Math 240 Tutorial
Solutions

August 1

Question 1. Find the unit vector in the direction of the given vectors (a) $\begin{pmatrix} -30 \\ 40 \end{pmatrix}$, (b) $\begin{pmatrix} 7/4 \\ 1/2 \\ 1 \end{pmatrix}$, and (c) $\begin{pmatrix} 8/3 \\ 2 \end{pmatrix}$.

(a) We have $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{(-30)^2 + 40^2} = 50$ so that the unit vector is given by $\frac{\vec{v}}{\|\vec{v}\|} = \begin{pmatrix} -3/5 \\ 4/5 \end{pmatrix}$.

(b) We have $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{(7/4)^2 + (1/2)^2 + 1^2} = \frac{1}{2}\sqrt{49 + 4 + 16} = \frac{\sqrt{69}}{2}$ so that the unit vector is given by $\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{69}} \begin{pmatrix} 7/2 \\ 1 \\ 2 \end{pmatrix}$.

(c) We have $\|v\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{(8/3)^2 + 2^2} = \frac{10}{3}$ so that the unit vector is given by $\frac{\vec{v}}{\|\vec{v}\|} = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$.

Question 2. (a) Let $\vec{u}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$, and $\vec{x} = \begin{pmatrix} 9 \\ -7 \end{pmatrix}$. Does $\{\vec{u}_1, \vec{u}_2\}$ form an orthogonal basis for $\mathbf{R}^2$? If it does, write $\vec{x}$ in terms of this basis. (b) Compute the orthogonal projection of $\vec{x} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$ onto the line through $\vec{y} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ and the origin.

(a) Note that $\langle \vec{u}_1, \vec{u}_2 \rangle = 12 - 12 = 0$ so that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis for $\mathbf{R}^2$. We have

$$\vec{x} = \frac{\langle \vec{x}, \vec{u}_1 \rangle}{\|\vec{u}_1\|}\vec{u}_1 + \frac{\langle \vec{x}, \vec{u}_2 \rangle}{\|\vec{u}_2\|}\vec{u}_2 = \frac{39}{\sqrt{13}}\vec{u}_1 + \frac{13}{\sqrt{13}}\vec{u}_2.$$

(b) The projection is given by

$$\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|}\vec{y} = \frac{-4 + 14}{\sqrt{16 + 4}} \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \frac{5}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Question 3. Let $\vec{y} \in \mathbf{R}^n$. Prove $\vec{x} \mapsto \langle \vec{x}, \vec{y} \rangle$ is a linear transformation $\mathbf{R}^n \to \mathbf{R}$.

Let $\vec{w}, \vec{x} \in \mathbf{R}^n$, and let $\alpha \in \mathbf{R}$. Then

$$\langle \vec{w} + \vec{x}, \vec{y} \rangle = \sum_{j=1}^{n} (w_j + x_j) y_j = \sum_{j=1}^{n} w_j y_j + \sum_{j=1}^{n} x_j y_j = \langle \vec{w}, \vec{y} \rangle + \langle \vec{x}, \vec{y} \rangle.$$

and

$$\langle \alpha\vec{w}, \vec{y} \rangle = \sum_{j=1}^{n} \alpha w_j y_j = \alpha \sum_{j=1}^{n} w_j y_j = \alpha \langle \vec{w}, \vec{y} \rangle.$$

It follows, therefore, that the map is a linear map.

Question 4. Let

$$\vec{y} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}, \quad \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Verify that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set, and find the orthogonal projection of $\vec{y}$ onto span$\{\vec{u}_1, \vec{u}_2\}$. Construct a nonzero vector $\vec{z}$ that is orthogonal to $\vec{u}_1$ and $\vec{u}_2$. Find the distance from $\vec{y}$ to span$\{\vec{u}_1, \vec{u}_2\}$.

