Question. Let m and n be positive integers such that $m \geqq n$, and let $a_1, \ldots, a_m, c \in \mathbf{R}$ with $c \neq 0$ and $\text{span}\{a_1, \ldots, a_m\} = \mathbf{R}^n$. Prove there exists an $i \in \{1, \ldots, m\}$ for which

$$\text{span}\{a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_m\} = \mathbf{R}^n.$$

Solution. We will prove the result by contradiction. The logical form of the proposition is

$$(\forall m \geqq n)(\forall a_1, \ldots, a_m, c \in \mathbf{R}^n)\Big[\text{span}\{a_1, \ldots, a_m\} = \mathbf{R}^n \Rightarrow \Big(c \neq 0 \Rightarrow (\exists i \in \{1, \ldots, m\})(\text{span}\{a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_m\} = \mathbf{R}^n)\Big)\Big]$$

To prove the result by contradiction, we assume the logical negation of the statement. This is given by

$$(\exists m \geqq n)(\exists a_1, \ldots, a_m, c \in \mathbf{R}^n)\Big[\text{span}\{a_1, \ldots, a_m\} = \mathbf{R}^n \ \& \ c \neq 0 \ \& \ (\forall i \in \{1, \ldots, m\})(\text{span}\{a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_m\} \neq \mathbf{R}^n)\Big]$$

So, to prove the result by contradiction, we assume there are positive integers $m \geqq n$ such that there exists $a_1, \ldots, a_m, c \in \mathbf{R}^n$ for which $\text{span}\{a_1, \ldots, a_m\} = \mathbf{R}^n$ and $c \neq 0$ and, for every index $i \in \{1, \ldots, m\}$,

$$\text{span}\{a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_m\} \neq \mathbf{R}^n.$$

Choose an arbitrary $i \in \{1, \ldots, m\}$. By assumption, there is some $v \in \mathbf{R}^n$ such that $v \notin \text{span}\{a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_m\}$. Since $\text{span}\{a_1, \ldots, a_m\} = \mathbf{R}^n$, there are scalars $\alpha_1, \ldots, \alpha_m \in \mathbf{R}$ such that

$$v = \alpha_1 a_1 + \cdots + \alpha_i a_i + \cdots + \alpha_m a_m.$$

If $\alpha_i = 0$, then

$$v \in \text{span}\{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m\} \subseteqq \text{span}\{a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_m\}$$

which contradicts our assumptions about v; so, $\alpha_i \neq 0$.

Since $c \in \mathbf{R}^n = \text{span}\{a_1, \ldots, a_m\}$, there are scalars $\beta_1, \ldots, \beta_m \in \mathbf{R}$ such that

$$c = \beta_1 a_1 + \cdots + \beta_i a_i + \cdots + \beta_m a_m.$$

Assume that $\beta_i \neq 0$. Then there is a unique $x \in \mathbf{R}$ such that $x \neq 0$ and $\alpha_i = x\beta_i$. But then

$$\begin{aligned}
&(\alpha_1 - x\beta_1)a_1 + \cdots + (\alpha_{i-1} - x\beta_{i-1})a_{i-1} + xc + (\alpha_{i+1} - x\beta_{i+1})a_{i+1} + \cdots + (\alpha_m - x\beta_m)a_m \\
&= \alpha_1 a_1 + \cdots + \alpha_{i-1}a_{i-1} + x\beta_i a_i + \alpha_{i+1}a_{i+1} + \cdots + \alpha_m a_m \\
&= \alpha_1 a_1 + \cdots + \alpha_{i-1}a_{i-1} + \alpha_i a_i + \alpha_{i+1}a_{i+1} + \cdots + \alpha_m a_m \\
&= v
\end{aligned}$$

which expresses v as a linear combination of $a_1, \ldots, a_{i-1}, c, a_{i+1} \ldots, a_m$, a contradiction. Thus, $\beta_i = 0$. Since i was arbitrary, $\beta_i = 0$ for every $i \in \{1, \ldots, m\}$. But this means that $c = 0$, contrary to our assumption. Therefore, our assumption that the proposition was false is incorrect, that is, it must be true.