# Math 240 Tutorial
# Solutions

**June 6**

## Systems of Linear Equations and Row Reduction

**Question 1.** Place the following augmented matrices into an echelon form. Does the corresponding system of linear equations admit any solutions?

(a)

$$\left(\begin{array}{cccc|c} 4 & 8 & 12 & 4 & 7 \\ 2 & 5 & 6 & 6 & 11 \\ 0 & 5 & 1 & 26 & 13 \\ 0 & 5 & 0 & 21 & 17 \end{array}\right).$$

An echelon form:

$$\left(\begin{array}{cccc|c} 2 & 5 & 6 & 6 & 11 \\ 0 & 2 & 0 & 8 & 15 \\ 0 & 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 2 & -41 \end{array}\right).$$

Every column except the last is a pivot column, so the system has a unique solution.

(b)

$$\left(\begin{array}{cccc|c} 4 & 8 & 12 & 4 & 0 \\ 2 & 5 & 6 & 6 & 0 \\ 0 & 5 & 1 & 25 & 0 \\ 0 & 5 & 0 & 20 & 0 \end{array}\right).$$

An echelon form:

$$\left(\begin{array}{cccc|c} 2 & 5 & 6 & 6 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The corresponding system has infinitely many solutions.

(c)

$$\left(\begin{array}{cccc|c} 4 & 8 & 12 & 4 & 7 \\ 2 & 5 & 6 & 6 & 11 \\ 0 & 5 & 1 & 25 & 13 \\ 0 & 5 & 0 & 20 & 17 \end{array}\right).$$

An echelon form:

$$\left(\begin{array}{cccc|c} 2 & 5 & 6 & 6 & 11 \\ 0 & 2 & 0 & 8 & 15 \\ 0 & 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 & 41 \end{array}\right).$$

Since the last column is a pivot column, the corresponding system of linear equations is inconsistent.

**Question 2.** Find the values of $k$ for which the system of equations

$$\begin{aligned} x + ky &= 1, \\ kx + y &= 1, \end{aligned}$$

has

(a) no solution,

The augmented matrix for the system of linear equations is given by

$$\left( \begin{array}{cc|c} 1 & k & 1 \\ k & 1 & 1 \end{array} \right).$$

If $k = 0$, then there is a unique solution given by $(x, y) = (1, 1)$; so, we assume that $k \neq 0$. The Gaussian form of the matrix is then

$$\left( \begin{array}{cc|c} 1 & k & 1 \\ 0 & \frac{1}{k} - k & \frac{1}{k} - 1 \end{array} \right)$$

If $\frac{1}{k} - k \neq 0$, that is, if $|k| \neq 1$, then there is a unique solution given by $(x, y) = ((1 + k)^{-1}, (1 + k)^{-1})$.

It remains to examine the case that $|k| = 1$. If $k = 1$, then we have the equation $x + y = 1$. This has infinitely many solutions. If $k = -1$, then we have the have the system

$$\begin{aligned} x - y &= 1, \\ -x + y &= 1. \end{aligned}$$

Adding these two equations gives $0 = 2$, a contradiction. Therefore, there is no solution only in the case that $k = -1$.

(b) a unique solution, and

From our work in part (a), there is a unique solution whenever $|k| \neq 1$.

(c) infinitely many solutions.

From our work in part (a), there are infinitely many solutions in the case that $k = 1$.

(d) When there is exactly one solution, what are the values of $x$ and $y$.

By part (a), this happens whenever $|k| \neq 1$. If $k = 0$, then $(x, y) = (1, 1)$. If $k \neq 0$ and $|k| \neq 1$, then

$$(x, y) = \left( \frac{1}{1 + k}, \frac{1}{1 + k} \right).$$

**Question 3.** Consider the following two systems of equations.

$$\begin{aligned} x+y+z &= 16, \\ x+2y+2z &= 11, \\ 2x+3y-4z &= 3, \end{aligned}$$

and

$$\begin{aligned} x+y+z &= 7, \\ x+2y+2z &= 10, \\ 2x+3y-4z &= 3. \end{aligned}$$

Solve both systems simultaneously by applying row reduction to an appropriate $3 \times 5$ matrix.

