Supplementary Material For: Attention! Dynamic Epistemic Logic Models of (In)attentive Agents

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Proposition 0.1. *$\mathcal{E}(\varphi)$ of Definition 3.1 and $\mathcal{E}'(\varphi)$ of Definition 3.2 are isomorphic.*

Proof. We already concluded that the two models have the same set of preconditions, and that all events have distinct preconditions. We then just need to show that for all $a \in Ag$ and all events $e, f \in E$ of $\mathcal{E}(\varphi)$, we have $(e, f) \in Q_a$ in $\mathcal{E}(\varphi)$ iff $(pre(e), pre(f)) \in Q_a$ in $\mathcal{E}'(\varphi)$. To see this, consider first an edge in Q_a of $\mathcal{E}(\varphi)$. It's either of the form $((i, J), (1, K))$ for some $i \in \{0, 1\}, J, K \subseteq Ag$ and $a \in J$ or it's of the form $((i, J), s_\top)$ for some $i \in \{0, 1\}, J \subseteq Ag$ and $a \notin J$. According to Definition 3.1, an edge of the first form is an edge from an event with precondition $\neg\varphi \wedge \bigwedge_{a\in J} \mathsf{h}_a \wedge \bigwedge_{a\notin J} \neg\mathsf{h}_a$ or $\varphi \wedge \bigwedge_{a\in J} \mathsf{h}_a \wedge \bigwedge_{a\notin J} \neg\mathsf{h}_a$ to an event with precondition $\varphi \wedge \bigwedge_{a\in K} \mathsf{h}_a \wedge \bigwedge_{a\notin K} \neg\mathsf{h}_a$. Such an edge clearly satisfies Basic Attentiveness (since φ is a conjunct of the target of the edge) and Inertia (the condition $a \in J$ for the source event implies that h_a is contained in the precondition of the source, and hence Inertia holds trivially). This shows that edges in $\mathcal{E}(\varphi)$ of the first type are also edges in $\mathcal{E}'(\varphi)$. The argument for edges of the second type is similar, but here the condition of the source is $a \notin J$, meaning that Basic Attentiveness instead is trivial, and we only need to show Inertia. According to Definition 3.1, an edge of the second type is an edge from an event with precondition $\neg\varphi \wedge \bigwedge_{a\in J} \mathsf{h}_a \wedge \bigwedge_{a\notin J} \neg\mathsf{h}_a$ or $\varphi \wedge \bigwedge_{a\in J} \mathsf{h}_a \wedge \bigwedge_{a\notin J} \neg\mathsf{h}_a$ (as before) to an event with precondition $\top$. Since $a \notin J$, we have that h_a is not contained in the precondition of the source event. Inertia then requires that the precondition of the target is $\top$, but that we already concluded. So Inertia holds, as required.

For the other direction, we start with an edge $(e, f) \in Q_a$ of $\mathcal{E}'(\varphi)$ satisfying both Basic Attentiveness and Inertia, and show that it is of one of the two types in $\mathcal{E}(\varphi)$. We split into cases depending on whether $\mathsf{h}_a \in e$ or not. If $\mathsf{h}_a \in e$, then by Basic Attentiveness, $\varphi \in f$. Let J denote the set of agents for which h_a occurs positively in e, and let K denote the same set for f. Since $\mathsf{h}_a \in e$, we get $a \in J$. Let $i = 0$ if $\neg\varphi$ occurs in e, otherwise let $i = 1$. Then $e = pre((i, J))$, using the notation from Definition 3.1. Since $\varphi \in f$, we have that $f = pre((1, K))$. By Definition 3.1, Q_a contains an edge from (i, J) to $(1, K)$. This covers the case where $\mathsf{h}_a \in e$. Consider now the case $\mathsf{h}_a \notin e$. By Inertia, $f = \top$. Define J and i as before from e. Then, as before, $e = pre((i, J))$. Since $f = \top$, $f = pre(s_\top)$. By Definition 3.1, Q_a contains an edge from (i, J) to $s_\top$, and we're done. ◻

Notation used in all the proofs that follow: In the following proofs, events containing all the announced literals will be called "maximal". We will use notation $Q_a[e]$ to indicate the states from a set E that are Q_a-accessible from e, i.e., $Q_a[e] = \{f \in E \colon (e, f) \in Q_a\}$. Lastly, if $\varphi, \psi \in \mathcal{L}$ are conjunctions of literals, we will say that $\psi \in \varphi$ iff $Lit(\psi) \subseteq Lit(\varphi)$. In that case, we will also say that φ contains ψ.

Lemma 0.2. *For any pointed Kripke model $(\mathcal{M}, w)$ with $(\mathcal{M}, w) \models \varphi$, and any $a \in Ag$, consider the unique $S \subseteq At(\varphi)$ that is such that $(\mathcal{M}, w) \models (\bigwedge_{p\in S} \mathsf{h}_a p \wedge \bigwedge_{p\in At(\varphi)\setminus S} \neg\mathsf{h}_a p)$. Then, the updated models $(\mathcal{M}, w) \otimes \mathcal{F}(\bigwedge_{p\in S} \ell(p)) = ((W^{\varphi_S}, R^{\varphi_S}, V^{\varphi_S}), (w, e'))$ and $(\mathcal{M}, w) \otimes \mathcal{F}(\varphi) = ((W^{\varphi}, R^{\varphi}, V^{\varphi}), (w, e))$ are such that*

