Math 240 Tutorial
Solutions

June 27

Question 1. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be an invertible linear transformation, and let $\vec{v}_1, \dots, \vec{v}_m \in \mathbf{R}^n$. Prove that $\{\vec{v}_1, \dots, \vec{v}_m\}$ is an independent set if and only if $\{T(\vec{v}_1), \dots, T(\vec{v}_m)\}$ is an independent set.

Suppose $\{\vec{v}_1, \dots, \vec{v}_m\}$ is an independent set but $\{T(\vec{v}_1), \dots, T(\vec{v}_m)\}$ is not. Then there are scalars $\alpha_1, \dots, \alpha_m \in \mathbf{R}$, not all 0, such that

$$\vec{0} = \alpha_1 T(\vec{v}_1) + \cdots + T(\vec{v}_m) = T(\alpha\vec{v}_1 + \cdots + \alpha_m\vec{v}_m).$$

Since T is invertible, it is 1-to-1. This means

$$\vec{0} = \alpha\vec{v}_1 + \cdots + \alpha_m\vec{v}_m.$$

This contradicts the assumed independence of $\{\vec{v}_1, \dots, \vec{v}_m\}$. So, $\{T(\vec{v}_1), \dots, T(\vec{v}_m)\}$ is also an independent set.

Conversely, suppose $\{\vec{v}_1, \dots, \vec{v}_m\}$ is not an independent set. Then there are scalars $\beta_1, \dots, \beta_m in \mathbf{R}$, not all 0, such that

$$\vec{0} = \alpha\vec{v}_1 + \cdots + \alpha_m\vec{v}_m.$$

Applying T to both sides,

$$\vec{0} = \alpha_1 T(\vec{v}_1) + \cdots + T(\vec{v}_m),$$

showing that $\{T(\vec{v}_1), \dots, T(\vec{v}_m)\}$ is also a dependent set.

Question 2. Define $T : \mathbf{R}^n \to \mathbf{R}^n$ by

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_2 - x_3 - x_4, -x_1 + x_2 - x_3 - x_4, -x_1 - x_2 + x_3 - x_4, -x_1 - x_2 - x_3 + x_4).$$

Is T linear? Is T invertible? If it is, what is its inverse?

The components of the image of $\vec{x}$ are linear combinations of the components of $\vec{x}$, so T is linear. The standard matrix for T is given by

$$T_A = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

which is readily verified to be invertible. Moreover, it is its own inverse. So $T^{-1} = T$.

Question 3. Show that if E and F are two $n \times n$ matrices such that $EF = I$, then E and F commute.

Since $EF = I$, E is invertible and $F = E^{-1}$ by the uniqueness of E^{-1}. Of course, $EE^{-1} = E^{-1}E = I$.

Question 4. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ and $U : \mathbf{R}^n \to \mathbf{R}^n$ be two linear transformations such that $T(U(\vec{x})) = \vec{x}$ for every $\vec{x} \in \mathbf{R}^n$. Show that T is invertible and $U = T^{-1}$.

By assumption, U is a right inverse of T. Since T is a linear operator of a finite dimensional vector space, it is invertible and $T^{-1} = U$.

Question 5. Show that if A is invertible, then $\det(A^{-1}) = 1/\det(A)$.

We rewrite the putative equality as $\det(A)\det(A^{-1}) = 1$. By the properties of the determinant,

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1}),$$

which proves the result.

Question 6. Let A, B, and P be $n \times n$ matrices where P is invertible and $B = P^{-1}AP$. Show that $\det(A) = \det(B)$.

Observe

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \frac{\det(A)\det(P)}{\det(P)} = \det(A)$$

where we have used the previous exercise.

Question 7. Let V be a vector space, and let H and K be subspaces of V. Show the following

(a) $H + K$ and $H \cap K$ are subapces.

Let $\vec{v}_1, \vec{v}_2 \in H + K$. Then there are vectors $\vec{h}_1, \vec{h}_2 \in H$ and $\vec{k}_1, \vec{k}_2 \in K$ such that $\vec{v}_1 = \vec{h}_1 + \vec{k}_1$ and $\vec{v}_2 = \vec{h}_2 + \vec{k}_2$. Then $\vec{v}_1 + \vec{v}_2 = (\vec{h}_1 + \vec{h}_2) + (\vec{k}_1 + \vec{k}_2)$. Since $\vec{h}_1 + \vec{h}_2 \in H$ and $\vec{k}_1 + \vec{k}_2 \in K$, we have $\vec{v}_1 + \vec{v}_2 \in H + K$. If $\alpha \in \mathbf{R}$, then $\alpha\vec{v} = \alpha\vec{h} + \alpha\vec{k}$. Since $\alpha\vec{h} \in H$ and $\alpha\vec{k} \in K$, $\alpha\vec{v} \in H + K$. Finally, $H + K \neq \emptyset$ since $\vec{0} \in H + K$. We have therefore shown that $H + K$ is a subspace of V.

