LECON2112 Advanced Microeconomics II

– Assignment 4 –

(SOLUTIONS)

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Academic Year: 2023/2024
Spring term

Exercises[1]

9Bd. Identify all the (pure and mixed strategy) subgame perfect Nash equilibria of the game shown the figure below.

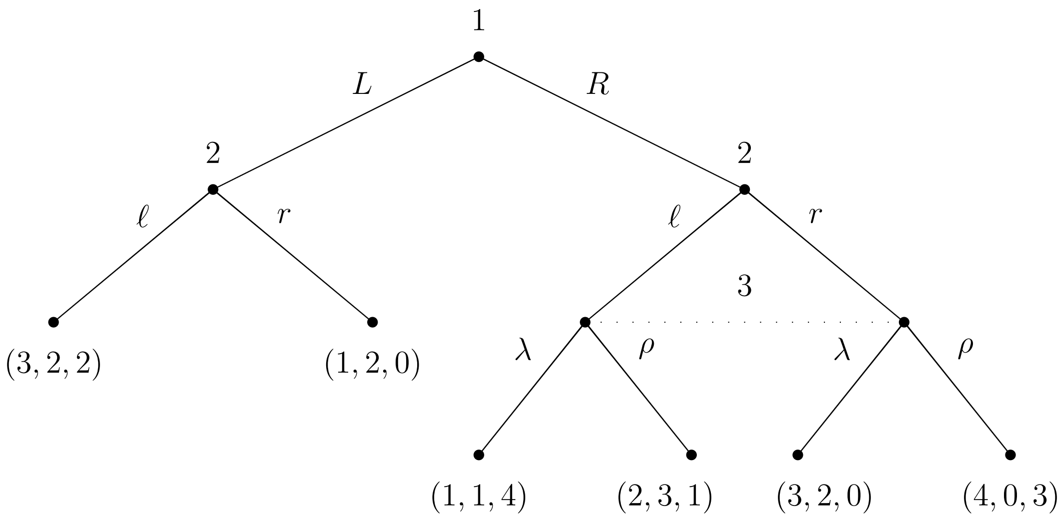


Figure 1: Exercise 9.B.d

SOLUTION. We proceed by backward induction and consider the equilibria in the subgames. First of all, consider the subgame that would be played if player 1 played L. In this subgame, player 2 is completely indifferent between playing l and r, as both strategies give her a payoff of 2. Denoting $\sigma_2 \in [0,1]$ the probability that player 2 plays l in this subgame, the equilibria of this subgame are therefore all the strategies:

$$\{\sigma_2 \in [0,1]\}$$

(i.e., the set of all possible mixed strategies of the subgame). Player 1's payoff, when playing L, will therefore be

$$U_1(L, \sigma_2) = 3\sigma_2 + (1 - \sigma_2) = 1 + 2\sigma_2$$

[1] Source: Mas-Colell, Whinston, & Green, 1995. "Microeconomic Theory," Oxford University Press.

Now consider the other possible subgame, reached if player 1 plays R. Given the information set of player 3, this subgame is a simultaneous move game, which we can represent in the following normal form game:

	λ	ρ
l	1,4	3, 1
r	2, 0	0, 3

(notice that we can disregard the payoff for player 1 as she is not a player in this subgame). First of all, notice that there is no pure strategy equilibrium (see colors indicating best replies in last table).

In this subgame, denote τ_2 the probability player 2 plays l and τ_3 the probability player 3 plays λ. The mixed strategy equilibrium of this subgame is

$$(\tau_2^*, \tau_3^*) = (\frac{1}{2}, \frac{3}{4})$$

These values are obtained by solving the following equations, which express the property that players are indifferent between the pure strategies that they play with positive probability in a *mixed* strategy equilibrium. Probability τ_2^* makes player 3 indifferent between playing λ or ρ:

$$U_3(R, \tau_2^*, \lambda) = U_3(R, \tau_2^*, \rho)$$
$$\tau_2^* 4 + (1 - \tau_2^*)0 = \tau_2^* + (1 - \tau_2^*)3$$

Probability τ_3^* makes player 2 indifferent between playing l or r:

$$U_2(R, l, \tau_3^*) = U_3(R, r, \tau_3^*)$$
$$\tau_3^* + (1 - \tau_3^*)3 = \tau_3^* 2 + (1 - \tau_3^*)0$$

