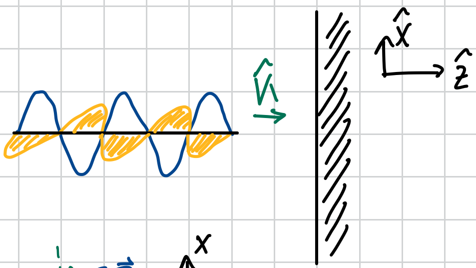
# HW 7

1.) $\hat{n} \perp$ to plane of incidence

$\hookrightarrow \tilde{E}$ in $\hat{y}$ $\quad \tilde{B} = \frac{1}{v}(\hat{k} \times \tilde{E})$

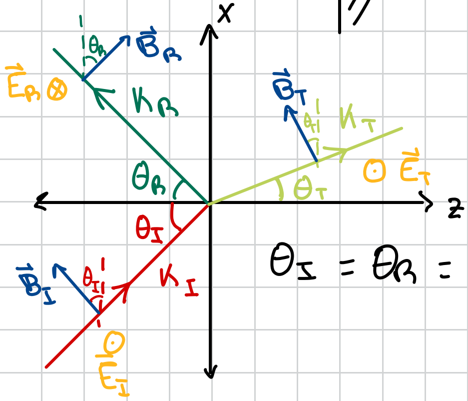



i.) $\varepsilon_1(\tilde{E}_{0I} + \tilde{E}_{0R})_z = \varepsilon_2(\tilde{E}_{0T})_z \rightarrow 0 = 0$

ii.) $(\tilde{B}_{0I} + \tilde{B}_{0R})_z = (\tilde{B}_{0T})_z$

iii.) $(\tilde{E}_{0I} + \tilde{E}_{0R})_{x,y} = (\tilde{E}_{0T})_{x,y}$

iv.) $\frac{(\tilde{B}_{0I} + \tilde{B}_{0R})_{x,y}}{\mu_1} = \frac{(\tilde{B}_{0T})_{x,y}}{\mu_2}$




$\theta_I = \theta_R = \pi/2$

ii.) $-B_{0I}\sin\theta_I + B_{0R}\sin\theta_R = B_T \sin\theta_T$

$$(E_{0I} - E_{0R})\frac{\sin\theta_I}{v_1} = E_{0T}\frac{\sin\theta_T}{v_2}$$

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \tilde{E}_{0T}\frac{v_1}{v_2}\frac{\sin\theta_T}{\sin\theta_I}$$

iii.) $\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$

iv.) $(-\tilde{B}_{0I} + \tilde{B}_{0R})\frac{\cos\theta_I}{\mu_1} = -\tilde{B}_{0T}\frac{\cos\theta_T}{\mu_2}$

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \tilde{E}_{0T}\frac{\cos\theta_T}{\cos\theta_I}\frac{\mu_1 v_1}{\mu_2 v_2} \qquad \alpha = \frac{\cos\theta_T}{\cos\theta_I} \qquad \beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

All together

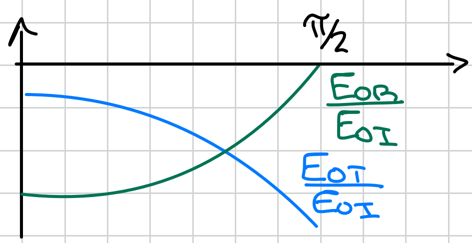
$$E_{0I} + E_{0R} = E_{0T}$$

$$\boxed{E_{0T} = \left(1 + \frac{1-\alpha\beta}{1+\alpha\beta}\right)E_{0I} = \frac{2}{1+\alpha\beta}E_{0I}}$$