We consider the following matrix augmented by two columns

$$\left(\begin{array}{ccc|cc} 1 & 1 & 1 & 16 & 7 \\ 1 & 2 & 2 & 11 & 10 \\ 2 & 3 & -4 & 3 & 3 \end{array}\right).$$

Its reduced row echelon form is given by

$$\left(\begin{array}{ccc|cc} 1 & 0 & 0 & 21 & 4 \\ 0 & 1 & 0 & -\frac{59}{7} & 1 \\ 0 & 0 & 1 & \frac{24}{7} & 2 \end{array}\right).$$

Therefore, both systems have a unique solution. The first is given by $x = 21$, $y = -59/7$, and $z = 24/7$; and the second is given by $x = 4$, $y = 1$, and $z = 2$.

**Question 4.** Consider the following homogeneous system of linear equations where $a, b \in \mathbf{R}$ are constants.

$$\begin{aligned} x+2y &= 0, \\ ax+8y+3z &= 0, \\ by+5z &= 0. \end{aligned}$$

(a) Find a value for $a$ which makes it necessary to interchange rows during row reduction.

$a = 4$. Of course, we would need $b \neq 0$ as well.

(b) Suppose that $a$ does not have the value you found in part (a). Find a value for $b$ so that the system has a nontrivial solution.

$b = 40, -10$.

(c) Suppose that $a$ does not have the value you found in part (a) and that $b = 100$. Suppose further that $a$ is chosen so that the solution to the system is not unique. The general solution to the system is $(\alpha^{-1}z, -\beta^{-1}z, z)$ where $\alpha$ and $\beta$ are what?

Since we assume that $a \neq 4$, we can row reduce the coefficient matrix to

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{3}{8-2a} \\ 0 & 0 & 5-\frac{300}{8-2a} \end{pmatrix}.$$

In order for the system to have nontrivial solutions, we require $5 - \frac{300}{8-2a} = 0$, that is, $a = -26$. Therefore, the general solution is given by

$$x = \frac{1}{10}z, \qquad y = -\frac{1}{20}z$$

where $z$ is free. So, $\alpha = 10$ and $\beta = -20$.

## Spans of Collections of Vectors

**Question 5.** Consider the following three vectors in $\mathbf{R}^3$

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Show that $\mathbf{R}^3 = \operatorname{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

We will prove this result in two ways. First, observing that

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the result follows. If, however, we required more information in how a vector of $R^3$ is decomposable as a linear combination of $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$, we could note the following

$$\begin{pmatrix} 1 & 0 & 1 & a_1 \\ 1 & 1 & 0 & a_2 \\ 0 & 1 & 1 & a_3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2}a_1 + \frac{1}{2}a_2 - \frac{1}{2}a_3 \\ 0 & 1 & 0 & -\frac{1}{2}a_1 + \frac{1}{2}a_2 + \frac{1}{2}a_3 \\ 0 & 0 & 1 & \frac{1}{2}a_1 - \frac{1}{2}a_2 + \frac{1}{2}a_3 \end{pmatrix}.$$

This means that if $\left(\begin{smallmatrix} a_1 \\ a_2 \\ a_3 \end{smallmatrix}\right) \in \mathbf{R}^3$, then

$$\left(\frac{1}{2}a_1 + \frac{1}{2}a_2 - \frac{1}{2}a_3\right)\vec{u}_1 - \left(\frac{1}{2}a_1 - \frac{1}{2}a_2 - \frac{1}{2}a_3\right)\vec{u}_2 + \left(\frac{1}{2}a_1 - \frac{1}{2}a_2 + \frac{1}{2}a_3\right)\vec{u}_3 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

**Question 6.** Consider the following four vectors in $\mathbf{R}^4$ given by

$$\vec{v}_1 = \begin{pmatrix} +1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ +1 \\ -1 \\ -1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -1 \\ -1 \\ +1 \\ -1 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ +1 \end{pmatrix}.$$

(a) Show whether $\vec{v}_1 \in \operatorname{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$ or not by solving the corresponding system of linear equations.

