LECON2112 Advanced Microeconomics II

– Assignment 1 –

(SOLUTIONS)

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Exercises[1]

7D1. In a game where player i has N information sets indexed $n = 1, ..., N$ and M_n possible actions at information set n, how many strategies does player i have?

SOLUTION. Player 1 has $M_1 \times M_2 \times M_3 \times ... \times M_N$ strategies.

8B2. **(a)** Argue that if a player has two weakly dominant strategies, then for every strategy choice by his opponents, the two strategies yield him equal payoffs.

SOLUTION. Let $s_i^1, s_i^2 \in S_i$ be two weakly dominant strategies for Player i. Then,

$$\begin{aligned} u_i(s_i^1, s_{-i}) &\geq u_i(s_i', s_{-i}) \quad \forall s_i' \in S_i, \text{ and } \forall s_{-i} \in S_{-i}, \text{ and} \\ u_i(s_i^2, s_{-i}) &\geq u_i(s_i', s_{-i}) \quad \forall s_i' \in S_i, \text{ and } \forall s_{-i} \in S_{-i}. \end{aligned}$$

Therefore, for all $s_{-i} \in S_{-i}$, it holds that $(s_i^1, s_{-i}) \geq (s_i^2, s_{-i})$ and $(s_i^2, s_{-i}) \geq (s_i^1, s_{-i})$, implying that

$$u_i(s_i^1, s_{-i}) = u_i(s_i^2, s_{-i}) \quad \forall s_{-i} \in S_{-i}.$$

(b) Provide an example of a two-player game in which a player has two weakly dominant pure strategies but his opponent prefers that he plays one of them rather than the other.

SOLUTION. Consider the following game:

		Player 2	
		L	R
Player 1	U	$1,4$	$2,5$
	D	$1,2$	$2,3$

Both of Player 1's strategies (U) and (D) are weakly dominant. However, Player 2 prefers that Player 1 chooses (U).

[1] Source: Mas-Colell, Whinston, & Green, 1995. "Microeconomic Theory," Oxford University Press.

8B3. Consider the following auction (known as a *second-price*, or *Vickerey*, auction). An object is auctioned off to I bidders. Bidder i's valuation of the object (in monetary terms) is v_i. The auction rules are that each player submits a bid (nonnegative number) in a sealed envelope. The envelopes are then opened, and the bidder who has submitted the highest bid gets the object but pays the auctioneer the amount of the *second-highest* bid. If more than one bidder submits the highest bid, each gets the object with equal probability. Show that submitting a bid of v_i with certainty is a weakly dominant strategy for bidder i. Also argue that this is bidder i's unique weakly dominant strategy.

SOLUTION. Suppose not. Assume bidder i bids $b_i > v_i$. Then, if some other bidder bids something larger than b_i, bidder i is just as well off as if he would have bid v_i. If all other players bid lower than v_i, then bidder i obtains the object and pays the amount of the second-highest bid. If the second-highest bid is $b_j < v_i$, this results in the same payoff for player i as if he bids v_i. However, suppose that the second-highest bid of another player is $b_j > v_i$. Then, by bidding b_i, bidder i will win the object and obtain a negative payoff. By bidding v_i, he will not win the object, and hence obtains a payoff equal to zero. Therefore, bidding $b_i > v_i$ is weakly dominated by bidding v_i.

Now, suppose bidder i bids $b_i < v_i$. Then if all other bidders bid something smaller than b_i, bidder i is just as well off as if he would have bid v_i. That is, he would win the object and pay the second-highest bid. If some other player bids higher than v_i, then bidder i does not win the object regardless of whether he bids b_i or v_i. However, suppose that nobody bids higher than v_i and the highest bid of the other players is b_j with $b_i < b_j < v_i$. Then by bidding b_i, player i will not win the object, therefore getting a payoff equal to zero. By bidding v_i, he would win the object paying $b_j < v_i$, and thus obtain a payoff of $v_i - b_j > 0$. Therefore, bidding $b_i < v_i$ is weakly dominated by bidding v_i. Therefore, bidding v_i is a weakly dominant strategy.

8Bd. Consider the following game. There are five players, three male singles 1, 2 and 3, and two female singles, a and b. The preferences $\succ_1$, $\succ_2$, $\succ_3$, $\succ_a$ and $\succ_b$ of those five players vis-à-vis a partner from the other gender and remaining single (which is denoted $\emptyset$) are as follows:

$$\begin{array}{lcl} b \succ_1 a \succ_1 \emptyset & \qquad & 1 \succ_a 2 \succ_a 3 \succ_a \emptyset \\ a \succ_2 b \succ_2 \emptyset & & 2 \succ_b 1 \succ_b 3 \succ_b \emptyset \\ a \succ_3 b \succ_3 \emptyset & & \end{array}$$

In stage 1, players 1, 2 and 3 propose the name of one female player, a or b. In stage 2, each player among a and b, if she got two or three propositions in stage 1, chooses one of them. Then, couples are formed. In case one female player did not get any proposition, she remains single. If she gets one proposition, she is married to the player having made the proposition. If she gets several propositions, she is married to the proposition she has chosen in stage 2.

(a) What is the strategy set of a male player?

