Math 240 Tutorial
Solutions

July 18

Question 1. For the following matrices, give a basis for their null space.

(a)

$$A = \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{pmatrix}.$$

We have

$$A \sim \begin{pmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 4 & -2 \end{pmatrix}.$$

So every vector in $\ker(A)$ can be written as

$$\begin{pmatrix} 7x - 6y \\ -4x + 2y \\ x \\ y \end{pmatrix} = x \begin{pmatrix} 7 \\ -4 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} -6 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, a basis is given by

$$\left\{ \begin{pmatrix} 7 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(b)

$$A = \begin{pmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

We have

$$A \sim \begin{pmatrix} 1 & -6 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

So every vector in $\ker(A)$ can be written as

$$\begin{pmatrix} 6x \\ x \\ 0 \\ y \end{pmatrix} = x \begin{pmatrix} 6 \\ 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, a basis is given by

$$\left\{ \begin{pmatrix} 6 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(c)

$$A = \begin{pmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix A is alread in reduced row echelon form, so we see that every vector in $\ker(A)$ can be written as

$$\begin{pmatrix} 2x - 4y \\ x \\ 9y \\ y \\ 0 \end{pmatrix} = x \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{pmatrix}.$$

Therefore, a basis is given by

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Question 2. Find a basis for the space spanned by

$$\begin{pmatrix} -8 \\ 7 \\ 6 \\ 5 \\ -7 \end{pmatrix}, \begin{pmatrix} 8 \\ -7 \\ -9 \\ -5 \\ 7 \end{pmatrix}, \begin{pmatrix} -8 \\ 7 \\ 4 \\ 5 \\ -7 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 9 \\ 6 \\ -7 \end{pmatrix}, \begin{pmatrix} -9 \\ 3 \\ -4 \\ -1 \\ 0 \end{pmatrix}.$$

We reduce the follow matrix

$$\begin{pmatrix} -8 & 8 & -8 & 1 & -9 \\ 7 & -7 & 7 & 4 & 3 \\ 6 & -9 & 4 & 9 & -4 \\ 5 & -5 & 5 & 6 & -1 \\ -7 & 7 & -7 & -7 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{5}{3} & 0 & \frac{4}{3} \\ 0 & 1 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

So the span of the vectors is the column space of the matrix which has the basis

$$\left\{ \begin{pmatrix} -8 \\ 7 \\ 6 \\ 5 \\ -7 \end{pmatrix}, \begin{pmatrix} 8 \\ -7 \\ -9 \\ -5 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 9 \\ 6 \\ -7 \end{pmatrix} \right\}.$$

Question 3. Given vectors $\vec{u}_1, \ldots, \vec{u}_p$ in a vector space V, show $\vec{x}$ is a linear combination of $\vec{u}_1, \ldots, \vec{u}_p$ if and only if $[\vec{x}]_B$ is a linear combination of $[\vec{u}_1]_B, \ldots, [\vec{u}_p]_B$.

This follows at once because change of basis is an invertible linear map.

Question 4. Find a basis for the vectors in $\mathbf{R}^4$ of the form

$$\begin{pmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{pmatrix}$$

where $a,\, b,\, c \in \mathbf{R}$.

We can write the vectors as

$$a + \begin{pmatrix} 3 \\ 6 \\ -9 \\ -3 \end{pmatrix} + b \begin{pmatrix} 6 \\ -2 \\ 5 \\ 1 \end{pmatrix} + c \begin{pmatrix} -1 \\ -2 \\ 3 \\ 1 \end{pmatrix}$$

A maximal linearly independent set of these three vectors is given by

$$\left\{ \begin{pmatrix} 3 \\ 6 \\ -9 \\ -3 \end{pmatrix}, \begin{pmatrix} 6 \\ -2 \\ 5 \\ 1 \end{pmatrix} \right\}.$$

Question 5. Find a basis for

$$H_1 = \{(a,\, b,\, c) : a - 3b + c = 0,\, b - 2c = 0,\, 2b - c = 0\}$$

and

$$H_2 = \{(a,\, b,\, c,\, d) : a - 3b + c = 0\}.$$

We recognize H_1 has the kernel of the matrix

$$A_1 = \begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & -1 \end{pmatrix}.$$

A_1 is invertible, so $\ker(A_1) = \{\vec{0}\}$ and $\dim \ker(A_1) = 0$.

H_2 is the kernel of a linear functional and has dimension $4 - 1 = 3$. It is the kernel of the matrix

$$A_2 = \begin{pmatrix} 1 & -3 & 1 & 0 \end{pmatrix}.$$

The vectors of this space are given by

$$\begin{pmatrix} 3b+c \\ b \\ c \\ d \end{pmatrix} = b \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Question 6. The the space $C(\mathbf{R})$ of all continuous functions on the real line is an infinite dimensional vector space.

We already have that the set $C(\mathbf{R})$ is a vector space under componentwise addition. That it is infinite dimensional follows from the fact that the space of all polynomials of finite degree $\mathbf{P}_\infty$ is a proper subspace.

Question 7. For an $n \times n$ matrix A, we use

$$A\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$$

to denote the determinant of the submatrix formed by choosing the rows $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$. Let $\vartheta_1, \ldots, \vartheta_n$ be the not necessarily distinct and possibly complex eigenvalues of A. Prove that

$$\sum_{1 \leqq i_1 < \cdots < i_k \leqq n} \vartheta_{i_1} \cdots \vartheta_{i_k} = \sum_{1 \leqq i_1 < \cdots < i_k \leqq n} A\begin{pmatrix} i_1 & \cdots & i_k \\ i_1 & \cdots & i_k \end{pmatrix}.$$

Use this to prove $\mathrm{tr}(A) = \sum_{i=1}^n A_{i,i} = \sum_{i=1}^n \vartheta_i$ and $\det(A) = \vartheta_1 \cdots \vartheta_n$. [Hint: You will need to consider the characteristic equation $\det(xI - A)$ and the multilinearity of the determinant.]