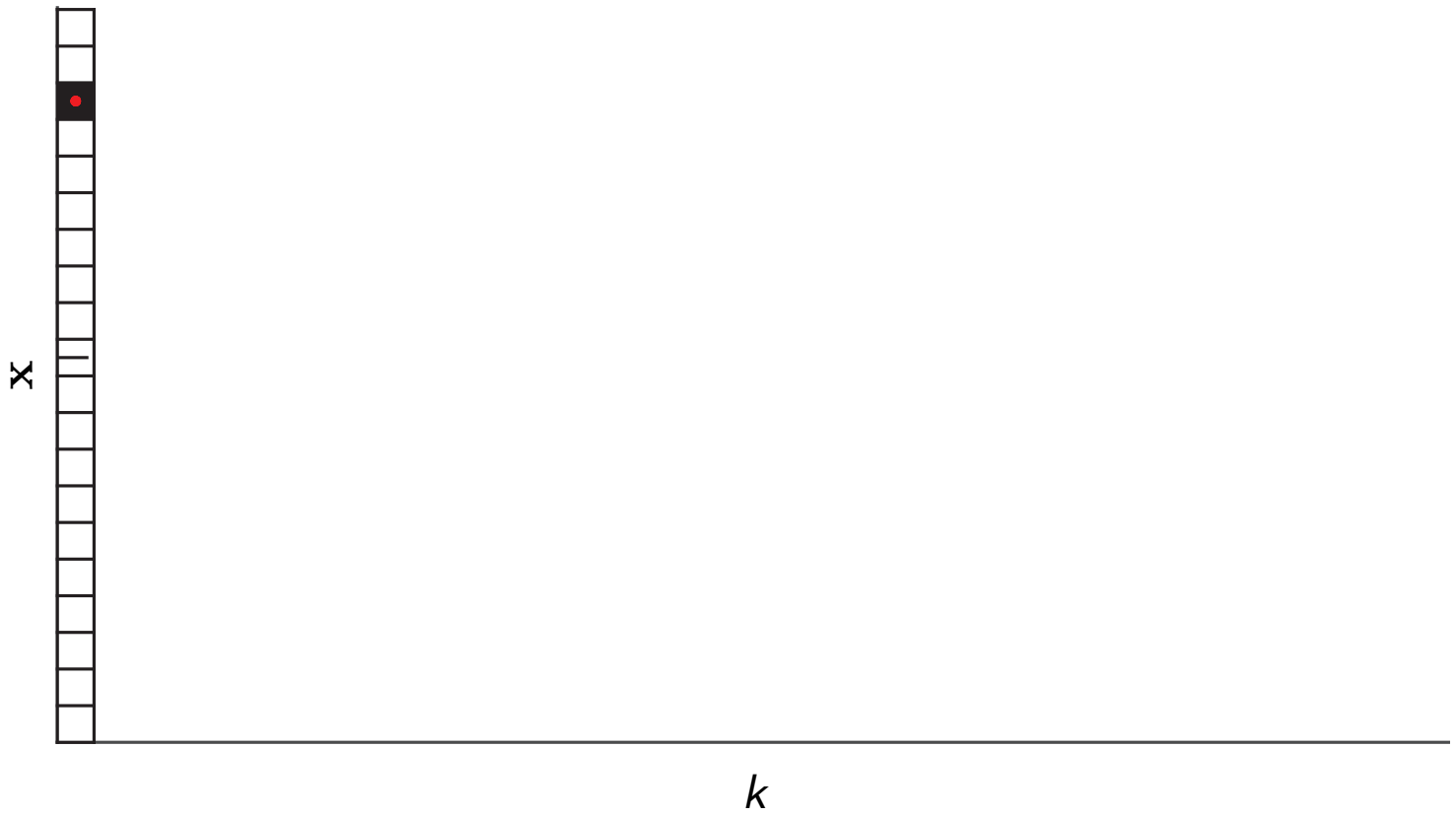
Outline



Let's use the following example (for illustration purposes)

- State $\mathbf{x} \in \mathbb{N}$ and input $\mathbf{u} \in \mathbb{N}$ locked on a grid
- Input $\mathbf{u}$ can only move $\mathbf{x}$ on the grid
- Stage cost: $L(\mathbf{x}, \mathbf{u}) = \frac{1}{2}\mathbf{x}^2 + \frac{1}{2}\mathbf{u}^2$
- Discount $\gamma = 0.9$

Dynamics: $\mathbf{x}_+ = \mathbf{x} + \mathbf{u} + \text{round}(\mathbf{e})$ where $\mathbf{e} \sim \mathcal{N}(0, 1)$



e.g. for
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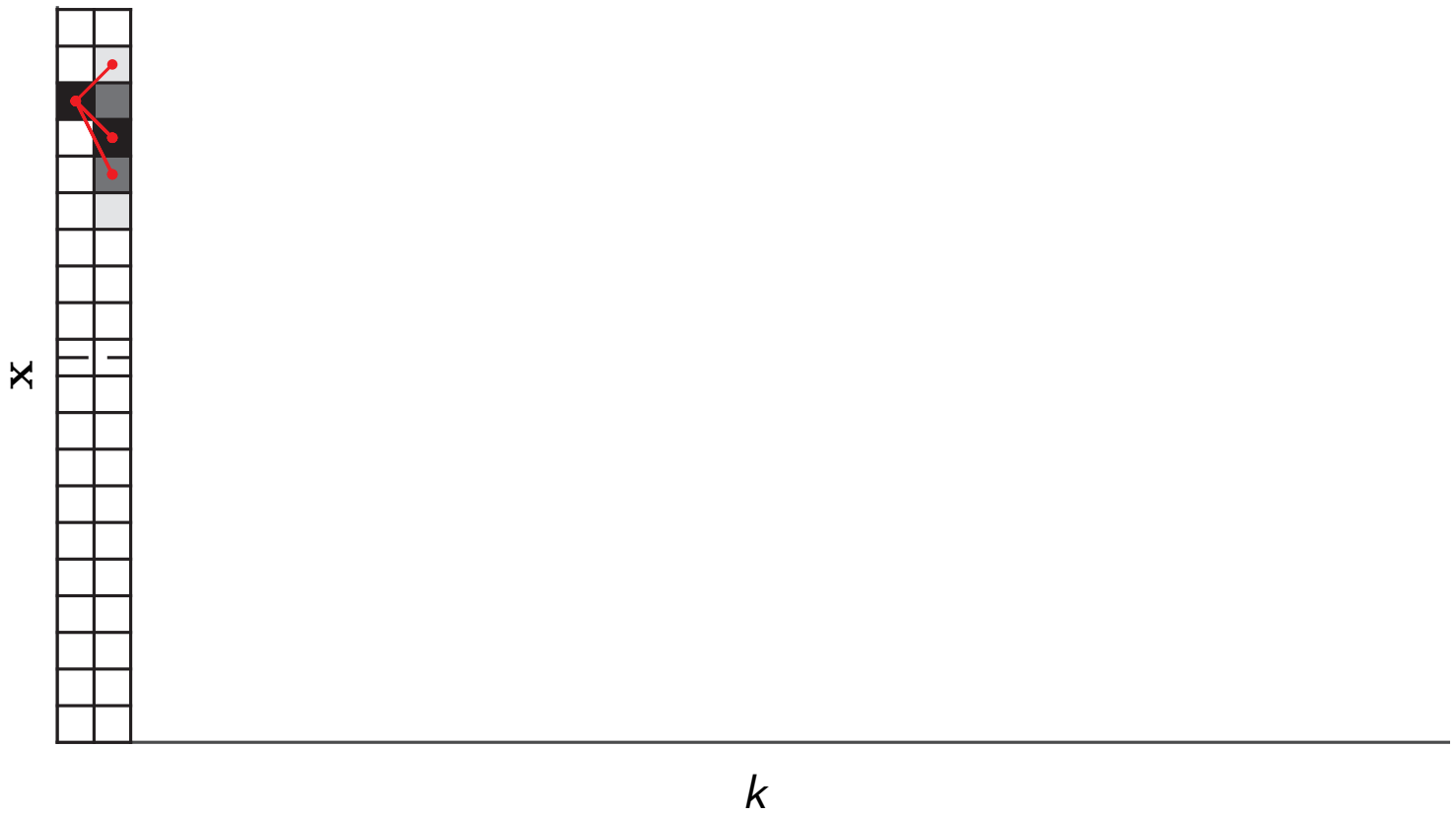

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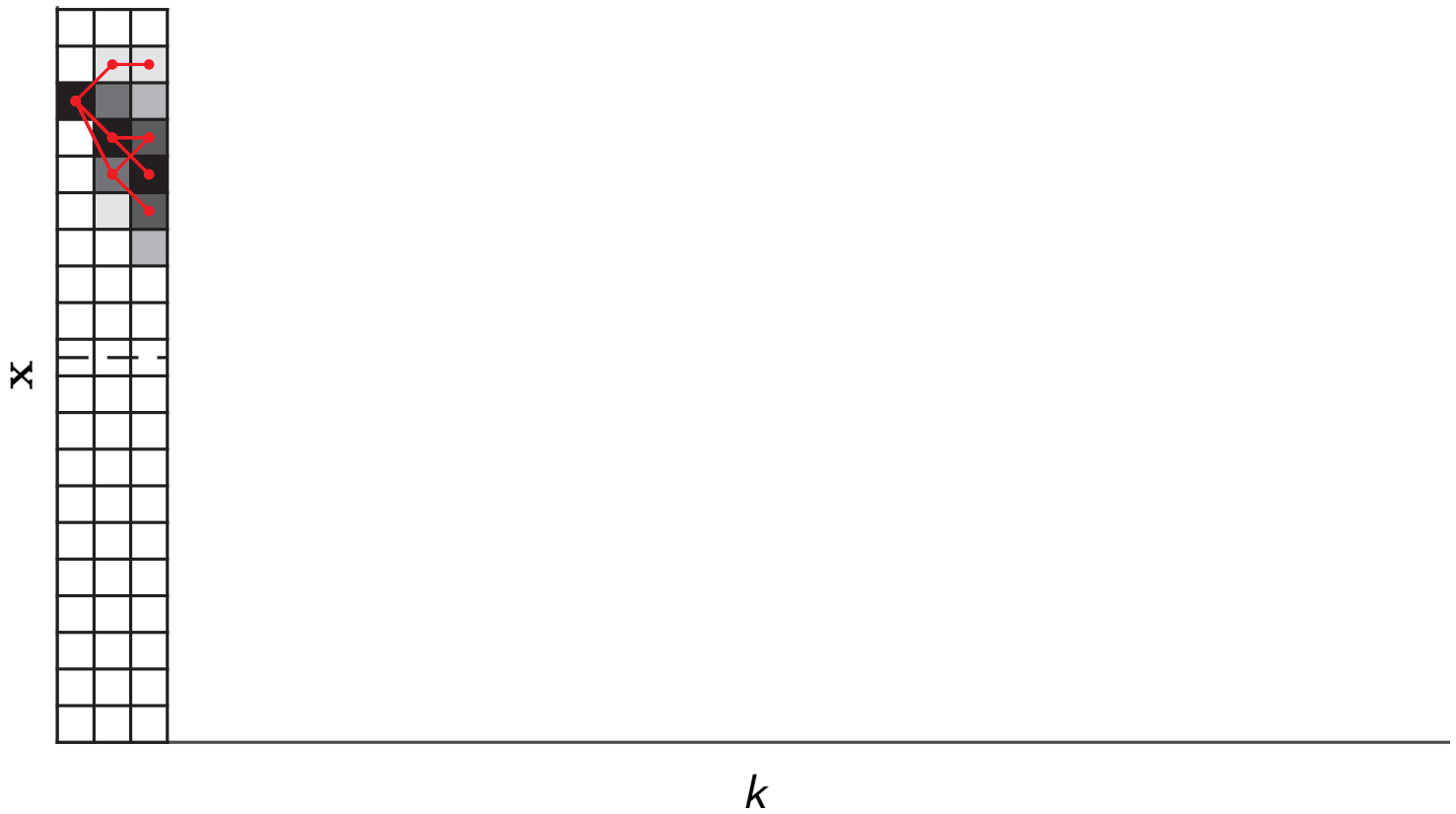

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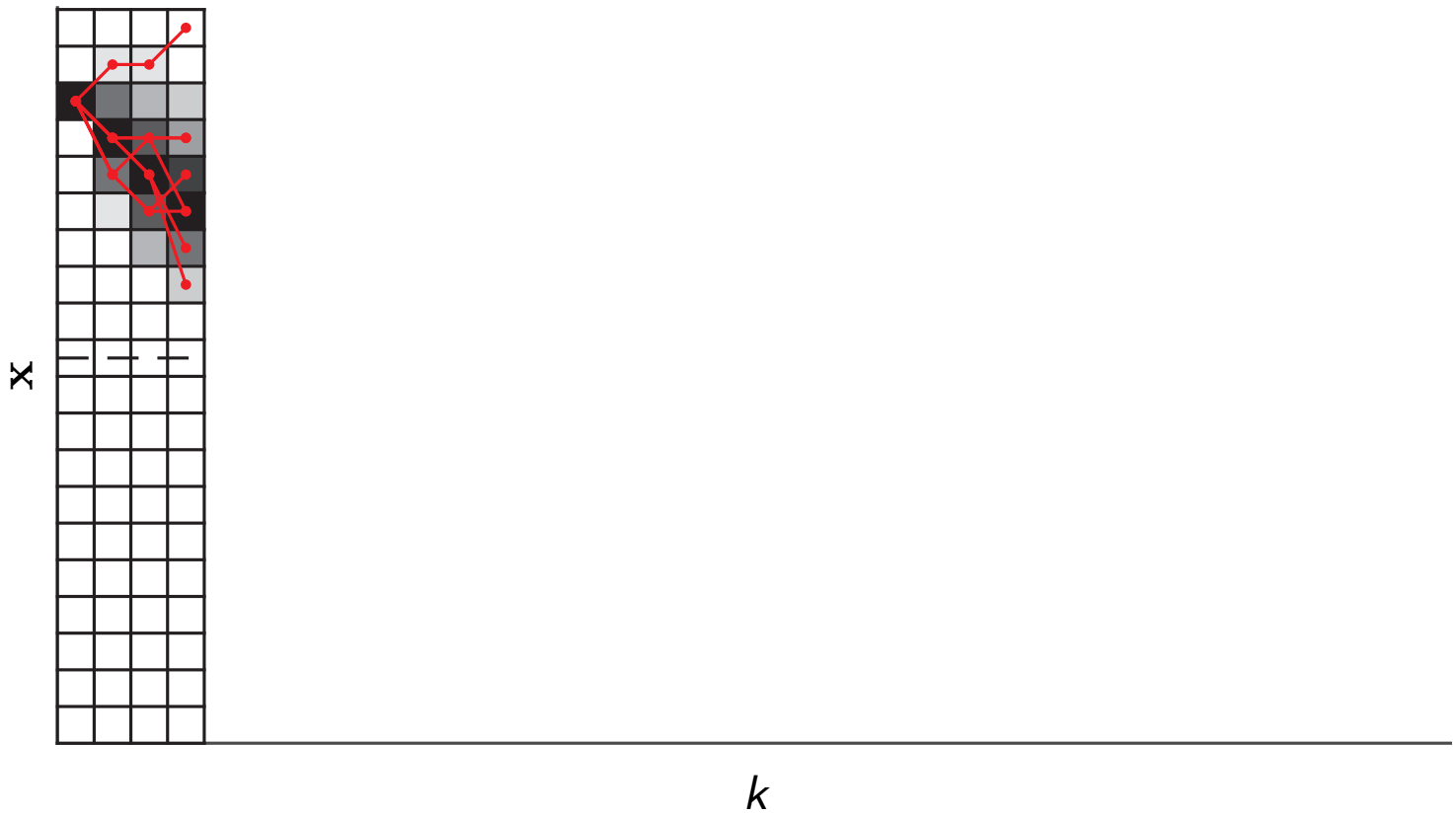

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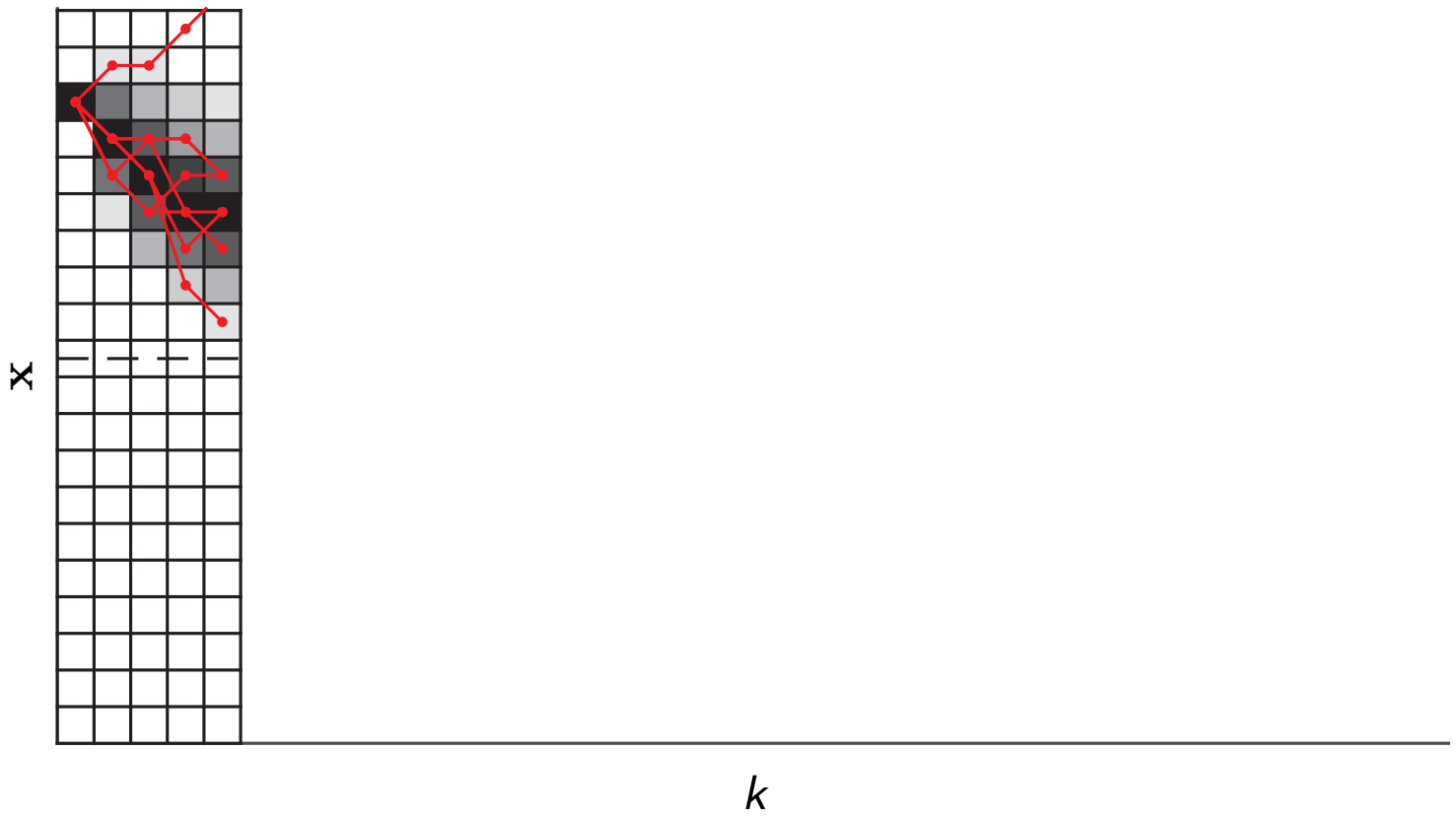

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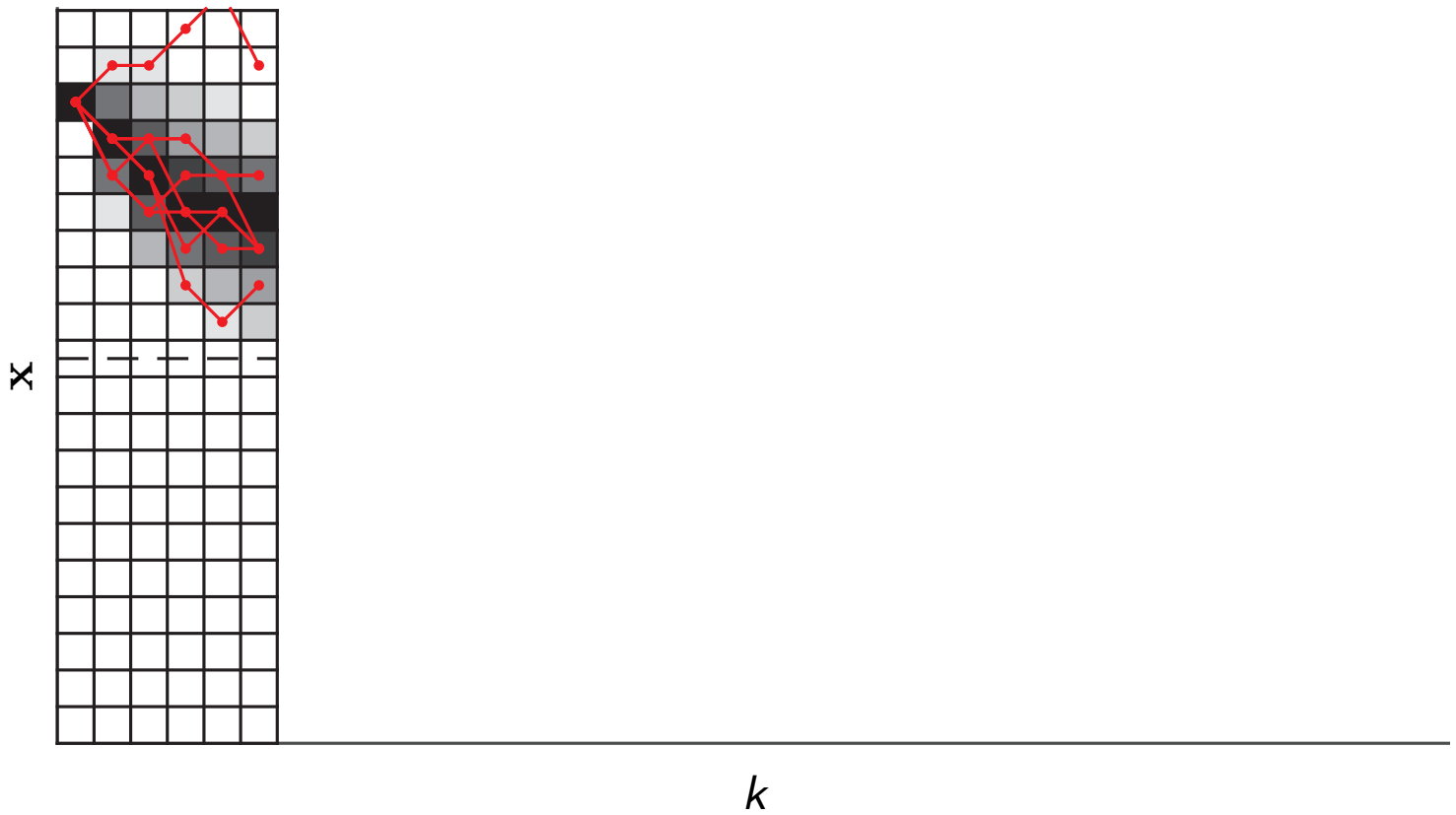

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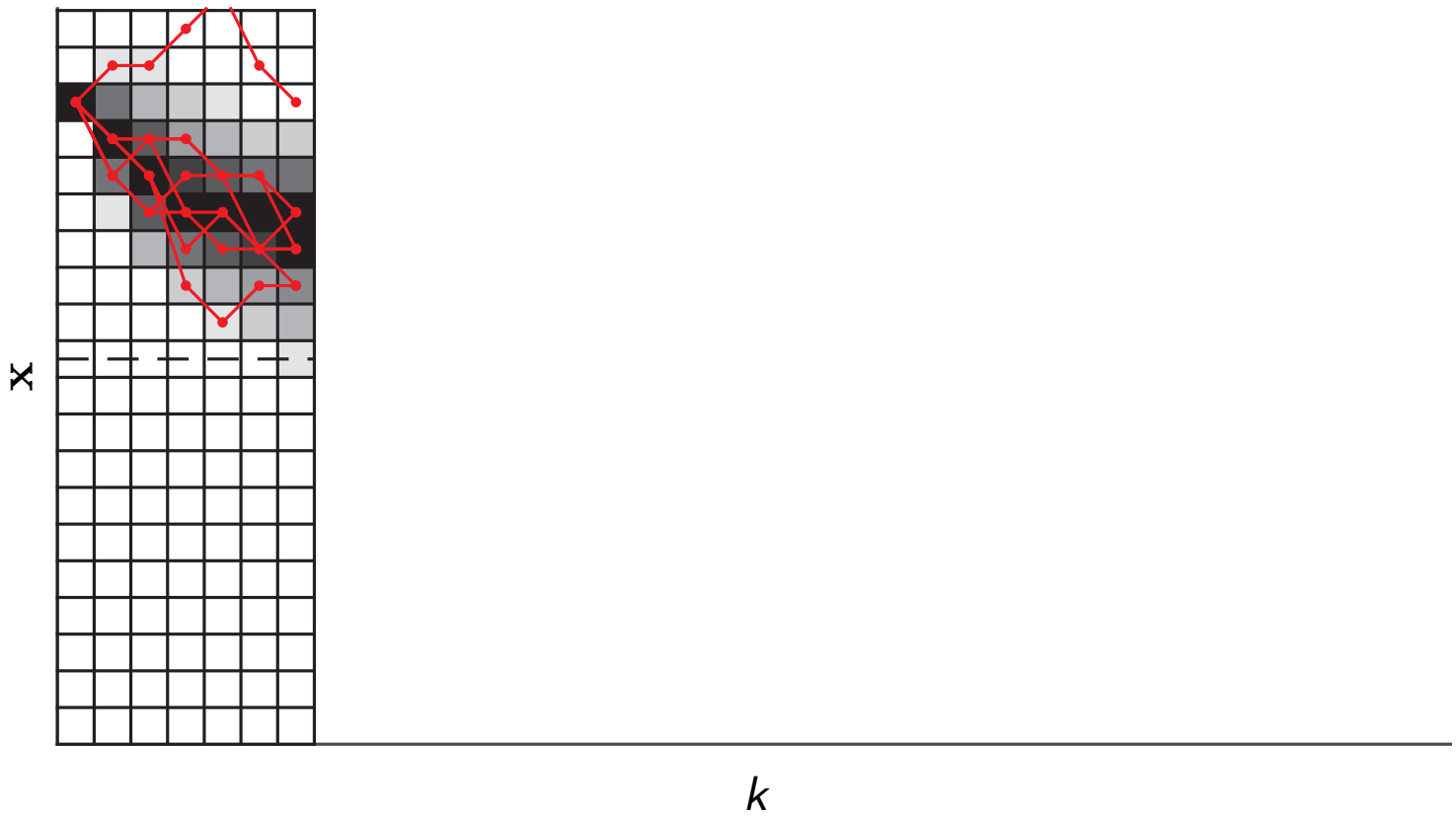

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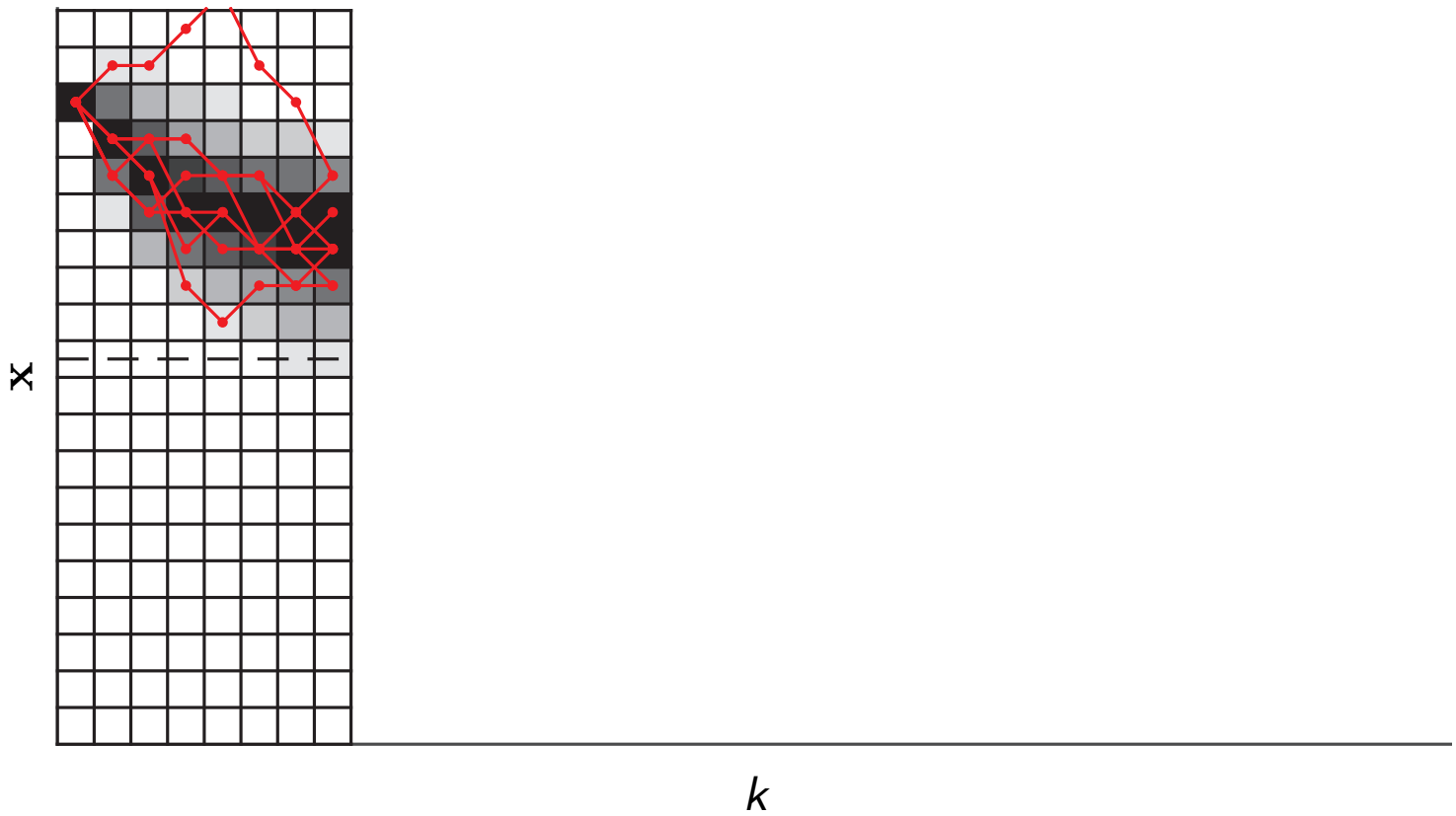

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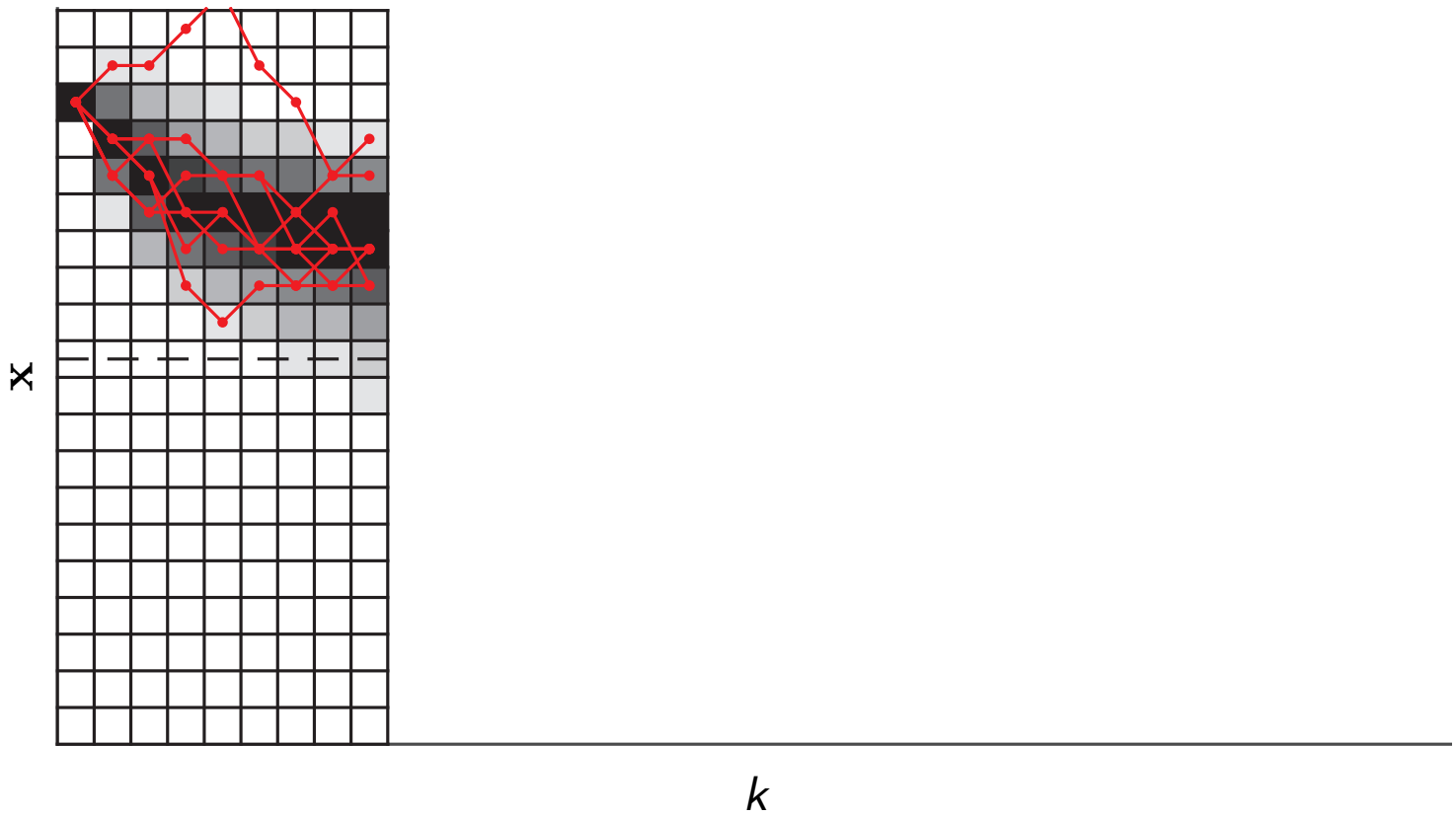

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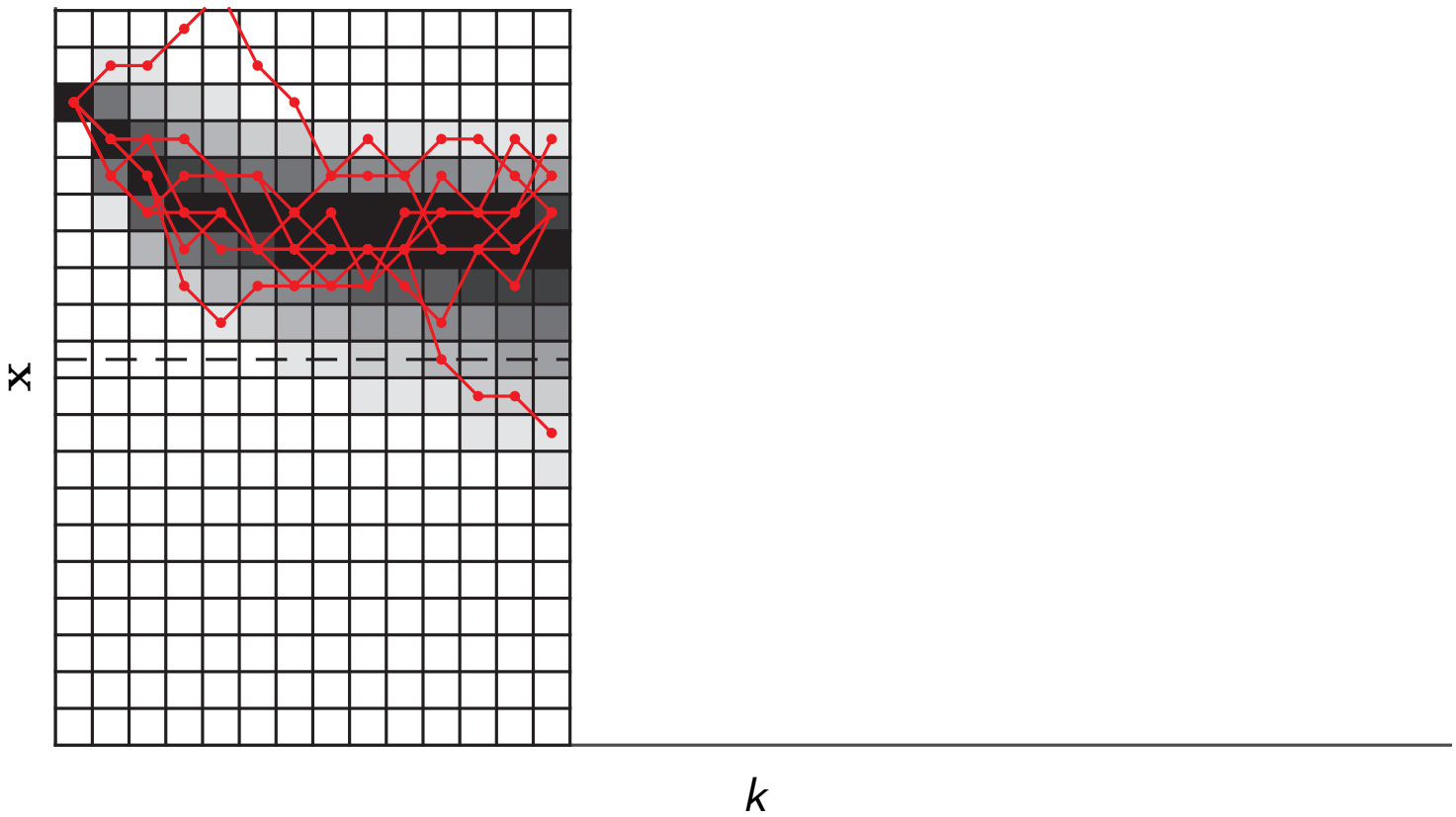

