A WEAK KAM SOLUTION

FOR THE OVERDAMPED LANGEVIN DYNAMICS HJE

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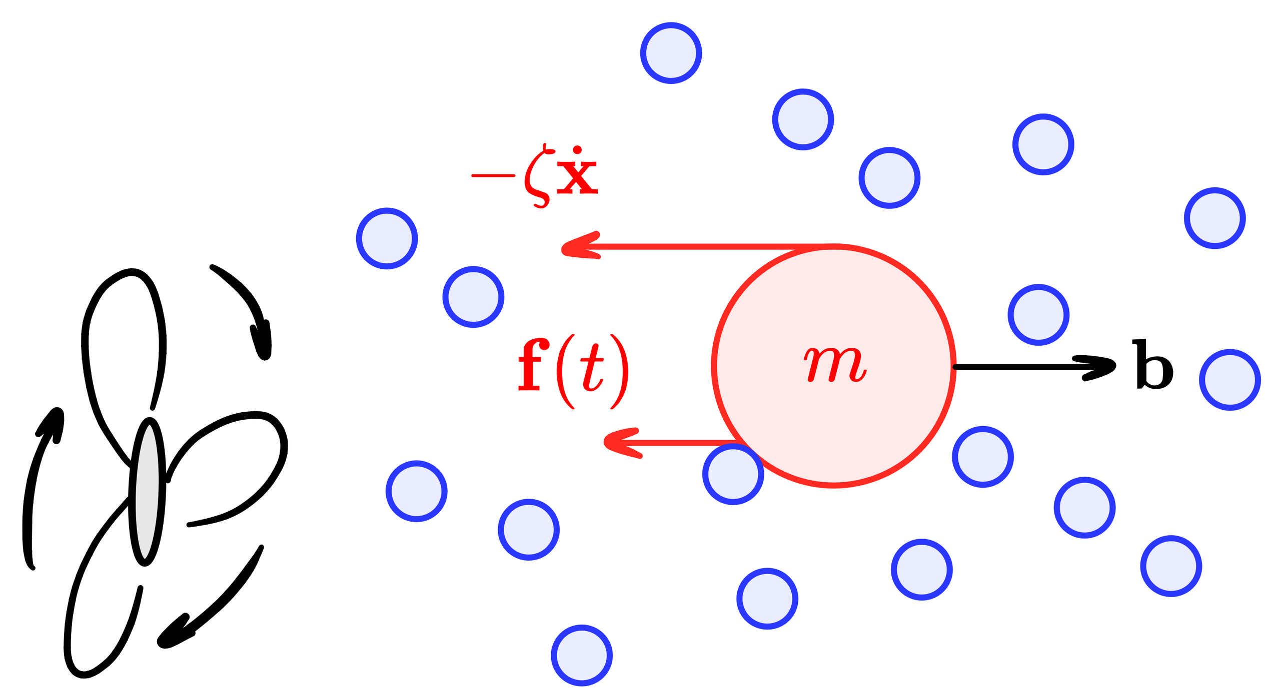
OVERVIEW

We use weak KAM theory to solve the Hamilton–Jacobi equation arising from overdamped Langevin dynamics on the one-dimensional torus, relaxing assumptions on the external field. We compute the Mañé critical value, Peierls barrier, and Aubry set explicitly, and show that the dynamic solution corresponds to the global energy landscape as an invariant of the Lax–Oleinik semigroup.



What about this HJE?

The objective of this article is to solve the HJE:

$$\partial_t \psi + D\psi \cdot (D\psi + \mathbf{b}(\mathbf{x})) = 0.$$

This PDE models the motion of a colloidal particle in a fluid and arises from Overdamped Langevin Dynamics in the low temperature limit.