1. *$R^{\varphi}_a[(w, e)] = R^{\varphi_S}_a[(w, e')]$;*
2. *For all $(v, f) \in R^{\varphi}_a[(w, e)]$, there exists a bisimulation between $(\mathcal{M}^{\varphi}, (v, f))$ and $(\mathcal{M}^{\varphi_S}, (v, f))$, notation $(\mathcal{M}^{\varphi}, (v, f)) \leftrightarrows (\mathcal{M}^{\varphi_S}, (v, f))$;[1]*

Proof. Let $(\mathcal{M}, w) = ((W, R, V), w)$ be a pointed Kripke model. We will use the same notation as in the previous proof for φ_S, for $\mathcal{F}(\varphi) = ((E, Q, pre), E_d)$ and $\mathcal{F}(\varphi_S) = ((E', Q', pre'), E'_d)$. For the φ- and φ_S-updates of $(\mathcal{M}, w)$ we will use the notation introduced in the statement of the lemma, if not otherwise stated.

Assume that $(\mathcal{M}, w) \models \varphi$. Then $\mathcal{F}(\varphi)$ and $\mathcal{F}(\varphi_S)$ are applicable to $(\mathcal{M}, w)$, so $(\mathcal{M}, w) \otimes \mathcal{F}(\varphi) = (\mathcal{M}^{\varphi}, (w, e))$ and $(\mathcal{M}, w) \otimes \mathcal{F}(\varphi_S) = (\mathcal{M}^{\varphi_S}, (w, e'))$ exist. Now let $S \subseteq At(\varphi)$ be the unique S that is such that $(\mathcal{M}, w) \models (\bigwedge_{p\in S} \mathsf{h}_a p \wedge \bigwedge_{p\in At(\varphi)\setminus S} \neg\mathsf{h}_a p)$, for some $a \in Ag$.

(1) We first show that $R^{\varphi}_a[(w, e)] = R^{\varphi_S}_a[(w, e')]$, proving the two inclusions separately.

($\Rightarrow$) Let $(v, f) \in R^{\varphi}[(w, e)]$. This means that $v \in R_a[w]$ and $f \in Q_a[e]$. Then, to reach the desired result that $(v, f) \in R^{\varphi_S}[(w, e')]$, we only need to show that $f \in Q'_a[e']$, as then we would have that $v \in R_a[w]$ and $f \in Q'_a[e']$, and since $(\mathcal{M}, v) \models pre(f)$ then $(v, f) \in W^{\varphi_S}$ and we could conclude that $(v, f) \in R^{\varphi_S}[(w, e')]$. We show that $f \in Q'_a[e']$ by showing that f is such that $f \in E'$ and that it satisfies the requirements that Attentiveness and Inertia pose to belong to $Q'_a[e']$, i.e., it contains the needed formulas.

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[1] The notion of bisimulation for Kripke model is standard, see e.g., [1].

So let's first see what formulas f contains. By initial assumption, $(\mathcal{M}, w) \models (\bigwedge_{p\in S} \mathsf{h}_a p \wedge \bigwedge_{p\in At(\varphi)\setminus S} \neg \mathsf{h}_a p)$. As $(w, e) \in W^{\varphi}$, then by product update definition and maximality of e, it holds that $(\bigwedge_{p\in S} \mathsf{h}_a p \wedge \bigwedge_{p\in At(\varphi)\setminus S} \neg \mathsf{h}_a p) \in e$. Then by Attentiveness and $\bigwedge_{p\in S} \mathsf{h}_a p \in e$, we know that $\bigwedge_{p\in S}(\ell(p) \wedge \mathsf{h}_a p) \in f$. Moreover, as $\bigwedge_{p\in At(\varphi)\setminus S} \neg \mathsf{h}_a p \in e$, then by def. of event model for propositional attention (in particular by definition of its set of events) for all $p \in At(\varphi) \setminus S$, $\mathsf{h}_a p \notin e$ and so by Inertia, f doesn't contain $\ell(p)$, for all $p \in At(\varphi) \setminus S$, which then also means that $\mathsf{h}_a p \notin f$ for all such $p \in At(\varphi) \setminus S$, by def. of event models for propositional attention. Hence, f is such that $\bigwedge_{p\in S}(\ell(p) \wedge \mathsf{h}_a p) \in f$ as well as, for all $p \in At(\varphi) \setminus S$, $\mathsf{h}_a p, \ell(p) \notin f$.

Now let's see what is required to belong to $Q'_a[e']$. Since by initial assumption $(\mathcal{M}, w) \models \bigwedge_{p\in S} \mathsf{h}_a p$ for some $S \subseteq At(\varphi)$ and since $(w, e') \in W^{\varphi_S}$, then by product update definition and maximality of e', it holds that $\bigwedge_{p\in S} \mathsf{h}_a p \in e'$. Then we can use Attentiveness to see that in order to belong to $Q'_a[e']$ an event must contain $\bigwedge_{p\in S}(\ell(p) \wedge \mathsf{h}_a p)$. Moreover, since all events in $Q'_a[e']$ are events from $\mathcal{F}(\varphi_S)$ then they contain only literals and attention atoms from φ_S. So to belong to $Q'_a[e']$, and thus to E', an event must not contain $\ell(p), \mathsf{h}_a p$, for all $p \in At(\varphi) \setminus S$. Hence, to belong to $Q'_a[e']$, an event f' must be such $\bigwedge_{p\in S}(\ell(p) \wedge \mathsf{h}_a p) \in f'$ as well as, for all $p \in At(\varphi) \setminus S$, $\mathsf{h}_a p \notin f'$ and $\ell(p) \notin f'$. This is exactly what we have with f and since Attentiveness and Inertia are the only requirements to satisfy to be part of $Q'_a[e']$, then and $f \in Q'_a[e']$.

