Math 240 Tutorial
Solutions

May 30

Question 1. Consider the set $S = \{(1,3,-4,2),(2,2,-4,0),(1,-3,2,-4),(-1,0,1,0)\}$ of vectors in $\mathbf{R}^4$. Show they form a linearly dependent set, and express one vector as a linear combination of the others.

We form a matrix whose columns are the given vectors

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & 2 & -3 & 0 \\ -4 & -4 & 2 & 1 \\ 2 & 0 & -4 & 0 \end{pmatrix}.$$

The reduced row echelon form is given by

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the number of pivot rows is less than the number of rows, the columns form a dependent set.

Since the third column is not a pivot column, we can express it as a linear combination of the other three. Reading the answer off from the reduced row echelon form given above, we have

$$\begin{pmatrix} 1 \\ -3 \\ 2 \\ -4 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 3 \\ -4 \\ 2 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 2 \\ 2 \\ -4 \\ 0 \end{pmatrix}.$$

Question 2. Consider the set $S = \{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1),(0,0,0,1)\}$ of vectors in $\mathbf{R}^4$. Show they form a linearly independent set. For a general vector $(a_1,a_2,a_3,a_4) \in \mathbf{R}^4$, derive the coefficients for this vector when it is expanded as a linear combination of the vectors in S.

We form a matrix whose columns are the given vectors

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{pmatrix}.$$

The reduced row echelon form is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This is the identity matrix of order four, which shows that the given vectors are linearly independent.

To find expressions for the coefficients a directed, we row reduce the following augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & 0 & a_3 \\ -1 & -1 & -1 & 1 & a_4 \end{array}\right).$$

The reduced row echelon form is given by

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & a_1 + a_2 + a_3 + a_4 \end{array}\right).$$

Therefore, an arbitrary vector in $\mathbf{R}^4$ can be expressed as

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} + (a_1 + a_2 + a_3 + a_4) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Question 3. Let S_1 and S_2 be finite subsets of $\mathbf{R}^n$, for some n, such that $S_1 \subsetneqq S_2$. Prove that if S_1 is a linearly dependent set, then so is S_2. Show that this is equivalent to if S_2 is a linearly independent set, then so is S_1.

Let S_1 and S_2 be finite subsets of $\mathbf{R}^n$ such that $S_1 \subsetneqq S_2$. Suppose that S_1 is linearly dependent, but S_2 is linearly independent. For definiteness, we write

$$S_1 = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$$

and

$$S_2 = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m, \vec{v}_{m+1}, \ldots, \vec{v}_k\}.$$

By assumption, there is a collection $a_1, a_2, \ldots, a_m$ of scalars such that $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m = \vec{0}$ but $a_i \neq 0$ for some $i \in \{1, 2, \ldots, m\}$. Set $a_{m+1} = a_{m+2} = \cdots = a_k = 0$ so that

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m + a_{m+1}\vec{v}_{m+1} + \cdots + a_k\vec{v}_k = \vec{0}.$$

By our assumption that S_2 is linear independent, we have

$$a_1 = a_2 = \cdots = a_m = a_{m+1} = \cdots = a_k = 0$$

by the definition of linearly independent. But this implies that each $a_i = 0$ for $i \in \{1, 2, \ldots, m\}$, contrary to what we have observed. This contradiction proves the result.

Question 4. Let S be a linearly independent set of $\mathbf{R}^n$, and let $\vec{v}$ be a vector in $\mathbf{R}^n$ that is not in S. Prove that $S \cup \{\vec{v}\}$ is linearly dependent if and only if $\vec{v} \in \operatorname{span}(S)$.

We may assume that $\vec{v} \neq \vec{0}$; for otherwise, the result is trivial. Write $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$. Suppose that $S \cup \{\vec{v}\}$ is linearly dependent but $\vec{v} \notin \vec{v} \in \operatorname{span}(S)$. Then there are scalars $a_1, a_2, \ldots, a_m, b$ not all zero such that

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m + b\vec{v} = \vec{0}.$$

If $b = 0$, then S is linearly dependent, contrary to assumption, hence $b \neq 0$. But then

$$\vec{v} = -\frac{a_1}{b}\vec{v}_1 - \frac{a_2}{b}\vec{v}_2 - \cdots - \frac{a_m}{b}\vec{v}_m$$

so that $\vec{v} \in \operatorname{span}(S)$, again, contrary to our assumption. Since we have shown that each possibility for b ends in a contradiction, we have that $\vec{v} \in \operatorname{span}(S)$ given $S \cup \{\vec{v}\}$ is dependent.

Conversely, assume that $S \cup \{\vec{v}\}$ is linearly independent, but $\vec{v} \in \text{span}(S)$. Then there are scalars $a_1, a_2, \ldots, a_m$ not all zero (recall $\vec{v} \neq \vec{0}$) for which

$$\vec{v} = a_1\vec{v}_1 + a_1\vec{v}_2 + \cdots + a_m\vec{v}_m.$$

But then

$$a_1\vec{v}_1 + a_1\vec{v}_2 + \cdots + a_m\vec{v}_m - \vec{v} = \vec{0}.$$

But then each $a_1 = a_2 = \cdots = a_m = 0$, contrary to assumption. Therefore, $\vec{v} \notin \text{span}(S)$ given $S \cup \{\vec{v}\}$ is linearly independent.

