

Math 240 Tutorial Solutions

August 1

Question 1. Find the unit vector in the direction of the given vectors (a) $\begin{pmatrix} -30 \\ 40 \end{pmatrix}$, (b) $\begin{pmatrix} 7/4 \\ 1/2 \\ 1 \end{pmatrix}$, and (c) $\begin{pmatrix} 8/3 \\ 2 \end{pmatrix}$.

(a) We have $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{(-30)^2 + 40^2} = 50$ so that the unit vector is given by $\frac{\vec{v}}{\|\vec{v}\|} = \begin{pmatrix} -3/5 \\ 4/5 \end{pmatrix}$.

(b) We have $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{(7/4)^2 + (1/2)^2 + 1^2} = \frac{1}{2}\sqrt{49 + 4 + 16} = \frac{\sqrt{69}}{2}$ so that the unit vector is given by $\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{69}} \begin{pmatrix} 7/2 \\ 1 \\ 2 \end{pmatrix}$.

(c) We have $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{(8/3)^2 + 2^2} = \frac{10}{3}$ so that the unit vector is given by $\frac{\vec{v}}{\|\vec{v}\|} = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$.

Question 2. (a) Let $\vec{u}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$, and $\vec{x} = \begin{pmatrix} 9 \\ -7 \end{pmatrix}$. Does $\{\vec{u}_1, \vec{u}_2\}$ form an orthogonal basis for \mathbf{R}^2 ? If it does, write \vec{x} in terms of this basis. (b) Compute the orthogonal projection of $\vec{x} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$ onto the line through $\vec{y} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ and the origin.

(a) Note that $\langle \vec{u}_1, \vec{u}_2 \rangle = 12 - 12 = 0$ so that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis for \mathbf{R}^2 . We have

$$\vec{x} = \frac{\langle \vec{x}, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1 + \frac{\langle \vec{x}, \vec{u}_2 \rangle}{\|\vec{u}_2\|^2} \vec{u}_2 = \frac{39}{13} \vec{u}_1 + \frac{13}{52} \vec{u}_2.$$

(b) The projection is given by

$$\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2} \vec{y} = \frac{-4 + 14}{16 + 4} \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Question 3. Let $\vec{y} \in \mathbf{R}^n$. Prove $\vec{x} \mapsto \langle \vec{x}, \vec{y} \rangle$ is a linear transformation $\mathbf{R}^n \rightarrow \mathbf{R}$.

Let $\vec{w}, \vec{x} \in \mathbf{R}^n$, and let $\alpha \in \mathbf{R}$. Then

$$\langle \vec{w} + \vec{x}, \vec{y} \rangle = \sum_{j=1}^n (w_j + x_j) y_j = \sum_{j=1}^n w_j y_j + \sum_{j=1}^n x_j y_j = \langle \vec{w}, \vec{y} \rangle + \langle \vec{x}, \vec{y} \rangle.$$

and

$$\langle \alpha \vec{w}, \vec{y} \rangle = \sum_{j=1}^n \alpha w_j y_j = \alpha \sum_{j=1}^n w_j y_j = \alpha \langle \vec{w}, \vec{y} \rangle.$$

It follows, therefore, that the map is a linear map.

Question 4. Let

$$\vec{y} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}, \quad \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Verify that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set, and find the orthogonal projection of \vec{y} onto $\text{span}\{\vec{u}_1, \vec{u}_2\}$. Construct a nonzero vector \vec{z} that is orthogonal to \vec{u}_1 and \vec{u}_2 . Find the distance from \vec{y} to $\text{span}\{\vec{u}_1, \vec{u}_2\}$.

