Math 240 Tutorial
Soltuions

June 13

Question 1. Show the following for $\mathbf{R}^n$.

(a) Show that scalar multiplication is a linear transformation.

Fix $a \in \mathbf{R}$, and let $T : \mathbf{R}^n \to \mathbf{R}^n$ be the map $T(\vec{v}) = a\vec{v}$. Then $T(b\vec{v}) = ab\vec{v} = ba\vec{v} = bT(\vec{v})$ and $T(\vec{v} + \vec{u}) = a(\vec{v} + \vec{u}) = a\vec{v} + a\vec{u} = T(\vec{v}) + T(\vec{u})$. We have shown that T is linear.

(b) When is this linear map invertible?

This map is invertible precisely in the case the scalar by which we are multiplying is nonzero.

(c) Is its inverse a linear transformation?

Let T be as in part (a), and assume that $a \neq 0$. Then T is invertible and T^{-1} is given by multiplcation by a^{-1}. Since this is multiplcation by a scalar, it is linear.

(d) Fix an element $a \in \mathbf{R}^n$. What is the matrix corresponding to the linear transformation $\vec{v} \mapsto a\vec{v}$ with respect to the standard spanning vectors?

Recall the standard spanning vectors are $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ where $\vec{e}_i$ is the vector with a 1 in position i and zeros everywhere else. Then the matrix corresponding to multiplication by a is givn by

$$[T(\vec{e}_1)|T(\vec{e}_2)|\cdots|T(\vec{e}_n)] = [a\vec{e}_1|a\vec{e}_2|\cdots|a\vec{e}_n] = aI.$$

Question 2. Give the matrix for the transformation that rotates vectors in $\mathbf{R}^2$ by $2\pi/3$ radians.

We consider the standard basis vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The vector e_1 maps to $(-1/2, \sqrt{3}/2)$, and the vector e_2 maps to $(-1/2, -\sqrt{3}/2)$. So the matrix for the transformation is given by

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}.$$

Question 3. Fix $a \in \mathbf{R}$ and $\vec{u} \in \mathbf{R}^n$ with $\vec{u} \neq \vec{0}$. Is the map given by $\vec{v} \mapsto a\vec{v} + \vec{u}$, linear? Why or why not?

No; it is not a linear map. Let $\vec{v}_1, \vec{v}_2 \in \mathbf{R}^n$. Then $\vec{v}_1 + \vec{v}_2 \mapsto a\vec{v}_1 + a\vec{v}_2 + \vec{u}$. But $\vec{v}_1 \mapsto a\vec{v}_1 + \vec{u}$ and $\vec{v}_2 \mapsto a\vec{v}_2 + \vec{u}$. However, $(a\vec{v}_1 + \vec{u}) + (a\vec{v}_2 + \vec{u}) = a\vec{v}_1 + a\vec{v}_2 + 2\vec{u} \neq a\vec{v}_1 + a\vec{v}_2 + \vec{u}$.

Question 4. Consider a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$, and define $\text{Ker}(T) = \{\vec{v} \in \mathbf{R}^n : T(\vec{v}) = \vec{0}\}$. This is the kernel of the linear transformation T. For $\vec{v} \in \mathbf{R}^n$, define $\vec{v} + \text{Ker}(T) = \{\vec{v} + \vec{u} : \vec{u} \in \text{Ker}(T)\}$. Show the following.

(a) $\text{Ker}(T)$ is closed under scalar multiplcation and vector addition.

Let $\vec{v}_1,\ \vec{v}_2 \in \text{Ker}(T)$. Then $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{0} + \vec{0} = \vec{0}$, so $\vec{v}_1 + \vec{v}_2 \in \text{Ker}(T)$. Similarly, $T(a\vec{v}) = aT(\vec{v}) = a\vec{0} = \vec{0}$ whenever $\vec{v} \in \text{Ker}(T)$; so, $a\vec{v} \in \text{Ker}(T)$. This shows that $\text{Ker}(T)$ is closed under vector addition and scalar multiplication.

(b) For $\vec{v} \in \mathbf{R}^n$, show that $\vec{v} + \text{Ker}(T)$ consists of all and only those elements of $\mathbf{R}^n$ that map to $T(\vec{v})$ under T.

