Math 240 Tutorial
Solutions

June 6

Systems of Linear Equations and Row Reduction

Question 1. Place the following augmented matrices into an echelon form. Does the corresponding system of linear equations admit any solutions?

(a)

$$\left(\begin{array}{cccc|c} 4 & 8 & 12 & 4 & 7 \\ 2 & 5 & 6 & 6 & 11 \\ 0 & 5 & 1 & 26 & 13 \\ 0 & 5 & 0 & 21 & 17 \end{array}\right).$$

An echelon form:

$$\left(\begin{array}{cccc|c} 2 & 5 & 6 & 6 & 11 \\ 0 & 2 & 0 & 8 & 15 \\ 0 & 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 2 & -41 \end{array}\right).$$

Every column except the last is a pivot column, so the system has a unique solution.

(b)

$$\left(\begin{array}{cccc|c} 4 & 8 & 12 & 4 & 0 \\ 2 & 5 & 6 & 6 & 0 \\ 0 & 5 & 1 & 25 & 0 \\ 0 & 5 & 0 & 20 & 0 \end{array}\right).$$

An echelon form:

$$\left(\begin{array}{cccc|c} 2 & 5 & 6 & 6 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The corresponding system has infinitely many solutions.

(c)

$$\left(\begin{array}{cccc|c} 4 & 8 & 12 & 4 & 7 \\ 2 & 5 & 6 & 6 & 11 \\ 0 & 5 & 1 & 25 & 13 \\ 0 & 5 & 0 & 20 & 17 \end{array}\right).$$

An echelon form:

$$\left(\begin{array}{cccc|c} 2 & 5 & 6 & 6 & 11 \\ 0 & 2 & 0 & 8 & 15 \\ 0 & 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 & 41 \end{array}\right).$$

Since the last column is a pivot column, the corresponding system of linear equations is inconsistent.

Question 2. Find the values of k for which the system of equations

$$\begin{aligned} x + ky &= 1, \\ kx + y &= 1, \end{aligned}$$

has

(a) no solution,

The augmented matrix for the system of linear equations is given by

$$\left(\begin{array}{cc|c} 1 & k & 1 \\ k & 1 & 1 \end{array}\right).$$

If $k = 0$, then there is a unique solution given by $(x, y) = (1, 1)$; so, we assume that $k \neq 0$. The Gaussian form of the matrix is then

$$\left(\begin{array}{cc|c} 1 & k & 1 \\ 0 & \frac{1}{k} - k & \frac{1}{k} - 1 \end{array}\right)$$

If $\frac{1}{k} - k \neq 0$, that is, if $|k| \neq 1$, then there is a unique solution given by $(x, y) = ((1 + k)^{-1}, (1 + k)^{-1})$.

It remains to examine the case that $|k| = 1$. If $k = 1$, then we have the equation $x + y = 1$. This has infinitely many solutions. If $k = -1$, then we have the have the system

$$\begin{aligned} x - y &= 1, \\ -x + y &= 1. \end{aligned}$$

Adding these two equations gives $0 = 2$, a contradiction. Therefore, there is no solution only in the case that $k = -1$.

(b) a unique solution, and

From our work in part (a), there is a unique solution whenever $|k| \neq 1$.

(c) infinitely many solutions.

From our work in part (a), there are infinitely many solutions in the case that $k = 1$.

(d) When there is exactly one solution, what are the values of x and y.

By part (a), this happens whenever $|k| \neq 1$. If $k = 0$, then $(x, y) = (1, 1)$. If $k \neq 0$ and $|k| \neq 1$, then

$$(x, y) = \left(\frac{1}{1 + k}, \frac{1}{1 + k}\right).$$

Question 3. Consider the following two systems of equations.

$$\begin{aligned} x + y + z &= 16, \\ x + 2y + 2z &= 11, \\ 2x + 3y - 4z &= 3, \end{aligned}$$

and

$$\begin{aligned} x + y + z &= 7, \\ x + 2y + 2z &= 10, \\ 2x + 3y - 4z &= 3. \end{aligned}$$

Solve both systems simultaneously by applying row reduction to an appropriate 3×5 matrix.

