Math 240 Tutorial
Solutions

July 11

Question 1. Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}.$$

(a) Calculate the determinant of A using (1) cofactor expansion, and (2) row reduction.

Using cofactor expansion along the first row, we have

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \end{vmatrix} \\
&= \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} + 3\begin{vmatrix} -1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & -1 \end{vmatrix} \\
&= 8\begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix} \\
&= 8(-2) \\
&= -16.
\end{aligned}$$

Using row reduction,

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \\ 0 & -2 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & 2 \end{vmatrix} \\
&= \begin{vmatrix} 1 & -1 & -1 & -1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 4 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -1 & -1 & -1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 4 \end{vmatrix} \\
&= (-1)(-2)(-2)(4) \\
&= -16.
\end{aligned}$$

(b) Calculate the inverse using (1) the adjugate of A, and (2) row reduction.

Omitting the tedious details, we have

$$\operatorname{adj}(A) = \begin{pmatrix} -4 & 4 & 4 & 4 \\ 4 & -4 & 4 & 4 \\ 4 & 4 & -4 & 4 \\ 4 & 4 & 4 & -4 \end{pmatrix}$$

so that

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$$
$$= \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}.$$

(c) Do the columns (rows) of A form a basis for $\mathbf{R}^4$? If they do, give the change of basis matrix from the standard basis of $\mathbf{R}^4$ to the columns of A.

Since the matrix is invertible, the columns of A form a basis for $\mathbf{R}^4$. The change of basis matrix is simply A^{-1}.

Question 2. Consider the matrices

$$A = \begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}.$$

The matrix P is invertible. Find P^{-1} by any means and calculate $P^{-1}AP = D$. Prove that A is invertible if and only if D is invertible. If A is invertible, find its inverse by first finding the inverse of D and then multiplying D^{-1} by P^{-1} and P in some order.

We have

$$P^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -3 & 4 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

so that

$$P^{-1}AP = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since D is a diagonal matrix with each diagonal entry nonzero, it follows that it is invertible with inverse

$$D^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

From the properties of matrix inversion, we have

$$P^{-1}A^{-1}P = D^{-1}$$

so that

$$P^{-1}D^{-1}P = A^{-1}.$$

Doing the calculation, we find

$$A^{-1} = \begin{pmatrix} 2 & -3/2 & 1/2 \\ 3/2 & -1 & 1/2 \\ 3/2 & -3/2 & 1 \end{pmatrix}.$$

Question 3. Prove that the linear transformations of $\mathbf{R}^2$ consisting of compositions of reflections and rotations have determinants ± 1.

Using the geometry of $\mathbf{R}^2$, we see that every such transformation can be written as a product

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}.$$

The determinant of the right factor is 1, while the determinant of the left factor is -1.

Question 4. An isomorphism is an invertible linear transformation from one vector space onto another. Give two distinct isomorphisms $\mathbf{P}_3 \to \mathbf{R}^3$. NB: $\mathbf{R}^n$ (or $\mathbf{C}^n$) are often referred to as the coordinate spaces. This question shows that the coordinate representation of a vector is not in general unique; it depends on the choice of basis.

One isomorphism is the standard one given by $e_i \leftrightarrows x^i$. For another isomorphism, we simply need another basis for $\mathbf{P}_3$. One such basis is given by $\{1, x, x(x-1), x(x-1)(x-2)\}$. The isomorphism isn't difficult to find in this case as well.

Question 5. Define

$$H = \left\{ \begin{pmatrix} u & -u-x \\ 0 & x \end{pmatrix} : u, x \in \mathbf{R} \right\},$$
$$K = \left\{ \begin{pmatrix} v & 0 \\ w & -v \end{pmatrix} \right\}.$$

Do the following.

(a) H and K are subspaces of $M_{2\times 2}(\mathbf{R})$.

Note that $O \in H$ so that $H \neq \emptyset$. Define

$$A_1 = \begin{pmatrix} u_1 & -u_1 - x_1 \\ 0 & x_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} u_2 & -u_2 - x_2 \\ 0 & x_2 \end{pmatrix}.$$

Then

$$A_1 + A_2 = \begin{pmatrix} u_1 + u_2 & -(u_1+u_2) - (x_1+x_2) \\ 0 & x_1 + x_2 \end{pmatrix} \in H,$$

and

$$\alpha A_1 = \begin{pmatrix} \alpha u_1 & -\alpha u_1 - \alpha x_1 \\ 0 & \alpha x_1 \end{pmatrix} \in H.$$

Hence, H is a subspace of $M_{2\times 2}(\mathbf{R})$, Similarly, K is also a subspace of $M_{2\times 2}(\mathbf{R})$.

(b) Construct bases for H, K, $H + K$, and $H \cap K$.

Let $E_{i,j}$ be as in the solution to Question 3. Then $\{E_{1,1} - E_{1,2}, E_{2,2} - E_{1,2}\}$ is a basis for H, $\{E_{1,1} - E_{2,2}, E_{2,1}\}$ is a basis for K, and $\{E_{1,1} - E_{2,2}\}$ is a basis for $H \cap K$. Finally, we note that

$$H+K = \operatorname{span}\{E_{1,1}-E_{1,2}, E_{2,2}-E_{1,2}, E_{1,1}-E_{2,2}, E_{2,1}\} = \operatorname{span}\{E_{1,1}-E_{1,2}, E_{2,2}-E_{1,2}, E_{2,1}\}.$$

Since $\{E_{1,1} - E_{1,2}, E_{2,2} - E_{1,2}, E_{2,1}\}$ is linearly independent, this is a basis for $H + K$.

Question 6. Answer whether the following are subspaces of $\mathbf{R}^3$. The set of points $(x,\, y,\, z) \in \mathbf{R}^3$ such that

(a) $x + 2y - 3z = 4$,

No.

(b) $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z}{4}$,

No.

(c) $x + y + z = 0$ and $x - y + z = 1$,

No.

(d) $x = -z$ and $x = z$,

Yes.

(e) $x^2 + y^2 = z$, or

No.

(f) $\frac{x}{2} = \frac{y-3}{5}$.

No.