Math 240 Tutorial
Questions

June 6

Systems of Linear Equations and Row Reduction

Question 1. Place the following augmented matrices into an echelon form. Does the corresponding system of linear equations admit any solutions?

(a)

$$\left(\begin{array}{cccc|c} 4 & 8 & 12 & 4 & 7 \\ 2 & 5 & 6 & 6 & 11 \\ 0 & 5 & 1 & 26 & 13 \\ 0 & 5 & 0 & 21 & 17 \end{array}\right).$$

(b)

$$\left(\begin{array}{cccc|c} 4 & 8 & 12 & 4 & 0 \\ 2 & 5 & 6 & 6 & 0 \\ 0 & 5 & 1 & 25 & 0 \\ 0 & 5 & 0 & 20 & 0 \end{array}\right).$$

(c)

$$\left(\begin{array}{cccc|c} 4 & 8 & 12 & 4 & 7 \\ 2 & 5 & 6 & 6 & 11 \\ 0 & 5 & 1 & 25 & 13 \\ 0 & 5 & 0 & 20 & 17 \end{array}\right).$$

Question 2. Find the values of k for which the system of equations

$$\begin{aligned} x + ky &= 1, \\ kx + y &= 1, \end{aligned}$$

has

(a) no solution,

(b) a unique solution, and

(c) infinitely many solutions.

(d) When there is exactly one solution, what are the values of x and y.

Question 3. Consider the following two systems of equations.

$$\begin{aligned} x + y + z &= 16, \\ x + 2y + 2z &= 11, \\ 2x + 3y - 4z &= 3, \end{aligned}$$

and

$$\begin{aligned} x + y + z &= 7, \\ x + 2y + 2z &= 10, \\ 2x + 3y - 4z &= 3. \end{aligned}$$

Solve both systems simultaneously by applying row reduction to an appropriate 3×5 matrix.

Question 4. Consider the following homogeneous system of linear equations where $a, b \in \mathbf{R}$ are constants.

$$\begin{aligned} x + 2y &= 0, \\ ax + 8y + 3z &= 0, \\ by + 5z &= 0. \end{aligned}$$

(a) Find a value for a which makes it necessary to interchange rows during row reduction.

(b) Suppose that a does not have the value you found in part (a). Find a value for b so that the system has a nontrivial solution.

(c) Suppose that a does not have the value you found in part (a) and that $b = 100$. Suppose further that a is chosen so that the solution to the system is not unique. The general solution to the system is $(\alpha^{-1}z, -\beta^{-1}z, z)$ where α and β are what?

Spans of Collections of Vectors

Question 5. Consider the following three vectors in $\mathbf{R}^3$

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Show that $\mathbf{R}^3 = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

Question 6. Consider the following four vectors in $\mathbf{R}^4$ given by

$$\vec{v}_1 = \begin{pmatrix} +1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ +1 \\ -1 \\ -1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -1 \\ -1 \\ +1 \\ -1 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ +1 \end{pmatrix}.$$

(a) Show whether $\vec{v}_1 \in \text{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$ or not by solving the corresponding system of linear equations.

(b) Let $a_1, a_2, a_3, a_4 \in \mathbf{R}$. Under what conditions on a_1, a_2, a_3, a_4 is $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + a_4\vec{v}_4 = \vec{0}$ true?

(c) How can we use part (b) to provide a second proof of part (a)? Can you generalize to answer the following question: Is $\vec{v}_i \in \text{span}\{\vec{v}_j, \vec{v}_k, \vec{v}_l\}$ for i, j, k, l distinct?

Question 7. Let V_1 and V_2 be two subsets of $\mathbf{R}^n$, and define $V_1 + V_2 = \{\vec{v}_1 + \vec{v}_2 : \vec{v}_1 \in V_1 \text{ and } \vec{v}_2 \in V_2\}$. Show (a) $\text{span}(V_1 \cup V_2) = \text{span}(V_1) + \text{span}(V_2)$, and (b) $\text{span}(V_1 \cap V_2) \subseteq \text{span}(V_1) \cap \text{span}(V_2)$. Further, give an example of subsets V_1 and V_2 of $\mathbf{R}^n$, for some n, for which $\text{span}(V_1 \cap V_2) \subsetneq \text{span}(V_1) \cap \text{span}(V_2)$.

Linear Independence

Question 8. Show that in $\mathbf{R}^3$, the vectors $\vec{x} = (1, 1, 0)$, $\vec{y} = (0, 1, 2)$, and $\vec{z} = (3, 1, -4)$ are linearly dependent by finding scalars α and β such that $\alpha\vec{x} + \beta\vec{y} + \vec{z} = \vec{0}$.