We consider the following matrix augmented by two columns

$$\left(\begin{array}{ccr|cc} 1 & 1 & 1 & 16 & 7 \\ 1 & 2 & 2 & 11 & 10 \\ 2 & 3 & -4 & 3 & 3 \end{array}\right).$$

Its reduced row echelon form is given by

$$\left(\begin{array}{ccc|cc} 1 & 0 & 0 & 21 & 4 \\ 0 & 1 & 0 & -\frac{59}{7} & 1 \\ 0 & 0 & 1 & \frac{24}{7} & 2 \end{array}\right).$$

Therefore, both systems have a unique solution. The first is given by $x = 21$, $y = -59/7$, and $z = 24/7$; and the second is given by $x = 4$, $y = 1$, and $z = 2$.

Question 4. Consider the following homogeneous system of linear equations where $a, b \in \mathbf{R}$ are constants.

$$\begin{aligned} x + 2y &= 0, \\ ax + 8y + 3z &= 0, \\ by + 5z &= 0. \end{aligned}$$

(a) Find a value for a which makes it necessary to interchange rows during row reduction.

The augmented matrix for the system is

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ a & 8 & 3 & 0 \\ 0 & b & 5 & 0 \end{array}\right).$$

If $a = 4$, then -4 times the first row added to the second gives

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & b & 5 & 0 \end{array}\right)$$

Therefore, if $b \neq 0$, then we would be required to interchange the second and third rows. It isn't difficult to see that this is the only case in which we would need to interchange rows.

(b) Suppose that a does not have the value you found in part (a). Find a value for b so that the system has a nontrivial solution.

Since $a \neq 4$, we have

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ a & 8 & 3 & 0 \\ 0 & b & 5 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & \frac{3}{8-2a} & 0 \\ 0 & 0 & 5-\frac{3b}{8-2a} & 0 \end{array}\right).$$

This system has a nontrivial solution only in the case $5 - 3b/(8-2a) = 0$, that is, only in the case $b = (40-10a)/3$.

(c) Suppose that a does not have the value you found in part (a) and that $b = 100$. Suppose further that a is chosen so that the solution to the system is not unique. The general solution to the system is $(\alpha^{-1}z, -\beta^{-1}z, z)$ where α and β are what?

Since we assume that $a \neq 4$, we can row reduce the coefficient matrix to

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{3}{8-2a} \\ 0 & 0 & 5-\frac{300}{8-2a} \end{pmatrix}.$$

In order for the system to have nontrivial solutions, we require $5 - \frac{300}{8-2a} = 0$, that is, $a = -26$. Therefore, the general solution is given by

$$x = \frac{1}{10}z, \qquad y = -\frac{1}{20}z$$

where z is free. So, $\alpha = 10$ and $\beta = -20$.

Spans of Collections of Vectors

Question 5. Consider the following three vectors in $\mathbf{R}^3$

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Show that $\mathbf{R}^3 = \operatorname{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

We will prove this result in two ways. First, observing that

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the result follows. If, however, we required more information in how a vector of R^3 is decomposable as a linear combination of $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$, we could note the following

$$\begin{pmatrix} 1 & 0 & 1 & a_1 \\ 1 & 1 & 0 & a_2 \\ 0 & 1 & 1 & a_3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2}a_1 + \frac{1}{2}a_2 - \frac{1}{2}a_3 \\ 0 & 1 & 0 & -\frac{1}{2}a_1 + \frac{1}{2}a_2 + \frac{1}{2}a_3 \\ 0 & 0 & 1 & \frac{1}{2}a_1 - \frac{1}{2}a_2 + \frac{1}{2}a_3 \end{pmatrix}.$$