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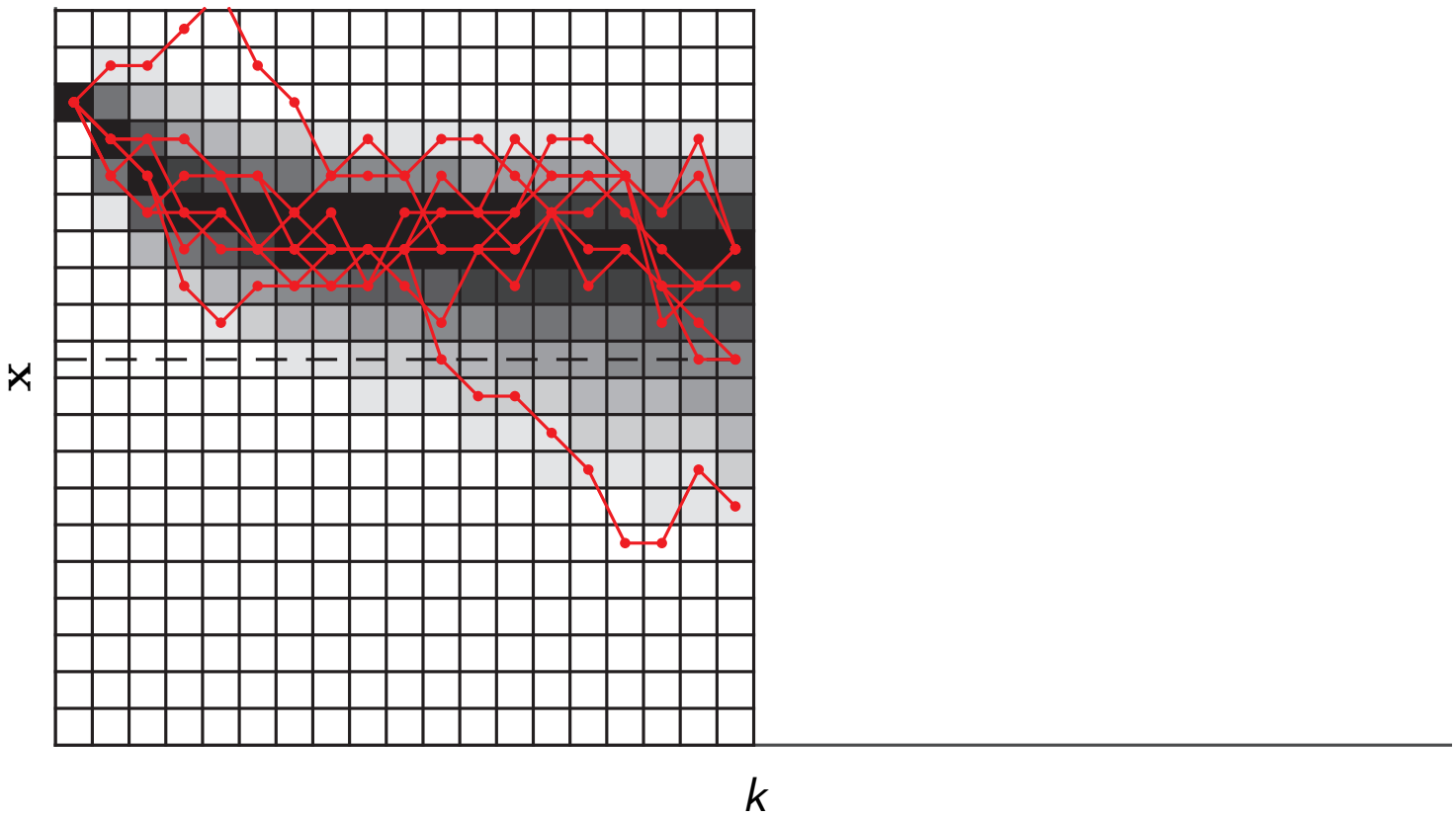

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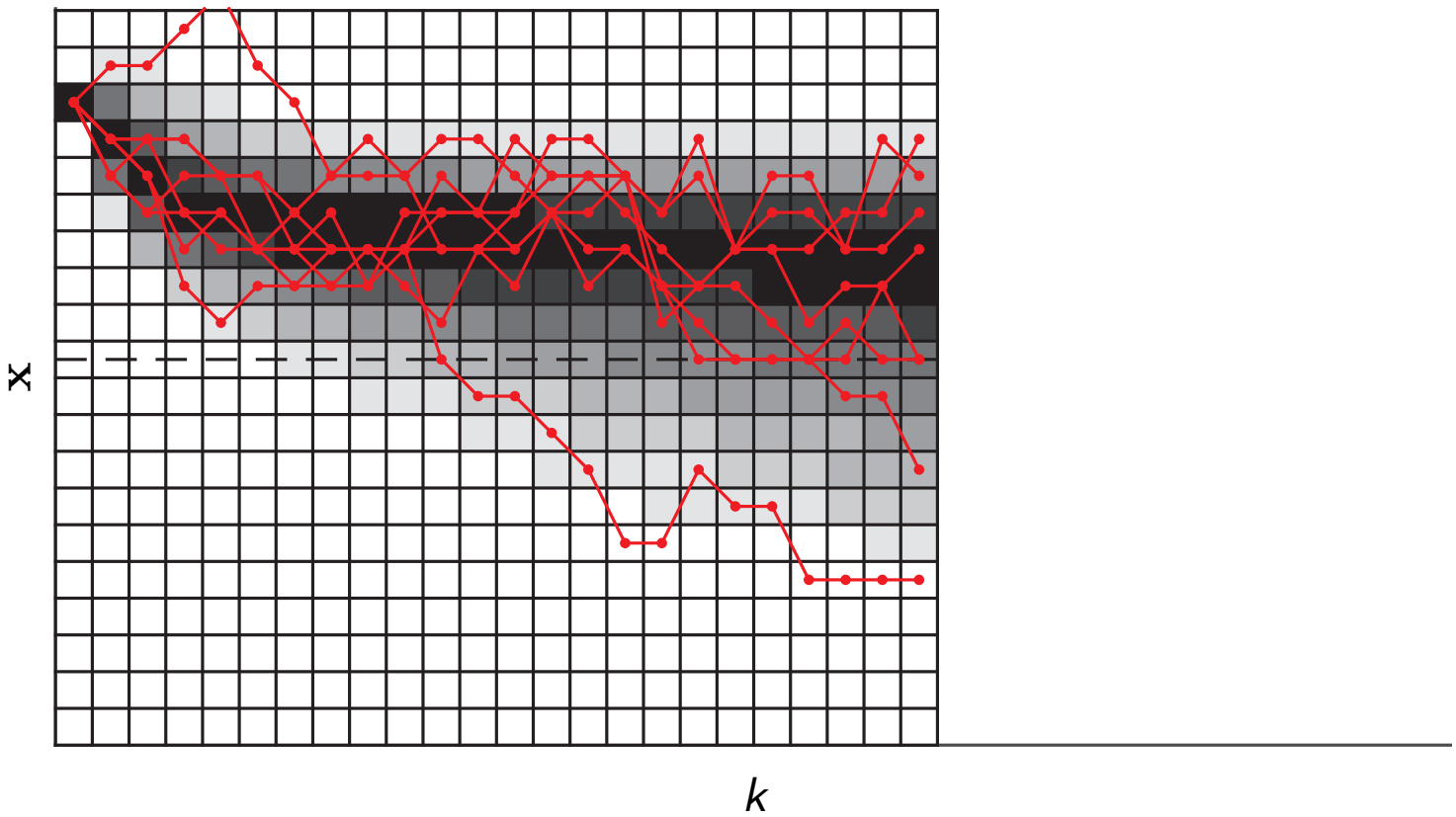

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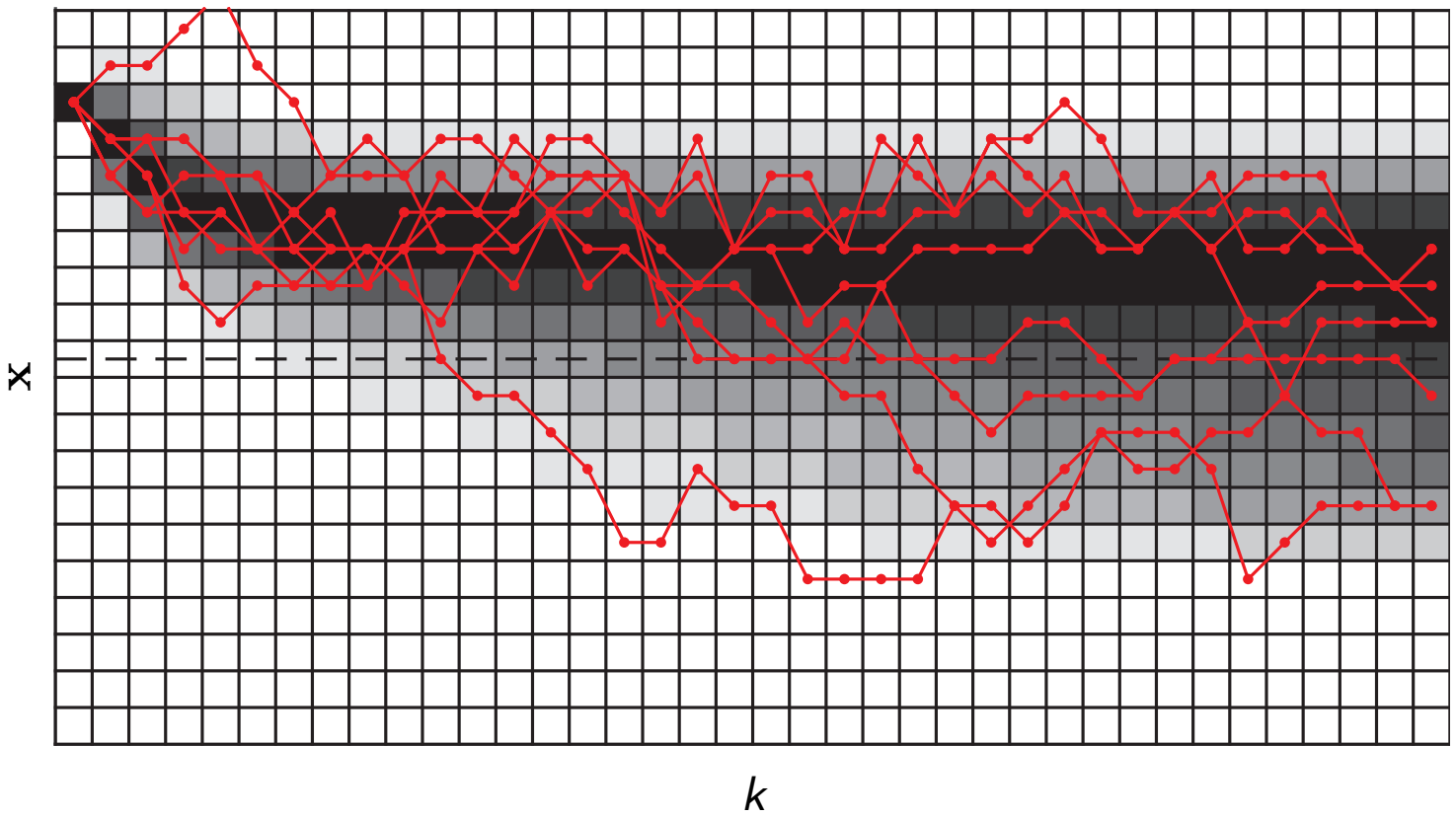

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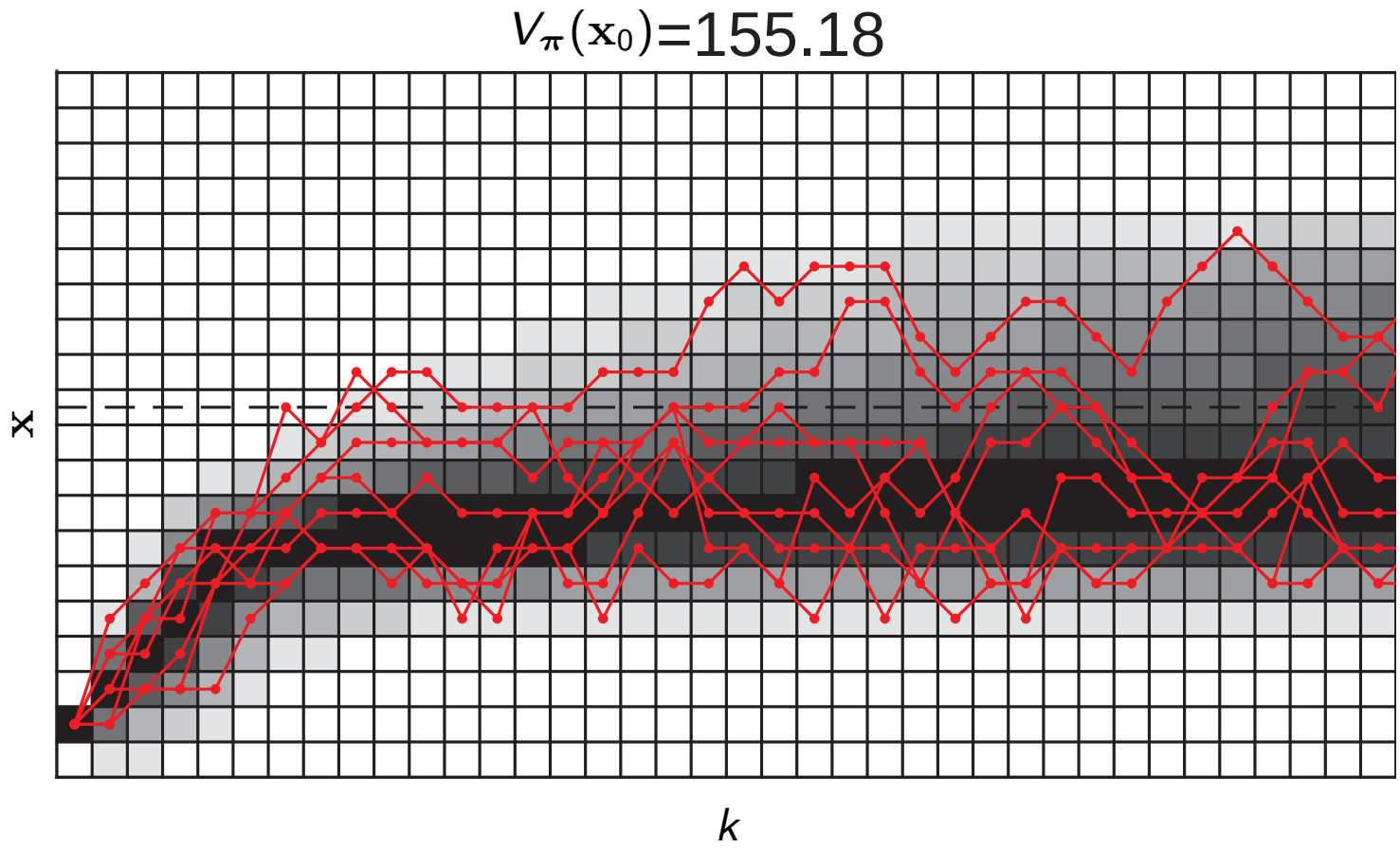
Value function - How good is it to be in a given state?

Value function for policy π:

$$V_{\pi}(\mathbf{x}) = \mathbb{E}\left[\sum_{k=0}^{\infty} \gamma^k L(\mathbf{x}_k, \mathbf{u}_k) \,\middle|\, \mathbf{u}_k = \boldsymbol{\pi}(\mathbf{x}_k),\ \mathbf{x}_0 = \mathbf{x}\right]$$



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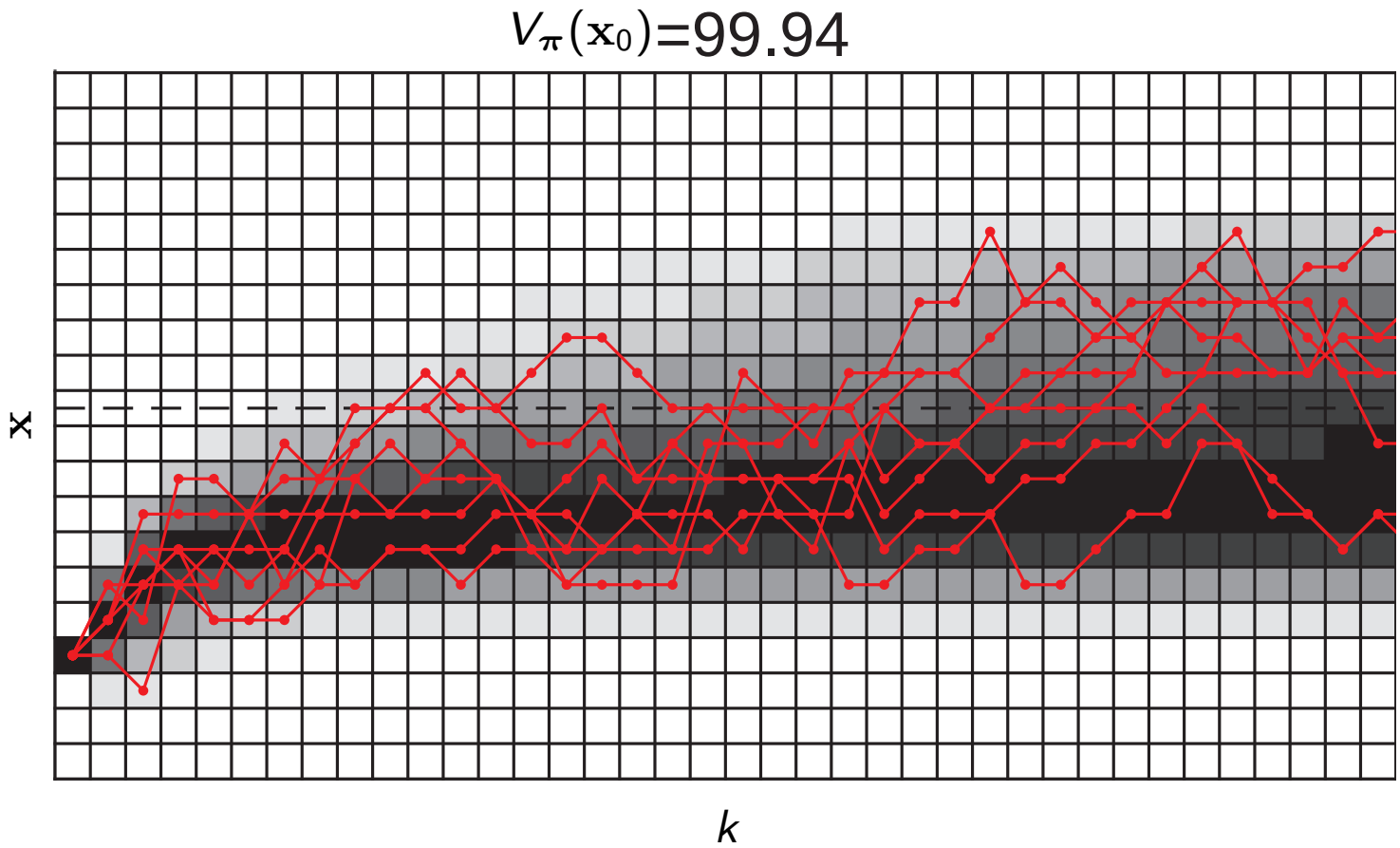

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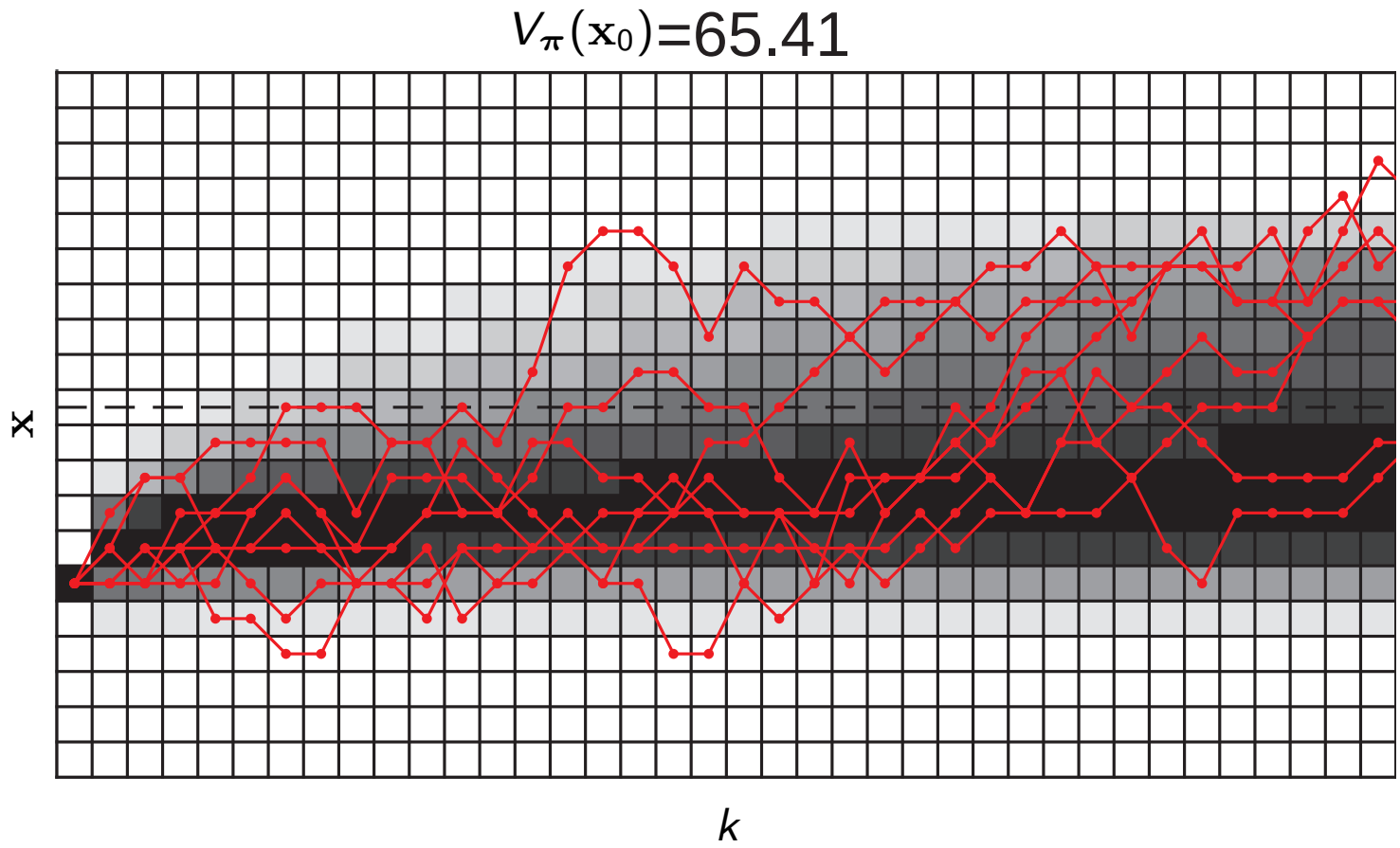

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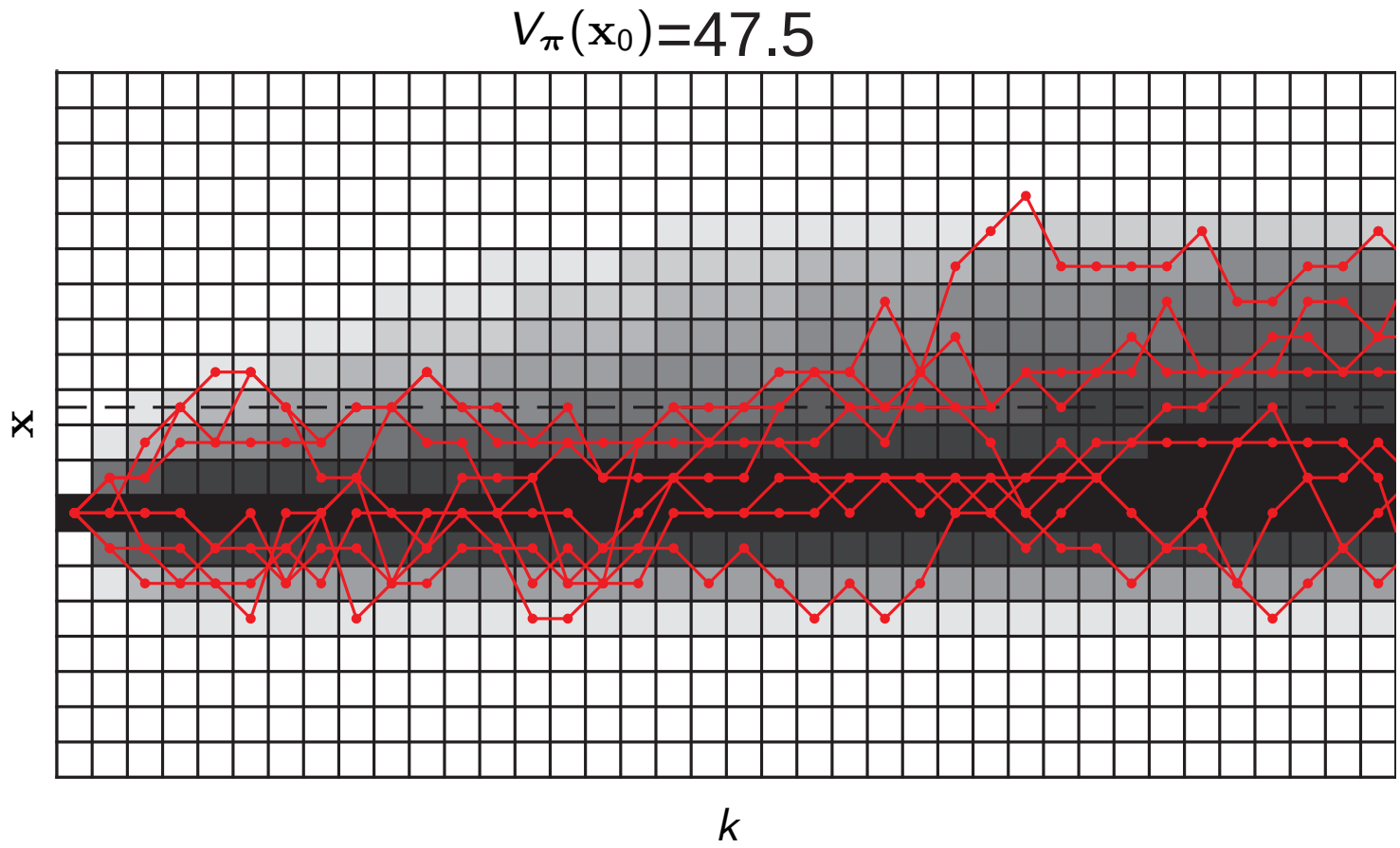

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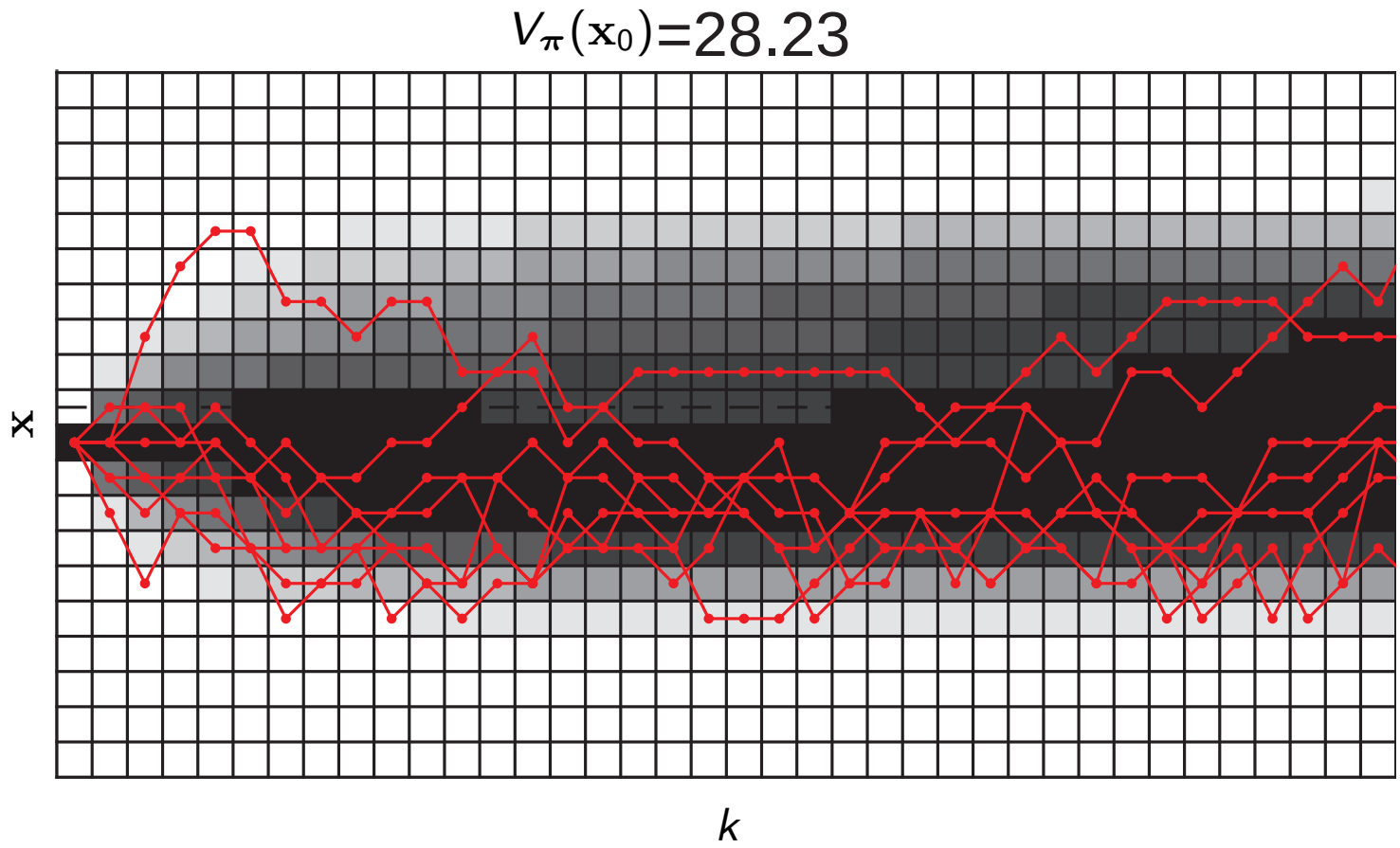

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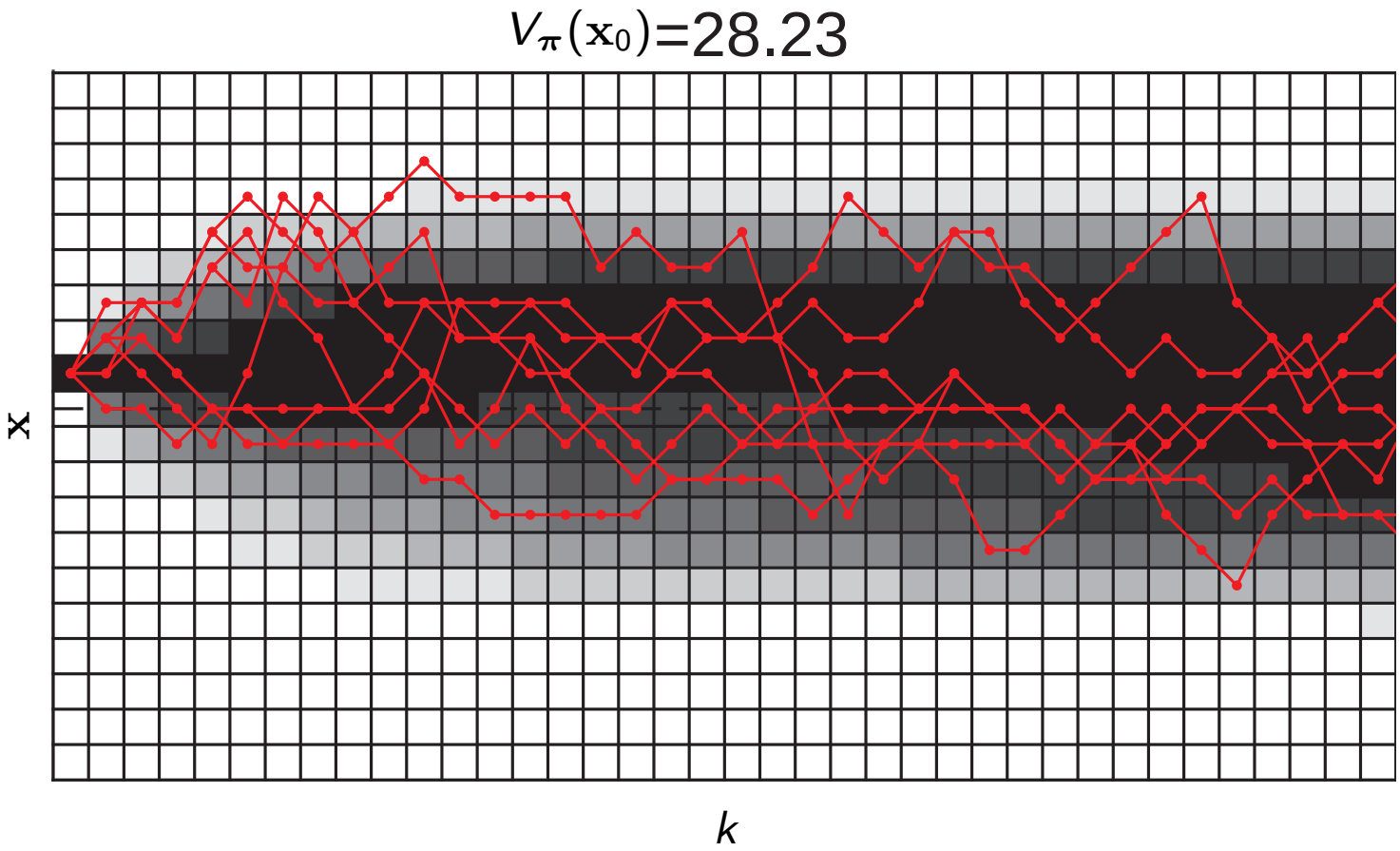

