

Math 240 Tutorial Questions

June 6

Systems of Linear Equations and Row Reduction

Question 1. Place the following augmented matrices into an echelon form. Does the corresponding system of linear equations admit any solutions?

(a)

$$\left(\begin{array}{cccc|c} 4 & 8 & 12 & 4 & 7 \\ 2 & 5 & 6 & 6 & 11 \\ 0 & 5 & 1 & 26 & 13 \\ 0 & 5 & 0 & 21 & 17 \end{array} \right).$$

(b)

$$\left(\begin{array}{cccc|c} 4 & 8 & 12 & 4 & 0 \\ 2 & 5 & 6 & 6 & 0 \\ 0 & 5 & 1 & 25 & 0 \\ 0 & 5 & 0 & 20 & 0 \end{array} \right).$$

(c)

$$\left(\begin{array}{cccc|c} 4 & 8 & 12 & 4 & 7 \\ 2 & 5 & 6 & 6 & 11 \\ 0 & 5 & 1 & 25 & 13 \\ 0 & 5 & 0 & 20 & 17 \end{array} \right).$$

Question 2. Find the values of k for which the system of equations

$$x + ky = 1,$$

$$kx + y = 1,$$

has

(a) no solution,

(b) a unique solution, and

(c) infinitely many solutions.

(d) When there is exactly one solution, what are the values of x and y .

Question 3. Consider the following two systems of equations.

$$x + y + z = 16,$$

$$x + 2y + 2z = 11,$$

$$2x + 3y - 4z = 3,$$

and

$$x + y + z = 7,$$

$$x + 2y + 2z = 10,$$

$$2x + 3y - 4z = 3.$$

Solve both systems simultaneously by applying row reduction to an appropriate 3×5 matrix.

Question 4. Consider the following homogeneous system of linear equations where $a, b \in \mathbf{R}$ are constants.

$$\begin{aligned}x + 2y &= 0, \\ax + 8y + 3z &= 0, \\by + 5z &= 0.\end{aligned}$$

- Find a value for a which makes it necessary to interchange rows during row reduction.
- Suppose that a does not have the value you found in part (a). Find a value for b so that the system has a trivial solution.
- Suppose that a does not have the value you found in part (a) and that $b = 100$. Suppose further that a is chosen so that the solution to the system is not unique. The general solution to the system is $(\alpha^{-1}z, -\beta^{-1}z, z)$ where α and β are what?

Spans of Collections of Vectors

Question 5. Consider the following three vectors in \mathbf{R}^3

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Show that $\mathbf{R}^3 = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

Question 6. Consider the following four vectors in \mathbf{R}^4 given by

$$\vec{v}_1 = \begin{pmatrix} +1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ +1 \\ -1 \\ -1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -1 \\ -1 \\ +1 \\ -1 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ +1 \end{pmatrix}.$$

- Show whether $\vec{v}_1 \in \text{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$ or not by solving the corresponding system of linear equations.
- Let $a_1, a_2, a_3, a_4 \in \mathbf{R}$. Under what conditions on a_1, a_2, a_3, a_4 is $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + a_4\vec{v}_4 = \vec{0}$ true?
- How can we use part (b) to provide a second proof of part (a)? Can you generalize to answer the following question: Is $\vec{v}_i \in \text{span}\{\vec{v}_j, \vec{v}_k, \vec{v}_l\}$ for i, j, k, l distinct?

Question 7. Let V_1 and V_2 be two subsets of \mathbf{R}^n , and define $V_1 + V_2 = \{\vec{v}_1 + \vec{v}_2 : \vec{v}_1 \in V_1 \text{ and } \vec{v}_2 \in V_2\}$. Show (a) $\text{span}(V_1 \cup V_2) = \text{span}(V_1) + \text{span}(V_2)$, and (b) $\text{span}(V_1 \cap V_2) \subseteq \text{span}(V_1) \cap \text{span}(V_2)$. Further, give an example of subsets V_1 and V_2 of \mathbf{R}^n , for some n , for which $\text{span}(V_1 \cap V_2) \subsetneq \text{span}(V_1) \cap \text{span}(V_2)$.

Linear Independence

Question 8. Show that in \mathbf{R}^3 , the vectors $\vec{x} = (1, 1, 0)$, $\vec{y} = (0, 1, 2)$, and $\vec{z} = (3, 1, -4)$ are linearly dependent by finding scalars α and β such that $\alpha\vec{x} + \beta\vec{y} + \vec{z} = \vec{0}$.

