Math 240 Tutorial
Questions

May 23

Question 1. For each part, explain whether or not the stated matrix–vector multiplication can be carried out. If it can, do the multiplication.

(a)

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Question 2. Write the following linear system first as a vector equation and then as a matrix equation

$$\begin{aligned} u + 2v - w - 2x + 3y &= b_1, \\ x - y + 2z &= b_2, \\ 2u + 4v - 2w - 4x + 7y - 4z &= b_3, \\ -x + y - 2z &= b_4, \\ 3u + 6v - 3w - 6x + 7y + 8z &= b_5, \end{aligned}$$

where $b_1,\ b_2,\ b_3,\ b_4,\ b_5 \in \mathbf{R}$.

Question 3. For each of the following lists of row vectors in $\mathbf{R}^3$, determine whether the first vector can be expressed as a linear combination of the other two vectors.

(a) $(-2, 0, 3)$, $(1, 3, 0)$, $(2, 4, -1)$.

(b) $(1, 2, -3)$, $(-3, 2, 1)$, $(2, -1, -1)$.

(c) $(3, 4, 1)$, $(1, -2, 1)$, $(-2, -1, 1)$.

(d) $(2, -1, 0)$, $(1, 2, -3)$, $(1, -3, 2)$.

(e) $(5, 1, -5)$, $(1, -2, -3)$, $(-2, 3, -4)$.

(f) $(-2, 2, 2)$, $(1, 2, -1)$, $(-3, -3, 3)$.

Question 4. Consider the following three vectors in $\mathbf{R}^3$

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Show that $\mathbf{R}^3 = \operatorname{span}\{\vec{u}_1,\ \vec{u}_2,\ \vec{u}_3\}$.

Question 5. Show that $\operatorname{span}\{\vec{u},\ \vec{v},\ \vec{w}\} = \operatorname{span}\{\vec{u},\ \vec{v} + \vec{w},\ \vec{v} - \vec{w}\}$.

Question 6. Consider the following four vectors in $\mathbf{R}^4$ given by

$$\vec{v}_1 = \begin{pmatrix} +1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ +1 \\ -1 \\ -1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -1 \\ -1 \\ +1 \\ -1 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ +1 \end{pmatrix}.$$

(a) Show whether $\vec{v}_1 \in \text{span}\{\vec{v}_2,\, \vec{v}_3,\, \vec{v}_4\}$ or not by solving the corresponding system of linear equations.

(b) Let $a_1,\, a_2,\, a_3,\, a_4 \in \mathbf{R}$. Under what conditions on $a_1,\, a_2,\, a_3,\, a_4$ is $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + a_4\vec{v}_4 = \vec{0}$ true?

(c) How can we use part (b) to provide a second proof of part (a)? Can you generalize to answer the following question: Is $\vec{v}_i \in \text{span}\{\vec{v}_j,\, \vec{v}_k,\, \vec{v}_l\}$ for $i,\, j,\, k,\, l$ distinct?

Question 7. Let $\vec{v}_1,\, \vec{v}_2, \ldots,\, \vec{v}_n \in \mathbf{R}^n$ be such that if $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \vec{0}$ then $a_1 = a_2 = \cdots = a_n = 0$. Show this implies that every vector in $\text{span}\{\vec{v}_1,\, \vec{v}_2, \ldots,\, \vec{v}_n\}$ can be written *uniquely* as a linear combination of $\vec{v}_1,\, \vec{v}_2, \ldots,\, \vec{v}_n$.

Question 8. Let V_1 and V_2 be two subsets of $\mathbf{R}^n$, and define $V_1 + V_2 = \{\vec{v}_1 + \vec{v}_2 : \vec{v}_1 \in V_1 \text{ and } \vec{v}_2 \in V_2\}$. Show (1) $\text{span}(V_1 \cup V_2) = \text{span}(V_1) + \text{span}(V_2)$, and (2) $\text{span}(V_1 \cap V_2) \subseteq \text{span}(V_1) \cap \text{span}(V_2)$. Further, give an example of subsets V_1 and V_2 of $\mathbf{R}^n$, for some n, for which $\text{span}(V_1 \cap V_2) \subsetneq \text{span}(V_1) \cap \text{span}(V_2)$.