Math 240 Tutorial
Solutions

July 25

Question 1. Let a finite dimensional vector space V have two bases β and β', and let Q be the transformation matrix from β'-coordinates to β-coordinates. Show that for any linear transformation $T : V \to V$, it holds that

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q.$$

We may write $Q = [I]_{\beta'}^{\beta}$. Then

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That is, $[T]_{\beta'} = Q^{-1}[T]_\beta Q$.

Question 2. A scalar matrix is a matrix of the form λI for some scalar λ.

(a) Prove that is a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.

Assume there is an invertible matrix P such that $P^{-1}AP = \lambda I$. Multiplying on the left by P and on the right by P^{-1}, we have $A = P(\lambda I)P^{-1} = \lambda I(PP^{-1}) = \lambda I$.

(b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.

If A has only one eigenvalue, then A is similar to a scalar matrix λI. From part (a), it follows that $A = \lambda I$.

(c) Prove that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is not diagonalizable.

The matrix has only one eigenvalue (namely, $\lambda = 1$). Therefore, if it were diagonalizable, it must be that it is the identity matrix from part (b). But clearly this is false, so the matrix is not diagonalizable.

Question 3. For each of the following linear operators T on a vector space V and ordered basis β, compute $[T]_\beta$ and determine whether β is a basis consisting of eigenvectors of T.

(a) $V = \mathbf{R}^2$, $T\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \left(\begin{smallmatrix} 10a-6b \\ 17a-10b \end{smallmatrix}\right)$, and $\beta = \{\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)\}$.

Let $\vec{v}_1 = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ and $\vec{v}_2 = \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$. Then

$$\vec{v}_1 \mapsto \begin{pmatrix} -2 \\ -3 \end{pmatrix} = -\vec{v}_2, \qquad \vec{v}_2 \mapsto \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\vec{v}_1.$$

So $[T]_\beta = \left(\begin{smallmatrix} 0 & 2 \\ -1 & 0 \end{smallmatrix}\right)$, and β is not a basis of eigenvalues for T.

(b) $V = \mathbf{P}_1(\mathbf{R})$, $T(a+bx) = (6a-6b)+(12a-11b)x$, and $\beta = \{3+4x, 2+3x\}$.

Let $\vec{v}_1 = 3+4x$ and $\vec{v}_2 = 2+3x$. Then

$$\vec{v}_1 \mapsto -6-8x = -2\vec{v}_1, \qquad \vec{v}_2 \mapsto -6-9x = -3\vec{v}_2.$$

So $[T]_\beta = \left(\begin{smallmatrix} -2 & 0 \\ 0 & -3 \end{smallmatrix}\right)$, and β is a basis of eigenvalues for T.

(c) $V = \mathbf{R}^3$, $T\left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right) = \left(\begin{smallmatrix} 3a+2b-2c \\ -4a-3b+2c \\ -c \end{smallmatrix}\right)$, and $\beta = \left\{\left(\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ -1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \\ 2 \end{smallmatrix}\right)\right\}$.

Let $\vec{v}_1 = \left(\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right)$, $\vec{v}_2 = \left(\begin{smallmatrix} 1 \\ -1 \\ 0 \end{smallmatrix}\right)$, and $\vec{v}_3 = \left(\begin{smallmatrix} 1 \\ 0 \\ 2 \end{smallmatrix}\right)$. Then

$$\vec{v}_1 \mapsto \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = -\vec{v}_1, \qquad \vec{v}_2 \mapsto \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \vec{v}_2, \qquad \vec{v}_3 \mapsto \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} = -\vec{v}_3.$$

So $[T]_\beta = \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$, and β is a basis of eigenvectors for T.

(d) $V = \mathbf{P}_2(\mathbf{R})$, $T(a+bx+cx^2) = (-4a+2b-2c)-(7a+3b+7c)x+(7a+b+5c)x^2$, and $\beta = \{x-x^2, -1+x^2, -1-x+x^2\}$.

