

Math 240 Tutorial Questions

August 1

Question 1. Find the unit vector in the direction of the given vectors (a) $\begin{pmatrix} -30 \\ 40 \end{pmatrix}$, (b) $\begin{pmatrix} 7/4 \\ 1/2 \\ 1 \end{pmatrix}$, and (c) $\begin{pmatrix} 8/3 \\ 2 \end{pmatrix}$.

Question 2. (a) Let $\vec{u}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$, and $\vec{x} = \begin{pmatrix} 9 \\ -7 \end{pmatrix}$. Does $\{\vec{u}_1, \vec{u}_2\}$ form an orthogonal basis for \mathbf{R}^2 ? If it does, write \vec{x} in terms of this basis. (b) Compute the orthogonal projection of $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ onto the line through $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$ and the origin.

Question 3. Let $\vec{y} \in \mathbf{R}^n$. Prove $\vec{x} \mapsto \langle \vec{x}, \vec{y} \rangle$ is a linear transformation $\mathbf{R}^n \rightarrow \mathbf{R}$.

Question 4. Let

$$\vec{y} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}, \quad \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Verify that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set, and find the orthogonal projection of \vec{y} onto $\text{span}\{\vec{u}_1, \vec{u}_2\}$. Construct a nonzero vector \vec{z} that is orthogonal to \vec{u}_1 and \vec{u}_2 . Find the distance from \vec{y} to $\text{span}\{\vec{u}_1, \vec{u}_2\}$.

Question 5. Let W be a subspace of \mathbf{R}^n with an orthogonal basis $\beta_1 = \{\vec{w}_1, \dots, \vec{w}_p\}$, and let $\beta_2 = \{\vec{v}_1, \dots, \vec{v}_q\}$ be an orthogonal basis for W^\perp .

- (a) Explain why $\beta_1 \cup \beta_2$ is an orthogonal set.
- (b) Explain why the set in part (a) spans \mathbf{R}^n .
- (c) Show that $\dim(W) + \dim(W^\perp) = n$.

Question 6. Let A be an $m \times n$ matrix with linearly independent columns, and let $A = QR$ be its QR -factorization. Prove that R is invertible with positive eigenvalues.

Question 7. Do the following.

- (a) Verify that $\langle p, q \rangle \equiv \int_{-1}^1 pq \, dx$ is an inner product on $\mathbf{P}_n(\mathbf{R})$.
- (b) Recall the standard basis $\beta = \{1, x, x^2\}$ for $\mathbf{P}_2(\mathbf{R})$. Use the Gram–Schmidt Process to find an orthogonal basis β' for $\mathbf{P}_2(\mathbf{R})$.
- (c) Let T be the linear map $\mathbf{P}_2(\mathbf{R}) \rightarrow \mathbf{P}_1(\mathbf{R})$ defined by $p \mapsto \frac{dp}{dx}$. Find the QR -factorization of $[T]_{\beta'}$.