Observe

$$\left(\begin{array}{rrr|r} -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

so that $\vec{v}_1 \notin \operatorname{span}\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$.

(b) Let $a_1, a_2, a_3, a_4 \in \mathbf{R}$. Under what conditions on $a_1, a_2, a_3, a_4$ is $a_1\vec{v}_1+a_2\vec{v}_2+a_3\vec{v}_3+a_4\vec{v}_4 = \vec{0}$ true?

We have

$$\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so that only the solution $a_1 = a_2 = a_3 = a_4 = 0$ exists.

(c) How can we use part (b) to provide a second proof of part (a)? Can you generalize to answer the following question: Is $\vec{v}_i \in \operatorname{span}\{\vec{v}_j, \vec{v}_k, \vec{v}_l\}$ for $i, j, k, l$ distinct?

We show only the second question (the first being a special case of the second). We will use proof by contradiction. Assume to the contrary that there exists $a_j, a_k, a_l \in \mathbf{R}$ not all zero such that $\vec{v}_i = a_j\vec{v}_j + a_k\vec{v}_k + a_l\vec{v}_l$. But then $\vec{v}_i - a_j\vec{v}_j - a_k\vec{v}_k - a_l\vec{v}_l = \vec{0}$. From our answer to part (b), it follows that $a_j = a_k = a_l = 0$ and $1 = 0$. But $1 = 0$ is a contradiction. Therefore, our original assumption that $\vec{v}_i \in \operatorname{span}\{\vec{v}_j, \vec{v}_k, \vec{v}_l\}$ is incorrect. That is, we must have $\vec{v}_i \notin \operatorname{span}\{\vec{v}_j, \vec{v}_k, \vec{v}_l\}$

**Question 7.** Let $V_1$ and $V_2$ be two subsets of $\mathbf{R}^n$, and define $V_1+V_2 = \{\vec{v}_1+\vec{v}_2 : \vec{v}_1 \in V_1 \text{ and } \vec{v}_2 \in V_2\}$. Show (a) $\operatorname{span}(V_1 \cup V_2) = \operatorname{span}(V_1) + \operatorname{span}(V_2)$, and (b) $\operatorname{span}(V_1 \cap V_2) \subseteq \operatorname{span}(V_1) \cap \operatorname{span}(V_2)$. Further, give an example of subsets $V_1$ and $V_2$ of $\mathbf{R}^n$, for some $n$, for which $\operatorname{span}(V_1 \cap V_2) \subsetneq \operatorname{span}(V_1) \cap \operatorname{span}(V_2)$.

Throughout, let $V_1 \cap V_2 = \{w_1, w_2, \ldots, w_p\}$, $V_1 = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-p}, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_p\}$, $V_2 = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{m-p}, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_p\}$.

(a) Let $\vec{x} \in \operatorname{span}(V_1 \cup V_2)$. Then there exists scalars $a_1, a_2, \ldots, a_{n-p} \in \mathbf{R}$ and $b_1, b_2, \ldots, b_{m-p} \in \mathbf{R}$ and $c_1, c_2, \ldots, c_p \in \mathbf{R}$ such that

$$\vec{x} = \sum_{i=1}^{n-p} a_i\vec{v}_i + \sum_{j=1}^{m-p} b_j\vec{u}_j + \sum_{k=1}^{p} c_k\vec{w}_k.$$

