

Contents

Source: <https://github.com/thomabir/astrophysics-ii>

1 Cosmological Background

1.1 Cosmological Principle

The Cosmological principle states that the universe is homogeneous and isotropic on sufficiently large scales, larger than about 100 Mpc. This is a generalization of the Copernican principle, according to which there is no special place and no special direction in the universe.

In the following, the evolution of the universe will be described, assuming that it is isotropic and homogeneous. In the next chapter, local perturbations about this uniform background will be taken into account.

1.2 Elements of General Relativity

The theoretical basis of cosmology is General Relativity (gr), which is a description of gravity that describes the attraction between masses as a result of the curvature of spacetime, rather than the gravitational force in Newton's description. The curvature of space-time is described by a metric $g_{\mu\nu}$, which is used to define the notion of distance on curved spaces.

As an illustration we consider a two-dimensional space with a coordinate system x^μ , where $\mu = 1, 2$. The physical squared distance is then $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, where dx^μ and dx^ν are the coordinate distances. Assuming that the space is homogeneous and isotropic, the distance can be written as

$$ds^2 = a^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\phi^2 \right),$$

where k describes the curvature and a is the scale factor, which can also be time-dependent. If the scale factor is time dependent, the surface is uniformly expanding or contracting. The curvature k for a homogeneous and isotropic surface is either $k = 0$ for a plane, $k = 1$ for a sphere or $k = -1$ for a hyperbolic surface. In order to describe the sphere the coordinate transformation $r = \sin \chi$ is used and to describe the hyperbolic surface $r = \sinh \chi$.

The discussion of the two-dimensional space can be adapted to our universe, which is in four dimensional space-time described by the Minkowsky metric $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. In this space-time, the physical square distance for a flat universe would be $ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$. In general, the dynamics of space-time is described by the Einstein field equations:

$$G_{\mu\nu} = \frac{8\pi G}{\rho} T_{\mu\nu}.$$

$G_{\mu\nu}$ is the Einstein tensor, which includes the curvature of the space-time. $T_{\mu\nu}$ is the stress energy tensor and describes the content of the space-time. For an ideal fluid the stress energy tensor is

$$T_{\mu\nu} = \text{diag}(\rho c^2, p, p, p),$$

where ρc^2 is the energy density and p the pressure.

The trajectories of particles are geodesics, which means they take the straightest paths possible. As an example for photons (massless particles) $ds^2 = 0$.

1.3 FRW metric

The Friedmann-Robertson-Walker (frw) metric is the metric of a homogeneous and isotropic universe. It is defined as

$$ds^2 = c^2 dt^2 - a(t)^2 [d\chi^2 + r(\chi)^2 d\Omega^2],$$

where

- $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the solid angle element,
- χ is comoving radius,
- $r(\chi) = f_K(\chi) = \begin{cases} \sin \chi & \text{closed case, positive curvature} \\ \chi & \text{flat case} \\ \sinh \chi & \text{open case, negative curvature} \end{cases}$
- $a(t)$ is the scale factor, which describes the expansion (or contraction) of space.

1.3.1 Redshift

Almost all observations about astronomical objects are made through light signals, therefore it is important to know what happens to the light until it reaches us.

Consider a photon emitted from e.g. a galaxy, which travels to the observer. We know from section 1.2 that photons travel along null geodesics, which satisfy $ds^2 = 0$. Since the photon wavelength λ is a length scale, we can assume it scales with scale factor a . The redshift is then defined through

$$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = \frac{a_{\text{obs}}}{a_{\text{emit}}} = \frac{a_0}{a_{\text{emit}}},$$

where z is the redshift, λ_{obs} and $a_{\text{obs}} = a_0$ are the wavelength and scale factor at the observer, and λ_{emit} and a_{emit} are at emission.

For an expanding universe, we find $a_{\text{emit}}/a_0 \leq 1$, and therefore $z \geq 0$. The object appears redder.

1.3.2 Hubble parameter

- Hubble parameter: $H := \dot{a}/a$
- The Hubble parameter describes how fast the proper distance between two fundamental observers changes at time t .
- today's value gets a subscript zero: H_0
- Because it is hard to measure H accurately, we write it as

$$H_0 = 100h \frac{\text{km}}{\text{s Mpc}},$$

where $h \approx 0.7$ is the dimensionless Hubble parameter.

- $H_0^{-1} \approx 10 \text{ Gyr}$ is about the age of the universe
- $cH_0^{-1} \approx 4 \text{ Gpc}$ is about the size of the observable universe

1.4 Friedmann equation

The Friedmann equations are derived by plugging the FRW metric into Einstein's equations and assuming a uniform and ideal fluid to find the stress tensor (see section 1.2):

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} \quad \text{Friedmann equation}$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \quad 2^{nd} \text{ Friedmann equation ,}$$

where the density $\rho = \rho_m + \rho_r + \rho_\Lambda$ and $\rho_m = \rho_{m,0} \left(\frac{a_0}{a}\right)^3 \approx a^{-3}$, $\rho_r = \rho_{r,0} \left(\frac{a_0}{a}\right)^4 \approx a^{-4}$ and $\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$.

The Friedmann equation is a differential equation for the scale factor $a(t)$ and it is also possible to derive it from Newtonian calculations.

The critical density is defined as

$$\rho_{\text{crit}}(t) = \frac{3H(t)^2}{8\pi G}$$

Today, the critical density is about five hydrogen atoms per cubic metre, or one galaxy per Mpc^3 .

Density parameters:

- The subscript i describes one component of the universe ($i = \text{matter, radiation, dark matter ...}$)
- density parameter: $\Omega_i(t) = \rho_i(t)/\rho_{\text{crit}}(t)$
- total energy density: $\rho(t) = \sum_i \rho_i(t)$
- total density parameter: $\Omega(t) = \rho(t)/\rho_{\text{crit}}(t)$
- curvature density parameter: $\Omega_{K,0} = 1 - \Omega_0 = -\frac{Kc^2}{H_0^2 a_0^2}$

With these definitions, the (first) Friedmann equation can be rewritten as

$$\frac{H}{H_0} = \sqrt{\frac{\rho}{\rho_{\text{crit},0}} + \Omega_{K,0} \left(\frac{a_0}{a}\right)^2}$$

1.5 Solutions

To solve the Friedmann equation, $\rho(t)$ or $\rho(a)$ need to be known. It can be calculated as

$$\rho = n\epsilon$$

where n is the particle number per unit volume and ϵ the energy per particle. Since the particle number is proportional to the inverse volume and the volume is $V \propto a^3(t)$, we find that $n \propto a^{-3}$.

