Math 240 Tutorial
Solutions

June 20

Question 1. Show the following for $\mathbf{R}^n$.

(a) Show that scalar multiplication is a linear transformation.

Fix $a \in \mathbf{R}$, and let $T : \mathbf{R}^n \to \mathbf{R}^n$ be the map $T(\vec{v}) = a\vec{v}$. Then $T(b\vec{v}) = ab\vec{v} = ba\vec{v} = bT(\vec{v})$ and $T(\vec{v} + \vec{u}) = a(\vec{v} + \vec{u}) = a\vec{v} + a\vec{u} = T(\vec{v}) + T(\vec{u})$. We have shown that T is linear.

(b) When is this linear map invertible?

This map is invertible precisely in the case the scalar by which we are multiplying is nonzero.

(c) Is its inverse a linear transformation?

Let T be as in part (a), and assume that $a \neq 0$. Then T is invertible and T^{-1} is given by multiplcation by a^{-1}. Since this is multiplcation by a scalar, it is linear.

(d) Fix an element $a \in \mathbf{R}^n$. What is the matrix corresponding to the linear transformation $\vec{v} \mapsto a\vec{v}$ with respect to the standard spanning vectors?

Recall the standard spanning vectors are $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ where $\vec{e}_i$ is the vector with a 1 in position i and zeros everywhere else. Then the matrix corresponding to multiplication by a is givn by

$$[T(\vec{e}_1)|T(\vec{e}_2)|\cdots|T(\vec{e}_n)] = [a\vec{e}_1|a\vec{e}_2|\cdots|a\vec{e}_n] = aI.$$

Question 2. Construct the standard matrix for the transformation that rotates the vectors of $\mathbf{R}^2$ by $-\pi/6$ radians.

Note that

$$e_1 = (1, 0) \mapsto (\cos(-\pi/6), \sin(-\pi/6)) = (\sqrt{3}/2, -1/2),$$
$$e_2 = (0, 1) \mapsto (\cos(\pi/3), \sin(\pi/3)) = (1/2, \sqrt{3}/2).$$

It follows that the standard matrix is given by

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

Question 3. Define $T : \mathbf{R}^3 \to \mathbf{R}^4$ by

$$T(\vec{x}) = (x_1 - x_3,\ x_1 + x_2,\ x_3 - x_2,\ x_1 - 2x_2).$$

(a) Is T linear?

It is easily verified that $T(\vec{x}_1 + \vec{x}_2) = T(\vec{x}_1) + T(\vec{x}_2)$ and $T(\alpha\vec{x}) = \alpha T(\vec{x})$ for every $\alpha \in \mathbf{R}$ and $x_1, \vec{x}_2 \in \mathbf{R}^3$.

(b) What is $T(1, -2, 3)$?

By definition, $T(1, -2, 3) = (-2, -1, 5, 5)$.

(c) Find a vector $\vec{x} \in \mathbf{R}^3$ such that $T(\vec{x}) = (8, 9, -5, 0)$.

We solve the system

$$\begin{aligned} x_1 - x_3 &= 8, \\ x_1 + x_2 &= 9, \\ x_3 - x_2 &= -5, \\ x_1 - 2x_2 &= 0 \end{aligned}$$

to find the unique preimage is $(6, 3, -2)$.

(d) What is the standard basis for T?

Observe

$$\begin{aligned} e_1 &\mapsto (1, 1, 0, 1), \\ e_2 &\mapsto (0, 1, -1, -2), \\ e_3 &\mapsto (-1, 0, 1, 0) \end{aligned}$$

so the standard matrix is

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}.$$

Question 4. Calculate the following determinants.

(a)

$$\det \begin{pmatrix} 6 & 9 & 39 & 49 \\ 5 & 7 & 32 & 37 \\ 3 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Note

$$\begin{pmatrix} 6 & 9 & 39 & 49 \\ 5 & 7 & 32 & 37 \\ 3 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 30 & 40 \\ 0 & 0 & 0 & -\frac{10}{3} \end{pmatrix}$$

where we have made 2 row swaps. So the determinant is $30(-\frac{10}{3}) = -100$.

(b)

$$\det \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 2 & 0 \\ 2 & -1 & 3 & 1 \\ 4 & 17 & 0 & -5 \end{pmatrix}.$$

Note

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 2 & 0 \\ 2 & -1 & 3 & 1 \\ 4 & 17 & 0 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & -1 \\ 4 & 17 & 0 & -5 \end{pmatrix}.$$

So the determinant is 0 (there are two equal rows in a row equivalent matrix).

(c)

$$\det \begin{pmatrix} 13 & 3 & -8 & 6 \\ 0 & 0 & -4 & 0 \\ 1 & 0 & 7 & -2 \\ 3 & 0 & 2 & 0 \end{pmatrix}.$$

Note

$$\begin{pmatrix} 13 & 3 & -8 & 6 \\ 0 & 0 & -4 & 0 \\ 1 & 0 & 7 & -2 \\ 3 & 0 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 7 \\ 0 & 3 & 32 & -99 \\ 0 & 0 & 6 & -19 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

where we have had to do 3 row swaps and 1 column swap. Therefore, the determinant is given by $(3)(6)(-4) = -72$.

Question 5. Solve the following equation for x.

$$\det \begin{pmatrix} 3 & -4 & 7 & 0 & 6 & -2 \\ 2 & 0 & 1 & 8 & 0 & 0 \\ 3 & 4 & -8 & 3 & 1 & 2 \\ 27 & 6 & 5 & 0 & 0 & 3 \\ 3 & x & 0 & 2 & 1 & -1 \\ 1 & 0 & -1 & 3 & 4 & 0 \end{pmatrix} = 0.$$

Evaluating the determinant, we have $3685x + 7370 = 0$ so that $x = -2$.

Question 6. Let M be the matrix

$$\begin{pmatrix} 5 & 4 & -2 & 3 \\ 5 & 7 & -1 & 8 \\ 5 & 7 & 6 & 10 \\ 5 & 7 & 1 & 9 \end{pmatrix}.$$

The following hold.

(a) det M can be expressed as the constant 5 times the determinant of

$$\begin{pmatrix} 3 & 1 & 5 \\ 3 & & \\ 3 & & \end{pmatrix}.$$

(b) The determinant of the 3×3 is part (a) can be expressed as the constant 3 times the determinant of

$$\begin{pmatrix} 7 & 2 \\ 2 & \end{pmatrix}.$$

The determinant of the 2×2 matrix in part (b) is what? Thus the detmerinant of M is what?

The 2×2 matrix is $\left(\begin{smallmatrix} 7 & 2 \\ 2 & 1 \end{smallmatrix}\right)$. The determinant of the 2×2 matrix is $7 - 4 = 3$, so the determinant of M is $(5)(3)(3) = 45$.

Question 7. Consider again the vector space $\mathbf{P}_3$, and let $Q \subseteq \mathbf{P}_3$ be the subset of polynomials of degree at most 3 that vanish when $x = 3$. Is Q a subspace? If it is, give a spanning set of Q.

Let $p(x) \in \mathbf{P}_3$ have 3 as a root. Then $p(x) = (x-3)q(x)$ for some polynomial q of degree at most 2. So, the subspace under consideration has dimension at most 3. In fact, $\{x - 3, (x-3)^2, (x-3)^3\}$ are linearly independent, so their span is Q.