Player 1's payoff, when playing R, will therefore be

$$U_1(R, \tau_2^*, \tau_3^*) = \tau_2^* \tau_3^* + \tau_2^*(1 - \tau_3^*)2 + (1 - \tau_2^*)\tau_3^* 3 + (1 - \tau_2^*)(1 - \tau_3^*)4 = \frac{9}{4}$$

Consider Player 1 now. She will strictly prefer L if

$$U_1(L, \sigma_2) > U_1(R, \tau_2^*, \tau_3^*)$$
$$1 + 2\sigma_2 > \frac{9}{4}$$

that is if $\sigma_2 > \frac{5}{8}$. Thus, a first set of subgame perfect Nash equilibria of the game is defined as follows:

$$\Big(\underbrace{L}_{s_1}, \underbrace{\{\sigma_2 > \frac{5}{8} \text{ if L }, \tau_2^* \text{ if R }\}}_{s_2}, \underbrace{\tau_3^*}_{s_3}\Big).$$

If $\sigma_2 < \frac{5}{8}$, then player 1 will always prefer to play R. Then another set of subgame perfect Nash equilibria is

$$\Big(\underbrace{R}_{s_1}, \underbrace{\{\sigma_2 < \frac{5}{8} \text{ if L }, \tau_2^* \text{ if R }\}}_{s_2}, \underbrace{\tau_3^*}_{s_3}\Big).$$

Finally if $\sigma_2 = \frac{5}{8}$, then player 1 is indifferent between her two pure strategies. Denoting σ_1 the probability player 1 plays L, a final set of subgame perfect NE is

$$\Big(\underbrace{\sigma_1 \in [0,1]}_{s_1}, \underbrace{\{\sigma_2 = \frac{5}{8} \text{ if L }, \tau_2^* \text{ if R }\}}_{s_2}, \underbrace{\tau_3^*}_{s_3}\Big).$$

We draw your attention on the fact that the following strategy profile (s_1, s_2, s_3) is a Nash Equilibrium but not a subgame perfect Nash equilibrium:

$$(s_1, s_2, s_3) = \Big(\underbrace{L}_{s_1}, \underbrace{\{\sigma_2 = 1 \text{ if L }, \tau_2 = 1 \text{ if R }\}}_{s_2}, \underbrace{\tau_3 = 1}_{s_3}\Big).$$

In effect, we have $3 = U_1(L, \sigma_2 = 1) > U_1(R, \tau_2 = 1, \tau_3 = 1) = 1$ and player 1 prefers to play L given the strategies played by players 2 and 3 in the two subgames. Given player 1 plays L, the subgame for which player 2 and 3 play a simultaneous game is never reached and those players have hence no incentive to change their strategy for that subgame. So (s_1, s_2, s_3) qualify as a Nash Equilibrium in the total game. Nevertheless, the additional requirement of the solution concept "Subgame Perfect Nash Equilibrium" is precisely that the strategy profile consists of Nash Equilibria in all subgames, even those never reached.

9Bi. Consider the following game. There are three players. There are two outcomes, a and b. All players strictly prefer a to b. At stage 1, all players vote for either a or b. If there is a unanimity in favor of one outcome, then the game stops and that is the outcome of the game. Without a unanimity, there is a second stage. At stage 2, all players vote again for either a or b. The outcome of the game is the outcome having received the largest number of votes.

(a) How many strategies does a player have?

SOLUTION. They need to choose between a and b. Which means that at each information set, they get 2 possibilities. Let's count the number of information sets: The first stage, the second stage if they voted a first stage and b gets one vote, second stage if they voted a first stage and b gets 2 votes, second stage if they voted b first stage and a get one vote, and finally second stage if they voted b and a gets two votes.

We have 5 information sets in total, 2 possibilities in each, meaning the number of pure strategies is $2^5 = 32$ for each agent.

(b) Is there a subgame perfect Nash equilibrium for which the outcome is b?

SOLUTION. Yes. If we assume all agents have utility like $u(a) > u(b) > u(0)$,we have the equilibrium

$$((a, b), (b, b), (b, b))$$

from which no one has any incentive to deviate from because it wouldn't change the outcome (b).

