Math 240 Tutorial
Solutions

May 23

Question 1. For each part, explain whether or not the stated matrix–vector multiplication can be carried out. If it can, do the multiplication.

(a)

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

The matrix and the vector are not conformable; to be specific, the number of columns of the matrix does not equal the number of entries in the column. So, the multiplication cannot be carried out.

(b)

$$\begin{pmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Here the matrix and the vector are conformable. Their product is given by

$$\begin{pmatrix} x \\ 1 \\ 1 \end{pmatrix}.$$

Question 2. Write the following linear system first as a vector equation and then as a matrix equation

$$\begin{aligned} u + 2v - w - 2x + 3y &= b_1, \\ x - y + 2z &= b_2, \\ 2u + 4v - 2w - 4x + 7y - 4z &= b_3, \\ -x + y - 2z &= b_4, \\ 3u + 6v - 3w - 6x + 7y + 8z &= b_5, \end{aligned}$$

where $b_1,\ b_2,\ b_3,\ b_4,\ b_5 \in \mathbf{R}$.

The vector equation is given by

$$u \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{pmatrix} + v \begin{pmatrix} 2 \\ 0 \\ 4 \\ 0 \\ 6 \end{pmatrix} + w \begin{pmatrix} -1 \\ 0 \\ -2 \\ 0 \\ -3 \end{pmatrix} + x \begin{pmatrix} -2 \\ 1 \\ -4 \\ -1 \\ -6 \end{pmatrix} + y \begin{pmatrix} 3 \\ -1 \\ 7 \\ 1 \\ 7 \end{pmatrix} + z \begin{pmatrix} 0 \\ 2 \\ -4 \\ -2 \\ 8 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}.$$

The matrix equation is given by

$$\begin{pmatrix} 1 & 2 & -1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 2 & 4 & -2 & -4 & 7 & -4 \\ 0 & 0 & 0 & -1 & 1 & -2 \\ 3 & 6 & -3 & -6 & 7 & 8 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}.$$

Question 3. For each of the following lists of row vectors in $\mathbf{R}^3$, determine whether the first vector can be expressed as a linear combination of the other two vectors.

(a) $(-2, 0, 3)$, $(1, 3, 0)$, $(2, 4, -1)$.

We have
$$\left(\begin{array}{rr|r} 1 & 2 & -2 \\ 3 & 4 & 0 \\ 0 & -1 & 3 \end{array}\right) \sim \left(\begin{array}{rr|r} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right).$$
Therefore, $(-2, 0, 3) \in \text{span}\{(1, 3, 0),\ (2, 4, -1)\}$.

(b) $(1, 2, -3)$, $(-3, 2, 1)$, $(2, -1, -1)$.

We have
$$\left(\begin{array}{rr|r} -3 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & -1 & -3 \end{array}\right) \sim \left(\begin{array}{rr|r} 1 & 0 & 5 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{array}\right).$$
Therefore, $(1, 2, -3) \in \text{span}\{(-3, 2, 1),\ (2, -1, -1)\}$.

(c) $(3, 4, 1)$, $(1, -2, 1)$, $(-2, -1, 1)$.

We have
$$\left(\begin{array}{rr|r} 1 & -2 & 3 \\ -2 & -1 & 4 \\ 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right).$$
Therefore, $(1, 2, -3) \notin \text{span}\{(-3, 2, 1),\ (2, -1, -1)\}$.

(d) $(2, -1, 0)$, $(1, 2, -3)$, $(1, -3, 2)$.

We have
$$\left(\begin{array}{rr|r} 1 & 1 & 2 \\ 2 & -3 & -1 \\ -3 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right).$$
Therefore, $(2, -1, 0) \notin \text{span}\{(1, 2, -3),\ (1, -3, 2)\}$.

(e) $(5, 1, -5)$, $(1, -2, -3)$, $(-2, 3, -4)$.

We have
$$\left(\begin{array}{rr|r} 1 & -2 & 5 \\ -2 & 3 & 1 \\ -3 & -4 & -5 \end{array}\right) \sim \left(\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right).$$
Therefore, $(5, 1, -5) \notin \text{span}\{(1, -2, -3),\ (-2, 3, -4)\}$.

(f) $(-2, 2, 2)$, $(1, 2, -1)$, $(-3, -3, 3)$.

We have

$$\left(\begin{array}{cc|c} 1 & -3 & -2 \\ 2 & -3 & 2 \\ -1 & 3 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right).$$

Therefore, $(-2, 2, 2) \in \text{span}\{(1, 2, -1), (-3, -3, 3)\}$.