Question 9. Let $\vec{w} = (1, 1, 0, 0)$, $\vec{x} = (1, 0, 1, 0)$, $\vec{y} = (0, 0, 1, 1)$, and $\vec{z} = (0, 1, 0, 1)$, and let $S = \{\vec{w}, \vec{x}, \vec{y}, \vec{z}\}$.

(a) Show that S is not a spanning set for $\mathbf{R}^4$ by finding a vector $\vec{u}$ in $\mathbf{R}^4$ such that $\vec{u} \notin \text{span}(S)$. One such vector is $\vec{u} = (1, 2, 3, a)$ where a is any real number execpt what?

(b) Show that S is a linearly dependent set of vectors by finding scalars α, γ, and δ such that $\alpha\vec{w} + \vec{x} + \gamma\vec{y} + \delta\vec{z} = \vec{0}$.

(c) Show that S is a linear dependent set by writing $\vec{z}$ as a linear combination of the remaining vectors in S.

Question 10. Let S_1 and S_2 be finite subsets of $\mathbf{R}^n$, for some n, such that $S_1 \subsetneqq S_2$. Prove that if S_1 is a linearly dependent set, then so is S_2. Show that this is equivalent to if S_2 is a linearly independent set, then so is S_1.

Question 11. Do the following.

(a) Let $\vec{u}$ and $\vec{v}$ be distinct vectors in $\mathbf{R}^n$. Prove that $\{\vec{u}, \vec{v}\}$ is linearly independent if and only if $\{\vec{u} + \vec{v}, \vec{u} - \vec{v}\}$ is linearly independent.

(b) Let $\vec{u}, \vec{v}, \vec{w}$ be distinct vectors in $\mathbf{R}^n$. Prove that $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent if and only if $\{\vec{u} + \vec{v}, \vec{u} + \vec{w}, \vec{v} + \vec{w}\}$ is linear independent.

Linear Transformations

Question 12. Show the following for $\mathbf{R}^n$.

(a) Show that scalar multiplication is a linear transformation.

(b) When is this linear map invertible?

(c) Is its inverse a linear transformation?

(d) Fix an element $a \in \mathbf{R}^n$. What is the matrix corresponding to the linear transformation $\vec{v} \mapsto a\vec{v}$?

Question 13. Fix $a \in \mathbf{R}$ and $\vec{u} \in \mathbf{R}^n$ with $\vec{u} \neq \vec{0}$. Is the map given by $\vec{v} \mapsto a\vec{v} + \vec{u}$, linear? Why or why not?

Question 14. Consider a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$, and define $\text{Ker}(T) = \{\vec{v} \in \mathbf{R}^n : T(\vec{v}) = \vec{0}\}$. This is the kernel of the linear transformation T. For $\vec{v} \in \mathbf{R}^n$, define $\vec{v} + \text{Ker}(T) = \{\vec{v} + \vec{u} : \vec{u} \in \text{Ker}(T)\}$. Show the following.

(a) $\text{Ker}(T)$ is closed under scalar multiplcation and vector addition.

(b) For $\vec{v} \in \mathbf{R}^n$, show that $\vec{v} + \text{Ker}(T)$ consists of all and only those elements of $\mathbf{R}^n$ that map to $T(\vec{v})$ under T.

(c) For $\vec{v}_1, \vec{v}_2 \in \mathbf{R}^n$, show that either $\vec{v}_1 + \text{Ker}(T) = \vec{v}_2 + \text{Ker}(T)$ or $\vec{v}_1 + \text{Ker}(T) \cap \vec{v}_2 + \text{Ker}(T) = \emptyset$.

Matrix Operations

Question 15. Give an example of a nonzero matrix A such that $A^2 = O$.

Question 16. The trace of a square matrix A of dimensions $N \times N$ is defined as $\mathrm{tr}(A) = \sum_{k=1}^{N} A_{k,k}$, i.e., the sum of the diagonal entries of the matrix. For any other $N \times N$ matrix B, show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Question 17. An $N \times N$ matrix A is circulant if it is of the form

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_N \\ a_N & a_1 & a_2 & \cdots & a_{N-1} \\ a_{N-1} & a_N & a_1 & \cdots & a_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}.$$

Show that if B is any other $N \times N$ circulant matrix, then $AB = BA$.

Question 18. A diagonal matrix is one for which every entry not on the main diagonal is zero. Let A and B be $N \times N$ matrices such that there exists and invertible $N \times N$ matrix P for which $D_A = P^{-1}AP$ and $D_B = P^{-1}BP$ are diagonal matrices. Show that A and B commute.