9Bk. Jane and Kate must share 6 Euros. They have to use the following procedure. First, Jane makes a proposal to divide the money in even integers, with both shares *strictly* positive, that is, she proposes either the division (2,4) or the division (4,2) (the first number refers to Jane's share, the second one to Kate's). In the former case, the game stops and Jane's proposal is the outcome. In the latter case, Kate either accepts the proposal, and it is the end of the game and Jane's proposal is the outcome, or Kate makes a counter-proposal. The counter-proposal is a division in odd integers of 4 Euros. In all cases following Kate's counter-proposal, Jane can either accept the counter-proposal, in which case it is the outcome of the game, or refuse it, in which case the outcome is (0,0). We assume the utility of either player is equal to the quantity of money she gets.

(a) What is the strategy set of Jane?

SOLUTION. See solutions from class.

(b) What is the strategy set of Kate?

SOLUTION. See solutions from class.

(c) Does one of the player have a weakly dominated strategy? A dominant strategy? Explain your answer.

SOLUTION. See solutions from class.

(d) Can you find an equilibrium of this game by iteratively deleting strictly dominated strategies?

SOLUTION. See solutions from class.

(e) Can you find an equilibrium of this game by iteratively deleting weakly dominated strategies? Does it depend on the order of deletion of the weakly dominated strategies?

SOLUTION. See solutions from class.

(f) Does there exist Nash equilibria in pure strategies?

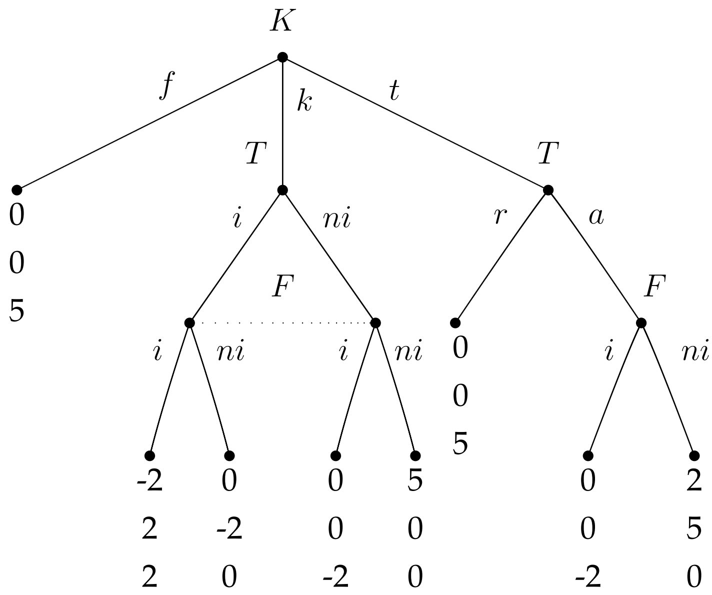
SOLUTION. See solutions from class.

(g) Briefly comment on the different equilibria you have found, if any.

SOLUTION. See solutions from class.

9Bn. A presidential election between three candidates K, T and F just took place. The rumor is that F won the election. K is the person responsible for announcing the results. K has three options: announcing that the rumor is correct ("f"), announcing that he himself won ("k") or proposing a deal to T ("t"). Because it is obviously false, announcing that he won might push T and F to (simultaneously) decide to start an insurrection ("i"). If they both start an insurrection, they seize power for sure. K may alternatively propose to T the following deal: he announces that T has won the election and T will guarantee K's peaceful retirement. If T accepts ("a"), F might start an insurrection, but he might lack enough support to be successful. If T refuses ("r"), then K announces that F has won and F becomes president.

The extended-form representation of this game is presented below.



(a) Give an example of a strategy for player F.

SOLUTION. See solutions from class.

(b) Give a weakly dominated strategy of player F.

SOLUTION. See solutions from class.

(c) Does a player have a strictly dominated strategy?

SOLUTION. See solutions from class.

(d) Find all Nash equilibria in pure strategies of the simultaneous subgame that follows move "k".

SOLUTION. See solutions from class.

(e) Is there a Subgame Perfect Nash equilibrium leading to a payoff of 5 for player K? Motivate your answer.

SOLUTION. See solutions from class.

(f) Is there a Subgame Perfect Nash equilibrium leading to a payoff of 5 for player T? Motivate your answer.

SOLUTION. See solutions from class.

(g) Is there a Subgame Perfect Nash equilibrium leading to a payoff of 5 for player F? Motivate your answer.

SOLUTION. See solutions from class.