Hence, we have that if $f \in Q_a[e]$ then $f \in Q'_a[e']$. Above we assumed that $(v, f) \in R^{\varphi}[(w, e)]$, i.e., that $v \in R_a[w]$ and $f \in Q_a[e]$. This now implies that $v \in R_a[w]$ and $f \in Q'_a[e']$, and since $(\mathcal{M}, v) \models pre(f)$ and so $(v, f) \in W^{\varphi_S}$, then by def. of product update that $(v, f) \in R^{\varphi_S}_a[(w, e')]$.

($\Leftarrow$) This proof proceed analogously to the above proof of the other inclusion.

We can conclude that $R^{\varphi}_a[(w, e)] = R^{\varphi_S}_a[(w, e')]$.

(2) We now show that for all $(v, f) \in R^{\varphi}_a[(w, e)]$, $(\mathcal{M}^{\varphi}, (v, f)) \leftrightarrows (\mathcal{M}^{\varphi_S}, (v, f))$. Consider a bisimulation $\mathcal{Z} \subseteq (W^{\varphi} \times W^{\varphi_S})$ defined by $(u', g') \in \mathcal{Z}[(u, g)]$ iff $u = u'$ and $g = g'$ (recall that events are formulas, so $g = g'$ means that their preconditions are the same). We show that it satisfies the three requirements of bisimulations for Kripke models. Let $(u', g') \in \mathcal{Z}[(u, g)]$.

[Atom]: Since $u = u'$, then clearly $(u, g), (u', g')$ satisfy the same atomic formulas, by def. of product update.

[Forth]: Let $(t, h) \in R^{\varphi}_b[(u, g)]$, for some $b \in Ag$. We want to show that there exists a state $(t', h') \in W^{\varphi_S}$ such that $(t', h') \in R^{\varphi_S}_b[(u', g')]$ and $(t', h') \in \mathcal{Z}[(t, h)]$. By def. of product update, since $(t, h) \in R^{\varphi}_b[(u, g)]$, then $t \in R_b[u]$ and $h \in Q_b[g]$. As by initial assumption $(u', g') \in \mathcal{Z}[(u, g)]$, then $g = g'$, that is, g and g' are the same formula. This implies, by Attentiveness and Inertia and by $h \in Q_b[g]$, that $h \in Q'_b[g']$ (the argument to see that this holds proceeds analogously to the argument given in (1), to show that if $f \in Q_a[e]$ then $f \in Q'_a[e']$). Moreover, as $(u', g') \in \mathcal{Z}[(u, g)]$ then $u = u'$, and since $t \in R_b[u]$ then clearly $t \in R_b[u']$. Since $(t, h) \in R^{\varphi}_b[(u, g)]$ then $(t, h) \in W^{\varphi}$, implying that the precondition of h is satisfied in t. We now have $h \in Q'_b[g']$, $t \in R_b[u']$ and that the precondition of h is satisfied in t which, by product update definition, implies $(t, h) \in W^{\varphi_S}$ and $(t, h) \in R^{\varphi_S}_b[(u', g')]$. Letting $t' = t$ and $h' = h$, this proves the required.

[Back]: Analogous to the Forth condition.

As by (1) we have $R^{\varphi}_a[(w, e)] = R^{\varphi_S}_a[(w, e')]$, then by choice of bisimulation relation $\mathcal{Z}$ we can conclude that for all $(v, f) \in R^{\varphi}_a[(w, e)]$, $(\mathcal{M}^{\varphi}, (v, f)) \leftrightarrows (\mathcal{M}^{\varphi_S}, (v, f))$. $\square$

Table 1: The logic of propositional attention Λ. It is assumed that $\varphi = \ell(p_1) \wedge \cdots \wedge \ell(p_n)$ for some literals $\ell(p_i)$, $i = 1, \ldots, n$.

All propositional tautologies
$B_a(\varphi \to \psi) \to (B_a\varphi \to B_a\psi)$
$[\mathcal{F}(\varphi)]p \leftrightarrow (\varphi \to p)$
$[\mathcal{F}(\varphi)]\neg\psi \leftrightarrow (\varphi \to \neg[\mathcal{F}(\varphi)]\psi)$
$[\mathcal{F}(\varphi)](\psi \wedge \chi) \leftrightarrow ([\mathcal{F}(\varphi)]\psi \wedge [\mathcal{F}(\varphi)]\chi)$
$[\mathcal{F}(\varphi)]B_a\psi \leftrightarrow (\varphi \to \bigvee_{S\subseteq At(\varphi)}((\bigwedge_{p\in S} \mathsf{h}_a p \wedge \bigwedge_{p\in At(\varphi)\setminus S} \neg \mathsf{h}_a p) \to B_a([\mathcal{F}(\bigwedge_{p\in S} \ell(p))]\psi)))$
From φ and $\varphi \to \psi$, infer ψ
From φ infer $B_a\varphi$
From $\varphi \leftrightarrow \psi$, infer $\chi[\varphi/p] \leftrightarrow \chi[\psi/p]$

Proof of Theorem 4.2. We report here the table with axioms and inference rules of the Logic for Propositional Attention (Table 1), as they will be needed in the proofs below.