$$E_{0I} - E_{0R} = (E_{0I} + E_{0R})\alpha\beta$$

$$E_{0R}(1+\alpha\beta) = -E_{0I}(\alpha\beta - 1)$$

$$\boxed{E_{0R} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)E_{0I}}$$




**Problem 9.17** Analyze the case of polarization *perpendicular* to the plane of incidence (i.e. electric fields in the $y$ direction, in Fig. 9.15). Impose the boundary conditions (Eq. 9.101), and obtain the Fresnel equations for $\tilde{E}_{0_R}$ and $E_{0_T}$. Sketch $(\tilde{E}_{0_R}/\tilde{E}_{0_I})$ and $(\tilde{E}_{0_T}/\tilde{E}_{0_I})$ as functions of $\theta_I$, for the case $\beta = n_2/n_1 = 1.5$. (Note that for this $\beta$ the reflected wave is *always* 180° out of phase.) Show that there is no Brewster's angle for *any* $n_1$ and $n_2$: $\tilde{E}_{0_R}$ is *never* zero (unless, of course, $n_1 = n_2$ and $\mu_1 = \mu_2$, in which case the two media are optically indistinguishable). Confirm that your Fresnel equations reduce to the proper forms at normal incidence. Compute the reflection and transmission coefficients, and check that they add up to 1.

normal incidence $\Rightarrow \theta_I = \theta_R = 0$

$$\alpha = \cos\theta_T = \sqrt{1 - (n_1/n_2)^2\sin^2(0)} = 1$$

$$\boxed{E_{0T} = \frac{2}{1+\beta}E_{0I} \qquad E_{0R} = \frac{1-\beta}{1+\beta}E_{0I} \quad \text{for normal incidence}}$$

$$R = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2 \qquad T = \left(\frac{2}{1+\alpha\beta}\right)^2\alpha\beta$$

$$R + T = \frac{(1-\alpha\beta)^2}{(1+\alpha\beta)^2} + \frac{4\alpha\beta}{(1+\alpha\beta)^2} \rightarrow 1 - 2\alpha\beta + \alpha^2\beta^2 + 4 = 1 + 2\alpha\beta + \alpha^2\beta^2$$

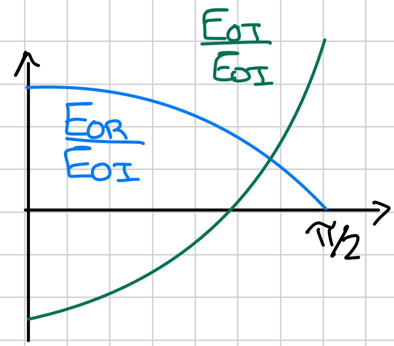
Brewster's angle $\Rightarrow E_{0R} = 0 \Rightarrow \alpha\beta = 1$

$$\alpha = \frac{1}{\beta} \rightarrow \frac{\sqrt{1-\left(\frac{v_2}{v_1}\right)^2 \sin^2\theta_I}}{\cos\theta_I} = \frac{\mu_2 v_2}{\mu_1 v_1} \rightarrow 1 = \left(\frac{v_2}{v_1}\right)^2 \left(\sin^2\theta + \underbrace{\left(\frac{\mu_2}{\mu_1}\right)^2}_{\approx 1} \cos^2\theta\right)$$

$$1 = \left(\frac{v_2}{v_1}\right)^2$$

if $v_2 = v_1$ then there's never any reflection so no specific brewster's angle

**Problem 9.18** The index of refraction of diamond is 2.42. Construct the graph analogous to Fig. 9.16 for the air/diamond interface. (Assume $\mu_1 = \mu_2 = \mu_0$.) In particular, calculate (a) the amplitudes at normal incidence, (b) Brewster's angle, and (c) the "crossover" angle, at which the reflected and transmitted amplitudes are equal.




2.) $\beta = \frac{\mu_1 (2.42)}{\mu_2 (1)} = 2.42$ $\quad \alpha = \left(1 - \left(\frac{\sin\theta_I}{2.42}\right)^2\right)^{1/2} \frac{1}{\cos\theta_I}$

$\hookrightarrow \mu_1 \approx \mu_2$

a.) $\theta_I = 0 \Rightarrow \alpha = 1$ $\quad \frac{E_{0R}}{E_{0I}} = \frac{1-2.42}{1+2.42} = \frac{-1.42}{3.42}$

$$\frac{E_{0T}}{E_{0I}} = \frac{2}{1+2.42} = \frac{2}{3.42}$$

$$\boxed{E_{0T} = 0.585\, E_{0I}} \qquad \boxed{E_{0R} = -0.415\, E_{0I}}$$

b.) $\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right) = \tan^{-1}(2.42) = \boxed{67.5^\circ}$

c.) $E_{0T} = E_{0R} \rightarrow \frac{\alpha - \beta}{\alpha + \beta} = \frac{2}{\alpha+\beta} \rightarrow \alpha - \beta = 2$