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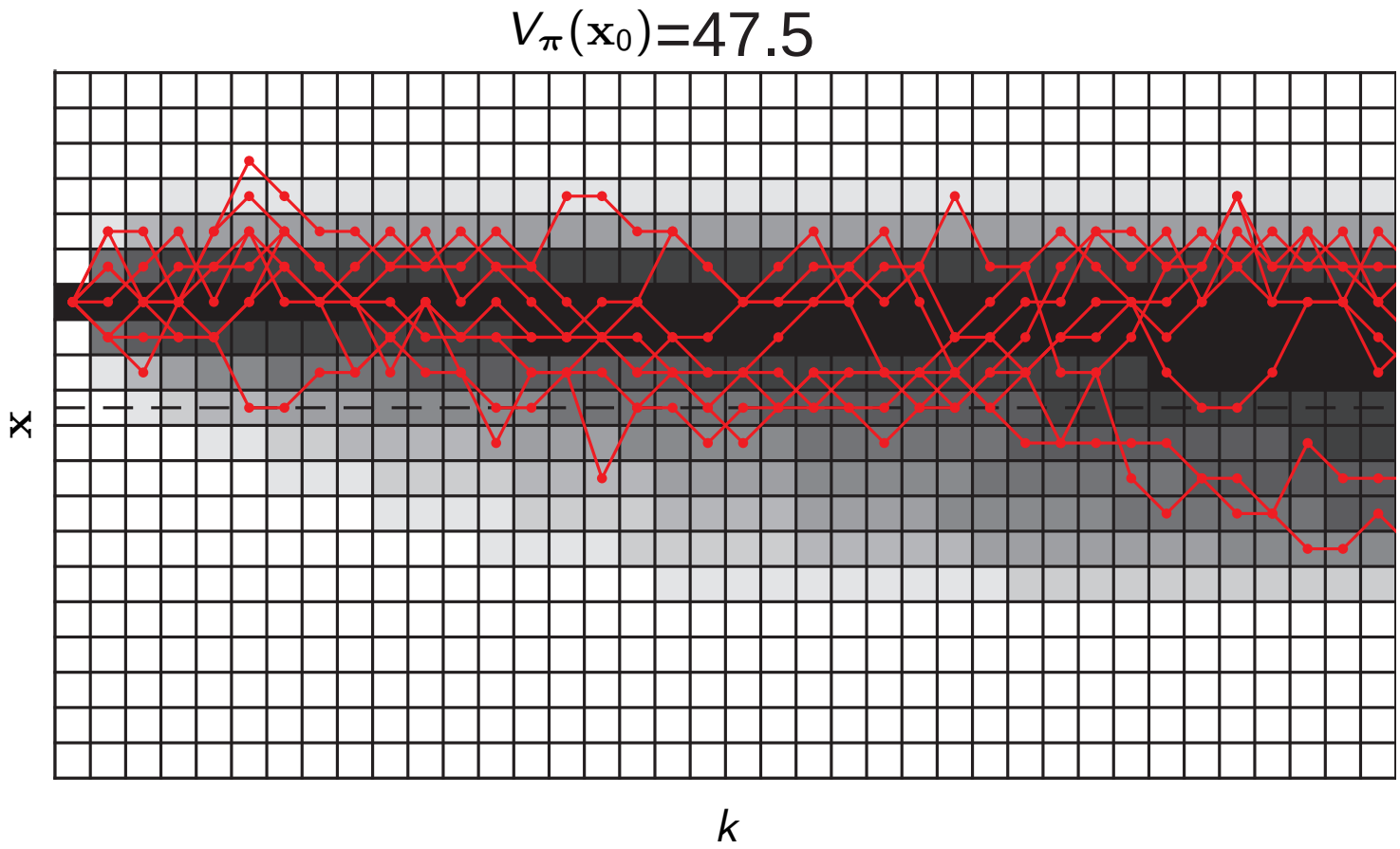

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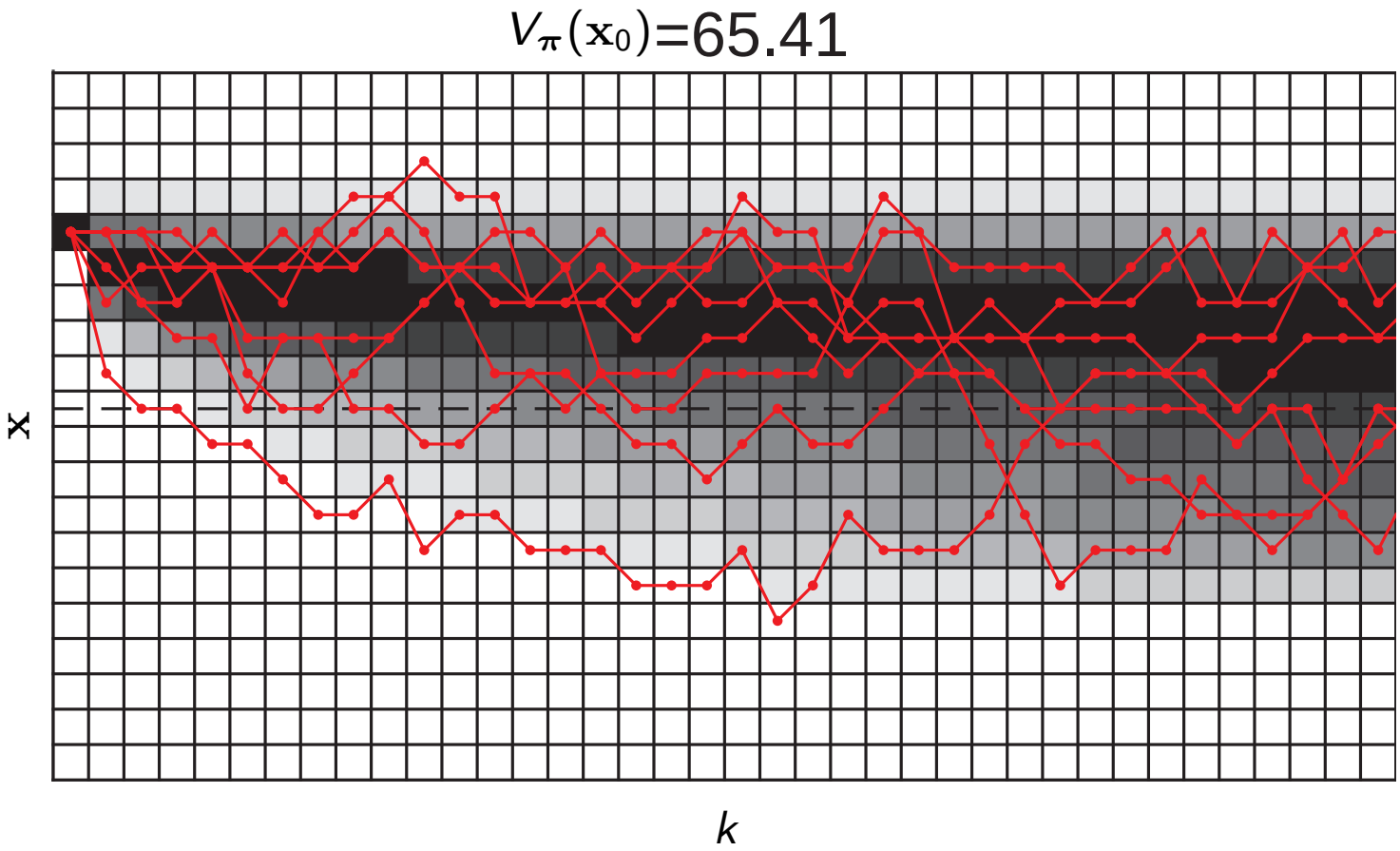

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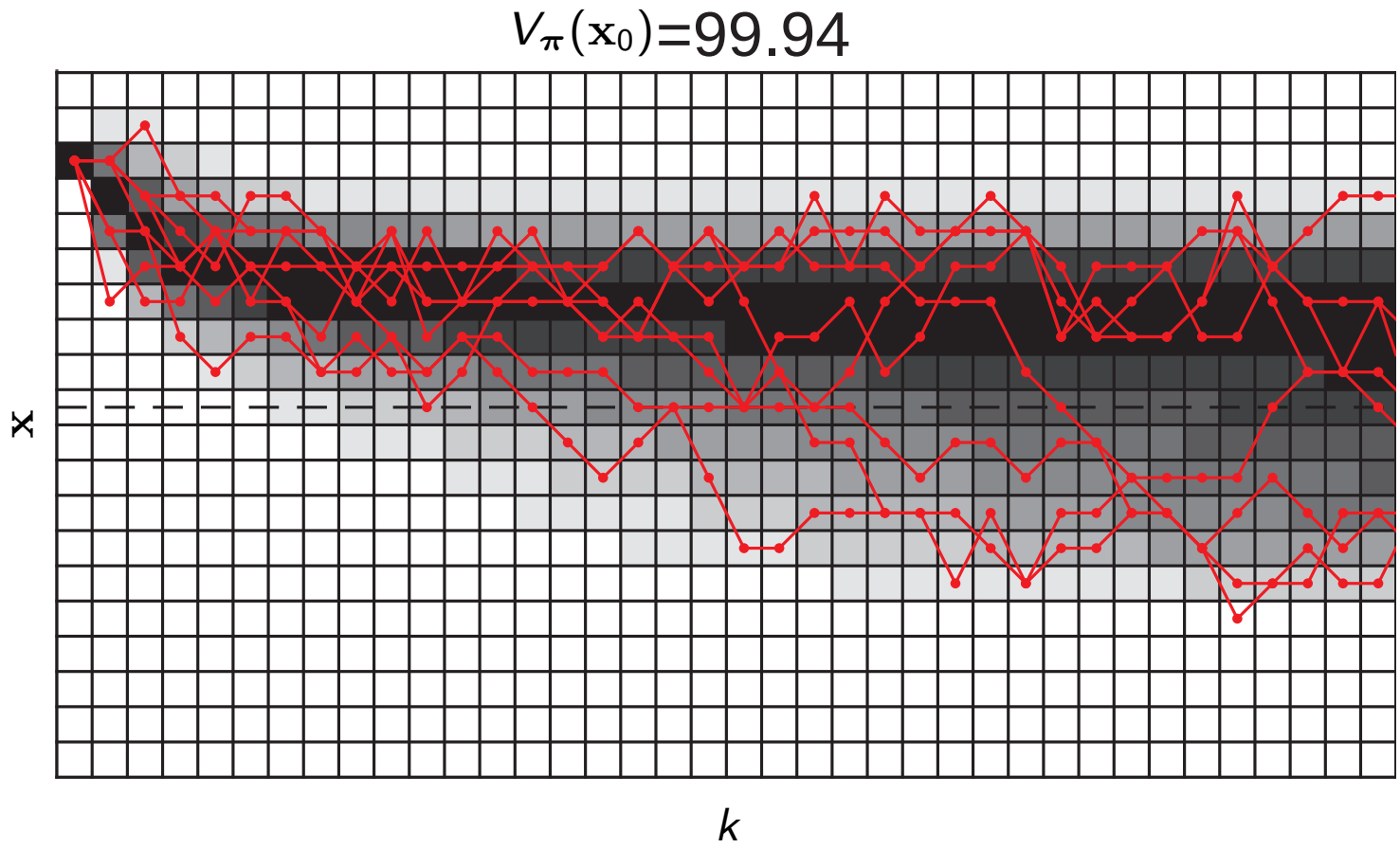

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Meaning: for a given state $\mathbf{x}$

- We follow policy $\boldsymbol{\pi}$ for all k (can be stochastic)
- State trajectory $\mathbf{x}_{1,\ldots,\infty}$ is stochastic as it follows $\mathbb{P}[\mathbf{x}_+ \,|\, \mathbf{x}, \mathbf{u}]$
- Initial state $\mathbf{x}_0 = \mathbf{x}$ is fixed
- Value function at $\mathbf{x}$ is the resulting cumulated (discounted) cost
- Expected value $\mathbb{E}[.]$ taken over trajectories, using trajectories distributions...

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where we use the distributions $\mathbf{x}_0 \to \mathbf{x}_k$, recall

$$\mathbb{P}\left[\mathbf{x}_k \,|\, \mathbf{x}_0\right] = \int \prod_{i=1}^{k} \mathbb{P}\left[\mathbf{x}_i \,|\, \mathbf{x}_{i-1},\, \boldsymbol{\pi}(\mathbf{x}_{i-1})\right] \cdot \mathrm{d}\mathbf{x}_1 \ldots \mathrm{d}\mathbf{x}_{k-1}$$

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Note that this construction has all the properties of an expected value (e.g. linearity)

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Relation $J(\boldsymbol{\pi})$ to $V_\pi(\mathbf{x})$?

$$J(\boldsymbol{\pi}) = \mathbb{E}[V_\pi(\mathbf{x})] = \int V_\pi(\mathbf{x}_0) \mathbb{P}[\mathbf{x}_0] \mathrm{d}\mathbf{x}_0$$

Note that this construction has all the properties of an expected value (e.g. linearity)

Bellman equation - Computing the value function

Value function for policy π (either $\mathbf{u} = \boldsymbol{\pi}(\mathbf{x})$ or $\boldsymbol{\pi}[\mathbf{u} \,|\, \mathbf{x}]$):

$$V_{\boldsymbol{\pi}}(\mathbf{x}) = \mathbb{E}\left[\sum_{k=0}^{\infty} \gamma^k L(\mathbf{x}_k, \mathbf{u}_k) \,\middle|\, \boldsymbol{\pi},\, \mathbf{x}_0 = \mathbf{x} \right]$$

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$$V_\pi(\mathbf{x}_0) = \mathbb{E}\left[\sum_{k=0}^{\infty} \gamma^k L(\mathbf{x}_k, \mathbf{u}_k) \,\middle|\, \mathbf{x}_0, \pi\right] = L(\mathbf{x}_0, \pi(\mathbf{x}_0)) + \mathbb{E}\left[\sum_{k=1}^{\infty} \gamma^k L(\mathbf{x}_k, \mathbf{u}_k) \,\middle|\, \mathbf{x}_0, \pi\right]$$

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$$V_\pi(\mathbf{x}_0) = \mathbb{E}\left[\sum_{k=0}^{\infty} \gamma^k L(\mathbf{x}_k, \mathbf{u}_k) \,\middle|\, \mathbf{x}_0, \pi \right] = L(\mathbf{x}_0, \pi(\mathbf{x}_0)) + \mathbb{E}\left[\sum_{k=1}^{\infty} \gamma^k L(\mathbf{x}_k, \mathbf{u}_k) \,\middle|\, \mathbf{x}_0, \pi \right]$$

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Bellman equation - Computing the value function

Value function for policy π (either $\mathbf{u} = \pi(\mathbf{x})$ or $\pi[\mathbf{u} \,|\, \mathbf{x}]$):

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where

$$\mathbb{E}\left[V_\pi(\mathbf{x}_1) \,|\, \mathbf{x}_0 \right] = \int V_\pi(\mathbf{x}_1) \overbrace{\mathbb{P}\left[\mathbf{x}_1 \,|\, \mathbf{x}_0, \pi(\mathbf{x}_0) \right]}^{\textit{Model}} \mathrm{d}\mathbf{x}_1$$

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We get:

Bellman equation

$$V_{\boldsymbol{\pi}}(\mathbf{x}) = L(\mathbf{x}, \boldsymbol{\pi}(\mathbf{x})) + \gamma \mathbb{E}\left[V_{\boldsymbol{\pi}}(\mathbf{x}_+) \,|\, \mathbf{x}, \boldsymbol{\pi} \right]$$

sometimes called optimality backup

Action-value function

Bellman equation for the value function

$$V_{\pi}(\mathbf{x}) = L(\mathbf{x}, \pi(\mathbf{x})) + \gamma \mathbb{E}\left[V_{\pi}(\mathbf{x}_+) \mid \mathbf{x}, \pi\right]$$

Define

Action-value function:

$$Q_{\pi}(\mathbf{x}, \mathbf{u}) = L(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}\left[V_{\pi}(\mathbf{x}_+) \mid \mathbf{x}, \mathbf{u}\right]$$

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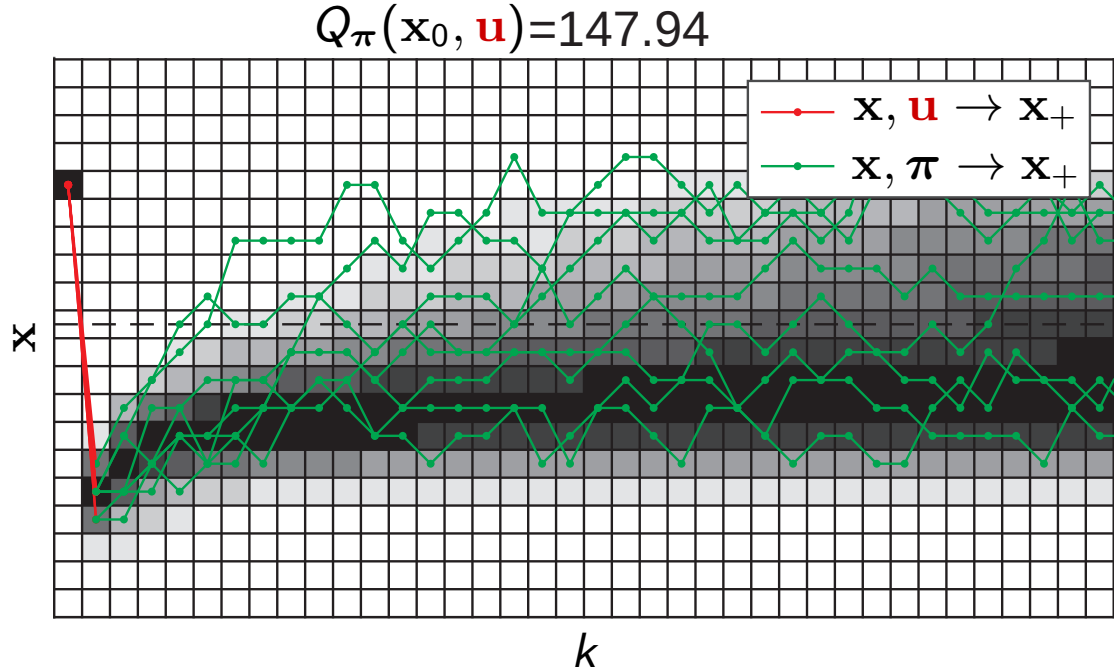

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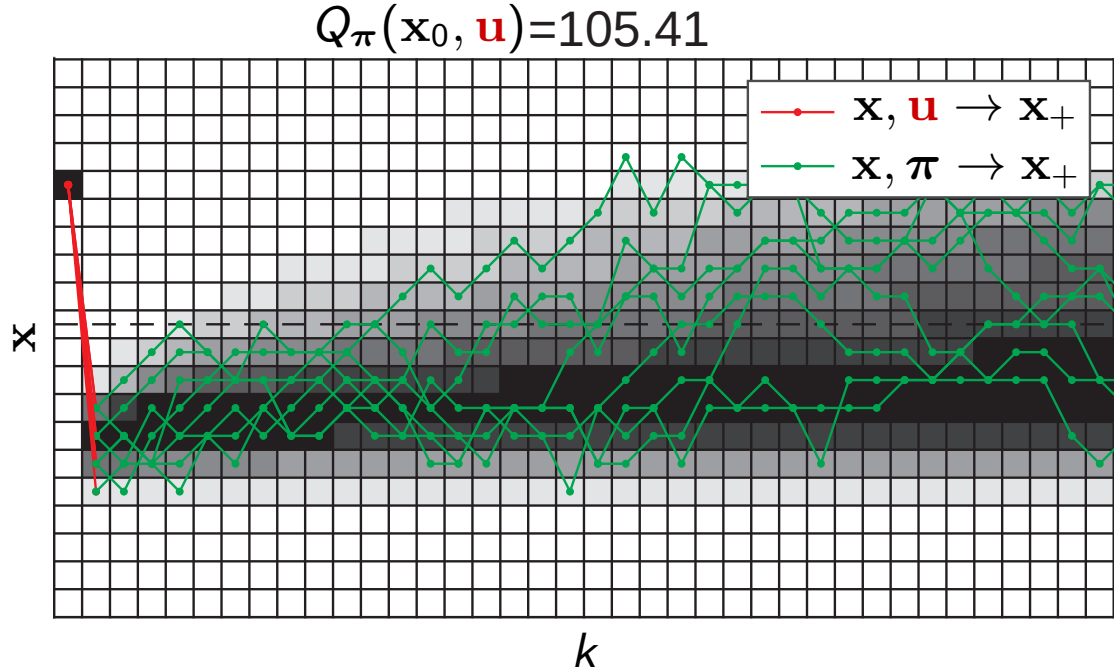

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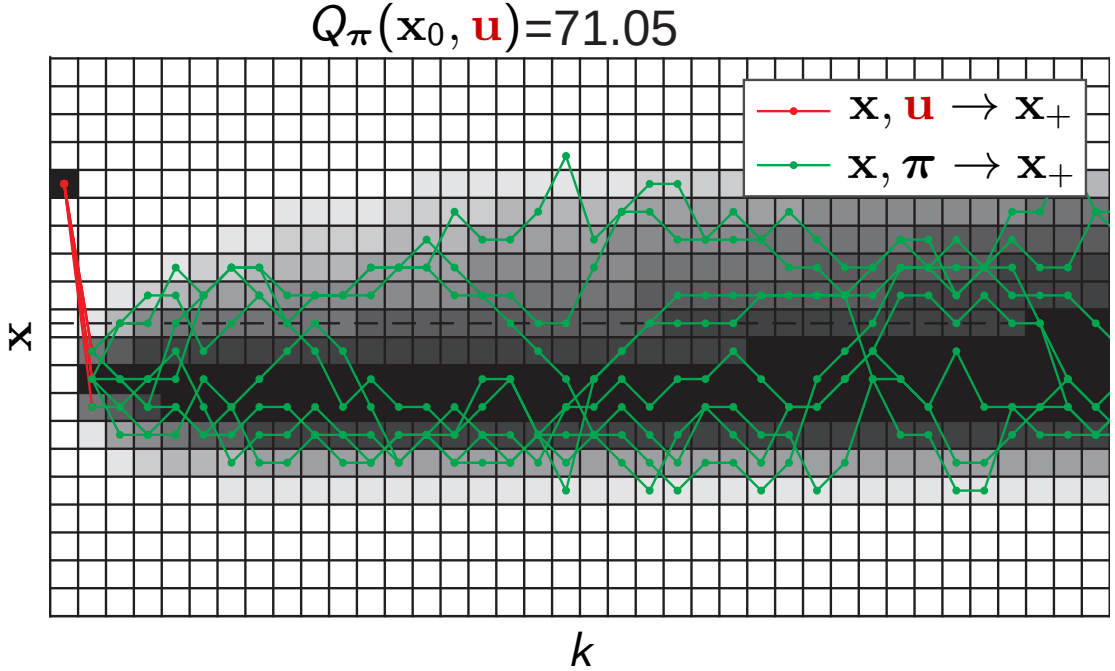

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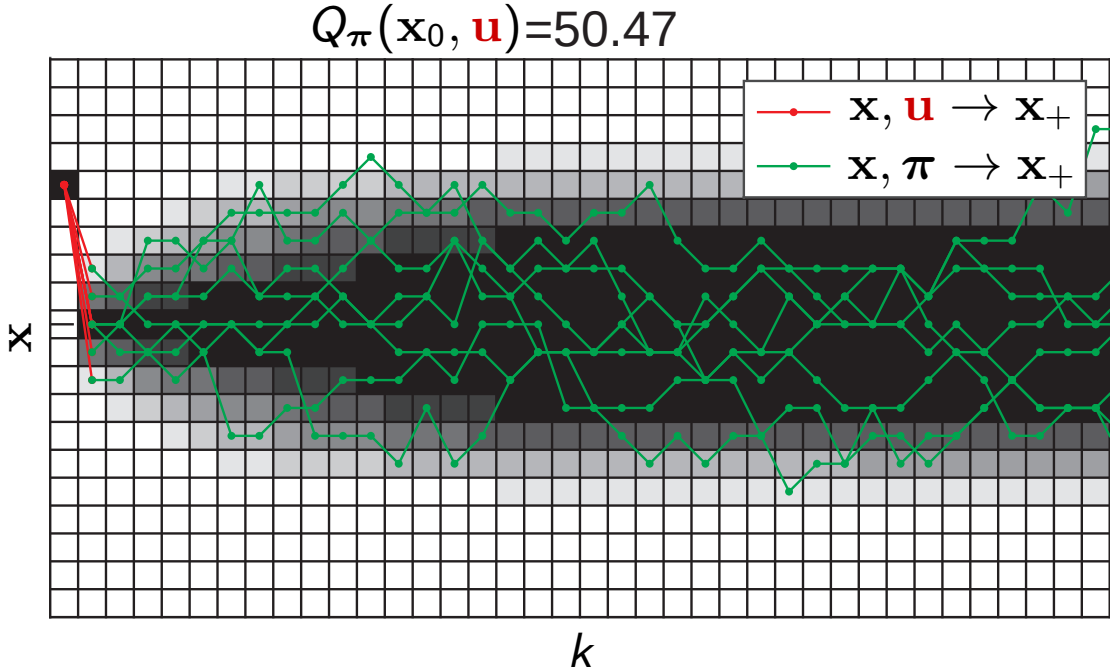

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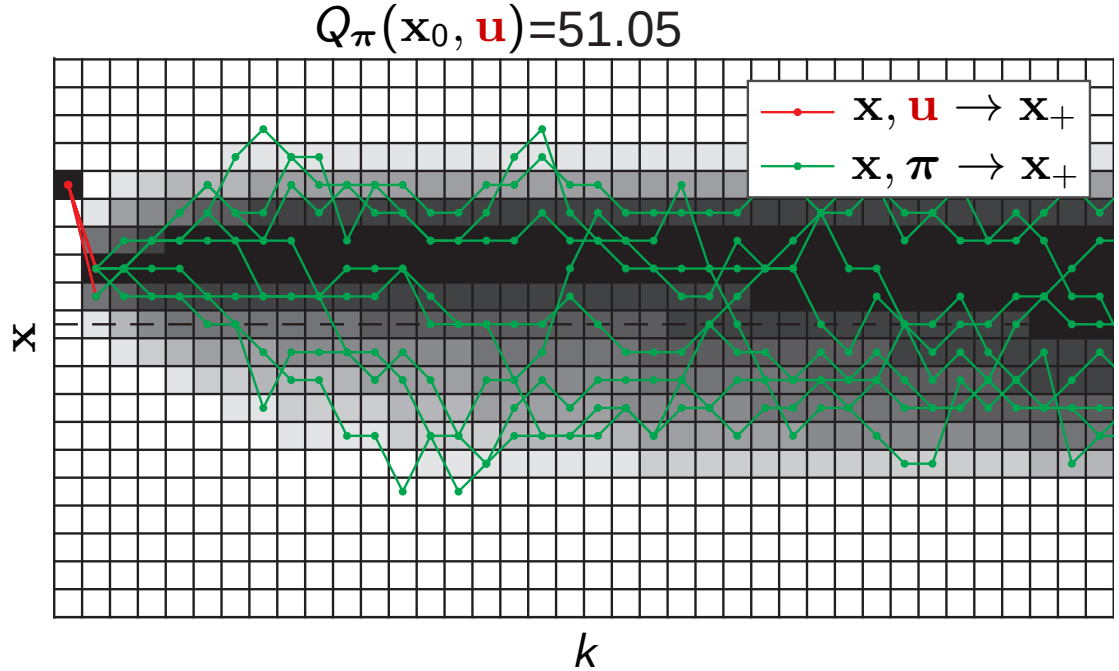

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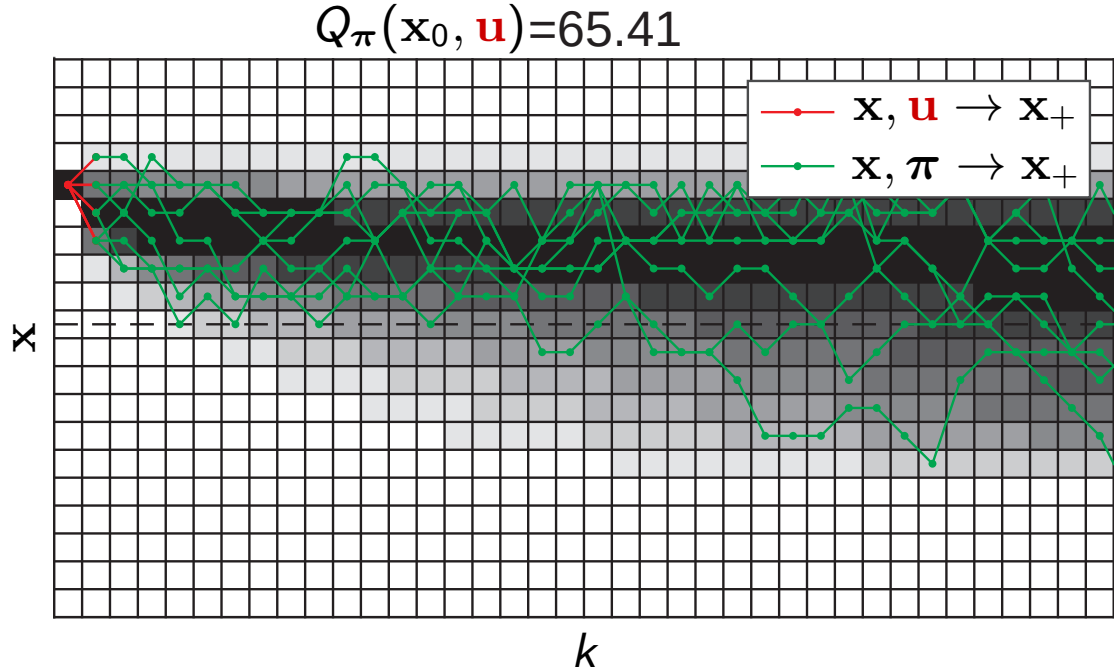

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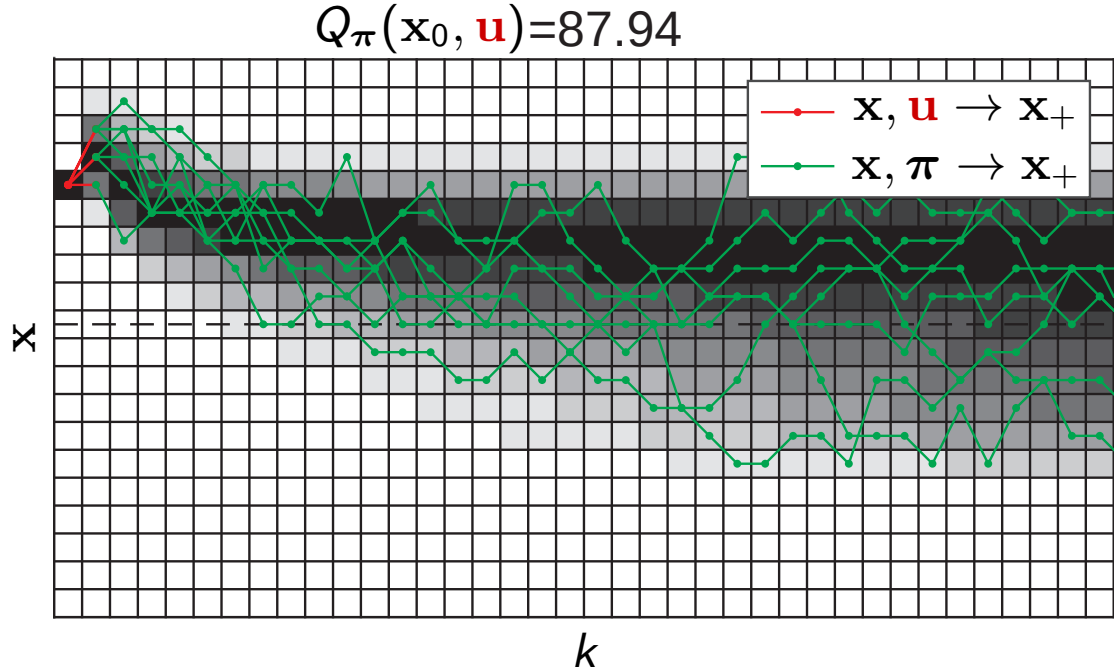

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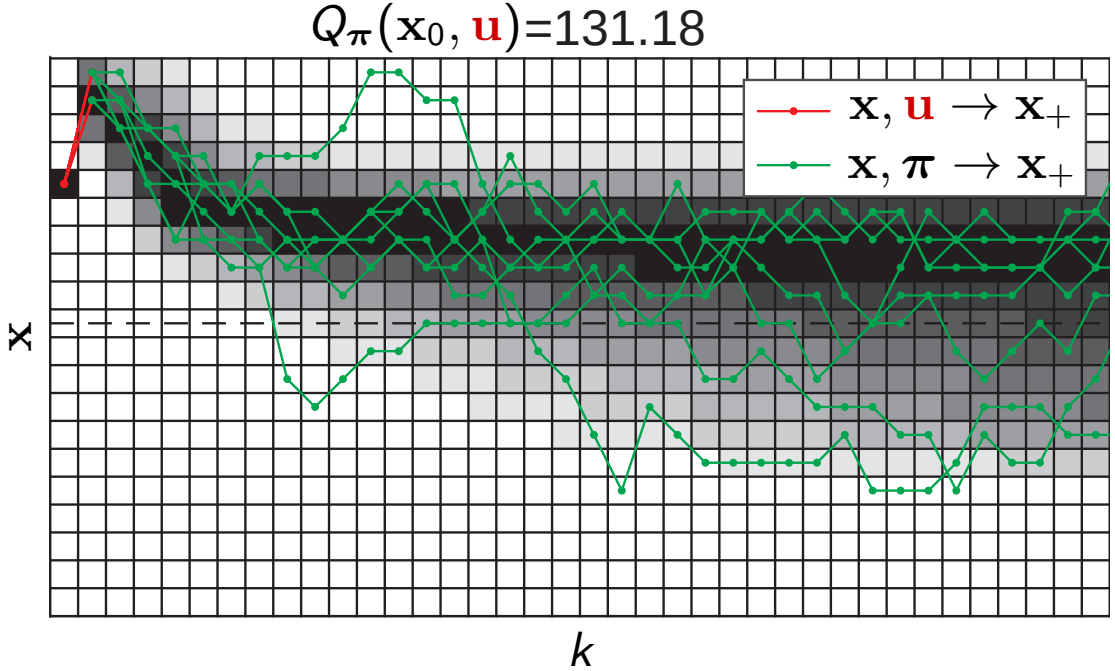

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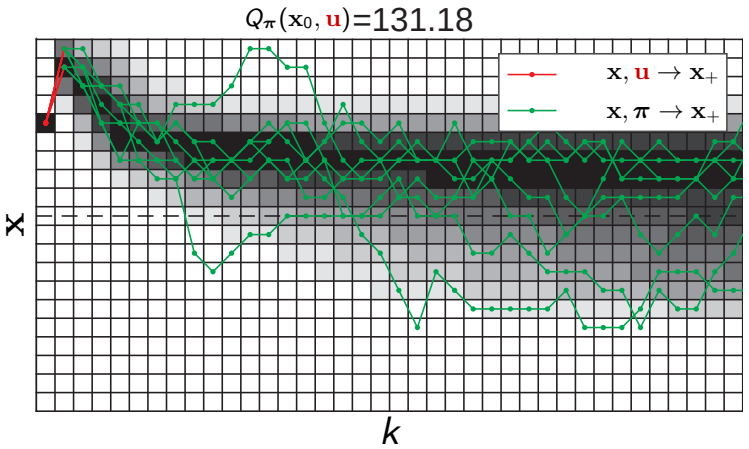

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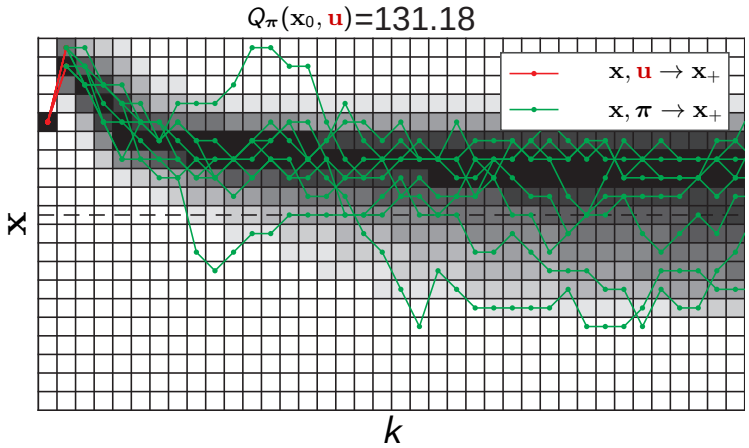

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Relationship Q_π and V_π

$$Q_\pi(\mathbf{x}, \pi(\mathbf{x})) = V_\pi(\mathbf{x})$$

Optimal policy & value function

- Ordering over policies: π is better or equal to π' if $V_{\pi}(\mathbf{x}) \leq V_{\pi'}(\mathbf{x})$ for all $\mathbf{x}$

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for all $\mathbf{x}, \mathbf{u}$ and π

$Q_{\pi_\star}(\mathbf{x},\mathbf{u}) \leq Q_\pi(\mathbf{x},\mathbf{u})$
Hence minimizer $\pi_\star$ of V_π is minimizer of Q_π

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Optimal action-value function

$$Q_\star(\mathbf{x},\mathbf{u}) = \min_{\pi} Q_\pi(\mathbf{x},\mathbf{u}) = Q_{\pi_\star}(\mathbf{x},\mathbf{u})$$

$Q_{\pi_\star}(\mathbf{x},\mathbf{u}) \leq Q_\pi(\mathbf{x},\mathbf{u})$
Hence minimizer $\pi_\star$ of V_π is minimizer of Q_π