Let $\vec{v}_1 = x-x^2$, $\vec{v}_2 = -1+x^2$, and $\vec{v}_3 = -1-x+x^2$. Then

$$\vec{v}_1 \mapsto 4+4x-4x^2 = -4\vec{v}_3, \quad \vec{v}_2 \mapsto 2-2x^2 = -2\vec{v}_2, \quad \vec{v}_3 \mapsto 3x-3x^2 = 3\vec{v}_1.$$

So $[T]_\beta = \left(\begin{smallmatrix} 0 & 0 & 3 \\ 0 & -2 & 0 \\ -4 & 0 & 0 \end{smallmatrix}\right)$, and β is not a basis of eigenvectors for T.

(e) $V = P_3(\mathbf{R})$, $T(a+bx+cx^2+dx^3) = -d+(-c+d)x+(a+b-2c)x^2+(-b+c-2d)x^3$, and $\beta = \{1-x+x^3, 1+x^2, 1, x+x^2\}$.

Let $\vec{v}_1 = 1-x+x^3$, $\vec{v}_2 = 2+x^2$, $\vec{v}_3 = 1$, and $\vec{v}_4 = x+x^2$. Then

$$\vec{v}_1 \mapsto -1+x-x^3 = -\vec{v}_1, \qquad \vec{v}_3 \mapsto x^2 = \vec{v}_2 - \vec{v}_3,$$
$$\vec{v}_2 \mapsto -x-x^2+x^3 = \vec{v}_1 - \vec{v}_2, \qquad \vec{v}_4 \mapsto -x-x^2 = -\vec{v}_4.$$

So

$$[T]_\beta = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and β is not a basis of eigenvectors for T.

(f) $V = \mathcal{M}_{2\times 2}(\mathbf{R})$, $T\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} -7a-4b+4c-4d & b \\ -8a-4b+5c-4d & d \end{smallmatrix}\right)$, and $\beta = \left\{\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 2 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 2 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & 2 \end{smallmatrix}\right)\right\}$.

Let $\vec{v}_1 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $\vec{v}_2 = \left(\begin{smallmatrix} -1 & 2 \\ 0 & 0 \end{smallmatrix}\right)$, $\vec{v}_3 = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 0 \end{smallmatrix}\right)$, and $\vec{v}_4 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. Then

$$\vec{v}_1 \mapsto \begin{pmatrix} -3 & 0 \\ -3 & 0 \end{pmatrix} = -3\vec{v}_1, \qquad \vec{v}_3 \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \vec{v}_3,$$
$$\vec{v}_2 \mapsto \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = \vec{v}_2, \qquad \vec{v}_4 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = \vec{v}_4.$$

So

$$[T]_\beta = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and β is a basis of eigenvectors of T.

Question 4. For each of the following matrices $A \in \mathcal{M}_{n\times n}(F)$:

(i) Determine all the eigenvalues of A.

(ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.

(iii) If possible, find a basis for F^n consisting of eigenvectors of A.

(iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(a) $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 2 \end{smallmatrix}\right)$ for $F = \mathbf{R}$.

We have $\det(tI - A) = t^2 - 3t - 4 = (t-4)(t+1)$, so the eigenvalues for A are $\lambda_1 = 4$ and $\lambda_2 = -1$.

For $\lambda_1 = 4$, we have

$$4I - A = \begin{pmatrix} 3 & -2 \\ -3 & 2 \end{pmatrix} \sim \begin{pmatrix} 3 & -2 \\ 0 & 0 \end{pmatrix}.$$

So $E_4 = \{\left(\begin{smallmatrix} 2a \\ 3a \end{smallmatrix}\right) : a \in \mathbf{R}\}$ is spanned by the eigenvector $\left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$.

For $\lambda_2 = -1$, we have

$$-I - A = \begin{pmatrix} -2 & -2 \\ -3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

So $E_{-1} = \{\left(\begin{smallmatrix} -a \\ a \end{smallmatrix}\right) : a \in \mathbf{R}\}$ is spanned by the eigenvector $\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)$.

We may take $Q = \left(\begin{smallmatrix} 2 & -1 \\ 3 & 1 \end{smallmatrix}\right)$ and $D = \left(\begin{smallmatrix} 4 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

(b) $A = \left(\begin{smallmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{smallmatrix}\right)$ for $F = \mathbf{R}$.

We have $\det(tI - A) = t^3 - 6t^2 + 11t - 6 = (t-3)(t-2)(t-1)$, so the eigenvalues for A are 1, 2, and 3.