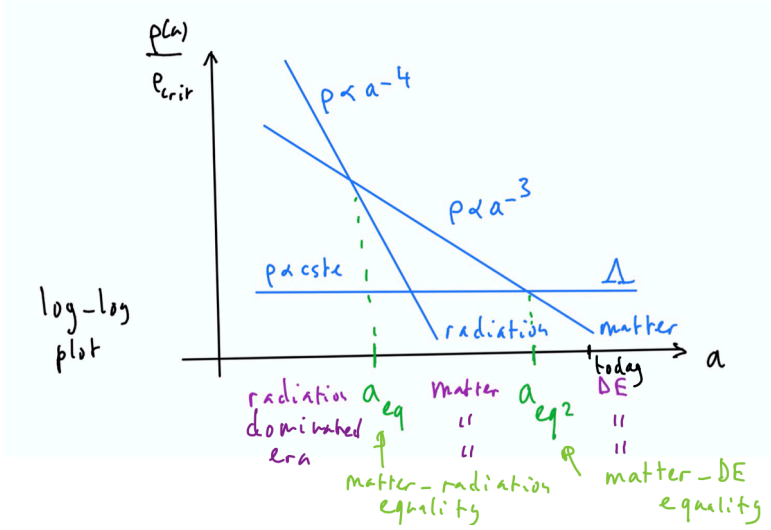


Figure 1.1: Domination of different components at different times

	w	ρ	p	T
Matter	0	a^{-3}	a^{-5}	a^{-2}
Radiation	1/3	a^{-4}	a^{-4}	a^{-1}
Radiation	-1	a^0	a^0	-

- Non-Relativistic matter: $\epsilon = mc^2$ is constant with a , while $n \propto a^{-3}$. Thus $\rho_m \propto a^{-3}$.
- Radiation: $\epsilon = h\nu = hc/\lambda \propto a^{-1}$. Thus $\rho_r \propto a^{-4}$.
- Vacuum energy: In this case the energy density only depends on the difference between the true and the false vacuum, therefore it is constant in a , $\rho_\Lambda \propto \text{constant}$.

There is a generalization for general fluids:

- equation of state: $p = w\rho c^2$, where w is the eq. of state parameter.
- If w is time-independent we find for the density: $\rho \propto a^{-3(1+w)}$
- $w = \begin{cases} 0 & \text{matter} \\ 1/3 & \text{radiation} \\ -1 & \text{vacuum energy} \end{cases}$

The results can be plugged into the Friedmann equation:

$$\begin{aligned} \frac{H}{H_0} &= \sqrt{\frac{\rho}{\rho_{\text{crit},0}} + \Omega_{K,0} \left(\frac{a_0}{a}\right)^2} \\ &= \sqrt{\Omega_{m,0} \left(\frac{a_0}{a}\right)^3 + \Omega_{r,0} \left(\frac{a_0}{a}\right)^4 + \Omega_{\Lambda,0} + \Omega_{K,0} \left(\frac{a_0}{a}\right)^2} \end{aligned}$$

This is a differential equation with $\Omega_{i,0}$ as parameters.

The standard cosmological model:

- $\Omega_{m,0} \approx 0.3$
- $\Omega_{r,0} \approx 10^{-5}$
- $\Omega_{\Lambda,0} \approx 0.7$

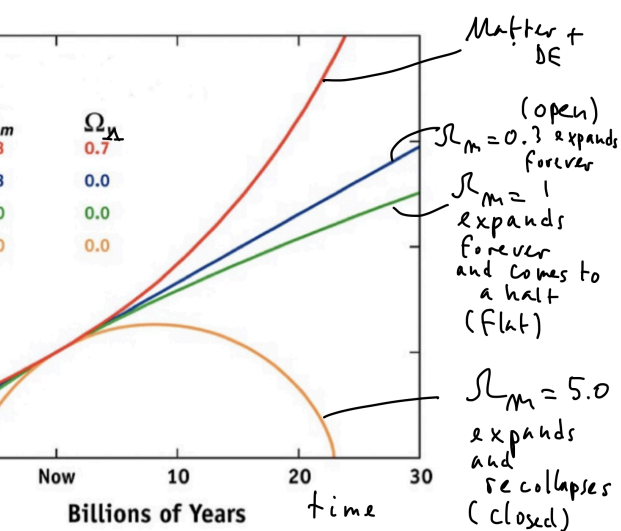


Figure 1.2: Possible evolution of the universe for different density parameters.

- $\Omega_{K,0} \approx 0$
- $\Omega_0 \approx 1$
- $h \approx 0.7$

At different times, the universe is dominated by different components. To get solutions to the Friedmann equations only the dominating component is considered:

- Matter dominated:

$$\frac{H}{H_0} = \sqrt{\Omega_{m,0} \left(\frac{a_0}{a}\right)^3} \implies a \propto t^{2/3}$$

- Radiation dominated:

$$\frac{H}{H_0} = \sqrt{\Omega_{r,0} \left(\frac{a_0}{a}\right)^4} \implies a \propto t^{1/2}$$

- Λ dominated:

$$\frac{H}{H_0} \propto \text{constant} \implies a \propto e^{Ht}$$

- General fluid ($w \neq -1$):

$$\rho \propto a^{-3(1+w)} \implies a \propto t^{\frac{2}{3(1+w)}}$$

1.6 Distances and times

1.6.1 Comoving radius

We measured the redshift of a photon that has travelled to us on a radial trajectory. How far away (in comoving distance) is the source?

$$\begin{aligned}
 0 &= ds^2 && \text{photon} \\
 &= c^2 dt^2 - a(t)^2 [d\chi^2 + r(\chi)^2 d\Omega^2] && \text{FRW metric} \\
 \Rightarrow c dt &= a(t) d\chi && d\Omega^2 = 0 \text{ on a radial trajectory} \\
 \Rightarrow d\chi &= \frac{c dt}{a(t)} \\
 &= \frac{c da}{a^2 H(a)} && H = \frac{\dot{a}}{a}, \text{ so } dt = \frac{da}{aH(a)} \\
 \Rightarrow \chi(a) &= c \int_a^{a_0} \frac{da'}{a'^2 H(a')} && \chi(a_0) = 0
 \end{aligned}$$

$H(a)$ has to be obtained from the Friedmann equations. As a result, we will get $\chi(a, a_0)$. We can use $a/a_0 = 1/(1+z)$ to get $\chi(z, a_0)$. Since a_0 can be defined arbitrarily (for example, $a_0 = 1$), we get $\chi(z)$.

