Math 240 Tutorial
Solutions

July 5

Question 1. Consider the vector space $\mathbf{R}^3$, and let

$$H = \left\{ \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} : a \in \mathbf{R} \right\}.$$

Answer the following.

(a) Show that H is a subspace of $\mathbf{R}^3$.

This is a clear consequence of the facts that H is nonempty and

$$\begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \beta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha+\beta \\ 0 \\ 0 \end{pmatrix} \in H,$$
$$\alpha \begin{pmatrix} \beta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha\beta \\ 0 \\ 0 \end{pmatrix} \in H.$$

(b) What is the dimension of H?

$\dim(H) = 1$.

(c) Construct a basis for H.

H is spanned by $\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$.

Question 2. Let $\mathbf{P}_3$ be the vector space of all polynomials of degree at most 3, and let

$$H = \{p(x) \in \mathbf{P}_3 : p(3) = 0\}.$$

Answer the following.

(a) Show that H is a subspace of $\mathbf{P}_3$.

If $p, q \in H$, then $(p+q)(3) = p(3) + q(3) = 0$ so that $p + q \in H$. Furthermore, $\alpha p(3) = 0$, so that $\alpha p \in H$ for every $\alpha \in \mathbf{R}$. Since $0 \in H$, H is nonempty. All this shows that H is a subspace of $\mathbf{P}_3$.

(b) What is the dimension of H?

If $p \in H$, then p can be written as $p(x) = (x-3)(a + bx + cx^2)$ for some $a, b, c \in \mathbf{R}$. From this, it is clear that $\dim(H) = 3$.

(c) Construct a basis for H.

Note that every $p \in H$ can be written as $(x-3)q$ where $q \in \mathbf{P}_2$. Conversely, every element of $\mathbf{P}_2$ gives an element of H upon multiplication by $x-3$. Since $\mathbf{P}_2 = \text{span}\{1, x, x^2\}$, we see that $H = \text{span}\{x-3, x^2-3x, x^3-3x^2\}$. Since the degrees of each of $\{x-3, x^2-3x, x^3-3x^2\}$ are distinct, they are independent.

(d) Let $\mathbf{P}_2$ be the vector space of polynomials of degree at most 2. $\mathbf{P}_2$ is a subspace of $\mathbf{P}_3$ (why?). Give an invertible linear transformation that maps $\mathbf{P}_2$ onto H. What is the matrix for the transformation with respect to the standard basis of $\mathbf{P}_3$?

The transformation is simply multiplication by $x-3$. The standard matrix is given by

$$T_A = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Question 3. Let m and n be positive integers. Show the following.

(a) The set $M_{m\times n}(\mathbf{R})$ of $m \times n$ matrices with real entries is a vector space.

Let $A, B \in M_{m\times n}(\mathbf{R})$. Then $A+B = B+A \in M_{m\times n}(\mathbf{R})$. If O is the $m \times n$ matrix of all zeros, then $A+O = O+A = A$. If $A \in M_{m\times n}(\mathbf{R})$, then $-A \in M_{m\times n}(\mathbf{R})$ and $A-A = -A+A = O$. We next note that matrix mutiplication is associative and $M_{m\times n}(\mathbf{R})$ is clearly nonempty. Now, if $\alpha, \beta \in \mathbf{R}$, then we know that $\alpha A \in M_{m\times n}$, $(\alpha\beta)A = \alpha(\beta A)$, $\alpha O = O$, and $1A = A$. All this shows that $M_{m\times n}(\mathbf{R})$ is a vector space.

(b) What is the dimension of $M_{m\times n}(\mathbf{R})$?

From below, $\text{dim}(M_{m\times n}(\mathbf{R})) = nm$.

(c) Construct a basis for $M_{m\times n}(\mathbf{R})$.

Let $E_{i,j}$ be the matrix whose (i,j)-th entry is 1 and every other is 0. Then $M_{m\times n}(\mathbf{R}) = \text{span}\{E_{i,j} : 1 \leqq i \leqq m,\ 1 \leqq j \leqq n\}$. Since these matrices are disjoint, they are independent.

(d) Show the subset of matrices with trace 0 forms a subspace of $M_{n\times n}(\mathbf{R})$. Use the Dimension Theorem to find the dimension of this subset. You will first need to show that the trace map is a linear map. Construct a basis for this subspace.

