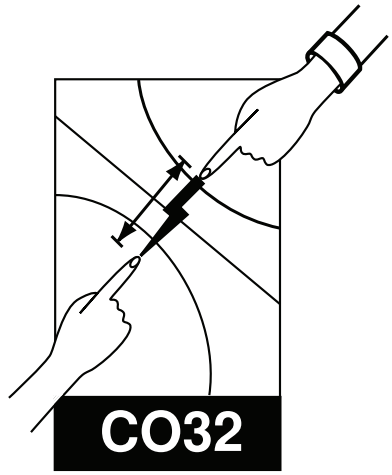
Fourier optics



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1 Introduction

Real light sources are, in practice, almost never monochromatic. They are composed of separate frequencies superimposed on each other. These frequencies can be separated out using Fourier Series, which can split a periodic function into a large number of sine and cosine waves with certain amplitudes and frequencies. The frequencies are chosen so that they are harmonics (multiple integrals) of the original periodic function.

As the period of the function gets larger the separation between frequencies in the frequency spectrum gets smaller and eventually a new continuous function can be introduced that expresses the frequency spectrum. This is known as the *Fourier Transform* and is used to describe the diffraction patterns in optics.

In this practical you will be investigating the *Discrete Fourier Transform*, a form of the Fourier Transform that is particularly suited to computational physics, and using it to investigate various diffraction patterns[1].

2 The Discrete Fourier Transform

The Fourier transform of a continuous function, $f(x)$, is written as

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x u} dx \tag{1}$$

and its inverse as

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{-2\pi i x u} du \tag{2}$$

The (1D) Fourier transform can be approximated as a sum over discrete values

$$F(u) = \frac{1}{2N} \sum_{x=-N}^{N-1} \left(f(x) e^{-\frac{\pi i x u}{N}} \right). \tag{3}$$

This sum is a form of what is known as the Discrete Fourier Transform (DFT).' If we write the function $f(x)$ in complex form $f(x) = \mathcal{R}f(x) + i\mathcal{I}f(x)$ we can write an expression for the complex transform (remembering that $e^{i\phi} = cos\phi + isin\phi$).

$$\mathcal{R}F(u) = \frac{1}{2N} \sum_{x=-N}^{N-1} \left[\mathcal{R}f(x) \cos\left(-\frac{\pi x u}{N}\right) - \mathcal{I}f(x) \sin\left(-\frac{\pi x u}{N}\right) \right] \tag{4}$$

$$\mathcal{I}F(u) = \frac{1}{2N} \sum_{x=-N}^{N-1} \left[\mathcal{R}f(x) \sin\left(-\frac{\pi x u}{N}\right) - \mathcal{I}f(x) \cos\left(-\frac{\pi x u}{N}\right) \right] \tag{5}$$

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3 Calculations

3.1 The DFT

Tackle *either* problem A or problem B. You will need to create arrays to hold the (discrete) values of the input function and the real and imaginary parts of the DFT.

For these problems, set N to 250. Note, however, that the array you will need to store will have 500 elements. Since the index of a C array always starts from zero, you will have to transform your x (or u) variable such that the value of a function at $x = -250$ will be stored in the zeroth element of the array.

Problem A

Create a single slit centred on the origin (the centre of your array) with width 10 and height 1. The array containing the imaginary parts will be zero and the array containing the real parts will be 1 for the 10 elements at either side of the centre of the array and zero otherwise.

Calculate the DFT of this single slit function and plot the real part and the amplitude of the transform. Don't forget to rearrange the array(s) containing the transform.

Move the slit so that it is centred at -10, with a height of 1. Plot the amplitude of the transform and compare it to the single slit centred on the origin. Comment on the graphs.

For the single slit centred on the origin, double the width of the slit leaving the height 1. Plot the amplitude of the transform.

Now halve the height of the last slit.

▶ Compare the amplitudes of the transforms and discuss your results with a demonstrator.

Problem B

Create 2 slits (a double slit), one centred at -15 and the other at $+15$ each of width 20 and height 1. The array containing the imaginary parts will be zero and the array containing the real parts will be 1 for the 10 elements at either side of the centre of the array and zero otherwise. Plot the amplitude of this transform. Remember to rearrange the array(s) containing your transforms.

Now move the slits to be at ± 25. Compare the amplitude of the transform to the one centred at ± 15.

Double the width of the previous slits and plot the amplitude of the transform.

Halve the height of the slits and plot the amplitude of the transform.

▶ Compare and explain the differences between the transforms of the 4 double slits you have plotted.

3.2 Convolution

The convolution of two functions $f(x)$ and $g(x)$ is given by the following integral

$$h(X) = f(x) * g(x) = \int_{-\infty}^{\infty} f(x) g(X - x)\, \mathrm{d}x \tag{6}$$

If we denote the Fourier transforms of $f(x)$, $g(x)$ and $h(x)$ by $F(u)$, $G(u)$, and $H(u)$, respectively, and if $h(x) = f(x) * g(x)$ the convolution theorem states that

$$\mathrm{FT}\{f * g\} = \mathrm{FT}\{f\} \times \mathrm{FT}\{g\} \tag{7}$$

$$H(u) = F(u) \times G(u) \tag{8}$$

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i.e. forming the convolution of two functions in real space is equivalent to taking the product of their Fourier transforms in frequency space.

Convolve a single slit (centred on the origin, width 20 and height 1) with itself.

Calculate the transform of the single slit and the convolution. Square the Fourier transform of the slit and compare it to the transform of the convolution verifying the convolution theorem.

This practical is assessed via a report (refer to AD34 — *the art of scientific report writing*). When you have written your report, take it to a demonstrator for marking when the lab is open (Monday and Tuesday 1000–1700 for all of Michaelmas Term and weeks 3–4 in Trinity Term — marking is not possible in Hilary Term).

Don't forget that you will also need to keep good records of your progress during the practical, usually by a combination of commenting your code and notes in your logbook.

Optional
1 day credit

4 The two-dimensional DFT

In 2-D (an $N \times M$ grid) the DFT is written

$$F(u,v) = \frac{1}{4NM} \sum_{x=-N}^{N-1} \sum_{y=-M}^{M-1} f(x,y) e^{-\pi i(xu/N + yv/M)} \tag{9}$$

Tackle any *two* of the following problems.

Problem A

Create a single source in the centre of a square (50×50) lattice. For simplicity create a cross shaped source i.e. $[25][25] = [25][26] = [25][24] = [24][25] = [26][25] = 1.0$. You can change the strength of the source if you like.

Plot both the real part of the source and the amplitude of the Fourier transform.

Problem B

Create 2 single (2×2 square) sources in a square (50×50) lattice. You can place the sources anywhere in the lattice you wish and set the source strength at 1.0.

Plot both the real part of the source and the amplitude of the Fourier transform.

Problem C

Create a single slit in a square (50×50) lattice. The slit should have a width of 3 and a length of 20 and should be centred at the centre of the lattice. Set the source strength at 1.0.

Plot both the real part of the source and the amplitude of the Fourier transform.

Problem D

Create a double slit in a square (50×50) lattice. Each slit should have a width of 3 and a length of 20 and should be at a distance ± 10 from the centre of the lattice. Set the source strength at 1.0.

Plot both the real part of the source and the amplitude of the Fourier transform.

References

[1] E. Hecht, *Optics*, 4th edition, Addison Wesley

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