1.6.2 Angular distance & Luminosity distance

The comoving distance χ and the proper distance $a\chi$ to a source are not directly observable. However, the angular size θ and the flux F of an object can be measured directly.

Intrinsic properties of the source:

- its size D
- its luminosity L

Properties of space-time:

- the comoving distance to the source χ
- the proper distance to the source $a(t)\chi$

Measurable quantities for an observer:

- the angular size θ
- the flux F

In Euclidean space, the following relations hold:

$$\theta = \frac{D}{d} \qquad F = \frac{L}{4\pi d^2}$$

where d is the distance to the source. In FRW-space, we define the following:

- The angular-diameter distance d_A satisfies $\theta = D/d_A$. One can show $d_A = a \cdot r(\chi)$, where $r(\chi) = D/(a\theta) = D_{\text{comoving}}/\theta$ is the comoving angular diameter distance.
- The luminosity distance d_L satisfies $F = L/(4\pi d_L^2)$. One can show that $d_L = a_0 r(\chi)(1+z) = d_A(1+z)^2$. Taking $a_0 = 1$ yields $d_L = r(\chi)/a = d_A/a^2$.

In a static space $d = d_A = d_L$ would hold.

1.6.3 Comoving Horizon

Suppose a (hypothetical, non-interacting) photon was emitted at the Big Bang. How far (in comoving distance) could it have travelled until now? This comoving distance is called the comoving horizon. We plug into the equation of the comoving radius/distance, with $a = 0$ at the start:

$$\chi_h = \chi(r_h) = c \int_0^{a_0} \frac{da'}{a'^2 H(a')}$$

1.6.4 Age of the Universe

How old is the universe?

$$\begin{aligned} t_0 &= \int_0^{t_0} dt \\ &= \int_0^{a_0} \frac{da}{aH(a)} \end{aligned} \quad H(a) = \dot{a}/a$$

$H(a)$ is again found from the Friedmann equation. With the standard cosmological model, $t_0 \approx 14$ Gyr.

1.7 Thermal history of the Universe

According to the Big Bang paradigm, the universe was once hot and dense, and now it expands and cools down. Today, it is far from thermal equilibrium, but it must have been in thermal equilibrium at some point in the past if it continuously expands.

The matter content at each epoch depends on:

- Thermodynamics
- Particle, nuclear and atomic physics

A system is in thermal equilibrium if $\Gamma \gg H$

- Γ = interactions/time is the interaction rate
- $H = \dot{a}/a$ is the Hubble constant

Similarly, a system is in thermal equilibrium if $\tau_\Gamma \ll \tau_H$

- $\tau_\Gamma = 1/\Gamma$ is the characteristic timescale of interactions
- $\tau_H = 1/H$ is the characteristic timescale of expansion

We already know about H . Γ is defined as

$$\Gamma = nv\sigma$$

- n number density, particles/volume
- v velocity of particles
- σ scattering cross-section, has units of area.

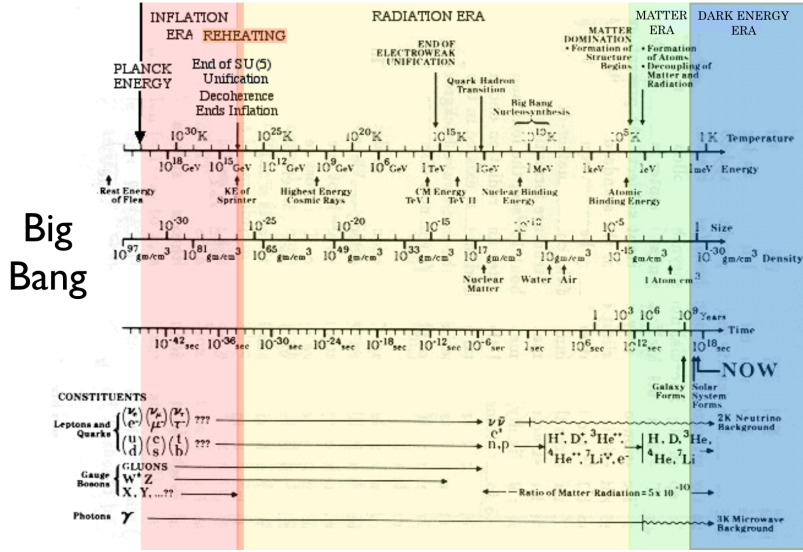


Figure 1.3: History of the Big Bang

Adapted from Kolb & Turner 1990

At early times, $\Gamma \gg H$. Particles are in thermal equilibrium with the plasma and coupled to photons. This scenario will be treated in ??.

At later times $\Gamma \ll H$. Particles are not in thermal equilibrium and are decoupled from photons. See ??

The decoupling or “freeze out” happens when $\Gamma \approx H$. This transition is described by the Boltzmann equation in ??.

In ?? and ??, an overview of the thermal history of the universe is given.

1.8 Thermodynamics

To describe the evolution of the universe quantitatively, a few definitions are required:

- The probability that a particle is in a volume $d^3x \, d^3p$ at time t is given by the (phase space) distribution function:

$$f(\mathbf{x}, \mathbf{p}, t) \, d^3x \, d^3p$$

In a homogeneous and isotropic universe, we have $f(\mathbf{x}, \mathbf{p}, t) = f(p, t)$.

