

## Math 240 Tutorial Solutions

June 27

**Question 1.** Let  $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be an invertible linear transformation, and let  $\vec{v}_1, \dots, \vec{v}_m \in \mathbf{R}^n$ . Prove that  $\{\vec{v}_1, \dots, \vec{v}_m\}$  is an independent set if and only if  $\{T(\vec{v}_1), \dots, T(\vec{v}_m)\}$  is an independent set.

**Question 2.** Define  $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$  by

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_2 - x_3 - x_4, -x_1 + x_2 - x_3 - x_4, -x_1 - x_2 + x_3 - x_4, -x_1 - x_2 - x_3 + x_4).$$

Is  $T$  linear? Is  $T$  invertible? If it is, what is its inverse?

The components of the image of  $\vec{x}$  are linear combinations of the components of  $\vec{x}$ , so  $T$  is linear. The standard matrix for  $T$  is given by

$$T_A = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

which is readily verified to be invertible. Moreover, it is its own inverse. So  $T^{-1} = T$ .

**Question 3.** Show that if  $E$  and  $F$  are two  $n \times n$  matrices such that  $EF = I$ , then  $E$  and  $F$  commute.

Since  $EF = I$ ,  $E$  is invertible and  $F = E^{-1}$  by the uniqueness of  $E^{-1}$ . Of course,  $EE^{-1} = E^{-1}E = I$ .

**Question 4.** Let  $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$  and  $U : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be two linear transformations such that  $T(U(\vec{x})) = \vec{x}$  for every  $\vec{x} \in \mathbf{R}^n$ . Show that  $T$  is invertible and  $U = T^{-1}$ .

By assumption,  $U$  is a right inverse of  $T$ . Since  $T$  is a linear operator of a finite dimensional vector space, it is invertible and  $T^{-1} = U$ .

**Question 5.** Show that if  $A$  is invertible, then  $\det(A^{-1}) = 1/\det(A)$ .

We rewrite the putative equality as  $\det(A)\det(A^{-1}) = 1$ . By the properties of the determinant,

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1}),$$

which proves the result.

**Question 6.** Let  $A$ ,  $B$ , and  $P$  be  $n \times n$  matrices where  $P$  is invertible and  $B = P^{-1}AP$ . Show that  $\det(A) = \det(B)$ .

Observe

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \frac{\det(A)\det(P)}{\det(P)} = \det(A)$$

where we have used the previous exercise.

**Question 7.** Let  $V$  be a vector space, and let  $H$  and  $K$  be subspaces of  $V$ . Show the following

- (a)  $H + K$  and  $H \cap K$  are subspaces.

Let  $\vec{v}_1, \vec{v}_2 \in H + K$ . Then there are vectors  $\vec{h}_1, \vec{h}_2 \in H$  and  $\vec{k}_1, \vec{k}_2 \in K$  such that  $\vec{v}_1 = \vec{h}_1 + \vec{k}_1$  and  $\vec{v}_2 = \vec{h}_2 + \vec{k}_2$ . Then  $\vec{v}_1 + \vec{v}_2 = (\vec{h}_1 + \vec{h}_2) + (\vec{k}_1 + \vec{k}_2)$ . Since  $\vec{h}_1 + \vec{h}_2 \in H$  and  $\vec{k}_1 + \vec{k}_2 \in K$ , we have  $\vec{v}_1 + \vec{v}_2 \in H + K$ . If  $\alpha \in \mathbf{R}$ , then  $\alpha\vec{v} = \alpha\vec{h} + \alpha\vec{k}$ . Since  $\alpha\vec{h} \in H$  and  $\alpha\vec{k} \in K$ ,  $\alpha\vec{v} \in H + K$ . Finally,  $H + K \neq \emptyset$  since  $\vec{0} \in H + K$ . We have therefore shown that  $H + K$  is a subspace of  $V$ .

Let  $\vec{v}_1, \vec{v}_2 \in H \cap K$ . Then  $\vec{v}_1, \vec{v}_2 \in H$  and  $\vec{v}_1, \vec{v}_2 \in K$ . Therefore,  $\vec{v}_1 + \vec{v}_2 \in H$  and  $\vec{v}_1 + \vec{v}_2 \in K$  so that  $\vec{v}_1 + \vec{v}_2 \in H \cap K$ . Similarly, for  $\alpha \in \mathbf{R}$ ,  $\alpha\vec{v} \in H$  and  $\alpha\vec{v} \in K$  so that  $\alpha\vec{v} \in H \cap K$ . Since  $\vec{0} \in H$  and  $\vec{0} \in K$ , we have  $\vec{0} \in H \cap K$  so that  $H \cap K \neq \emptyset$ . We have shown that  $H \cap K$  is a subspace of  $V$ .

