A Weak KAM Solution for the Overdamped Langevin Dynamics HJE

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In this work, we present a rigorous and fully explicit variational construction of weak KAM solutions for the Hamilton–Jacobi equation (HJE) associated with overdamped Langevin dynamics on the one-dimensional torus. We begin by deriving the HJE corresponding to this specific dynamics. Then, assuming an external drift field $\mathbf{b}(x)$ with at least one root, we prove that the Mañé critical value is $c^* = 0$. Furthermore, we explicitly compute the Peierls barrier, the Aubry set, and the global energy landscape in the one-dimensional conservative case, and we discuss extensions to higher dimensions. Finally, we show that the dynamic solution to the proposed HJE is given by the global energy landscape W, as it is an invariant solution of the Lax–Oleinik semigroup.

Overdamped Langevin Dynamics | Weak KAM Theory | Global Energy Landscape

1. Introduction

Overdamped Langevin dynamics* (see Eq. (2.1)) describe the motion of a colloidal particle in a fluid subject to an external conservative force $\mathbf{b}(\mathbf{x}) = -DU(\mathbf{x})$, where $\mathbf{x} = \mathbf{x}(t) \in \mathbb{R}^d$ or $\mathbb{T}^d$ † is the position vector of the particle, and D denotes the gradient operator. In the thermodynamic limit $T \to 0$, this stochastic dynamics leads to a deterministic Hamilton–Jacobi equation (HJE), which captures the most probable path under rare fluctuations.

The HJE is a reformulation of classical mechanics [10], replacing Newton's second-order equations with a first-order nonlinear PDE. It is typically expressed in terms of the action function‡ as the unknown. Although the equation is simpler in form, it is nonlinear and classical solutions usually do not exist [2, 14].

Let $\Omega \subset \mathbb{R}^d$ be an open set. The Cauchy problem for a general HJE is given by [2]:

$$\begin{cases} u_t + H(\mathbf{x}, t, u, Du) = 0 & \text{in } \Omega \times (0, \mathcal{T}], \\ u = z & \text{on } \partial\Omega \times (0, \mathcal{T}], \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) & \text{on } \Omega, \end{cases} \tag{1.1}$$

where $H : \Omega \times [0, \mathcal{T}] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is the Hamiltonian, u is the unknown function, and z, u_0 are the boundary and initial data, t is the time coordinate and $\mathcal{T}$ is the stopping time. Since classical solutions to Eq. (1.1) often fail to exist even when H is smooth, the theory of viscosity solutions is used. This framework, introduced by Evans [3] and developed by Crandall and Lions [2], ensures well-posedness of the problem under general conditions.

To study the global properties of solutions to the stationary HJE, we adopt *Weak KAM theory*, a variational framework introduced by Fathi [5] that generalizes the viscosity solution approach to analyze long-time behavior, minimizing orbits, and critical energy levels. This theory introduces key concepts such as the Mañé critical value, the Peierls barrier, and the projected Aubry set, which encode information about the invariant sets and energy landscape of the system. These concepts allow us to rigorously define solutions even when classical methods fail due to non-smoothness. Some classical texts or notes on the subject include [15, 4, 5, 14].

In this work, we construct an weak KAM solution to a stationary HJE arising from overdamped Langevin dynamics. We focus on the case where the drift field $\mathbf{b}(\mathbf{x})$ satisfies specific structural assumptions (see Section 2C).

The outline of the article is as follows. We begin in Section 2 by deriving the HJE from Langevin dynamics and introducing the relevant weak KAM structures. Section 2C in particular states our assumptions. In Section 3, we outline our weak

*Also called Brownian dynamics or Langevin dynamics with drift / advection / convective / velocity / transport term.

†The use of $\mathbb{T}^d$ throughout the article es customary in weak KAM theory due to the compactness of the space.

