Math 240 Tutorial
Questions

July 25

Question 1. Let a finite dimensional vector space V have two bases β and β', and let Q be the transformation matrix from β'-coordinates to β-coordinates. Show that for any linear transformation $T : V \to V$, it holds that

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q.$$

Question 2. A scalar matrix is a matrix of the form λI for some scalar λ.

(a) Prove that is a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.

(b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.

(c) Prove that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

Question 3. For each of the following linear operators T on a vector space V and ordered basis β, compute $[T]_\beta$ and determine whether β is a basis consisting of eigenvectors of T.

(a) $V = \mathbf{R}^2$, $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10a-6b \\ 17a-10b \end{pmatrix}$, and $\beta = \left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$.

(b) $V = \mathbf{P}_1(\mathbf{R})$, $T(a+bx) = (6a-6b) + (12a-11b)x$, and $\beta = \{3+4x, 2+3x\}$.

(c) $V = \mathbf{R}^3$, $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a+2b-2c \\ -4a-3b+2c \\ -c \end{pmatrix}$, and $\beta = \left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right\}$.

(d) $V = \mathbf{P}_2(\mathbf{R})$, $T(a+bx+cx^2) = (-4a+2b-2c) - (7a+3b+7c)x + (7a+b+5c)x^2$, and $\beta = \{x-x^2, -1+x^2, -1-x+x^2\}$.

(e) $V = P_3(\mathbf{R})$, $T(a+bx+cx^2+dx^3) = -d + (-c+d)x + (a+b-2c)x^2 + (-b+c-2d)x^3$, and $\beta = \{1-x+x^3, 1+x^2, 1, x+x^2\}$.

(f) $V = \mathcal{M}_{2\times 2}(\mathbf{R})$, $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -7a-4b+4c-4d & b \\ -8a-4b+5c-4d & d \end{pmatrix}$, and $\beta = \left\{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\right\}$.

Question 4. For each of the following matrices $A \in \mathcal{M}_{n\times n}(F)$:

(i) Determine all the eigenvalues of A.

(ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.

(iii) If possible, find a basis for F^n consisting of eigenvectors of A.

(iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ for $F = \mathbf{R}$.

(b) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ for $F = \mathbf{R}$.

(c) $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$ for $F = \mathbf{C}$.

(d) $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$ for $F = \mathbf{R}$.

Question 5. Prove the geometric multiplicity of an eigenvalue is at most the algebraic multiplicity.