$\alpha = 2 + \beta$

$$\left(1 - \left(\frac{\sin\theta_I}{2.42}\right)^2\right)^{1/2} \frac{1}{\cos\theta_I} = 4.42$$

$$1 - \left(\frac{\sin\theta_I}{2.42}\right)^2 = 4.42^2 \cos^2\theta_I$$

$$1 = 4.42^2 \cos^2\theta_I + \left(\frac{1}{2.42^2}\right)\sin^2\theta_I$$

$$1 = 4.42^2 \cos^2\theta_I + \left(\frac{1}{2.42^2}\right)(1 - \cos^2\theta_I)$$

$$1 - \frac{1}{2.42^2} = \left(4.42^2 - \frac{1}{2.42^2}\right)\cos^2\theta_I$$

$$\frac{1 - \frac{1}{2.42^2}}{4.42^2 - \frac{1}{2.42^2}} = \cos^2\theta_I$$

$$\theta_I = \cos^{-1}\left(\left(\frac{1 - \frac{1}{2.42^2}}{4.42^2 - \frac{1}{2.42^2}}\right)^{1/2}\right) = \boxed{78.3^\circ}$$

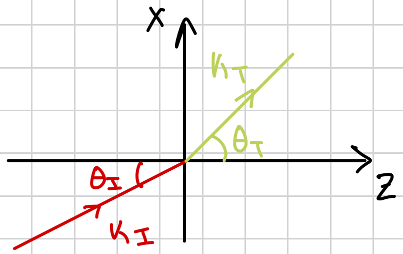
3.) $\vec{k}$ bends away from $\hat{n}$

$\theta_I = \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) \longrightarrow \theta_T = 90°$ $\qquad \theta_I > \theta_c \to$ tir

↳ get evanescent wave

$\theta_I > \theta_c \Rightarrow$ TIR $\qquad \vec{k}_T = k_T(\sin\theta_T\,\hat{x} + \cos\theta_T\,\hat{z})$ $\qquad \sin\theta_T = \frac{n_1}{n_2}\sin\theta_I$

$k_T = \frac{\omega n_2}{c}$ $\qquad \cos\theta_T = \sqrt{1-\sin^2\theta_T} = i\sqrt{\sin^2\theta_T - 1}$




a.) show that $\tilde{E}_T = \tilde{E}_{0T}\, e^{-\kappa z} e^{i(kx-\omega t)}$

$\tilde{E}_T = \tilde{E}_{0T}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$\vec{k}_T\cdot\vec{r} \longrightarrow k_T(x\sin\theta_T + z\cos\theta_T) = x k_T\sin\theta_T + k_T z i\sqrt{\sin^2\theta_T - 1}$

$\vec{k} = kx + i\kappa z$ $\qquad k = k_T\sin\theta_T = \frac{\omega \cancel{n_2}}{c}\frac{n_1\sin\theta_I}{\cancel{n_2}}$ $\qquad \boxed{k = \frac{\omega}{c} n_1\sin\theta_I}$

$\boxed{\tilde{E}_T = \tilde{E}_{0T}\, e^{-\kappa z} e^{i(kx-\omega t)}}$ $\qquad \boxed{\kappa = \frac{\omega n_2}{c}\sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_I - 1}}$

b.) note that $\alpha$ imag. calculate $R$ for $\hat{n} \parallel$ to plane

↳ $\alpha = ia$ $\qquad$ ↳ $\Rightarrow$ that $E$ in $\hat{x}$

$\beta = \frac{\mu_1 n_1}{\mu_2 n_2}$ $\qquad a = \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{\sin^2\theta_T - 1}}{\cos\theta_I}$

$$\boxed{R = \left|\frac{\alpha-\beta}{\alpha+\beta}\right|^2 = \left|\frac{ia-\beta}{ia+\beta}\right|^2 = \frac{a^2+\beta^2}{a^2+\beta^2} = 1}$$

c.) do same for polarization $\perp$ to plane of incidence

$$\boxed{R = \left|\frac{1-\alpha\beta}{1+\alpha\beta}\right|^2 = \left|\frac{1-ia\beta}{1+ia\beta}\right|^2 = \frac{1^2 + a^2\beta^2}{1^2 + a^2\beta^2} = 1}$$

d.) show real evanescent fields are $\vec{E} = \vec{E}_0\, e^{-\kappa z}\cos(kx-\omega t)\hat{y}$