- (b)  $H$  and  $K$  are subspaces of  $H + K$ .

Note that  $\vec{0} \in H$  and  $\vec{0} \in K$ . So, for  $\vec{h} \in H$  and  $\vec{k} \in K$ , we have  $\vec{h} + \vec{0} \in H + K$  and  $\vec{0} + \vec{k} \in H + K$ . This shows  $H \subseteq H + K$  and  $K \subseteq H + K$ .

- (c)  $H \cap K$  is a subspace of both  $H$  and  $K$ .

We have already verified  $H \cap K$  to be a subspace of the ambient space  $V$ , and clearly  $H \cap K \subseteq H$  and  $H \cap K \subseteq K$ , whereupon  $H \cap K$  is a linear subspace of both  $H$  and  $K$ .

**Question 8.** Let  $V$  be a vector space, and let  $W$  be a vector space of  $V$ . Recall that, for  $\vec{v} \in V$ ,  $\vec{v} + W = \{\vec{v} + \vec{w} : \vec{w} \in W\}$ . Show the following.

- (a) For distinct  $\vec{v}_1, \vec{v}_2 \in V$ ,  $\vec{v}_1 + W$  and  $\vec{v}_2 + W$  are either disjoint or equal.

Assume  $\vec{v}_1 + W \cap \vec{v}_2 + W \neq \emptyset$ , and let  $\vec{x} \in \vec{v}_1 + W \cap \vec{v}_2 + W$ . Then there are  $\vec{w}_1, \vec{w}_2 \in W$  such that  $\vec{x} = \vec{v}_1 + \vec{w}_1 = \vec{v}_2 + \vec{w}_2$ , whereupon  $\vec{v}_1 = \vec{v}_2 + \vec{w}_3$  where  $\vec{w}_3 = \vec{w}_2 - \vec{w}_1 \in W$ . Let  $\vec{v}_1 + \vec{w}$  be an arbitrary element of  $\vec{v}_1 + W$ . Then  $\vec{v}_1 + \vec{w} = \vec{v}_2 + (\vec{w}_3 + \vec{w})$  which shows that  $\vec{v}_1 + W \subseteq \vec{v}_2 + W$ . Similarly,  $\vec{v}_2 + W \subseteq \vec{v}_1 + W$  so that  $\vec{v}_1 + W = \vec{v}_2 + W$ , as desired.

- (b)  $\vec{v}_1 + W = \vec{v}_2 + W$  if and only if  $\vec{v}_1 - \vec{v}_2 \in W$ .

Assume  $\vec{v}_1 + W = \vec{v}_2 + W$ . In particular,  $\vec{v}_1 \in \vec{v}_2 + W$  so that there is a  $\vec{w} \in W$  for which  $\vec{v}_1 = \vec{v}_2 + \vec{w}$ ; in other words,  $\vec{v}_1 - \vec{v}_2 = \vec{w} \in W$ .

Conversely, suppose that  $\vec{v}_1 - \vec{v}_2 = \vec{w}$  for some  $\vec{w} \in W$ . Then  $\vec{v}_1 = \vec{v}_2 + \vec{w}$  so that  $\vec{v}_1 \in \vec{v}_2 + W$ . If  $\vec{v}_1 + \vec{w}' \in \vec{v}_1 + W$  is arbitrary, then  $\vec{v}_1 + \vec{w}' = \vec{v}_2 + (\vec{w} + \vec{w}') \in \vec{v}_2 + W$  since  $\vec{w} + \vec{w}' \in W$ . Therefore,  $\vec{v}_1 + W \subseteq \vec{v}_2 + W$ . Similarly,  $\vec{v}_2 + W \subseteq \vec{v}_1 + W$ , hence  $\vec{v}_1 + W = \vec{v}_2 + W$ .

- (c) Every  $\vec{v} \in V$  belongs to  $\vec{u} + W$  for some  $\vec{u} \in V$ .

Since  $\vec{0} \in W$ , it follows that  $\vec{v} = \vec{v} + \vec{0} \in \vec{v} + W$ .

We can define an arithmetic on  $H = \{\vec{v} + W : \vec{v} \in V\}$ . For  $\vec{v}_1 + W, \vec{v}_2 + W \in H$  and  $\alpha \in \mathbf{R}$ , define  $(\vec{v}_1 + W) + (\vec{v}_2 + W) = (\vec{v}_1 + \vec{v}_2) + W$  and  $\alpha(\vec{v}_1 + W) = (\alpha\vec{v}_1) + W$ . Then:

- (d)  $H$  is a vector under the arithmetic defined above. We call  $H$  the quotient space of  $V$  by  $W$ , and we denote it as  $H = V/W$ .