Let $A, B \in M_{n\times n}(\mathbf{R})$, and let $\alpha \in \mathbf{R}$. Then $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$ and $\text{tr}(\alpha A) = \alpha \text{tr}(A)$. This shows that $\text{tr} : M_{n\times n} \to \mathbf{R}$ is a linear map. The subset in question is then given by $\text{ker}(\text{tr})$, which we recognize as a linear subspace of $M_{n\times n}(\mathbf{R})$.

By the Dimension Theorem, $\text{nullity}(\text{tr}) + \text{rank}(\text{tr}) = \dim(M_{n\times n}(\mathbf{R}))$. Since $\alpha \in \mathbf{R}$ has the preimage

$$\begin{pmatrix} \alpha & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

the trace map is onto. Therefore, $\text{rank}(tr) = \dim(\mathbf{R}) = 1$. Furthermore, as $\dim(M_{n\times n}(\mathbf{R})) = n^2$, we obtain $\text{nullity}(\text{tr}) = n^2 - 1$.

With $E_{i,j}$ defined as above, we find that $\{E_{i,j}\}_{i\neq j} \cup \{E_{i,i} - E_{n,n}\}_{1\leq i<n}$ is a basis.

Question 4. Define

$$H = \left\{ \begin{pmatrix} u & -u-x \\ 0 & x \end{pmatrix} : u, x \in \mathbf{R} \right\},$$
$$K = \left\{ \begin{pmatrix} v & 0 \\ w & -v \end{pmatrix} \right\}.$$

Do the following.

(a) H and K are subspaces of $M_{2\times 2}(\mathbf{R})$.

Note that $O \in H$ so that $H \neq \emptyset$. Define

$$A_1 = \begin{pmatrix} u_1 & -u_1 - x_1 \\ 0 & x_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} u_2 & -u_2 - x_2 \\ 0 & x_2 \end{pmatrix}.$$

Then

$$A_1 + A_2 = \begin{pmatrix} u_1 + u_2 & -(u_1+u_2) - (x_1+x_2) \\ 0 & x_1 + x_2 \end{pmatrix} \in H,$$

and

$$\alpha A_1 = \begin{pmatrix} \alpha u_1 & -\alpha u_1 - \alpha x_1 \\ 0 & \alpha x_1 \end{pmatrix} \in H.$$

Hence, H is a subspace of $M_{2\times 2}(\mathbf{R})$, Similarly, K is also a subspace of $M_{2\times 2}(\mathbf{R})$.

(b) Construct bases for H, K, $H + K$, and $H \cap K$.

Let $E_{i,j}$ be as in the solution to Question 3. Then $\{E_{1,1} - E_{1,2}, E_{2,2} - E_{1,2}\}$ is a basis for H, $\{E_{1,1} - E_{2,2}, E_{2,1}\}$ is a basis for K, and $\{E_{1,1} - E_{2,2}\}$ is a basis for $H \cap K$. Finally, we note that

$$H+K = \text{span}\{E_{1,1}-E_{1,2}, E_{2,2}-E_{1,2}, E_{1,1}-E_{2,2}, E_{2,1}\} = \text{span}\{E_{1,1}-E_{1,2}, E_{2,2}-E_{1,2}, E_{2,1}\}.$$

Since $\{E_{1,1} - E_{1,2}, E_{2,2} - E_{1,2}, E_{2,1}\}$ is linearly independent, this is a basis for $H + K$.

Question 5. Your course text proves the Dimension Theorem by counting pivot positions in matrices. Prove the theorem by arguing from the general definitions, without recourse to matrices, in the following way. Given a linear transformation $T : V \to W$, take a basis for $\ker(T)$ and enlarge it to a basis for V. Apply T to the vectors that were added to the basis of $\ker(T)$, and argue that they form a linearly independent set that spans the range of T in W.

Let V and W be vector spaces where V has finite dimension, and let $T : V \to W$ be a linear transformation. Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis for $\ker(T)$, and enlarge it to a basis $B = \{\vec{v}_1, \dots, \vec{v}_k\, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ of V. We claim that $S = \{T(\vec{v}_{k+1}, \dots, T(\vec{v}_n))\}$ is a basis for $\text{range}(T)$.