$\vec{B} = \frac{E_0}{\omega} e^{-\kappa z}(\kappa\sin(kx-\omega t)\hat{x} + k\cos(kx-\omega t)\hat{z})$

E $\perp$ to plane

↳ $\Rightarrow$ E in $\hat{y}$ direction

$\tilde{E} = \tilde{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ $\qquad \tilde{B} = \frac{\tilde{E}_0}{v_2} e^{i(\vec{k}\cdot\vec{r} - \omega t)}(-\cos\theta_T\,\hat{x} + \sin\theta_T\,\hat{z})$

$\tilde{E} = \tilde{E}_0 e^{-\kappa z} e^{i(kx-\omega t)}$ $\qquad \tilde{B} = \frac{\tilde{E}_0}{v_2} e^{-\kappa z} e^{i(kx-\omega t)}\left(\frac{-ic\kappa}{\omega n_2}\hat{x} + \frac{ck}{\omega n_2}\hat{z}\right)$

note $v_2 = c/n_2$

$\boxed{\vec{E} = E_0\, e^{-\kappa z}\cos(kx-\omega t)\hat{y}}$ $\qquad \boxed{\vec{B} = \frac{E_0}{\omega} e^{-\kappa z}(\kappa\sin(kx-\omega t)\hat{x} + \cos(kx-\omega t)\hat{z})}$

e.) Check everything satisfies Maxwell's eqns

$$\nabla\cdot\vec{E}=0$$
$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$
$$\nabla\cdot\vec{B}=0$$
$$\nabla\times\vec{B}=\mu\varepsilon\frac{\partial\vec{E}}{\partial t}+\mu\sigma\vec{E}$$

$$\nabla\cdot\vec{B}=\frac{E_0}{\omega}e^{-Kz}K\cos(kx-\omega t)(k)+\frac{E_0}{\omega}e^{-Kz}(-K)\cos(kx-\omega t)(k)$$

$$\boxed{\nabla\cdot\vec{B}=0}$$

$$\boxed{\nabla\cdot\vec{E}=\frac{\partial(E_0e^{-Kz}\cos(kx-\omega t))}{\partial y}=0}$$

$$\nabla\times\vec{E}=\left(\underbrace{\cancel{\frac{\partial E_z}{\partial y}}}_{0}-\frac{\partial E_y}{\partial z}\right)\hat{x}-\left(\underbrace{\cancel{\frac{\partial E_z}{\partial x}}}_{0}-\underbrace{\cancel{\frac{\partial E_x}{\partial z}}}_{0}\right)\hat{y}+\left(\frac{\partial E_y}{\partial x}-\underbrace{\cancel{\frac{\partial E_x}{\partial y}}}_{0}\right)\hat{z}$$

$$\boxed{\nabla\times\vec{E}=E_0Ke^{-Kz}\cos(kx-\omega t)\hat{x}+E_0e^{-Kz}k\sin(kx-\omega t)\hat{z}=-\frac{\partial\vec{B}}{\partial t}}$$

$$\frac{\partial\vec{B}}{\partial t}=\frac{E_0}{\omega}e^{-Kz}\left(K(-\omega)\cos(kx-\omega t)\hat{x}+k\omega\sin(kx-\omega t)\hat{z}\right)$$

$$\boxed{-\frac{\partial\vec{B}}{\partial t}=E_0e^{-Kz}\left(K\cos(kx-\omega t)\hat{x}-k\sin(kx-\omega t)\hat{z}\right)=\nabla\times\vec{E}}\ \checkmark$$

$$\nabla\times\vec{B}=\left(\frac{\partial B_z}{\partial y}-\underbrace{\cancel{\frac{\partial B_y}{\partial z}}}_{0}\right)\hat{x}-\left(\frac{\partial B_z}{\partial x}-\frac{\partial B_x}{\partial z}\right)\hat{y}+\left(\underbrace{\cancel{\frac{\partial B_y}{\partial x}}}_{0}-\frac{\partial B_x}{\partial y}\right)\hat{n}$$

$$\nabla\times\vec{B}=-\frac{E_0}{\omega}e^{-Kz}\left(-k^2\sin(kx-\omega t)+K^2\sin(kx-\omega t)\right)\hat{y}$$

