Θ^2_{lm} in the Hydrogen atom

SphericalHarmonicY[l, m, θ, ϕ] gives the spherical harmonic $Y_l^m(\theta, \phi)$. It is the angular solution found in the Hydrogen atom.

The spherical harmonics are orthonormal with respect to integration over the surface of the unit sphere.

For $l \geq 0$, $Y_l^m(\theta, \phi) = \sqrt{(2\,l+1)/(4\,\pi)}\ \sqrt{(l-m)!/(l+m)!}\ P_l^m(\cos(\theta))\ e^{im\phi}$ where P_l^m is the associated Legendre function.

For $l \leq -1$, $Y_l^m(\theta, \phi) = Y_{-(l+1)}^m(\theta, \phi)$

Examples: Spherical Harmonics

```
Y00 = SphericalHarmonicY[0, 0, θ, ϕ];
Y10 = SphericalHarmonicY[1, 0, θ, ϕ];
Y11 = SphericalHarmonicY[1, 1, θ, ϕ]
Y20 = SphericalHarmonicY[2, 0, θ, ϕ];
Y21 = SphericalHarmonicY[2, 1, θ, ϕ];
Y22 = SphericalHarmonicY[2, 2, θ, ϕ];
Y30 = SphericalHarmonicY[3, 0, θ, ϕ];
Y31 = SphericalHarmonicY[3, 1, θ, ϕ];
Y32 = SphericalHarmonicY[3, 2, θ, ϕ];
Y33 = SphericalHarmonicY[3, 3, θ, ϕ];
```

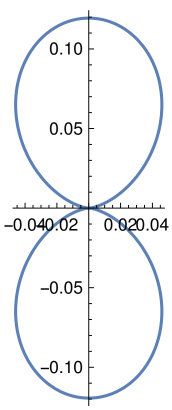
$$-\frac{1}{2}\, e^{i\,\phi}\, \sqrt{\frac{3}{2\,\pi}}\ \mathrm{Sin}[\theta]$$

The usual $\Theta_{lm}^2 = Y_{lm} * Y_{lm}^*$

```
Θ00 = Simplify[Y00 * Conjugate[Y00], Element[{θ, ϕ}, Reals]];
Θ10 = Simplify[Y10 * Conjugate[Y10], Element[{θ, ϕ}, Reals]];
Θ11 = Simplify[Y11 * Conjugate[Y11], Element[{θ, ϕ}, Reals]]
Θ20 = Simplify[Y20 * Conjugate[Y20], Element[{θ, ϕ}, Reals]];
Θ21 = Simplify[Y21 * Conjugate[Y21], Element[{θ, ϕ}, Reals]];
Θ22 = Simplify[Y22 * Conjugate[Y22], Element[{θ, ϕ}, Reals]];
Θ30 = Simplify[Y30 * Conjugate[Y30], Element[{θ, ϕ}, Reals]];
Θ31 = Simplify[Y31 * Conjugate[Y31], Element[{θ, ϕ}, Reals]];
Θ32 = Simplify[Y32 * Conjugate[Y32], Element[{θ, ϕ}, Reals]];
Θ33 = Simplify[Y33 * Conjugate[Y33], Element[{θ, ϕ}, Reals]];
```

$$\frac{3\,\mathrm{Sin}[\theta]^2}{8\,\pi}$$

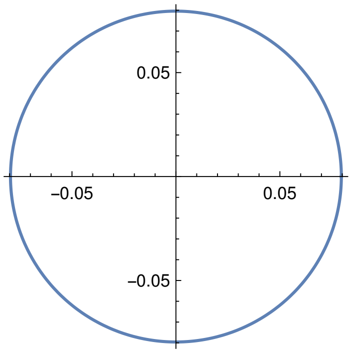
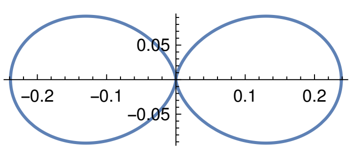
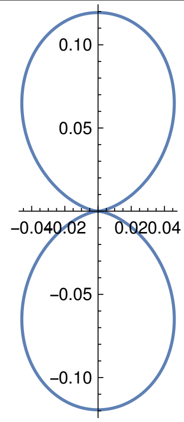
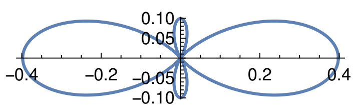
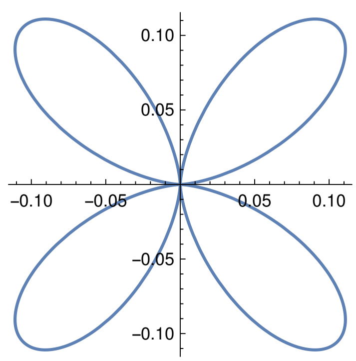
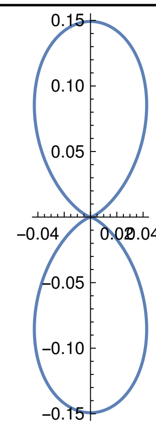
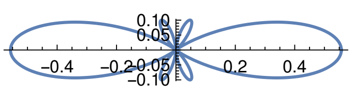
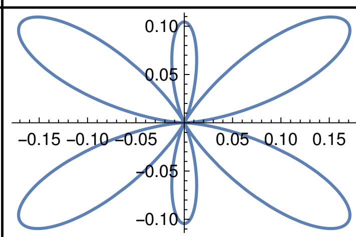
```
PolarPlot[Θ11, {θ, 0, 2 Pi}]
```




```
Grid[{{"l", Θ00, , ,}, {"m", Θ10, Θ11, ,},
  {, Θ20, Θ21, Θ22,}, {, Θ30, Θ31, Θ32, Θ33}}, Frame → All]
```