Completeness: It proceeds by usual reduction arguments [4].

Soundness: We show that axioms and inferences rules from Table 1 are valid. Axioms and inference rules for normal modal logic are valid in pointed Kripke models, by standard results [1]. As our product update is of the state-eliminating kind, the propositional reduction axioms are valid [4]. Thus, we only need to show the validity of the reduction axiom for attention-based belief updates. We prove the two directions separately.

Let $(\mathcal{M}, w) = ((W, R, V), w)$ be a pointed Kripke model. We use the same notation as in the previous proof for φ_S, for $\mathcal{F}(\varphi)$ and $\mathcal{F}(\varphi_S)$, and for the updates $(\mathcal{M}^{\varphi}, (w, e))$ and $(\mathcal{M}^{\varphi_S}, (w, e'))$.

($\Rightarrow$) In this direction we want to prove that if we assume $(\mathcal{M}, w) \models [\mathcal{F}(\varphi)]B_a\psi$ for some arbitrary $a \in Ag$, then it follows that $(\mathcal{M}, w) \models \varphi \to \bigvee_{S\subseteq At(\varphi)}((\bigwedge_{p\in S} \mathsf{h}_a p \wedge \bigwedge_{p\in At(\varphi)\setminus S} \neg \mathsf{h}_a p) \to B_a(\bigwedge_{p\in S} \ell(p) \to [\mathcal{F}(\varphi_S)]\psi))$. We will show that the claim follows straightforwardly from Lemma 0.2. Let $(\mathcal{M}, w) \models [\mathcal{F}(\varphi)]B_a\psi$ for some arbitrary $a \in Ag$, let $(\mathcal{M}, w) \models \varphi$ and let $S \subseteq At(\varphi)$ be the unique S such that $(\mathcal{M}, w) \models \bigwedge_{p\in S} \mathsf{h}_a p \wedge \bigwedge_{p\in At(\varphi)\setminus S} \neg \mathsf{h}_a p$. As $(\mathcal{M}, w) \models \varphi$ then $\mathcal{F}(\varphi)$ is applicable in $(\mathcal{M}, w)$ and $(\mathcal{M}^{\varphi}, (w, e))$ and $(\mathcal{M}^{\varphi_S}, (w, e'))$ exist. As $(\mathcal{M}, w) \models [\mathcal{F}(\varphi)]B_a\psi$, then we know, by semantics of the dynamic modality and by applicability of $\mathcal{F}(\varphi)$ to $(\mathcal{M}, w)$, that $(\mathcal{M}^{\varphi}, (w, e)) \models B_a\psi$, and so, by semantics of belief modality, for all $(v, f) \in R^{\varphi}_a[(w, e)]$, $(\mathcal{M}^{\varphi}, (v, f)) \models \psi$. As our assumptions here are the same assumptions made in Lemma 0.2, we can then use that lemma to obtain that $R^{\varphi}_a[(w, e)] = R^{\varphi_S}_a[(w, e')]$ and that for all $(v, f) \in R^{\varphi}_a[(w, e)]$, $(\mathcal{M}^{\varphi}, (v, f)) \leftrightarrows (\mathcal{M}^{\varphi_S}, (v, f))$. By standard results, bisimulation implies modal equivalence (see e.g., [1]). Hence, it follows that for all $(v, f) \in R^{\varphi_S}_a[(w, e')]$, $(\mathcal{M}^{\varphi_S}, (v, f)) \models \psi$. This means that for all $v \in R_a[w]$ and all $f \in Q'_a[e']$ that are such that $(v, f) \in W^{\varphi_S}$, $(\mathcal{M}^{\varphi_S}, (v, f)) \models \psi$.

Now we have two cases: for any $v \in R_a[w]$, either $\mathcal{F}(\varphi_S)$ is applicable in $(\mathcal{M}, v)$ or it is not. If it is not applicable, we can directly conclude that $(\mathcal{M}, v) \models [\mathcal{F}(\varphi_S)]\psi$, by semantics of dynamic modality, and since this holds for an arbitrary $v \in R_a[w]$, then $(\mathcal{M}, w) \models B_a([\mathcal{F}(\varphi_S)]\psi)$, by semantics of belief modality. Now consider the case in which $\mathcal{F}(\varphi_S)$ is applicable in $(\mathcal{M}, v)$. In this case, we need to show that for any $f \in Q'_a[e']$ with $(v, f) \in W^{\varphi}$, f is maximal, i.e., $f \in E'_d$, to then be able to infer, by semantics of dynamic modality, that for all $v \in R_a[w]$, $(\mathcal{M}, v) \models [\mathcal{F}(\varphi_S)]\psi$. To that goal notice that since $(\mathcal{M}, w) \models \bigwedge_{p \in S} \mathsf{h}_a p$, then by maximality of e' with respect to φ_S and product update definition, $\bigwedge_{p \in S} \mathsf{h}_a p \in e'$, and so by Attentiveness $\bigwedge_{p \in S} \ell(p) \in f$, for all $f \in Q'_a[e']$. So f is indeed maximal with respect to φ_S and thus $f \in E'_d$. Hence, we have that for all $v \in R_a[w]$, $(\mathcal{M}, v) \models [\mathcal{F}(\varphi_S)]\psi$, which by semantics of belief modality implies that $(\mathcal{M}, w) \models B_a([\mathcal{F}(\varphi_S)]\psi)$, as we wanted to conclude.