First, we prove that S generates $\text{range}(T)$. Indeed,

$$\begin{aligned}\text{range}(T) &= \text{span}\{T(\vec{v}_1), \dots, T(\vec{v}_n)\}\\ &= \text{span}\{T(\vec{v}_{k+1}), \dots, T(\vec{v}_n)\}\\ &= \text{span}(S),\end{aligned}$$

where we have used the fact that $T(\vec{v}_i) = \vec{0}$ for $1 \leqq i \leqq k$.

We next show that S is linearly independent. Suppose there are scalars $\alpha_{k+1}, \dots, \alpha_n \in \mathbf{R}$ for which

$$\sum_{i=k+1}^{n} \alpha_i T(\vec{v}_i) = \vec{0}.$$

Because T is linear,

$$T\left(\sum_{i=k+1}^{n} \alpha_i \vec{v}_i\right) = \vec{0}$$

so that $\sum_{i=k+1}^{n} \alpha_i \vec{v}_i \in \ker(T)$. Hence, there are scalars $\beta_1, \dots, \beta_k \in \mathbf{R}$ for which

$$\sum_{i=1}^{k}(-\beta_i)\vec{v}_i + \sum_{i=k+1}^{n} \alpha_i \vec{v}_i = \vec{0}.$$

By the independence of B, each $\alpha_i = 0$ and each $\beta_i = 0$ so that S is linearly independent. Furthermore, this argument shows that $T(\vec{v}_{k+1}), \dots, T(\vec{v}_n)$ are distinct, whereupon $\text{rank}(T) = n - k$.

Question 6. Use the Dimension Theorem to show a linear transformation $T : V \to V$ is invertible if and only if it is onto if and only if it is one-to-one.

T is invertible if and only if $\text{rank}(T) = n$ if and only if T is onto. Also, by the Dimension Theorem, $\text{rank}(T) = n$ if and only if $\text{nullity}(T) = 0$ if and only if $\ker(T) = \{\vec{0}\}$ if and only if T is one-to-one.

Question 7. Let V be a vector space of finite dimension, and let H and K be subspaces of V. Show

$$\dim(H + K) = \dim(H) + \dim(K) - \dim(H \cap K).$$

Let $B_0 = \{\vec{v}_1, \dots, \vec{v}_m\}$ be a basis for $H \cap K$. First, enlarge B_0 to a basis

$$B_1 = \{\vec{v}_1, \dots, \vec{v}_m, \vec{u}_1, \dots, \vec{u}_j\}$$

of H. Next, enlarge B_0 to a basis

$$B_2 = \{\vec{v}_1, \dots, \vec{v}_m, \vec{w}_1, \dots, \vec{w}_k\}$$

of K. Note that $\dim(H \cap K) = m$, $\dim(H) = m + j$, and $\dim(K) = m + k$.

We have $\text{span}(B_1 \cup B_2) = H + K$ (why?). So it remains to show that

$$B_1 \cup B_2 = \{\vec{v}_1, \dots, \vec{v}_m, \vec{u}_1, \dots, \vec{u}_j, \vec{w}_1, \dots, \vec{w}_k\}$$

is linearly independent. Indeed, suppose there are scalars $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_j, \gamma_1, \dots, \gamma_k \in \mathbf{R}$ for which

$$\sum_{i=1}^{m} \alpha_i \vec{v}_i + \sum_{i=1}^{j} \beta_i \vec{u}_i + \sum_{i=1}^{k} \gamma_i \vec{w}_i = \vec{0}.$$

Rearranging,

$$\sum_{i=1}^{k} \gamma_i \vec{w}_i = -\sum_{i=1}^{m} \alpha_i \vec{v}_i - \sum_{i=1}^{j} \beta_i \vec{u}_i \in H.$$

Since $\sum_{i=1}^{k} \gamma_i \vec{w}_i \in K$, we have it is in $H \cap K$. So, there are scalars $\delta_1, \ldots, \delta_m \in \mathbf{R}$ for which

$$\sum_{i=1}^{k} \gamma_i \vec{w}_i - \sum_{j=1}^{m} \delta_j \vec{v}_j = \vec{0}.$$

By the independence of B_2, each $\gamma_i = 0$ and each $\delta_i = 0$. Therefore, our original equation becomes

$$\sum_{i=1}^{m} \alpha_i \vec{v}_i + \sum_{i=1}^{j} \beta_i \vec{u}_i + = \vec{0}.$$

By the independence of B_1, each $\alpha_i = 0$ and each $\beta_i = 0$, thus showing the independence of $B_1 \cup B_2$.