This means that if $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbf{R}^3$, then

$$\left(\frac{1}{2}a_1 + \frac{1}{2}a_2 - \frac{1}{2}a_3\right)\vec{u}_1 - \left(\frac{1}{2}a_1 - \frac{1}{2}a_2 - \frac{1}{2}a_3\right)\vec{u}_2 + \left(\frac{1}{2}a_1 - \frac{1}{2}a_2 + \frac{1}{2}a_3\right)\vec{u}_3 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Question 6. Consider the following four vectors in $\mathbf{R}^4$ given by

$$\vec{v}_1 = \begin{pmatrix} +1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ +1 \\ -1 \\ -1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -1 \\ -1 \\ +1 \\ -1 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ +1 \end{pmatrix}.$$

(a) Show whether $\vec{v}_1 \in \text{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$ or not by solving the corresponding system of linear equations.

Observe

$$\left(\begin{array}{rrr|r} -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

so that $\vec{v}_1 \notin \text{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$.

(b) Let $a_1, a_2, a_3, a_4 \in \mathbf{R}$. Under what conditions on a_1, a_2, a_3, a_4 is $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + a_4\vec{v}_4 = \vec{0}$ true?

We have

$$\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so that only the solution $a_1 = a_2 = a_3 = a_4 = 0$ exists.

(c) How can we use part (b) to provide a second proof of part (a)? Can you generalize to answer the following question: Is $\vec{v}_i \in \text{span}\{\vec{v}_j, \vec{v}_k, \vec{v}_l\}$ for i, j, k, l distinct?

We show only the second question (the first being a special case of the second). We will use proof by contradiction. Assume to the contrary that there exists $a_j, a_k, a_l \in \mathbf{R}$ not all zero such that $\vec{v}_i = a_j\vec{v}_j + a_k\vec{v}_k + a_l\vec{v}_l$. But then $\vec{v}_i - a_j\vec{v}_j - a_k\vec{v}_k - a_l\vec{v}_l = \vec{0}$. From our answer to part (b), it follows that $a_j = a_k = a_l = 0$ and $1 = 0$. But $1 = 0$ is a contradiction. Therefore, our original assumption that $\vec{v}_i \in \text{span}\{\vec{v}_j, \vec{v}_k, \vec{v}_l\}$ is incorrect. That is, we must have $\vec{v}_i \notin \text{span}\{\vec{v}_j, \vec{v}_k, \vec{v}_l\}$

Question 7. Let V_1 and V_2 be two subsets of $\mathbf{R}^n$, and define $V_1 + V_2 = \{\vec{v}_1 + \vec{v}_2 : \vec{v}_1 \in V_1 \text{ and } \vec{v}_2 \in V_2\}$. Show (a) $\text{span}(V_1 \cup V_2) = \text{span}(V_1) + \text{span}(V_2)$, and (b) $\text{span}(V_1 \cap V_2) \subseteq \text{span}(V_1) \cap \text{span}(V_2)$. Further, give an example of subsets V_1 and V_2 of $\mathbf{R}^n$, for some n, for which $\text{span}(V_1 \cap V_2) \subsetneq \text{span}(V_1) \cap \text{span}(V_2)$.

Throughout, let $V_1 \cap V_2 = \{w_1, w_2, \ldots, w_p\}$, $V_1 = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-p}, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_p\}$, $V_2 = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{m-p}, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_p\}$.

(a) Let $\vec{x} \in \text{span}(V_1 \cup V_2)$. Then there exists scalars $a_1, a_2, \ldots, a_{n-p} \in \mathbf{R}$ and $b_1, b_2, \ldots, b_{m-p} \in \mathbf{R}$ and $c_1, c_2, \ldots, c_p \in \mathbf{R}$ such that

$$\vec{x} = \sum_{i=1}^{n-p} a_i \vec{v}_i + \sum_{j=1}^{m-p} b_j \vec{u}_j + \sum_{k=1}^{p} c_k \vec{w}_k.$$