Bellman optimality equation

Relationships

$$V_\star(\mathbf{x}) = \min_{\mathbf{u}} \; Q_\star(\mathbf{x}, \mathbf{u})$$

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Yield Bellman optimality equation for value function:

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Bellman optimality equation

Relationships

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Yield Bellman optimality equation for value function:

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Observe that $V_\star \to Q_\star$ exchanges $\mathbb{E}$ and $\min_{\mathbf{u}}$

Advantage function

Associated to a policy:

$$A_\pi(\mathbf{x}, \mathbf{u}) = Q_\pi(\mathbf{x}, \mathbf{u}) - V_\pi(\mathbf{x})$$

Properties:

$$A_\pi(\mathbf{x}, \boldsymbol{\pi}(\mathbf{x})) = 0$$

$$A_\pi(\mathbf{x}, \mathbf{u}) < 0 \Rightarrow \mathbf{u} \text{ better than } \boldsymbol{\pi}(\mathbf{x})$$

- Assesses the value of an input $\mathbf{u}$ compared to a base policy (for given state)
- When $A_\pi(\mathbf{x}, \boldsymbol{\pi}(\mathbf{x})) \geq 0$ for all $\mathbf{x}$, we have an optimal policy
- Will be useful for policy gradient methods

Note:

- $A_\pi(\mathbf{x}, \mathbf{u}) < 0 \Rightarrow \mathbf{u}$ better than $\boldsymbol{\pi}(\mathbf{x})$ for $\mathbf{x}$
- $A_\pi(\mathbf{x}, \mathbf{u}) > 0 \Rightarrow \mathbf{u}$ worse than $\boldsymbol{\pi}(\mathbf{x})$ for $\mathbf{x}$

allows for "comparing" arbitrary input $\mathbf{u}$ to baseline policy

Advantage function

Associated to a policy:

$$A_\pi(\mathbf{x}, \mathbf{u}) = Q_\pi(\mathbf{x}, \mathbf{u}) - V_\pi(\mathbf{x})$$

Properties:

$$A_\pi(\mathbf{x}, \boldsymbol{\pi}(\mathbf{x})) = 0$$

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Optimal:

$$A_\star(\mathbf{x}, \mathbf{u}) = Q_\star(\mathbf{x}, \mathbf{u}) - V_\star(\mathbf{x})$$

Properties:

$$A_\star(\mathbf{x}, \boldsymbol{\pi}_\star(\mathbf{x})) = 0$$

$$A_\star(\mathbf{x}, \mathbf{u}) \geq 0$$

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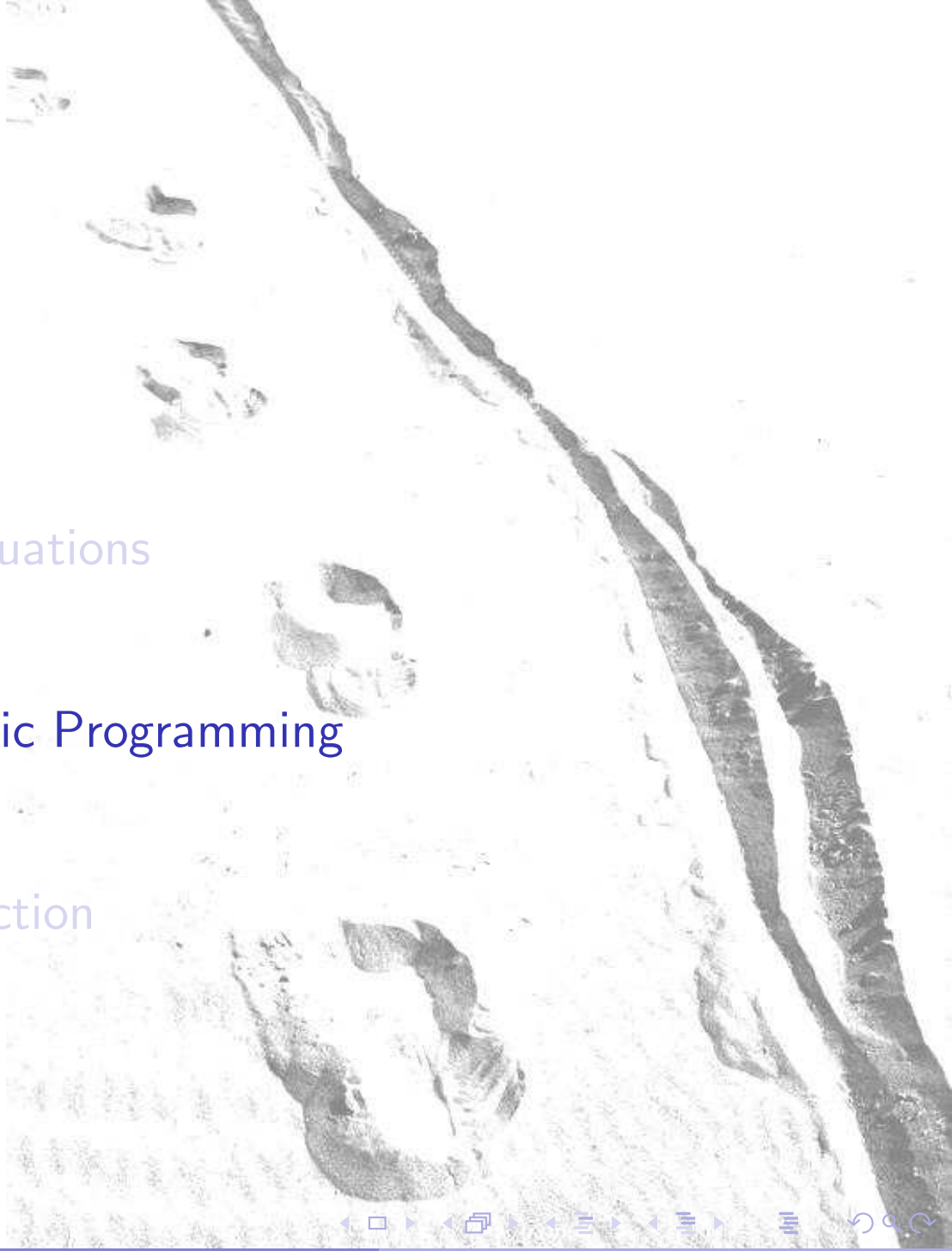
Outline



How to compute value functions?

How to solve the Bellman equations?

$$V_{\pi}(\mathbf{x}) = L(\mathbf{x}, \boldsymbol{\pi}(\mathbf{x})) + \gamma \mathbb{E}\left[V_{\pi}(\mathbf{x}_+) \,|\, \mathbf{x}, \boldsymbol{\pi}\right] \qquad V_{\pi}(\mathbf{x}) = Q_{\pi}(\mathbf{x}, \boldsymbol{\pi}(\mathbf{x}))$$

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Policy evaluation: for a given policy $\boldsymbol{\pi}$ start with arbitrary value function $V_0(\mathbf{x})$, iterate:

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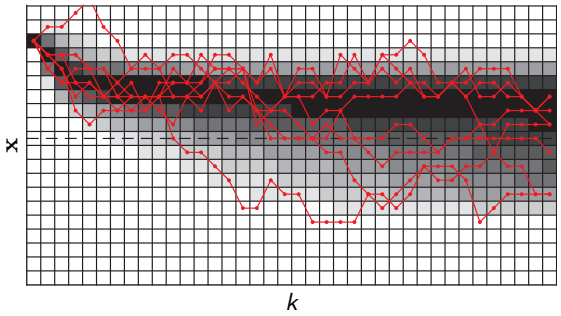
Note: convergence for $\gamma = 1$ needs extra assumptions!

Computing V_π - Illustration



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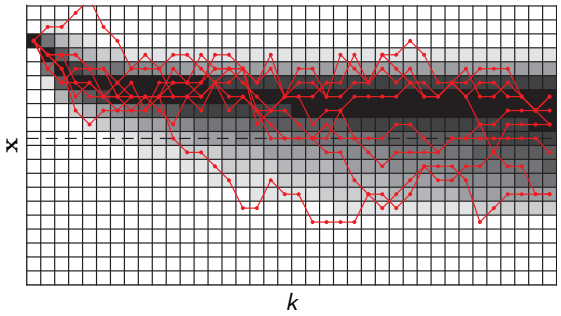

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Algorithm: π-evaluation

Input: V (e.g. $= 0$) and $\mathrm{tol} > 0$

$\mathrm{Iter} = \mathrm{True}$

while Iter **do**

  for *All* $\mathbf{x}$ **do**

    $V_+(\mathbf{x}) \leftarrow L(\mathbf{x}, \boldsymbol{\pi}(\mathbf{x})) + \gamma \mathbb{E}\left[V\left(\mathbf{x}_+\right) \mid \mathbf{x}, \boldsymbol{\pi}\right]$

  if $\|V_+ - V\|_\infty \leq \mathrm{tol}$ **then**

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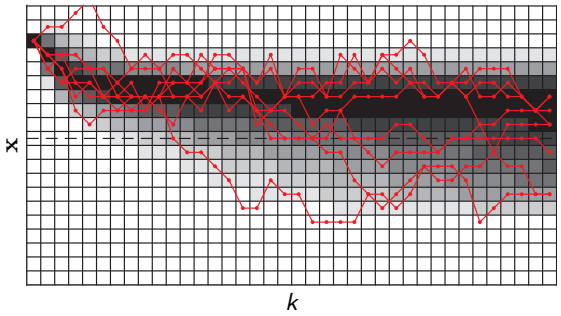
  $V \leftarrow V_+$

return $V_\pi \leftarrow V_+$

Computing V_π - Illustration



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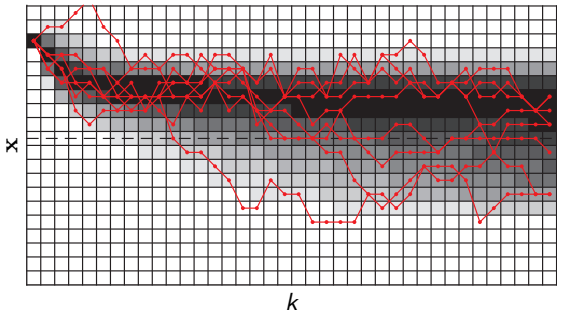
Remarks:

- Coding π-evaluation requires a discretized state-space $\mathbf{x} \in \mathbb{X}$
- V, V_+ are stored as tables (say of size n)
- Each iteration requires n^2 accesses
- Expected value for discrete state

$$\mathbb{E}\left[V\left(\mathbf{x}_+\right) \mid \mathbf{x}, \boldsymbol{\pi}\right] = \sum_{\mathbf{x}_+ \in \mathbb{X}} V\left(\mathbf{x}_+\right) \underbrace{\mathbb{P}[\mathbf{x}_+ \mid \mathbf{x}, \boldsymbol{\pi}(\mathbf{x})]}_{\textit{Model}}$$

Computing V_π - Illustration



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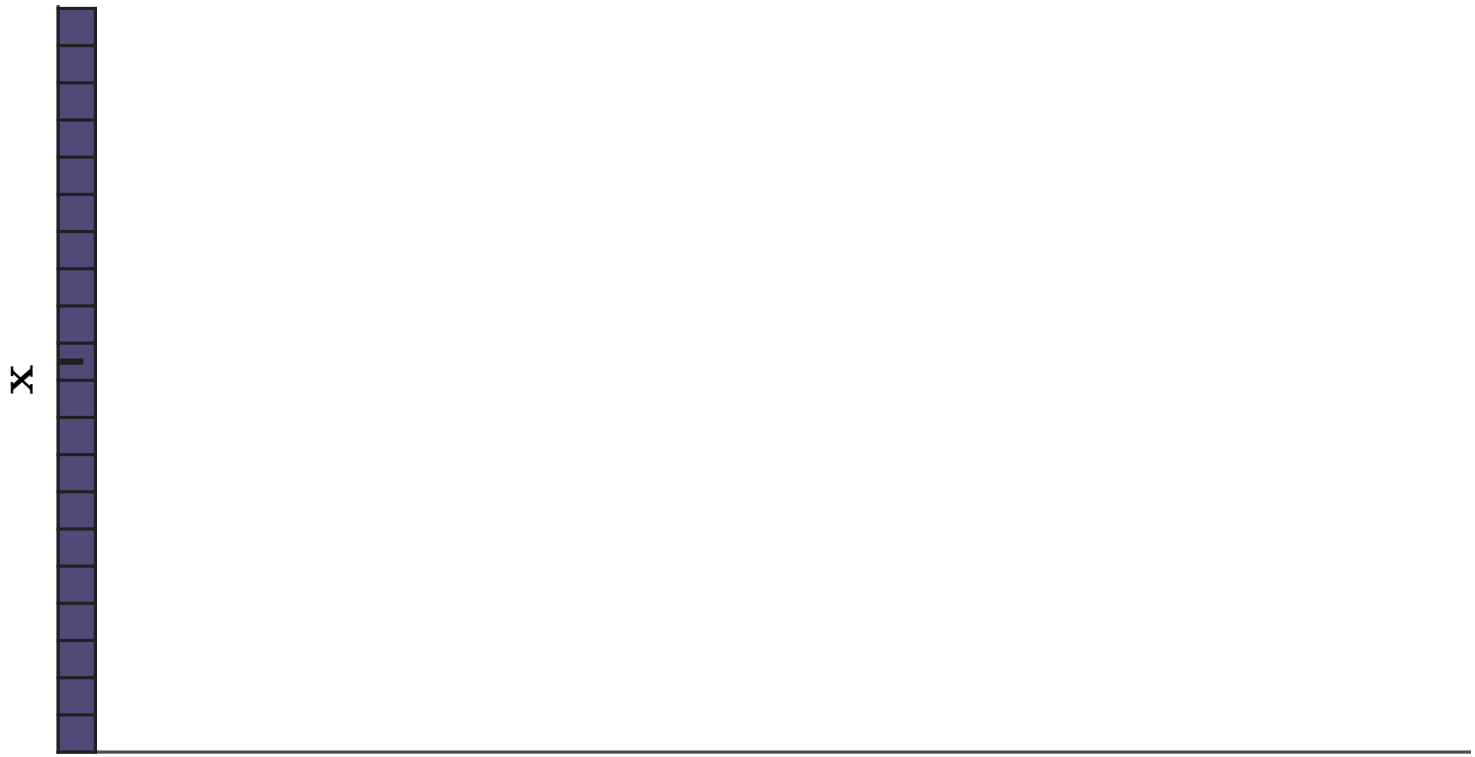

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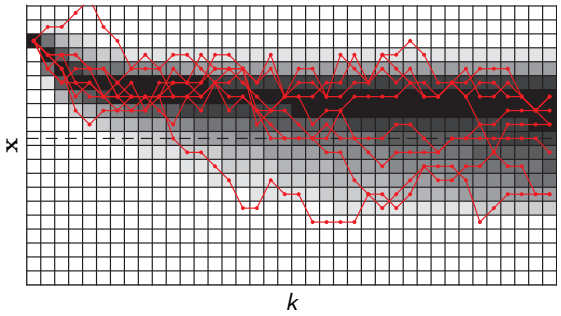

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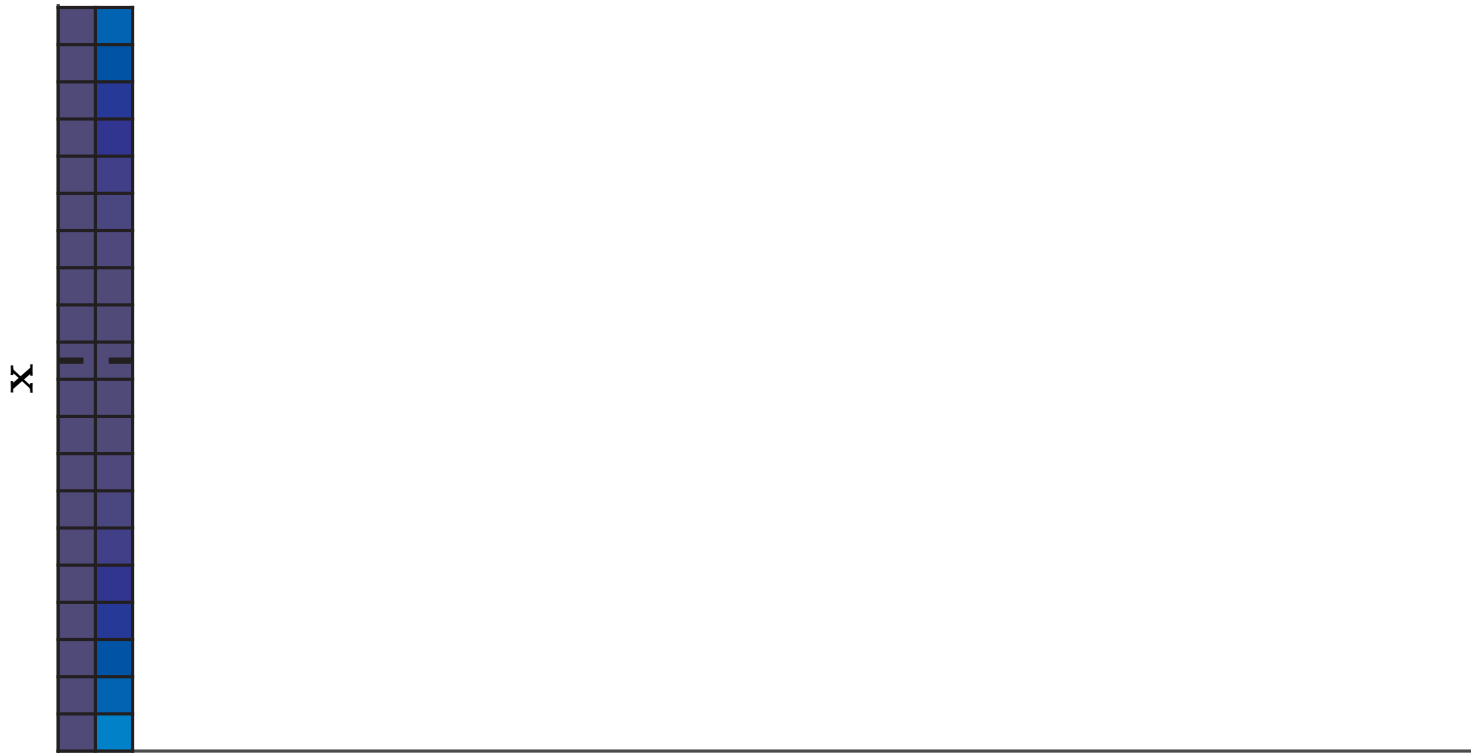

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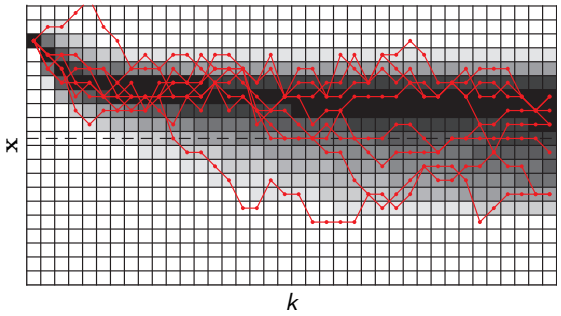

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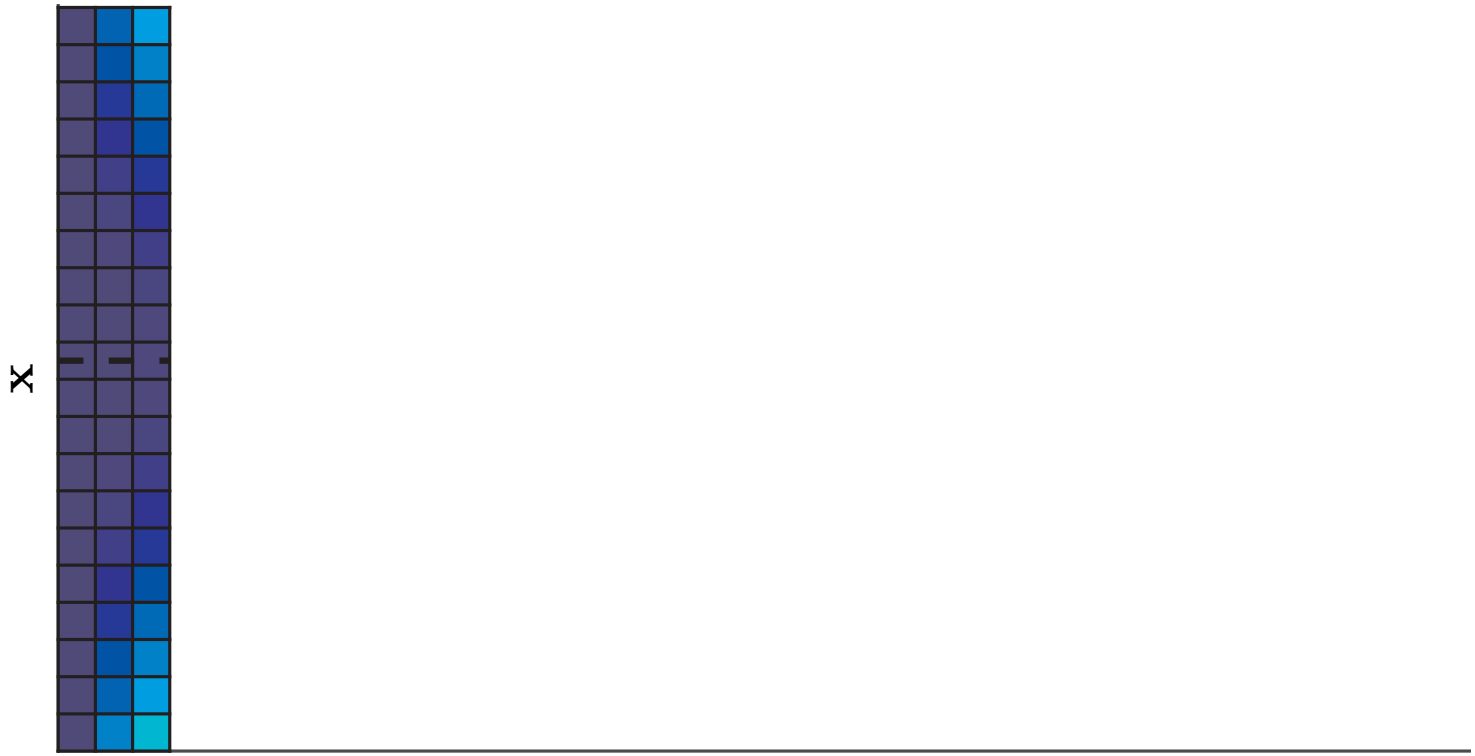

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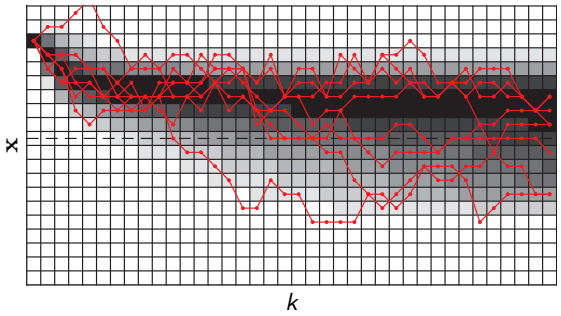

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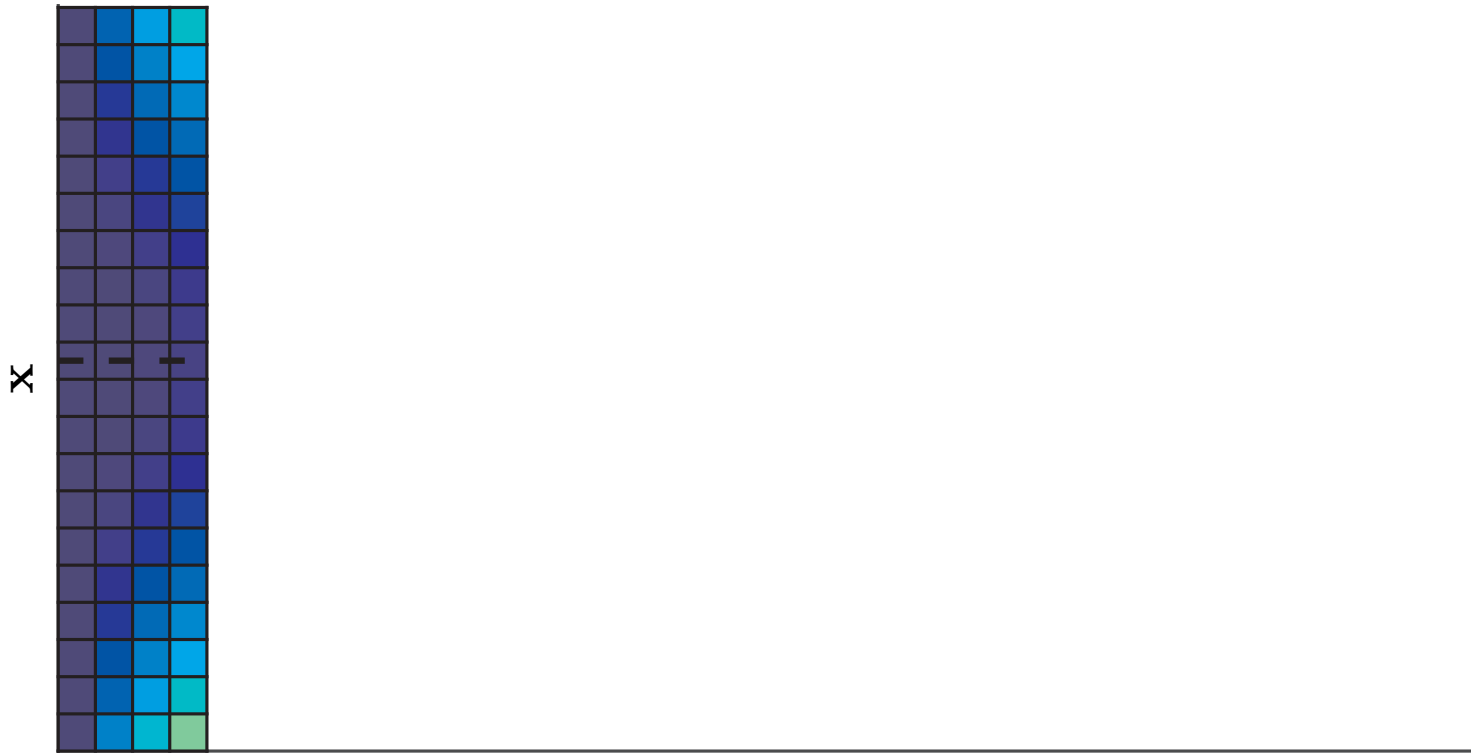

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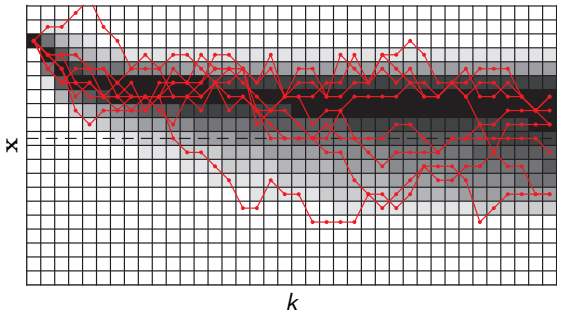

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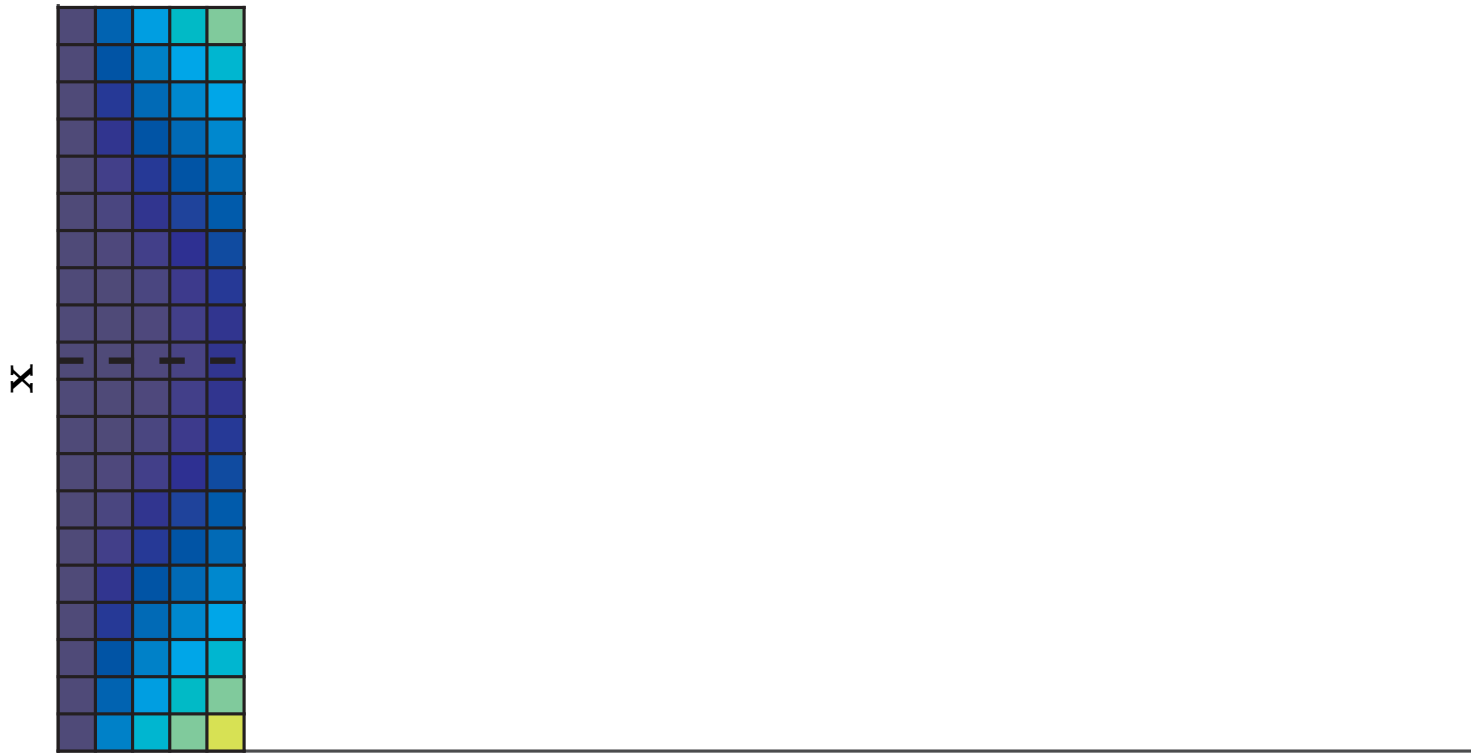

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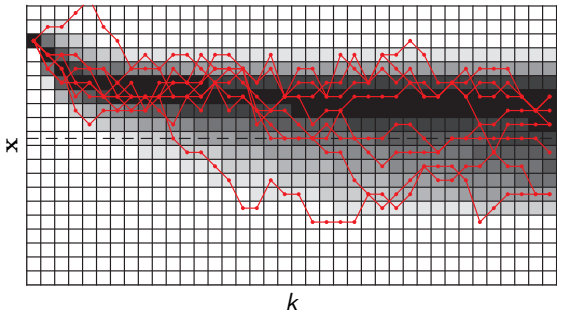

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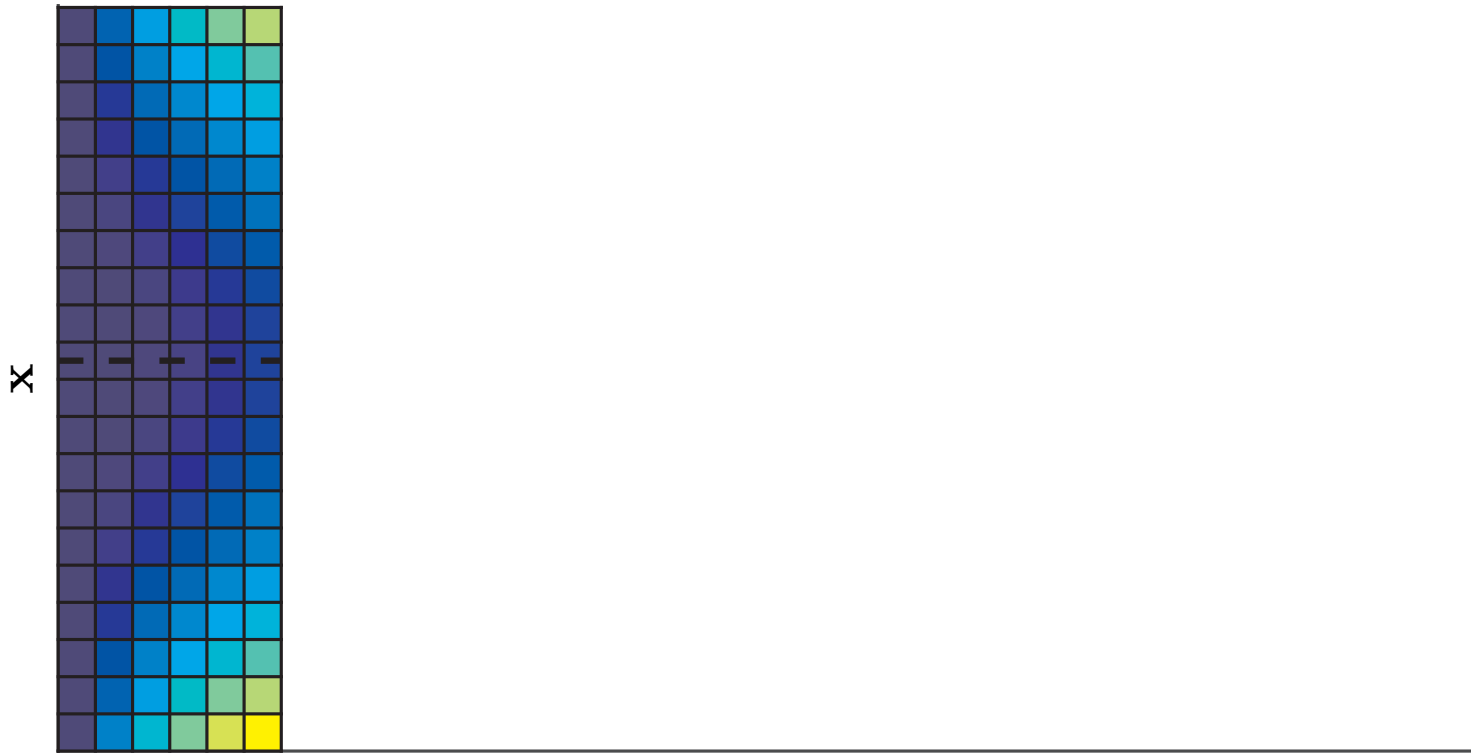

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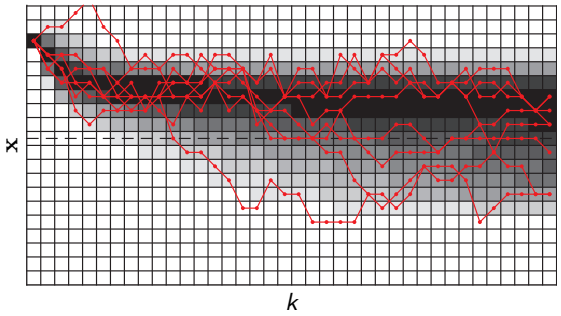