- The number of particles per unit volume is given by the **number density**:

$$n(t) = 4\pi \int f(p, t) p^2 \, dp$$

- The energy per unit volume is given by the **energy density**:

$$\rho(t) = 4\pi \int E(p) f(p, t) p^2 \, dp,$$

with $E(p) = \sqrt{p^2 + m^2}$.

	time	redshift	energy temp.
period of extremely rapid exponential expansion, caused phase transitions where the inflaton field emerged. Inflation explains properties of the universe which are difficult to account for otherwise.	10^{-34} s	?	?
Quarks and antiquarks (protons, neutrons) are formed from quarks. There are way more baryons formed than antibaryons. Matter-antimatter asymmetry.	?	?	?
Hadronization The universe has cooled sufficiently such that quarks and antiquarks (baryons and mesons) can form.	10^{-5} s	10^{12}	200 MeV $3 \cdot 10^{12}$ K
Hadronization and decay, the only hadrons left are nucleons (protons and neutrons).	10^{-4} s		50 MeV 10^{12} K
Dark Matter Freeze-out Dark Matter interacts very weakly with ordinary matter, so it decouples early on.	?	?	?
Neutrino decoupling and annihilation Electrons and positrons annihilate $e^+ + e^- \rightarrow 2\gamma$. Since the number of charged particles is conserved, neutrinos decouple. The n/p ratio freezes out at $\sim 1/7$.	4 s	$2 \cdot 10^9$	0.3 MeV $5 \cdot 10^9$ K
Big Bang Nucleosynthesis Light nuclei such as D and He get synthesized. Photons are still ionized.	3 min	$4 \cdot 10^8$	0.08 MeV 10^9 K
Matter-radiation equality	$6 \cdot 10^4$ yr	3400	0.75 eV 8700 K
Formation of neutral atoms through $e^- + p^+ \rightarrow H + \gamma$	$2 \cdot 10^5$ yr	1200	0.34 eV 4000 K
Photon decoupling The number density of charged particles has decreased enough for photons to decouple. These photons form the Cosmic Microwave Background (CMB).			
Stars form and re-ionize neutral hydrogen.	$2 \cdot 10^8$ yr	20	4 meV 50 K
Matter-dark matter equality	9 Gyr	0.4	0.33 meV 3.8 K
	13.8 Gyr	0	0.24 eV 2.7 K

Table 1.1: Thermal history of the universe

- The **pressure** is given by

$$P(t) = 4\pi c^2 \int \frac{p^2}{3E(p)} f(p, t) p^2 dp$$

Since we work in natural units, we drop the c .

1.8.1 At early times ($\Gamma > H$)

The distribution function of particles in thermodynamic equilibrium is the Bose-Einstein or the Fermi-Dirac distribution:

$$f_{\text{eq}}(p, t) = \frac{g}{(2\pi)^3} \left[\exp\left(\frac{E(p) - \mu}{T} \pm 1\right) \right]^{-1}$$

- $+$ is for fermions and $-$ for bosons
- g is a spin degeneracy factor. Examples: $g_\nu = 1$, $g_\gamma = 2$, $g_{\text{quark}} = 6$
- μ is the chemical potential, which is the response of a thermodynamics system to a change of particle number. Usually, $\mu = 0$ for our purposes.
- T is the temperature of the universe, which is time dependent.

For *non-relativistic particles*, $T \ll m$ and $E \approx m + p^2/2m$. Plugging this in yields

$$n = g \left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(\frac{p-m}{T}\right) \quad \rho = nm \quad P = nT$$

For *relativistic particles*, $T \gg m$ and $E \approx p$. For both fermions and bosons, this yields

$$n = gT^3 \quad \rho = gT^4 \quad P = \frac{\rho}{3}$$

In the case of a relativistic gas, $T \propto a^{-1}$, so $n \propto a^{-3}$ and $\rho \propto a^{-4}$. This is what we have already seen in ??.

1.8.2 At late times ($\Gamma < H$)

The transition between non-equilibrium and equilibrium takes place at the decoupling time of freeze-out time t_f . At $t > t_f$, the transition function is

$$f(p, t) = f\left(p \frac{a(t)}{a(t_f)}, t_f\right)$$

The shape of the function is “frozen in” at the freeze-out time t_f .

For a relativistic particle,

$$f(p, t) = \frac{g}{(2\pi)^3} \left[\exp\left(\frac{pa(t)}{T_f a(t_f)} \pm 1\right) \right]^{-1},$$

which is the same as for an equilibrium particle, but with $T_f := T_f a(t_f)/a(t)$.

1.8.3 Boltzmann equation

The Boltzmann equation

$$\frac{df_i}{dt} = c_i[f_i]$$

describes the time evolution of the distribution function. $c_i[f_i]$ is the collision term.

f only depends on p and t , so we can write

$$\frac{df_i}{dt} = \frac{\partial f_i}{\partial t} + \frac{\partial f_i}{\partial p} \frac{\partial p}{\partial t}$$

The last term can be simplified. Because $p = p_0 a^{-1}$, $dp/dt = -p_0 a^{-1} \dot{a} = -pH$. The Boltzmann equation then becomes

$$\begin{aligned} \frac{\partial f_i}{\partial t} - pH(t) \frac{\partial f_i}{\partial p} &= c_i[f_i] \\ \Rightarrow \frac{\partial}{\partial t} \int d^3 p f_i - H(t) \int d^3 p p \frac{\partial f_i}{\partial p} &= \int d^3 p c_i[f_i] \\ \Rightarrow \frac{dn_i}{dt} + 3H(t)n_i &= \int d^3 p c_i[f_i] \end{aligned}$$

In the last line, we used partial integration.¹ If the system is collisionless, $c_i[f_i] = 0$, and the solution of the Boltzmann equation is $n_i \propto a^{-3}$. The $3H(t)n_i$ is called the Hubble drag term.