But

$$\sum_{i=1}^{n-p} a_i \vec{v}_i + \sum_{k=1}^{p} c_k \vec{w}_k \in \text{span}(V_1)$$

and

$$\sum_{j=1}^{m-p} b_j \vec{u}_j \in \text{span}(V_2).$$

Therefore, $\vec{x} \in \text{span}(V_1) + \text{span}(V_2)$, and $\text{span}(V_1 \cup V_2) \subseteqq \text{span}(V_1) + \text{span}(V_2)$. Conversely, let $\vec{y} \in \text{span}(V_1) + \text{span}(V_2)$. Then there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbf{R}$ and $\beta_1, \beta_2, \ldots, \beta_m \in \mathbf{R}$ such that

$$y = \sum_{i=1}^{n-p} \alpha_i \vec{v}_i + \sum_{j=1}^{p} (\alpha_j + \beta_j) \vec{w}_j + \sum_{k=1}^{m-p} \beta_k \vec{u}_k.$$

Since $V_1 \cup V_2 = \{\vec{v}_1, \ldots, \vec{v}_n, \vec{u}_1, \ldots, \vec{u}_m, \vec{w}_1, \ldots, \vec{w}_p\}$, we have that $\vec{y} \in \text{span}(V_1 \cup V_2)$, and $\text{span}(V_1 \cup V_2) \supseteqq \text{span}(V_1) + \text{span}(V_2)$.

(b) Let $\vec{x} \in \text{span}(V_1 \cap V_2)$. Then there exists scalars $a_1, a_2, \ldots, a_p$ such that

$$\vec{x} = a_1 \vec{w}_1 + a_2 \vec{w}_2 + \cdots + a_p \vec{w}_p.$$

Since $V_1 \cap V_2 \subseteqq V_1$ and $V_1 \cap V_2 \subseteqq V_2$, we see at once that $\vec{x}$ is in both $\text{span}(V_1)$ and $\text{span}(V_2)$, that is, $\vec{x} \in \text{span}(V_1) \cap \text{span}(V_2)$, as desired. Any number of counter examples can be found to show that $\text{span}(V_1 \cap V_2) = \text{span}(V_1) \cap \text{span}(V_2)$ is not true in general. It isn't difficult to see that $V_1 = \{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$ and $V_2 = \{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\}$ give a counter example. Since then $\text{span}(V_1 \cap V_2) = \{\vec{0}\}$ and $\text{span}(V_1) \cap \text{span}(V_2) = \text{span}(V_2) \neq \{\vec{0}\}$.

Linear Independence

Question 8. Show that in $\mathbf{R}^3$, the vectors $\vec{x} = (1, 1, 0)$, $\vec{y} = (0, 1, 2)$, and $\vec{z} = (3, 1, -4)$ are linearly dependent by finding scalars α and β such that $\alpha\vec{x} + \beta\vec{y} + \vec{z} = \vec{0}$.

We solve the system $\alpha\vec{x} + \beta\vec{y} = -\vec{z}$ to find $\alpha = -3$ and $\beta = 2$.

Question 9. Let $\vec{w} = (1, 1, 0, 0)$, $\vec{x} = (1, 0, 1, 0)$, $\vec{y} = (0, 0, 1, 1)$, and $\vec{z} = (0, 1, 0, 1)$, and let $S = \{\vec{w}, \vec{x}, \vec{y}, \vec{z}\}$.

(a) Show that S is not a spanning set for $\mathbf{R}^4$ by finding a vector $\vec{u}$ in $\mathbf{R}^4$ such that $\vec{u} \notin \text{span}(S)$. One such vector is $\vec{u} = (1, 2, 3, a)$ where a is any real number execpt what?