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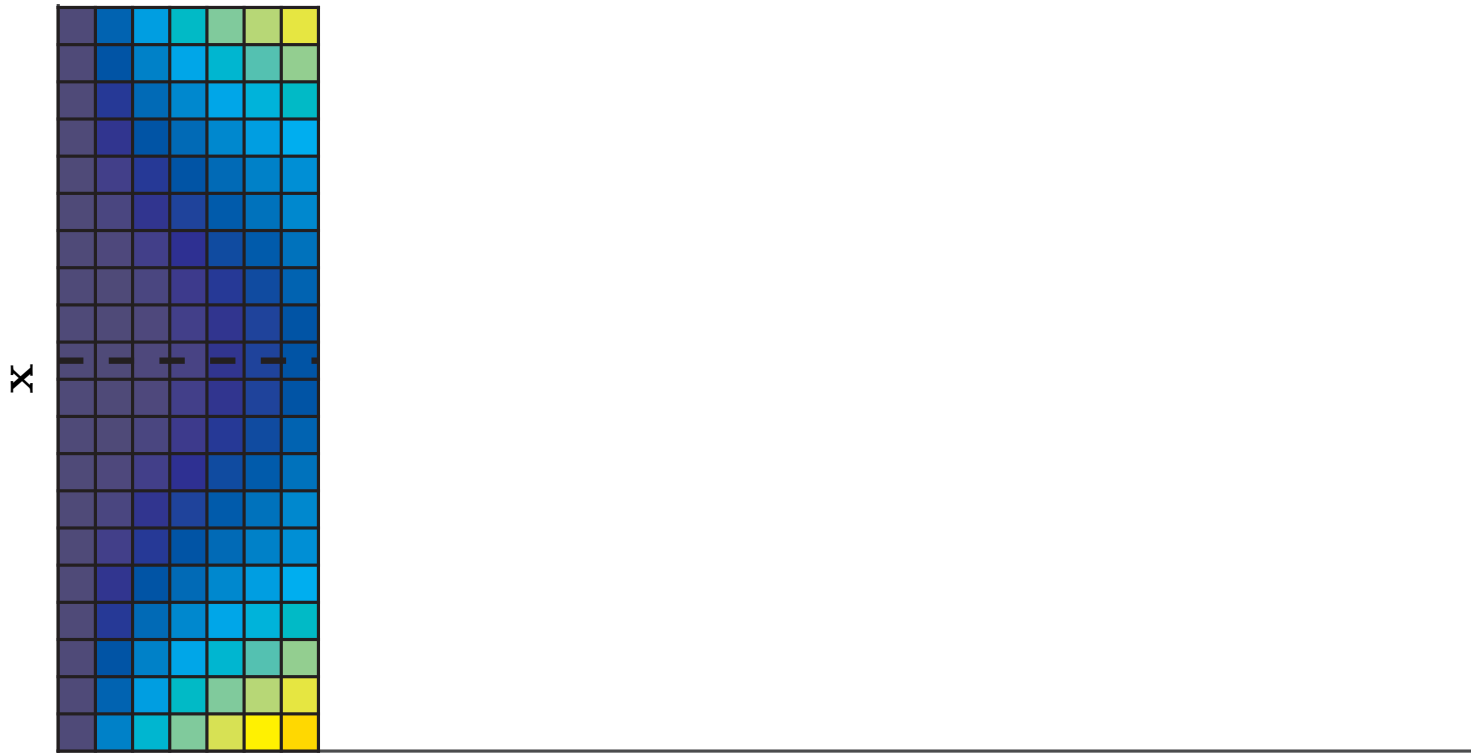

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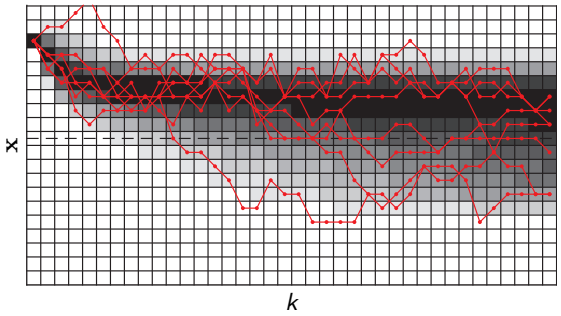

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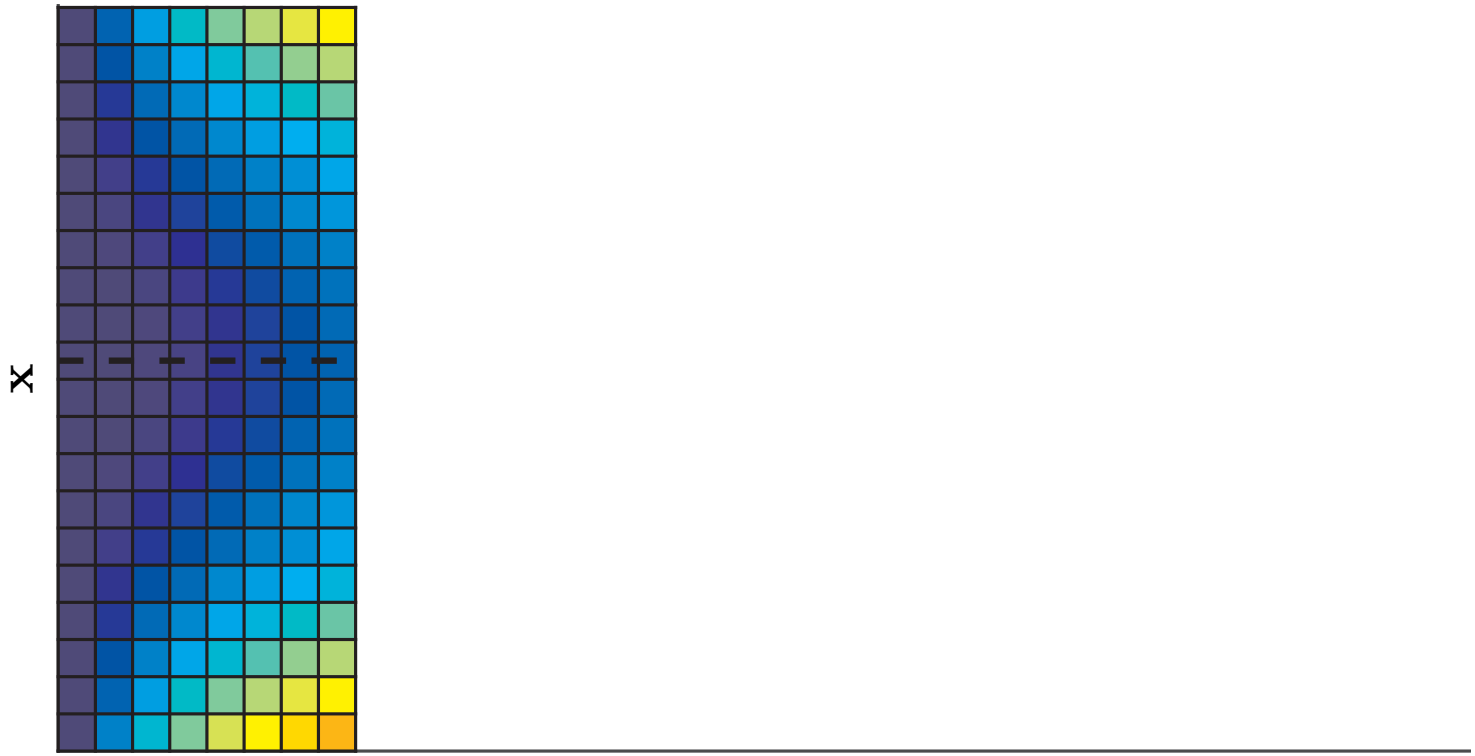

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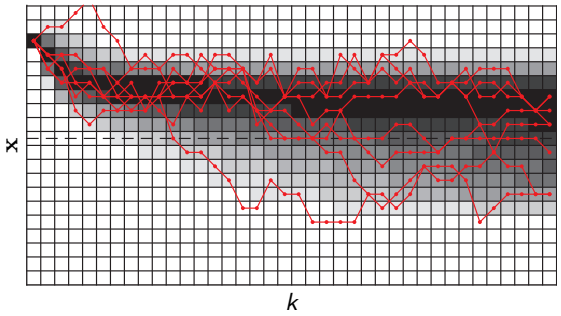

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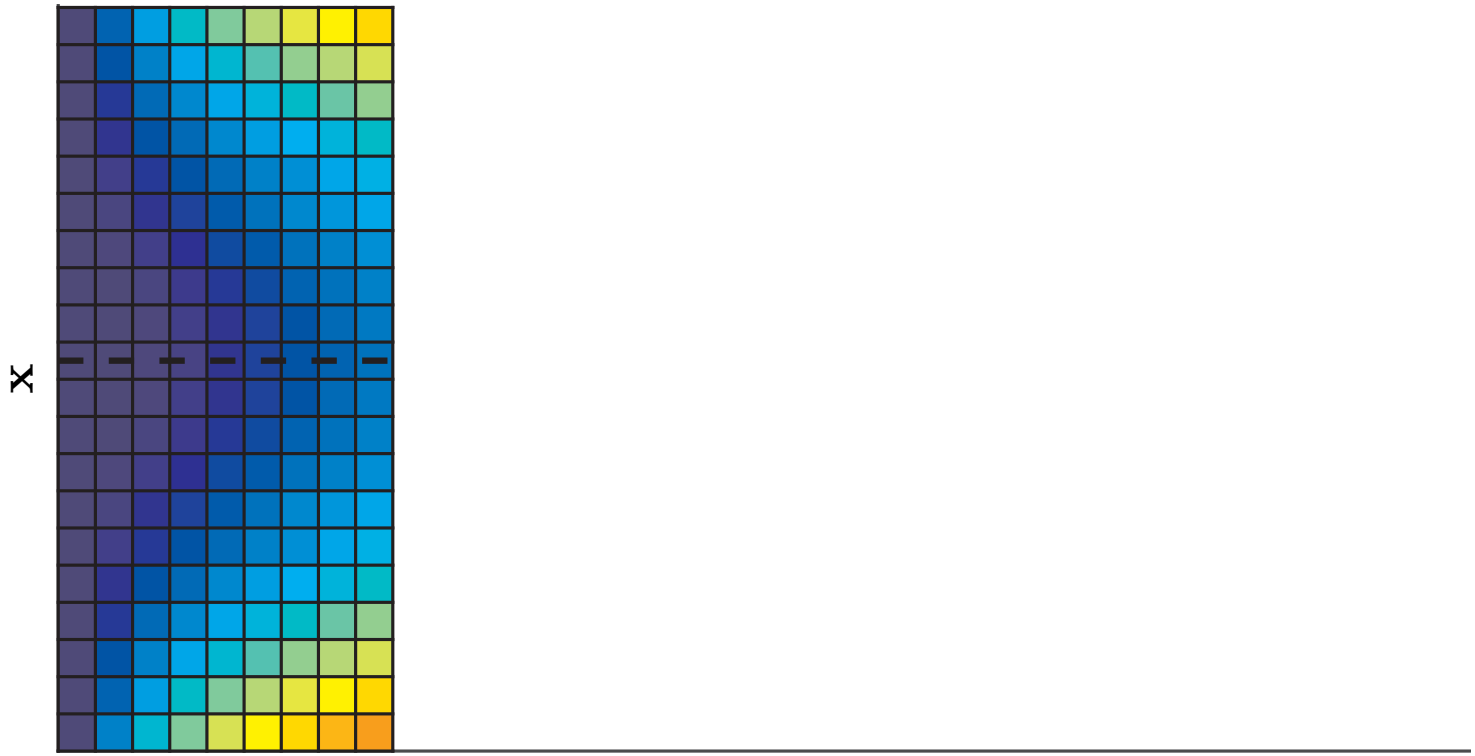

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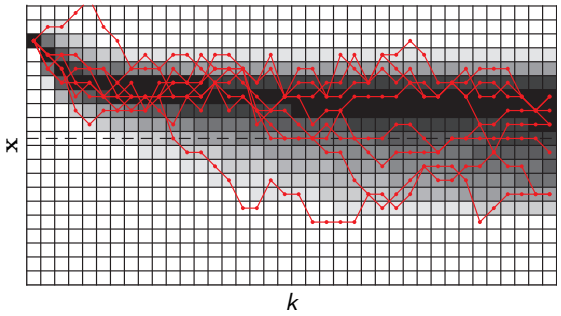

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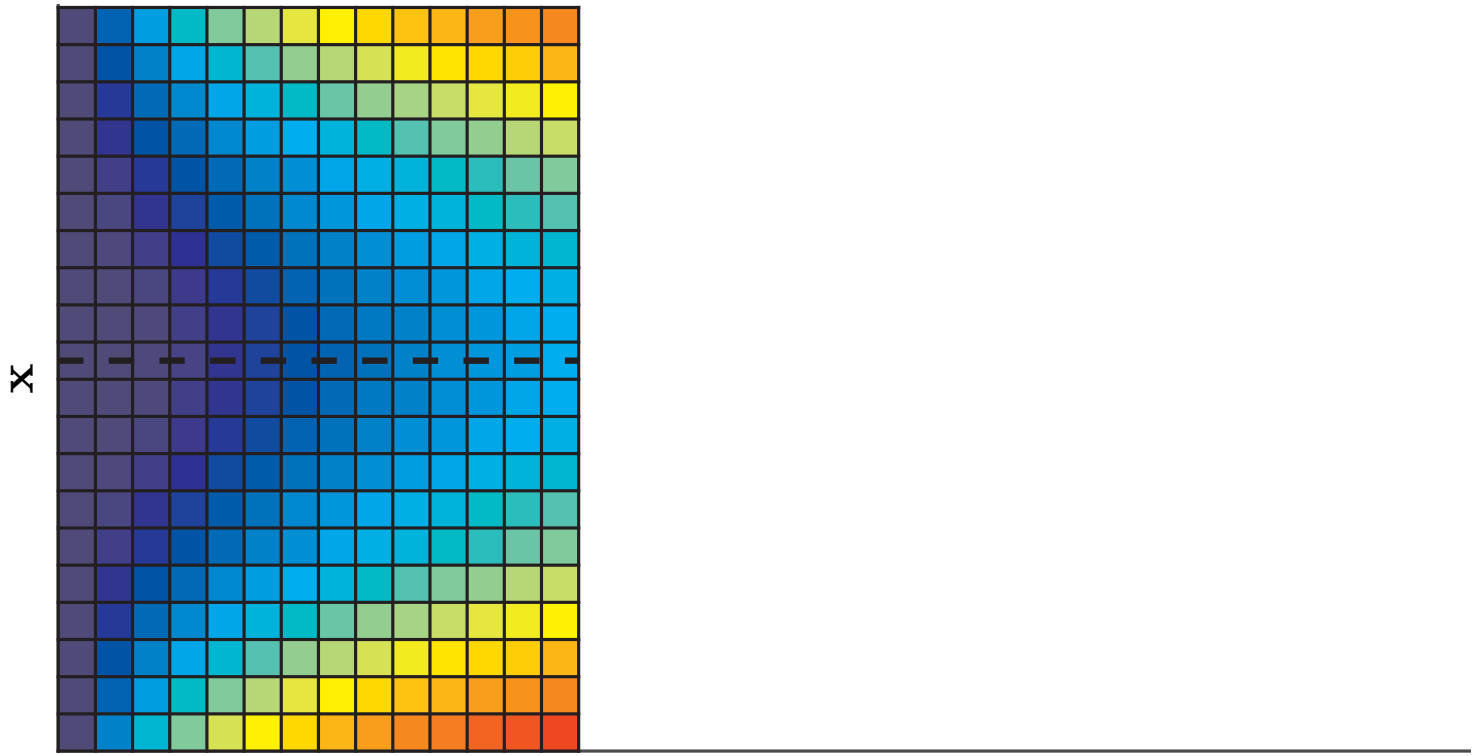
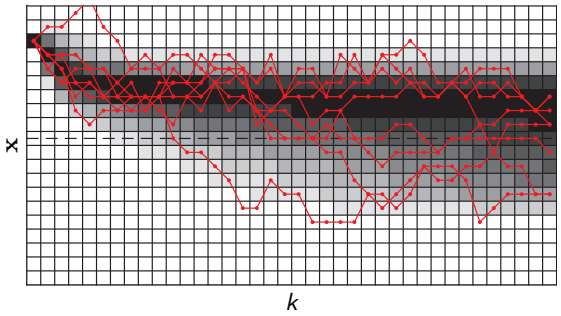


Iteration of $\boldsymbol{\pi}$ evaluation

Computing V_π - Illustration



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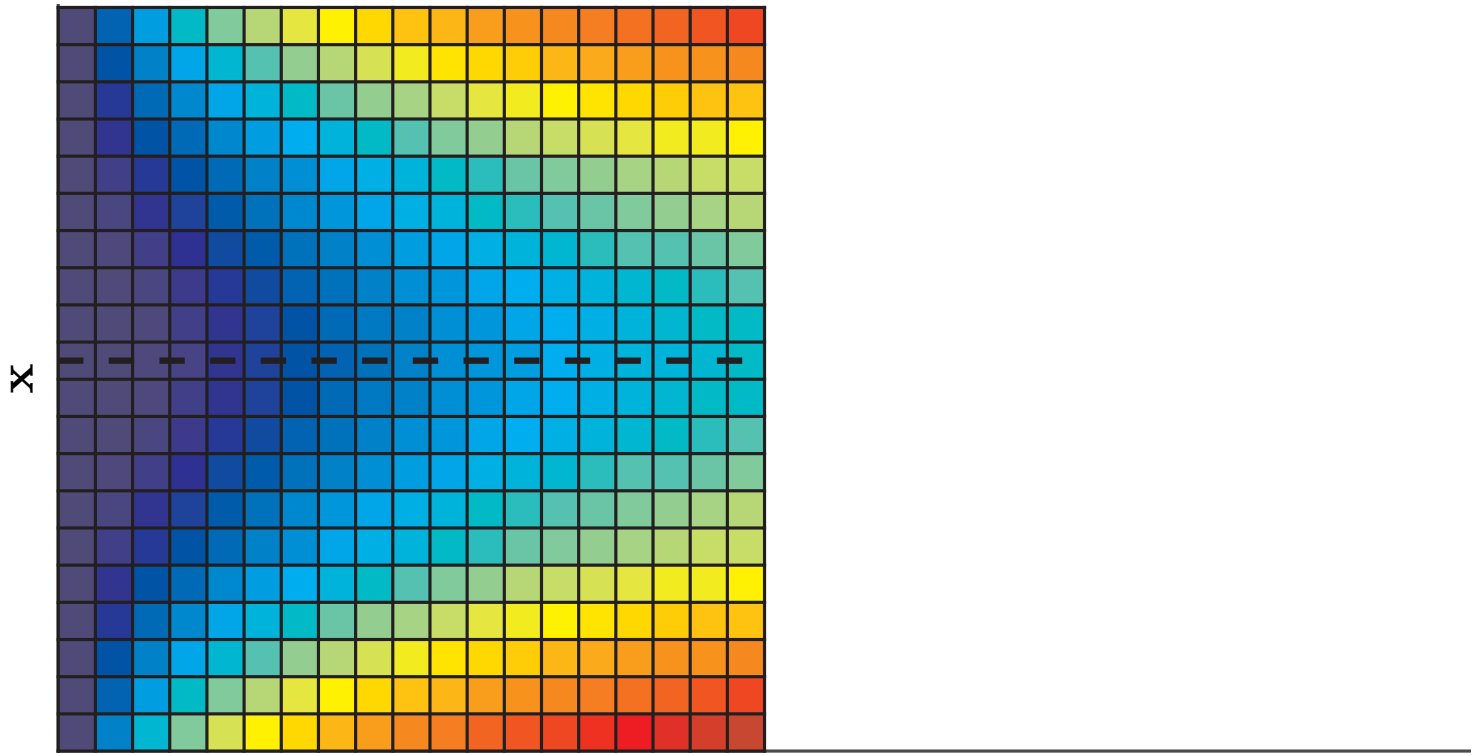

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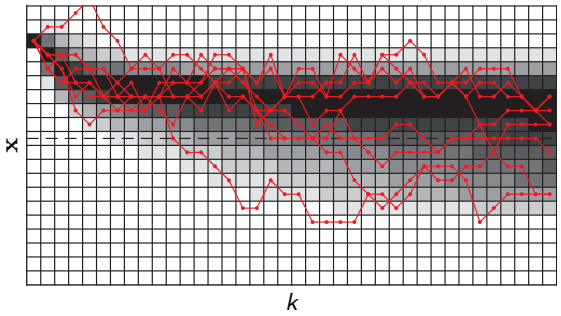

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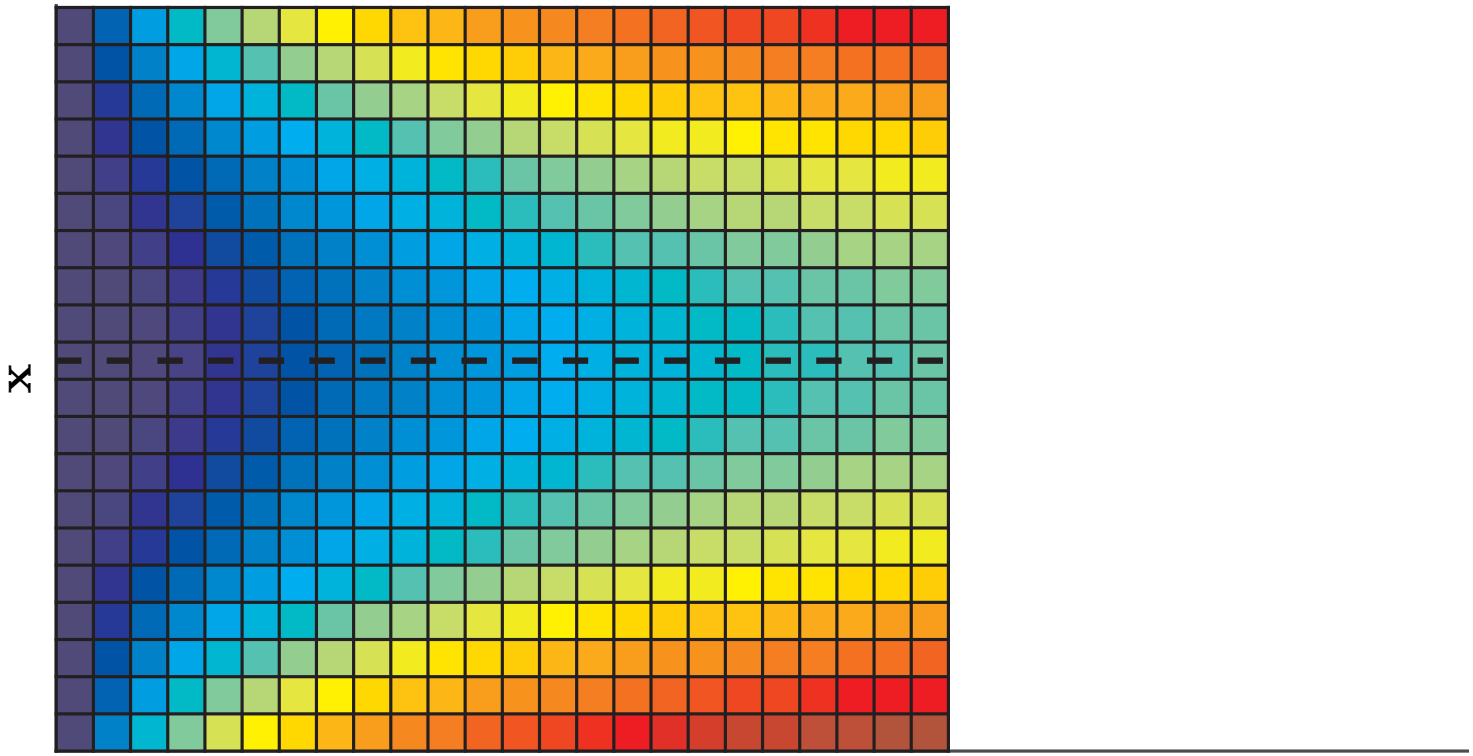
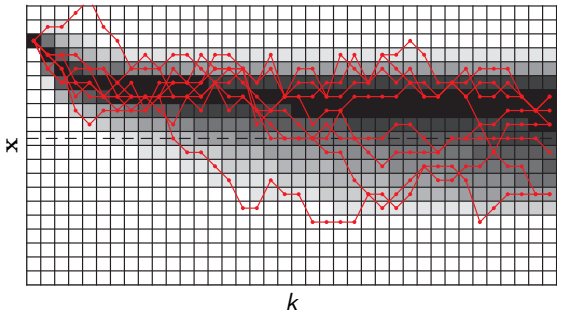


Iteration of $\boldsymbol{\pi}$ evaluation

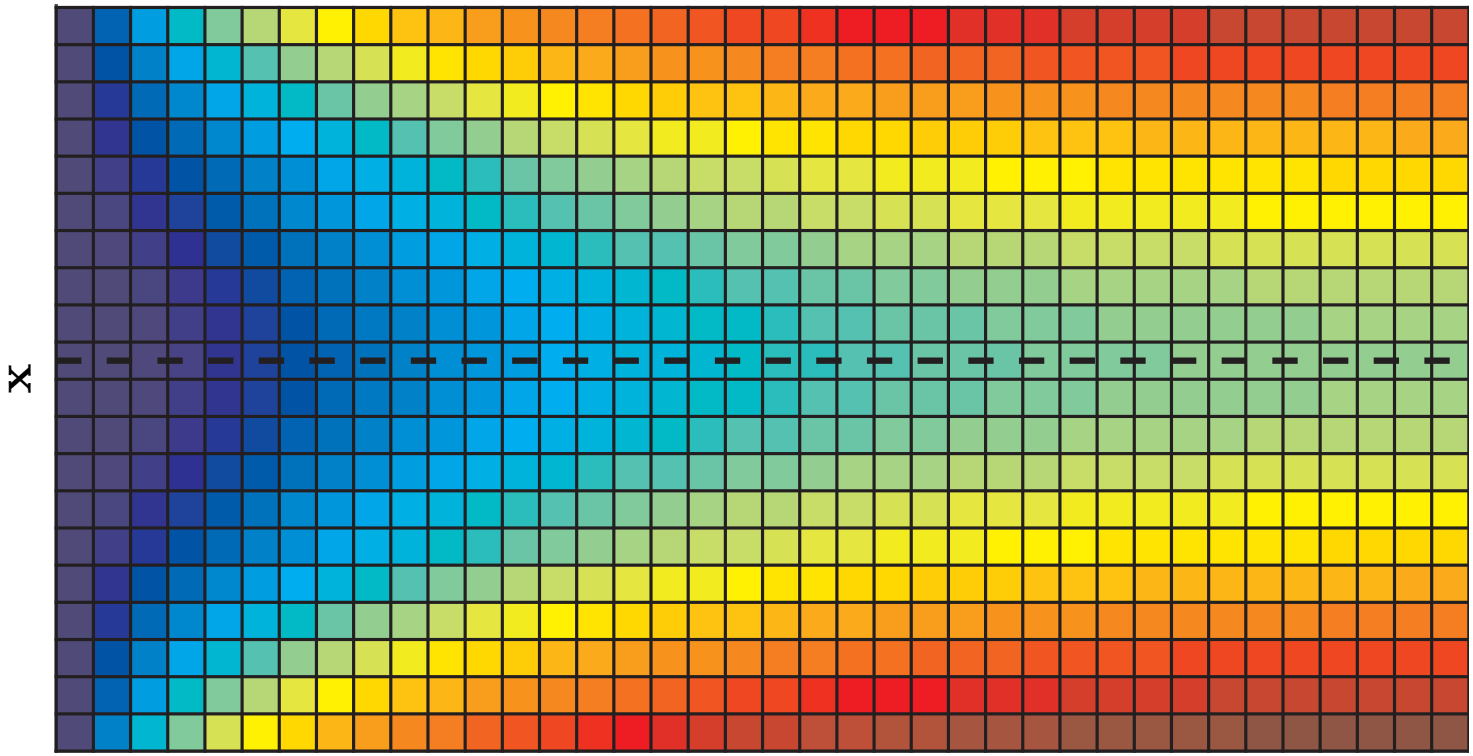
Computing V_π - Illustration



Start with e.g. $V(\mathbf{x}) = 0$ and iterate

$$V(\mathbf{x}) \leftarrow L(\mathbf{x}, \boldsymbol{\pi}(\mathbf{x})) + \gamma \mathbb{E}\left[V\left(\mathbf{x}_+\right) \mid \mathbf{x}, \boldsymbol{\pi}\right]$$

as a sweep over all $\mathbf{x}$



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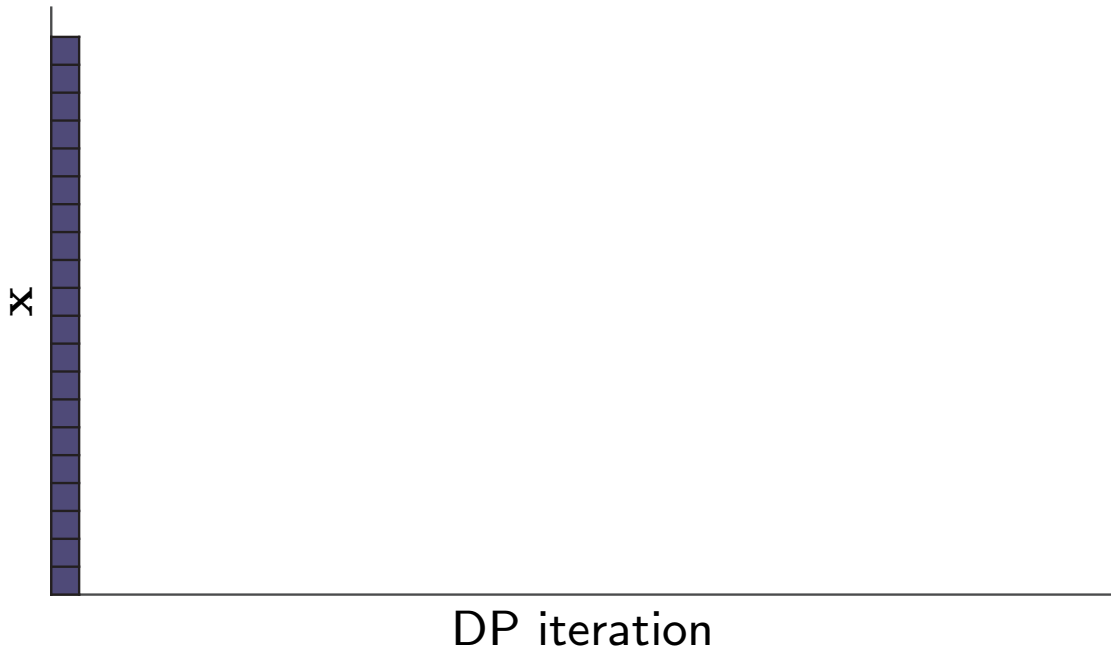
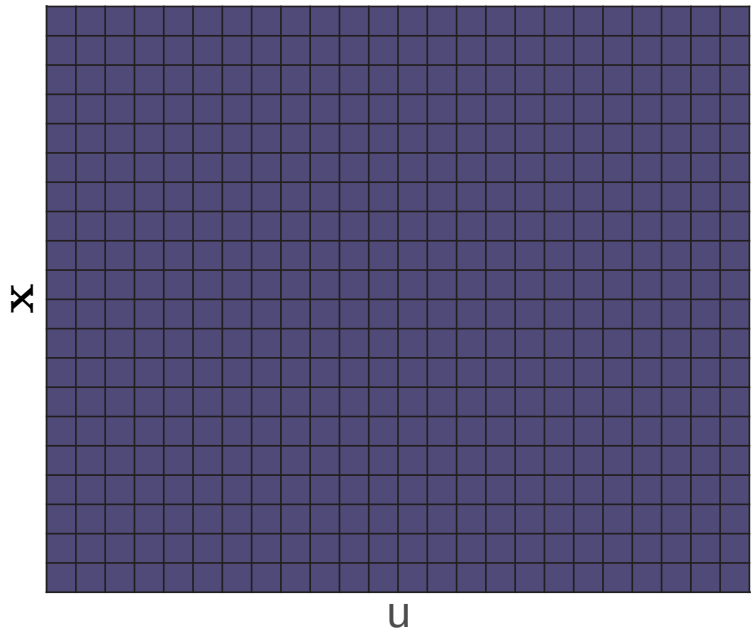
Need to sweep over $\mathbf{x}$ and $\mathbf{u}$ to build Q

V, Q converge to V_π, Q_π





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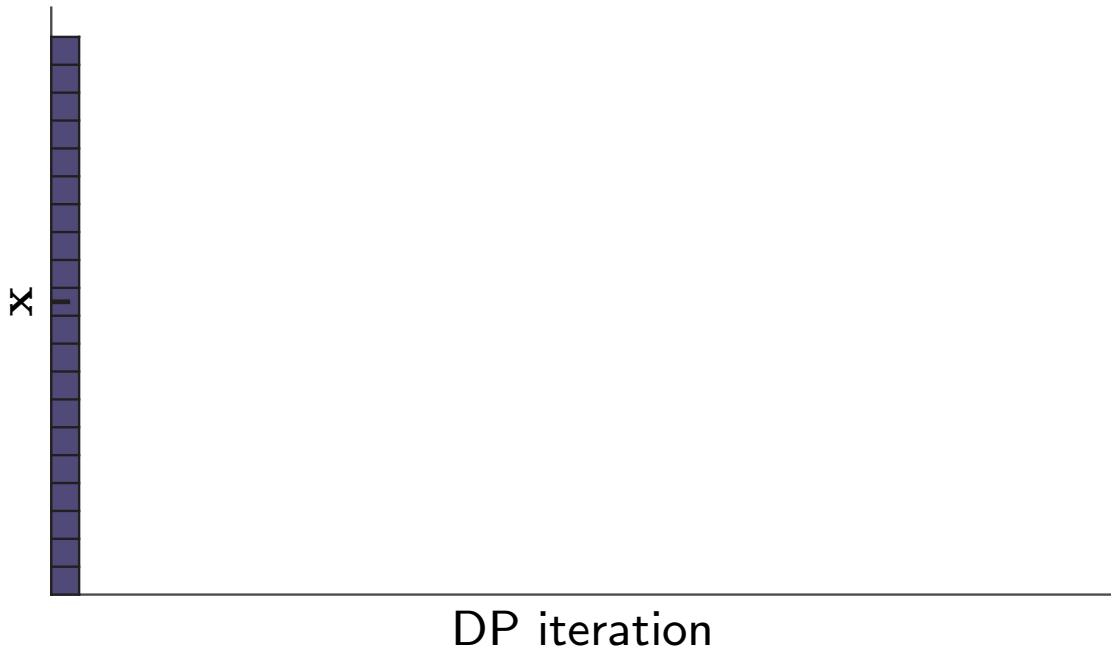

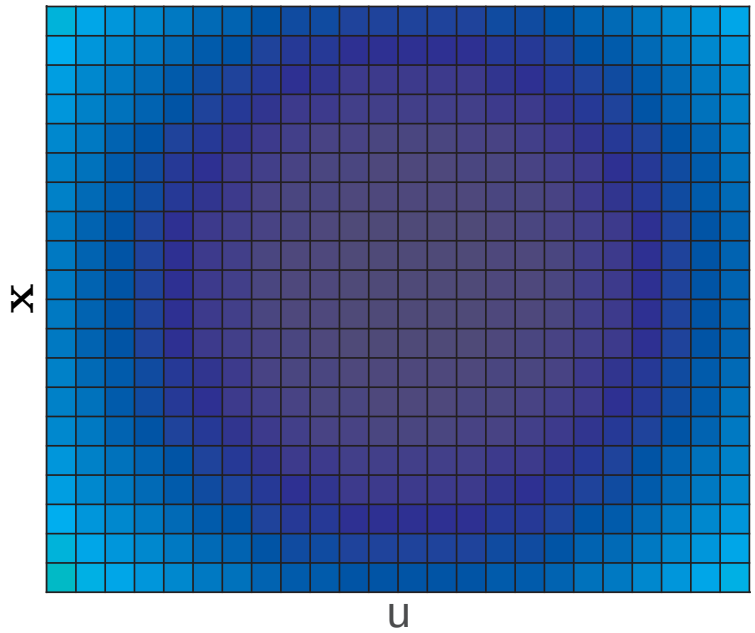

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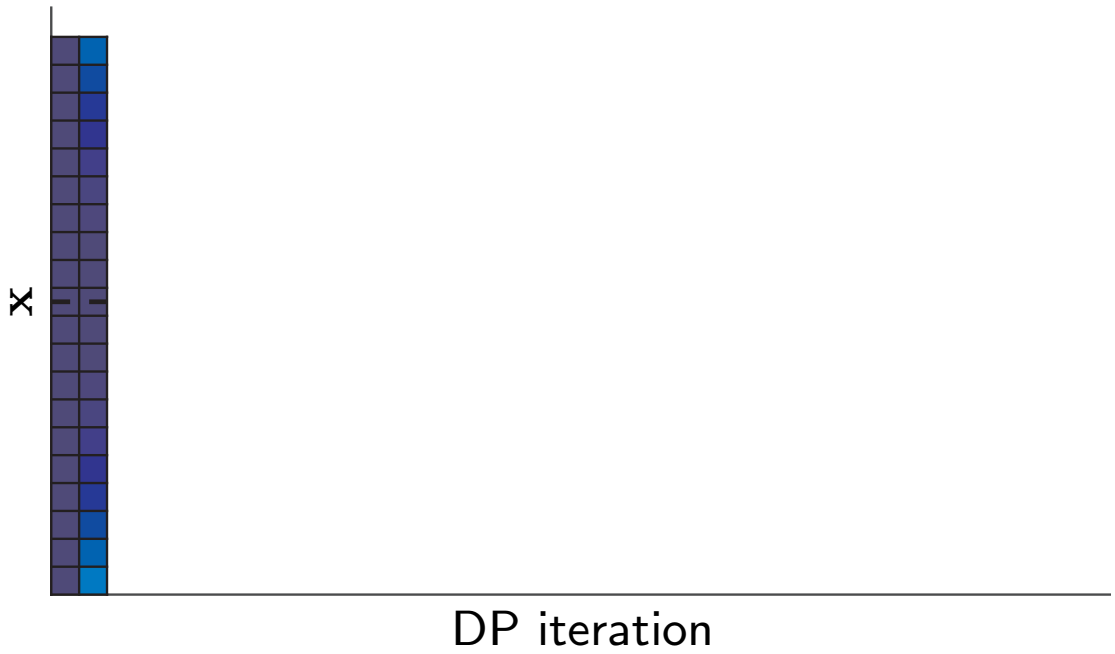