($\Leftarrow$) For this other direction, the goal is showing that by assuming $(\mathcal{M}, w) \models \varphi \rightarrow \bigvee_{S \subseteq At(\varphi)}((\bigwedge_{p \in S} \mathsf{h}_a p \wedge \bigwedge_{p \in At(\varphi) \setminus S} \neg \mathsf{h}_a p) \rightarrow B_a([\mathcal{F}(\varphi_S)]\psi))$ we can conclude that $(\mathcal{M}, w) \models [\mathcal{F}(\varphi)]B_a\psi$. Here we proceed by contraposition and so show that by assuming $(\mathcal{M}, w) \not\models [\mathcal{F}(\varphi)]B_a\psi$, i.e., by assuming that $\mathcal{F}(\varphi)$ is applicable in $(\mathcal{M}, w)$ but $(\mathcal{M}^{\varphi}, (w, e)) \not\models B_a\psi$, we can conclude that $(\mathcal{M}, w) \not\models \varphi \rightarrow \bigvee_{S \subseteq At(\varphi)}((\bigwedge_{p \in S} \mathsf{h}_a p \wedge \bigwedge_{p \in At(\varphi) \setminus S} \neg \mathsf{h}_a p) \rightarrow B_a([\mathcal{F}(\varphi_S)]\psi))$, i.e., we can conclude that if $(\mathcal{M}, w) \models \varphi$ then $(\mathcal{M}, w) \not\models \bigvee_{S \subseteq At(\varphi)}((\bigwedge_{p \in S} \mathsf{h}_a p \wedge \bigwedge_{p \in At(\varphi) \setminus S} \neg \mathsf{h}_a p) \rightarrow B_a([\mathcal{F}(\varphi_S)]\psi))$, which means concluding that if $(\mathcal{M}, w) \models \bigvee_{S \subseteq At(\varphi)}(\bigwedge_{p \in S} \mathsf{h}_a p \wedge \bigwedge_{p \in At(\varphi) \setminus S} \neg \mathsf{h}_a p)$ then $(\mathcal{M}, w) \not\models B_a([\mathcal{F}(\varphi_S)]\psi)$, which again means that there exists a $v \in R_a[w]$ with $(\mathcal{M}, v) \not\models [\mathcal{F}(\varphi_S)]\psi$, i.e., $\mathcal{F}(\varphi_S)$ is applicable in $(\mathcal{M}, v)$ but $(\mathcal{M}, v) \not\models \psi$. Also here the conclusion will follow straightforwardly by using Lemma 0.2.

So we start by making all the stated assumptions. Let $\mathcal{F}(\varphi)$ be applicable to $(\mathcal{M}, w)$ and let $(\mathcal{M}^{\varphi}, (w, e)) \not\models B_a\psi$, for some $a \in Ag$. Moreover, let $(\mathcal{M}, w) \models \varphi$ and let $S \subseteq At(\varphi)$ be the unique S such that $(\mathcal{M}, w) \models \bigwedge_{p \in S} \mathsf{h}_a p \wedge \bigwedge_{p \in At(\varphi) \setminus S} \neg \mathsf{h}_a p$. The goal is to show that for this particular S, we also have $(\mathcal{M}, w) \not\models B_a([\mathcal{F}(\varphi_S)]\psi)$. As by assumption the event model $\mathcal{F}(\varphi)$ is applicable in $(\mathcal{M}, w)$, then also the event model $\mathcal{F}(\varphi_S)$ is applicable in $(\mathcal{M}, w)$, and $(\mathcal{M}^{\varphi_S}, (w, e'))$ exists.

Now $(\mathcal{M}^{\varphi}, (w, e)) \not\models B_a\psi$ implies by semantics of belief modality that there exists some $(v, f) \in R^{\varphi}_a[(w, e)]$ such that $(\mathcal{M}^{\varphi}, (v, f)) \not\models \psi$. As the assumptions of Lemma 0.2 are satisfied here, then $R^{\varphi}_a[(w, e)] = R^{\varphi_S}_a[(w, e')]$, and all the $(v, f) \in R^{\varphi}_a[(w, e)]$ are such that $\mathcal{M}^{\varphi_S}, (v, f) \leftrightarrows \mathcal{M}^{\varphi}, (v, f)$. As modal equivalence follows by standard results on bisimulation and Kripke models (see e.g., [1]), then it follows that there exists some $(v, f) \in R^{\varphi_S}_a[(w, e')]$ such that $(\mathcal{M}^{\varphi_S}, (v, f)) \not\models \psi$. This means that there exists some $v \in R_a[w]$ and $f \in Q'_a[e']$ such that $(\mathcal{M}^{\varphi_S}, (v, f)) \not\models \psi$. As $(\mathcal{M}, w) \models \bigwedge_{p \in S} \mathsf{h}_a p$, then $\bigwedge_{p \in S} \mathsf{h}_a p \in e'$ and by Attentiveness $\bigwedge_{p \in S}(\mathsf{h}_a p \wedge \ell(p)) \in f$ for all $f \in Q'_a[e']$. So f is maximal with respect to φ_S and thus $f \in E'_d$. It was necessary to show maximality of f here as we now know that $\mathcal{F}(\varphi_S)$ is applicable in $(\mathcal{M}, v)$ and so we know, by semantics of dynamic modality, that there exists some $v \in R_a[w]$ that is such that $(\mathcal{M}, v) \not\models [\mathcal{F}(\varphi_S)]\psi$. So we have that $(\mathcal{M}, v) \models \bigwedge_{p \in S} \ell(p)$ and $(\mathcal{M}, v) \not\models [\mathcal{F}(\varphi_S)]\psi$, that is $(\mathcal{M}, v) \not\models \bigwedge_{p \in S} \ell(p) \rightarrow [\mathcal{F}(\varphi_S)]\psi$. Hence, by $v \in R_a[w]$, we can conclude that $(\mathcal{M}, w) \not\models B_a[\mathcal{F}(\varphi_S)]\psi$. $\square$