We form the matrix A whose columns are $\vec{w}$, $\vec{x}$, $\vec{y}$, and $\vec{z}$. The reduced row echelon form for A is then

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

whereupon we see that span$\{\vec{w},\ \vec{x},\ \vec{y},\ \vec{z}\} = $ span$\{\vec{w},\ \vec{x},\ \vec{y}\}$. We can then check that the system

$$\alpha\vec{w} + \beta\vec{x} + \gamma\vec{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ a \end{pmatrix}$$

is inconsistent precisely in the case that $a = 4$. Therefore, the vector $(1, 2, 3, 4)^t$ is not in the span of $\vec{w}$, $\vec{x}$, $\vec{y}$, and $\vec{z}$ so that they do not span $\mathbf{R}^4$.

(b) Show that S is a linearly dependent set of vectors by finding scalars α, γ, and δ such that $\alpha\vec{w} + \vec{x} + \gamma\vec{y} + \delta\vec{z} = \vec{0}$.

Solving the system $\alpha\vec{w} + \gamma\vec{y} + \delta\vec{z} = -\vec{x}$, we find that there is a unique solution given by $\alpha = \gamma = -1$ and $\delta = 1$.

(c) Show that S is a linear dependent set by writing $\vec{z}$ as a linear combination of the remaining vectors in S.

From the reduced row echelon form for the matrix A in part (a), we see that $\vec{z} = \vec{w} - \vec{x} + \vec{y}$.

Question 10. Let S_1 and S_2 be finite subsets of $\mathbf{R}^n$, for some n, such that $S_1 \subsetneq S_2$. Prove that if S_1 is a linearly dependent set, then so is S_2. Show that this is equivalent to if S_2 is a linearly independent set, then so is S_1.

Let S_1 and S_2 be finite subsets of $\mathbf{R}^n$ such that $S_1 \subsetneq S_2$. Suppose that S_1 is linearly dependent, but S_2 is linearly independent. For definiteness, we write

$$S_1 = \{\vec{v}_1,\ \vec{v}_2, \ldots,\ \vec{v}_m\}$$

and

$$S_2 = \{\vec{v}_1,\ \vec{v}_2, \ldots,\ \vec{v}_m,\ \vec{v}_{m+1}, \ldots,\ \vec{v}_k\}.$$

By assumption, there is a collection $a_1,\ a_2, \ldots,\ a_m$ of scalars such that $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m = \vec{0}$ but $a_i \neq 0$ for some $i \in \{1,\ 2, \ldots,\ m\}$. Set $a_{m+1} = a_{m+2} = \cdots = a_k = 0$ so that

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m + a_{m+1}\vec{v}_{m+1} + \cdots + a_k\vec{v}_k = \vec{0}.$$

By our assumption that S_2 is linear independent, we have

$$a_1 = a_2 = \cdots = a_m = a_{m+1} = \cdots = a_k = 0$$

by the definition of linearly independent. But this implies that each $a_i = 0$ for $i \in \{1,\ 2, \ldots,\ m\}$, contrary to what we have observed. This contradiction proves the result.

Question 11. Do the following.

(a) Let $\vec{u}$ and $\vec{v}$ be distinct vectors in $\mathbf{R}^n$. Prove that $\{\vec{u}, \vec{v}\}$ is linearly independent if and only if $\{\vec{u} + \vec{v}, \vec{u} - \vec{v}\}$ is linearly independent.

Suppose first that $\{\vec{u},\ \vec{v}\}$ is linearly independent, but $\vec{u} + \vec{v} = a(\vec{u} - \vec{v})$ for some $a \in \mathbf{R}$. If $a = 1$, then $\vec{v} = \vec{0}$, contradicting the fact that $\{\vec{u},\ \vec{v}\}$ is independent. So $a \neq 1$ and $\vec{u} = -\frac{1+a}{1-a}\vec{v}$, again, contradicting the fact that $\{\vec{u},\ \vec{v}\}$ is independent.