Observe $\langle \vec{u}_1, \vec{u}_2 \rangle = -1 + 1 = 0$, so they are orthogonal. To simplify the calculations, however, observe that $\text{span}\{\vec{u}_1, \vec{u}_2\} = \text{span}\{\hat{u}_1, \hat{u}_2\}$ where

$$\hat{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Applying Gram–Schmidt with this new basis gives that the projection is given by

$$\hat{y} = \langle y, \hat{u}_1 \rangle \hat{u}_1 + \langle y, \hat{u}_2 \rangle \hat{u}_2 = -\hat{u}_1 + 4\hat{u}_2 = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}.$$

A vector orthogonal to the span is given by

$$\vec{z} = \vec{y} - \hat{y} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$$

Therefore, the distance from y to $\text{span}\{\vec{u}_1, \vec{u}_2\}$ is given by $\|\vec{z}\| = 3$.

Question 5. Let W be a subspace of \mathbf{R}^n with an orthogonal basis $\beta_1 = \{\vec{w}_1, \dots, \vec{w}_p\}$, and let $\beta_2 = \{\vec{v}_1, \dots, \vec{v}_q\}$ be an orthogonal basis for W^\perp .

(a) Explain why $\beta_1 \cup \beta_2$ is an orthogonal set.

Every vector in β_1 is orthogonal to every other vector in β_1 as well as every vector in β_2 . Similarly, every vector in β_2 is orthogonal to every other vector in β_2 as well as every vector in β_1 .

(b) Explain why the set in part (a) spans \mathbf{R}^n .

Every vector \vec{x} in \mathbf{R}^n can be written uniquely as $\vec{x} = \vec{w}_1 + \vec{w}_2$ where $\vec{w}_1 \in W$ and $\vec{w}_2 \in W^\perp$.

(c) Show that $\dim(W) + \dim(W^\perp) = n$.

Part (b) shows that $\mathbf{R}^n = W \cup W^\perp$. Recall that $n = \dim(\mathbf{R}^n) = \dim(W \cup W^\perp) = \dim(W) + \dim(W^\perp) - \dim(W \cap W^\perp)$. But $W \cap W^\perp = \{0\}$ has dimension 0, so $\dim(W) + \dim(W^\perp) = n$.

Question 6. Let A be an $m \times n$ matrix with linearly independent columns, and let $A = QR$ be its QR -factorization. Prove that R is invertible with positive eigenvalues.

Note that if $R\vec{x} = \vec{0}$, then $A\vec{x} = QR\vec{x} = Q\vec{0} = \vec{0}$. Since A has linearly independent columns, it must be that $\vec{x} = \vec{0}$, and it follows that R is invertible.

The proof in the text shows that R is an upper triangular matrix with nonnegative diagonal entries. Since R is invertible, these entries are actually positive. The eigenvalues of a triangular matrix are its diagonal entries.

Question 7. Recall that $H = \text{span}\{x - 3, x^2 - 3x\}$ is the subspace of $\mathbf{P}_2(\mathbf{R})$ consisting of all those vectors divisible by $x - 3$. Do the following.

- (a) Verify that $\langle p, q \rangle \equiv \int_{-1}^1 pq \, dx$ is an inner product on $\mathbf{P}_n(\mathbf{R})$.

This simply follows by the linearity of the integral.

- (b) Use the Gram–Schmidt Process to find an orthogonal basis β for H .

We construct the orthogonal basis $\beta = \{\vec{f}_1, \vec{f}_2\}$. First, take $\vec{f}_1 = x - 3$. Then we calculate

$$\vec{f}_2 = x^2 - 3x - \frac{\langle x^2 - 3x, \vec{f}_1 \rangle}{\|\vec{f}_1\|^2} \vec{f}_1 = x^2 - \frac{39}{14}x - \frac{9}{14}.$$

- (c) Let T be the linear map $H \rightarrow \mathbf{P}_1(\mathbf{R})$ defined by $p \mapsto \frac{dp}{dx}$. Find the QR -factorization of $[T]_\beta^\gamma$ where γ is the standard basis $\{1, x\}$ of $\mathbf{P}_1(\mathbf{R})$.

We have

$$[T]_\beta^\gamma = \left[\left[\frac{df_1}{dx} \right]_\gamma \mid \left[\frac{df_2}{dx} \right]_\gamma \right] = \begin{pmatrix} 1 & -39/14 \\ 0 & 2 \end{pmatrix}.$$

But this is already in the required form.