Lemma 0.3. *For any pointed Kripke model $(\mathcal{M}, w)$ with $(\mathcal{M}, w) \models \varphi$, and for any $a \in Ag$, consider the $S \subseteq At(\varphi)$ that is such that $(\mathcal{M}, w) \models \bigwedge_{p \in S} \mathsf{h}_a p \wedge \bigwedge_{p \in At(\varphi) \setminus S} \neg \mathsf{h}_a p$. Let $\varphi_{Sd} = \bigwedge_{p \in S} \ell(p) \wedge \bigwedge_{p \in At(\varphi) \setminus S} d_a(p)$. The updated models $(\mathcal{M}, w) \otimes \mathcal{E}(\varphi_{Sd}, d) = ((W^{\varphi_{Sd}}, R^{\varphi_{Sd}}, V^{\varphi_{Sd}}), (w, e'))$ and $(\mathcal{M}, w) \otimes \mathcal{E}(\varphi, d) = ((W^{\varphi}, R^{\varphi}, V^{\varphi}), (w, e))$ are such that*

(1) $R^{\varphi}_a[(w, e)] = R^{\varphi_{Sd}}_a[(w, e')]$
(2) *For all $(v, f) \in R^{\varphi}_a[(w, e)]$, $(\mathcal{M}^{\varphi}, (v, f)) \leftrightarrows (\mathcal{M}^{\varphi_{Sd}}, (v, f))$.*

Proof. The proofs of both (1) and (2) proceed analogously to the proofs of (1) and (2) of Lemma 0.2, respectively. We hence only show left to right of (1). We follow similar notational conventions as in Lemma 0.2, letting $\mathcal{E}(\varphi, d) = ((E, Q, pre), E_d)$ and $\mathcal{E}(\varphi_{Sd}, d) = ((E', Q', pre'), E'_d)$.

Let $(v, f) \in R^{\varphi}[(w, e)]$. This means that $v \in R_a[w]$ and $f \in Q_a[e]$. Then, to reach the desired result that $(v, f) \in R^{\varphi_{Sd}}[(w, e')]$, we only need to show that $f \in Q'_a[e']$, as then we would have that $v \in R_a[w]$ and $f \in Q'_a[e']$, and since $(\mathcal{M}, v) \models pre(f)$ then $(v, f) \in W^{\varphi_S}$ and we could conclude that $(v, f) \in R^{\varphi_{Sd}}[(w, e')]$. Similarly to the proof above, we show this by showing that f is such that $f \in E'$ and that f satisfies the requirements that Attentiveness and Defaulting pose to belong to $Q'_a[e']$, i.e., it contains the needed formulas.

So let's first see what formulas f contains. By initial assumption, $(\mathcal{M}, w) \models (\bigwedge_{p \in S} \mathsf{h}_a p \wedge \bigwedge_{p \in At(\varphi) \setminus S} \neg \mathsf{h}_a p)$ for some $S \subseteq At(\varphi)$. As $(w, e) \in W^{\varphi}$, then by product update definition and maximality of e, it holds that $(\bigwedge_{p \in S} \mathsf{h}_a p \wedge \bigwedge_{p \in At(\varphi) \setminus S} \neg \mathsf{h}_a p) \in e$. Then by Attentiveness and $\bigwedge_{p \in S} \mathsf{h}_a p \in e$, we know that $\bigwedge_{p \in S}(\ell(p) \wedge \mathsf{h}_a p) \in f$. Moreover, as $\bigwedge_{p \in At(\varphi) \setminus S} \neg \mathsf{h}_a p \in e$, then by def. of event model for propositional attention with defaults (in particular by definition of its set of events) for all $p \in At(\varphi) \setminus S$, $\mathsf{h}_a p \notin e$ and so by Defaulting, f contains $\bigwedge_{p \in At(\varphi) \setminus S} d_a(p)$, which then implies that $\mathsf{h}_a p \notin f$ for all such $p \in At(\varphi) \setminus S$, by def. of event models for propositional attention with defaults. Hence, f is such that $\bigwedge_{p \in S}(\ell(p) \wedge \mathsf{h}_a p) \wedge \bigwedge_{p \in At(\varphi) \setminus S} d_a(p) \in f$ and, for all $p \in At(\varphi) \setminus S$, $\mathsf{h}_a p \notin f$.