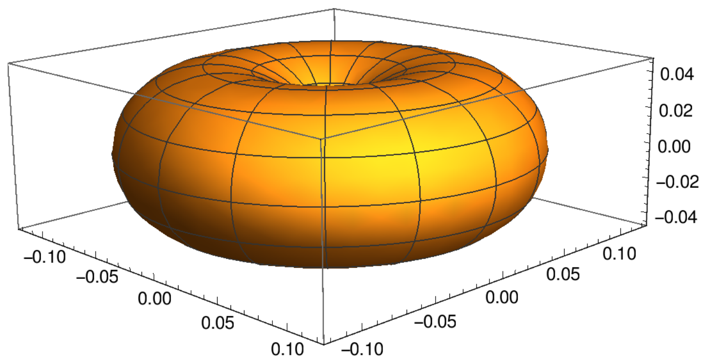
l	$\frac{1}{4\pi}$			
m	$\frac{3\,\mathrm{Cos}[\theta]^2}{4\pi}$	$\frac{3\,\mathrm{Sin}[\theta]^2}{8\pi}$		
	$\frac{5\,(1+3\,\mathrm{Cos}[2\theta])^2}{64\pi}$	$\frac{15\,\mathrm{Sin}[2\theta]^2}{32\pi}$	$\frac{15\,\mathrm{Sin}[\theta]^4}{32\pi}$	
	$\frac{7\,\mathrm{Cos}[\theta]^2\,(1-5\,\mathrm{Cos}[2\theta])^2}{64\pi}$	$\frac{21\,(3+5\,\mathrm{Cos}[2\theta])^2\,\mathrm{Sin}[\theta]^2}{256\pi}$	$\frac{105\,\mathrm{Cos}[\theta]^2\,\mathrm{Sin}[\theta]^4}{32\pi}$	$\frac{35\,\mathrm{Sin}[\theta]^6}{64\pi}$

```
Grid[{{"l", PolarPlot[Θ00, {θ, 0, 2 Pi}], , ,},
  {"m", PolarPlot[Θ10, {θ, 0, 2 Pi}], PolarPlot[Θ11, {θ, 0, 2 Pi}], ,},
  {, PolarPlot[Θ20, {θ, 0, 2 Pi}], PolarPlot[Θ21, {θ, 0, 2 Pi}],
   PolarPlot[Θ22, {θ, 0, 2 Pi}],},
  {, PolarPlot[Θ30, {θ, 0, 2 Pi}], PolarPlot[Θ31, {θ, 0, 2 Pi}], Θ32, Θ33}}, Frame → All]
```