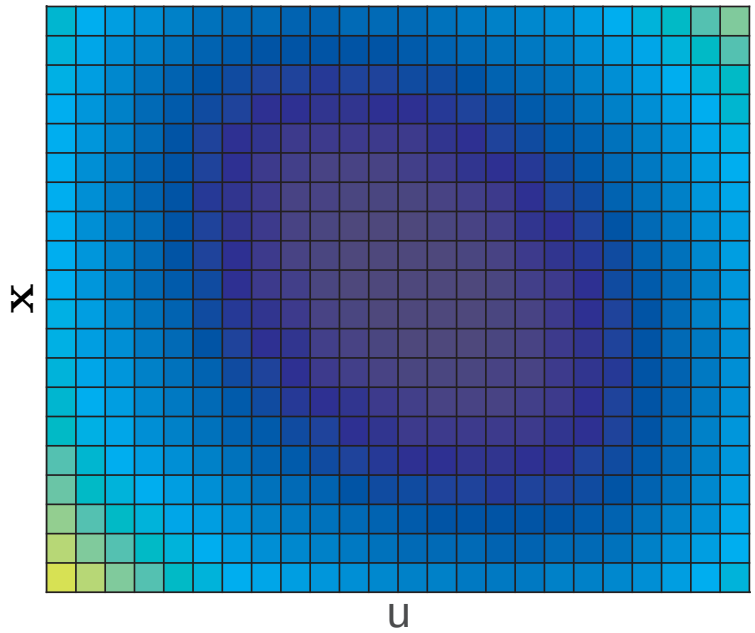

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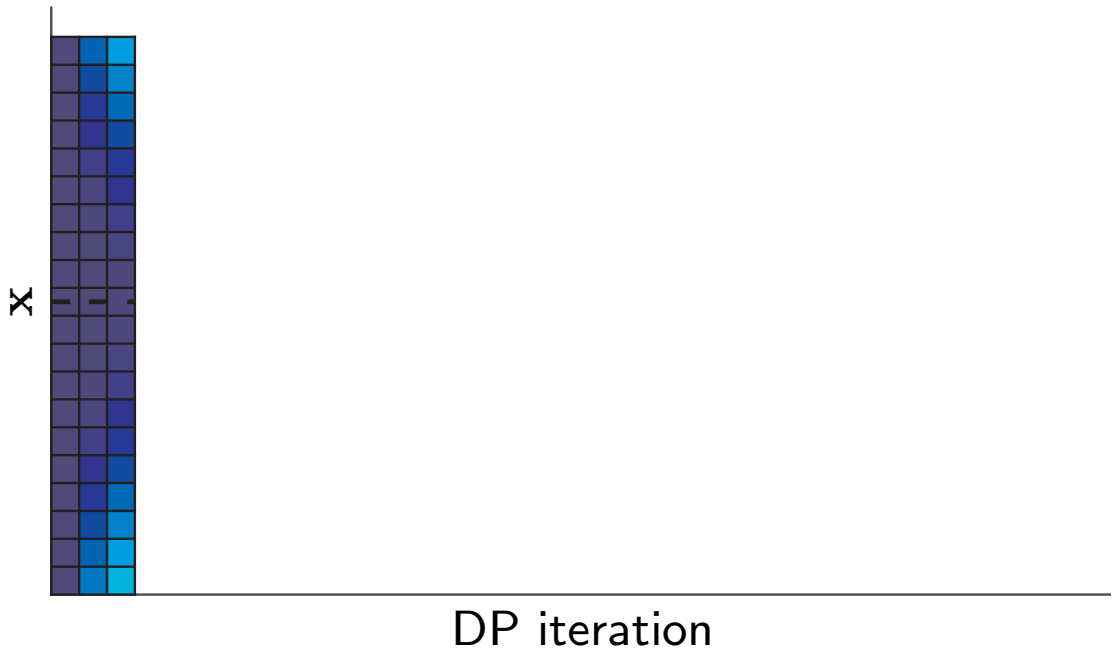

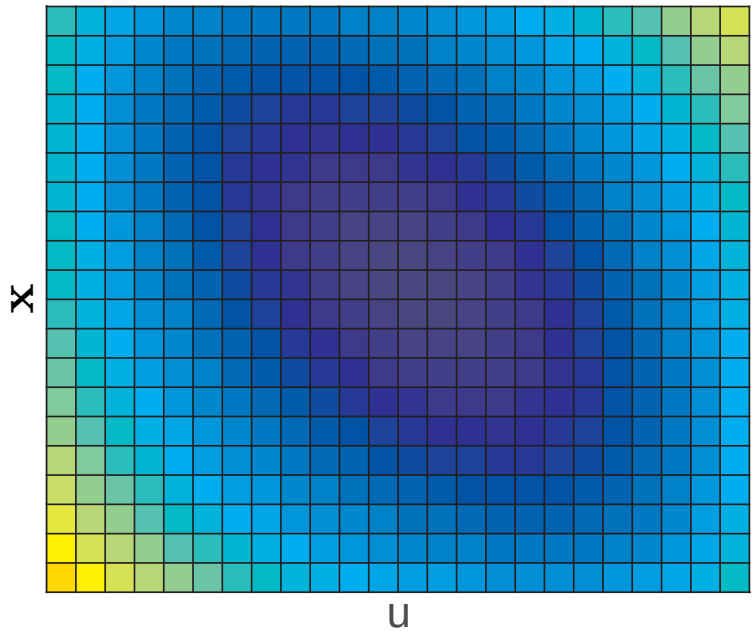

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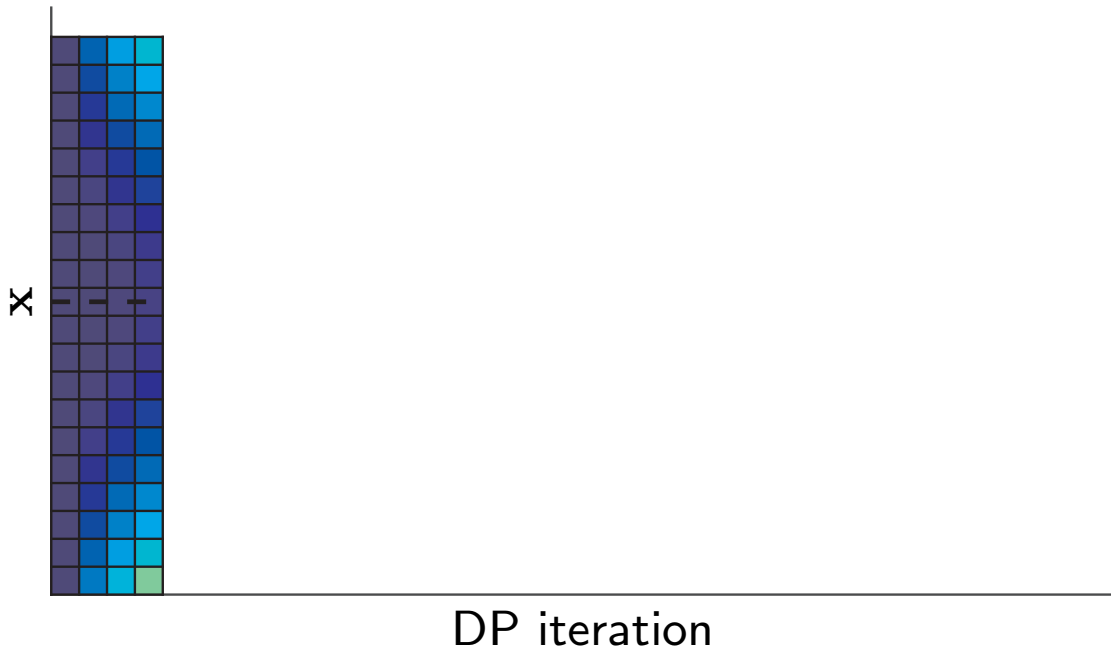

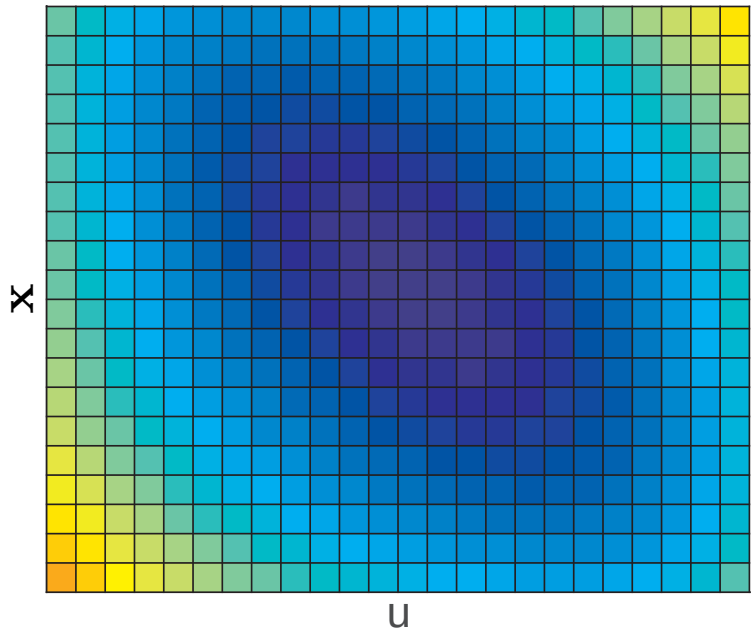

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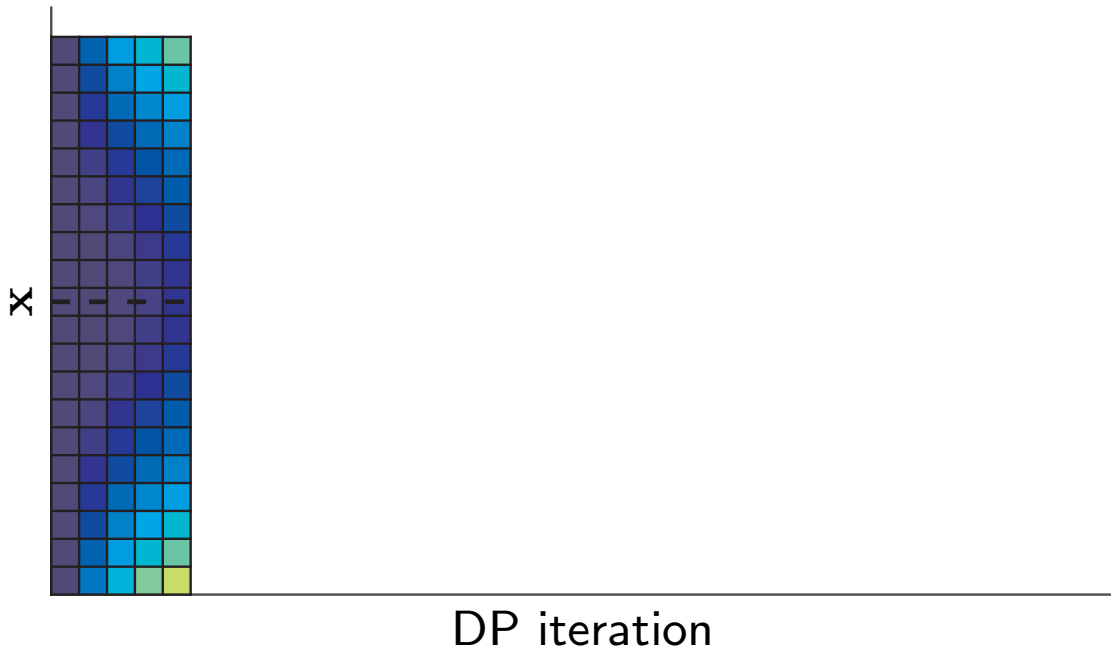

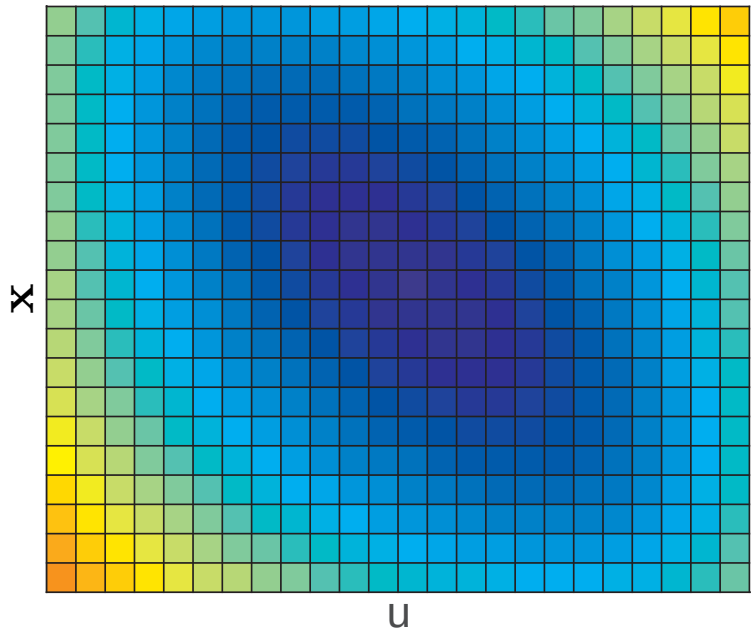

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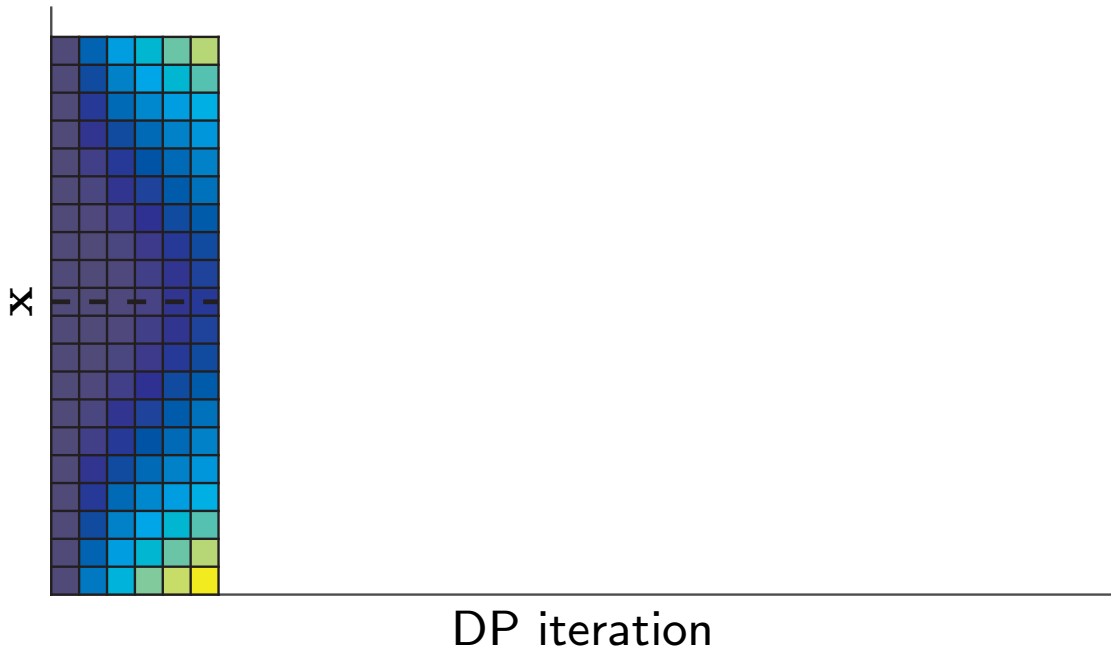

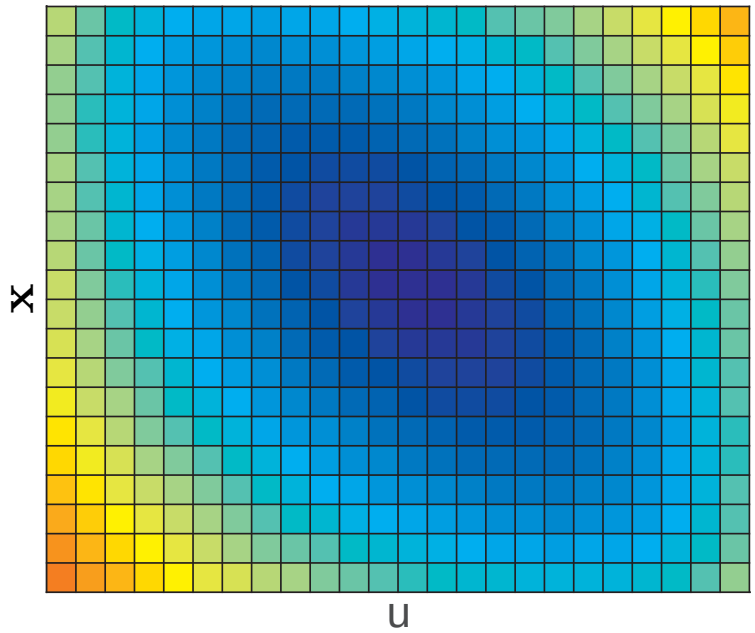

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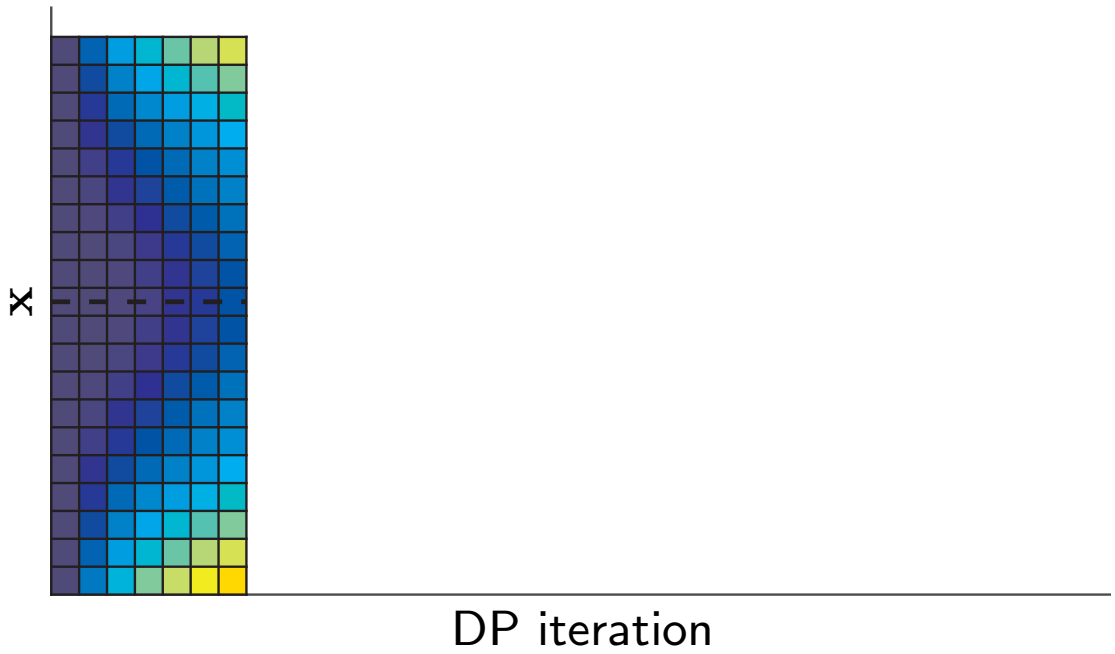

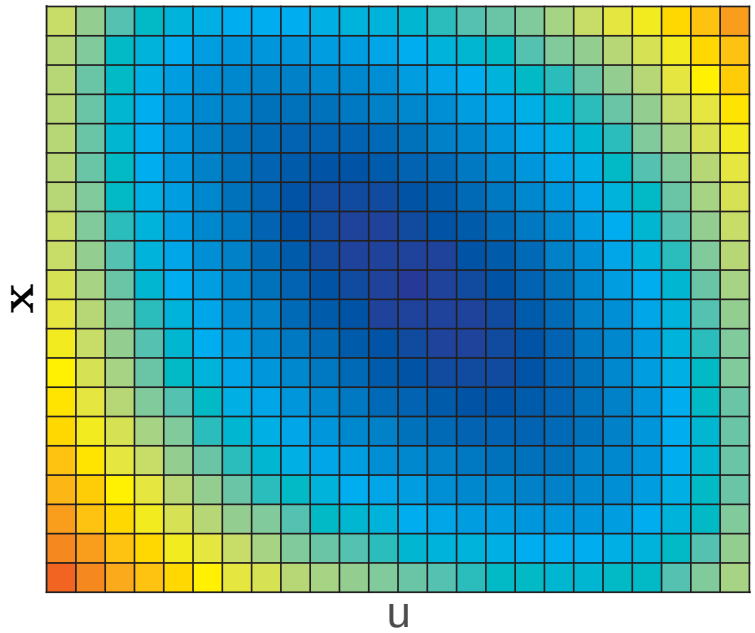

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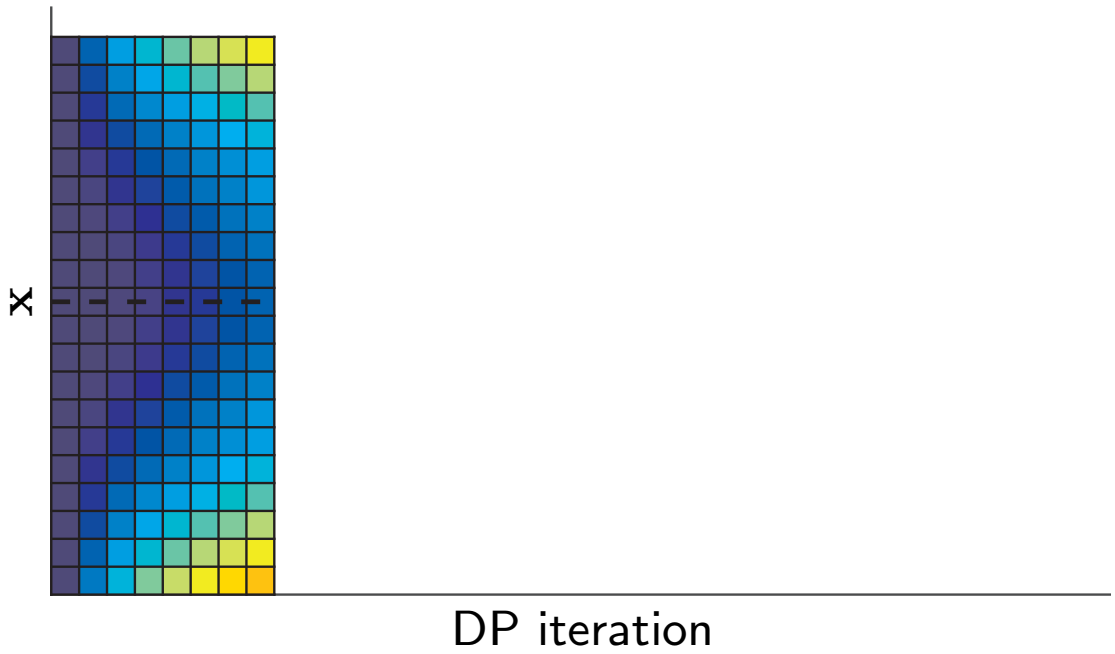

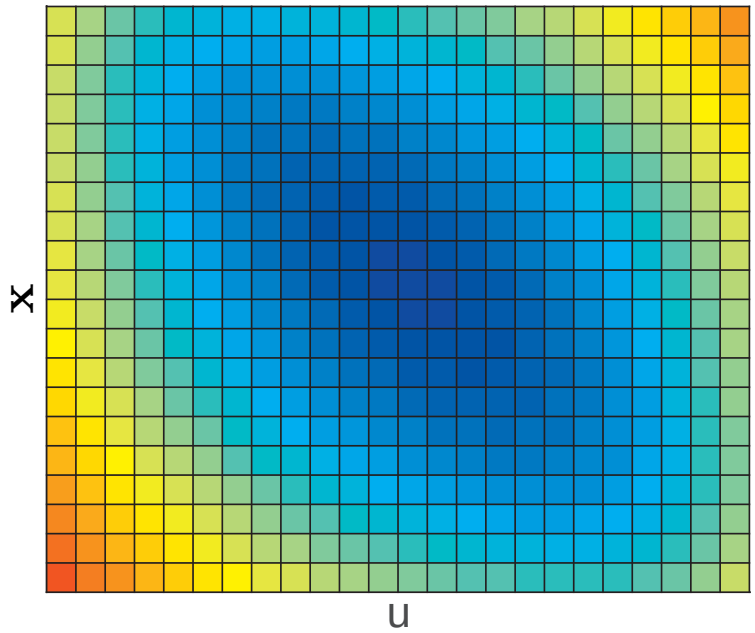

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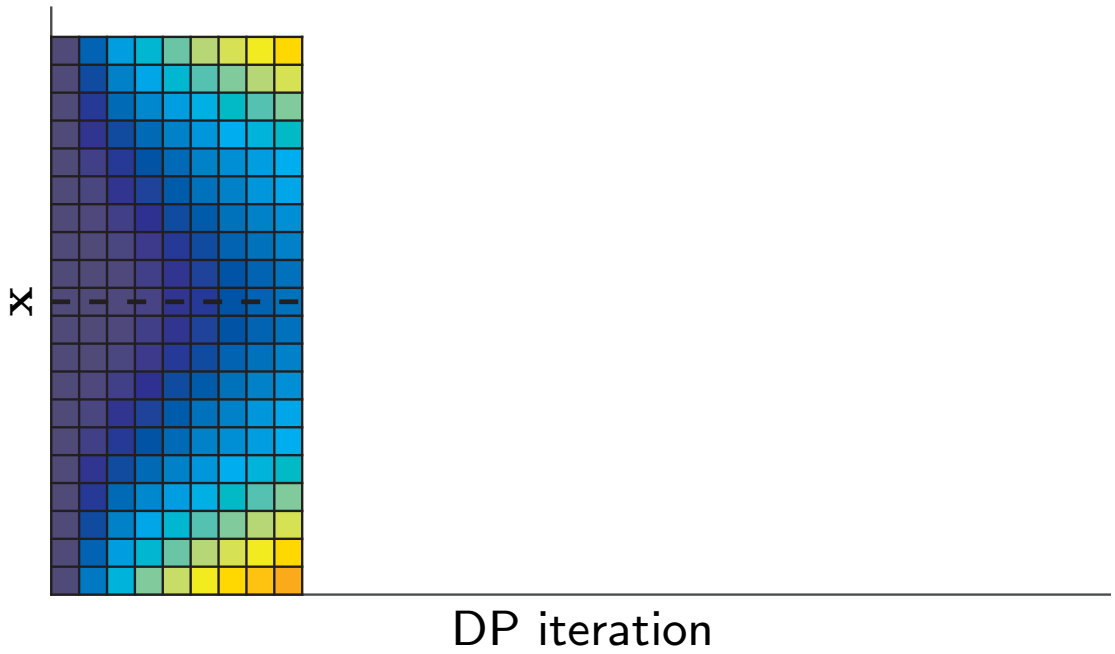

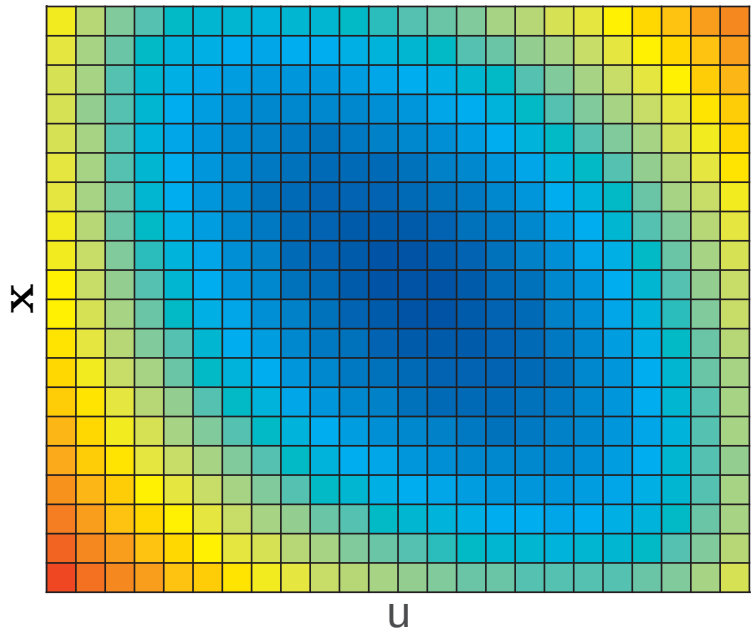

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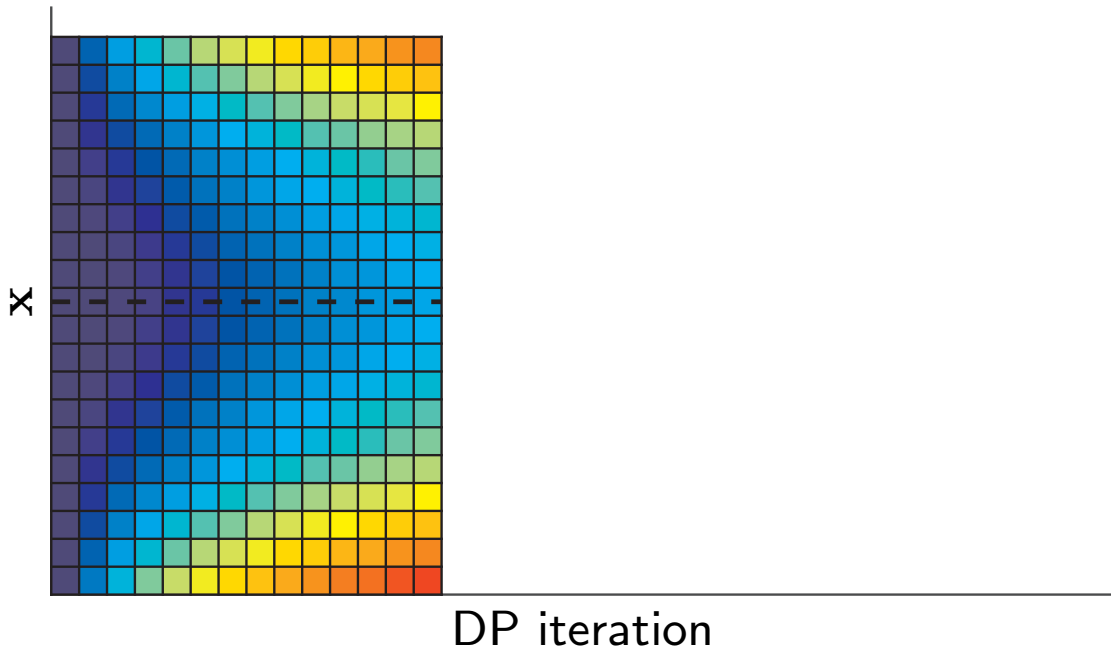

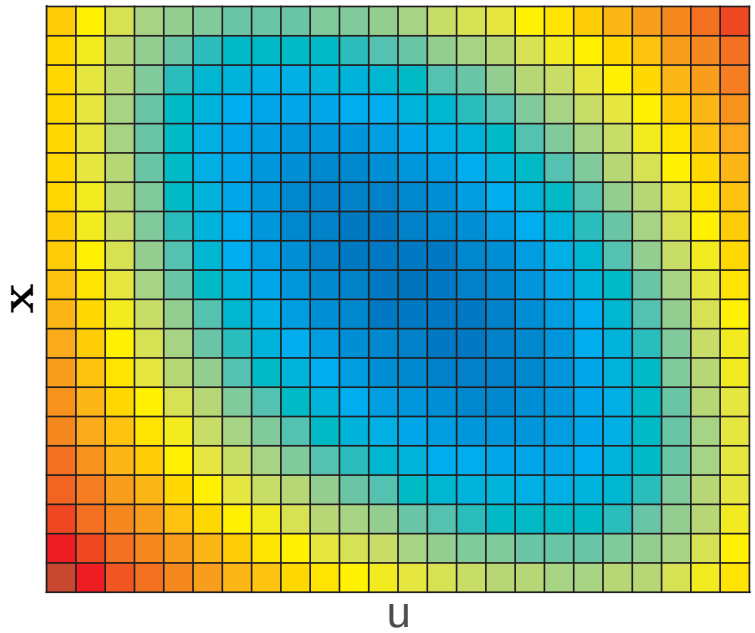

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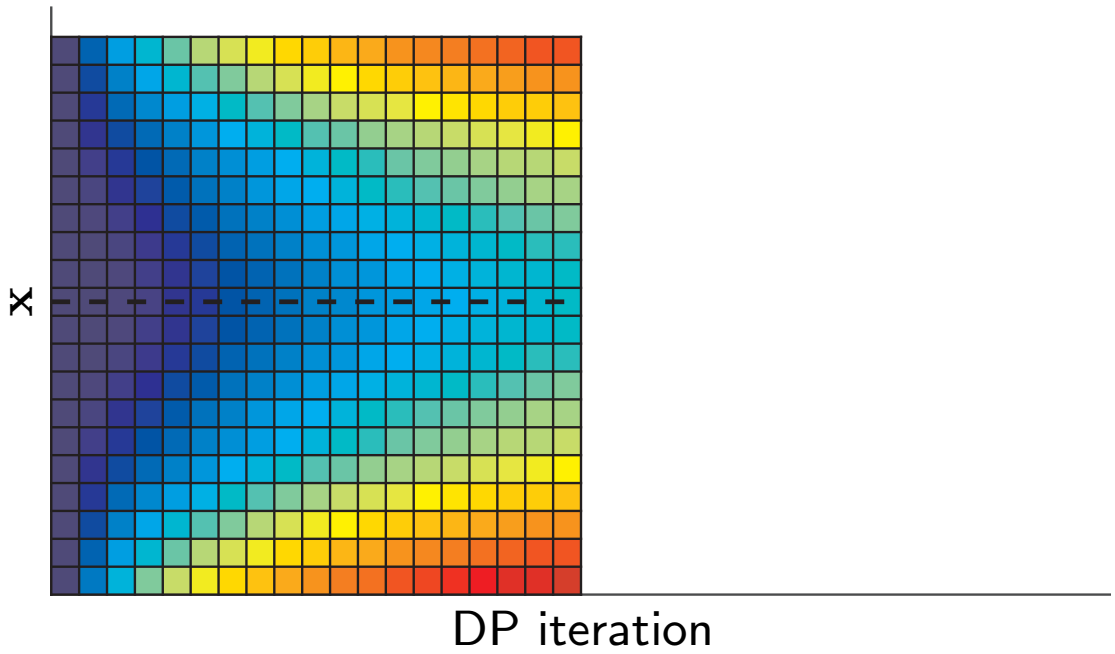

