

Math 240 Tutorial Questions

June 13

Question 1. Fix $a \in \mathbf{R}$ and $\vec{u} \in \mathbf{R}^n$ with $\vec{u} \neq \vec{0}$. Is the map given by $\vec{v} \mapsto a\vec{v} + \vec{u}$, linear? Why or why not?

Question 2. Consider a linear transformation $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$, and define $\text{Ker}(T) = \{\vec{v} \in \mathbf{R}^n : T(\vec{v}) = \vec{0}\}$. This is the kernel of the linear transformation T . For $\vec{v} \in \mathbf{R}^n$, define $\vec{v} + \text{Ker}(T) = \{\vec{v} + \vec{u} : \vec{u} \in \text{Ker}(T)\}$. Show the following.

- (a) $\text{Ker}(T)$ is closed under scalar multiplication and vector addition.
- (b) For $\vec{v} \in \mathbf{R}^n$, show that $\vec{v} + \text{Ker}(T)$ consists of all and only those elements of \mathbf{R}^n that map to $T(\vec{v})$ under T .
- (c) For $\vec{v}_1, \vec{v}_2 \in \mathbf{R}^n$, show that either $\vec{v}_1 + \text{Ker}(T) = \vec{v}_2 + \text{Ker}(T)$ or $\vec{v}_1 + \text{Ker}(T) \cap \vec{v}_2 + \text{Ker}(T) = \emptyset$.

Question 3. The trace of a square matrix A of dimensions $N \times N$ is defined as $\text{tr}(A) = \sum_{k=1}^N A_{k,k}$, i.e., the sum of the diagonal entries of the matrix. For any other $N \times N$ matrix B , show that $\text{tr}(AB) = \text{tr}(BA)$.

Question 4. An $N \times N$ matrix A is circulant if it is of the form

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_N \\ a_N & a_1 & a_2 & \cdots & a_{N-1} \\ a_{N-1} & a_N & a_1 & \cdots & a_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}.$$

Show that if B is any other $N \times N$ circulant matrix, then $AB = BA$.

Question 5. Let $N = \{1, 2, \dots, n\}$. A permutation of N is an invertible map $N \rightarrow N$. Write the $n \times n$ identity matrix as

$$I = [e_1 \mid e_2 \mid \cdots \mid e_n],$$

and let σ be a permutation of N . The matrix corresponding to σ is given by

$$P_\sigma = [e_{\sigma(1)} \mid e_{\sigma(2)} \mid \cdots \mid e_{\sigma(n)}].$$

Answer the following.

- (a) Derive an expression for the (i, j) entry of P_σ .
- (b) If A is any other $n \times n$ matrix, what does doing the multiplication AP_σ have?
- (c) If B is any other $n \times n$ matrix, what does doing the multiplication $P_\sigma B$ have?
- (d) Is P_σ invertible? If it is, what is its inverse?

Question 6. A diagonal matrix is one for which every entry not on the main diagonal is zero. Let A and B be $N \times N$ matrices such that there exists an invertible $N \times N$ matrix P for which $D_A = P^{-1}AP$ and $D_B = P^{-1}BP$ are diagonal matrices. Show that A and B commute.