LX208 — Homework 2 Discussion

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Exercise 2.5ai $A \subseteq B \leftrightarrow A \cap B = A$

Proof. $\rightarrow$: By definition, $A \subseteq B$ implies $\forall x[x \in A \rightarrow x \in B]$. Hence

$$\begin{aligned} A \cap B &:= \{x \mid x \in A \text{ and } x \in B\} \\ &= \{x \mid x \in A\} \\ &= A \end{aligned}$$

$\leftarrow$: As shown in exercise 2.4, $A \cap B \subseteq B$. Since $A \cap B = A$, $A \subseteq B$. □

Proof. $\rightarrow$: We know that $A \subseteq B$, so there is no $x \in A$ that lies outside of B. Recall that by the definition of intersection, $A \cap B := \{x \mid x \in A \text{ and } x \in B\}$. But since all members of A are members of B, $A \cap B$ has to contain all members of A, i.e. $A \cap B \subseteq A$. Moreover, it is easy to see that $A \cap B$ cannot be a proper subset of A, for there are no $x \in A$ that are not in B. It follows immediately that $A \cap B = A$.

$\leftarrow$: We are given that $A \cap B = A$. So all elements in A also have to lie within B. Otherwise, there would be an $x \in A$ not contained in $A \cap B$, contradicting our initial assumption. But if all $x \in A$ are also included in B, then by definition $A \subseteq B$. □

Proof. $\rightarrow$: We prove by contradiction. Suppose $A \cap B \neq A$. First consider the case $A \cap B \subsetneq A$. Then there is an $x \in A \cap B$ s.t. $x \notin A$, which violates the definition of intersection [different route: use exercise 2.4]. Alternatively, assume $A \cap B \supsetneq A$, i.e. that there is an $x \in A$ s.t. $x \notin A \cap B$. Clearly, this is not in accord with our initial assumption that $A \subseteq B$.

$\leftarrow$: We prove again by contradiction. Suppose $A \subsetneq B$. Then there is an $x \in A$ that is not a member of B. But by the definition of $\cap$, then, $A \cap B \neq A$. Contradiction. $\square$

Exercise 2.5aii $A \subseteq B \longleftrightarrow A - B = \emptyset$

Proof. $\rightarrow$: Note again that $A \subseteq B \rightarrow \forall x[x \in A \rightarrow x \in B]$, whence

$$\begin{aligned} A - B &:= \{x \mid x \in A \text{ and } x \notin B\} \\ &= \{x \mid x \in \emptyset\} \\ &= \emptyset \end{aligned}$$

$\leftarrow$: By definition, $A - B := \{x \mid x \in A \text{ and } x \notin B\}$. If $A - B = \emptyset$, then there is no x s.t. $x \in A$ and $x \notin B$, whence $A \subseteq B$. $\square$

Proof. $\rightarrow$: If $A \subseteq B$, then every element of A is contained in B, or equivalently, there is no $x \in A$ that is not a member of B. By the definition of relative complement, then, $A - B = \emptyset$.
$\leftarrow$: Assume $A \subsetneq B$, from which it follows that there is an $x \in A$ s.t. $x \notin B$. But given the definition of relative complement, then, $x \in A - B$. Contradiction. $\square$

Exercise 2.5bi $A \cap A = A$

Proof. $A \cap A := \{x \mid x \in A \text{ and } x \in A\} = \{x \mid x \in A\} = A$. $\square$

Proof. We prove by contradiction. First assume $A \cap A \subsetneq A$. But we already know from exercise 2.4 that $A \cap A \subseteq A$. So suppose $A \cap A \supsetneq A$. Then there is an $x \in A$ that is not in $A \cap A$, implying that one of the two arguments of the intersection function is different from A. Contradiction. $\square$

Exercise 2.5ci $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof. $\subseteq$: We prove by contradiction. For arbitrary $x \in E$, assume that $x \in A \cap (B \cup C)$ yet $x \notin (A \cap B) \cup (A \cap C)$, i.e. x is neither in $A \cap B$ nor in $A \cap C$. This is the case if $x \notin A$ or if $x \notin B$ and $x \notin C$. But we know $x \in A \cap (B \cup C)$, so in particular $x \in A$. Then it has to be the case that

$x \notin B$ and $x \notin C$. But then $x \notin B \cup C$, whence $x \notin A \cap (B \cup C)$. Contradiction.