But

$$\sum_{i=1}^{n-p} a_i\vec{v}_i + \sum_{k=1}^{p} c_k\vec{w}_k \in \operatorname{span}(V_1)$$

and

$$\sum_{j=1}^{m-p} b_j\vec{u}_j \in \operatorname{span}(V_2).$$

Therefore, $\vec{x} \in \text{span}(V_1) + \text{span}(V_2)$, and $\text{span}(V_1 \cup V_2) \subseteqq \text{span}(V_1) + \text{span}(V_2)$. Conversely, let $\vec{y} \in \text{span}(V_1)+\text{span}(V_2)$. Then there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbf{R}$ and $\beta_1, \beta_2, \ldots, \beta_m \in \mathbf{R}$ such that

$$y = \sum_{i=1}^{n-p} \alpha_i \vec{v}_i + \sum_{j=1}^{p} (\alpha_j + \beta_j)\vec{w}_j + \sum_{k=1}^{m-p} \beta_k \vec{u}_k.$$

Since $V_1 \cup V_2 = \{\vec{v}_1, \ldots, \vec{v}_n, \vec{u}_1, \ldots, \vec{u}_m, \vec{w}_1, \ldots, \vec{w}_p\}$, we have that $\vec{y} \in \text{span}(V_1 \cup V_2)$, and $\text{span}(V_1 \cup V_2) \supseteqq \text{span}(V_1) + \text{span}(V_2)$.

(b) Let $\vec{x} \in \text{span}(V_1 \cap V_2)$. Then there exists scalars $a_1, a_2, \ldots, a_p$ such that

$$\vec{x} = a_1\vec{w}_1 + a_2\vec{w}_2 + \cdots + a_p\vec{w}_p.$$

Since $V_1 \cap V_2 \subseteqq V_1$ and $V_1 \cap V_2 \subseteqq V_2$, we see at once that $\vec{x}$ is in both $\text{span}(V_1)$ and $\text{span}(V_2)$, that is, $\vec{x} \in \text{span}(V_1) \cap \text{span}(V_2)$, as desired. Any number of counter examples can be found to show that $\text{span}(V_1 \cap V_2) = \text{span}(V_1) \cap \text{span}(V_2)$ is not true in general. It isn't difficult to see that $V_1 = \{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$ and $V_2 = \{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\}$ give a counter example. Since then $\text{span}(V_1 \cap V_2) = \{\vec{0}\}$ and $\text{span}(V_1) \cap \text{span}(V_2) = \text{span}(V_2) \neq \{\vec{0}\}$.

## Linear Independence

**Question 8.** Show that in $\mathbf{R}^3$, the vectors $\vec{x} = (1, 1, 0)$, $\vec{y} = (0, 1, 2)$, and $\vec{z} = (3, 1, -4)$ are linearly dependent by finding scalars $\alpha$ and $\beta$ such that $\alpha\vec{x} + \beta\vec{y} + \vec{z} = \vec{0}$.

We solve the system $\alpha\vec{x} + \beta\vec{y} = -\vec{z}$ to find $\alpha = -3$ and $\beta = 2$.

**Question 9.** Let $\vec{w} = (1, 1, 0, 0)$, $\vec{x} = (1, 0, 1, 0)$, $\vec{y} = (0, 0, 1, 1)$, and $\vec{z} = (0, 1, 0, 1)$, and let $S = \{\vec{w}, \vec{x}, \vec{y}, \vec{z}\}$.

(a) Show that $S$ is not a spanning set for $\mathbf{R}^4$ by finding a vector $\vec{u}$ in $\mathbf{R}^4$ such that $\vec{u} \notin \text{span}(S)$. One such vector is $\vec{u} = (1, 2, 3, a)$ where $a$ is any real number execpt what?