We now look at reactions of the type $i + j \leftrightarrow a + b$. The collision term is then of the form

$$c_i[f_i] = \alpha(T)n_a n_b - \beta(T)n_i n_j$$

- $\alpha(T)$ is the production rate
- $\beta(T)$ is the destruction rate

To simplify the equation, we make a few assumptions:

- a and b are in equilibrium with a general plasma at temperature T
- $n_i = n_j$ (this is the case for antiparticles)
- radiation era: $a \propto t^{1/2} \propto T^{-1}$

The equation is then

$$\frac{dn_i}{dt} + 3H(t)n_i = \beta(T)(n_{i,\text{eq}}^2 - n_i^2)$$

To analyse the equation, we define

- $x = m_i/T$ is used as a time variable
- $y_i = n_i/S$, where S is the entropy
- $\Gamma(x) = n_{i,\text{eq}}(x)\beta(x)$

The equation is then

$$\frac{x}{y_{i,\text{eq}}} \frac{dy_i}{dx} = -\frac{\Gamma(x)}{H(x)} \left[\left(\frac{y_i}{y_{i,\text{eq}}} \right)^2 - 1 \right]$$

Initial conditions: $x \ll 1$ at early times, so $y_i = y_{i,\text{eq}}$.

No analytical solution is known. A set of numerical solutions is shown in ??

¹ In detail:

$$\begin{aligned} \int d^3 p p \frac{\partial f_i}{\partial p} &= \int dp p^3 \frac{\partial f_i}{\partial p} \\ &= - \int dp \frac{\partial p^3}{\partial p} f_i \\ &= -3n_i \end{aligned}$$

1.9 Particle relics

There are two types of relics:

- Hot relics freeze out when the particles are still relativistic. Since $x = m_i/T$, this means $x_f \ll 1$
- Cold relics freeze out when the particles are already non-relativistic, with $x_f \gg 1$

1.9.1 Hot relics

Hot relics are still relativistic today, so their rest mass is $m_i \ll T_0 = 2.4 \cdot 10^{-4}$ eV. An example would be massless neutrinos.

The solution of the Boltzmann equation is

$$\Omega_{i,0} h^2 = \frac{g_{i,\text{eff}}}{2} \left[\frac{g_{*s}(x_0)}{g_{*s}(x_f)} \right]^{4/3} \Omega_{\gamma,0} h^2$$

The g 's are degeneracy factors and satisfy $g_{*s}(x_0) \leq g_{*s}(x_f)$, and the photon density is $\Omega_{\gamma,0} h^2 = 2.5 \cdot 10^{-5}$. It follows that $\Omega_{i,0}$ is very small, which means that hot relic particles contribute very little to today's energy density.

1.9.2 WIMPs

Next we consider weakly interacting massive particles (WIMPs). Examples are massive neutrinos and stable, light supersymmetric particles. WIMPs can either be hot ($x_f \ll 1$) or cold ($x_f \gg 1$).

Hot WIMPs The solution of the Boltzmann equation yields

$$\Omega_{i,0} h^2 = 7.64 \cdot 10^{-2} \left(\frac{g_{i,\text{eff}}}{g_{*s}(x_f)} \right) \frac{m_i}{\text{eV}}$$

Since we know that $\Omega_{i,0} < 1$, we can get a constraint $m_i < 100$ eV on the mass of the hot WIMPs, which is possible for neutrinos. However, hot WIMPs are ruled out as dark matter by structure formation arguments.

Cold WIMPs Boltzmann says

$$\Omega_{i,0} h^2 = \begin{cases} 1.8 \left(\frac{m_i}{\text{GeV}} \right)^{-2} \left[1 + 0.17 \ln \left(\frac{m_i}{\text{GeV}} \right) \right] & \text{if } m_i < 100 \text{ GeV} \\ \left(\frac{m_i}{3 \text{ TeV}} \right) & \text{if } m_i > 100 \text{ GeV} \end{cases}$$

To get $\Omega_{i,0} < 1$, we need m_i to be between 1.4 GeV and 3 TeV. Cold Wimps are good candidates for dark matter.

1.10 Primordial nucleosynthesis

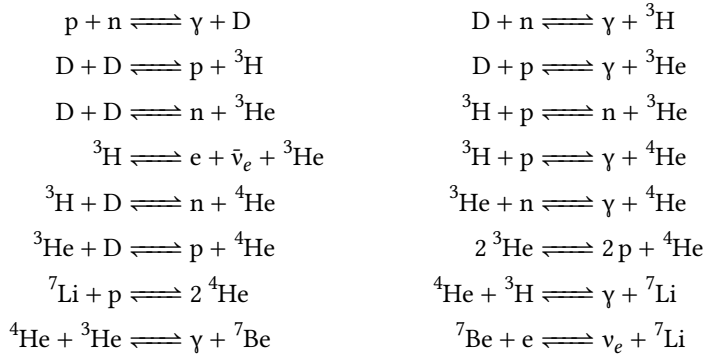
This is the epoch where protons and neutrons first combined to nuclei (not atoms) heavier than hydrogen. Heavier atoms can also be synthesized in stars through nuclear reactions, which has to be differentiated in observations.

Initial conditions

Initially, the temperature is $T < 10^{13}$ K and the associated energy scale is $k_B T \approx 0.8$ MeV. Protons and neutrons are in thermal equilibrium and interact weakly via the processes $p + e \leftrightarrow n + \bar{\nu}_e$ and $n + \bar{e} \leftrightarrow p + \bar{\nu}_e$. Since the mass of the nucleons is around 940 MeV, they are already non-relativistic. The masses of neutrons and protons are slightly different, so their abundances after freeze out are different.

Nuclear reactions

Once the temperature drops below 1 MeV, which is the binding energy of a typical nucleus, the nuclei start forming. A host of nuclear reactions can occur then:



For each of these reactions, there is a Boltzmann equation which describes the evolution of the number densities. This coupled system of equations can be solved numerically to find the abundances of nuclei. The solution for a few nuclei is shown in ??.

1.11 Recombination

As the universe cools down to a temperature that is lower than the binding energy of hydrogen (13.6 eV), some hydrogen atoms start to form via the reaction $p + e \rightarrow H + \gamma$. This epoch is called recombination, even though electrons and protons combine for the first time. Other atoms also start forming, but we only care about hydrogen for now.