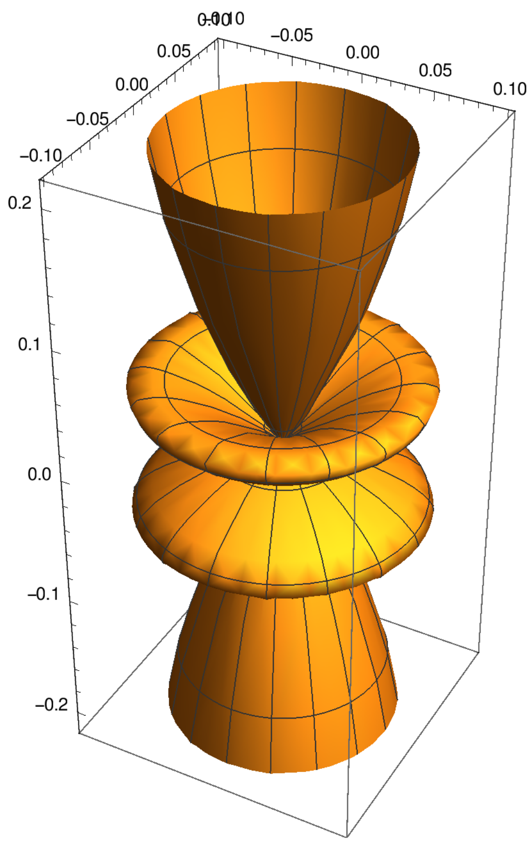
l				
m				
				
			$\frac{105\,\mathrm{Cos}[\theta]^2\,\mathrm{Sin}[\theta]^4}{32\,\pi}$	$\frac{35\,\mathrm{Sin}[\theta]^6}{64\,\pi}$

Polar 3 D

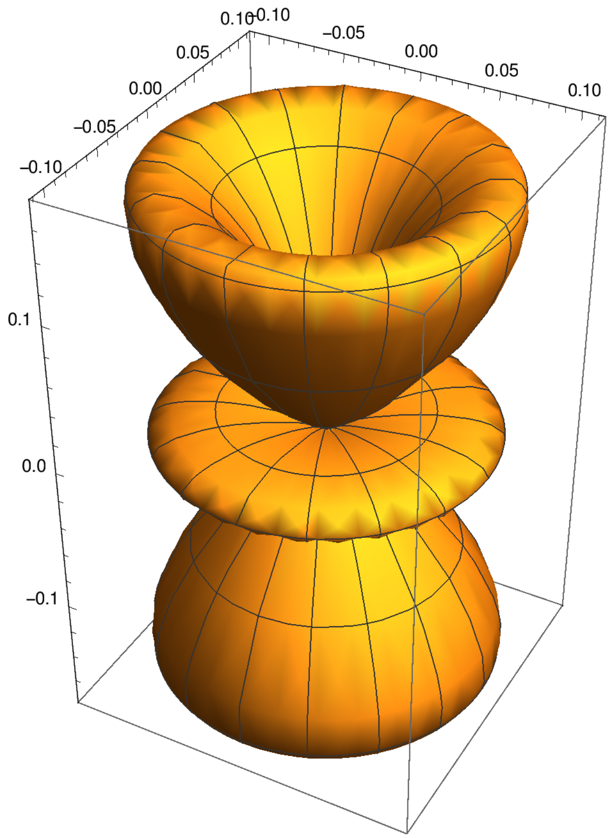
```
SphericalPlot3D[Θ11, {θ, 0, Pi}, {ϕ, 0, 2 Pi}]
```



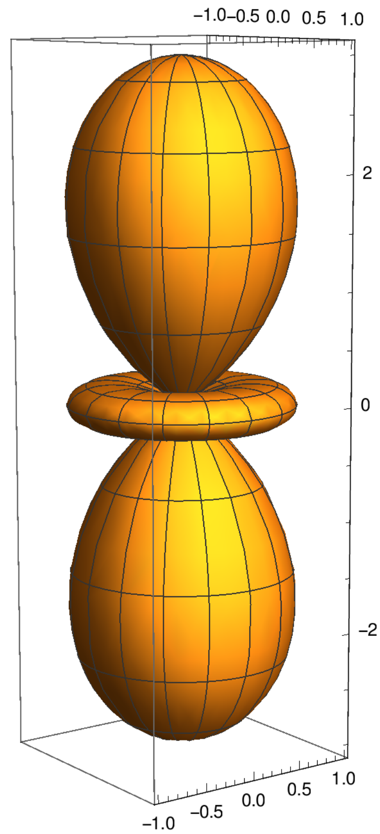

```
SphericalPlot3D[Θ30, {θ, 0, Pi}, {ϕ, 0, 2 Pi}]
```



```
SphericalPlot3D[Θ31, {θ, 0, Pi}, {ϕ, 0, 2 Pi}]
```



```
SphericalPlot3D[1 + 2 Cos[2 θ], {θ, 0, Pi}, {ϕ, 0, 2 Pi}]
```



Plot an eigenfunction to the Laplace equation in spherical coordinates :

```
GraphicsGrid@Table[
  SphericalPlot3D[Evaluate@Abs@SphericalHarmonicY[l, m, θ, ϕ], {θ, 0, Pi}, {ϕ, 0, 2 Pi},
   PlotRange → 0.6, Mesh → None, Boxed → False, Axes → None], {l, 0, 3}, {m, 0, l}]
```