Math 240 Tutorial
Questions

June 13

Question 1. Show the following for $\mathbf{R}^n$.

(a) Show that scalar multiplication is a linear transformation.

(b) When is this linear map invertible?

(c) Is its inverse a linear transformation?

(d) Fix an element $a \in \mathbf{R}^n$. What is the matrix corresponding to the linear transformation $\vec{v} \mapsto a\vec{v}$ with respect to the standard spanning vectors?

Question 2. Give the matrix for the transformation that rotates vectors in $\mathbf{R}^2$ by $2\pi/3$ radians.

Question 3. Fix $a \in \mathbf{R}$ and $\vec{u} \in \mathbf{R}^n$ with $\vec{u} \neq \vec{0}$. Is the map given by $\vec{v} \mapsto a\vec{v} + \vec{u}$, linear? Why or why not?

Question 4. Consider a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$, and define $\text{Ker}(T) = \{\vec{v} \in \mathbf{R}^n : T(\vec{v}) = \vec{0}\}$. This is the kernel of the linear transformation T. For $\vec{v} \in \mathbf{R}^n$, define $\vec{v} + \text{Ker}(T) = \{\vec{v} + \vec{u} : \vec{u} \in \text{Ker}(T)\}$. Show the following.

(a) $\text{Ker}(T)$ is closed under scalar multiplcation and vector addition.

(b) For $\vec{v} \in \mathbf{R}^n$, show that $\vec{v} + \text{Ker}(T)$ consists of all and only those elements of $\mathbf{R}^n$ that map to $T(\vec{v})$ under T.

(c) For $\vec{v}_1, \vec{v}_2 \in \mathbf{R}^n$, show that either $\vec{v}_1 + \text{Ker}(T) = \vec{v}_2 + \text{Ker}(T)$ or $\vec{v}_1 + \text{Ker}(T) \cap \vec{v}_2 + \text{Ker}(T) = \emptyset$.

Question 5. Find a matrix A such that $A^4 = O$, but no smaller positive power A is O.

Question 6. The trace of a square matrix A of dimensions $N \times N$ is defined as $\text{tr}(A) = \sum_{k=1}^{N} A_{k,k}$, i.e., the sum of the diagonal entries of the matrix. For any other $N \times N$ matrix B, show that $\text{tr}(AB) = \text{tr}(BA)$.

Question 7. An $N \times N$ matrix A is circulant if it is of the form

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_N \\ a_N & a_1 & a_2 & \cdots & a_{N-1} \\ a_{N-1} & a_N & a_1 & \cdots & a_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}.$$

Show that if B is any other $N \times N$ circulant matrix, then $AB = BA$.

Question 8. Let $N = \{1, 2, \ldots, n\}$. A permutation of N is an invertible map $N \to N$. Write the $n \times n$ identity matrix as

$$I = [e_1 \mid e_2 \mid \cdots \mid e_n],$$

and let σ be a permutation of N. The matrix corresponding to σ is given by

$$P_\sigma = [e_{\sigma(1)} \mid e_{\sigma(2)} \mid \cdots \mid e_{\sigma(n)}].$$

Answer the following.

(a) Derive an expression for the (i, j) entry of P_σ.

(b) If A is any other $n \times n$ matrix, what effect does doing the multiplcation AP_σ have?

(c) If B is any other $n \times n$ matrix, what effect does doing the multiplcation $P_\sigma B$ have?

(d) Is P_σ invertible? If it is, what is its inverse?

(e) How many columns(rows) are fixed by P_σ.

Question 9. A diagonal matrix is one for which every entry not on the main diagonal is zero. Let A and B be $N \times N$ matrices such that there exists and invertible $N \times N$ matrix P for which $D_A = P^{-1}AP$ and $D_B = P^{-1}BP$ are diagonal matrices. Show that A and B commute.