The ionization fraction x_e is the ratio between the number density of electrons and the number density of baryons (protons, hydrogen):

$$x_e := \frac{n_e}{n_b}$$

For given initial conditions, the ionization fraction can be calculated as a function of z by solving the Boltzmann equation, see ??

1.11.1 Recombination

The time of recombination is defined as the time when there are ten times more baryons than electrons: $x_e(z_{\text{rec}}) = 0.1$. This happens at a temperature of $T_{\text{rec}} \approx 0.3$ eV and a redshift of $z \approx 1300$. Note that T_{rec} is smaller than the binding energy of hydrogen. This is because recombination is delayed by the high abundance of photons.

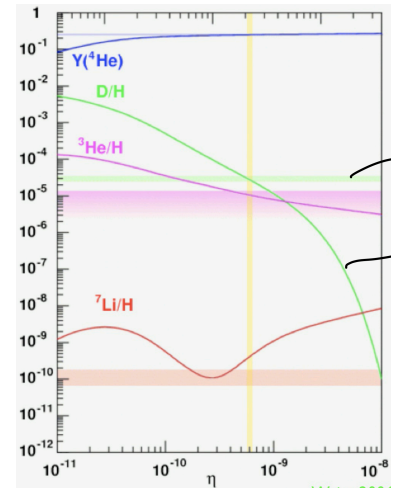


Figure 1.5: The abundances of D, ${}^4\text{He}$, ${}^3\text{He}$, and ${}^7\text{Li}$, as a function of the photon to baryon ratio η . The yellow vertical area indicates the measured value of η , while the shaded horizontal bars are the measured abundances. The modelled abundances agree well with the experiments, except for ${}^7\text{Li}$, which is probably due to uncertainties in how much ${}^7\text{Li}$ is destroyed in stars.

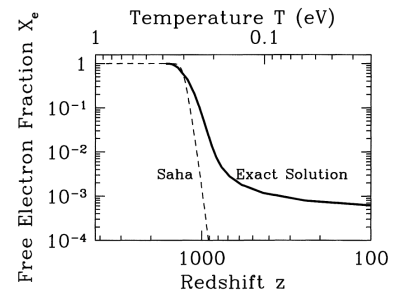


Figure 1.6: The ionization X_e as a function of redshift. Before recombination (at high redshift), the universe is fully ionized with $X_e = 1$. The Saha solution assumes thermal equilibrium, which is only valid until recombination. After the freeze-out, a residual ionization fraction ($X_e \approx 10^{-3}$) remains.

1.11.2 Decoupling

The interaction of electrons and photons (in the relevant energy regime) is described by Thomson scattering, with a scattering cross-section of $\sigma_T = 6.65 \cdot 10^{-25} \text{ cm}^2$. The reaction rate is then (see ??)

$$\Gamma_T = n_e \sigma_T c,$$

where n_e is the number density of electrons, and c the speed of photons. We know that strong coupling occurs as long as $\Gamma_T \gg H$, so we define the moment of decoupling such that

$$\Gamma_T(z_{\text{dec}}) = H(z_{\text{dec}})$$

This happens at $z \approx 1100$, $E \approx 0.26 \text{ eV}$, and $T \approx 3000 \text{ K}$, after the universe is $380\,000 \text{ yr}$ old. Note that $z_{\text{dec}} < z_{\text{rec}}$, so decoupling occurs soon after recombination.

After decoupling, the universe is transparent to photons. When an observer today stares into empty space, the photons he measures come from the surface of last scattering, where the photons interacted for the last time during decoupling. This is the cosmic microwave background (CMB), which has a temperature of

$$T_{\text{CMB}} = T_{\text{dec}} \frac{a_0}{a_{\text{dec}}} \approx 3 \text{ K}$$

Measurements of the CMB show a blackbody spectrum with remarkably deviations of $\Delta T/T \approx 10^{-5}$, as can be seen in ??.

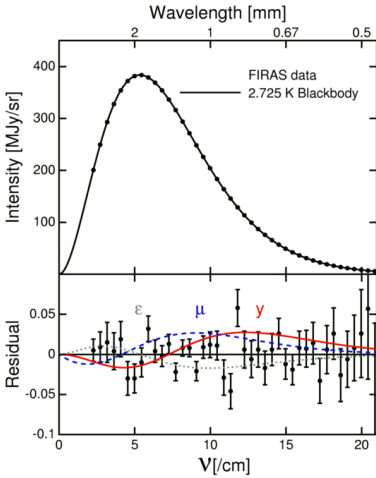


Figure 1.7: The measured spectrum of the cosmic microwave background and a fit with a blackbody spectrum. The residuals show an average deviation of only $\Delta T/T \approx 10^{-5}$.

1.12 Λ CDM model

The Λ CDM model is a refinement of the Big Bang model, and it is the standard model of cosmology. Three observational pillars justify this model:

- the expansion of the universe,
- the big bang nucleosynthesis, and
- the cosmic microwave background.

According to Λ CDM, the energy content of the universe today is made up of radiation, matter, dark matter, and dark energy. Their contributions today are:

- Photons and neutrinos: $\Omega_{\gamma,0} \approx \Omega_{\nu,0}/0.68 \approx 2.5 \cdot 10^{-5} h^{-2}$
- Baryons: $\Omega_{b,0} \approx 0.05$
- Dark Matter: $\Omega_{dm,0} \approx 0.25$
- Dark Energy: $\Omega_{\Lambda} \approx 0.7$

We now look at the constituents in more detail.

Radiation

As radiation, we classify those particles that are still relativistic today. Accordingly, they need to have very low mass.

- Most photons are part of the CMB and have a temperature $T \approx 2.73$ K. They contribute only very little to the energy density: $\Omega_{\gamma,0} = 2.5 \cdot 10^{-5}$.
- Massless² neutrinos contribute about the same as photons: $\Omega_{\nu,0} = 0.68\Omega_{\gamma,0} \approx 1.7 \cdot 10^{-5}$.

Baryons

Baryons from the Big Bang nucleosynthesis contribute $\Omega_b \approx 0.05$. They mostly occur in two forms:

- Even though only ten percent of baryons are found in stars, they are the largest fraction of visible matter.
- The rest of the baryons is in the form of various phases, such as cold gas, warm gas, and hot gas, both in the intergalactic and the interstellar medium.