Question 5. Do the following.

(a) Let $\vec{u}$ and $\vec{v}$ be distinct vectors in $\mathbf{R}^n$. Prove that $\{\vec{u}, \vec{v}\}$ is linearly independent if and only if $\{\vec{u} + \vec{v}, \vec{u} - \vec{v}\}$ is linearly independent.

Suppose first that $\{\vec{u},\ \vec{v}\}$ is linearly independent, but $\vec{u} + \vec{v} = a(\vec{u} - \vec{v})$ for some $a \in \mathbf{R}$. If $a = 1$, then $\vec{v} = \vec{0}$, contradicting the fact that $\{\vec{u},\ \vec{v}\}$ is independent. So $a \neq 1$ and $\vec{u} = -\frac{1+a}{1-a}\vec{v}$, again, contradicting the fact that $\{\vec{u},\ \vec{v}\}$ is independent.

Conversely, suppose that $\vec{u} = b\vec{v}$ for some $b \in \mathbf{R}$. Then $\vec{u} + \vec{v} = (1 + b)\vec{u}$ and $\vec{u} - \vec{v} = (1 - b)\vec{u}$. But this means that both $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are in $\text{span}\{\vec{u}\}$, whereupon $\{\vec{u} + \vec{v}, \vec{u} - \vec{v}\}$ is linearly dependent.

(b) Let $\vec{u}, \vec{v}, \vec{w}$ be distinct vectors in $\mathbf{R}^n$. Prove that $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent if and only if $\{\vec{u} + \vec{v}, \vec{u} + \vec{w}, \vec{v} + \vec{w}\}$ is linear independent.

First, we assume that $\{\vec{u},\ \vec{v},\ \vec{w}\}$ is linearly independent, and we assume there are scalars $a,\ b,\ c$ such that

$$\vec{0} = a(\vec{u} + \vec{v}) + b(\vec{u} + \vec{w}) + c(\vec{v} + \vec{w}) = (a + b)\vec{u} + (a + c)\vec{v} + (b + c)\vec{w}.$$

Since we are assuming that $\{\vec{u},\ \vec{v},\ \vec{w}\}$ is linearly independent, it must be that $a + b = a + c = b + c = 0$. This linear system has only the trivial solution $a = b = c = 0$, whereupon $\{\vec{u} + \vec{v}, \vec{u} + \vec{w}, \vec{v} + \vec{w}\}$ is linearly independent as well.

Next, we assume that $\{\vec{u}+\vec{v}, \vec{u}+\vec{w}, \vec{v}+\vec{w}\}$ is independent. Suppose there are scalars $a,\ b,\ c$ such that $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$. Then

$$(a - b + c)(\vec{u} + \vec{v}) + (a + b - c)(\vec{u} + \vec{w}) + (-a + b + c)(\vec{v} + \vec{w}) = \vec{0}.$$

By our assumed linear independence of $\{\vec{u} + \vec{v}, \vec{u} + \vec{w}, \vec{v} + \vec{w}\}$, we have $a - b + c = a + b - c = -a + b + c = 0$. Solving, we find this possesses only the trivial solution $a = b = c = 0$. It follows $\{\vec{u},\ \vec{v},\ \vec{w}\}$ is independent as well.

Question 6. Show the following for $\mathbf{R}^n$.

(a) Show that scalar multiplication is a linear transformation.

Fix $a \in \mathbf{R}$, and let $T : \mathbf{R}^n \to \mathbf{R}^n$ be the map $T(\vec{v}) = a\vec{v}$. Then $T(b\vec{v}) = ab\vec{v} = ba\vec{v} = bT(\vec{v})$ and $T(\vec{v} + \vec{u}) = a(\vec{v} + \vec{u}) = a\vec{v} + a\vec{u} = T(\vec{v}) + T(\vec{u})$. We have shown that T is linear.

(b) When is this linear map invertible?

This map is invertible precisely in the case the scalar by which we are multiplying is nonzero.

(c) Is its inverse a linear transformation?

Let T be as in part (a), and assume that $a \neq 0$. Then T is invertible and T^{-1} is given by multiplcation by a^{-1}. Since this is multiplcation by a scalar, it is linear.

(d) Fix an element $a \in \mathbf{R}^n$. What is the matrix corresponding to the linear transformation $\vec{v} \mapsto a\vec{v}$ with respect to the standard spanning vectors?