‡Also called Hamilton's principal function

Significance Statement

The stationary Hamilton–Jacobi equation (HJE) derived from overdamped Langevin dynamics presents a fundamental challenge: the absence of classical solutions (of class C^1) [2, 14].
This paper as an extension of [7] addresses that obstacle by constructing and analyzing weak KAM solutions, a framework that extends the reach of PDE analysis into the realm of nonsmooth dynamics. Weak KAM theory provides not only existence results, but also tools to describe physically meaningful structures such as invariant sets, action-minimizing trajectories, and critical energy levels, features that are not used in classical approaches.
Beyond its mathematical interest, this approach has various applications in diverse fields. In the context of chemical reactions, for example, Hamiltonians derived from the large deviation principle characterize most probable paths and transition states [8]. Similarly, in control theory and stochastic optimization, the techniques developed here offer new ways to understand long-term behavior and stability [9]. Thus, this project contributes both to the theoretical foundations and the applied utility of modern dynamical systems.

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KAM construction method. Section 4 contains the main results, including explicit formulas for the Peierls barrier, Aubry set, and the weak KAM solutions. We close with conclusions in Section 5.

2. Mathematical Foundations

A. Overdamped Langevin Dynamics in $\mathbb{T}^d$ and the HJE. Consider a spherical colloidal particle immersed in a viscous fluid and subject to an external conservative force field $\mathbf{b}(\mathbf{x})$, where $\mathbf{x} = \mathbf{x}(t) \in \mathbb{T}^d$ denotes the position vector of the particle.

The motion of such a particle, accounting for friction and random thermal forces, is governed by the Langevin equation [13]:

$$\mathbf{b}(\mathbf{x}) - \zeta\dot{\mathbf{x}} + \mathbf{f}(t) = m\ddot{\mathbf{x}}, \tag{2.1}$$

where $\mathbf{f}(t)$ is the stochastic Langevin force representing thermal noise, ζ is the friction coefficient (assumed constant), and m is the mass of the particle.

In the overdamped limit, where inertial effects are negligible compared to friction ($m \to 0$), and using the fluctuation–dissipation relation [16] together with the equipartition theorem $\langle E\rangle = \frac{d}{2}k_B T$ [11], Eq. (2.1) reduces to the stochastic differential equation (SDE):

$$\dot{\mathbf{x}} = \frac{1}{\zeta}\mathbf{b}(\mathbf{x}) + \sqrt{\frac{2k_BT}{\zeta}}\,\mathbf{B}_t, \tag{2.2}$$

where k_B is Boltzmann's constant, T is the temperature, and where $\mathbf{B}_t$ denotes standard Brownian motion in $\mathbb{T}^d$. Defining the diffusion coefficient as $\varepsilon := k_BT/\zeta$, the SDE becomes[§]:

$$d\mathbf{x} = \mathbf{b}(\mathbf{x})\,dt + \sqrt{2\varepsilon}\,d\mathbf{B}. \tag{2.3}$$

The probability density function $\rho(\mathbf{x}, t)$ associated with the solution of this SDE satisfies the Fokker–Planck equation [12, 16]:

$$\partial_t\rho = -D\cdot(\rho\,\mathbf{b}) + \varepsilon D^2\rho. \tag{2.4}$$

Notice that in one dimension, this simplifies to the ODE in [7]:

$$\partial_t\rho = -(\rho b)' + \varepsilon\rho''. \tag{2.5}$$

Now, following a WKB-type ansatz, we look for solutions of the form:

$$\rho_\varepsilon(\mathbf{x}, t) = e^{-\frac{\psi_\varepsilon(\mathbf{x},t)}{\varepsilon}}, \tag{2.6}$$

where $\psi_\varepsilon : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ is a smooth function and ε is considered a free parameter.

Substituting Eq. (2.6) into the Fokker–Planck equation and taking the limit as $\varepsilon \to 0$, we obtain the following HJE[¶]:

$$\partial_t\psi + D\psi\cdot(D\psi + \mathbf{b}(\mathbf{x})) = 0. \tag{2.7}$$

This nonlinear first-order PDE captures the dominant contribution to the particle distribution in the small-noise limit ($T \to 0$), corresponding to the most probable paths or large deviation minimizers.