For $t = 1$, we have

$$I - A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ -2 & -2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

So $E_1 = \left\{\left(\begin{smallmatrix} -a \\ -a \\ a \end{smallmatrix}\right) : a \in R\right\}$ is spanned by $\left(\begin{smallmatrix} -1 \\ -1 \\ 1 \end{smallmatrix}\right)$.

For $t = 2$, we have

$$2I - A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 1 & 1 \\ -2 & -2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

So $E_2 = \left\{ \begin{pmatrix} -a \\ a \\ 0 \end{pmatrix} : a \in \mathbf{R} \right\}$ is spanned by $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

For $t = 3$, we have

$$3I - A = \begin{pmatrix} 3 & 2 & 3 \\ 1 & 2 & 1 \\ -2 & -2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So $E_3 = \left\{ \begin{pmatrix} -a \\ 0 \\ a \end{pmatrix} : a \in \mathbf{R} \right\}$ is spanned by $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

We may take $Q = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(c) $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$ for $F = \mathbf{C}$.

We have $\det(tI - A) = t^2 - 1 = (t+1)(t-1)$, so the eigenvalues for A are 1 and -1.

For $t = 1$, we have

$$I - A = \begin{pmatrix} 1-i & -1 \\ -2 & 1+i \end{pmatrix} \sim \begin{pmatrix} 2 & -1-i \\ 0 & 0 \end{pmatrix}.$$

So $E_1 = \left\{ \begin{pmatrix} (1+i)a \\ 2a \end{pmatrix} : a \in \mathbf{R} \right\}$ is spanned by $\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$.

For $t = -1$, we have

$$-I - A = \begin{pmatrix} -1-i & -1 \\ -2 & -1+i \end{pmatrix} \sim \begin{pmatrix} 2 & 1-i \\ 0 & 0 \end{pmatrix}.$$

So $E_{-1} = \left\{ \begin{pmatrix} (1-i)a \\ 2a \end{pmatrix} : a \in \mathbf{R} \right\}$ is spanned by $\begin{pmatrix} 1-i \\ 2 \end{pmatrix}$.

We may take $Q = \begin{pmatrix} 1+i & 1-i \\ 2 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(d) $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$ for $F = \mathbf{R}$.

We have $\det(tI - A) = t^3 - 2t^2 + t = t(t-1)^2$, so the eigenvalues for A are 0 and 1 with multiplicities 1 and 2, respectively.

For $t = 0$, we have

$$-A = \begin{pmatrix} -2 & 0 & 1 \\ -4 & -1 & 4 \\ -2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

So $E_0 = \left\{ \begin{pmatrix} -a \\ 2a \\ a \end{pmatrix} : a \in \mathbf{R} \right\}$ is spanned by $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$.

For $t = 1$, we have

$$I - A = \begin{pmatrix} -1 & 0 & 1 \\ -4 & 0 & 4 \\ -2 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

so $E_1 = \left\{ \begin{pmatrix} a \\ b \\ a \end{pmatrix} : a, b \in \mathbf{R} \right\}$ is spanned by $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

We may take $Q = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Question 5. Prove the geometric multiplicity of an eigenvalue is at most the algebraic multiplicity.

Let A be an $n \times n$ matrix with eigenvalue λ where $\dim(E_\lambda) = p$. Let $\{\vec{v}_1, \ldots, \vec{v}_p\}$ be a basis of eigenvectors for the eigenspace E_λ, and enlarge this to a basis $\beta = \{\vec{v}_1, \ldots, \vec{v}_p, \vec{v}_{p+1}, \ldots, \vec{v}_n\}$ for $\mathbf{R}^n$ (or $\mathbf{C}^n$). Then

$$tI - [A]_\beta = \begin{pmatrix} (t-\lambda)I & A_2 \\ O & tI - A_3 \end{pmatrix}$$

It then follows (by induction) that $\det(tI - A) = (t-\lambda)^p \det(tI - A_3) = (t-\lambda)^p g(t)$, where $g(t)$ is a polynomial. If λ has algebraic multiplicty m, then since $(t-\lambda)^p$ is a factor of $\det(tI - A)$ it follows that $p \leqq m$.