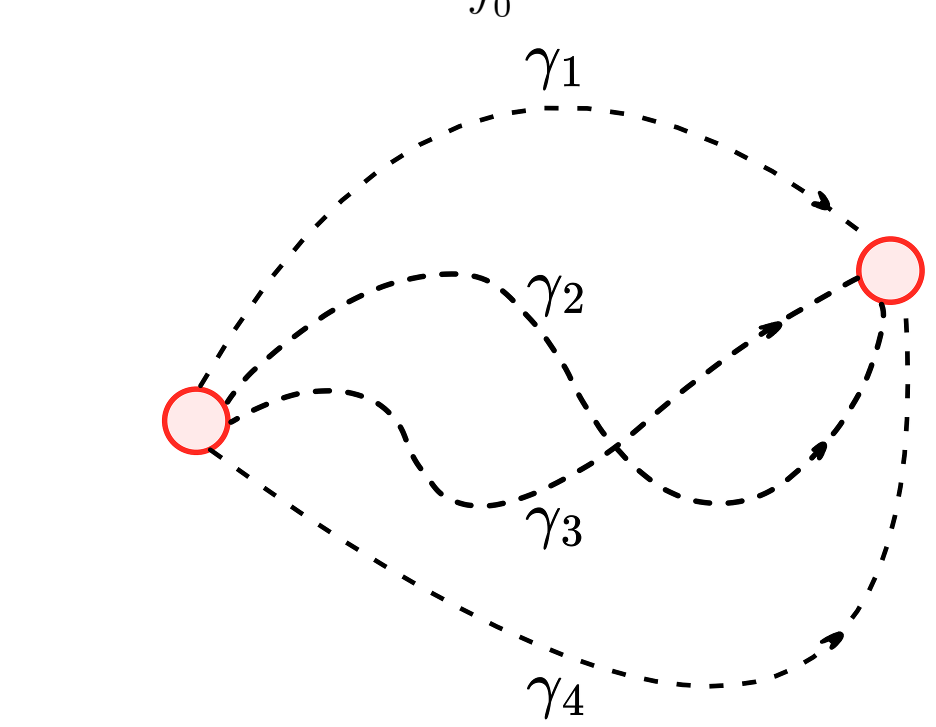
The HJE is a reformulation of classical mechanics [1] replacing Newton's second-order equations with a first-order nonlinear PDE. It is typically expressed in terms of the action function as the unknown. Although the equation is simpler in form, it is nonlinear and classical solutions usually do not exist [2, 3].

Weak KAM theory is a variational framework introduced by Fathi [4] that generalizes the viscosity solution approach to solve this kinds of equations.

Weak KAM theory is built around 4 vital concepts. Given those four explicitly, one can build a general solution for the HJE.