§Notice the $\mathbf{b}$ in this equation does not have force units anymore. Let us assume natural units.

¶We use u or ψ for this unknown function, following Eq. (1.1)

A.1. The Hamiltonian. By comparing Eq. (2.7) with the general Hamilton–Jacobi formulation Eq. (1.1), we identify the Hamiltonian as:

$$H(\mathbf{x}, \mathbf{p}) = \mathbf{p}\cdot(\mathbf{p} + \mathbf{b}(\mathbf{x})) = |\mathbf{p}|^2 + \mathbf{p}\cdot\mathbf{b}(\mathbf{x}), \tag{2.8}$$

where $\mathbf{p} = D\psi$ represents the momentum variable in phase space.

This Hamiltonian consists of a quadratic kinetic term $|\mathbf{p}|^2$ and a linear coupling with the drift field $\mathbf{b}(\mathbf{x})$, representing the influence of the external potential. Also notice it is convex in $\mathbf{p} \in \mathbb{R}^d$, and that it is of class $C^m(\mathbb{T}^d \times \mathbb{R}^d)$ only if $\mathbf{b}$ is continuous up to m derivatives.

A.2. The Lagrangian. In our context, the Lagrangian is defined as the convex conjugate [15] of the Hamiltonian. That is,

$$\begin{aligned} L(\mathbf{x}, \mathbf{v}) &:= \sup_{\mathbf{p}\in\mathbb{R}^d} (\mathbf{v}\cdot\mathbf{p} - H(\mathbf{x}, \mathbf{p})) \\ &= \sup_{\mathbf{p}\in\mathbb{R}^d} (\mathbf{v}\cdot\mathbf{p} - \mathbf{p}\cdot(\mathbf{p} + \mathbf{b})). \end{aligned} \tag{2.9}$$

The expression to be maximized is a concave quadratic function of $\mathbf{p}$ (a paraboloid opening downwards). Hence, the set in Eq. (2.9) is clearly bounded above, and its supremum is really its maximum. Therefore it is easy to verify that:

$$L(\mathbf{x}, \mathbf{v}) = \frac{1}{4}||\mathbf{v} - \mathbf{b}(\mathbf{x})||^2 \tag{2.10}$$

This Lagrangian inherits several important structural properties. The function is convex with respect to $\mathbf{v}$, ensuring the existence of a unique minimizer at each point. It is also coercive in $\mathbf{v}$, in the sense that $L(\mathbf{x}, \mathbf{v}) \to \infty$ as $\|\mathbf{v}\| \to \infty$, and remains invariant under affine transformations of the drift. Also it is as smooth as the drift field, it is non-negative and $L(\mathbf{x}, \mathbf{v}) = 0 \iff \mathbf{v} = \mathbf{b}(\mathbf{x})$.

The strict convexity, coercivity, and superlinearity qualify L as a Tonelli Lagrangian, ensuring the well-posedness of associated variational problems central to weak KAM theory [5].

B. Weak KAM Concepts. Weak KAM theory provides a variational framework to study the long-time behavior of solutions to Hamilton–Jacobi equations. At its core, the theory revolves around minimizing the action functional, which encodes the total cost of a trajectory over time.

Definition 2.1. *Let $L : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a Tonelli Lagrangian. The **action** of a curve $\gamma : [0, T] \to \mathbb{T}^d$ is defined as:*

$$A_T(\gamma) := \int_0^T L(\gamma(t), \dot\gamma(t))\,\mathrm{d}t. \tag{2.11}$$

The central problem in weak KAM theory is to understand which curves minimize this action, especially over long time intervals. This leads to the study of critical energy levels, recurrent behaviors, and the geometry of minimizers. Three key concepts in this theory are the Mañé critical value, the Peierls barrier, and the Aubry set.