Dark Matter

Dark matter (dm) contributes $\Omega_{\text{dm}} \approx 0.25$. We know of its existence through its gravitational effects. There is evidence for dark matter in galaxies, galaxy clusters, large scale structure, and many others.

Dark matter is mostly non-baryonic, because baryons are constrained to $\Omega_b \approx 0.05$ by Big Bang nucleosynthesis. Also, the structure formation constraints below rule out baryons as dark matter candidates.

- Dark matter is cold (non-relativistic). It is thus called Cold Dark Matter (CDM)
- It interacts very weakly, so we can approximate it as non-collisional.

A good candidate for dark matter are particles beyond the standard model, such as WIMPs.

Massive Compact Halo Objects (MACHOs), such as black holes, are not good candidates, since objects with masses from 10^{-6} to $15 M_{\odot}$ are ruled out.

Dark Energy

Dark Energy contributes $\Omega_{\text{de}} \approx 0.7$. It is needed to get a flat geometry for the universe, and to explain the recent acceleration of the expansion of the universe.

From the first Friedmann equation, we know that a constant energy density results in a scale factor $a \propto e^{Ht}$ that is exponentially growing. Consider an effective fluid with equation of state $p_{\text{DE}} = w\rho_{\text{DE}}c^2$. We demand $\ddot{a} > 0$, and the second Friedmann equation then yields $w < -1/3$.

There are a few possible candidates:

- The cosmological constant Λ corresponds to $w = -1$, independent of time.

- A model called *quintessence*, which is a dynamical scalar field, would give rise to another form of energy. Since it is dynamical, you can map it to an effective fluid, with a w that varies in time.
- There could be a theory of gravity that can improve upon or replace general relativity.

Current constraints demand $w = -1$ with an uncertainty of about 5 %. The cosmological constant is thus a model that is consistent with observations, and it is part of the Λ CDM model.

Summary

In total, $\Omega_0 \approx 1$, so the universe has a flat geometry. The exact nature of dark energy and dark matter are some of the most pressing questions in fundamental physics.

There are other ingredients to Λ CDM, such as the model of gravity and the choice of initial conditions. A process called inflation is also added, as we will see later.

The Λ CMD model is very successful to fit current observations, but there are some tensions that are starting to emerge with more detailed measurements from cosmological probes.

1.13 Problems with Λ CDM

The Λ CDM model without inflation has a few intrinsic fundamental problems.

1.13.1 The horizon problem

We have defined the particle horizon

$$\chi_h(t) = \int_0^t \frac{c \, dt'}{a(t')},$$

which is the maximal comoving distance that a photon could have travelled from the big bang ($t = 0$) until time t . Let's look at the size of the horizon at decoupling time, where photons scattered for the last time. This corresponds to a redshift $z_{\text{dec}} \approx 1100$. The integral evaluates to

$$\chi_h(t_{\text{dec}}) \approx 180h^{-1} \text{ Mpc}$$

The corresponding CMB angular scale is

$$\theta_h = \frac{\chi_h(t_{\text{dec}})}{r(\chi_{\text{dec}})} \approx 1.8 \text{ deg},$$

where r is the comoving angular diameter distance. This is unexpected, because the CMB is extremely homogeneous across the whole sky, even though the photons from different areas of the sky could not have been in causal contact when they scattered! Something must be wrong here.

1.13.2 Flatness problem

We have observationally determined $\Omega_0 \approx 1$, so we have a flat geometry. Let's look at how Ω varies with time. For this, we can use the Friedmann equation:

$$H(a)^2 = \frac{8\pi G}{3} \rho(a) - \frac{Kc^2}{a^2}$$

We look at the deviation of Ω from 1:

$$\begin{aligned} \frac{1 - \Omega(a)}{\Omega(a)} &= \Omega(a)^{-1} - 1 \\ &= -\frac{3Kc^2}{8\pi G \rho(a) a^2} \\ &\propto \begin{cases} a^2 & \text{in the radiation era } \rho \propto a^{-4} \\ a & \text{in the matter era } \rho \propto a^{-3} \end{cases} \end{aligned}$$

Consider time t_i in the radiation era. Then

$$\begin{aligned} \frac{\Omega_i^{-1} - 1}{\Omega_0^{-1} - 1} &= \frac{\Omega_i^{-1} - 1}{\Omega_{\text{eq}}^{-1} - 1} \frac{\Omega_{\text{eq}}^{-1} - 1}{\Omega_0^{-1} - 1} \\ &= \left(\frac{a_i}{a_{\text{eq}}} \right)^2 \left(\frac{a_{\text{eq}}}{a_0} \right) \\ &= \left(\frac{T_{\text{eq}}}{T_i} \right)^2 \frac{T_0}{T_{\text{eq}}} \quad T \propto a^{-1} \text{ for a relativistic gas} \end{aligned}$$

We choose the Planck time as our starting point:

$$\begin{aligned} t_{\text{init}} &= t_{\text{Planck}} := \left(\frac{\hbar G}{c^5} \right)^{1/2} \\ T_{\text{Planck}} &= 10^{32} \text{ K} \\ T_0 &= 3 \text{ K} \\ T_{\text{eq}} &= 10^4 \text{ K} \end{aligned}$$

Then

$$\frac{\Omega_{\text{Planck}}^{-1} - 1}{\Omega_0^{-1} - 1} \approx 10^{-60}$$

This means that any deviation of Ω from 1 at the Planck time has been amplified by 60 orders of magnitude until today. This is also called the fine tuning problem, which requires $\Omega = 1$ with extreme accuracy at early times. What is the physical explanation of this?

1.13.3 Monopole problem

Consider $T \approx 10^{14} \text{ GeV}$, which was the case during the grand unification (GUT) epoch, where the fundamental forces were unified. At these temperatures, there is a phase transition. Before the transition, the forces are unified, and during the transition, there is a spontaneous symmetry breaking. A prediction for topological defects that happen during this transition requires that there should be magnetic monopoles with an enormous density $\Omega_{\text{monopole},0} \approx 10^{11}$. If you still exist, this scenario is obviously ruled out by observations.