SOLUTION. The strategy set of the male player 1 is composed of two strategies:

$$S_1 = \{a, b\}.$$

(b) What is the strategy set of a female player?

SOLUTION. One particular strategy of female a is the following:

$$s_a = \big((1|1,2,3),(1|1,2),(3|1,3),(2|2,3)\big).$$

This strategy specifies one choice that female a could make at any information set she might reach. Strategy s_a is such that she chooses male 1 in the case that all three males propose to her, she chooses male 1 in the case that males 1 and 2 propose to her, etc. Given that she has no choice whenever a single male proposes to her (she is forced to marry him) or when none proposes to her, strategy s_a needs not specify anything for these cases. For notational simplicity, we pick the convention to write:

$$s_a = (1,1,3,2) \equiv \big((1|1,2,3),(1|1,2),(3|1,3),(2|2,3)\big).$$

The strategy set of female a is very large. Using our terminology convention:

$$S_a = \big\{(i,j,k,l)_{i\in 1,2,3,\ j\in 1,2,\ k\in 1,3,\ l\in 2,3}\big\}.$$

(c) Does player 1 have a weakly dominated strategy? A dominant strategy? Motivate your answer.

SOLUTION. Player 1 does not have a weakly dominated strategy. For a given strategy profile

$$s = (s_1, s_2, s_3, s_a, s_b),$$

let $\theta_i(s)$ be the individual to which i is married when s is played. Letting $S = S_1 \times S_2 \times S_3 \times S_a \times S_b$ be the set of all strategy profiles, the function $\theta_i : S \to 1,2,3,a,b,\emptyset$ is the outcome function for player i. In order to show that none of the two strategies in the strategy set of male 1 satisfies the definition of a weakly dominated strategy, we need to show that:

1. there exists $s_{-1} = (s_2, s_3, s_a, sb_)$ such that $\theta_1(a, s_{-1}) \succ_1 \theta_1(b, s_{-1})$; and
2. there exists s'_{-1} such that $\theta_1(a, s'_{-1}) \succ_1 \theta_1(b, s'_{-1})$.

Part 1 shows that strategy b does not weakly dominate strategy a. As there are only two strategies for player 1, this proves that strategy a is not weakly dominated. For symmetric reason, part 2 shows that strategy b is not weakly dominated. Two such profiles s_{-1} and s'_{-1} are for example:

$$s_{-1} = \begin{cases} s_2 = b, \\ s_3 = a, \\ s_a = (1,2,1,2), \\ s_b = (2,2,3,2) \end{cases} \qquad \text{and} \qquad s'_{-1} = \begin{cases} s'_2 = a, \\ s'_3 = a, \\ s'_a = (2,2,3,2), \\ s'_b = (2,2,3,2) \end{cases} .$$

Profile s_{-1} is s.t. $\theta_1(a, s_{-1}) = a$ and $\theta_1(b, s_{-1}) = \emptyset$. Profile s'_{-1} is s.t. $\theta_1(a, s'_{-1}) = \emptyset$ and $\theta_1(b, s'_{-1}) = b$.

Observe that profiles s_{-1} and s'_{-1} need not be composed of undominated strategies for players 2, 3, a and b (as our purpose is to show that strategies a and b are not weakly dominated).

Player 1 does not have a dominant strategy. Part 1 shows that strategy b is not weakly dominant (and hence not strictly dominant). Part 2 shows that strategy a is not dominant.

(d) Does there exist a strategy profile of weakly *undominated* strategies whose outcome is that 3 gets married with a? If yes, give one such strategy profile, otherwise, explain. (Note: An undominated strategy is a strategy that is not dominated.)

SOLUTION. Here is a profile of weakly undominated strategies whose outcome features that male 3 is married to female a:

$$s = \begin{cases} s_1 = b, \\ s_2 = b, \\ s_3 = a, \\ s_a = (1,1,1,2), \\ s_b = (2,2,1,2) \end{cases}.$$

We have indeed that $\theta_3(s) = a$, as required. We still need to prove that each strategy in the profile is weakly undominated.

We proved in **8Bd.(c)** that strategy $s_1 = b$ is not weakly dominated. This implies that this strategy is weakly undominated (in the strategy set of a player, the set of weakly undominated strategies is the complement of the set of weakly dominated strategies). A similar proof shows that $s_2 = b$ and $s_3 = a$ are weakly undominated.

Strategy $s_a = (1,1,1,2)$ is such that she picks her preferred male inside each choice set. Strategy s_a is a weakly dominant strategy for player a. This can be proved by showing that this strategy satisfies the definition of a weakly dominant strategy: for all $s_{-a} \in S_{-a}$ and all $s'_a \in S_a$, we have

$$\theta_a(s_a, s_{-a}) \succsim_a \theta_a(s'_a, s_{-a}).$$

Proof by contradiction: assume there exists s'_a and s_{-a} such that $\theta_a(s_a, s_{-a}) \not\succsim_a \theta_a(s_a, s'_{-a})$. This implies that $\theta_a(s'_a, s_{-a})$ is preferred by female a to her preferred element in the choice set reached by a given s_{-a}, a contradiction.

This implies that s_a is weakly undominated, since any weakly dominant strategies is by definition weakly undominated. The proof that s_b is weakly undominated is analogous.