Math 240 Tutorial
Solutions

May 16

Question 1. Place the following augmented matrices into an echelon form. Does the corresponding system of linear equations admit any solutions?

(a)

$$\left(\begin{array}{cccc|c} 4 & 8 & 12 & 4 & 7 \\ 2 & 5 & 6 & 6 & 11 \\ 0 & 5 & 1 & 26 & 13 \\ 0 & 5 & 0 & 21 & 17 \end{array}\right).$$

An echelon form:

$$\left(\begin{array}{cccc|c} 2 & 5 & 6 & 6 & 11 \\ 0 & 2 & 0 & 8 & 15 \\ 0 & 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 2 & -41 \end{array}\right).$$

Every column except the last is a pivot column, so the system has a unique solution.

(b)

$$\left(\begin{array}{cccc|c} 4 & 8 & 12 & 4 & 0 \\ 2 & 5 & 6 & 6 & 0 \\ 0 & 5 & 1 & 25 & 0 \\ 0 & 5 & 0 & 20 & 0 \end{array}\right).$$

An echelon form:

$$\left(\begin{array}{cccc|c} 2 & 5 & 6 & 6 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The corresponding system has infinitely many solutions.

(c)

$$\left(\begin{array}{cccc|c} 4 & 8 & 12 & 4 & 7 \\ 2 & 5 & 6 & 6 & 11 \\ 0 & 5 & 1 & 25 & 13 \\ 0 & 5 & 0 & 20 & 17 \end{array}\right).$$

An echelon form:

$$\left(\begin{array}{cccc|c} 2 & 5 & 6 & 6 & 11 \\ 0 & 2 & 0 & 8 & 15 \\ 0 & 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 & 41 \end{array}\right).$$

Since the last column is a pivot column, the corresponding system of linear equations is inconsistent.

Question 2. Consider the following system of equations

$$\begin{aligned} w + x + y + z &= 6, \\ w + y + z &= 4, \\ w + y &= 2. \end{aligned}$$

(a) List the leading variables.

The augmented matrix for the linear system of equations is given by

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 1 & 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 & 2 \end{array}\right). \tag{1}$$

The Gauss–Jordan form of this augmented matrix is then seen to be

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right). \tag{2}$$

The leading variables corresponding to the pivot columns are then $w,\ y,$ and z.

(b) List the free variables.

Recall the free variables are those variables of the system of linear equations that do not correspond to pivot columns. By (2), we have the single free variable y.

(c) Write the general solution to the equation (expressed in terms of the free variables).

By the Gauss–Jordan form (1) of the augmented matrix, we have the general solution is then

$$(w, x, y, z) = (2 - y, 2, y, 2).$$

(d) Suppose a fourth equation $-2w + y = 5$ is added to the system. What is the solution of the resulting system?

The augmented matrix for the new system is

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 1 & 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 & 2 \\ -2 & 0 & 1 & 0 & 5 \end{array}\right).$$

The corresponding Gauss–Jordan form is then seen to be

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

Therefore, this system has a unique solution given by

$$(w, x, y, z) = (-1, 2, 3, 2).$$

(e) Suppose the fourth equation is $-2w - 2y = -3$. What can we say about the solutions for the resulting system?

The augmented matrix for the new system is

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 1 & 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 & 2 \\ -2 & 0 & -2 & 0 & -3 \end{array}\right).$$

In order for this system to be consistent, the third and fourth rows must be proportional. Clearly, they are not; so, the system is inconsistent. We can also see this by the Gaussian form which is given by

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Since the last column is a pivot column, the system is inconsistent.

Question 3. Find the values of k for which the system of equations

$$\begin{aligned} x + ky &= 1, \\ kx + y &= 1, \end{aligned}$$

has

(a) no solution,

The augmented matrix for the system of linear equations is given by

$$\left(\begin{array}{cc|c} 1 & k & 1 \\ k & 1 & 1 \end{array}\right).$$

If $k = 0$, then there is a unique solution given by $(x, y) = (1, 1)$; so, we assume that $k \neq 0$. The Gaussian form of the matrix is then

$$\left(\begin{array}{cc|c} 1 & k & 1 \\ 0 & \frac{1}{k} - k & \frac{1}{k} - 1 \end{array}\right)$$

If $\frac{1}{k} - k \neq 0$, that is, if $|k| \neq 1$, then there is a unique solution given by $(x, y) = ((1 + k)^{-1}, (1 + k)^{-1})$.

