Math 240 Tutorial Questions

July 11

Question 1. Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}.$$

(a) Calculate the determinant of A using (1) cofactor expansion, and (2) row reduction.

(b) Calculate the inverse using (1) the adjugate of A, and (2) row reduction.

(c) Do the columns (rows) of A form a basis for $\mathbf{R}^4$? If they do, give the change of basis matrix from the standard basis of $\mathbf{R}^4$ to the columns of A.

Question 2. Consider the matrices

$$A = \begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}.$$

The matrix P is invertible. Find P^{-1} by any means and calculate $P^{-1}AP = D$. Prove that A is invertible if and only if D is invertible. If A is invertible, find its inverse by first finding the inverse of D and then multiplying D^{-1} by P^{-1} and P in some order.

Question 3. Prove that the linear transformations of $\mathbf{R}^2$ consisting of compositions of reflections and rotations have determinants ± 1.

Question 4. An isomorphism is an invertible linear transformation from one vector space onto another. Give two distinct isomorphisms $\mathbf{P}_3 \to \mathbf{R}^3$. NB: $\mathbf{R}^n$ (or $\mathbf{C}^n$) are often referred to as the coordinate spaces. This question shows that the coordinate representation of a vector is not in general unique; it depends on the choice of basis.

Question 5. Define

$$H = \left\{ \begin{pmatrix} u & -u-x \\ 0 & x \end{pmatrix} : u, x \in \mathbf{R} \right\},$$
$$K = \left\{ \begin{pmatrix} v & 0 \\ w & -v \end{pmatrix} \right\}.$$

Do the following.

(a) H and K are subspaces of $M_{2\times 2}(\mathbf{R})$.

(b) Construct bases for H, K, $H + K$, and $H \cap K$.

Question 6. Answer whether the following are subspaces of $\mathbf{R}^3$. The set of points $(x, y, z) \in \mathbf{R}^3$ such that

(a) $x + 2y - 3z = 4$,

(b) $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z}{4}$,

(c) $x + y + z = 0$ and $x - y + z = 1$,

(d) $x = -z$ and $x = z$,

(e) $x^2 + y^2 = z$, or

(f) $\frac{x}{2} = \frac{y-3}{5}$.