Question 9. Let $\vec{w} = (1, 1, 0, 0)$, $\vec{x} = (1, 0, 1, 0)$, $\vec{y} = (0, 0, 1, 1)$, and $\vec{z} = (0, 1, 0, 1)$, and let $S = \{\vec{w}, \vec{x}, \vec{y}, \vec{z}\}$.

- (a) Show that S is not a spanning set for \mathbf{R}^4 by finding a vector \vec{u} in \mathbf{R}^4 such that $\vec{u} \notin \text{span}(S)$. One such vector is $\vec{u} = (1, 2, 3, a)$ where a is any real number except what?
- (b) Show that S is a linearly dependent set of vectors by finding scalars α , γ , and δ such that $\alpha\vec{w} + \vec{x} + \gamma\vec{y} + \delta\vec{z} = \vec{0}$.
- (c) Show that S is a linear dependent set by writing \vec{z} as a linear combination of the remaining vectors in S .

Question 10. Let S_1 and S_2 be finite subsets of \mathbf{R}^n , for some n , such that $S_1 \subseteq S_2$. Prove that if S_1 is a linearly dependent set, then so is S_2 . Show that this is equivalent to if S_2 is a linearly independent set, then so is S_1 .

Question 11. Do the following.

- (a) Let \vec{u} and \vec{v} be distinct vectors in \mathbf{R}^n . Prove that $\{\vec{u}, \vec{v}\}$ is linearly independent if and only if $\{\vec{u} + \vec{v}, \vec{u} - \vec{v}\}$ is linearly independent.
- (b) Let $\vec{u}, \vec{v}, \vec{w}$ be distinct vectors in \mathbf{R}^n . Prove that $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent if and only if $\{\vec{u} + \vec{v}, \vec{u} + \vec{w}, \vec{v} + \vec{w}\}$ is linear independent.

Linear Transformations

Question 12. Show the following for \mathbf{R}^n .

- (a) Show that scalar multiplication is a linear transformation.
- (b) When is this linear map invertible?
- (c) Is its inverse a linear transformation?
- (d) Fix an element $a \in \mathbf{R}$. What is the matrix corresponding to the linear transformation $\vec{v} \mapsto a\vec{v}$?

Question 13. Fix $a \in \mathbf{R}$ and $\vec{u} \in \mathbf{R}^n$ with $\vec{u} \neq \vec{0}$. Is the map given by $\vec{v} \mapsto a\vec{v} + \vec{u}$, linear? Why or why not?

Question 14. Consider a linear transformation $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$, and define $\text{Ker}(T) = \{\vec{v} \in \mathbf{R}^n : T(\vec{v}) = \vec{0}\}$. This is the kernel of the linear transformation T . For $\vec{v} \in \mathbf{R}^n$, define $\vec{v} + \text{Ker}(T) = \{\vec{v} + \vec{u} : \vec{u} \in \text{Ker}(T)\}$. Show the following.

- (a) $\text{Ker}(T)$ is closed under scalar multiplication and vector addition.
- (b) For $\vec{v} \in \mathbf{R}^n$, show that $\vec{v} + \text{Ker}(T)$ consists of all and only those elements of \mathbf{R}^n that map to $T(\vec{v})$ under T .
- (c) For $\vec{v}_1, \vec{v}_2 \in \mathbf{R}^n$, show that either $\vec{v}_1 + \text{Ker}(T) = \vec{v}_2 + \text{Ker}(T)$ or $\vec{v}_1 + \text{Ker}(T) \cap \vec{v}_2 + \text{Ker}(T) = \emptyset$.

Matrix Operations

Question 15. Give an example of a nonzero matrix A such that $A^2 = O$.

Question 16. The trace of a square matrix A of dimensions $N \times N$ is defined as $\text{tr}(A) = \sum_{k=1}^N A_{k,k}$, i.e., the sum of the diagonal entries of the matrix. For any other $N \times N$ matrix B , show that $\text{tr}(AB) = \text{tr}(BA)$.

Question 17. An $N \times N$ matrix A is circulant if it is of the form

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_N \\ a_N & a_1 & a_2 & \cdots & a_{N-1} \\ a_{N-1} & a_N & a_1 & \cdots & a_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}.$$

Show that if B is any other $N \times N$ circulant matrix, then $AB = BA$.

Question 18. A diagonal matrix is one for which every entry not on the main diagonal is zero. Let A and B be $N \times N$ matrices such that there exists an invertible $N \times N$ matrix P for which $D_A = P^{-1}AP$ and $D_B = P^{-1}BP$ are diagonal matrices. Show that A and B commute.