Definition 2.2. *The **Mañé critical value** is defined as [1]:*

$$\begin{aligned} c^* := \sup\{c \in \mathbb{R} : \exists \text{ closed curve } \mathbf{x}(\cdot) \text{ s.t.} \\ \int_0^T [L(\mathbf{x}(t), \dot{\mathbf{x}}(t)) + c]\ \mathrm{d}t < 0\}. \end{aligned} \tag{2.12}$$

Equivalently, it can be characterized by subsolutions to the stationary HJE [5]:

$$c^* := \inf\{c \in \mathbb{R} : \exists u \in \mathrm{Lip}(\mathbb{T}^d) \ s.t. \ H(\mathbf{x}, Du) \le c \ a.e.\}. \tag{2.13}$$

The Mañé critical value is interpreted as the minimum energy needed for a particle to traverse any action minimizing path.

Definition 2.3. *The* ***Peierls barrier*** *function measures the asymptotic minimal cost of transitioning from a point* $\mathbf{x}$ *to a point* $\mathbf{y}$*:*

$$h(\mathbf{y};\mathbf{x}) := \liminf_{T\to+\infty} \inf_{\substack{\gamma(0)=\mathbf{x}\\ \gamma(T)=\mathbf{y}}} \int_0^T L(\gamma(t),\dot\gamma(t))\,\mathrm{d}t + c^*T. \tag{2.14}$$

where the infimum is taken over all the absolute continous curves γ *that start in* $\mathbf{x}$ *and end in* $\mathbf{y} \in \mathbb{T}^d$*.*

Definition 2.4. *The* ***Aubry set*** *is the set of energetically recurrent points:*

$$\boldsymbol{\mathcal{A}} := \{\mathbf{x} \in \mathbb{T}^d : h(\mathbf{x};\mathbf{x}) = 0\}. \tag{2.15}$$

This is in particular, an invariant set [15].

C. Setting. In this subsection we specify the different sets of hypotheses under which our results will be derived. Each result in this work will be explicitly associated with the hypotheses it depends on, indicating the level of generality or restriction required for its validity.

Hypothesis 2.5 (General)**.** *Let* $\mathbf{b} : \mathbb{T}^d \to \mathbb{R}^d$ *be a continuous vector field, not necessarily conservative. We assume the dimension* $d \in \mathbb{N}$*, and that* $\mathbf{b}$ *has at least one zero (i.e., there exists* $\mathbf{x} \in \mathbb{T}^d$ *such that* $\mathbf{b}(\mathbf{x}) = 0$*).*

Hypothesis 2.6 (Conservative field)**.** *Assume* $\mathbf{b}$ *is a conservative field. That is, there exists a function* $U \in C^1(\mathbb{T}^d)$ *such that* $\mathbf{b} = -DU$*. Furthermore, we assume* U *is* $\mathbb{Z}^d$ *skew-periodic and that it has finite critical points.*

Hypothesis 2.7 (One-dimensional conservative case)**.** *Throughout most of this work, we focus on the case* $d = 1$*. Assume* $U \in C^2(\mathbb{T}^1)$[‖] *has a finite number of critical points indexed as follows. Let* $x_1, x_2, \dots, x_k$ *be the stable local minima, interleaved with unstable local maxima* $x_{\frac12}, x_{1+\frac12}, \dots, x_{k+\frac12}$*, satisfying without loss of generality*

$$0 = x_{\frac12} < x_1 < x_{1+\frac12} < x_2 < \dots < x_k < x_{k+\frac12} = 1. \tag{2.16}$$

We denote periodic copies by $x_{i+\ell k} := x_i + \ell \in \mathbb{R}$ *for any* $\ell \in \mathbb{Z}$[**]*. This configuration defines a skew-periodic structure over* $\mathbb{R}$ *consistent with a periodic potential on* $\mathbb{T}^1$*.*

3. Methodology

Our approach builds on the variational framework of weak KAM theory, particularly extending the ideas developed in [7]. While that work operates largely under the restrictive setting of Hypothesis 2.7, our objective is to relax these assumptions and broaden the applicability of the theory. The main steps of our methodology are as follows:

1. Computing the Mañé critical value c^*.
2. Constructing the Peierls barrier $h(y;x)$.
3. Identifying the projected Aubry set $\mathcal{A}$.
4. Defining the global energy landscape $W(x)$ verifying that it is a weak KAM solution for the stationary version of Eq. (2.7) [7],
$$H(\mathbf{x},\mathbf{p}) = c^*. \tag{3.1}$$
5. Translating from the solution of the stationary HJE to a solution of the dynamic HJE (Eq. (2.7)) using the Lax-Oleinik semigroup.

4. Results

A. Mañé Critical Value. Let us begin with the most general result.

Theorem 4.1. *Under Hypothesis 2.5, the Mañé critical value satisfies* $c^* = 0$*.*

Proof. Since the Lagrangian is non-negative,

$$L(\mathbf{x},\mathbf{v}) = \frac14\|\mathbf{v} - \mathbf{b}(\mathbf{x})\|^2 \ge 0, \tag{4.1}$$

then for any absolutely continuous curve $\mathbf{x}(t)$,

$$\int_0^T (L(\mathbf{x}(t),\dot{\mathbf{x}}(t)) + c)\,\mathrm{d}t \ge cT. \tag{4.2}$$

Thus, no $c > 0$ can satisfy the required inequality in Eq. (2.12)[††], so

$$c^* \le 0. \tag{4.3}$$

Now assume, for contradiction, that $c^* < 0$. Since $\mathbf{b}$ has a root by Hypothesis 2.5, let $\mathbf{x}_i \in \mathbb{T}^d$ such that $\mathbf{b}(\mathbf{x}_i) = 0$. Consider the constant path $\mathbf{x}(t) \equiv \mathbf{x}_i$, for which $\dot{\mathbf{x}}(t) = 0$, and therefore

$$L(\mathbf{x}(t),\dot{\mathbf{x}}(t)) = \frac14\|0-0\|^2 = 0.$$

This gives:

$$\int_0^T (L(x(t),\dot x(t)) + c^*)\,\mathrm{d}t = \int_0^T c^*\mathrm{d}t = c^*T < 0, \tag{4.4}$$

which contradicts the definition of c^*[‡‡]. Thus, we saw $c^* \ge 0$, and combining both bounds, we conclude that $c^* = 0$. □

B. Peierls Barrier Construction. In general if one knows the critical points of U (the potential function), one computes trajectories connecting them [7]. Hence we first build the Peierls barrier in the most restrictive case. For an illustrative example of the graph of a Peierls barrier, see [7, Figure 2].

Theorem 4.2. *Assume the setting of Hypothesis 2.7. Then, for any fixed local minimum* x_i*, the Peierls barrier* $h(y;x_i)$ *satisfies the following:*

1. *The function* $h(y;x_i) \ge 0$ *is Lipschitz continuous and 1-periodic.*

‖ The 1D torus is homeomorphic to the cyclic interval [0,1), or to the circle $\mathbb{S}^1$.

** Note that the variables x in the previous expressions are not boldfaced, indicating that we are working in one dimension. Throughout this work, the absence of bold symbols denotes scalar quantities.

†† Easy to see by contradiction.

‡‡ Since c^* is the infimum of all such c's.

2. There exists $x^* \in \mathbb{T}^1$ such that $h(y;x_i)$ is nonincreasing in (x^*, x_i) to zero and then nondecreasing in (x_i, x^*+1) back to the same level $h(x^*;x_i) = h(x^*+1;x_i)$;

3. The only possible nondifferentiable point of $h(y;x_i)$ lies at x^*, where two monotone pieces are joined. Away from this point, $h(y;x_i)$ is C^1. Thus, $h(y;x_i)$ is piecewise C^1, with at most one kink.