Recall the standard spanning vectors are $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ where $\vec{e}_i$ is the vector with a 1 in position i and zeros everywhere else. Then the matrix corresponding to multiplication by a is givn by

$$[T(\vec{e}_1)|T(\vec{e}_2)|\cdots|T(\vec{e}_n)] = [a\vec{e}_1|a\vec{e}_2|\cdots|a\vec{e}_n] = aI.$$

Question 7. Fix $a \in \mathbf{R}$ and $\vec{u} \in \mathbf{R}^n$ with $\vec{u} \neq \vec{0}$. Is the map given by $\vec{v} \mapsto a\vec{v} + \vec{u}$ linear? Why or why not?

No; it is not a linear map. Let $\vec{v}_1, \vec{v}_2 \in \mathbf{R}^n$. Then $\vec{v}_1 + \vec{v}_2 \mapsto a\vec{v}_1 + a\vec{v}_2 + \vec{u}$. But $\vec{v}_1 \mapsto a\vec{v}_1 + \vec{u}$ and $\vec{v}_2 \mapsto a\vec{v}_2 + \vec{u}$. However, $(a\vec{v}_1 + \vec{u}) + (a\vec{v}_2 + \vec{u}) = a\vec{v}_1 + a\vec{v}_2 + 2\vec{u} \neq a\vec{v}_1 + a\vec{v}_2 + \vec{u}$.

Question 8. Consider a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$, and define $\text{Ker}(T) = \{\vec{v} \in \mathbf{R}^n : T(\vec{v}) = \vec{0}\}$. This is the kernel of the linear transformation T. For $\vec{v} \in \mathbf{R}^n$, define $\vec{v} + \text{Ker}(T) = \{\vec{v} + \vec{u} : \vec{u} \in \text{Ker}(T)\}$. Show the following.

(a) $\text{Ker}(T)$ is closed under scalar multiplcation and vector addition.

Let $\vec{v}_1, \vec{v}_2 \in \text{Ker}(T)$. Then $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{0} + \vec{0} = \vec{0}$, so $\vec{v}_1 + \vec{v}_2 \in \text{Ker}(T)$. Similarly, $T(a\vec{v}) = aT(\vec{v}) = a\vec{0} = \vec{0}$ whenever $\vec{v} \in \text{Ker}(T)$; so, $a\vec{v} \in \text{Ker}(T)$. This shows that $\text{Ker}(T)$ is closed under vector addition and scalar multiplication.

(b) For $\vec{v} \in \mathbf{R}^n$, show that $\vec{v} + \text{Ker}(T)$ consists of all and only those elements of $\mathbf{R}^n$ that map to $T(\vec{v})$ under T.

Let $V = \{u \in \mathbf{R}^n : T(\vec{u}) = T(\vec{v})\}$. We show $\vec{v} + \text{Ker}(T) = V$. Indeed, let $\vec{u} \in V$. Then $T(\vec{v}) = T(\vec{u})$ so that $T(u - v) = \vec{0}$. It follows that there is an $\vec{x} \in \text{Ker}(T)$ such that $\vec{u} - \vec{v} = \vec{x}$, that is, $\vec{u} = \vec{v} + \vec{x}$. This means $\vec{u} \in \vec{v} + \text{Ker}(T)$.

Conversely, let $\vec{u} \in \vec{v} + \text{Ker}(T)$. Then there is some $\vec{x} \in \text{Ker}(T)$ for which $\vec{u} = \vec{v} + \vec{x}$. It follows that $T(\vec{u}) = T(\vec{v} + \vec{x}) + T(\vec{v}) + T(\vec{x}) = T(\vec{v}) + \vec{0} = T(\vec{v})$. So, $\vec{u} \in V$, as required.

(c) For $\vec{v}_1, \vec{v}_2 \in \mathbf{R}^n$, show that either $\vec{v}_1 + \text{Ker}(T) = \vec{v}_2 + \text{Ker}(T)$ or $\vec{v}_1 + \text{Ker}(T) \cap \vec{v}_2 + \text{Ker}(T) = \emptyset$.

Let $\vec{v}_1, \vec{v}_2 \in \mathbf{R}^n$, and suppose that $\vec{v}_1 + \text{Ker}(T) \cap \vec{v}_2 + \text{Ker}(T) \neq \emptyset$. Then there is some $\vec{u} \in v_1 + \text{Ker}(T) \cap \vec{v}_2 + \text{Ker}(T)$. By definition, there must be $\vec{u}_1, \vec{u}_2 \in \text{Ker}(T)$ such that $\vec{u} = \vec{v}_1 + \vec{u}_1 =$

$\vec{v}_2 = \vec{u}_2$. But then $\vec{v}_1 - \vec{v}_2 = \vec{u}_2 - \vec{u}_1 = \vec{u}_3$ for some $\vec{u}_3 \in \text{Ker}(T)$. It follows that $\vec{v}_1 = \vec{v}_2 + \vec{u}_3$, so $\vec{v}_1 \in \vec{v}_2 + \text{Ker}(T)$. Similarly, $\vec{v}_2 = \vec{v}_1 - \vec{u}_3$, so $\vec{v}_2 \in \vec{v}_1 + \text{Ker}(T)$. It follows, therefore, that $\vec{v}_1 + \text{Ker}(T) = \vec{v}_2 + \text{Ker}(T)$.