Let $V = \{u \in \mathbf{R}^n : T(\vec{u}) = T(\vec{v})\}$. We show $\vec{v} + \text{Ker}(T) = V$. Indeed, let $\vec{u} \in V$. Then $T(\vec{v}) = T(\vec{u})$ so that $T(u - v) = \vec{0}$. It follows that there is an $\vec{x} \in \text{Ker}(T)$ such that $\vec{u} - \vec{v} = \vec{x}$, that is, $\vec{u} = \vec{v} + \vec{x}$. This means $\vec{u} \in \vec{v} + \text{Ker}(T)$.

Conversely, let $\vec{u} \in \vec{v} + \text{Ker}(T)$. Then there is some $\vec{x} \in \text{Ker}(T)$ for which $\vec{u} = \vec{v} + \vec{x}$. It follows that $T(\vec{u}) = T(\vec{v} + \vec{x}) + T(\vec{v}) + T(\vec{x}) = T(\vec{v}) + \vec{0} = T(\vec{v})$. So, $\vec{u} \in V$, as required.

(c) For $\vec{v}_1, \vec{v}_2 \in \mathbf{R}^n$, show that either $\vec{v}_1 + \text{Ker}(T) = \vec{v}_2 + \text{Ker}(T)$ or $\vec{v}_1 + \text{Ker}(T) \cap \vec{v}_2 + \text{Ker}(T) = \emptyset$.

Let $\vec{v}_1,\ \vec{v}_2 \in \mathbf{R}^n$, and suppose that $\vec{v}_1 + \text{Ker}(T) \cap \vec{v}_2 + \text{Ker}(T) \neq \emptyset$. Then there is some $\vec{u} \in v_1 + \text{Ker}(T) \cap \vec{v}_2 + \text{Ker}(T)$. By definition, there must be $\vec{u}_1,\ \vec{u}_2 \in \text{Ker}(T)$ such that $\vec{u} = \vec{v}_1 + \vec{u}_1 = \vec{v}_2 = \vec{u}_2$. But then $\vec{v}_1 - \vec{v}_2 = \vec{u}_2 - \vec{u}_1 = \vec{u}_3$ for some $\vec{u}_3 \in \text{Ker}(T)$. It follows that $\vec{v}_1 = \vec{v}_2 + \vec{u}_3$, so $\vec{v}_1 \in \vec{v}_2 + \text{Ker}(T)$. Similarly, $\vec{v}_2 = \vec{v}_1 - \vec{u}_3$, so $\vec{v}_2 \in \vec{v}_1 + \text{Ker}(T)$. It follows, therefore, that $\vec{v}_1 + \text{Ker}(T) = \vec{v}_2 + \text{Ker}(T)$.

Question 5. Find a matrix A such that $A^4 = O$, but no smaller positive power A is O.

The matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

satisfies the stated requirements.

Question 6. The trace of a square matrix A of dimensions $N \times N$ is defined as $\text{tr}(A) = \sum_{k=1}^{N} A_{k,k}$, i.e., the sum of the diagonal entries of the matrix. For any other $N \times N$ matrix B, show that $\text{tr}(AB) = \text{tr}(BA)$.

Observe

$$\begin{aligned} \text{tr}(AB) &= \sum_{k=1}^{N} (AB)_{k,k} \\ &= \sum_{k=1}^{N} \sum_{j=1}^{N} A_{k,j} B_{j,k} \\ &= \sum_{k=1}^{N} \sum_{j=1}^{N} A_{j,k} B_{k,j} \\ &= \text{tr}(BA) \end{aligned}$$

where the second to last equality follows because

$$\{(k,j,j,k) : 1 \leqq j,k \leqq N\} = \{(j,k,k,j) : 1 \leqq j,k \leqq N\}.$$

Question 7. An $N \times N$ matrix A is circulant if it is of the form

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_N \\ a_N & a_1 & a_2 & \cdots & a_{N-1} \\ a_{N-1} & a_N & a_1 & \cdots & a_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}.$$

Show that if B is any other $N \times N$ circulant matrix, then $AB = BA$.

Define the $N \times N$ matrix G by

$$G = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

If we multiply an $N \times N$ matrix C on the right by G, the resulting matrix is the one obtained by cyclically shifting the columns of C to the right. In particular, $G^N = I$ and $G^j \neq I$ for any $j \in \{1, \ldots, N-1\}$. We also note that we can write A and B by

$$A = \sum_{i=1}^{N} a_i G^{i-1}, \qquad B = \sum_{i=1}^{N} b_i G^{i-1},$$

that is, A and B are polynomials in G. Since they are each polynomials in G, it is easy to see that they must commute.