We form the matrix $A$ whose columns are $\vec{w}$, $\vec{x}$, $\vec{y}$, and $\vec{z}$. The reduced row echelon form for $A$ is then

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

whereupon we see that $\text{span}\{\vec{w}, \vec{x}, \vec{y}, \vec{z}\} = \text{span}\{\vec{w}, \vec{x}, \vec{y}\}$. We can then check that the system

$$\alpha\vec{w} + \beta\vec{x} + \gamma\vec{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ a \end{pmatrix}$$

is inconsistent precisely in the case that $a = 4$. Therefore, the vector $(1, 2, 3, 4)^t$ is not in the span of $\vec{w}$, $\vec{x}$, $\vec{y}$, and $\vec{z}$ so that they do not span $\mathbf{R}^4$.

(b) Show that $S$ is a linearly dependent set of vectors by finding scalars $\alpha$, $\gamma$, and $\delta$ such that $\alpha\vec{w} + \vec{x} + \gamma\vec{y} + \delta\vec{z} = \vec{0}$.

Solving the system $\alpha\vec{w} + \gamma\vec{y} + \delta\vec{z} = -\vec{x}$, we find that there is a unique solution given by $\alpha = \gamma = -1$ and $\delta = 1$.

(c) Show that $S$ is a linear dependent set by writing $\vec{z}$ as a linear combination of the remaining vectors in $S$.

From the reduced row echelon form for the matrix $A$ in part (a), we see that $\vec{z} = \vec{w} - \vec{x} + \vec{y}$.

**Question 10.** Let $S_1$ and $S_2$ be finite subsets of $\mathbf{R}^n$, for some $n$, such that $S_1 \subseteqq S_2$. Prove that if $S_1$ is a linearly dependent set, then so is $S_2$. Show that this is equivalent to if $S_2$ is a linearly independent set, then so is $S_1$.

Let $S_1$ and $S_2$ be finite subsets of $\mathbf{R}^n$ such that $S_1 \subseteqq S_2$. Suppose that $S_1$ is linearly dependent, but $S_2$ is linearly independent. For definiteness, we write

$$S_1 = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$$

and

$$S_2 = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m, \vec{v}_{m+1}, \dots, \vec{v}_k\}.$$

By assumption, there is a collection $a_1, a_2, \dots, a_m$ of scalars such that $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m = \vec{0}$ but $a_i \neq 0$ for some $i \in \{1, 2, \dots, m\}$. Set $a_{m+1} = a_{m+2} = \cdots = a_k = 0$ so that

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m + a_{m+1}\vec{v}_{m+1} + \cdots + a_k\vec{v}_k = \vec{0}.$$

By our assumption that $S_2$ is linear independent, we have

$$a_1 = a_2 = \cdots = a_m = a_{m+1} = \cdots = a_k = 0$$

by the definition of linearly independent. But this implies that each $a_i = 0$ for $i \in \{1, 2, \dots, m\}$, contrary to what we have observed. This contradiction proves the result.

**Question 11.** Do the following.

(a) Let $\vec{u}$ and $\vec{v}$ be distinct vectors in $\mathbf{R}^n$. Prove that $\{\vec{u}, \vec{v}\}$ is linearly independent if and only if $\{\vec{u} + \vec{v}, \vec{u} - \vec{v}\}$ is linearly independent.

Suppose first that $\{\vec{u}, \vec{v}\}$ is linearly independent, but $\vec{u} + \vec{v} = a(\vec{u} - \vec{v})$ for some $a \in \mathbf{R}$. If $a = 1$, then $\vec{v} = \vec{0}$, contradicting the fact that $\{\vec{u}, \vec{v}\}$ is independent. So $a \neq 1$ and $\vec{u} = -\frac{1+a}{1-a}\vec{v}$, again, contradicting the fact that $\{\vec{u}, \vec{v}\}$ is independent.

Conversely, suppose that $\vec{u} = b\vec{v}$ for some $b \in \mathbf{R}$. Then $\vec{u} + \vec{v} = (1 + b)\vec{u}$ and $\vec{u} - \vec{v} = (1 - b)\vec{u}$. But this means that both $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are in span$\{\vec{u}\}$, whereupon $\{\vec{u} + \vec{v}, \vec{u} - \vec{v}\}$ is linearly dependent.