Now let's see what is required to belong to $Q'_a[e']$. Since by initial assumption $(\mathcal{M}, w) \models \bigwedge_{p \in S} \mathsf{h}_a p$ and since $(w, e') \in W^{\varphi_{Sd}}$, then by product update definition and maximality of e', it holds that $\bigwedge_{p \in S} \mathsf{h}_a p \in e'$. Then we can use Attentiveness to see that in order to belong to $Q'_a[e']$ an event must contain $\bigwedge_{p \in S}(\ell(p) \wedge \mathsf{h}_a p)$. Moreover, as $\bigwedge_{p \in At(\varphi) \setminus S} \neg \mathsf{h}_a p \in e$, then by Defaulting, all events in $Q'_a[e']$ must contain $\bigwedge_{p \in At(\varphi)} d_a(p)$ which implies, by the way events with defaults are defined, that they must not contain $\mathsf{h}_a p$ for all such $p \in At(\varphi) \setminus S$. Hence, to belong to $Q'_a[e']$, an event f' must be such $\bigwedge_{p \in S}(\ell(p) \wedge \mathsf{h}_a p) \wedge \bigwedge_{p \in At(\varphi) \setminus S} d_a(p) \in f'$ as well as, for all $p \in At(\varphi) \setminus S$, $\mathsf{h}_a p \notin f'$. As this is exactly what we have with f, then $f \in E'$ and $f \in Q'_a[e']$.

Hence, we have that if $f \in Q_a[e]$ then $f \in Q'_a[e']$. Above we assumed that $(v, f) \in R^{\varphi}[(w, e)]$, i.e., that $v \in R_a[w]$ and $f \in Q_a[e]$. This now implies that $v \in R_a[w]$ and $f \in Q'_a[e']$, and by def. of product update that $(v, f) \in R^{\varphi_{Sd}}_a[(w, e')]$, which is what we wanted to conclude. $\square$

Proof of Theorem 5.2. The axiomatization of the logic for propositional attention with defaults, for which we now want to prove soundness and completeness, is given by the same axioms as in Table 1, except for the axiom for belief dynamics which is replaced by the following axiom where inattentive agents adopt the default option for the unattended atoms (where the announced formula is $\varphi = \ell(p_1) \wedge \cdots \wedge \ell(p_n)$). Where $\varphi_{Sd} = \bigwedge_{p\in S} \ell(p) \wedge \bigwedge_{p\in At(\varphi)\setminus S} d_a(p)$, we call the resulting table 'Table 2':

$$[\mathcal{E}(\varphi,d)]B_a\psi \leftrightarrow (\varphi \rightarrow \bigvee_{S\subseteq At(\varphi)} ((\bigwedge_{p\in S} \mathsf{h}_a p \wedge \bigwedge_{p\in At(\varphi)\setminus S} \neg \mathsf{h}_a p) \rightarrow B_a([\mathcal{E}(\varphi_{Sd},d)]\psi)))$$

Completeness: It proceeds by usual reduction arguments [4].

Soundness: We show that axioms and inferences rules from Table 2 are valid. Using the same reasoning as in the previous soundness proof, we only show here the validity of the reduction axiom for attention-based belief updates with defaults. We prove the two directions separately.

Let $(\mathcal{M}, w) = ((W, R, V), w)$ be a pointed Kripke model. We will use φ_{Sd} in the same way as above, and we will use also the same notation for $\mathcal{E}(\varphi, d)$ and $\mathcal{E}(\varphi_{Sd}, d)$, as well as for $(\mathcal{M}^\varphi, (w, e))$ and $(\mathcal{M}^{\varphi_{Sd}}, (w, e))$.

($\Rightarrow$) We want to prove that if we assume $(\mathcal{M}, w) \models [\mathcal{E}(\varphi, d)]B_a\psi$, $(\mathcal{M}, w) \models \varphi$ and $(\mathcal{M}, w) \models \bigwedge_{p\in S} \mathsf{h}_a p \wedge \bigwedge_{p\in At(\varphi)\setminus S} \neg\mathsf{h}_a p$, then it follows that $(\mathcal{M}, w) \models B_a[\mathcal{E}(\varphi_{Sd}, d)]\psi$. The proof strategy is analogous to the strategy of the previous soundness proof in the left to right direction.

So assume $(\mathcal{M}, w) \models [\mathcal{E}(\varphi, d)]B_a\psi$ and $(\mathcal{M}, w) \models \varphi$ and consider the unique $S \subseteq At(\varphi)$ that is such that $(\mathcal{M}, w) \models \bigwedge_{p\in S} \mathsf{h}_a p \wedge \bigwedge_{p\in At(\varphi)\setminus S} \neg\mathsf{h}_a p$. As $(\mathcal{M}, w) \models \varphi$ then $(\mathcal{M}^\varphi, (w, e))$ exists. As $(\mathcal{M}, w) \models [\mathcal{E}(\varphi, d)]B_a\psi$ then by semantics of the dynamic modality and by applicability of $\mathcal{E}(\varphi, d)$ to (M, w), $(\mathcal{M}^\varphi, (w, e)) \models B_a\psi$, which implies, by semantics of belief modality, that for all $(v, f) \in R_a^\varphi[(w, e)]$, $(\mathcal{M}^\varphi, (v, f)) \models \psi$. By Lemma 0.3, we know that $R_a^\varphi[(w, e)] = R_a^{\varphi_{Sd}}[(w, e')]$ and that for all $(v, f) \in R_a^\varphi[(w, e)]$, $(\mathcal{M}^\varphi, (v, f)) \leftrightarroweq (\mathcal{M}^{\varphi_{Sd}}, (v, f))$. By standard modal logic results, bisimulation implies modal equivalence, and so it follows that also for all $(v, f) \in R_a^{\varphi_{Sd}}[(w, e')]$, it is the case that $(\mathcal{M}^{\varphi_{Sd}}, (v, f)) \models \psi$, which is equivalent to saying that for all $v \in R_a[w]$ and for all $f \in Q'_a[e']$ that are such that $(v, f) \in W^{\varphi_{Sd}}$, $(\mathcal{M}^{\varphi_{Sd}}, (v, f)) \models \psi$.

