Math 240 Tutorial
Questions

May 30

Question 1. Consider the set $S = \{(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4), (-1, 0, 1, 0)\}$ of vectors in $\mathbf{R}^4$. Show they form a linearly dependent set, and exress on vector as a linear combination of the others.

Question 2. Consider the set $S = \{(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1), (0, 0, 0, 1)\}$ of vectors in $\mathbf{R}^4$. Show they form a linearly independent set. For a general vector $(a_1, a_2, a_3, a_4) \in \mathbf{R}^4$, derive the coefficients for this vector when it is expanded as a linear combination of the vectors in S.

Question 3. Let S_1 and S_2 be finite subsets of $\mathbf{R}^n$, for some n, such that $S_1 \subsetneq S_2$. Prove that if S_1 is a linearly dependent set, then so is S_2. Show that this is equivalent to if S_2 is a linearly independent set, then so is S_1.

Question 4. Let S be a linearly independent set of $\mathbf{R}^n$, and let $\vec{v}$ be a vector in $\mathbf{R}^n$ that is not in S. Prove that $S \cup \{\vec{v}\}$ is linearly dependent if and only if $\vec{v} \in \text{span}(S)$.

Question 5. Do the following.

(a) Let $\vec{u}$ and $\vec{v}$ be distinct vectors in $\mathbf{R}^n$. Prove that $\{\vec{u}, \vec{v}\}$ is linear independent if and only if $\{\vec{u} + \vec{v}, \vec{u} - \vec{v}\}$ is linearly independent.

(b) Let $\vec{u}, \vec{v}, \vec{w}$ be distinct vectors in $\mathbf{R}^n$. Prove that $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent if and only if $\{\vec{u} + \vec{v}, \vec{u} + \vec{w}, \vec{v} + \vec{w}\}$ is linear independent.

Question 6. Show the following for $\mathbf{R}^n$.

(a) Show that scalar multiplication is a linear transformation.

(b) When is this linear map invertible?

(c) Is its inverse a linear transformation?

(d) Fix an element $a \in \mathbf{R}^n$. What is the matrix corresponding to the linear transformation $\vec{v} \mapsto a\vec{v}$?

Question 7. Fix $a \in \mathbf{R}$ and $\vec{u} \in \mathbf{R}^n$. Is the map given by $\vec{v} \mapsto a\vec{v} + \vec{u}$ linear? Why or why not?

Question 8. Consider a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$, and define $\text{Ker}(T) = \{\vec{v} \in \mathbf{R}^n : T(\vec{v}) = \vec{0}\}$. This is the kernel of the linear transformation T. For $\vec{v} \in \mathbf{R}^n$, define $\vec{v} + \text{Ker}(T) = \{\vec{v} + \vec{u} : \vec{u} \in \text{Ker}(T)\}$. Show the following.

(a) $\text{Ker}(T)$ is closed under scalar multiplcation and vector addition.

(b) For $\vec{v} \in \mathbf{R}^n$, show that $\vec{v} + \text{Ker}(T)$ consists of all and only those elements of $\mathbf{R}^n$ that map to $\vec{v}$ under T.

(c) For $\vec{v}_1, \vec{v}_2 \in \mathbf{R}^n$, show that either $\vec{v}_1 + \text{Ker}(T) = \vec{v}_2 + \text{Ker}(T)$ or $\vec{v}_1 + \text{Ker}(T) \cap \vec{v}_2 + \text{Ker}(T) = \emptyset$.