It remains to examine the case that $|k| = 1$. If $k = 1$, then we have the equation $x + y = 1$. This has infinitely many solutions. If $k = -1$, then we have the have the system

$$\begin{aligned} x - y &= 1, \\ -x + y &= 1. \end{aligned}$$

Adding these two equations gives $0 = 2$, a contradiction. Therefore, there is no solution only in the case that $k = -1$.

(b) a unique solution, and

From our work in part (a), there is a unique solution whenever $|k| \neq 1$.

(c) infinitely many solutions.

From our work in part (a), there are infinitely many solutions in the case that $k = 1$.

(d) When there is exactly one solution, what are the values of x and y.

By part (a), this happens whenever $|k| \neq 1$. If $k = 0$, then $(x, y) = (1, 1)$. If $k \neq 0$ and $|k| \neq 1$, then

$$(x, y) = \left(\frac{1}{1+k}, \frac{1}{1+k} \right).$$

Question 4. Consider the following system of linear equations

$$\begin{aligned} u + 2v - w - 2x + 3y &= b_1, \\ x - y + 2z &= b_2, \\ 2u + 4v - 2w - 4x + 7y - 4z &= b_3, \\ -x + y - 2z &= b_4, \\ 3u + 6v - 3w - 6x + 7y + 8z &= b_5, \end{aligned}$$

where $b_1,\ b_2,\ b_3,\ b_4,\ b_5 \in \mathbf{R}$.

(a) What are the leading and free variables?

The augmented matrix for the system of linear equations is given by

$$\left(\begin{array}{rrrrrr|l} 1 & 2 & -1 & -2 & 3 & 0 & b_1 \\ 0 & 0 & 0 & 1 & -1 & 2 & b_2 \\ 2 & 4 & -2 & -4 & 7 & -4 & b_3 \\ 0 & 0 & 0 & -1 & 1 & -2 & b_4 \\ 3 & 6 & -3 & -6 & 7 & 8 & b_5 \end{array}\right).$$

The Gaussian form is then seen to be

$$\left(\begin{array}{rrrrrr|r} 1 & 2 & -1 & -2 & 3 & 0 & b_1 \\ 0 & 0 & 0 & 1 & -1 & 2 & b_2 \\ 0 & 0 & 0 & 0 & 1 & -4 & b_3 - 2b_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_2 + b_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2b_3 + b_5 - 7b_1 \end{array}\right). \tag{3}$$

The leading variables correspond to the pivot columns. So, the leading variables are given by $u,\ x,$ and y. The free variables, which correspond the non-pivot columns, are given by $v,\ w,$ and z.

(b) What conditions must the real constants $b_1,\ b_2,\ b_3,\ b_4,\ b_5$ satisfy in order that the system be consistent?

From (3), we see that $b_2 = -b_4$ and $2b_3 + b_5 = 7b_1$.

(c) Do the numbers $b_1 = 1$, $b_2 = -3$, $b_3 = 2$, $b_4 = b_5 = 3$ satisfy the conditions of part (b)? If so, find the general solution in terms of the free variables.

With the given values, we have that $b_2 = -3 = -b_4$ and $2b_3 + b_5 = 4 + 3 = 7(1) = 7b_1$; so, the given constant satisfy the requirements given in part (b). The Gaussian form (3) becomes

$$\left(\begin{array}{rrrrrr|r} 1 & 2 & -1 & -2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{4}$$

The Gauss–Jordan form is then

$$\left(\begin{array}{rrrrrr|r} 1 & 2 & -1 & 0 & 0 & 8 & -5 \\ 0 & 0 & 0 & 1 & 0 & -2 & -3 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{5}$$

It follows from (5) that the general solution is given by

$$(u, v, w, x, y, z) = (w - 2v - 8z - 5, v, w, 2z - 3, 4z, z).$$