ACTION

$$A_T(\gamma) := \int_0^T L(\gamma(t), \dot{\gamma}(t))\, \mathrm{d}t.$$

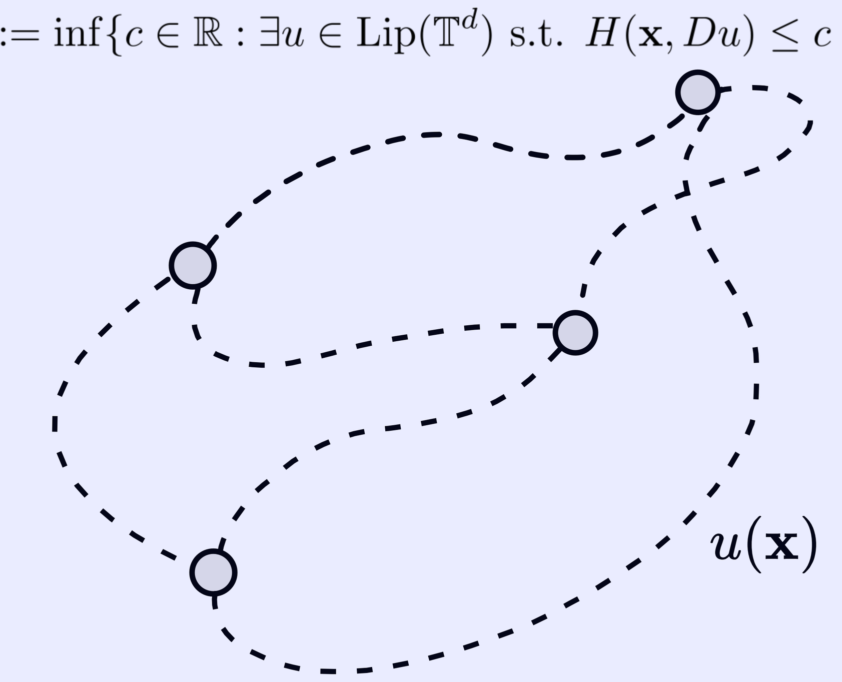


Is the cost of traversing a path between two points.

MAÑÉ CRITICAL VALUE

$\mathbb{T}^d$

$$c^* := \inf\{c \in \mathbb{R} : \exists u \in \mathrm{Lip}(\mathbb{T}^d) \text{ s.t. } H(\mathbf{x}, Du) \le c \text{ a.e.}\}.$$

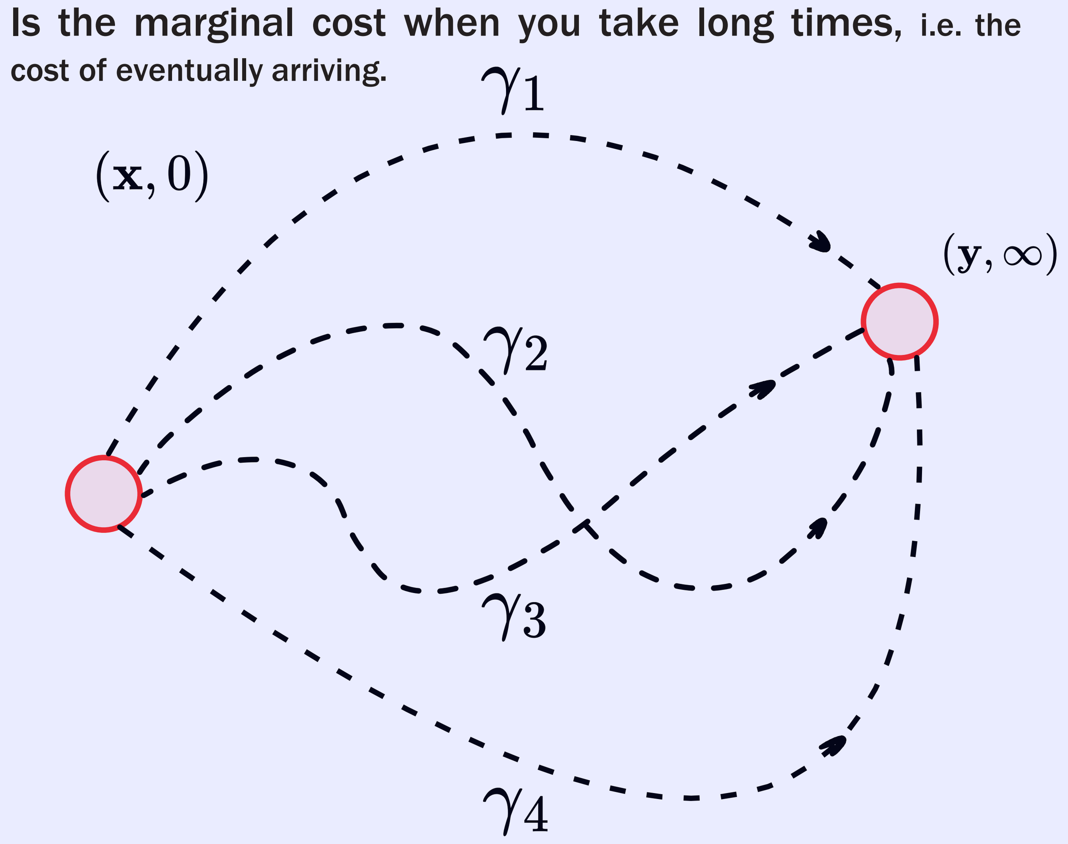


Is the minimal amount of "gas" you should have in the tank to visit all the points.