Question 8. Let $N = \{1, 2, \ldots, n\}$. A permutation of N is an invertible map $N \to N$. Write the $n \times n$ identity matrix as

$$I = [e_1 \mid e_2 \mid \cdots \mid e_n],$$

and let σ be a permutation of N. The matrix corresponding to σ is given by

$$P_\sigma = [e_{\sigma(1)} \mid e_{\sigma(2)} \mid \cdots \mid e_{\sigma(n)}].$$

Answer the following.

(a) Derive an expression for the (i,j) entry of P_σ.

Recall the so-called Kronecker delta function defined by

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

By definition, the (i,j) entry of P_σ is 1 exactly in the case that $i = \sigma^{-1}(j)$. So, we write $P_{\sigma_{i,j}} = \delta_{i,\sigma^{-1}j}$.

(b) If A is any other $n \times n$ matrix, what effect does doing the multiplcation AP_σ have?

By definition, we have

$$AP_\sigma = A[e_{\sigma(1)} \mid e_{\sigma(2)} \mid \cdots \mid e_{\sigma(n)}] = [Ae_{\sigma(1)} \mid Ae_{\sigma(2)} \mid \cdots \mid Ae_{\sigma(n)}],$$

but Ae_j is simply the j-th column of A; so, the effect of multiplying on the right by P_σ is simply to apply the permutation σ to the columns of A.

We could also infer this from part (a). Observe

$$(AP_\sigma)_{i,j} = \sum_k A_{i,k} P_{\sigma_{k,j}} = A_{i,\sigma^{-1}(j)}.$$

(c) If B is any other $n \times n$ matrix, what effect does doing the multiplcation $P_\sigma B$ have?

Here, we need to employ part (a). We have

$$(P_\sigma B)_{i,j} = \sum_k P_{\sigma_{i,k}} B_{k,j} = B_{\sigma(i),j}.$$

This means that the permutation σ^{-1} is applied to the rows of B.

(d) Is P_σ invertible? If it is, what is its inverse?

Yes; it is invertible. The inverse of P_σ is given by $P_{\sigma^{-1}}$. From part (a), we note that the (i, j) entry of P_σ^t is $\delta_{j,\sigma^{-1}(i)} = \delta_{i,\sigma(j)}$, which is applying the permutation σ^{-1} to the columns of I. We have shown $P_\sigma^{-1} = P_{\sigma^{-1}} = P_\sigma^t$.

(e) How many columns(rows) are fixed by P_σ.

If we are multiyipling by P_σ on the right, then we are interested in the number of fixed columns. This is equal to the number of indices i such that $\sigma(i) = i$ which implies $e_{\sigma(i)} = e_i$. Note that this is the same as counting the number of 1s along the diagonal. Since the entries of P_σ different than 1 are 0, it follows that the number of columns fixed by P_σ is given by $\mathrm{tr}(P_\sigma)$. For this reason, $\mathrm{tr}(P_\sigma)$ is often called the permutation character of P_σ.

Question 9. A diagonal matrix is one for which every entry not on the main diagonal is zero. Let A and B be $N \times N$ matrices such that there exists and invertible $N \times N$ matrix P for which $D_A = P^{-1}AP$ and $D_B = P^{-1}BP$ are diagonal matrices. Show that A and B commute.

Since $D_A = P^{-1}AP$ and $D_B = P^{-1}BP$, we have that $A = PD_AP^{-1}$ and $B = PD_BP^{-1}$. Then

$$\begin{aligned}
AB &= (PD_AP^{-1})(PD_BP^{-1}) \\
&= PD_A(P^{-1}P)D_BP^{-1} \\
&= PD_AID_BP^{-1} \\
&= PD_AD_BP^{-1} \\
&= PD_BD_AP^{-1} \\
&= PD_BID_AP^{-1} \\
&= PD_B(P^{-1}P)D_AP^{-1} \\
&= (PD_BP^{-1})(PD_AP^{-1}) \\
&= BA
\end{aligned}$$

where we have used the fact that diagonal matrices commute.