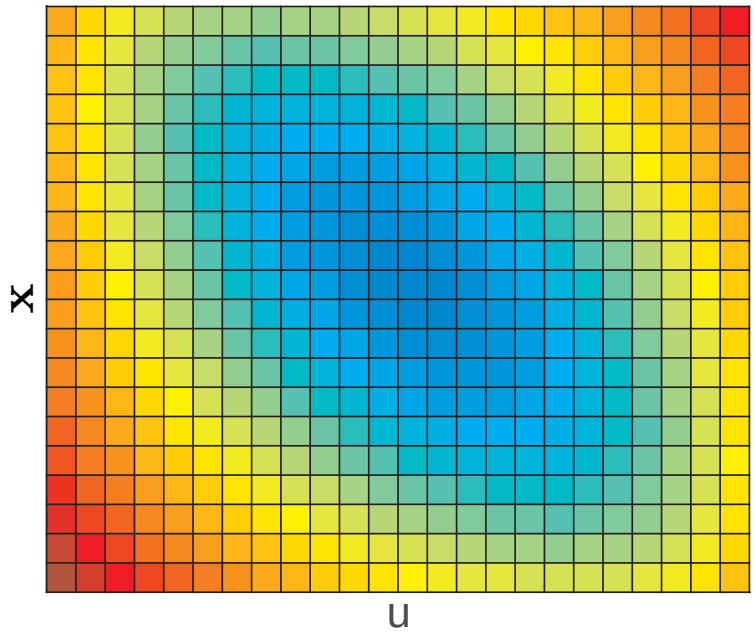

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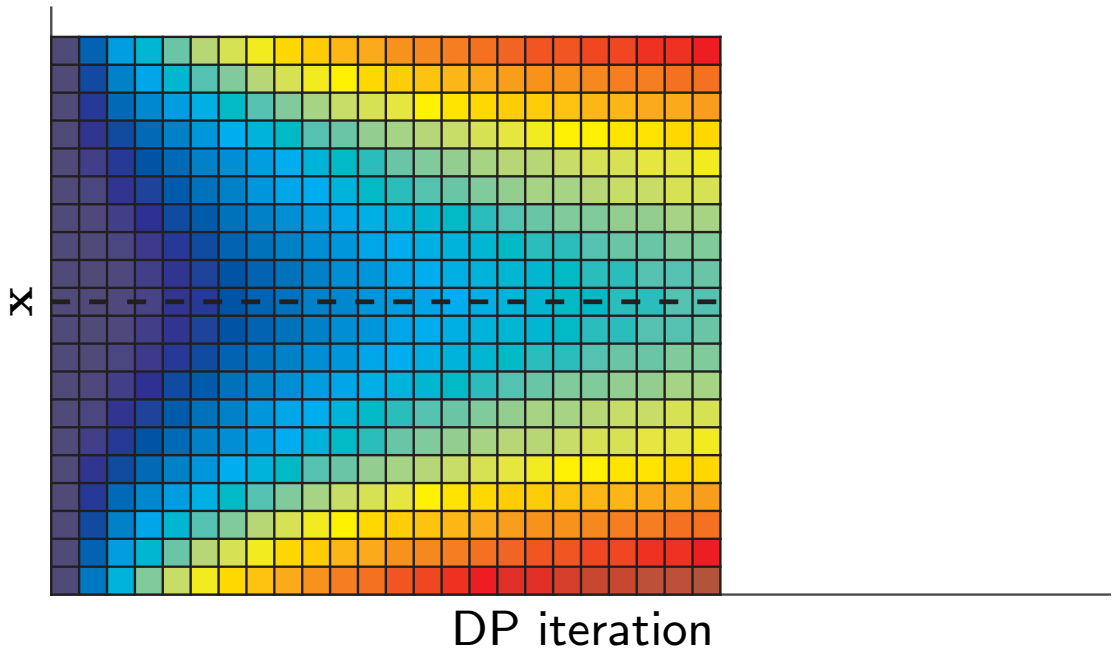

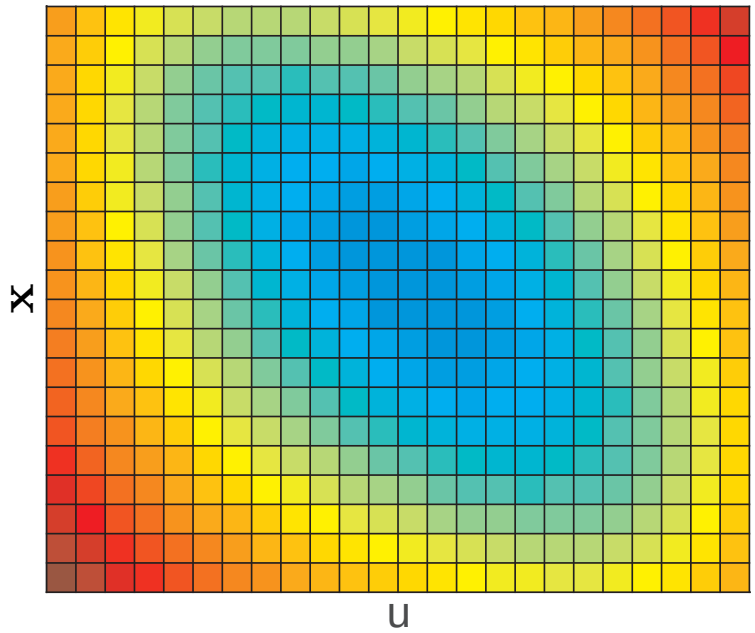

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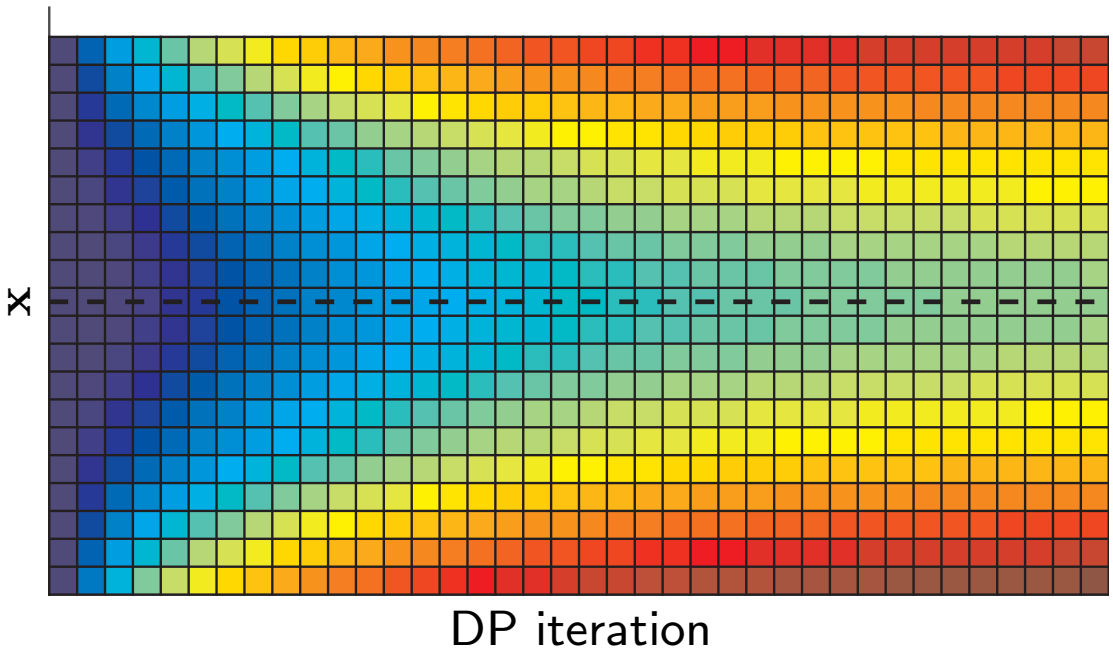

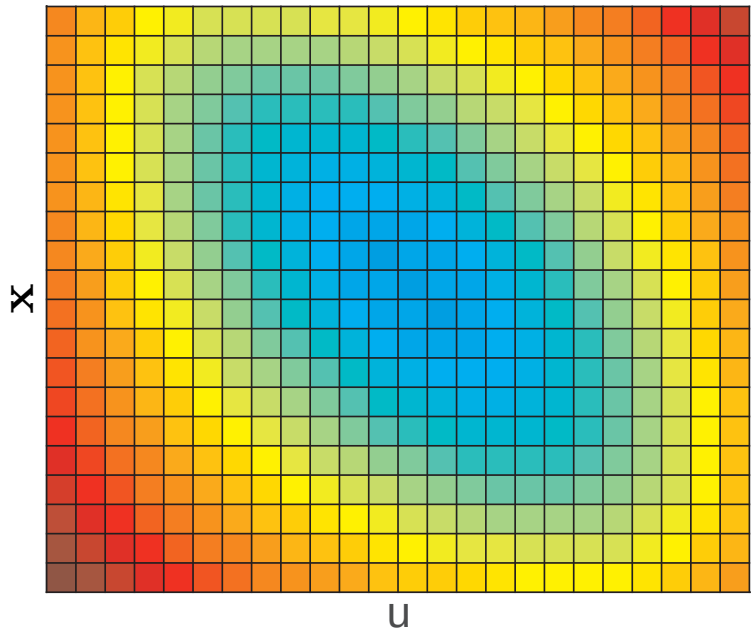

How to compute optimal policies?

How to solve the Bellman optimality equations?

$$V_\star(\mathbf{x}) = \min_{\mathbf{u}} \; L(\mathbf{x},\mathbf{u}) + \gamma \mathbb{E}\left[V_\star(\mathbf{x}_+) \,|\, \mathbf{x},\mathbf{u}\right] \qquad V_\star(\mathbf{x}) = \min_{\mathbf{u}} \; Q_\star(\mathbf{x},\mathbf{u}),$$

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π-improvement: select a $\boldsymbol{\pi}(\mathbf{x})$, alternate

1. Compute $Q_\pi(\mathbf{x})$
2. Update policy

$$\boldsymbol{\pi}(\mathbf{x}) \leftarrow \arg\min_{\mathbf{u}} \; Q_\pi(\mathbf{x},\mathbf{u})$$

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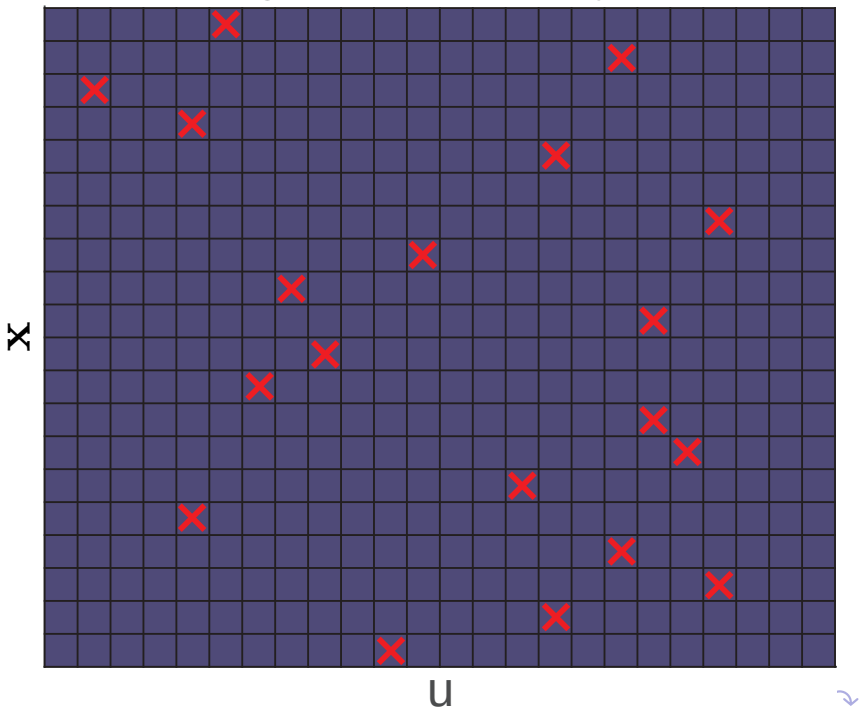
Algorithm: π-improvement

Input: π and $\mathrm{tol} > 0$
$\mathrm{Iter} = \mathrm{True}$
while Iter **do**
  Compute Q_π
  Update $\pi_+(\mathbf{x}) \leftarrow \arg\min_{\mathbf{u}} \; Q_\pi(\mathbf{x},\mathbf{u})$
  if $\|\pi_+ - \pi\| < \mathrm{tol}$ **then**
    $\mathrm{Iter} = \mathrm{False}$
  $\pi_+ \leftarrow \pi$
return $\pi_\star \leftarrow \pi_+$

E.g. random policy π



How to compute optimal policies?

How to solve the Bellman optimality equations?

$$V_\star(\mathbf{x}) = \min_{\mathbf{u}} \; L(\mathbf{x},\mathbf{u}) + \gamma \mathbb{E}\left[V_\star(\mathbf{x}_+) \,|\, \mathbf{x},\mathbf{u}\right] \qquad V_\star(\mathbf{x}) = \min_{\mathbf{u}} \; Q_\star(\mathbf{x},\mathbf{u}),$$

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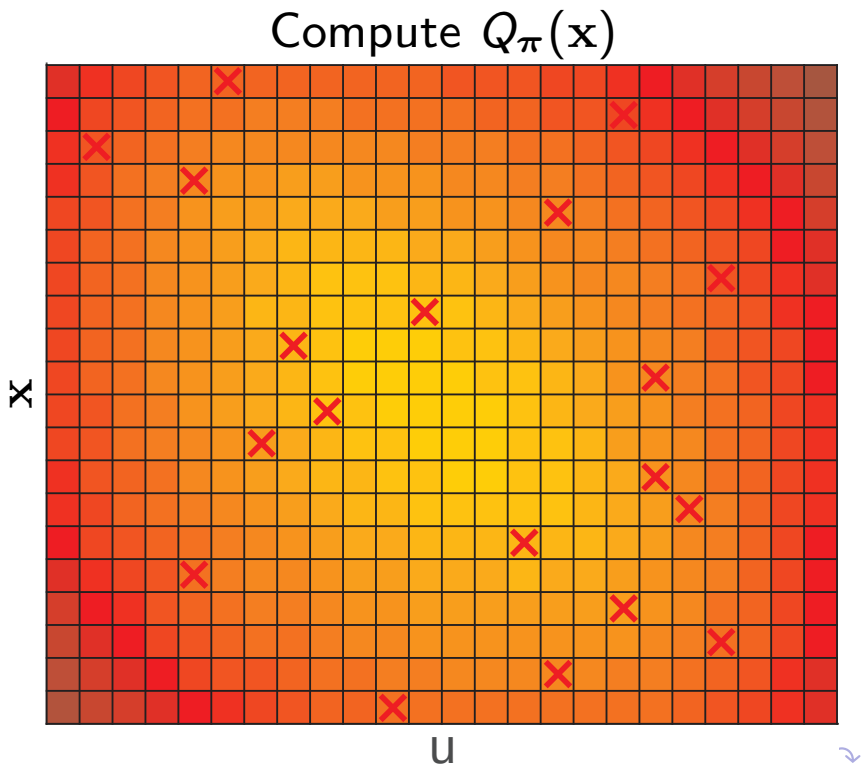
Algorithm: $\boldsymbol{\pi}$-improvement

Input: $\boldsymbol{\pi}$ and $\mathrm{tol} > 0$
$\mathrm{Iter} = \mathrm{True}$
while Iter **do**
  Compute $Q_{\boldsymbol{\pi}}$
  Update $\boldsymbol{\pi}_+(\mathbf{x}) \leftarrow \arg\min_{\mathbf{u}} \; Q_{\boldsymbol{\pi}}(\mathbf{x},\mathbf{u})$
  if $\|\boldsymbol{\pi}_+ - \boldsymbol{\pi}\| < \mathrm{tol}$ **then**
    $\mathrm{Iter} = \mathrm{False}$
  $\boldsymbol{\pi}_+ \leftarrow \boldsymbol{\pi}$
return $\boldsymbol{\pi}_\star \leftarrow \boldsymbol{\pi}_+$



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Input: $\boldsymbol{\pi}$ and $\mathrm{tol} > 0$
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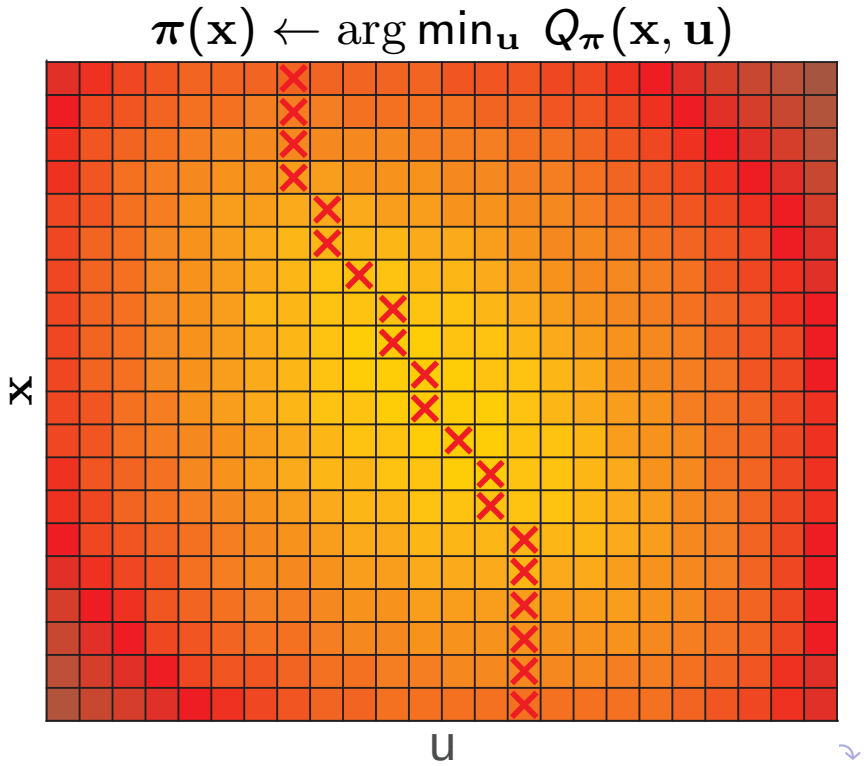

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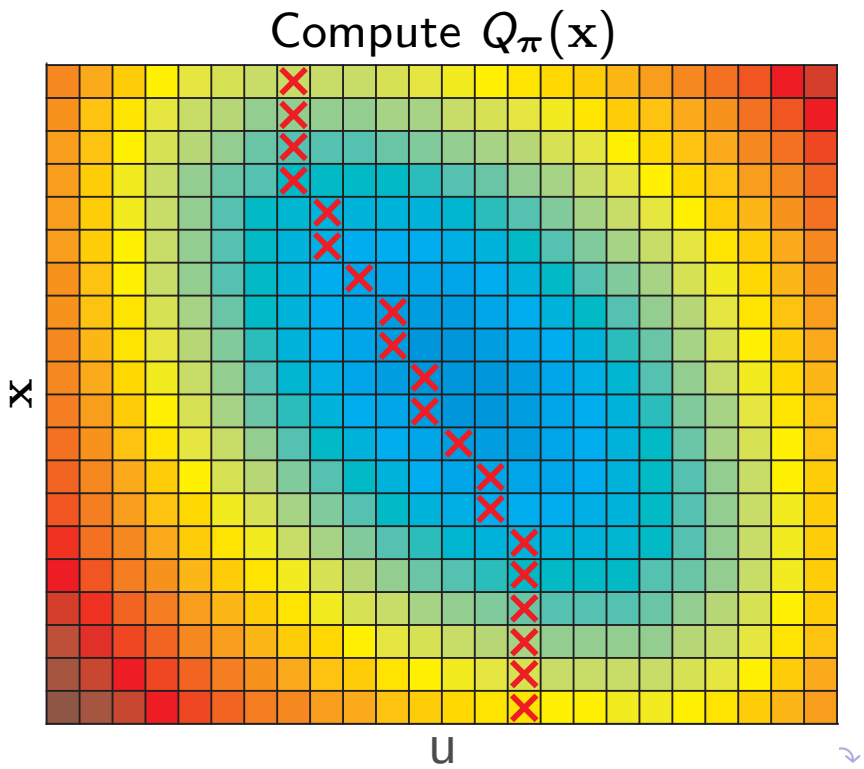

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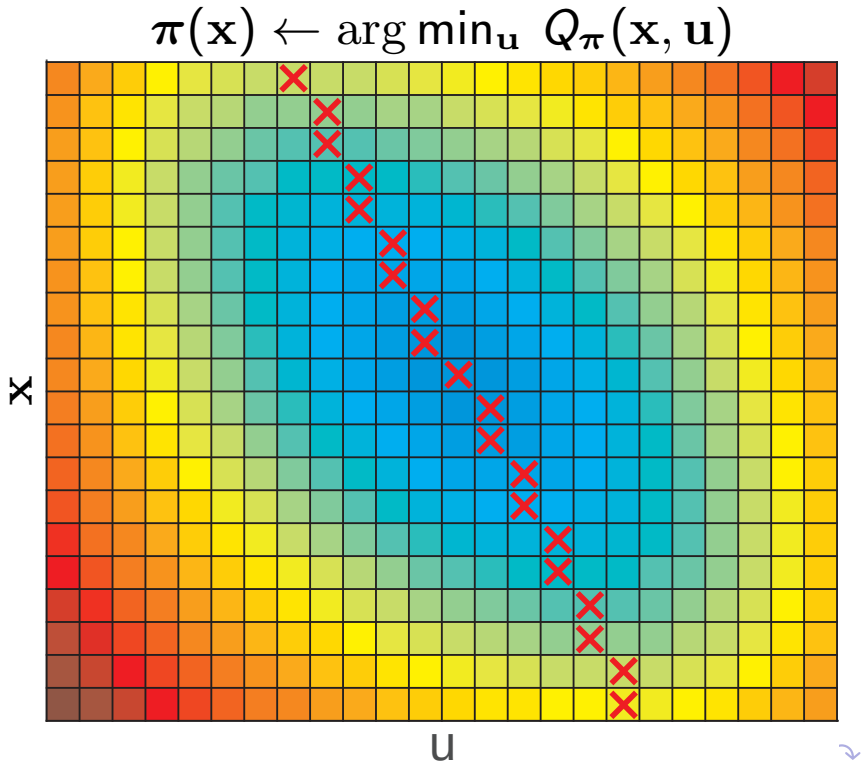

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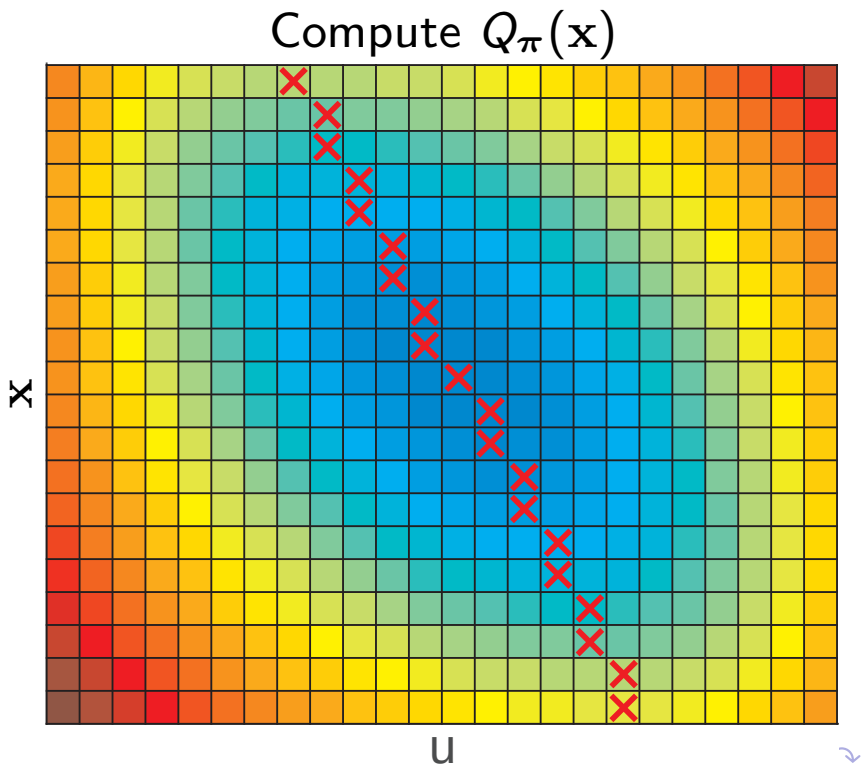

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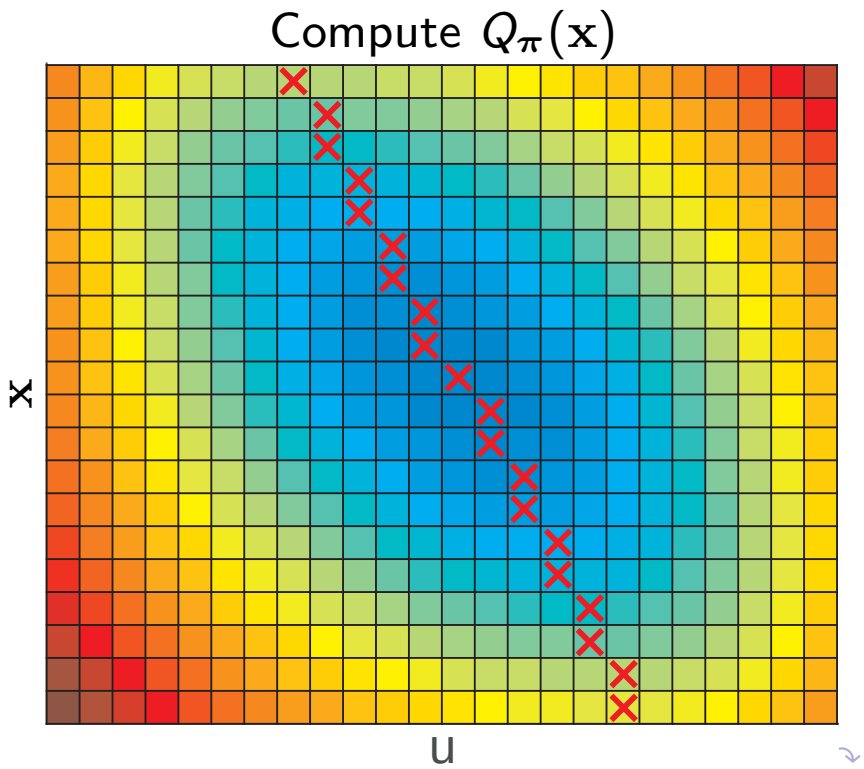
  $\boldsymbol{\pi}_+ \leftarrow \boldsymbol{\pi}$
return $\boldsymbol{\pi}_\star \leftarrow \boldsymbol{\pi}_+$

Converges to $\boldsymbol{\pi}_\star$ if initial policy yields $Q_{\boldsymbol{\pi}} < \infty$ for some $\mathbf{x}, \mathbf{u}$



Dynamic Programming

Algorithm: π-improvement

Input: π and $\text{tol} > 0$

$\text{Iter} = \text{True}$

while Iter **do**

  Compute Q_π

  Update

  $\pi_+(\mathbf{x}) \leftarrow \arg\min_{\mathbf{u}} \; Q_\pi(\mathbf{x}, \mathbf{u})$

  if $\|\pi_+ - \pi\| < \text{tol}$ **then**

    $\text{Iter} = \text{False}$

  $\pi_+ \leftarrow \pi$

return $\pi_\star \leftarrow \pi_+$

Dynamic Programming

Algorithm: π-improvement

Input: π and $\text{tol} > 0$
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  $\pi_+(\mathbf{x}) \leftarrow \arg\min_{\mathbf{u}} \ Q_\pi(\mathbf{x}, \mathbf{u})$
  if $\|\pi_+ - \pi\| < \text{tol}$ **then**
    $\text{Iter} = \text{False}$
  $\pi_+ \leftarrow \pi$
return $\pi_\star \leftarrow \pi_+$

Algorithm: Value iteration

Input: V and $\text{tol} > 0$
$\text{Iter} = \text{True}, \ Q = 0$
while Iter **do**
  Compute
  $Q_+(\mathbf{x}, \mathbf{u}) \leftarrow L(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}\left[V\left(\mathbf{x}_+\right) \mid \mathbf{x}, \mathbf{u}\right]$
  $V(\mathbf{x}) \leftarrow \min_{\mathbf{u}} \ Q_+(\mathbf{x}, \mathbf{u})$
  if $\|Q_+ - Q\| < \text{tol}$ **then**
    $\text{Iter} = \text{False}$
  $Q \leftarrow Q_+$
return $\pi_\star \leftarrow \arg\min_{\mathbf{u}} \ Q_+(\mathbf{x}, \mathbf{u})$

Dynamic Programming

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Input: π and $\text{tol} > 0$
$\text{Iter} = \text{True}$
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  $\pi_+(\mathbf{x}) \leftarrow \arg\min_{\mathbf{u}} \; Q_\pi(\mathbf{x}, \mathbf{u})$
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Value iteration

- Iteration often written as $V(\mathbf{x}) \leftarrow \min_{\mathbf{u}} \; L(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}\left[V\left(\mathbf{x}_+\right) \mid \mathbf{x}, \mathbf{u}\right]$
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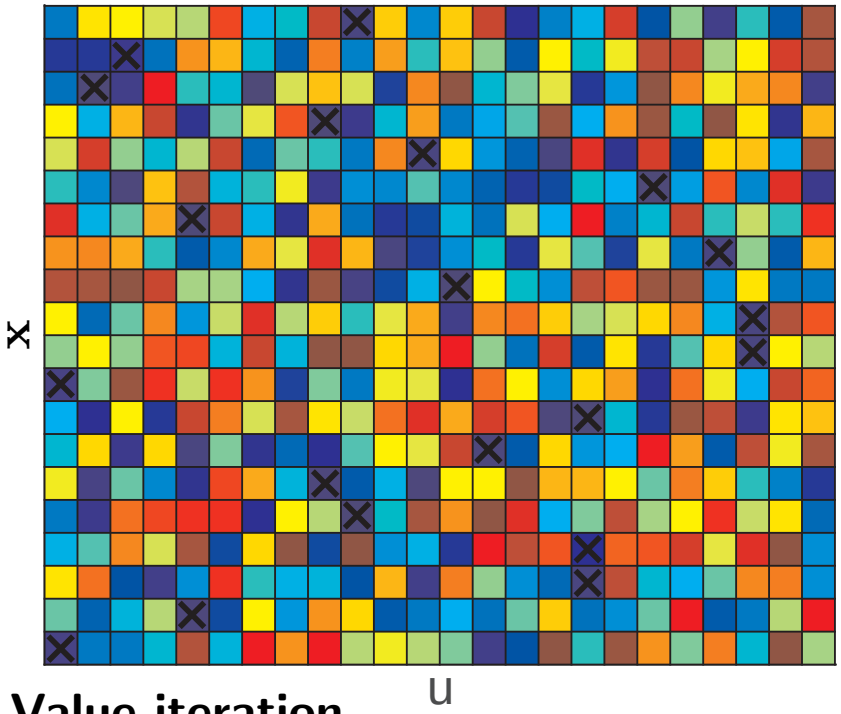
Dynamic Programming



Value iteration

Algorithm: Value iteration

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while Iter **do**

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$$Q_+(\mathbf{x}, \mathbf{u}) \leftarrow L(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}\left[V\left(\mathbf{x}_+\right) \mid \mathbf{x}, \mathbf{u}\right]$$

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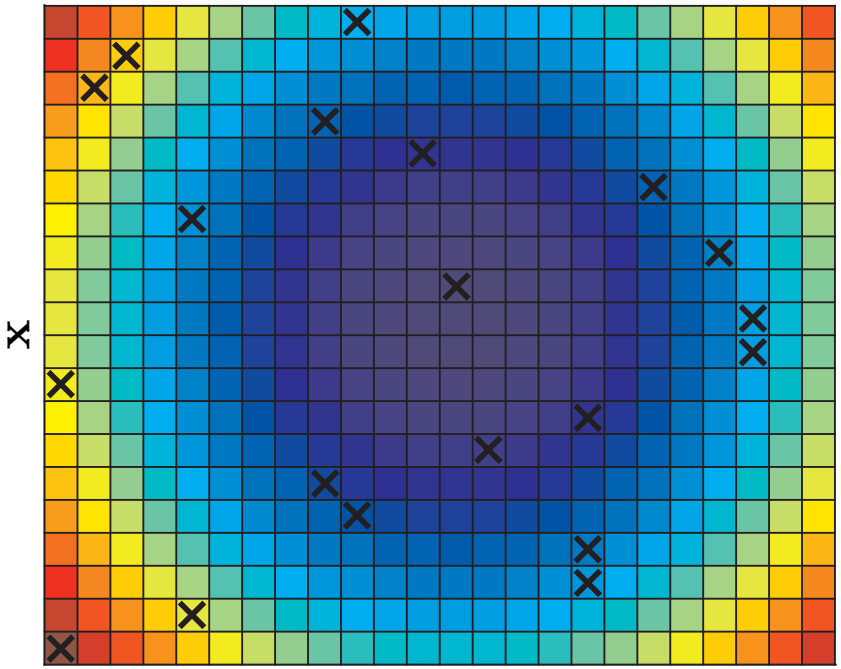

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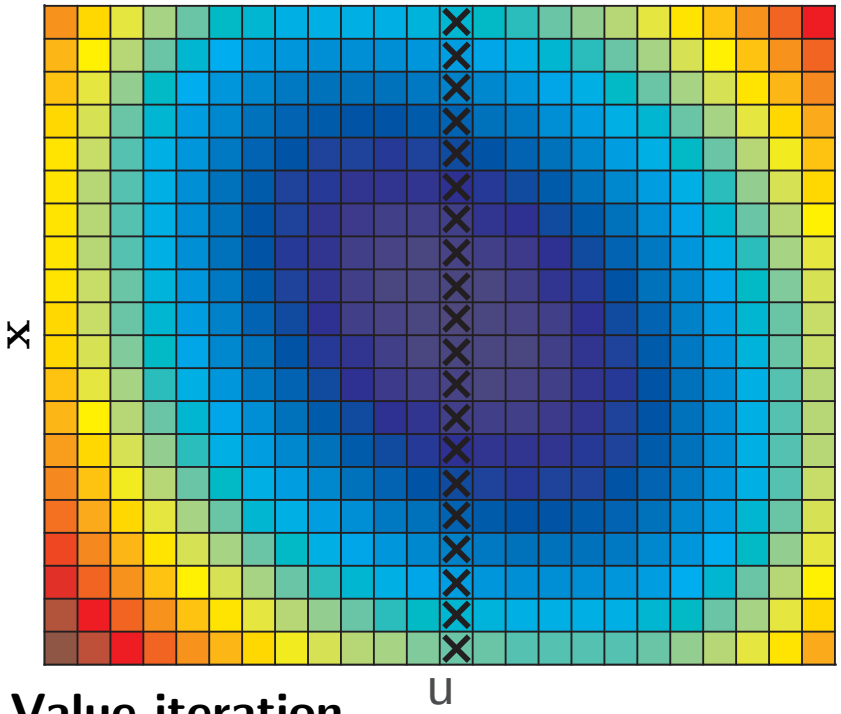

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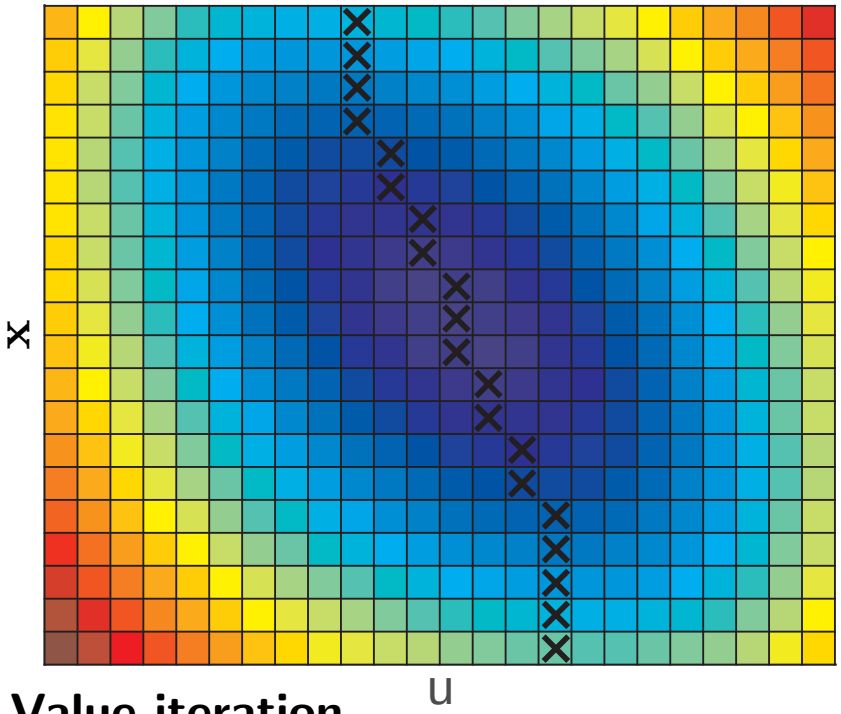

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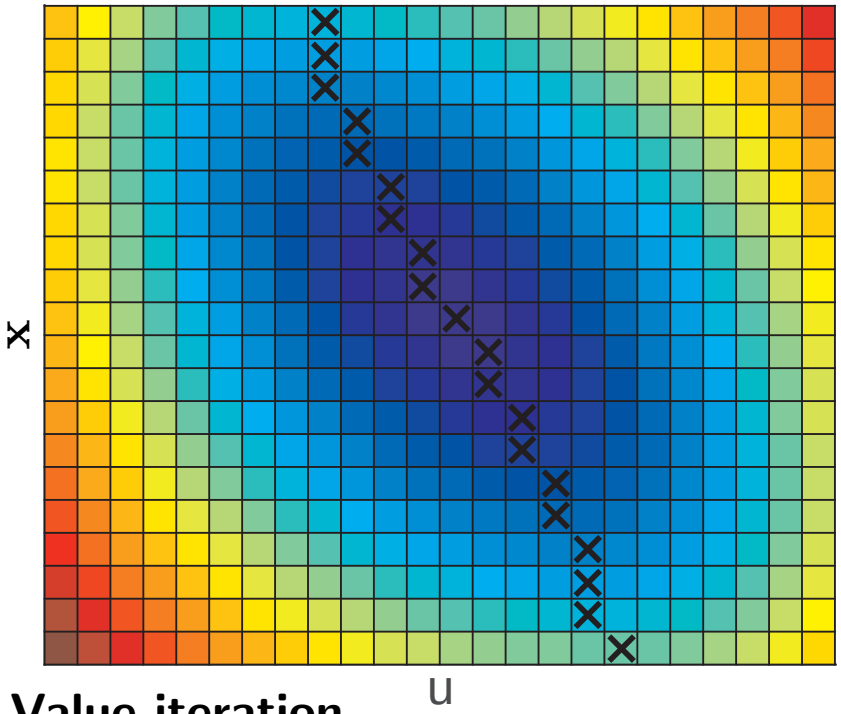

