Math 240 Tutorial
Questions

June 27

Question 1. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be an invertible linear transformation, and let $\vec{v}_1, \ldots, \vec{v}_m \in \mathbf{R}^n$. Prove that $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is an independent set if and only if $\{T(\vec{v}_1), \ldots, T(\vec{v}_m)\}$ is an independent set.

Question 2. Define $T : \mathbf{R}^n \to \mathbf{R}^n$ by

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_2 - x_3 - x_4, -x_1 + x_2 - x_3 - x_4, -x_1 - x_2 + x_3 - x_4, -x_1 - x_2 - x_3 + x_4).$$

Is T linear? Is T invertible? If it is, what is its inverse?

Question 3. Show that if E and F are two $n \times n$ matrices such that $EF = I$, then E and F commute.

Question 4. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ and $U : \mathbf{R}^n \to \mathbf{R}^n$ be two linear transformations such that $T(U(\vec{x})) = \vec{x}$ for every $\vec{x} \in \mathbf{R}^n$. Show that T is invertible and $U = T^{-1}$.

Question 5. Show that if A is invertible, then $\det(A^{-1}) = 1/\det(A)$.

Question 6. Let A, B, and P be $n \times n$ matrices where P is invertible and $B = P^{-1}AP$. Show that $\det(A) = \det(B)$.

Question 7. Let V be a vector space, and let H and K be subspaces of V. Show the following

(a) $H + K$ and $H \cap K$ are subapces.

(b) H and K are subspaces of $H + K$.

(c) $H \cap K$ is a subspace of both H and K.

Question 8. Let V be a vector space, and let W be a vector space of V. Recall that, for $\vec{v} \in V$, $\vec{v} + W = \{\vec{v} + \vec{w} : \vec{w} \in W\}$. Show the following.

(a) For distinct $\vec{v}_1, \vec{v}_2 \in V$, $\vec{v}_1 + W$ and $\vec{v}_2 + W$ are either disjoint or equal.

(b) $\vec{v}_1 + W = \vec{v}_2 + W$ if and only if $\vec{v}_1 - \vec{v}_2 \in W$.

(c) Every $\vec{v} \in V$ belongs to $\vec{u} + W$ for some $\vec{u} \in V$.

We can define an arithmetic on $H = \{\vec{v} + W : \vec{v} \in V\}$. For $\vec{v}_1 + W, \vec{v}_2 + W \in H$ and $\alpha \in \mathbf{R}$, define $(\vec{v}_1 + W) + (\vec{v}_2 + W) = (\vec{v}_1 + \vec{v}_2) + W$ and $\alpha(\vec{v}_1 + W) = (\alpha\vec{v}_1) + W$. Then:

(d) H is a vector under the arithmetic defined above. We call H the quotient space of V by W, and we denote it as $H = V/W$.