Let $\vec{v}_1, \vec{v}_2 \in H \cap K$. Then $\vec{v}_1, \vec{v}_2 \in H$ and $\vec{v}_1, \vec{v}_2 \in K$. Therefore, $\vec{v}_1 + \vec{v}_2 \in H$ and $\vec{v}_1 + \vec{v}_2 \in K$ so that $\vec{v}_1 + \vec{v}_2 \in H \cap K$. Similarly, for $\alpha \in \mathbf{R}$, $\alpha\vec{v} \in H$ and $\alpha\vec{v} \in K$ so that $\alpha\vec{v} \in H \cap K$. Since $\vec{0} \in H$ and $\vec{0} \in K$, we have $\vec{0} \in H \cap K$ so that $H \cap K \neq \emptyset$. We have shown that $H \cap K$ is a subsapce of V.

(b) H and K are subspaces of $H + K$.

Note that $\vec{0} \in H$ and $\vec{0} \in K$. So, for $\vec{h} \in H$ and $\vec{k} \in K$, we have $\vec{h} + \vec{0} \in H + K$ and $\vec{0} + \vec{k} \in H + K$. This shows $H \subsetneq H + K$ and $K \subsetneq H + K$.

(c) $H \cap K$ is a subspace of both H and K.

We have already verified $H \cap K$ to be a subspace of the ambient space V, and clearly $H \cap K \subsetneq H$ and $H \cap K \subsetneq K$, whereupon $H \cap K$ is a linear subspace of both H and K.

Question 8. Let V be a vector space, and let W be a vector space of V. Recall that, for $\vec{v} \in V$, $\vec{v} + W = \{\vec{v} + \vec{w} : \vec{w} \in W\}$. Show the following.

(a) For distinct $\vec{v}_1, \vec{v}_2 \in V$, $\vec{v}_1 + W$ and $\vec{v}_2 + W$ are either disjoint or equal.

Assume $\vec{v}_1 + W \cap \vec{v}_2 + W \neq \emptyset$, and let $\vec{x} \in \vec{v}_1 + W \cap \vec{v}_2 + W$. Then there are $\vec{w}_1, \vec{w}_2 \in W$ such that $\vec{x} = \vec{v}_1 + \vec{w}_1 = \vec{v}_2 + \vec{w}_2$, whereupon $\vec{v}_1 = \vec{v}_2 + \vec{w}_3$ where $\vec{w}_3 = \vec{w}_2 - \vec{w}_1 \in W$. Let $\vec{v}_1 + \vec{w}$ be an arbitary element of $\vec{v}_1 + W$. Then $\vec{v}_1 + \vec{w} = \vec{v}_2 + (\vec{w}_3 + \vec{w})$ which shows that $\vec{v}_1 + W \subsetneq \vec{v}_2 + W$. Similarly, $\vec{v}_2 + W \subsetneq \vec{v}_1 + W$ so that $\vec{v}_1 + W = \vec{v}_2 + W$, as desired.

(b) $\vec{v}_1 + W = \vec{v}_2 + W$ if and only if $\vec{v}_1 - \vec{v}_2 \in W$.

Assume $\vec{v}_1 + W = \vec{v}_2 + W$. In particular, $\vec{v}_1 \in \vec{v}_2 + W$ so that there is a $\vec{w} \in W$ for which $\vec{v}_1 = \vec{v}_2 + \vec{w}$; in other words, $\vec{v}_1 - \vec{v}_2 = \vec{w} \in W$.

Conversely, suppose that $\vec{v}_1 - \vec{v}_2 = \vec{w}$ for some $\vec{w} \in W$. Then $\vec{v}_1 = \vec{v}_2 + \vec{w}$ so that $\vec{v}_1 \in \vec{v}_2 + W$. If $\vec{v}_1 + \vec{w}' \in \vec{v}_1 + W$ is arbitrary, then $\vec{v}_1 + \vec{w}' = \vec{v}_2 + (\vec{w} + \vec{w}') \in \vec{v}_2 + W$ since $\vec{w} + \vec{w}' \in W$. Therefore, $\vec{v}_1 + W \subseteq \vec{v}_2 + W$. Similarly, $\vec{v}_2 + W \subseteq \vec{v}_1 + W$, hence $\vec{v}_1 + W = \vec{v}_2 + W$.

(c) Every $\vec{v} \in V$ belongs to $\vec{u} + W$ for some $\vec{u} \in V$.

Since $\vec{0} \in W$, it follows that $\vec{v} = \vec{v} + \vec{0} \in \vec{v} + W$.

We can define an arithmetic on $H = \{\vec{v} + W : \vec{v} \in V\}$. For $\vec{v}_1 + W, \vec{v}_2 + W \in H$ and $\alpha \in \mathbf{R}$, define $(\vec{v}_1 + W) + (\vec{v}_2 + W) = (\vec{v}_1 + \vec{v}_2) + W$ and $\alpha(\vec{v}_1 + W) = (\alpha\vec{v}_1) + W$. Then:

(d) H is a vector under the arithmetic defined above. We call H the quotient space of V by W, and we denote it as $H = V/W$.