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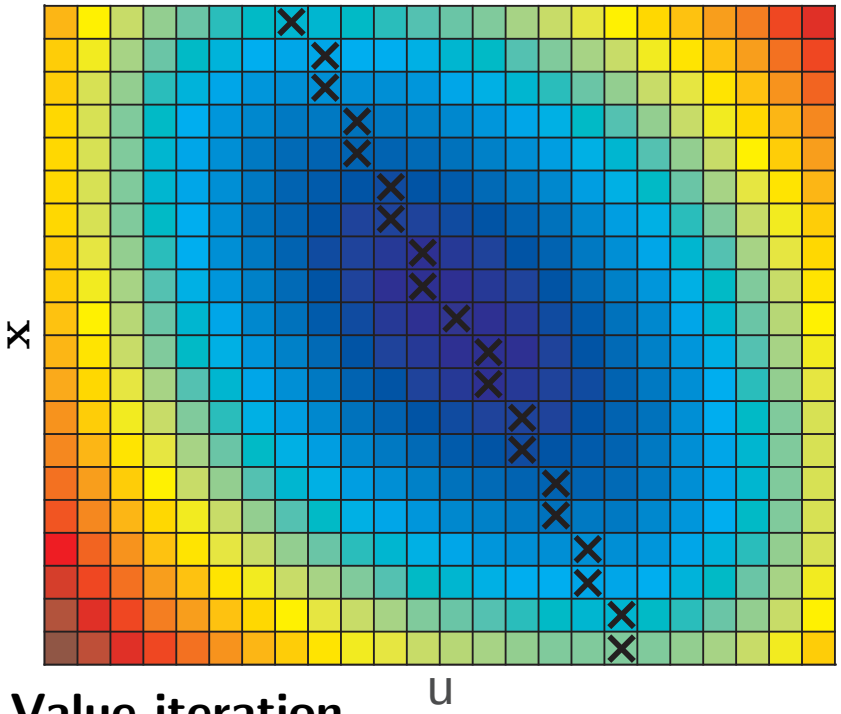

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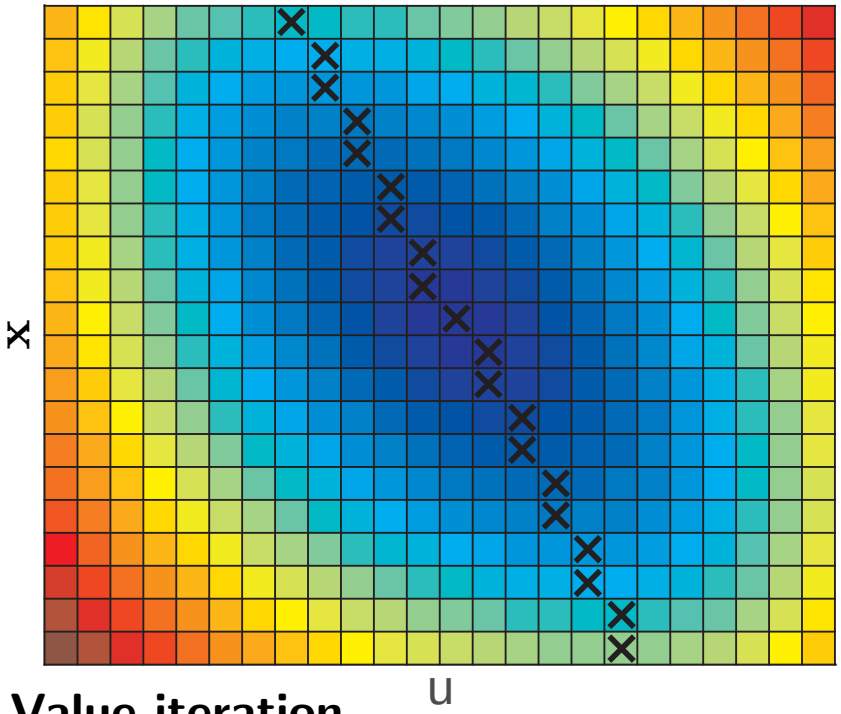

Value iteration

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Input: V and $\text{tol} > 0$

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while Iter **do**

  Compute

$$Q_+(\mathbf{x}, \mathbf{u}) \leftarrow L(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}\left[V\left(\mathbf{x}_+\right) \mid \mathbf{x}, \mathbf{u}\right]$$

$$V(\mathbf{x}) \leftarrow \min_{\mathbf{u}} \ Q_+(\mathbf{x}, \mathbf{u})$$

  if $\|Q_+ - Q\| < \text{tol}$ **then**

    $\text{Iter} = \text{False}$

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return $\pi_\star \leftarrow \arg\min_{\mathbf{u}} \ Q_+(\mathbf{x}, \mathbf{u})$

- Iteration often written as $V(\mathbf{x}) \leftarrow \min_{\mathbf{u}} \ L(\mathbf{x}, \mathbf{u}) + \gamma \mathbb{E}\left[V\left(\mathbf{x}_+\right) \mid \mathbf{x}, \mathbf{u}\right]$
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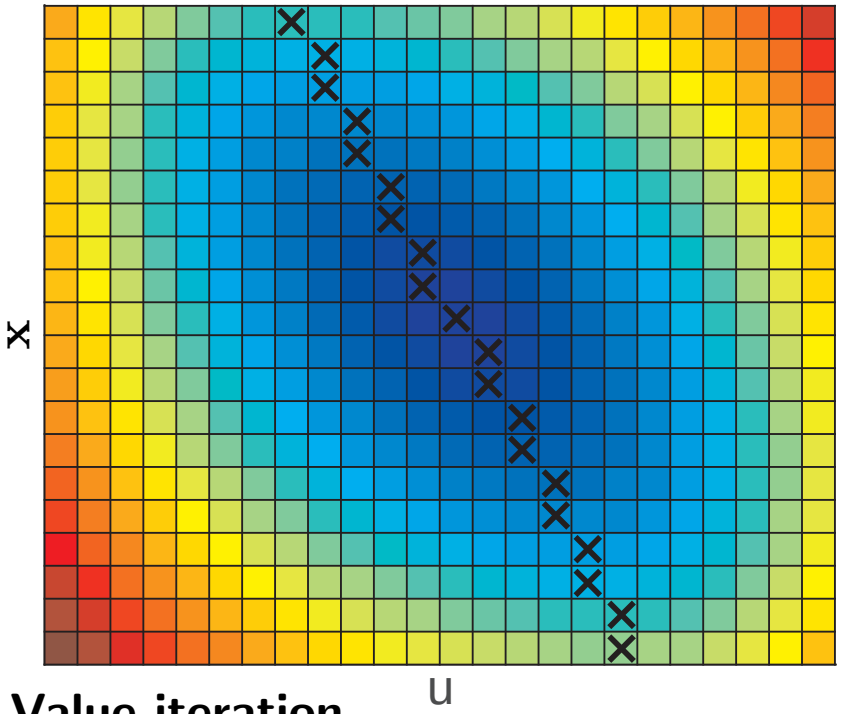
Dynamic Programming



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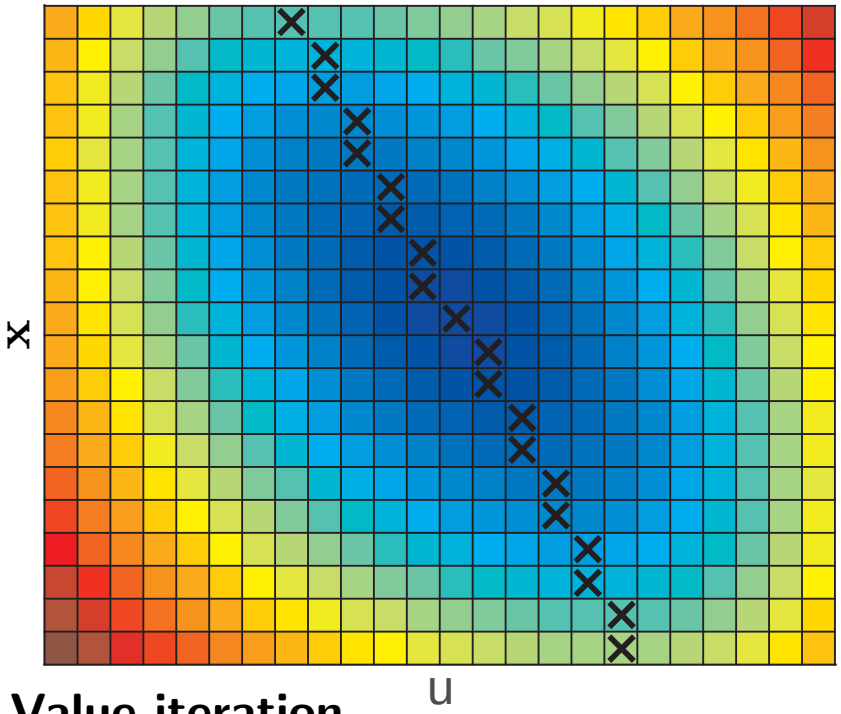

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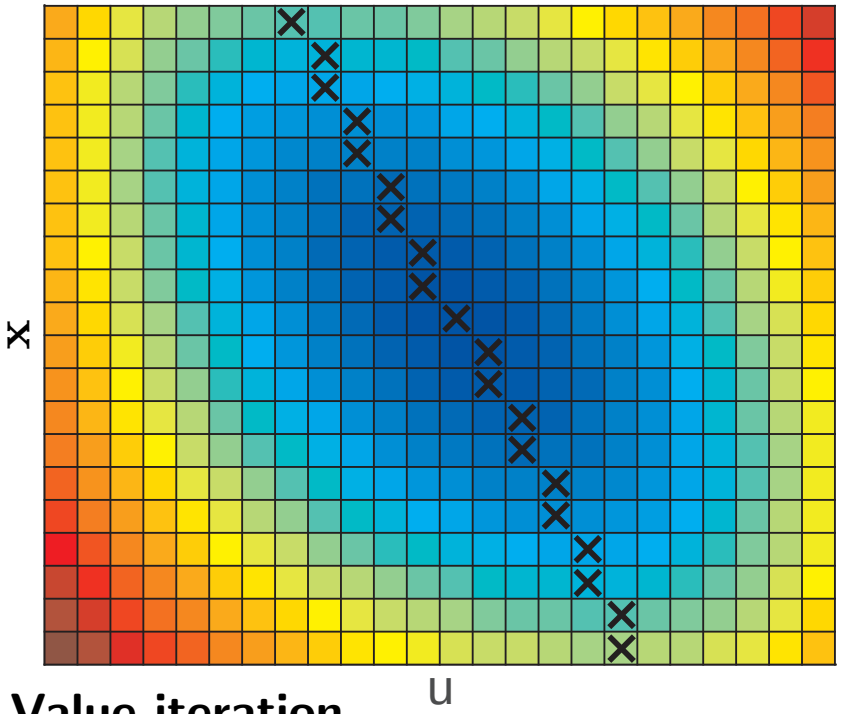

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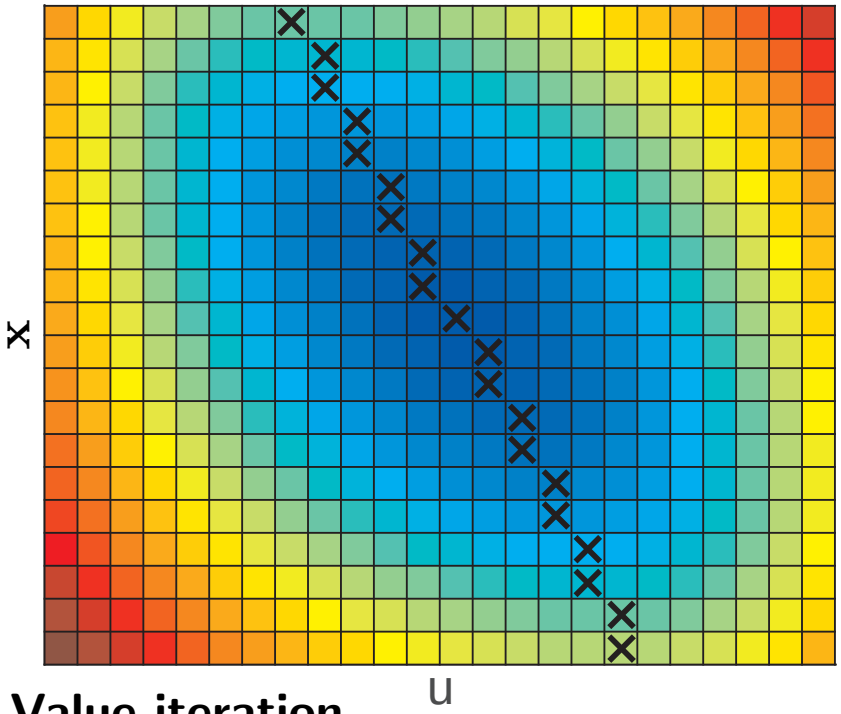

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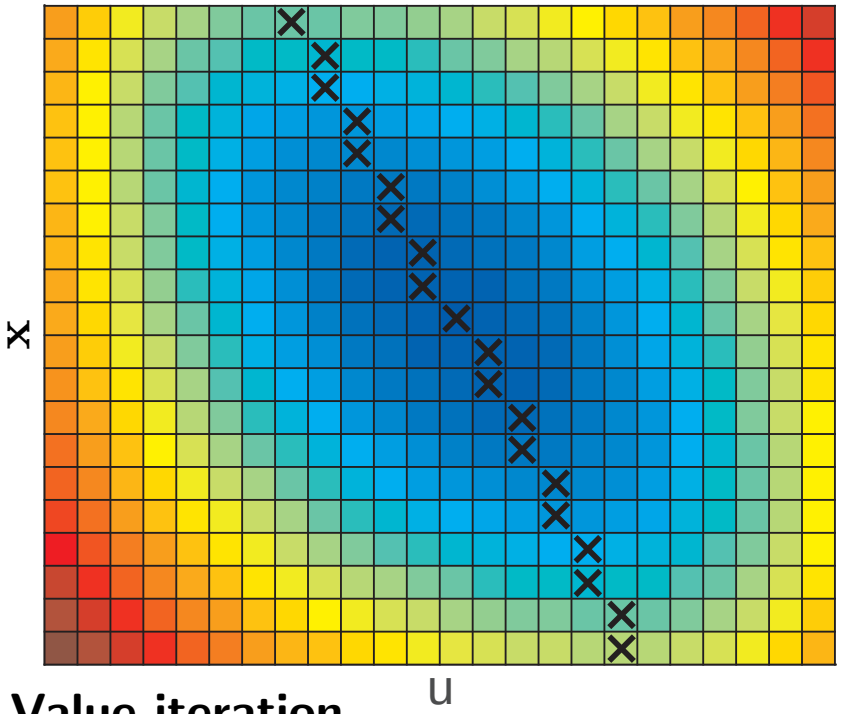

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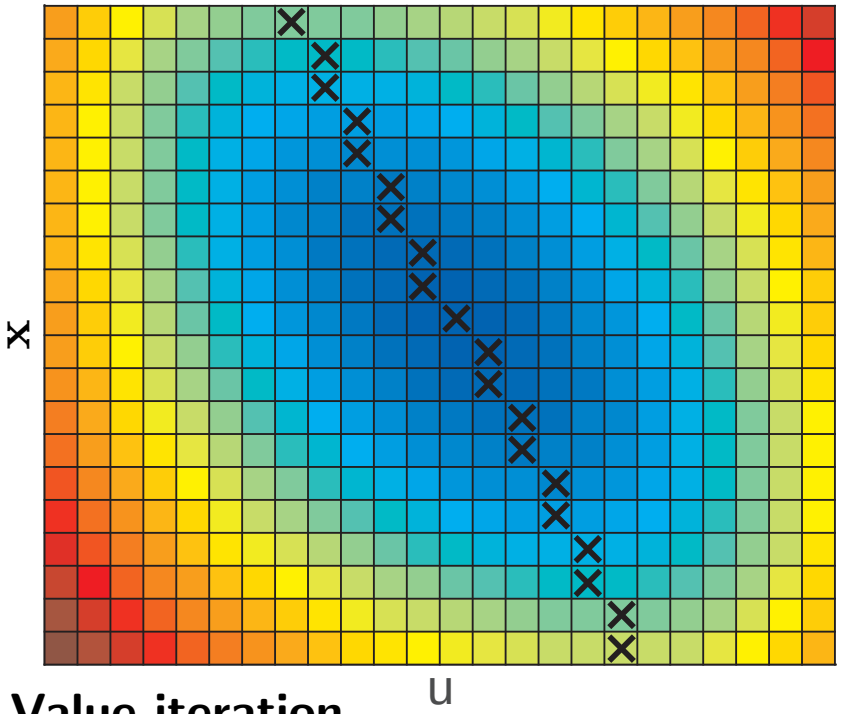

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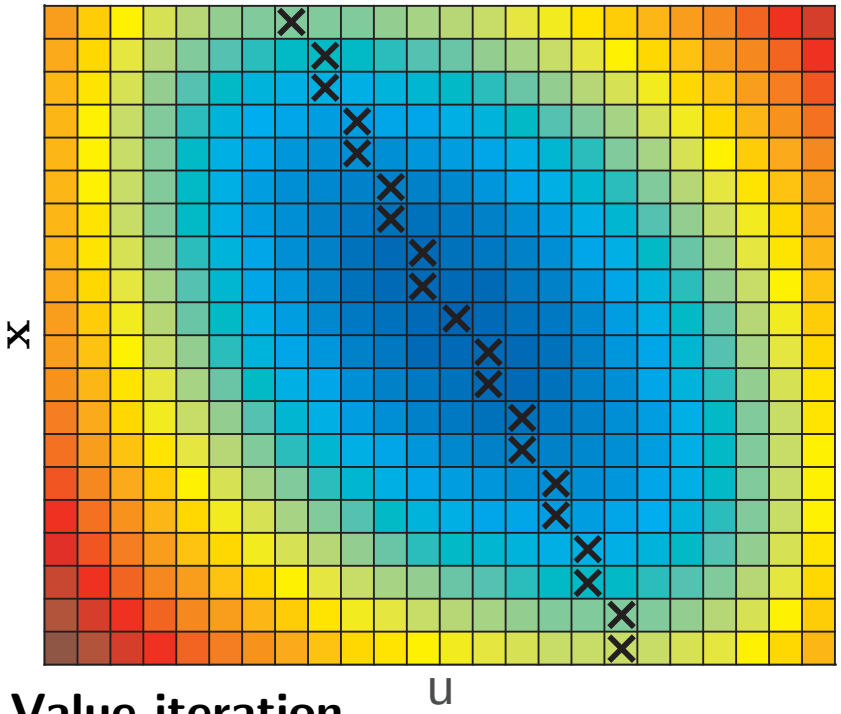

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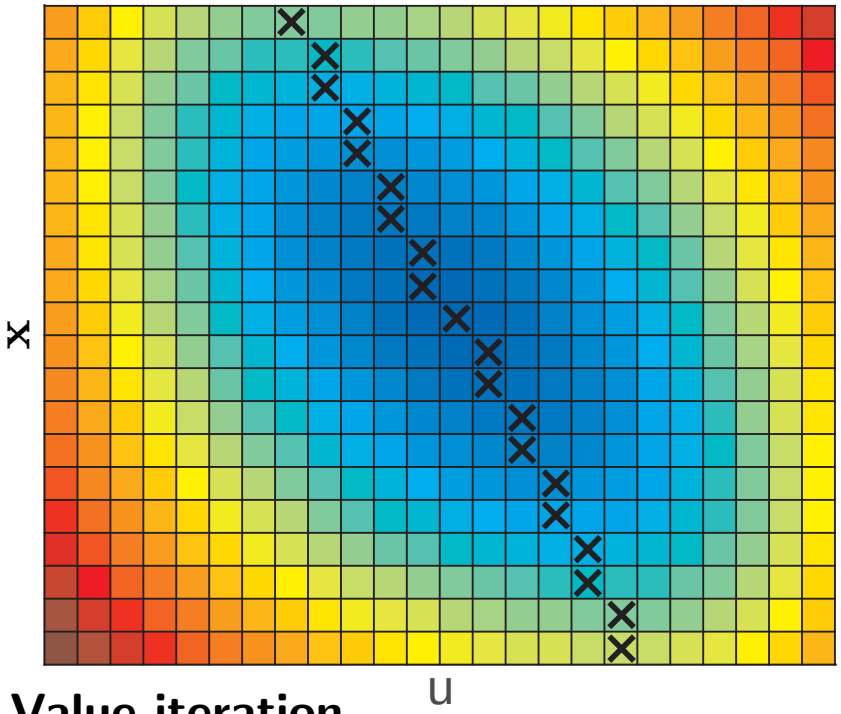

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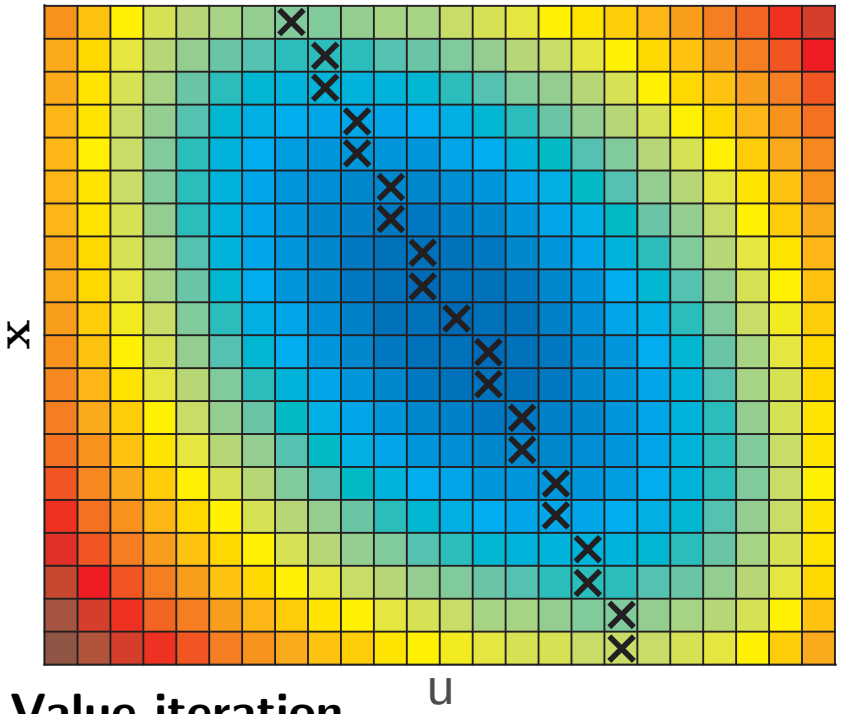

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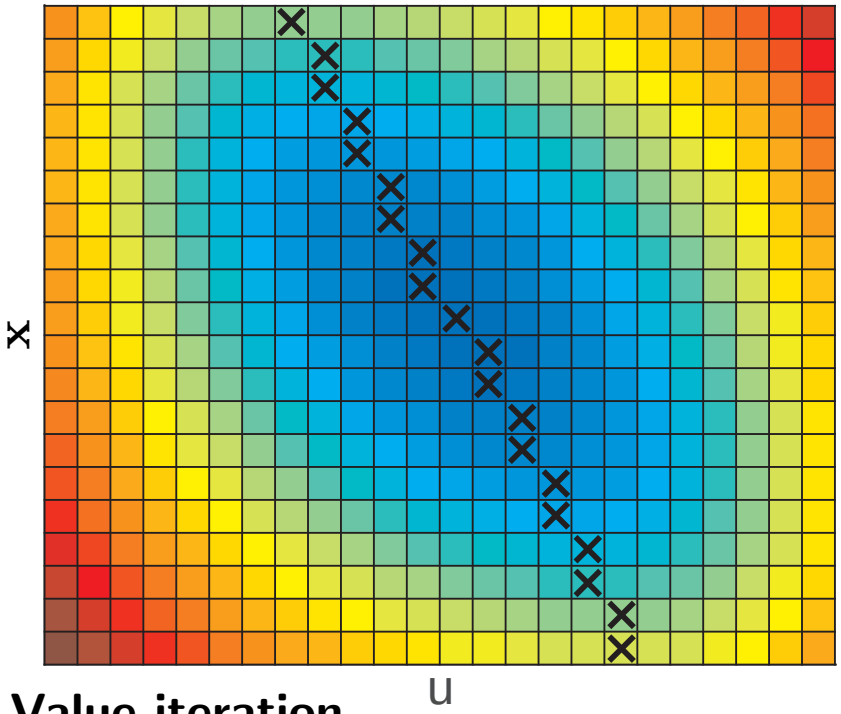

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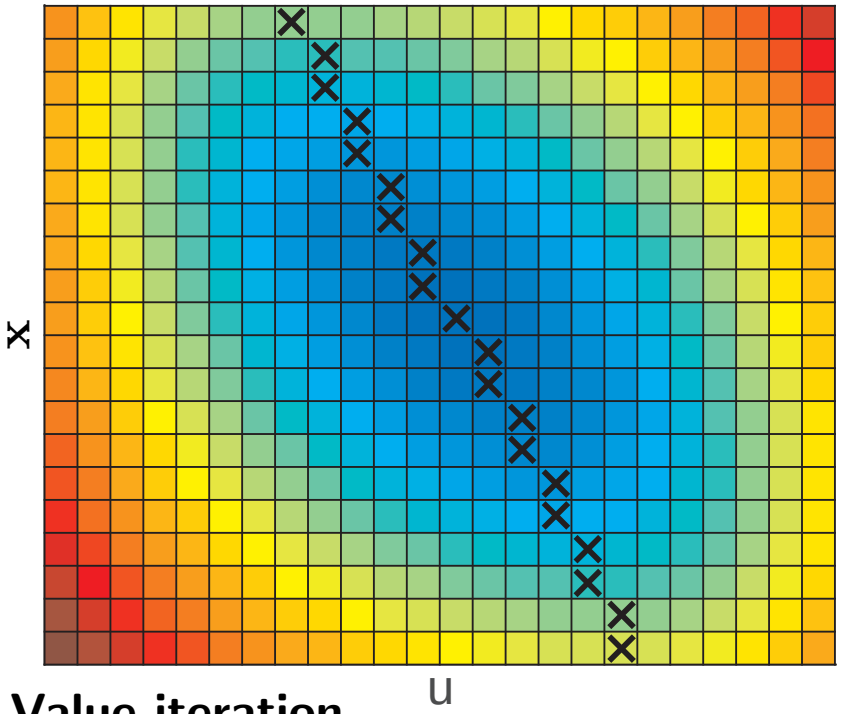

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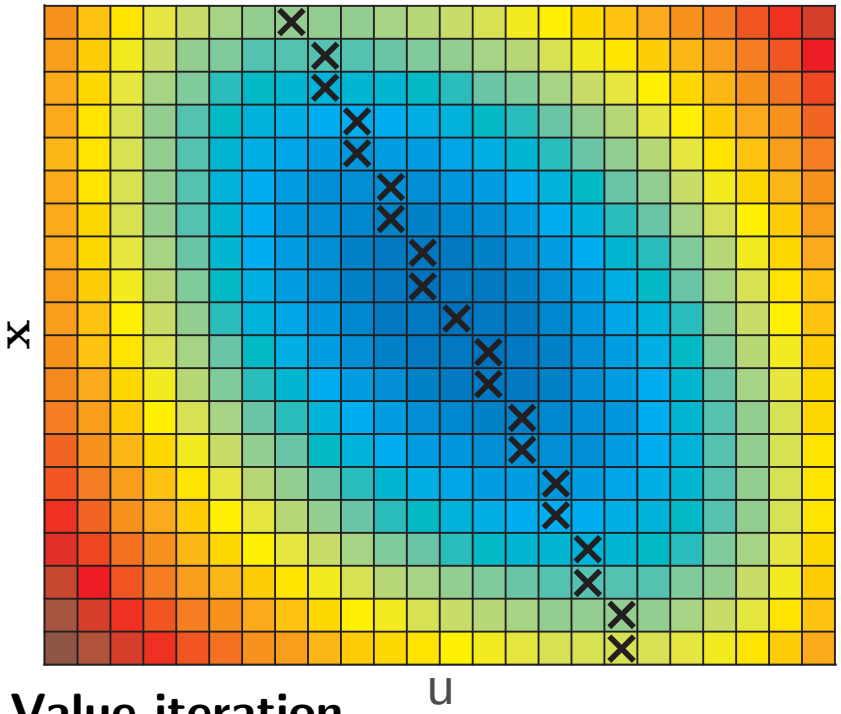

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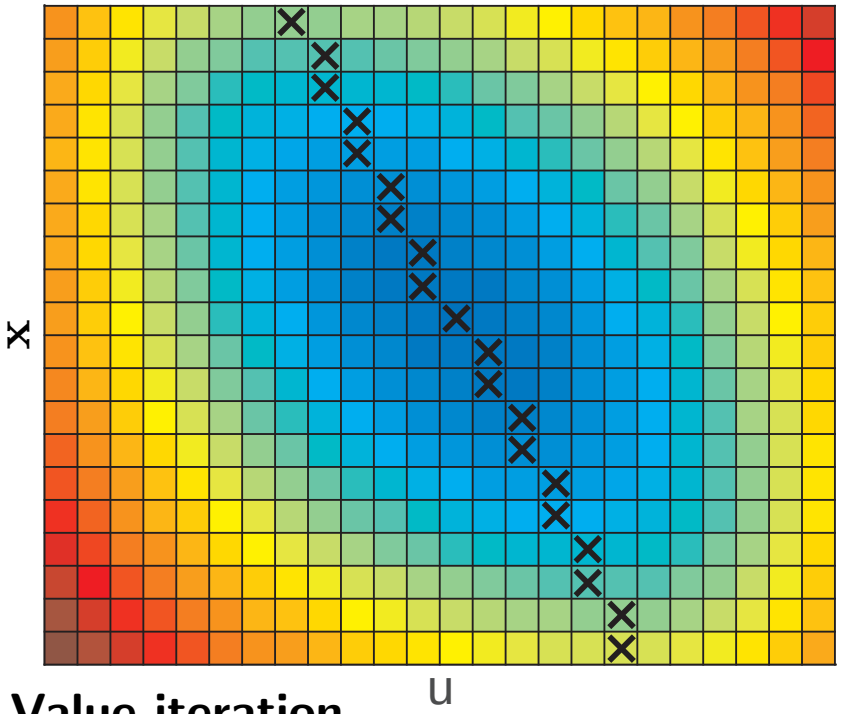

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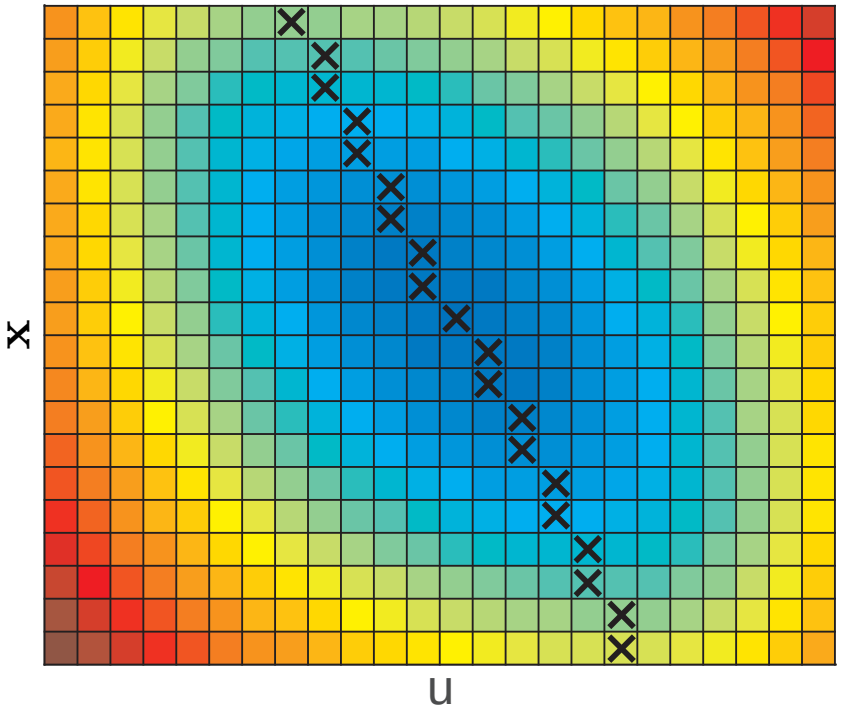

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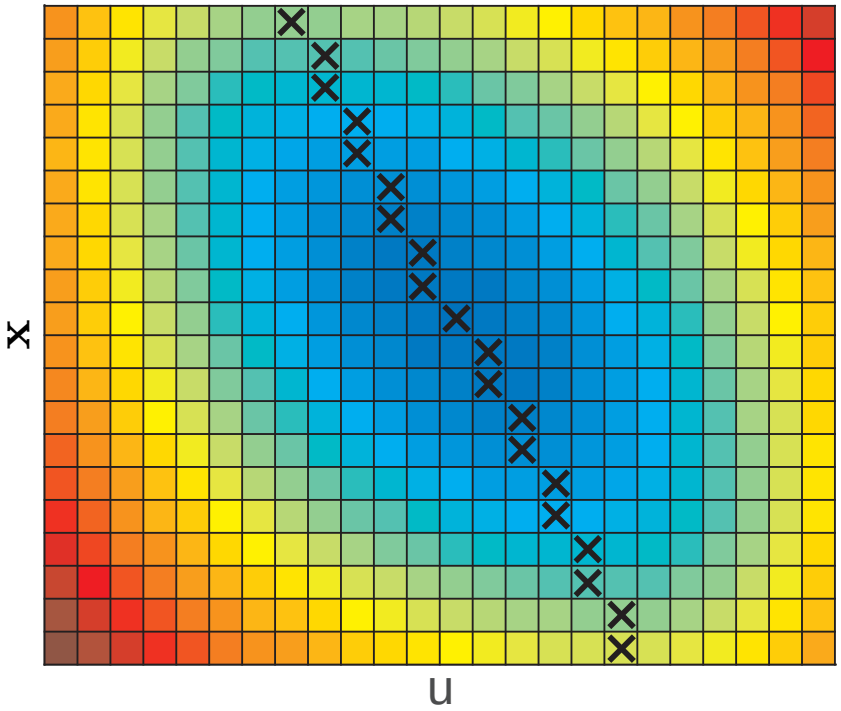

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Value iteration often labelled Dynamic Programming, for $\gamma < 1$ convergence to optimal policy is guaranteed for any (finite) starting V

Curse of dimensionality

The Bellman equations & DP are very powerful mathematical tools

- One can treat *any* (possibly) stochastic optimal control problem
- The global optimum is achieved
- Policy $\mathbf{u} = \boldsymbol{\pi}(\mathbf{x})$ provides a feedback law, no extra online work needed

why don't we just use that?!

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E.g.

- Imagine a system with **6 states, 2 inputs**
- Create **grids of 100 discrete values** for each

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- Imagine a system with **6 states, 2 inputs**
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then e.g. storing $Q(\mathbf{x}, \mathbf{u})$ requires:

$$100^8 = 10^{16} \text{ numbers } \equiv 10000 \text{ terabytes of memory}$$

and each DP iteration requires going through 10^{32} numbers

Outline

Contraction of Bellman backup

- We can define the Bellman backup operator $\mathcal{T}_{\pi}$:

$$\mathcal{T}_{\pi}(V) = L(\,.\,, \pi(\,.\,)) + \mathbb{E}[V(\mathbf{x}_{+}) \,|\, .\,, \pi(\,.\,)]$$

that transforms a function V into a new V. E.g. π-evaluation iterates $V \leftarrow \mathcal{T}_{\pi}(V)$

[1] otherwise we need a bit of measure theory

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Then for V bounded[1]...

The operator $\mathcal{T}_{\boldsymbol{\pi}}$ is a γ-**contraction**, i.e.

$$\left\|\mathcal{T}_{\boldsymbol{\pi}}(V) - \mathcal{T}_{\boldsymbol{\pi}}(V')\right\|_\infty \leq \gamma \left\|V - V'\right\|_\infty$$

This follows from observing that for any function V and any distribution, $\mathbb{E}[V] \leq \|V\|_\infty$

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Consequences

- For $\gamma < 1$ π-evaluation techniques converge to V_{π} regardless of the starting V
- Convergence is linear

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Contraction of Bellman Optimiality backup

We can define the Bellman optimality backup operator $\mathcal{T}_\star$:

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