1.13.4 Structure formation problem

Today, even though the cosmological principle holds for large distance scales (a few 100 Mpc), we still see many structures on various smaller scales in the universe. What were the “seeds”, the initial perturbations that are responsible for the creation of these structures?

1.13.5 Initial condition problem

All of these problems can be resolved by postulating very special and finely tuned initial conditions. These could arise from physics in the quantum gravity era. The benefit of the inflation model is that all these problems could be resolved without having to resort to quantum gravity.

1.14 Inflation

Inflation is a possible solution for the disparities that we have seen in the previous section. We will tackle them one by one from here.

1.14.1 Horizon problem

We define two things to reformulate the horizon problem:

- The size of the forward light cone:

$$\chi_f = \int_0^{t_s} \frac{c \, dt'}{a(t')}$$

- The size of the past light cone:

$$\chi_p = \int_{t_s}^{t_0} \frac{c \, dt'}{a(t')}$$

This is the maximum comoving distance that photons could have travelled since last scattering.

The horizon problem can be stated mathematically as $\chi'_f < \chi_p$: Regions which are observable today to have similar temperatures were apparently not in causal contact at last scattering. A solution is to increase χ'_f through a period of accelerated expansion called inflation. A sketch of this is shown in ??.

Let t_i and t_e be the start and end time of inflation, with $\Delta t = t_e - t_i$. We know

$$\chi_f = \chi_h(t_s) = \int_0^{t_s} \frac{c \, dt'}{a(t')}$$

For vacuum energy, we know that $\rho_{\text{vac}}(a)$ is constant. We assume that, at early times, there was some kind of vacuum energy which gives us

$$a \propto e^{Ht} \text{ with } H = \sqrt{8\pi G \rho_{\text{vac}}/3} = \text{constant},$$

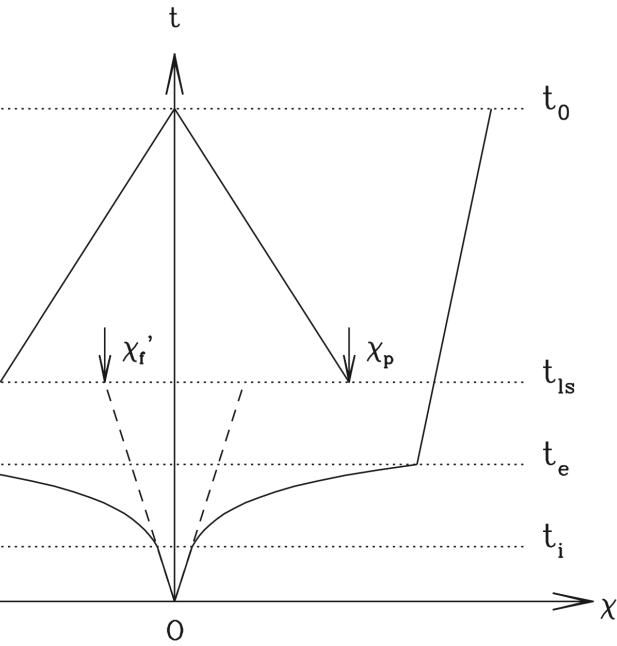


Figure 1.8: The light cone structure in an inflationary universe. Without inflation, the forward light cone (dashed line) would be smaller than our past light cone, χ_p , at the last scattering surface, resulting in causality problems. With a period of inflation from t_i to t_e , however, the forward light cone can be larger than the past light cone at t_0 .

exactly like our derivations for the cosmological constant. The contribution to χ_f during inflation is then

$$\begin{aligned}\chi_f(t_i, t_e) &= \int_{t_i}^{t_e} \frac{c \, dt'}{a(t')} \\ &\propto \frac{1}{Ha(t_e)} (e^{H\Delta t} - 1)\end{aligned}$$

We see that χ_f grows exponentially during inflation. For how long does inflation have to last to solve our problems? We need $\Delta t > 60H^{-1}$, or $e^{H\Delta t} < 10^{25}$, if we assume $t_e \approx t_{\text{GUT}}$. We thus require $a_e/a_i > e^{60}$, or in other words, we need 60 “ e -foldings”.

1.14.2 Flatness problem

One can show that

$$\frac{\Omega^{-1}(t_e) - 1}{\Omega^{-1}(t_i) - 1} = \frac{a(t_i)}{a(t_e)} < 10^{-52}$$

assuming 60 e -foldings. As a result, any curvature that was originally there gets flattened by inflation.

1.14.3 Monopole problem

The monopoles are diluted by the expansion during inflation by a factor

$$\left(\frac{a_e}{a_i}\right)^3 \approx 10^{78},$$

so the monopole density after inflation is practically zero.

1.14.4 Structure formation problem

As we will see later, inflation provides a mechanism to generate primordial fluctuations. Microscopic quantum fluctuations are turned into macroscopic classical fluctuations by the rapid expansion.

1.14.5 Initial condition problem

Inflation avoids having to set finely tuned initial conditions, since they will be diluted by the expansion. Quantum gravity is not required to solve this problem.

1.14.6 Realization

Inflation requires vacuum energy, which is realized as a scalar field $\phi(\mathbf{x}, t)$, called the *inflaton*. The potential of the scalar field is sketched in ??.

During inflation, ϕ is in a false vacuum state at a local minimum or a flat part of the potential. During the end of inflation, ϕ reaches the true vacuum state, which has a lower potential energy. This generates vacuum energy, which in turn drives accelerated expansion.

We impose a *slow roll condition*, which impedes the field from changing too quickly:

$$\dot{\phi}^2 \ll V(\phi)$$

There are several models, some of which are old inflation, new inflation, and chaotic inflation. There are many other models proposed, indicating that inflation is still an ongoing field of research.

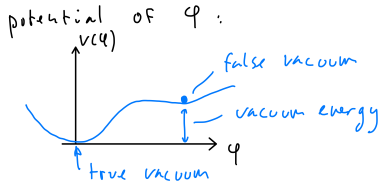


Figure 1.9: The potential of the inflaton field. During inflation, the field moves from its initial false vacuum state to a true vacuum state, which generates vacuum energy.

Acronyms

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