Proof. We prove the first and second numerals by construction of h. The third is shown in [7]. We construct $h(y;x_i)$ by explicitly building the one-sided barrier functions h_R and h_L, corresponding to minimal action paths that move rightward and leftward from a given local minimum x_i, respectively. The key object is the Lagrangian

$$L(x,v) = \frac{1}{4}\|v + U'(x)\|^2, \tag{4.5}$$

and the energy cost of a trajectory γ from x_i to y is

$$h_R(y;x_i) := \inf_{\substack{T\geq 0 \\ \gamma(0)=x_i,\, \gamma(T)=y}} \int_0^T \frac{1}{4}|\dot\gamma + U'(\gamma)|^2 \, \mathrm{d}t. \tag{4.6}$$

For $y = x_{i+1/2}$, an "uphill" point, the optimal path is governed by $\dot\gamma = U'(\gamma)$, yielding

$$h_R(x_{i+1/2};x_i) = U(x_{i+1/2}) - U(x_i). \tag{4.7}$$

Then, from $x_{i+1/2}$ to the next minimum x_{i+1}, the "downhill" path $\dot\gamma = -U'(\gamma)$ costs zero, giving

$$h_R(x_{i+1};x_i) = h_R(x_{i+1/2};x_i). \tag{4.8}$$

This pattern repeats: increasing on each ascent to a maximum, constant over each descent to the next minimum. Therefore, the full right barrier is given by the piecewise expression:

$$h_R(y;x_i) = \\ \begin{cases} U(y) - U(x_i), & y \in [x_i, x_{i+\frac12}], \\ U(x_{i+\frac12}) - U(x_i), & y \in [x_{i+\frac12}, x_{i+1}], \\ U(x_{i+\frac12}) - U(x_i) + U(y) - U(x_{i+1}), & y \in [x_{i+1}, x_{i+\frac32}], \\ \vdots & \\ \sum_{j=i}^{i+k-1} \left[U(x_{j+\frac12}) - U(x_j) \right], & y \in [x_{i+k-\frac12}, x_{i+k}]. \end{cases} \tag{4.9}$$

Similarly, the left barrier $h_L(y;x_i)$ is defined by minimizing the action along paths to the left of x_i. It is given by:

$$h_L(y;x_i) = \\ \begin{cases} \sum_{j=i-k+1}^{i} \left[U(x_{j-\frac12}) - U(x_j) \right], & y \in [x_{i-k}, x_{i-k+\frac12}], \\ \vdots & \\ U(x_{i-\frac12}) - U(x_i) + U(y) - U(x_{i-1}), & y \in [x_{i-\frac32}, x_{i-1}], \\ U(x_{i-\frac12}) - U(x_i), & y \in [x_{i-1}, x_{i-\frac12}], \\ U(y) - U(x_i), & y \in [x_{i-\frac12}, x_i]. \end{cases} \tag{4.10}$$

Because U is skew-periodic, we can extend both h_R and h_L globally using:

$$h_R(y \pm 1, x_{i\pm k}) = h_R(y, x_i), \quad h_L(y \pm 1, x_{i\pm k}) = h_L(y, x_i). \tag{4.11}$$

The Peierls barrier is then constructed by comparing these two [7, 15]:

$$h(y;x_i) := \begin{cases} \min\{h_L(y;x_i),\ h_R(y+1;x_i)\}, & y \leq x_i, \\ \min\{h_L(y-1;x_i),\ h_R(y;x_i)\}, & y > x_i. \end{cases} \tag{4.12}$$

Now, because h_R is nondecreasing and h_L is nonincreasing, there exists a unique $x^* \in \mathbb{S}^1$ such that

$$h_L(x^*;x_i) = h_R(x^*+1;x_i). \tag{4.13}$$

This implies

$$h(y;x_i) = \begin{cases} h_L(y;x_i), & y \in [x^*, x_i], \\ h_R(y;x_i), & y \in [x_i, x^*+1], \end{cases} \tag{4.14}$$

which is continuous, piecewise C^1, and Lipschitz. The only nondifferentiable point may occur at x^*, where the minimum switches.