Conversely, suppose that $\vec{u} = b\vec{v}$ for some $b \in \mathbf{R}$. Then $\vec{u} + \vec{v} = (1+b)\vec{u}$ and $\vec{u} - \vec{v} = (1-b)\vec{u}$. But this means that both $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are in span$\{\vec{u}\}$, whereupon $\{\vec{u} + \vec{v}, \vec{u} - \vec{v}\}$ is linearly dependent.

(b) Let $\vec{u}, \vec{v}, \vec{w}$ be distinct vectors in $\mathbf{R}^n$. Prove that $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent if and only if $\{\vec{u} + \vec{v}, \vec{u} + \vec{w}, \vec{v} + \vec{w}\}$ is linear independent.

First, we assume that $\{\vec{u},\ \vec{v},\ \vec{w}\}$ is linearly independent, and we assume there are scalars $a,\ b,\ c$ such that

$$\vec{0} = a(\vec{u} + \vec{v}) + b(\vec{u} + \vec{w}) + c(\vec{v} + \vec{w}) = (a+b)\vec{u} + (a+c)\vec{v} + (b+c)\vec{w}.$$

Since we are assuming that $\{\vec{u},\ \vec{v},\ \vec{w}\}$ is linearly independent, it must be that $a + b = a + c = b + c = 0$. This linear system has only the trivial solution $a = b = c = 0$, whereupon $\{\vec{u} + \vec{v}, \vec{u} + \vec{w}, \vec{v} + \vec{w}\}$ is linearly independent as well.

Next, we assume that $\{\vec{u}+\vec{v}, \vec{u}+\vec{w}, \vec{v}+\vec{w}\}$ is independent. Suppose there are scalars $a,\ b,\ c$ such that $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$. Then

$$(a-b+c)(\vec{u} + \vec{v}) + (a+b-c)(\vec{u} + \vec{w}) + (-a+b+c)(\vec{v} + \vec{w}) = \vec{0}.$$

By our assumed linear independence of $\{\vec{u} + \vec{v}, \vec{u} + \vec{w}, \vec{v} + \vec{w}\}$, we have $a - b + c = a + b - c = -a + b + c = 0$. Solving, we find this possesses only the trivial solution $a = b = c = 0$. It follows $\{\vec{u},\ \vec{v},\ \vec{w}\}$ is independent as well.

Linear Transformations

Question 12. Show the following for $\mathbf{R}^n$.

(a) Show that scalar multiplication is a linear transformation.

Fix $a \in \mathbf{R}$, and let $T : \mathbf{R}^n \to \mathbf{R}^n$ be the map $T(\vec{v}) = a\vec{v}$. Then $T(b\vec{v}) = ab\vec{v} = ba\vec{v} = bT(\vec{v})$ and $T(\vec{v} + \vec{u}) = a(\vec{v} + \vec{u}) = a\vec{v} + a\vec{u} = T(\vec{v}) + T(\vec{u})$. We have shown that T is linear.

(b) When is this linear map invertible?

This map is invertible precisely in the case the scalar by which we are multiplying is nonzero.

(c) Is its inverse a linear transformation?

Let T be as in part (a), and assume that $a \neq 0$. Then T is invertible and T^{-1} is given by multiplcation by a^{-1}. Since this is multiplcation by a scalar, it is linear.

(d) Fix an element $a \in \mathbf{R}^n$. What is the matrix corresponding to the linear transformation $\vec{v} \mapsto a\vec{v}$ with respect to the standard spanning vectors?

Recall the standard spanning vectors are $\{\vec{e}_1,\ \vec{e}_2, \ldots,\ \vec{e}_n\}$ where $\vec{e}_i$ is the vector with a 1 in position i and zeros everywhere else. Then the matrix corresponding to multiplication by a is givn by

$$[T(\vec{e}_1)|T(\vec{e}_2)|\cdots|T(\vec{e}_n)] = [a\vec{e}_1|a\vec{e}_2|\cdots|a\vec{e}_n] = aI.$$

Question 13. Fix $a \in \mathbf{R}$ and $\vec{u} \in \mathbf{R}^n$ with $\vec{u} \neq \vec{0}$. Is the map given by $\vec{v} \mapsto a\vec{v} + \vec{u}$, linear? Why or why not?