(b) Let $\vec{u}, \vec{v}, \vec{w}$ be distinct vectors in $\mathbf{R}^n$. Prove that $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent if and only if $\{\vec{u} + \vec{v}, \vec{u} + \vec{w}, \vec{v} + \vec{w}\}$ is linear independent.

First, we assume that $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent, and we assume there are scalars $a, b, c$ such that

$$\vec{0} = a(\vec{u} + \vec{v}) + b(\vec{u} + \vec{w}) + c(\vec{v} + \vec{w}) = (a + b)\vec{u} + (a + c)\vec{v} + (b + c)\vec{w}.$$

Since we are assuming that $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent, it must be that $a + b = a + c = b + c = 0$. This linear system has only the trivial solution $a = b = c = 0$, whereupon $\{\vec{u} + \vec{v}, \vec{u} + \vec{w}, \vec{v} + \vec{w}\}$ is linearly independent as well.

Next, we assume that $\{\vec{u}+\vec{v}, \vec{u}+\vec{w}, \vec{v}+\vec{w}\}$ is independent. Suppose there are scalars $a$, $b$, $c$ such that $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$. Then

$$(a - b + c)(\vec{u} + \vec{v}) + (a + b - c)(\vec{u} + \vec{w}) + (-a + b + c)(\vec{v} + \vec{w}) = \vec{0}.$$

By our assumed linear independence of $\{\vec{u} + \vec{v}, \vec{u} + \vec{w}, \vec{v} + \vec{w}\}$, we have $a - b + c = a + b - c = -a + b + c = 0$. Solving, we find this possesses only the trivial solution $a = b = c = 0$. It follows $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent as well.

## Linear Transformations

**Question 12.** Show the following for $\mathbf{R}^n$.

(a) Show that scalar multiplication is a linear transformation.

Fix $a \in \mathbf{R}$, and let $T : \mathbf{R}^n \to \mathbf{R}^n$ be the map $T(\vec{v}) = a\vec{v}$. Then $T(b\vec{v}) = ab\vec{v} = ba\vec{v} = bT(\vec{v})$ and $T(\vec{v} + \vec{u}) = a(\vec{v} + \vec{u}) = a\vec{v} + a\vec{u} = T(\vec{v}) + T(\vec{u})$. We have shown that $T$ is linear.

(b) When is this linear map invertible?

This map is invertible precisely in the case the scalar by which we are multiplying is nonzero.

(c) Is its inverse a linear transformation?

Let $T$ be as in part (a), and assume that $a \neq 0$. Then $T$ is invertible and $T^{-1}$ is given by multiplpcation by $a^{-1}$. Since this is multiplpcation by a scalar, it is linear.

(d) Fix an element $a \in \mathbf{R}^n$. What is the matrix corresponding to the linear transformation $\vec{v} \mapsto a\vec{v}$ with respect to the standard spanning vectors?

Recall the standard spanning vectors are $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ where $\vec{e}_i$ is the vector with a 1 in position $i$ and zeros everywhere else. Then the matrix corresponding to multiplication by $a$ is givn by

$$[T(\vec{e}_1)|T(\vec{e}_2)| \cdots |T(\vec{e}_n)] = [a\vec{e}_1|a\vec{e}_2| \cdots |a\vec{e}_n] = aI.$$

**Question 13.** Fix $a \in \mathbf{R}$ and $\vec{u} \in \mathbf{R}^n$ with $\vec{u} \neq \vec{0}$. Is the map given by $\vec{v} \mapsto a\vec{v} + \vec{u}$, linear? Why or why not?