$$\mu\varepsilon\frac{\partial\vec{E}}{\partial t}+\mu\sigma\vec{E}=\mu\varepsilon E_0e^{-Kz}\omega\sin(kx-\omega t)\hat{y}+\mu\sigma E_0e^{-Kz}\cos(kx-\omega t)\hat{y}$$

$$\mu\varepsilon\frac{\partial\vec{E}}{\partial t}+\mu\sigma\vec{E}=E_0e^{-Kz}\hat{y}\left(\mu\varepsilon\omega\sin(kx-\omega t)+\mu\underbrace{\cancel{\sigma}}_{0}\cos(kx-\omega t)\right)$$

b/c total internal reflection

$$k^2-K^2=\omega^2\varepsilon\mu\qquad\boxed{\frac{\partial\vec{B}}{\partial t}=E_0e^{-Kz}\omega\mu\varepsilon\sin(kx-\omega t)\hat{y}=\mu\varepsilon\frac{\partial\vec{E}}{\partial t}}$$

f.) Construct $\vec{S}$ & show no energy transported on avg

$$\boxed{\vec{S}=\frac{\vec{E}\times\vec{B}}{\mu}=\frac{E_0^2}{\mu\omega}e^{-2Kz}\left(k\cos^2(kx-\omega t)\hat{x}-K\sin(kx-\omega t)\cos(kx-\omega t)\hat{z}\right)}$$

$$\langle\cos^2\theta\rangle=\tfrac{1}{2}\qquad\langle\cos\theta\sin\theta\rangle=0$$

$$\therefore\ \boxed{\langle\vec{S}\rangle=\frac{E_0^2k}{2\mu\omega}e^{-2Kz}\hat{x}}$$ no energy transported in $\hat{z}$!

# Numerical HW

ex 1.) $m=1$ $n=0$ show the real fields of the $TE_{10}$ wave

$B_z = X(x)\,Y(y)$ $\quad \frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2 \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2 \quad -k_x^2 - k_y^2 + \left(\frac{\omega}{c}\right)^2 - k^2 = 0$

$X(x) = A\sin(k_x x) + B\cos(k_x x) \qquad Y(y) = C\sin(k_y y) + D\cos(k_y y)$

boundary conditions

1.) $B_x(0) = 0 \Rightarrow \frac{dX}{dx} = 0 \rightarrow A = 0$

$B_y(0) = 0 \Rightarrow \frac{dY}{dy} = 0 \rightarrow C = 0$

2.) $B_x(a) = 0 \Rightarrow \frac{dX}{dx} = 0 \rightarrow k_x = \frac{m\pi}{a}$

$B_y(b) = 0 \Rightarrow \frac{dY}{dy} = 0 \rightarrow k_y = \frac{n\pi}{b}$

$\boxed{B_z = B_0\cos(m\pi x/a)\cos(n\pi y/b)}$

Get other fields from this

ex 2.) show the real fields for general $TE_{mn}$ waves

same as before but have time consideration & different $\omega$

$\hookrightarrow \boxed{\vec{B}_z = (B_0\cos(m\pi x/a)\cos(n\pi y/b))(\cos(kz-\omega t))}$

get other fields from this

ex 4.)

$X = A\sin(k_x x) + B\cos(k_x x) \qquad Y = C\sin(k_y y) + D\cos(k_y y)$

$E_x$ maxima @ $x=0$ & @ $x=a$ $\longrightarrow E_x = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\, k\frac{\partial X}{\partial x}$

$E_x = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\, k\left(Ak_x\cos(k_x x) - Bk_x\sin(k_x x)\right) \longrightarrow k_x = \frac{m\pi}{a} \quad A=0$

$E_y$ maxima @ $y=0$ & @ $y=b$ $\longrightarrow E_y = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\, k\frac{\partial Y}{\partial y}$

$E_y = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(Ck_y\cos(k_y y) - Dk_y\sin(k_y y)\right) \rightarrow k_y = \frac{n\pi}{b} \quad C=0$

$\boxed{E_z = E_0\sin(m\pi/a\, x)\sin(n\pi/b\, y)}$

$k = \sqrt{-\left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 + \left(\frac{\omega}{c}\right)^2}$

$\omega > \sqrt{\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right)c^2}$ $\qquad \boxed{\omega_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$