Question 4. Consider the following three vectors in $\mathbf{R}^3$

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Show that $\mathbf{R}^3 = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

We will prove this result in two ways. First, observing that

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the result follows. If, however, we required more information in how a vector of R^3 is decomposable as a linear combination of $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$, we could note the following

$$\begin{pmatrix} 1 & 0 & 1 & a_1 \\ 1 & 1 & 0 & a_2 \\ 0 & 1 & 1 & a_3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2}a_1 + \frac{1}{2}a_2 - \frac{1}{2}a_3 \\ 0 & 1 & 0 & -\frac{1}{2}a_1 + \frac{1}{2}a_2 + \frac{1}{2}a_3 \\ 0 & 0 & 1 & \frac{1}{2}a_1 - \frac{1}{2}a_2 + \frac{1}{2}a_3 \end{pmatrix}.$$

This means that if $\left(\begin{smallmatrix} a_1 \\ a_2 \\ a_3 \end{smallmatrix}\right) \in \mathbf{R}^3$, then

$$\left(\frac{1}{2}a_1 + \frac{1}{2}a_2 - \frac{1}{2}a_3\right)\vec{u}_1 - \left(\frac{1}{2}a_1 - \frac{1}{2}a_2 - \frac{1}{2}a_3\right)\vec{u}_2 + \left(\frac{1}{2}a_1 - \frac{1}{2}a_2 + \frac{1}{2}a_3\right)\vec{u}_3 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Question 5. Show that $\text{span}\{\vec{u}, \vec{v}, \vec{w}\} = \text{span}\{\vec{u}, \vec{v} + \vec{w}, \vec{v} - \vec{w}\}$.

Since each of u, $v + w$, $v - w$ are in $\text{span}\{u, v, w\}$, it follows that $\text{span}\{u, v + w, v - w\} \subseteq \text{span}\{u, v, w\}$. Conversely, $\frac{1}{2}\Big((v + w) + (v - w)\Big) = v$ and $\frac{1}{2}\Big((v + w) - (v - w)\Big) = w$ so that $\text{span}\{u, v, w\} \subseteq \text{span}\{u, v + w, v - w\}$. These two containments show the desired equality.

Question 6. Consider the following four vectors in $\mathbf{R}^4$ given by

$$\vec{v}_1 = \begin{pmatrix} +1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ +1 \\ -1 \\ -1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -1 \\ -1 \\ +1 \\ -1 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ +1 \end{pmatrix}.$$

(a) Show whether $\vec{v}_1 \in \text{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$ or not by solving the corresponding system of linear equations.

Observe

$$\left(\begin{array}{ccc|c} -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

so that $\vec{v}_1 \notin \text{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$.

(b) Let $a_1, a_2, a_3, a_4 \in \mathbf{R}$. Under what conditions on a_1, a_2, a_3, a_4 is $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + a_4\vec{v}_4 = \vec{0}$ true?

We have

$$\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so that only the solution $a_1 = a_2 = a_3 = a_4 = 0$ exists.

(c) How can we use part (b) to provide a second proof of part (a)? Can you generalize to answer the following question: Is $\vec{v}_i \in \text{span}\{\vec{v}_j, \vec{v}_k, \vec{v}_l\}$ for i, j, k, l distinct?

We show only the second question (the first being a special case of the second). We will use proof by contradiction. Assume to the contrary that there exists $a_j, a_k, a_l \in \mathbf{R}$ not all zero such that $\vec{v}_i = a_j\vec{v}_j + a_k\vec{v}_k + a_l\vec{v}_l$. But then $\vec{v}_i - a_j\vec{v}_j - a_k\vec{v}_k - a_l\vec{v}_l = \vec{0}$. From our answer to part (b), it follows that $a_j = a_k = a_l = 0$ and $1 = 0$. But $1 = 0$ is a contradiction. Therefore, our original assumption that $\vec{v}_i \in \text{span}\{\vec{v}_j, \vec{v}_k, \vec{v}_l\}$ is incorrect. That is, we must have $\vec{v}_i \notin \text{span}\{\vec{v}_j, \vec{v}_k, \vec{v}_l\}$

Question 7. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in \mathbf{R}^n$ be such that if $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \vec{0}$ then $a_1 = a_2 = \cdots = a_n = 0$. Show this implies that every vector in $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ can be written *uniquely* as a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

Let $\vec{u} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Then there are scalars $a_1, a_2, \ldots, a_n$ such that $\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$. Assume there is another collection $b_1, b_2, \ldots, b_n$ of scalars such that $\vec{u} = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n$. Then

$$\vec{0} = \vec{u} - \vec{u} = (a_1 - b_1)\vec{v}_1 + (a_2 - b_2)\vec{v}_2 + \cdots + (a_n - b_n)\vec{v}_n.$$

By assumption,

$$a_1 - b_1 = a_2 - b_2 = \cdots = a_n - b_n = 0$$

so that

$$a_1 = b_1, \quad a_2 = b_2, \ \ldots \quad , a_n = b_n$$

as desired.