Observe $\langle \vec{u}_1, \vec{u}_2 \rangle = -1 + 1 = 0$, so they are orthogonal. To simplify the calculations, however, observe that span$\{\vec{u}_1, \vec{u}_2\} =$ span$\{\hat{u}_1, \hat{u}_2'\}$ where

$$\hat{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \hat{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Applying Gram–Schmidt with this new basis gives that the projection is given by

$$\hat{y} = \langle y, \hat{u}_1 \rangle \hat{u}_1 + \langle y, \hat{u}_2 \rangle \hat{u}_2 = -\hat{u}_1 + 4\hat{u}_2 = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}.$$

A vector orthogonal to the span is given by

$$\vec{z} = \vec{y} - \hat{y} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$$

Therefore, the distance from y to span$\{\vec{u}_1, \vec{u}_2\}$ is given by $\|\vec{z}\| = 3$.

Question 5. Let W be a subspace of $\mathbf{R}^n$ with an orthogonal basis $\beta_1 = \{\vec{w}_1, \dots, \vec{w}_p\}$, and let $\beta_2 = \{\vec{v}_1, \dots, \vec{v}_q\}$ be an orthogonal basis for $W^\perp$.

(a) Explain why $\beta_1 \cup \beta_2$ is an orthogonal set.

Every vector in β_1 is orthogonal to every other vector in β_1 as well as every vector in β_2. Similarly, every vector in β_2 is orthogonal to every other vector in β_2 as well as every vector in β_1.

(b) Explain why the set in part (a) spans $\mathbf{R}^n$.

Every vector $\vec{x}$ in $\mathbf{R}^n$ can be written uniquely as $\vec{x} = \vec{w}_1 + \vec{w}_2$ where $\vec{w}_1 \in W$ and $\vec{w}_2 \in W^\perp$.

(c) Show that $\dim(W) + \dim(W) = n$.

Part (b) shows that $\mathbf{R}^n = W \cup W^\perp$. Recall that $n = \dim(\mathbf{R}^n) = \dim(W \cup W^\perp) = \dim(W) + \dim(W^\perp) - \dim(W \cap W^\perp)$. But $W \cap W^\perp = \{0\}$ has dimension 0, so $\dim(W) + \dim(W^\perp) = n$.

Question 6. Let A be an $m \times n$ matrix with linearly independent columns, and let $A = QR$ be its QR-factorization. Prove that R is invertible with positive eigenvalues.

Note that if $R\vec{x} = \vec{0}$, then $A\vec{x} = QR\vec{x} = Q\vec{0} = \vec{0}$. Since A has linearly independent columns, it must be that $\vec{x}\vec{0}$, and it follows that R is invertible.

The proof in the text shows that R is an upper triangular matrix with nonnegative diagonal entries. Since R is invertible, these entries are actually positive. The eigenvalues of a triangular matrix are its diagnal entries.

Question 7. Recall that $H = \operatorname{span}\{x-3,\, x^2-3x\}$ is the subspace of $\mathbf{P}_2(\mathbf{R})$ consisting of all those vectors divisible by $x-3$. Do the following.

(a) Verify that $\langle p, q\rangle \equiv \int_{-1}^{1} pq \,\mathrm{d}x$ is an inner product on $\mathbf{P}_n(\mathbf{R})$.

This simply follows by the linearity of the integral.

(b) Use the Gram–Schmidt Process to find an orthogonal basis β for H.

We construct the orthogonal basis $\beta = \{\vec{f}_1,\, \vec{f}_2\}$. First, take $\vec{f}_1 = x-3$. Then we calculate

$$\vec{f}_2 = x^2 - 3x - \frac{\langle x^2-3x,\, \vec{f}_1\rangle}{\|\vec{f}_1\|}\vec{f}_1 = x^2 - \left(3 - \frac{6}{\sqrt{14}}\right)x - \frac{9}{\sqrt{14}}.$$

(c) Let T be the linear map $H \to \mathbf{P}_1(\mathbf{R})$ defined by $p \mapsto \frac{\mathrm{d}p}{\mathrm{d}x}$. Find the QR-factorization of $[T]_\beta^\gamma$ where γ is the standard basis $\{1,\, x\}$ of $\mathbf{P}_1(\mathbf{R})$.

We have

$$[T]_\beta^\gamma = \left[\left[\frac{\mathrm{d}f_1}{\mathrm{d}x}\right]_S \,\middle|\, \left[\frac{\mathrm{d}f_2}{\mathrm{d}x}\right]\right] = \begin{pmatrix} 1 & -3 + \frac{6}{\sqrt{14}} \\ 0 & 2 \end{pmatrix}.$$

But this is already in the required form.