DS203 Assignment 1 Solutions

Thomas Jacob, 190070068

22nd August 2020

Exercise 1 Solution:
P(A) = 0.8, P(B) = 0.2, P(defective |A) = 0.3, P(defective |B) = 0.1

1. The total probability of being defective is:
   P(defective) = P(defective |A) * P(A) + P(defective |B) * P(B)
   = 0.3 * 0.8 + 0.1 * 0.2
   = 0.26

2. The probability that a defective sample in the market is manufactured at company A:

$$\begin{aligned} \mathrm{P}(\mathrm{A} \mid \text{defective}) &= \frac{\mathrm{P}(\text{defective} \mid \mathrm{A}) * \mathrm{P}(\mathrm{A})}{\mathrm{P}(\text{defective})} \\ &= \frac{0.3 * 0.8}{0.26} \\ &= 0.923 \end{aligned}$$

Exercise 2 Solution:
P(working) = 0.8, P(not working) = 0.2, P(success |working) = 0.9, P(failure |working) = 0.1

1. P(first attempt fails) = P(failure |working) * P(working) + P(failure |not working) * P(not working)
   = 0.1 * 0.8 + 1 * 0.2
   = 0.28

2. $$\begin{aligned} \mathrm{P}(\text{server is working} \mid \text{first attempt fails}) &= \frac{\mathrm{P}(\text{first attempt fails} \mid \text{server is working}) * \mathrm{P}(\text{server is working})}{\mathrm{P}(\text{first attempt fails})} \\ &= \frac{0.1 * 0.8}{0.28} \\ &= 0.2857 \end{aligned}$$

3. $$\begin{aligned} \mathrm{P}(\text{second attempt fails} \mid \text{first attempt fails}) &= \frac{\mathrm{P}(\text{second attempt fails} \cap \text{first attempt fails})}{\mathrm{P}(\text{first attempt fails})} \\ &= \frac{0.1 * 0.1 * 0.8 + 1 * 1 * 0.2}{0.28} \\ &= 0.7428 \end{aligned}$$

4. $$\begin{aligned} \mathrm{P}(\text{server is working} \mid \text{first and second attempts fail}) &= \frac{\mathrm{P}(\text{first and second attempts fail} \mid \text{server works}) * \mathrm{P}(\text{server works})}{\mathrm{P}(\text{first and second attempts fail})} \\ &= \frac{0.1 * 0.1 * 0.8}{0.1 * 0.1 * 0.8 + 1 * 1 * 0.2} \\ &= 0.0385 \end{aligned}$$

Exercise 3 Solution:

1. Total possible outcomes = 36
   Favourable outcomes with at least one 6 = 11
   P(at least one 6) = 11/36 = 0.305

2. Total possible outcomes with different faces = 30
   Favourable outcomes with at least one 6 = 10

P(at least one 6) = 10/30 = 0.333

Exercise 4 Solution:

$$
\begin{aligned}
\text{P(male |color-blind)} &= \frac{\text{P(color-blind} \mid \text{male)} * \text{P(male)}}{\text{P(color-blind)}} \\
&= \frac{0.05 * 0.5}{0.05 * 0.5 + 0.01 * 0.5} \\
&= 5/6 = 0.833
\end{aligned}
$$

Exercise 5 Solution:

(a) P (E $\cap$ E) = $(\text{P(E)})^2$
P(E) = $(\text{P(E)})^2$
P(E) = 0 or P(E) = 1

(b) If A, B independent: P(A $\cap$ B) = P(A) * P(B) = 0.12
P(A $\cup$ B) = P(A) + P(B) - P(A $\cap$ B)
= 0.3 + 0.4 - 0.12 = 0.58

If A, B mutually exclusive: P(A $\cap$ B) = 0
P(A $\cup$ B) = P(A) + P(B) - P(A $\cap$ B)
= 0.3 + 0.4 = 0.7

(c) If A, B independent: P(A $\cap$ B) = P(A) * P(B) = 0.6 * 0.8 = 0.48
P(A $\cup$ B) = P(A) + P(B) - P(A $\cap$ B)
= 0.6 + 0.8 - 0.48 = 0.92 (<= 1) **Possible.**

If A, B mutually exclusive: P(A $\cap$ B) = 0
P(A $\cup$ B) = P(A) + P(B) - P(A $\cap$ B)
= 0.6 + 0.8 = 1.4 (> 1) **Not Possible.**

Exercise 6 Solution:

Property - All CDF's must be non-decreasing
3. **Valid** as the function satisfies all required properties.
P(X^2 > 5) = P(X > $\sqrt{5}$) + P(X < -$\sqrt{5}$)
= 1$-$ P(X <= $\sqrt{5}$) + P(X < -$\sqrt{5}$)
= $1 - (1 - e^{-5}/4) + e^{-5}/4 = 0.0034$

2. **Not valid** as the function dosen't satisfy the above property and has negative slope in the region $[0, 3]$

3. **Valid** as the function satisfies all required properties.
P(X^2 > 5) = P(X > $\sqrt{5}$) + P(X < -$\sqrt{5}$)
= P(X > $\sqrt{5}$) + 0
= 1 - P(X <= $\sqrt{5}$)
= 1 - (0.5 + $\sqrt{5}$/20) = 0.388