No; it is not a linear map. Let $\vec{v}_1$, $\vec{v}_2 \in \mathbf{R}^n$. Then $\vec{v}_1 + \vec{v}_2 \mapsto a\vec{v}_1 + a\vec{v}_2 + \vec{u}$. But $\vec{v}_1 \mapsto a\vec{v}_1 + \vec{u}$ and $\vec{v}_2 \mapsto a\vec{v}_2 + \vec{u}$. However, $(a\vec{v}_1 + \vec{u}) + (a\vec{v}_2 + \vec{u}) = a\vec{v}_1 + a\vec{v}_2 + 2\vec{u} \neq a\vec{v}_1 + a\vec{v}_2 + \vec{u}$.

**Question 14.** Consider a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$, and define $\mathrm{Ker}(T) = \{\vec{v} \in \mathbf{R}^n : T(\vec{v}) = \vec{0}\}$. This is the kernel of the linear transformation $T$. For $\vec{v} \in \mathbf{R}^n$, define $\vec{v} + \mathrm{Ker}(T) = \{\vec{v} + \vec{u} : \vec{u} \in \mathrm{Ker}(T)\}$. Show the following.

(a) $\text{Ker}(T)$ is closed under scalar multiplcation and vector addition.

Let $\vec{v}_1,\ \vec{v}_2 \in \text{Ker}(T)$. Then $T(\vec{v}_1+\vec{v}_2) = T(\vec{v}_1)+T(\vec{v}_2) = \vec{0}+\vec{0} = \vec{0}$, so $\vec{v}_1+\vec{v}_2 \in \text{Ker}(T)$. Similarly, $T(a\vec{v}) = aT(\vec{v}) = a\vec{0} = \vec{0}$ whenever $\vec{v} \in \text{Ker}(T)$; so, $a\vec{v} \in \text{Ker}(T)$. This shows that $\text{Ker}(T)$ is closed under vector addition and scalar multiplication.

(b) For $\vec{v} \in \mathbf{R}^n$, show that $\vec{v}+\text{Ker}(T)$ consists of all and only those elements of $\mathbf{R}^n$ that map to $T(\vec{v})$ under $T$.

Let $V = \{u \in \mathbf{R}^n : T(\vec{u}) = T(\vec{v})\}$. We show $\vec{v}+\text{Ker}(T) = V$. Indeed, let $\vec{u} \in V$. Then $T(\vec{v}) = T(\vec{u})$ so that $T(u-v) = \vec{0}$. It follows that there is an $\vec{x} \in \text{Ker}(T)$ such that $\vec{u}-\vec{v} = \vec{x}$, that is, $\vec{u} = \vec{v}+\vec{x}$. This means $\vec{u} \in \vec{v}+\text{Ker}(T)$.

Conversely, let $\vec{u} \in \vec{v}+\text{Ker}(T)$. Then there is some $\vec{x} \in \text{Ker}(T)$ for which $\vec{u} = \vec{v}+\vec{x}$. It follows that $T(\vec{u}) = T(\vec{v}+\vec{x}) + T(\vec{v}) + T(\vec{x}) = T(\vec{v})+\vec{0} = T(\vec{v})$. So, $\vec{u} \in V$, as required.

(c) For $\vec{v}_1, \vec{v}_2 \in \mathbf{R}^n$, show that either $\vec{v}_1+\text{Ker}(T) = \vec{v}_2+\text{Ker}(T)$ or $\vec{v}_1+\text{Ker}(T)\cap\vec{v}_2+\text{Ker}(T) = \emptyset$.