PEIERLS BARRIER

$\mathbb{T}^d$

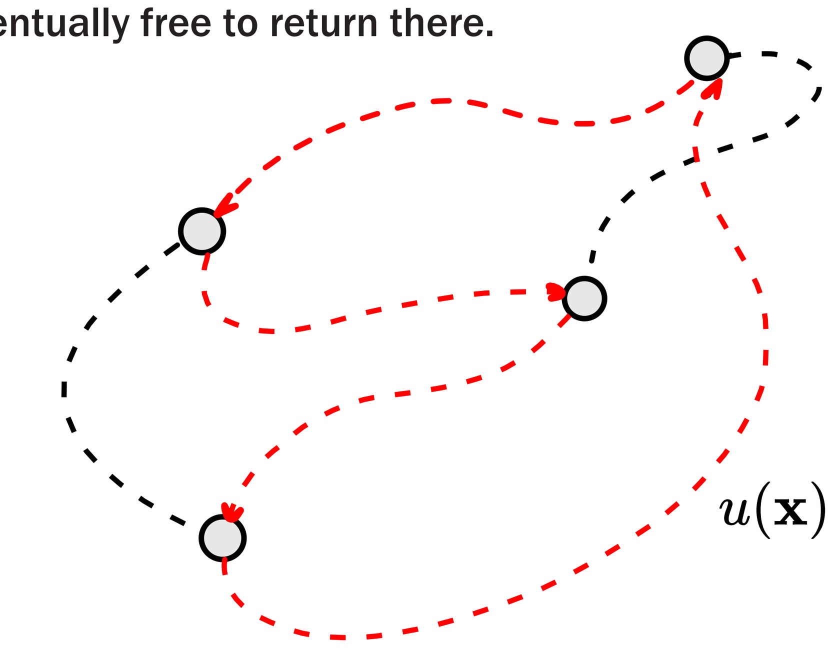
Is the marginal cost when you take long times, i.e. the cost of eventually arriving.



$$h(\mathbf{y};\mathbf{x}) := \liminf_{T\to+\infty} \inf_{\substack{\gamma(0)=\mathbf{x}\\ \gamma(T)=\mathbf{y}}} \int_0^T L(\gamma(t), \dot{\gamma}(t))\, \mathrm{d}t + c^* T.$$

AUBRY SET

Is the set of all recursive points. Meaning that it is eventually free to return there.



$$\mathcal{A} := \{\mathbf{x} \in \mathbb{T}^d : h(\mathbf{x}; \mathbf{x}) = 0\}.$$

SETTING AND HYPOTHESIS

What we were studying was, how general could b be such that we could construct the general HJE solution?

H1 (general setting): Let **b** be a continuous vector field defined on the d-dimensional torus. The field is not necessarily conservative.

H2 (conservative case): Assume **b** is a conservative vector field. This means there exists a continuously differentiable function U with finite critical points on the torus such that **b** is equal to minus the gradient of U.

H3 (1D case): Assume further that $d = 1$, and that there are no saddle points of U.

In the literature [5] shows the explicit solution under H3. [6] shows an application of Weak KAM method in chemical reactions, and [7, 8, 9] show the four concepts for other hamiltonians. No one has tried to use only H1 and H2 for this dynamics.

RESULTS

1. Mañé critical value was found to be zero under H1.

2. The Peierls barrier was the most complicated one. The results of [5] under H3 were replicated.

3. The Aubry set was found to be exactly the set of critical points of U (i.e. the roots of **b**) under H2.

4. Standard procedure as in [5] and [7] was used to construct the variational solution using the Lax-Oelinik semigroup.

REFERENCES AND EXTRAS

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