Math 240 Tutorial
Questions

July 25

Question 1. Let a finite dimensional vector space V have two bases β and β', and let Q be the transformation matrix from β'-coordinates to β-coordinates. Show that for any linear transformation $T : V \to V$, it holds that

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q.$$

Question 2. A scalar matrix is a matrix of the form λI for some scalar λ.

(a) Prove that is a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.

(b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.

(c) Prove that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is not diagonalizable.

Question 3. For each of the following linear operators T on a vector space V and ordered basis β, compute $[T]_\beta$ and determine whether β is a basis consisting of eigenvectors of T.

(a) $V = \mathbf{R}^2$, $T\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \left(\begin{smallmatrix} 10a-6b \\ 17a-10b \end{smallmatrix}\right)$, and $\beta = \{\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)\}$.

(b) $V = \mathbf{P}_1(\mathbf{R})$, $T = (a + bx) = (6a - 6b) + (12a - 11b)x$, and $\beta = \{3 + 4x, 2 + 3x\}$.

(c) $V = \mathbf{R}^3$, $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a+2b-2c \\ -4a-3b+2c \\ -c \end{pmatrix}$, and $\beta = \left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right\}$.

(d) $V = \mathbf{P}_2(\mathbf{R})$, $T(a + bx + cx^2) = (-4a + 2b - 2c) - (7a + 3b + 7c)x + (7a + b + 5c)x^2$, and $\beta = \{x - x^2, -1 + x^2, -1 - x + x^2\}$.

(e) $V = P_3(\mathbf{R})$, $T(a + bx + cx^2 + dx^3) = -d + (-c + d)x + (a + b - 2c)x^2 + (-b + c - 2d)x^3$, and $\beta = \{1 - x + x^3, 1 + x^2, 1, x + x^2\}$.

(f) $V = \mathcal{M}_{2\times 2}(\mathbf{R})$, $T\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} -7a-4b+4c-4d & b \\ -8a-4b+5c-4d & d \end{smallmatrix}\right)$, and $\beta = \left\{\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 2 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 2 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & 2 \end{smallmatrix}\right)\right\}$.

Question 4. For each of the following matrices $A \in \mathcal{M}_{n\times n}(F)$:

(i) Determine all the eigenvalues of A.

(ii) For each eigenvalue λ of A, find the set og eigenvectors corresponding to λ.

(iii) If possible, find a basis for F^n consisting of eigenvectors of A.

(iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(a) $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 2 \end{smallmatrix}\right)$ for $F = \mathbf{R}$.

(b) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ for $R = \mathbf{R}$.

(c) $A = \left(\begin{smallmatrix} i & 1 \\ 2 & -i \end{smallmatrix}\right)$ for $F = \mathbf{C}$.

(d) $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$ for $F = \mathbf{R}$.

Question 5. Prove the geometric multiplicity of an eigenvalue is at most the algebraic multiplicity.