No; it is not a linear map. Let $\vec{v}_1,\ \vec{v}_2 \in \mathbf{R}^n$. Then $\vec{v}_1 + \vec{v}_2 \mapsto a\vec{v}_1 + a\vec{v}_2 + \vec{u}$. But $\vec{v}_1 \mapsto a\vec{v}_1 + \vec{u}$ and $\vec{v}_2 \mapsto a\vec{v}_2 + \vec{u}$. However, $(a\vec{v}_1 + \vec{u}) + (a\vec{v}_2 + \vec{u}) = a\vec{v}_1 + a\vec{v}_2 + 2\vec{u} \neq a\vec{v}_1 + a\vec{v}_2 + \vec{u}$.

Question 14. Consider a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$, and define $\text{Ker}(T) = \{\vec{v} \in \mathbf{R}^n : T(\vec{v}) = \vec{0}\}$. This is the kernel of the linear transformation T. For $\vec{v} \in \mathbf{R}^n$, define $\vec{v} + \text{Ker}(T) = \{\vec{v} + \vec{u} : \vec{u} \in \text{Ker}(T)\}$. Show the following.

(a) $\text{Ker}(T)$ is closed under scalar multiplcation and vector addition.

Let $\vec{v}_1,\ \vec{v}_2 \in \text{Ker}(T)$. Then $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{0} + \vec{0} = \vec{0}$, so $\vec{v}_1 + \vec{v}_2 \in \text{Ker}(T)$. Similarly, $T(a\vec{v}) = aT(\vec{v}) = a\vec{0} = \vec{0}$ whenever $\vec{v} \in \text{Ker}(T)$; so, $a\vec{v} \in \text{Ker}(T)$. This shows that $\text{Ker}(T)$ is closed under vector addition and scalar multiplication.

(b) For $\vec{v} \in \mathbf{R}^n$, show that $\vec{v} + \text{Ker}(T)$ consists of all and only those elements of $\mathbf{R}^n$ that map to $T(\vec{v})$ under T.

Let $V = \{u \in \mathbf{R}^n : T(\vec{u}) = T(\vec{v})\}$. We show $\vec{v} + \text{Ker}(T) = V$. Indeed, let $\vec{u} \in V$. Then $T(\vec{v}) = T(\vec{u})$ so that $T(u - v) = \vec{0}$. It follows that there is an $\vec{x} \in \text{Ker}(T)$ such that $\vec{u} - \vec{v} = \vec{x}$, that is, $\vec{u} = \vec{v} + \vec{x}$. This means $\vec{u} \in \vec{v} + \text{Ker}(T)$.

Conversely, let $\vec{u} \in \vec{v} + \text{Ker}(T)$. Then there is some $\vec{x} \in \text{Ker}(T)$ for which $\vec{u} = \vec{v} + \vec{x}$. It follows that $T(\vec{u}) = T(\vec{v} + \vec{x}) + T(\vec{v}) + T(\vec{x}) = T(\vec{v}) + \vec{0} = T(\vec{v})$. So, $\vec{u} \in V$, as required.

(c) For $\vec{v}_1, \vec{v}_2 \in \mathbf{R}^n$, show that either $\vec{v}_1 + \text{Ker}(T) = \vec{v}_2 + \text{Ker}(T)$ or $\vec{v}_1 + \text{Ker}(T) \cap \vec{v}_2 + \text{Ker}(T) = \emptyset$.