Now as in the previous soundness proof we have two cases: either $\mathcal{E}(\varphi_{Sd}, d)$ is applicable to $(\mathcal{M}, v)$ or it is not. If it is not, then $(\mathcal{M}, v) \models [\mathcal{E}(\varphi_{Sd}, d)]\psi$. If instead $\mathcal{E}(\varphi_{Sd}, d)$ is applicable to $(\mathcal{M}, v)$ we need to show maximality of f for all such $f \in Q_a[e']$, i.e., $f \in E'_d$, to then infer by semantics of the dynamic modality, that $(\mathcal{M}, v) \models [\mathcal{E}(\varphi_{Sd}, d)]\psi$ for all $v \in R_a[w]$. The argument proceed similarly to the previous proof, namely, since $(\mathcal{M}, w) \models \bigwedge_{p\in S} \mathsf{h}_a p \wedge \bigwedge_{p\in At(\varphi)\setminus S} \neg\mathsf{h}_a p$, then $\bigwedge_{p\in S} \mathsf{h}_a p \wedge \bigwedge_{p\in At(\varphi)\setminus S} \neg\mathsf{h}_a p \in e'$. By $\bigwedge_{p\in S} \mathsf{h}_a p \in e'$ we know that by Attentiveness, $\bigwedge_{p\in S} \ell(p) \in f$, and by $\bigwedge_{p\in At(\varphi)\setminus S} \neg\mathsf{h}_a p \in e'$ we know that by Defaulting, $\bigwedge_{p\in At(\varphi)\setminus S} d_a(p) \in f$, for all $f \in Q_a[e']$. Hence, $\bigwedge_{p\in S} \ell(p) \wedge \bigwedge_{p\in At(\varphi)\setminus S} d_a(p) \in f$, for all $f \in Q_a[e']$. This means that all such f are indeed maximal with respect to φ_{Sd} and so $f \in E'_d$. Hence, by semantics of dynamic modality, we now get that for all $v \in R_a[w]$, $(\mathcal{M}, v) \models [\mathcal{E}(\varphi_{Sd}, d)]\psi$, and thus also that $(\mathcal{M}, w) \models B_a([\mathcal{E}(\varphi_{Sd}, d)]\psi)$, as we wanted to conclude.

($\Leftarrow$) The right to left direction proceeds similar to the right to left in the proof of Theorem 4.2, i.e., by using contraposition and Lemma 0.3 we can conclude the desired result. ◻

Proof of Proposition 6.5. For each $n \geq 1$, let $\mathcal{G}(n)$ denote the syntactic event model of Example 6.4 with $Ag = \{1, \ldots, n\}$. Let $\mathcal{H}(n)$ denote the semantic event model induced by $\mathcal{G}(n)$. The induced event model $\mathcal{H}(n)$ is the one defined in Definition 3.2 that we already concluded to have at least 2^n events (due to there being one event per subset of $\{\mathsf{h}_a : a \in Ag\}$, the subset containing the h_a that occur positively in the event precondition). In [2], it is proven that $\mathcal{H}(n)$ is not equivalent to a semantic event model with less than 2^n events. However, $\mathcal{G}(n)$ is of size $O(n)$, as we will now see. The formula ψ_E is of size $O(n)$: the inner-most disjunction is repeated once for each agent, but everything else is of fixed size. The formula ψ_a is also of fixed size, it simply has size (length) 22. We however need one of these formulas for each agent, so in total $(\psi_a)_{a\in Ag}$ also has size $O(n)$. This proves the required, except for the last point about using only one agent. To use only one agent, we need to turn to a different succinctness result, the one about arrow updates in [3]. For all $n \geq 1$, let $\mathcal{L}(n)$ be the language with atomic propositions $P = \{1, \ldots, n\}$ and a single agent a. For each $n \geq 1$, let $\mathcal{H}'(n)$ be the semantic event model over $\mathcal{L}(n)$ in which each subset of P is an event, and there is an a-edge from event $P' \subseteq P$ to event $P'' \subseteq P$ if for some i, $p_i \in P'' \setminus P'$. The event model $\mathcal{H}'(n)$ clearly has 2^n events. In [3], it is shown that there exists no semantic event model equivalent to $\mathcal{H}'(n)$ having less than 2^n events. To complete our proof, we then only need to show that we can represent $\mathcal{H}'(n)$ using a syntactic event model of size $O(n)$. Let $\mathcal{G}'(n) = (\psi'_E, \psi'_a)$ be the syntactic event model over $\mathcal{L}(n)$ defined by $\psi'_E = \top$ and $\psi'_a = \bigvee_{1\leq i\leq n}(\neg \mathsf{e} \Rightarrow p_i \rightarrow \Box\mathsf{e} \Rightarrow p_i)$. It is simple to check that the semantic event model induced by $\mathcal{G}'(n)$ is exactly $\mathcal{H}'(n)$. Also, clearly $\mathcal{G}'(n)$ has size $O(n)$. ◻

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