Let $\vec{v}_1,\ \vec{v}_2 \in \mathbf{R}^n$, and suppose that $\vec{v}_1+\text{Ker}(T)\cap\vec{v}_2+\text{Ker}(T) \neq \emptyset$. Then there is some $\vec{u} \in v_1+\text{Ker}(T)\cap\vec{v}_2+\text{Ker}(T)$. By definition, there must be $\vec{u}_1,\ \vec{u}_2 \in \text{Ker}(T)$ such that $\vec{u} = \vec{v}_1+\vec{u}_1 = \vec{v}_2 = \vec{u}_2$. But then $\vec{v}_1-\vec{v}_2 = \vec{u}_2-\vec{u}_1 = \vec{u}_3$ for some $\vec{u}_3 \in \text{Ker}(T)$. It follows that $\vec{v}_1 = \vec{v}_2+\vec{u}_3$, so $\vec{v}_1 \in \vec{v}_2+\text{Ker}(T)$. Similarly, $\vec{v}_2 = \vec{v}_1-\vec{u}_3$, so $\vec{v}_2 \in \vec{v}_1+\text{Ker}(T)$. It follows, therefore, that $\vec{v}_1+\text{Ker}(T) = \vec{v}_2+\text{Ker}(T)$.

## Matrix Operations

**Question 15.** Give an example of a nonzero matrix $A$ such that $A^2 = O$.

Take $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

**Question 16.** The trace of a square matrix $A$ of dimensions $N \times N$ is defined as $\text{tr}(A) = \sum_{k=1}^{N} A_{k,k}$, i.e., the sum of the diagonal entries of the matrix. For any other $N \times N$ matrix $B$, show that $\text{tr}(AB) = \text{tr}(BA)$.

Observe

$$\begin{aligned}
\text{tr}(AB) &= \sum_{k=1}^{N} (AB)_{k,k} \\
&= \sum_{k=1}^{N}\sum_{j=1}^{N} A_{k,j}B_{j,k} \\
&= \sum_{k=1}^{N}\sum_{j=1}^{N} A_{j,k}B_{k,j} \\
&= \text{tr}(BA)
\end{aligned}$$

where the second to last equality follows because

$$\{(k,j,j,k) : 1 \leqq j,k \leqq N\} = \{(j,k,k,j) : 1 \leqq j,k \leqq N\}.$$

**Question 17.** An $N \times N$ matrix $A$ is circulant if it is of the form

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_N \\ a_N & a_1 & a_2 & \cdots & a_{N-1} \\ a_{N-1} & a_N & a_1 & \cdots & a_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}.$$

Show that if $B$ is any other $N \times N$ circulant matrix, then $AB = BA$.

Define the $N \times N$ matrix $G$ by

$$G = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

If we multiply an $N \times N$ matrix $C$ on the right by $G$, the resulting matrix is the one obtained by cyclically shifting the columns of $C$ to the right. In particular, $G^N = I$ and $G^j \neq I$ for any $j \in \{1, \ldots, N-1\}$. We also note that we can write $A$ and $B$ by

$$A = \sum_{i=1}^{N} a_i G^{i-1}, \qquad B = \sum_{i=1}^{N} b_i G^{i-1},$$

that is, $A$ and $B$ are polynomials in $G$. Since they are each polynomials in $G$, it is easy to see that they must commute.

**Question 18.** A diagonal matrix is one for which every entry not on the main diagonal is zero. Let $A$ and $B$ be $N \times N$ matrices such that there exists and invertible $N \times N$ matrix $P$ for which $D_A = P^{-1}AP$ and $D_B = P^{-1}BP$ are diagonal matrices. Show that $A$ and $B$ commute.

Since $D_A = P^{-1}AP$ and $D_B = P^{-1}BP$, we have that $A = PD_AP^{-1}$ and $B = PD_BP^{-1}$. Then

$$\begin{aligned} AB &= (PD_AP^{-1})(PD_BP^{-1}) \\ &= PD_A(P^{-1}P)D_BP^{-1} \\ &= PD_AID_BP^{-1} \\ &= PD_AD_BP^{-1} \\ &= PD_BD_AP^{-1} \\ &= PD_BID_AP^{-1} \\ &= PD_B(P^{-1}P)D_AP^{-1} \\ &= (PD_BP^{-1})(PD_AP^{-1}) \\ &= BA \end{aligned}$$

where we have used the fact that diagonal matrices commute.