Let $\vec{v}_1,\ \vec{v}_2 \in \mathbf{R}^n$, and suppose that $\vec{v}_1 + \text{Ker}(T) \cap \vec{v}_2 + \text{Ker}(T) \neq \emptyset$. Then there is some $\vec{u} \in v_1 + \text{Ker}(T) \cap \vec{v}_2 + \text{Ker}(T)$. By definition, there must be $\vec{u}_1,\ \vec{u}_2 \in \text{Ker}(T)$ such that $\vec{u} = \vec{v}_1 + \vec{u}_1 = \vec{v}_2 = \vec{u}_2$. But then $\vec{v}_1 - \vec{v}_2 = \vec{u}_2 - \vec{u}_1 = \vec{u}_3$ for some $\vec{u}_3 \in \text{Ker}(T)$. It follows that $\vec{v}_1 = \vec{v}_2 + \vec{u}_3$, so $\vec{v}_1 \in \vec{v}_2 + \text{Ker}(T)$. Similarly, $\vec{v}_2 = \vec{v}_1 - \vec{u}_3$, so $\vec{v}_2 \in \vec{v}_1 + \text{Ker}(T)$. It follows, therefore, that $\vec{v}_1 + \text{Ker}(T) = \vec{v}_2 + \text{Ker}(T)$.

Matrix Operations

Question 15. Give an example of a nonzero matrix A such that $A^2 = O$.

Take $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

Question 16. The trace of a square matrix A of dimensions $N \times N$ is defined as $\text{tr}(A) = \sum_{k=1}^{N} A_{k,k}$, i.e., the sum of the diagonal entries of the matrix. For any other $N \times N$ matrix B, show that $\text{tr}(AB) = \text{tr}(BA)$.

Observe

$$\begin{aligned}
\operatorname{tr}(AB) &= \sum_{k=1}^{N} (AB)_{k,k} \\
&= \sum_{k=1}^{N} \sum_{j=1}^{N} A_{k,j} B_{j,k} \\
&= \sum_{k=1}^{N} \sum_{j=1}^{N} A_{j,k} B_{k,j} \\
&= \operatorname{tr}(BA)
\end{aligned}$$

where the second to last equality follows because

$$\{(k, j, j, k) : 1 \leqq j, k \leqq N\} = \{(j, k, k, j) : 1 \leqq j, k \leqq N\}.$$

Question 17. An $N \times N$ matrix A is circulant if it is of the form

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_N \\ a_N & a_1 & a_2 & \cdots & a_{N-1} \\ a_{N-1} & a_N & a_1 & \cdots & a_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}.$$

Show that if B is any other $N \times N$ circulant matrix, then $AB = BA$.

Define the $N \times N$ matrix G by

$$G = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

If we multiply an $N \times N$ matrix C on the right by G, the resulting matrix is the one obtained by cyclically shifting the columns of C to the right. In particular, $G^N = I$ and $G^j \neq I$ for any $j \in \{1, \ldots, N-1\}$. We also note that we can write A and B by

$$A = \sum_{i=1}^{N} a_i G^{i-1}, \qquad B = \sum_{i=1}^{N} b_i G^{i-1},$$

that is, A and B are polynomials in G. Since they are each polynomials in G, it is easy to see that they must commute.

Question 18. A diagonal matrix is one for which every entry not on the main diagonal is zero. Let A and B be $N \times N$ matrices such that there exists and invertible $N \times N$ matrix P for which $D_A = P^{-1}AP$ and $D_B = P^{-1}BP$ are diagonal matrices. Show that A and B commute.

Since $D_A = P^{-1}AP$ and $D_B = P^{-1}BP$, we have that $A = PD_AP^{-1}$ and $B = PD_BP^{-1}$. Then

$$\begin{aligned}
AB &= (PD_AP^{-1})(PD_BP^{-1}) \\
&= PD_A(P^{-1}P)D_BP^{-1} \\
&= PD_AID_BP^{-1} \\
&= PD_AD_BP^{-1} \\
&= PD_BD_AP^{-1} \\
&= PD_BID_AP^{-1} \\
&= PD_B(P^{-1}P)D_AP^{-1} \\
&= (PD_BP^{-1})(PD_AP^{-1}) \\
&= BA
\end{aligned}$$

where we have used the fact that diagonal matrices commute.