Exercise 7 Solution:

P(0) = 0.5 (from the graph and will be used later)
1. P(X <= 0.8) = $F_X(0.8)$ = 0.5

2. E(X) = 0 * P(0) + $\int_1^2 x f(x) dx$ where $f(x)$ is the PDF for the region [1, 2]

We know, $\frac{d}{dx} F_X(x) = f(x)$

In the region (1, 2), $F_X(x) = 0.5x$

Hence, $f(x) = 0.5 \; \forall x \in (1, 2)$

$$\mathrm{E(X)} = \int_1^2 0.5x dx = 0.75$$

3. $\mathrm{Var(X)} = (0 \text{ - } \mathrm{E(X)})^2\mathrm{P}(0) + \int_1^2 (x - \mathrm{E(X)})^2 f(x)dx$ where $f(x)$ is the PDF for the region [1, 2]

   Similar to the above case, $f(x) = 0.5 \; \forall x \in (1, 2)$

$$\mathrm{Var(X)} = 0.75^2 * 0.5 + \int_1^2 0.5(x - 0.75)^2 dx = 0.6042$$

Exercise 8 Solution:

For a given PDF $f(x)$, $\int_{-\infty}^{\infty} f(x)dx = 1$

Hence, $\int_0^{\infty} ce^{-2x}dx = 1$

or, c = 2.

$$\mathrm{P(X > 2)} = \int_2^{\infty} 2e^{-2x}dx = 0.0183$$

Exercise 9 Solution:

PMF values are the following:
For X = 0, $(0.3)^3 = 0.027$

For X = 1, $\binom{3}{1} * 0.7 * 0.3^2 = 0.189$

For X = 2, $\binom{3}{2} * 0.7^2 * 0.3 = 0.441$

For X = 3, $(0.7)^3 = 0.343$

Exercise 10 Solution:

$$\begin{aligned}
\mathrm{P(X >= 0.4 \mid X <= 0.8)} &= \frac{\mathrm{P(X >= 0.4 \cap X <= 0.8)}}{\mathrm{P(X <= 0.8)}} \\
&= \frac{\mathrm{P(0.4 <= X <= 0.8)}}{\mathrm{P(X <= 0.8)}} \\
&= \frac{\int_{0.4}^{0.8} 2x dx}{\int_0^{0.8} 2x dx} = 0.75
\end{aligned}$$

Exercise 11 Solution:

$$\begin{aligned}
\mathrm{P(X > (a{+}b) \mid X > a)} &= \frac{\mathrm{P(X > (a{+}b) \cap X > a)}}{\mathrm{P(X > a)}} \\
&= \frac{\mathrm{P(X > (a{+}b))}}{\mathrm{P(X > a)}} \\
&= \frac{\int_{(\mathrm{a+b})}^{\infty} \lambda e^{-\lambda x}dx}{\int_{\mathrm{a}}^{\infty} \lambda e^{-\lambda x}dx} = e^{-\lambda b}
\end{aligned}$$

Exercise 12 Solution:

$\mathrm{I_E} = 1$ only if event E occurs, that is, five heads (HHHHH)
Total possible outcomes = $2^5 = 32$
Favourable outcomes = 1

$P(I_E = 1) = 1/32 = 0.03125$

Exercise 13 Solution:

Since F(b) is constant before $b = 0$, has a sudden jump at 0 and remains constant till just before $b = 1$, PMF at $b = 0$ is 1/2. Similarly, since F(b) has a sudden jump at $b = 1$ and is constant thereafter, PMF at $b = 1$ is 1/2. Since these values sum upto 1, we are done.

Exercise 14 Solution:

The probability of drawing either a white or a black ball is the same and equal to 1/2 (= 3/6).
P(two white out of first four drawn) = Ways of choosing two out of four balls * P(two white) * P(two black)

$$= \binom{4}{2} * 0.5^4 = 0.375$$

Exercise 15 Solution:

Since we have a total number of n flips and a total of r heads appear in the n flips with the rth head coming in the nth flip, this means that there were $r - 1$ heads in the remaining $n - 1$ flips.

P($r - 1$ heads in n-1 flips) = Ways of choosing r-1 out of n-1 * P(r-1 heads) * P($n - r$ tails)

$$= \binom{n-1}{r-1} p^{r-1}(1-p)^{n-r}$$

P(r heads in n flips with final flip head) = P($r - 1$ heads in $n - 1$ flips) * P(head)

$$= \binom{n-1}{r-1} p^{r-1}(1-p)^{n-r} p$$
$$= \binom{n-1}{r-1} p^{r}(1-p)^{n-r}$$

Exercise 16 Solution:

The formula for a typical Poisson distribution is: $P(X = i) = \dfrac{e^{-\lambda}\lambda^i}{i!}, i = 0, 1, 2, 3...$

For $\lambda = 1$, $P(X = i) = \dfrac{e^{-1}}{i!}$

$$P(i >= 1) = \frac{e^{-1}}{1!} + \frac{e^{-1}}{2!} + \frac{e^{-1}}{3!} + ...$$

$$= e^{-1}(e - 1)$$

$$= 0.632$$

XXXXXX