□

Remark 4.3. *The one-dimensional hypothesis 2.7 can be partially relaxed to the higher-dimensional setting of 2.6 as follows. Assume $d > 1$, and let $U \in C^1(\mathbb{T}^d)$ have finitely many critical points $\mathbf{x}^{(i)}$. Choose one coordinate direction (e.g., x_1) such that all critical points have distinct values along that coordinate. This induces a strict ordering analogous to the 1D case.*

The main complication arises when two saddle points are adjacent in this ordering, making the "uphill/downhill" interpretation ambiguous. To avoid this, we assume no two consecutive critical points are saddles. Under this condition, the construction of barrier functions extends as in the 1D case.

C. Projected Aubry Set. Given the Peierls barrier, the Aubry set can be characterized explicitly.

Theorem 4.4. *Assume Hypothesis 2.7. Then, for the Hamiltonian (2.8), the projected Aubry set is $\mathcal{A} = \{x_i, x_{i+\frac12} : i = 1, \ldots, k\}$.*

Proof. Recall that the projected Aubry set is defined as $\mathcal{A} = \{y \in \mathbb{T}^1 : h(y;y) = 0\}$. From the structure of the Peierls barrier and the Lagrangian Eq. (2.10), we observe:

$$L(x,v) = \frac{1}{4}\|v + U'(x)\|^2 = 0 \iff v = -U'(x). \tag{4.15}$$

Therefore, $L(x, -U'(x)) = 0$ if and only if $U'(x) = 0$, i.e., x is a critical point. For such x, the minimal action over any time horizon is zero, yielding $h(x;x) = 0$. Hence, the Aubry set consists precisely of all local minima and maxima of U due to the biconditional statement. □

Remark 4.5. *This result extends to the higher-dimensional setting described in Hypothesis 2.6, where the Aubry set consists of all critical points of U.*

D. Global Energy Landscape Construction. The construction of the global energy landscape $W(x)$ relies fundamentally on the Peierls barrier and the projected Aubry set. Under Hypothesis 2.7, all long-time minimizing dynamics concentrate around the critical points of $U(x)$, particularly the local minima. These minima form the core of the projected Aubry set and serve as natural candidates for boundary conditions in the variational formulation of the Hamilton–Jacobi equation.

Definition 4.6. *Given boundary values $W_i = W(x_i) \in \mathbb{R}$ prescribed at each stable minimum $x_i \in \mathcal{A}$, the global energy landscape is defined by:*

$$W(x) := \min_{j=1,\dots,k} \{W_j + h(x;x_j)\}, \quad \forall x \in \mathbb{T}^1, \tag{4.16}$$

where $h(x;x_j)$ is the Peierls barrier from x_j to x.

This formula, introduced in [6, Chapter 6, Theorem 4.3], synthesizes the local energy costs (encoded in h) into a global solution W, which satisfies the stationary Hamilton–Jacobi equation in the weak KAM sense.

Let S_t denote the Lax–Oleinik semigroup associated with the dynamic Hamilton–Jacobi Eq. (2.7), where the semigroup acts on any terminal function $u_T(x)$ by

$$(S_t u_T)(y) := \inf_{x \in \mathbb{T}^1} \left\{ u_T(x) + \inf_{\substack{\gamma(0)=x \\ \gamma(t)=y}} \int_0^t L(\dot\gamma(s), \gamma(s))\, \mathrm{d}s \right\}. \tag{4.17}$$

We now state a direct consequence of the representation of weak KAM solutions due to Fathi [5, Proposition 4.6.7]:

Corollary 4.7. *Let $w(x)$ be any weak KAM solution to the stationary Hamilton–Jacobi equation,*

$$H(x, w'(x)) = c^*, \tag{4.18}$$

with the hamiltonian given by Eq. (2.8)*. Then $w(x)$ is an invariant solution of the Lax–Oleinik semigroup, i.e.,*

$$S_t w = w \quad \forall t \geq 0, \tag{4.19}$$

and admits the variational representation

$$w(y) = \inf_{x \in \mathbb{S}^1} \{w(x) + v(y;x)\} = \inf_{x_i \in \mathcal{A}} \{w(x_i) + h(y;x_i)\}. \tag{4.20}$$

In particular, the function $W(x)$ defined by

$$W(x) = \min_{j=1,\dots,k} \{W_j + h(x;x_j)\} \tag{4.21}$$

is an invariant solution of S_t.

