Math 240 Tutorial
Questions

August 1

Question 1. Find the unit vector in the direction of the given vectors (a) $\begin{pmatrix} -30 \\ 40 \end{pmatrix}$, (b) $\begin{pmatrix} 7/4 \\ 1/2 \\ 1 \end{pmatrix}$, and (c) $\begin{pmatrix} 8/3 \\ 2 \end{pmatrix}$.

Question 2. (a) Let $\vec{u}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$, and $\vec{x} = \begin{pmatrix} 9 \\ -7 \end{pmatrix}$. Does $\{\vec{u}_1,\, \vec{u}_2\}$ form an orthogonal basis for $\mathbf{R}^2$? If it does, write $\vec{x}$ in terms of this basis. (b) Compute the orthogonal projection of $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ onto the line through $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$ and the origin.

Question 3. Let $\vec{y} \in \mathbf{R}^n$. Prove $\vec{x} \mapsto \langle \vec{x},\, \vec{y} \rangle$ is a linear transformation $\mathbf{R}^n \to \mathbf{R}$.

Question 4. Let

$$\vec{y} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}, \quad \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Verify that $\{\vec{u}_1,\, \vec{u}_2\}$ is an orthogonal set, and find the orthogonal projection of $\vec{y}$ onto $\operatorname{span}\{\vec{u}_1,\, \vec{u}_2\}$. Construct a nonzero vector $\vec{z}$ that is orthogonal to $\vec{u}_1$ and $\vec{u}_2$. Find the distance from $\vec{y}$ to $\operatorname{span}\{\vec{u}_1,\, \vec{u}_2\}$.

Question 5. Let W be a subspace of $\mathbf{R}^n$ with an orthogonal basis $\beta_1 = \{\vec{w}_1, \ldots, \vec{w}_p\}$, and let $\beta_2 = \{\vec{v}_1, \ldots, \vec{v}_q\}$ be an orthogonal basis for $W^\perp$.

(a) Explain why $\beta_1 \cup \beta_2$ is an orthogonal set.

(b) Explain why the set in part (a) spans $\mathbf{R}^n$.

(c) Show that $\dim(W) + \dim(W) = n$.

Question 6. Let A be an $m \times n$ matrix with linearly independent columns, and let $A = QR$ be its QR-factorization. Prove that R is invertible with positive eigenvalues.

Question 7. Recall that $H = \operatorname{span}\{x - 3,\, x^2 - 3x\}$ is the subspace of $\mathbf{P}_2(\mathbf{R})$ consisting of all those vectors divisible by $x - 3$. Do the following.

(a) Verify that $\langle p, q \rangle \equiv \int_{-1}^{1} pq \,\mathrm{d}x$ is an inner product on $\mathbf{P}_n(\mathbf{R})$.

(b) Use the Gram–Schmidt Process to find an orthogonal basis β for H.

(c) Let T be the linear map $H \to \mathbf{P}_1(\mathbf{R})$ defined by $p \mapsto \frac{\mathrm{d}p}{\mathrm{d}x}$. Find the QR-factorization of $[T]_\beta^\gamma$ where $\gamma = \{1,\, x\}$ is the standard basis for $\mathbf{P}_1(\mathbf{R})$.