Question 8. Let V_1 and V_2 be two subsets of $\mathbf{R}^n$, and define $V_1 + V_2 = \{\vec{v}_1 + \vec{v}_2 : \vec{v}_1 \in V_1 \text{ and } \vec{v}_2 \in V_2\}$. Show (a) $\text{span}(V_1 \cup V_2) = \text{span}(V_1) + \text{span}(V_2)$, and (b) $\text{span}(V_1 \cap V_2) \subseteq \text{span}(V_1) \cap \text{span}(V_2)$. Further, give an example of subsets V_1 and V_2 of $\mathbf{R}^n$, for some n, for which $\text{span}(V_1 \cap V_2) \subsetneq \text{span}(V_1) \cap \text{span}(V_2)$.

Throughout, let $V_1 \cap V_2 = \{w_1, w_2, \ldots, w_p\}$, $V_1 = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-p}, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_p\}$, $V_2 = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{m-p}, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_p\}$.

(a) Let $\vec{x} \in \text{span}(V_1 \cup V_2)$. Then there exists scalars $a_1,\, a_2, \ldots,\, a_{n-p} \in \mathbf{R}$ and $b_1,\, b_2, \ldots,\, b_{m-p} \in \mathbf{R}$ and $c_1,\, c_2, \ldots,\, c_p \in \mathbf{R}$ such that

$$\vec{x} = \sum_{i=1}^{n-p} a_i \vec{v}_i + \sum_{j=1}^{m-p} b_j \vec{u}_j + \sum_{k=1}^{p} c_k \vec{w}_k.$$

But

$$\sum_{i=1}^{n-p} a_i \vec{v}_i + \sum_{k=1}^{p} c_k \vec{w}_k \in \text{span}(V_1)$$

and

$$\sum_{j=1}^{m-p} b_j \vec{u}_j \in \text{span}(V_2).$$

Therefore, $\vec{x} \in \text{span}(V_1) + \text{span}(V_2)$, and $\text{span}(V_1 \cup V_2) \subseteqq \text{span}(V_1) + \text{span}(V_2)$. Conversely, let $\vec{y} \in \text{span}(V_1) + \text{span}(V_2)$. Then there exists scalars $\alpha_1,\, \alpha_2, \ldots,\, \alpha_n \in \mathbf{R}$ and $\beta_1,\, \beta_2, \ldots,\, \beta_m \in \mathbf{R}$ such that

$$y = \sum_{i=1}^{n-p} \alpha_i \vec{v}_i + \sum_{j=1}^{p} (\alpha_j + \beta_j) \vec{w}_j + \sum_{k=1}^{m-p} \beta_k \vec{u}_k.$$

Since $V_1 \cup V_2 = \{\vec{v}_1,\, \ldots,\, \vec{v}_n,\, \vec{u}_1, \ldots,\, \vec{u}_m,\, \vec{w}_1, \ldots,\, \vec{w}_p\}$, we have that $\vec{y} \in \text{span}(V_1 \cup V_2)$, and $\text{span}(V_1 \cup V_2) \supseteqq \text{span}(V_1) + \text{span}(V_2)$.

(b) Let $\vec{x} \in \text{span}(V_1 \cap V_2)$. Then there exists scalars $a_1,\, a_2, \ldots,\, a_p$ such that

$$\vec{x} = a_1 \vec{w}_1 + a_2 \vec{w}_2 + \cdots + a_p \vec{w}_p.$$

Since $V_1 \cap V_2 \subseteqq V_1$ and $V_1 \cap V_2 \subseteqq V_2$, we see at once that $\vec{x}$ is in both $\text{span}(V_1)$ and $\text{span}(V_2)$, that is, $\vec{x} \in \text{span}(V_1) \cap \text{span}(V_2)$, as desired. Any number of counter examples can be found to show that $\text{span}(V_1 \cap V_2) = \text{span}(V_1) \cap \text{span}(V_2)$ is not true in general. It isn't difficult to see that $V_1 = \{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\, \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$ and $V_2 = \{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\}$ give a counter example. Since then $\text{span}(V_1 \cap V_2) = \{\vec{0}\}$ and $\text{span}(V_1) \cap \text{span}(V_2) = \text{span}(V_2) \neq \{\vec{0}\}$.