Proof. From [5, Proposition 4.6.7], any weak KAM solution w is invariant under the Lax–Oleinik semigroup:

$$w(y) = \inf_{x \in \mathbb{S}^1} \left\{ w(x) + \inf_{\gamma(0)=x, \gamma(t)=y} \int_0^t L(\dot\gamma, \gamma)\, \mathrm{d}s \right\}. \tag{4.22}$$

Taking the infimum over t and rearranging, one obtains:

$$w(y) = \inf_{x \in \mathbb{S}^1} \{w(x) + v(y;x)\}, \tag{4.23}$$

where $v(y;x)$ denotes the minimal action from x to y over arbitrary time horizons [7]. By definition of the Peierls barrier and the fact that weak KAM solutions are uniquely determined by their values on the projected Aubry set [5, Theorem 4.12.6], one recovers the explicit representation:

$$w(y) = \inf_{x_i \in \mathcal{A}} \{w(x_i) + h(y;x_i)\}. \tag{4.24}$$

Since $W(x)$ is defined using this representation with fixed boundary values $W_j = w(x_j)$, it follows that W is a weak KAM solution and hence an invariant solution of S_t. □

Remark 4.8. *Although invariant solutions of the Lax–Oleinik semigroup always exist on compact manifolds [5], they are not unique. The same holds for weak KAM solutions, whose non-uniqueness reflects the freedom in prescribing values on the Aubry set.*

E. Weak KAM solution to the HJE. There is only one thing missing, how do you get a solution to the dynamic HJE from the static solution $W(x)$?

Fathi [5] showed that it is only needed to use the Lax-Oleinik semigroup S_t to propagate the solution forward in time:

$$(S_t W)(y) := \inf_{x \in \mathbb{T}^1} \left\{ W(x) + \inf_{\substack{\gamma(0)=x \\ \gamma(t)=y}} \int_0^t L(\gamma(s), \dot\gamma(s))\, ds \right\}. \tag{4.25}$$

Now, the key is, if $W(x)$ is a weak KAM solution, then it is an invariant solution under the Lax–Oleinik semigroup:

$$S_t W = W \quad \forall t \geq 0. \tag{4.26}$$

This means one can define the dynamic solution as:

$$\psi(x,t) := W(x), \tag{4.27}$$

i.e., constant in time. It solves the dynamic HJE in the viscosity sense[§§].

5. Conclusions

In this work, we developed a variational approach to solve the stationary Hamilton–Jacobi equation derived from overdamped Langevin dynamics in the low-temperature limit. Building upon and relaxing the hypotheses of [7], we generalized the construction of weak KAM solutions beyond the one-dimensional setting, accommodating more general drift fields with multiple critical points.

Our main results include proving that the Mañé critical value is zero under mild conditions, explicitly constructing the Peierls barrier and projected Aubry set, and showing how these structures determine the global energy landscape. We confirmed that this energy landscape yields a weak KAM solution, which is also an invariant solution under the Lax–Oleinik semigroup and hence solves the dynamic HJE.

Looking ahead, this framework invites several extensions. Visualizing solutions through rich graphical examples would clarify the interpretation of the theory. Moreover, applying the method to specific drift fields from fluid mechanics could validate the approach by comparing it to numerical PDE models. These directions could bridge theoretical insights with practical applications in stochastic processes, metastability, and optimal control.

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[§§] The title "variational" for this kind of solutions is given by the fact that it is expressed in terms of an optimization process (min/inf or max/sup). Machines are particularly good to evaluate this type of solutions.

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