$\supseteq$: Suppose that $x \in (A \cap B) \cup (A \cap C)$. So x has to be in $A \cap B$ or in $A \cap C$. Notice that in either case, x has to be a member of A. Assume that $x \in A$. Then in order to be contained in $(A \cap B) \cup (A \cap C)$, x must also be in B or in C, i.e. it must hold that $x \in B \cup C$. Combining these two conditions, we see that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. □

Proof. Investigating the truth tables for "and" and "or", we see that (x and (y or z)) is equivalent to ((x and y) or (x and z)). Therefore

$$\begin{aligned} A \cap (B \cup C) &:= \{x \mid x \in A \text{ and } (x \in B \text{ or } x \in C)\} \\ &= \{x \mid (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)\} \\ &= (A \cap B) \cup (A \cap C) \end{aligned}$$ □

Exercise 2.5cii $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof. Analogous to 2.5ci. □

Exercise 2.5di $\neg(A \cap B) = \neg A \cup \neg B$

Proof. Observe first that $\neg(A \cap B) = E - (A \cap B) = \{x \mid x \in E \text{ and } x \notin (A \cap B)\}$. Clearly, if $x \notin A$ or $x \notin B$, then $x \notin (A \cap B)$. For the other direction, suppose neither $x \notin A$ nor $x \notin B$ holds, so both $x \in A$ and $x \in B$, contradicting our initial assumption that $x \notin (A \cap B)$. We conclude that $x \notin A$ or $x \notin B$ iff $x \notin (A \cap B)$, so $\neg(A \cap B) = \{x \mid x \in E \text{ and } (x \notin A \text{ or } x \notin B)\}$. Using the definition of complement, this is equivalent to the set $\{x \mid x \in E \text{ and } (x \in \neg A \text{ or } x \in \neg B)\}$, which, by the definition of union, is in turn equivalent to $\neg A \cup \neg B$ (after dropping the redundant $x \in E$ clause), proving equality of $\neg(A \cap B)$ and $\neg A \cup \neg B$. □

Proof. It is easy to see from the definition of intersection that ($x \notin A$ or $x \notin B$) implies $x \notin (A \cap B)$.

An easy proof by contradiction shows that the converse holds, too. Thus

$$
\begin{aligned}
x \in \neg(A \cap B) &\longleftrightarrow x \notin (A \cap B) && \text{(definition of complement)} \\
&\longleftrightarrow x \notin A \text{ or } x \notin B && \text{(shown above)} \\
&\longleftrightarrow x \in \neg A \text{ or } x \in \neg B && \text{(definition of complement)} \\
&\longleftrightarrow x \in \neg A \cup \neg B && \text{(definition of union)}
\end{aligned}
$$

□

Proof. $\subseteq$: We prove by contradiction. Assume there was an $x \in \neg(A \cap B)$ s.t. $x \notin \neg A \cup \neg B$. By the definition of complement, $x \in \neg(A \cap B) = x \notin A \cap B$. But then $x \notin A$ or $x \notin B$, or equivalently, $x \in \neg A$ or $x \in \neg B$, implying $x \in \neg A \cup \neg B$. Contradiction.

$\supseteq$: Inspecting the definition of complement and union, we see that $\neg(A \cap B)$ is the set of elements that are not in both A and B. It is easy to see that $x \in \neg(A \cap B)$ if $x \notin A$ or $x \notin B$. By the definition of union, this yields $(\neg A \cup \neg B) \subseteq \neg(A \cap B)$. □

Exercise 2.5dii $\neg(A \cup B) = \neg A \cap \neg B$

Proof. Analogous to 2.5di. □