Math 240 Tutorial
Questions

July 5

Question 1. Consider the vector space $\mathbf{R}^3$, and let

$$H = \left\{ \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} : a \in \mathbf{R} \right\}.$$

Answer the following.

(a) Show that H is a subspace of $\mathbf{R}^3$.

(b) What is the dimension of H?

(c) Construct a basis for H.

Question 2. Let $\mathbf{P}_3$ be the vector space of all polynomials of degree at most 3, and let

$$H = \{p(x) \in \mathbf{P}_3 : p(3) = 0\}.$$

Answer the following.

(a) Show that H is a subspace of $\mathbf{P}_3$.

(b) What is the dimension of H?

(c) Construct a basis for H.

(d) Let $\mathbf{P}_2$ be the vector space of polynomials of degree at most 2. $\mathbf{P}_2$ is a subspace of $\mathbf{P}_3$ (why?). Give an invertible linear transformation that maps $\mathbf{P}_2$ onto H. What is the matrix for the transformation with respect to the standard basis of $\mathbf{P}_3$?

Question 3. Let m and n be positive integers. Show the following.

(a) The set $M_{m\times n}(\mathbf{R})$ of $m \times n$ matrices with real entries is a vector space.

(b) What is the dimension of $M_{m\times n}(\mathbf{R})$?

(c) Construct a basis for $M_{m\times n}(\mathbf{R})$.

(d) Show the subset of matrices with trace 0 forms a subspace of $M_{n\times n}(\mathbf{R})$. Use the Dimension Theorem to find the dimension of this subset. You will first need to show that the trace map is a linear map. Construct a basis for this subspace.

Question 4. Define

$$H = \left\{ \begin{pmatrix} u & -u-x \\ 0 & x \end{pmatrix} : u, x \in \mathbf{R} \right\},$$
$$K = \left\{ \begin{pmatrix} v & 0 \\ w & -v \end{pmatrix} \right\}.$$

Do the following.

(a) H and K are subspaces of $M_{2\times 2}(\mathbf{R})$.

(b) Construct bases for H, K, $H+K$, and $H\cap K$.

Question 5. Your course text proves the Dimension Theorem by counting pivot positions in matrices. Prove the theorem by arguing from the general definitions, without recourse to matrices, in the following way. Given a linear transformation $T : V \to W$, take a basis for ker(T) and enlarge it to a basis for V. Apply T to the vectors that were added to the basis of ker(T), and argue that they form a linearly independent set that spans the range of T in W.

Question 6. Use the Dimension Theorem to show a linear transformation $T : V \to V$ is invertible if and only if it is onto if and only if it is one-to-one.

Question 7. Let V be a vector space of finite dimension, and let H and K be subspaces of V. Show

$$\dim(H+K) = \dim(H) + \dim(K) - \dim(H\cap K).$$